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Value-ranges, Julius Caesar and Indeterminacy

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VALUE-RANGES, JULIUS CAESAR AND INDETERMINACY

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For Raul

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"My efforts to throw light on the questions surrounding
the word 'number' and the words and signs for
individual numbers seem to have ended in complete failure.
Still these efforts have not been wholly in vain.
Precisely because they have failed,
we can learn something from them."
(Frege, *Ns*)

Resumo

As *Grundgesetze der Arithmetik* de Frege é o livro que contém a versão final do sistema formal desenvolvido para provar a sua tese de que aritmética é redutível à lógica. A fim de evitar a indeterminação levantada pelo problema Julius Caesar, a mais fundamental questão filosófica encontrada pelo seu logicismo, Frege é levado a definir os números como extensões de conceitos, e, com isto, introduzir o Axioma V em seu sistema para governar a noção de percurso de valores. Porém, no parágrafo 10 do livro, Frege encontra um novo problema de indeterminação, a saber, o fato de que o Axioma V não determina a referência dos nomes de percurso de valores. Para resolver este problema, Frege executa a identificação trans-sortal, que é a identificação dos valores de verdade com percursos de valores de funções particulares. Entretanto, porque a identificação não nos fornece uma determinação tão completa quanto a que deveríamos esperar de seu famoso princípio da completa determinação (ela não permite decidir se Julius Caesar é um percurso de valores), estudiosos como, principalmente, Dummett (1981) e Wright (1983), têm afirmado que Frege foi, afinal, incapaz de resolver o problema Julius Caesar em uma versão persistente. O objetivo desta dissertação se assenta em duas vertentes. Primeiro, queremos propor uma interpretação, sugerida por Greimann (2003), para conciliar a identificação trans-sortal de Frege com o seu comprometimento com o princípio da completa determinação. Segundo, queremos concluir, acompanhando Ruffino (2002), que não há problema Julius Caesar para percurso de valores.

Palavras-chaves: Percurso de Valores. Problema Julius Caesar. Problema da Indeterminação.

Abstract

Frege's *Grundgesetze der Arithmetik* is the book that sets forth the final version of the formal system developed to prove his thesis that arithmetic is reducible to logic. In order to avoid the indeterminacy raised by the Julius Caesar problem, the most fundamental philosophical problem found in his logicism, Frege is compelled to define numbers as extensions of concepts, and, with this, introduces the Axiom V into his system to rule the notion of value-ranges. However, in *Grundgesetze*'s section 10, Frege finds a new problem of indeterminacy, namely, the issue that Axiom V does not determine the reference of the value-range names. In order to solve this problem, Frege performs the trans-sortal identification, which is the identification of the truth values with the value-ranges of particular functions. Nevertheless, because the identification does not provide a determination as complete as we would expect it would from his famous principle of complete determination (it does not decide whether Julius Caesar is a value-range), scholars such as Dummett (1981) and Wright (1983) have claimed that Frege was, after all, unable to solve the Julius Caesar in a persistent version. The aim of this dissertation is twofold. First, we aim to propose an interpretation, suggested by Greimman (2003), to accommodate Frege's trans-sortal identification with his subscription to the principle of complete determination. Second, we aim to conclude, following Ruffino (2002), that there is no Julius Caesar problem for value-ranges.

Key-words: Value-ranges. Julius Caesar problem. Problem of indeterminacy.

List of Abbreviations

Bs	*Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* Halle a. S.: Louis Nebert, 1879. Republicado em Begriffsschrift und andere Aufsätze. Hildesheim, Zurich and New York: Georg Olms Verlag, 2007.
Gl	*Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl.* Breslau: w. Koebner, 1884.
Gg I/II	*Grundgesetze der Arithmetik.* Band I/II, Jena: Verlag Herman Pohle, 1893/1903.
Ns	*Nachgelassene Schriften.* H. Hermes, F. Kambartel, F. Kaulbach (Hrsg), Felix Meiner Verlag, Hamburg, 1969.
BuG	*Über Begriff und Gegenstand.* Vierteljahresschrift für wissenschaftliche Philosophie, XVI, p.192-205, 1892.

List of Symbols

$\leftrightarrow$	Biconditional
$\wedge$	Conjunction
$\vdash$	Derivability/Provability
$\vee$	Disjunction
$\emptyset$	Empty Set
$=$	Equality/Identity
$\equiv$	Equivalence
$\exists$	Existential Quantifier
$\neq$	Inequality
λ	Lambda Operator
$\rightarrow$	Material Implication
$\neg$	Negation
$\parallel$	Parallelism relation
$\forall$	Universal Quantifier

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1 Introduction

Philosophers of mathematics are now familiar with the puzzling problem of identity brought by Gottlob Frege (1848–1925) when he was attempting to introduce numbers as abstract objects in the second half of the 19th century. By analogy with the introduction of directions, which were also regarded to be abstract objects, it can be put as follows. Consider the sentences 'line a is parallel to line b' and 'the direction of a = the direction of b'. Because parallelism typically states identity of direction, we may replace the relation of parallelism for the more generic relation of identity, restating the content of the former in a different way to obtain a new one. It thus seems plausible to introduce directions by proposing the following definition

$$Dir(a) = Dir(b) \leftrightarrow a \parallel b \tag{D}$$

where we employ the operator '$Dir(x)$' to mean our *definiendum* 'the direction of x'. Thus, the concept of direction is obtained from the concept of parallelism (as much as the criterion of identity for directions), so that to grasp precisely what a direction is, is to know that the direction of a and the direction of b refer to the same entity whenever the lines a and b are parallel. What is today known as the Julius Caesar problem, which is the topic of the first part of this dissertation, is a complaint connecting the notions of object and identity made by Frege (1884) that, by means of such definition, we cannot say whether "England is the same as the direction of the Earth's axis"(*Gl*, §66) is to be affirmed or denied, or—more accurately—the objection that it does not fix the truth value of mixed-identity statements, that is, statements where a direction is identical to another object that may not be described in terms of directions ('$Dir(a) = q$').

Putting aside the matter of directions, the Julius Caesar problem is originally addressed by Frege while attempting his definition of the concept of cardinal number. In one of his most famous works, the book *Die Grundlagen der Arithmetik*, or *The Foundations of Arithmetic* (1884), Frege attempts to introduce numbers as abstract

[0] In the present dissertation we follow the existing English translations of Frege's works that are mentioned in the list of references, with the exception that we have replaced Bauer-Mengelberg's 'ideography' (*Begriffsschrift*) for 'concept writing', Austin's 'Number' (*Anzahl*) for 'cardinal number', 'equal' (*Gleichzahlig*) for 'equinumerous' and Furth's 'course-of-values' (*Werthverlauf*) for 'value-range'. The choice of 'value-range' is perhaps the most simple and literal translation and, by all means, the most widely adopted translation of *Werthverlauf* present in the recent literature (Moore, Rein, 1986; Dummett, 1991; Demoupolos, 1995; Boolos, 1998; Heck, 1999; Ruffino, 2002). In the following we refer to Frege's books in italics and to the axiomatic systems set forth in the corresponding books in roman letters. Frege's books are cited by section numbers. Frege's *Nachlass* is cited by reference to the English translation. We prescind the fact that Grundgesetze is inconsistent.

objects by carrying out an admittedly similar definition. In Frege's own words, it goes as follows. "If a waiter wishes to be certain of laying exactly as many knives on a table of plates, he has no need to count either of them", he recalls, "all he has to do is to lay immediately to the right of every plate a knife, taking care that every knive on the table lies immediately to the right of a plate"(*Gl*, §70). Thus, the number of plates is the same as the number of knives whenever plates and knives are correlated one-to-one, and, this, somewhat generalized, suggests that the sentences 'the number belonging to the concept F = the number belonging to the concept G' and 'there is a one-to-one correspondence between the concepts F and G' have the same meaning. Motivated by this fact Frege proposes to introduce numbers by what became known as the Hume's Principle (HP) in the literature

$$Nx : Fx = Nx : Gx \leftrightarrow F1-1G \tag{HP}$$

where we make use of the cardinality operator '$Nx : \phi x$' to mean our *definiendum* 'the number belonging to the concept ϕ'. Similarly, we use '$F1-1G$' is to mean 'F and G are equinumerous', which is a short for 'there is an one-to-one correspondence between the concepts F and G'. But as much as D did not decide whether England was the same as the direction of the Earth's axis, the problem regarding the HP is that it does not decide, for example, whether Julius Caesar is the number of moons of Jupiter.

The way Frege found to solve this curious difficulty in the book was not to thoroughly addressing it directly, but proposing for numbers what he considered at that time a convincingly unproblematic definition instead. Then, he defined numbers by what he called *extensions of concepts* (*Gl*, §68), that was thought to behave much like the intuitive notion of a set in naïve set theory (*Ns*, p.269). Consider any concept, say, the concept *horse*. Intuitively, the extension of the concept *horse* was thought as a new object, namely, the object a that represents the logical conglomeration of all objects that falls into this concept, namely, all horses. Hence, extensions of concepts were, for Frege, some kind of a *logical object*[1]. Thus, insofar as extensions of concepts are concerned, the task of constructing numbers becomes easier: 0 was roughly constructed as the class of all concepts in which no object falls; 1 as the class of all concepts in which exactly one objects fall, 2 as the class of all concepts in which exactly two objects fall and so on. With this, HP follows as a theorem. But with this it is not obvious how Frege is supposed to decide whether Julius Caesar is the number of moons of Jupiter or not, one might just observe that we do not know whether Julius Caesar is an extension of some concept either, at least not until we know exactly what extensions of concepts are (Schirn, 2001).

It is plausible to expect that the substantial solution to the problem will be left to Frege's two-volume *Grundgesetze der Arithmetik*, or *Basic Laws of Arithmetic* (1893 and 1903), the book in which extensions of concepts are ultimately defined by what he

[1] For further discussion on this matter, see Ruffino, 2002.

called *value-ranges of functions* (*Gg*, x). But before proceeding any further, why should the problem of deciding, say, whether Julius Caesar is the extension of some concept, be taken as a serious problem after all? This is, at first glance, not obvious at all. This question is best answered after some study of the problem's context, which leads us to examine some of the most relevant aspects of Frege's writings.

1.1 Problem background

Under the influence of the arithmetization of analysis[2], one of the main research programs in the foundations of mathematics at the second half of the 19th century, Frege was convinced that the most pressing demand for the foundations of mathematics of his time was to provide arithmetic with a firm basis. Roughly speaking, the program that Frege intended to carry out was to set forth a body of axioms and rules of inference from which all arithmetical propositions could be derived as theorems. Against most of the mathematicians of his time, however, Frege was convinced that he could articulate a more drastic characterization of the foundations of arithmetic, now known as logicism. Frege believed that arithmetic had to be shown to have no basis in intuition, and, therefore, there was no need for non-logical axioms. More specifically, the goal of Frege's program was to vindicate that all arithmetic is reducible to logic, so that:

a) each theorem is derivable via general logical laws alone.

b) each fundamental concept is translatable via general logical definitions.

Thus, motivated by Leibniz's idea for a *calculus ratiocinator*, a theoretical universal logical calculation framework for expressing mathematical, scientific and metaphysical concepts, Frege is driven to the publication of *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, or *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought* (1897).

Begriffsschrift is the book that sets out the technical framework for the logicism, a highly idiosyncratic two-dimensional formal language called the concept writing. The language is introduced along with the consolidation of a set of nine axioms plus modus ponens that will compose the axiomatization of the classical bivalent system of second-order logic with identity that he will employ to set forth definitions and demonstrations in order to claim that arithmetic is reducible to logic. *Begriffsschrift* also serves to carry out some part of the program, by representing a general theory of sequences

[2] In the arithmetization of analysis, some great mathematicians such as Dedekind (1872) played a fundamental role, when he found that the theory of real numbers is reducible to the rational numbers system by means of näive set theory. This and related discoveries motivated Dedekind to believe that arithmetic were reducible to the natural numbers system plus a kind of "logic theory"of sets. Nevertheless, this initial period of expansion of the program has a turning point with the discovery of the paradoxes of set theory—particularly, Russell's Paradox—which, as we shall remark later, leads Frege to abandon his logicist view.

into the concept writing and showing that the theory of sequences is reducible to the concept of logical implication[3]. However, the argument for logicism was not complete. Frege was so aware of this that he promised a subsequent publication on this matter, having in mind his *Grundgesetze*:

> To proceed farther along the path indicated, to elucidate the concepts of number, magnitude, and so forth—all this will be the object of further investigations, which I shall publish immediately after this booklet. (*Bs*, preface)

But, ironically enough, Frege and his works did not enjoy much academic prestige during his lifetime, if it is to compare him to such notable mathematicians of his time as Karl Weierstrass (1815-1897), Felix Klein (1849-1925) or Henri Poincaré (1854-1912)[4]. Although *Begriffsschrift*, the first and brilliant monograph of a young 31 year old early career mathematician, was published with high expectations, its public reception was unfavorable, even tragic[5]. It had no more than six journals and one book reviews until 1880. Thus, encouraged by his colleague Carl Stumpf (1848-1936), which proposed him to write a book explaining his ideas with the absence of his laborious two-dimensional notation, Frege writes in ordinary language his second book and informal investigation on the foundations of arithmetic, *Grundlagen*.

The aim of *Grundlagen* was to made it probable that the laws of arithmetic are analytic judgements and consequently *a priori* (*Gl*, §87). Hence, the argument for logicism in the book takes the form of an apparent contrast with Kant's view that arithmetical propositions are synthetic *a priori*. In fact, Frege's conception of aprioricity and analyticity are a reformulation of the Kantian ones. In Frege's sense, primarily mathematical, the distinction regards not to the content of the judgement, but to its justification (*Gl*, §3), what drove him to his reformulation of the original conception of those terms[6]. It can be seen as follows. For Frege, the matter of a truth being either *a priori* or *a posteriori* and either analytic or synthetic turns to finding the proof of a proposition and analyze it right back to its primitive truths. More accurately, a truth is (i) analytic if one comes only on general logical laws and definitions; (ii) synthetic if one cannot prove it without making use of a general law belonging to the sphere of some special science; (iii) *a posteriori* if it

[3] See *Bs* §§23-31. This is the third part of the book, which is designed to show the reader how to manipulate the formal language (*Bs*, §23). The first part of the book is devoted to an introduction to the symbols of his language, while the second aims to present the logical laws (judgements of pure thought) and some derivations of them.

[4] Cf. Vilkko, 1998, p.413.

[5] This is not an uncontroversial view, cf. Vikko, 1998.

[6] Frege asserts that he just wants to state accurately what Kant has meant (*Gl*, §3). Moreover, since the conceptions of aprioricity and analyticity of these authors are not the same, it is mistaken to claim that Frege's position on arithmetic goes directly against Kant's. For further reading on this matter, see MacFarlane, 2002.

is impossible to construct a proof of it without including an appeal to facts; (iv) *a priori* if its proof can be derived exclusively from general laws.

In this sense of analyticity, then Frege is correct about the laws of arithmetic being analytic judgements whenever the truth of every arithmetical proposition is, in his standards, uncontroversially shown to come from general logical laws or definitions. Then arithmetic becomes an expansion of logic. To be sure, some arithmetical propositions take the form of mixed-identity statements such as '$Nx : Fx = q$', as confirmed by Frege's remark for the case of directions:

> Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction. That says nothing as to whether the proposition "the direction of a is identical with q"should be affirmed or denied, except for the one case where q is given in the form of "the direction of b"(*Gl*, §66).

The worry expressed in the quoted passage is certainly legitimate. Since some arithmetical propositions take the form of mixed-identity statements, and since HP was not able to fix the truth-value of them, he felt like he had to define numbers as extensions, which is ultimately expected to decide that objects such as Julius Caesar, the Moon or the England are not identical to the number belonging to any concept.

Now, before proceeding further, we want to pause for a moment and discuss three important points about Frege's project. First, in a well-known episode, on June 16, 1902, a letter addressed to Frege by Russell arrived while the second volume of *Grundgesetze*, funded by Frege's own grants, was in press. The incoming letter informed Frege about the famous Russell's paradox:

> I find myself in complete agreement with you in all essentials ... There is just one point where I have encountered a difficulty. You state (p. 17 [p. 23 above]) that a function too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality. (Russell, 1902)

Frege responded to Russell immediately, on June 22, informing that he will add an appendix in the second volume, which appeared in 1903, but Frege's solution was later considered to be unsatisfactory and to even lead to new contradictions[7]. It is today well-known that the derivability of the contradiction was caused by the inconsistency of the

[7] See Geach, 1956; Hintikka, 1956, p.239; Kneale and Kneale, 1962, p.654; Geach 1966, p.243 n.1; Sternfeld, 1966, p.119 and Linsky and Schum, 1971, p.5, for studies on this subject.

Axiom V in his system, which is the axiom introduced to rule the notion of value-ranges. This leads Frege to reluctantly abandon the axiom and his convictions about extensions, culminating in the end of the logicist program in 1924[8].

Second, recent research has shown that the axiom V plays no essential role in the derivation all the arithmetical propositions that Frege proves to be theorems in Grundgesetze. In fact, at least in the derivation of almost all of them, with the exception of the proof of HP in both directions[9], that is, (i) if the number belonging to the concept F is the same as the number belonging to the concept G then F and G are equinumerous

$$Nx : Fx = Nx : Gx \longrightarrow F1-1G \tag{HP$\longrightarrow$}$$

and (ii) if F and G are equinumerous then the number belonging to the concept F is the same as the number belonging to the concept G

$$F1-1G \longrightarrow Nx : Fx = Nx : Gx. \tag{HP$\longleftarrow$}$$

An immediate conclusion is that, the program could be formally carried out by simply removing the Axiom V and adding HP as axiom instead, in this way articulating a theory called Fregean Arithmetic, a sound second-order logic system, where second-order arithmetic is interpretable[10].

Third, in a notable letter in response to Russell in 1902, where he discusses about how he might avoid using the Axiom V, Frege has explicitly considered doing a similar procedure, but he had soon discarded this possibility by the claim that some difficulties would persist (Heck, 2005):

> We can also try the following expedient, and I hinted at this in my Foundations of Arithmetic. If we have a relation $\Phi(\xi, \eta)$ for which the following propositions hold: (1) from $\Phi(a, b)$ we can infer$\Phi(b, a)$, and (2) from $\Phi(a, b)$ and $\Phi(b, c)$ we can infer $\Phi(a, c)$; then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by writing, e.g., '§a =§b'. If the relation is, e.g., that of geometrical similarity, then 'a is similar to b' can be replaced by saying 'the shape of a is the same as the shape of b'. This is perhaps what you call 'definition by

8 Cf. Gabriel, 2004, pp.2-10.

9 Cf. Heck 2005, p.162.

10 Cf. Heck, 1997, p. 1. This claim that second-order logic plus HP suffices for derivation of second-order arithmetic has come to be known as *Frege's Theorem* (Heck, 1999b). Notably, this proof is sketched in *Gl*, SS82-83 and expounded in Wright's most influential neo-Fregean study *Frege's Conception of Numbers as Objects* (1983, p. 158-169). As a matter of fact, over the last few decades, there have been some efforts to rescue Frege's logicism. The first study is advanced by Wright (1983). Shortly, the Neo-Fregean's idea is that the Fregean Arithmetic is analytic in Frege's sense, thus resulting Wright's vindication that HP is an analytic principle as well (Wright, 1999). Indeed, this claim presupposes that the Julius Caesar problem is solvable, and for this purpose it was presented a solution by using a method involving sortals (Hale and Wright, 2001). In spite of the fruitfulness of this debate, we shall not discuss this matter in the present dissertation, since our subject here is not the Neo-Fregean viewpoint of the Julius Caesar problem.

> abstraction'. But the difficulties here are ...[11] the same as in transforming the generality of an identity into an identity of value-ranges (Frege 1980b, p. 141, letter xxxvi/6)

As a matter of fact, the procedure suggested by Frege here is the adoption of an *abstraction principle* in a similar way to his attempted introduction of the concept of direction (D) and numbers (HP). These principles had failed to decide whether some objects should be identified as one of the instances of the concepts that they were supposed to introduce, in this way giving rise to the Julius Caesar problem. Therefore, it is plausible to state that what ultimately forces Frege to abandon his project is his inability to solve the Julius Caesar problem, that is, his inability, without making use of extensions of concepts, to explain how we apprehend logical objects[12]. Now, let us note that an abstraction principle is any instance of the scheme

$$f(x) = f(y) \leftrightarrow Rxy \tag{AP}$$

where R is an equivalence relation such as being parallel, being equinumerous or having the same values for the same arguments[13]. Indeed, the introduction of the notion of value-ranges by the Axiom V is an instance of this scheme, too. More accurately, value-ranges are abstract objects introduced by the Axiom V of the system

$$\acute{\epsilon}f(\epsilon) = \acute{\alpha}g(\alpha) \longleftrightarrow \forall x(f(x) = g(x)) \tag{V}$$

where we use the abstraction operator '$\acute{\epsilon}\phi(\epsilon)$' to mean 'the value-range of the function ϕ'. By similarity of form, it is easy to see that the Axiom V will analogously fail to fix the truth value of mixed-identity sentences of the form '$\acute{\epsilon}f(\epsilon) = q$', statements where a value-range is identical to another object that may not be described in terms of value-ranges.

1.1.1 Problem definition

There has been some debate in the Fregean literature about whether Frege was, after all, able to fix the truth value of those mixed-identity statements involving value-ranges. This question is important since Frege defines numbers as extensions of concepts, which turn out to be just particular cases of value-ranges. If there are some indeterminacy for the notion of value-ranges, then there will be indeterminacy for the notion of numbers too, which would be incompatible with his claim that arithmetic is reducible to logic. In this sense of determination, if there is some residual indeterminacy

[11] As Heck recalls, the English translation contains at this point the word 'not', which is not found in the original German edition.

[12] For a defense of this view, see Heck, 2005, p.164-166.

[13] Strictly speaking, being parallel, being equinumerous and having the same values for the same arguments are not similar relations, in the sense that parallelism is a first level and the other two are second level relations. For simplicity sake, we are ignoring this fact and, similarly, ignoring the level distinction between the direction, cardinality and abstraction operators.

in the notion of value-ranges, and hence in the notion of number, then we would have no means to decide, for example, whether Julius Caesar is the number of moons of Jupiter.

This problem of the indeterminacy of the value-ranges (henceforth problem of indeterminacy) is ultimately expounded in Frege's *Grundgesetze*, at one of the most fruitful and widely studied sections in the book, namely, *Gg* I §10, which will be the subject of the second part of this dissertation. This is the place where Frege observes the indeterminacy and, moreover, attempts to articulate a general solution for it, carrying out a further determination of the notion of value-ranges. Briefly, Frege's way of solving this indeterminacy consists in what is usually called his trans-sortal identification, that is, the identification of truth values with particular value-ranges. More accurately, Frege's determination of the notion of value-ranges makes no mention of other objects, which is surprising, since his inability to determine whether Julius Caesar is the number of moons of Jupiter was the ultimate reason for his advance of the very notion of value-ranges itself. Motivated by this fact that Frege makes no reference to those objects when determining the notion of value-ranges, Parsons (1965) initially suspected that, despite all his efforts, Frege was ultimately unable to solve what he saw as a 'persistent version' of the Julius Caesar problem for value-ranges. Few years later, this suspicion was embraced by Dummett (1981), then, Wright (1983) and subsequently overspread by some part of the Fregean scholarship (Ricketts, 1997; Heck, 1997, 1999a, 2005; Schirn, 2001).

These suspicions are strongly suggested by the following reasons. First, it is commonly taken for granted that Frege subscribed to something that has come to be called the 'principle of complete determination':

> It must be determinate for every object whether it falls under a concept or not; a concept word which does not meet this requirement on its referent, has no referent(*Ns*, p.133).

Having this principle in mind, then any strict determination of the notion of value-ranges would need to decide whether Julius Caesar is a value-range or not. Second, Frege's way of solving the indeterminacy by means of the trans-sortal identification does not provide the 'determination' of value-ranges that one would expect from Frege's commitment with the principle of complete determination stated above: it does not rule out the possibility of Julius Caesar being a value-range.

Is there some way to accommodate Frege's commitment with the principle of complete determination with his claim that the trans-sortal identification contains all that is necessary to unequivocally determine the notion of value-ranges? This question is closely related with the suggestion of a Julius Caesar problem for value-ranges. In the light of these considerations, the aim of the present dissertation is twofold. First, we want to develop an alternative view of looking up to Frege's principle of complete determination, suggested by Greimann (2003), that does not allow for any conflict with the trans-sortal identification. This view does not depend on any dubious assumption on the domain of first-order variables of Frege's system. Second, we want to consolidate the view, suggested by Ruffino (2002), that the Julius Caesar problem poses for Frege the matter of establishing without any doubt the nature of numbers as logical objects, so that there is no Julius Caesar problem for value-ranges insofar as numbers are defined as extensions.

This work is organized as follows. In the chapter 2 we study in details Frege's heuristical attempts of definition of the concept of number in his *Grundlagen*, namely, the attempt of an inductive definition and the attempt of a contextual definition. Then, we examine more thoroughly the Julius Caesar problem and investigate the explicit definition of numbers as extensions. In the chapter 3 we move towards the formalism of Grundgesetze. We examine the problem of indeterminacy, study Frege's way of solving it by means of the trans-sortal identification and we study Frege's footnote proposal of extending it. In the chapter 4 we expound the semantics of Grundgesetze to make plausible our ways of looking at the principle of complete determination and at the Julius Caesar problem.

2 The Julius Caesar problem

2.1 The concept of number in Grundlagen

Frege's *Grundlagen* is usually regarded to be his masterpiece. Planned carefully, the book presents in high literary merits the majority of the philosophical pillars of the system of foundations of arithmetic to be expounded later in *Grundgesetze*. It contains: (i) an overpowering and impetuous criticism of the view of previous writers such as Kant, Leibniz, Grassmann, Mill, Lipschitz, Hankel, Jevons, Cantor, Schröder, Hobbes and others about the metaphysics and epistemology of arithmetic; (ii) an analysis of the concept of cardinal number, provided by proposed answers to his own critiques; (iii) a definition of the concept of number, targeting the analyticity of the arithmetical propositions; and (iv) informal proof sketches of some fundamental principles of number based on that definition, which, notably, includes the proof sketch of HP (Heck, 2012).

Remarkably, the greater part of the book is devoted to the articulation of ingenious arguments against the psychological, empiricist, formalist and other accounts of arithmetic, where he lays the cornerstone of his own ideas. Indeed, the book is guided by Frege's three fundamental principles, employed along the investigation:

a) Always separate sharply the psychological from the logical, the subjective from the objective;

b) Never ask for the meaning of a word in the isolation, but only in the context of the proposition;

c) Never to lose sight of the distinction between concept and object.

Admittedly, Frege remarks that the first two principles are interrelated, in the sense that if the second one is not observed, then one is almost forced to take representations or ideas as the meaning of the number names, violating the first one. Famously, the second principle is Frege's so-called *context principle* according to which it is only in the context of a proposition that a word acquires its meaning, which has figured prominently in the work of notable analytic philosophers such as Wittgenstein or Quine[1]. Finally, the third principle points out the categorical distinction between concept and objects which must be respected in order to avoid errors of reasoning, such a reason for which the ontological argument for the existence of God breaks down (*Gl*, §53).

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[1] See Wittgenstein, *Tractatus Logico-Philosophicus*, 1922, particularly 3.3. and 3.114, and Quine, *Two Dogmas of Empiricism*, 1953

Taking this all into account, the fourth part of the book is the place where Frege will ultimately propose his definitions of the concept of number, based on his preceding analysis that numbers (i) are not subjective product of mental processes; (ii) are not anything physical or a collection or aggregation of things; (iii) are not a property of things or abstracted from things such as weight or color. Ultimately, he concludes that we should consider number in the context of the ordinary language to achieve a fair understanding of what numeral statements stand for. At the same time, he proposes that numerical statements, such as 'the Earth has two poles' or 'Jupiter has four moons', contains an assignment of numbers to concepts[2]:

> It should throw some light on the matter to consider number in the context of a judgement which brings out its basic use. While looking at one and the same external phenomenon, I can say which equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men". Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another. This suggests as the answer to the first of the questions left open in our last paragraph, that the content of a statement of number is an assertion about a concept. (*Gl*, §46)

This suggestion will be used all along the book and, initially, in his attempt to complete the Leibnizian definitions of the individual numbers by giving his own definitions of the 0 and 1.

Indeed, Frege stresses the fact that numerical formulae are provable was rightly beforehand asserted by some philosophers and mathematicians, too (*Gl*, §6). Leibniz says that it is not an immediate truth that 2+2 is 4 and provides a demonstration for it. Leibniz defines (i) 2 as 1+1, (ii) 3 as 2+1, (iii) 4 as 3+1 and postulates one axiom, namely, 'if equals be substituted for equals, the equality remains'. He thus carries out his proof: '2+2=2+1+1 (by Def. i) = 3+1 (by Def. ii) = 4 (by Def. iii)'. According to Frege, however, this Leibnizian proof is failed whenever the associative law of addition $a + (b + c) = (a + b) + c$ is not made explicit. As he points out, the proof, which assumes the proposition $2 + (1 + 1) = (2 + 1) + 1$, contains a gap.

2.1.1 Numbers as second-order concepts?

Having in mind to complete the Leibzian definitions, Frege proceeds to give his first heuristic attempt of definition, carried out in *Gl* §55, which is to define the numbers by the inductive definition[3]:

[2] The elaboration of the idea that a statement of number represents an assertion about a concept is actually admittedly the most fundamental result of *Grundlagen*. Cf. *Gg* ix

[3] By making use of the modern notation, we may write Frege's definitions as follows:

(0) The number 0 belongs to the concept F iff for all x: not $F(x)$;

(1) The number 1 belongs to the concept F iff not for all x: not $F(x)$ and for all y and z: if Fy and Fz, then $y = z$;

(n+1) The number $n + 1$ belongs to the concept F iff there is an x such that $F(x)$ and the number n belongs to the concept of being a y such that $F(y)$ and $y \neq x$.

for which he immediately writes: "These definitions suggest themselves so spontaneously in the light of our previous results, that we shall have to go into reasons why they cannot be reckoned satisfactory"(§56) and raises three objections to it.

But before we go too far into it, its worth noting something about the definitions. A second glance might reveal that these definitions are not unambiguous as they initially seem to be. As Greimann (2003) remarks, it is unclear what their *definienda* should be. More specifically, the definitions can be taken either as (i) contextual definitions of the singular terms 'the number 0', 'the number 1' and 'the number n', or as (ii) explicit definitions of the second-order predicates 'There are 0 Fs', 'There is just 1 F' and 'There is just $n + 1$ F'. Let us look at the latter first. By following the explicit definitions of those second-order predicates we have what Dummett (1991) has called the *adjectival strategy*, whose goal, as the name says, is to grasp the adjectival use of the number names as predicates in the ordinary language[4]:

(0') There are 0 Fs iff for all x: not $F(x)$;

(1') There is just 1 F iff not for all x: not $F(x)$ and for all y and z: if Fy and Fz, then $y = z$;

(n+1') There is just 0 F iff not for all x: not $F(x)$ and for all y and z: if Fy and Fz, then $y = z$;

But, although those definitions seem really technically appropriate, to take the adjectival strategy this way would be going too far against Frege's philosophical convictions. For whenever we take those definitions as an attempt to explicitly define the second-order predicates, numbers should be considered second-order concepts or properties of concepts, and this would obviously contradict both his sharp distinction between concepts and objects and his view that numbers are abstract objects (although no conclusive argument for this thesis is given up to this part of the book). His objection to this attempt is thus confirmed by the claim:

(0*) The number 0 belongs to the concept $F \equiv \forall x \neg Fx$;

(1*) The number 1 belongs to the concept $F \equiv \neg(\forall x \neg Fx) \wedge (\forall x \forall y (Fx \wedge Fy \rightarrow x = y))$;

(n+1*) The number $n + 1$ belongs to the concept $F \equiv \exists x (Fx \wedge \exists_n y (Fy \wedge x \neq y))$.

[4] If we use the notation such as '$\exists_n x$' to mean 'there are just n x such that' (usually known in the literature as 'numerically definite quantifiers'), we further symbolize Frege's attempt as follows:

(0') $\exists_0 x Fx \equiv \forall x \neg Fx$;

(1') $\exists_1 x Fx \equiv \neg(\forall x \neg Fx) \wedge (\forall x \forall y (Fx \wedge Fy \rightarrow x = y))$;

(n+1') $\exists_{n+1} x Fx \equiv \exists x (Fx \wedge \exists_n y (Fy \wedge x \neq y))$.

> It is time to get a clearer view of what we mean by our expression "the content of a statement of number is an assertion about a concept". In the proposition "the number 0 belongs to the concept F", 0 is only an element in the predicate (taking the concept F to be the real subject). For this reason I have avoided calling a number such as 0 or 1 or 2 a *property* of a concept. Precisely because it forms only an element in what it is, a self-subsistent object ... we should not, therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. (*Gl*, §57)

Therefore, since for Frege the adjectival strategy is out of the question, the remaining alternative is to take Dummett's (1991) *substantival strategy* which, in turn, aims to regard the definitions as contextual definitions of number names as the singular terms 'the number 0','the number 1','the number 1+1', 'the number 1+1+1' and so forth (since singular terms stand for objects).

Now, let us return to the three objections. The first objection raised by Frege is the complaint that the definition of the number name 'the number $n+1$' depends on the fact 'the number n' being defined. Indeed, this is true—and since we have not defined what 'the number n' is supposed to be, we have no right to take for granted what 'the number $n+1$' stands for, too. Intuitively, Frege himself points out a solution for this objection, which is to depart from the definitions of the already known number names 'the number 0' and 'the number 1' to define the following number names by succession 'the number 1+1', 'the number 1+1+1', 'the number 1+1+1+1' and so forth. But in order to proceed this way, we would have to overcome the second objection.

The second objection consolidates the first appearance of the Julius Caesar problem. As mentioned above, the adjectival definition is impossible and we are compelled to advocate the substantival strategy taking the inductive definitions as contextual definitions of the number names 'the number 0', 'the number 1' and 'the number n' (otherwise we would define second-order predicates instead, which correspond to second-order concepts). But, according to Frege, even if we take the substantival strategy this way, the definitions do not determine *the* number that corresponds to those singular terms 'the number 0', 'the number 1', 'the number 1+1', 'the number 1+1+1' and so forth. Frege hastily raises the objection this way (and because the complaint is raised in such a *krasses Beispiel* as asking whether the Roman Emperor was a number belonging to some concept, the objection became known as the 'Julius Caesar problem', term coined by Wright in 1983[5]):

> we can never - to take a crude example - decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or is not. (*Gl*, §56)

[5] See Wright, *Frege's Conception of Numbers as Objects*, 1983, §14.

Firstly, Frege's inept words should not be mistaken. The allusion to the famous Roman Emperor is just his way to stress his objection, that, as we shall see later, raises a complex of problems. Up to the present moment, the problem can be seen as follows.

Consider if it would be possible that Julius Caesar be a prime number. Well, the answer is certainly negative. As we learned by taking History classes, the general was no doubt a man and nobody could ever mistake people with numbers (being prime is a predicate applicable to numbers and nothing else). Obviously, what holds for persons applies for all other concrete objects. Thus natural language sentences like 'trees be the same when multiplied by 1' or 'mountains are a root of some quadratic polynomial' are actually meaningless, since it is does not make sense to apply the predicates 'be the same when multiplied by 1' and 'be a root of some quadratic polynomial' to mountains or trees. But now what is presupposed here? It is the claim that numbers and concrete objects are different sort of things. Because, as far as we know, concrete objects should not be confused with numbers. Concrete objects such as people, trees or mountains enjoy the benefits of being spatiotemporal beings, they come into being and perish, and they are subject to the laws of matter. Numbers do not. They stand for operations such as successor function and have their own arithmetical rules. They are not temporal beings and they have no extension. We cannot point at a number, nor can we spill our coffee on them (Wright, 1983; Strawson, 1950).

Thus, the Julius Caesar problem points out that a definition of numbers should provide a discrimination of numbers and concrete objects in terms of pure logic. Clearly, this demand is inseparable from the context of the logicist program that requires that numbers be translated in terms of a purely logical vocabulary and that every arithmetical proposition should be derived by making use of general logic laws and definitions. But here comes the problem. We propose numbers a definition and, save for the propositions that can be inferred from it, we are left just like Locke's *tabula rasa* whenever further information (any information which cannot be decided by the formalism, such as asking whether a particular number is equal to Julius Caesar) about them are concerned. We are not allowed to make appeals to experience anymore. Therefore, if the fact that Julius Caesar is the number of moons of Jupiter can neither be affirmed nor refuted by the proposed definition, then we are in trouble. Then the given definition was not suitable for our purposes after all. Indeed, what seems interesting about the Julius Caesar problem is its ambition to provide a purely logical explanation of our ability of distinguishing concrete objects from numbers that, at its source, is rooted in the appeal to experience. Up to this point, at *Gl* §56, the Julius Caesar problem would compel Frege to try another way, which is carried out in his second attempt of definition in *Gl* §63.

Finally, the last objection raised by Frege is that (taken as an attempt to define number names as singular terms) we would be unable to demonstrate that $a = b$ holds if the number a belongs to the concept F and the number b belongs to the same concept, so that we would not be able to demonstrate arithmetical equations. Thus, Frege concludes that the inductive definitions could only serve as an explicit definition of the second-order predicates 'There are 0 Fs', 'There is just 1 F' and 'There is just $n+1$ F':

> It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases "the number 0 belongs to"[and] "the number 1 belongs to"; but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again. (*Gl*, §57)

However, is this complaint that strong? As a solution, suggested by Wright[6], we could add the following definition $(m = n) =_{df} (\forall F((\exists_m x)Fx \longleftrightarrow ((\exists_n x)Fx)$ to establish that the equality $m = n$ is true iff m is the number belonging to the concept F then the n is the number belonging to the concept F and *vice versa*. However, it is easy to see how we would make the same mistake which the first objection is based on, namely, we would illegitimately assume that the numbers m and n were already defined—when they were not. Of course, we could obtain the numbers m and n by succession, but, still, we would not overcome the second objection, which is to know if one of those numbers is identical to the general that crossed the Rubicon. Again, the definitions are unable to determine *the* number that belongs to a particular concept (and we can only expect using them as an explicit definition of second-order predicates).

Let us make a last comment about these definitions. As observed by Dummett (1991) and recalled by Heck (1999), even if Frege decided to define numbers as second-order concepts (hence abandoning his view of numbers as objects), he would most likely have to proceed just like Russell and Whitehead in *Principia Mathematica*, namely, postulating an axiom of infinity. This can be seen as follows. Since the number 0 is defined as the number belonging to the concept F, such that for all x: not $F(x)$, the number 0 can be defined as the number belonging to a concept in which no object falls whatsoever, say, as the number belonging to the concept 'not-identical to itself'. Now, for the definition of the number 1 the existence of at least one object in the domain is required (the number 0 could not play such role since, for the sake of argument, it was defined as a second-order predicate), since the number 1 is the number belonging to the concept F such that it is false that for all x: not F(x) and for all y and z: if Fy and Fz, then $y = z$. But for the sake of argument, suppose that there exists exactly one object in the domain. Let it be denoted by a. Then we can proceed by defining that the number 1 is the number belonging to the concept 'identical to a'. But now for the number 1+1 the existence of exactly two objects in the domain is required. Suppose that there exist exactly two objects in the domain, a and b. Then the number 1+1 is now definable but not the number 1+1+1, which requires

[6] Cf. Wright, 1983, p.37.

the assumption of exactly three objects and so on. Therefore, it is easy so see that Frege could not claim the existence of infinite natural numbers without postulating the existence of an infinite collection of objects (so that there would exist a corresponding concept in which all those objects fall, and a corresponding infinite number belonging to it), which, however, is not a general logical law. Thus Frege simply could not carry out his logicist program this way. In contrast, there is, as far as we can see it, at least one good technical advantage in treating numbers as objects. For, there is no need to postulate the existence of an infinite collection of objects. The number 0 could be defined similarly, that is, as the number belonging to the concept 'not-identical to itself'. Now for the number 1, which requires the existence of at least one object, we can use the number 0 itself. And for the remaining numbers similarly: for the number 1+1 we can use the preceding numbers 0 and 1 etc. As we shall see later, Frege would proceed for the definition of the individual numbers this way.

2.1.2 The contextual definition

After the failure of his first attempt of definition, Frege lays the ground for his second attempt along *Gl* §§57-61, where he stakes his last claims for the argument that numbers are abstract objects. There he establishes that not every object is given to us by appealing to our sensibility (*Sinnlichkeit*) and that, in spite of the fact that numbers are self-subsistent objects, we can have no ideas (*Vorstellungen*) or intuitions (*Anschauungen*) of them, which are neither in time nor in space. So, he raises the question: "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?"(*Gl*, §62), a question which he answers by recurring to the context principle:

> Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. (*Gl*, §62)

However, since this would still leave us a wide and arbitrary choice as to what this proposition should be, Frege recalls that number names must be seen as singular terms corresponding to self-subsistent objects, therefore, being enough to "give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again"(§62). All of this having being said, his natural suggestion is to pick out the identity statement among number names, so that he proposes to define the sense of the proposition 'the number belonging to the concept F is identical to the number belonging to the concept G'

$$Nx : Fx = Nx : Gx \tag{2.1}$$

which is essentially, in his view, to reformulate (*zerspalten*) its content (*Inhalt*) in other terms, without, obviously, using the expression 'the number belonging to the concept ...'

being defined. And, as seen earlier, Frege proposes to reproduce the equality by means of the relation of one-to-one correspondence of two concepts, proceeding in *Gl* §63 to his second attempt of definition, which is to define contextually the number belonging to the concept F by HP.

Recall that HP states that the number belonging to the concept F = the number belonging to the concept G iff there is a one-to-one correspondence between the concepts F and G. Now a question might arise whether the fact that 'there is a one-to-one correspondence between the concepts F and G' ($F1-1G$) can be represented in terms of a pure logic vocabulary. Indeed, although the sentence requires the use of second-order logic to express it (it quantifies over a relation), the existence of an one-to-one correspondence between the concepts F and G is defined as the existence of an one-to-one function R such that for all x: $F(x)$ if and only if $G(x)$

$$\exists R[\forall x(Fx \rightarrow \exists y(Gy \wedge \forall z(Rxz \rightarrow z = y)) \wedge \forall y(Gy \rightarrow \exists x(Fx \wedge \forall w(Rwy \rightarrow w = x)))] \tag{F1-1G}$$

Regardless of these facts, however, Frege raises the well-known inept objection against this attempt of definition, too. The objection is not directly addressed for the case of numbers this time. Instead, Frege moves the discussion to an example of directions[7], where he states:

> In the proposition "the direction of a is identical with the direction of b"the direction of a plays the part of an object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b. But this means does not provide for all cases. it will not, for instance, decide for us whether England is the same as the direction of the Earth axis - if I may be forgiven an example wich looks nonsensical. Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction. That says nothing as to whether the proposition "the direction of a is identical with q"is to be affirmed or denied, except for the one case where q is given in the form of "the direction of b". What we lack is the concept of direction; for if we had that, then we could lay it down that, if q is not a direction, our proposition is to be denied, while if it is a direction, our original definition will decide whether it is to be affirmed or denied. (*Gl*, §66)

And this objection indicates the emergence of the Julius Caesar problem once more. It appears to be the following. As numbers are (abstract) objects, Frege observes that they should be identified and re-identified as the same again on different occasions and in different guises. Then number names must have associated with them "a class of propositions which must have a sense, namely those which express our recognition of a number as the same again"(*Gl*, §62), number names must have a recognition statement,

[7] In this dissertation we restrict ourselves to the Julius Caesar Problem in the case of cardinal numbers. Cf. Hale and Wright (2001, p.352-66) for an extended discussion of the problem in the case of directions.

which is the expression that establishes the criteria of identity for them. The contextual definition by HP tells us that the associated recognition statement for number names is the one-to-one correspondence, which, as we remarked earlier, actually does a good job, say, in distinguishing 'the number of the moons of Jupiter' from 'the number of the poles of the Earth'. But recall that it fails to determine whether mixed-identity statements of the form

$$Nx : Fx = q$$

are true or false, that is, it does not determine whether a name q refer to a number regardless of being described in terms of "the number belonging to the concept F". Obviously, if q eventually stands for Julius Caesar, we have no means to know whether the conqueror of Gaul is the number belonging to the concept F. Why is this so? It is easy to observe that although one is able to determine what is the corresponding concept of the name described in such terms, say '$Nx : Fx$' (for which the corresponding concept is F) one has no means to determine what is the corresponding concept of 'q', what is mandatory in order to recognize that $Nx : Fx$ and q are identical by HP, namely, the fact that a one-to-one mapping between their corresponding concepts holds.

2.2 The dimensions of the Julius Caesar problem

Frege's first two attempts of definition of the concept of numbers turned out being inadequate, since they were not able to overcome the Julius Caesar problem. But, more accurately, what is so problematic about this objection in the first place? Let us make some further remarks. It is easy to confirm the true philosophical significance of the Julius Caesar problem by taking a look into the Fregean literature over the past few years. Substantial contributions in higher or lower degree were made by scholars such as Parsons (1965), Dummett (1981b), Wright (1983), Schirn (1996, 2001), Heck (1997, 1999, 2005), Wagner (1983), Uzquiano (2000), Wright and Hale (2001), Ruffino (2002), Greimann (2003), MacBride (2003, 2005, 2006), Cook and Ebert (2005), Kemp (2005), Weiner (2007) and, recently, Pendersen (2009). Over the past three decades, researchers came to the conclusion that, despite its apparent naivety, the Julius Caesar problem does not pose a single problem, but at least a complex of four of them. Heck (1997) seems to pioneer this further discrimination of the Julius Caesar problem, stressing that the objection involves both an epistemical and a semantic dimension of the problem. Greimann (2003) reinforces this interpretation, by remarking that the objection brings an ontological dimension as well (that Heck suggests, but did not develop it), and recalling that the objection should be seen as a logical problem as well. Ultimately, MacBride (2006) distinguishes an ontological, epistemical and semantic dimension of the problem as well, although he seems to overlook its logical dimension.

By and large, the greater part of the Fregean literature is now likely to recognize that the Julius Caesar problem poses for Frege a complex of four distinct but interrelated problems[8], namely, the ontological, epistemical, logical and semantical, which can be summarized in saying that a definition of number is defective because:

a) *The ontological problem*: it does not inform us what kind of entities numbers really are.

b) *The epistemological problem*: it does not afford us means of recognizing numbers as the same again if they are not given as number names.

c) *The semantic problem*: it does not fix the reference of the number names.

d) *The logical problem*: it does not establish a sharp distinction of the concept boundaries.

(a) According to the ontological problem, embraced by Schirn (1996), Wright and Hale (2001) and the assumption made in Benacerraf (1965, 1981) and Ruffino (2002), the Julius Caesar problem is addressed to the fact that HP does not establish what kind of thing numbers really are, since it is not able to distinguish numbers from people, mountains, the Moon and so on. Indeed, this ontological dimension seems to be the most evident one: Frege has long been celebrated for his Platonist view of mathematics[9], so that it seems reasonable to conclude that for him the most pressing demand of a definition of the concept of number is to establish their true nature. Frege's Platonism is usually supported by his famous claims that the truth of arithmetic are eternal truths, that the existence of numbers are independent from any person thinking about them and that numbers do not have a place in space and time (*Gl*, §§57-61). This view is usually compressed to the claim that there is a 'third realm' containing those objects. Now, given that there are objective facts determining the ontology of numbers, it follows that the structurally equivalent but ontologically distinct set-theoretical definitions of numbers cannot be both adequate.

A good example is to take Zermelo's and von Neumann's definition. According to Zermelo's, 0 is the empty set and the successor function takes x to the singleton of x, resulting in a progression such like $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$ ... Von Neumann defined 0 as the empty set, but, in its turn, the successor function takes x to the union of x and the singleton of x, obtaining the progression $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ ... As we can see, Zermelo's and von Neumann's number 0 are identical, that is, the number zero is defined as the empty set $\emptyset$. But the number 3, for example, would be regarded

8 Heck (1999) is explicitly reluctant to recognize a logical dimension of the Julius Caesar problem. But we show in (d) below that Heck's arguments are weak.

9 Frege's realism is nevertheless not an uncontroversial point in the literature. For some debate, see Sluga (1975), Dummett (1976; 1981b), Currie (1978), Resnik (1979), Weiner (1995). This is not intended to be an exhaustive list. Since this debate would lead us too far from our objective, we abstain ourselves of the matter here. Let us just remark that MacBride (2006) emphasizes that the ontological, epistemological and semantic problems may not be necessarily related, but turn out to be so when considering Frege's realist view.

as $\{\{\{\emptyset\}\}\}$ in Zermelo's and as $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ in von Neumann's. Because those are different objects, the definition of 3 cannot be regarded as both. But also, because they are structurally equivalent[10], there is no way of telling which one indeed portrays the supposed true ontology of numbers (Greimman, 2003). But despite of all this discussion about Frege's Platonism, there is, according to Greimann (2003), some evidence that he endorsed a moderate form of structuralism. As he points out, in *Grundlagen* Frege curiously admits "I attach no decisive importance even to bringing in the extensions of concepts at all"(*Gl*, §107). Also, as we shall see in the next chapter, in order to solve the Julius Caesar problem for value-ranges in *Grundgesetze* Frege demonstrates no interest in the ontological nature of the value-ranges when he makes stipulations in order to determine the truth value of sentences of the form '$\acute{\epsilon} f(\epsilon) = q$', he just seems concerned about how truth values and value-ranges relate to one another. However, he also notices that, in many passages, particularly *Gg* II §§146-147, Frege shows concern about the construction of mathematical objects, remarking that, whenever possible, they should be constructed as value-ranges.

(b) Frege's contextual attempt of definition by HP says that we recognize numbers as referents of names of the form 'the number belonging to the concept F' and in order to fully understand what these names mean is necessary and sufficient our grasp of HP (Heck, 1997). Construed as an epistemological problem, the Julius Caesar problem points out that HP does not explain our common understanding of numbers, and since we know for sure that objects such as England or Julius Caesar are not a number, a definition should provide identity criteria for numbers that settle the falsity of such expected identity statements. So, adequate identity criteria for numbers should establish that Julius Caesar is not a number (corresponding to our common sense belief and determine that Julius Caesar is not identical to any number). But HP fails to supply such identity criteria, so that it cannot be used to introduce numbers as earlier proposed (MacBride, 2006). It is important to note how this epistemological concern is pertinent in *Grundlagen*, given that Frege is explicitly attempting to refute Kant's epistemological doctrine that without the sensibility no object can be given to us:

> Objects are *given* to us by means of sensibility, and it alone yields us *intuitions*; they are *thought* through the understanding, and from the understanding arise *concepts*. But all thought must, directly or indirectly, by way of certain characters, relate ultimately to intuitions, and therefore, with us, to sensibility, because in no other way can an object be given to us. (Critique of Pure Reason, §1)

As previously stated, Frege holds the view that numbers are objects, which are non-spatial and non-temporal, and from which we cannot have any ideas or intuitions of them. Since

[10] In a structuralist point of view both are actually correct, since it only matters that it be the number that immediately succeeds the number 2.

Frege has precluded sensibility from the explanation of how numbers are given to us, he is compelled to provide an alternative account, which, in this case, should be conveyed by pure logic. As we saw, this explanation was supposed to be given by HP:

> How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. That, obviously, leaves us still a very wide choice. But we have already settled that number words are to be understood as standing for self-subsistent objects. And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion. (*Gl*, §62)

But we know already that HP cannot be used for this purpose since it does not afford us a general identity criterion to recognize numbers as self-subsistent objects, and it does not allow us to identify and re-identify numbers as the same again if the number belonging to the concept G is not given by a description of the form 'the number belonging to the concept ϕ' (Greimann, 2003). In fact, an old story told by Benacerraf (1965) can help us to understand how the epistemic problem works. Imagine two children, Ernie and Johnny, ignorant about arithmetic, but which were pretty skilled in logic. Instead of counting and begining their mathematical training as it is ordinarily done, Ernie and Johnny were taught HP instead. The children would begin to distinguish different numbers among them, until come to prove the Peano Postulates from HP plus second-order logic, a result which is usually called *Frege's Theorem* (Wright, 1983; Boolos, 1987; MacBride, 2006). But after proving some theorems, the children soon had a dispute about whether 3 belongs to 17. Earnie, for which his arithmetical progression was $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ ..., said that it does. Johnny, for which it was $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$ ..., said that it does not. Indeed, HP implies that Earnie and Johnny are right, but, from this follows that it does not provide them a full understanding of arithmetic (since the divergence is found in what objects they regard as numbers, and, as the ontological problem points out, HP does not determine what numbers are).

(c) In 1892, Frege publishes the paper *Über Sinn und Bedeutung*, or *On Sense and Reference*, the work in which he abandons his notion of 'content' to introduce the distinction between sense and reference[11]. Taking this into account, he establishes that a

[11] For Ruffino (1997) the introduction of the distinction between sense and reference in the *Begriffsschrift* is most likely encouraged by technical interest in providing the view of value-ranges as objects. According to him, the statement that every proposition have the same reference—either the True or the False—can be obtained in Grundgesetze in a step of Universal Generalization under Rule 5 (*Gg*, §48) applied to the Axiom IV. In this sense every clearly distinct but true propositions such as '2+2=4' and 'Snow is white' would be - without the distinction between sense and reference - identified as essentially the same (Cf. *Über die Begriffsschrift des Hernn Peano und meine eigene*, 1896, p.269).

proper name (singular terms or propositions) expresses a sense and refers to an object. Thus, proper names must refer to one and only one object: singular terms may refer to any object whatsoever, whereas the reference of the propositions is confined to the two truth values the True or the False (hence, truth values are objects for Frege). Similarly, proper names must express a sense: singular terms express their mode of presentation, and propositions express a thought, namely, the thought that the truth-conditions of the proposition are fulfilled. Taking into account all these facts, we recall that Frege's answer to the question of "how number can be given to us"is provided by "defining the sense of a proposition in which a number word occurs", which, in turn, is supposed to be given by HP, which reproduces the *content* of '$Nx : Fx = Nx : Gx$' by making use of '$F1 - 1G$'. Then, the semantic problem is, at least, ambiguous. If 'content' means sense, then HP is supposed to have both sides with the same sense, i.e. the truth-condition given by the one-to-one correspondence, and the problem is that it fails to take an account of '$Nx : Fx = q$'. Thus it raises a doubt whether the truth-condition truly fix the sense of the sentence '$Nx : Fx = Nx : Gx$', since the less complex predicate '$x = Nx : Gx$' would have any sense either. On the other hand, assuming that 'content' means reference, then HP should have both sides as coreferential (hence, having the same truth value). In this case, HP does not provide '$Nx : Fx = q$' a reference (truth value), since there is no way to determine its reference by the reference of a correspondence one-to-one. Then the problem is that HP does not determine *the* object a which is the referent of the number names. In any case, it is easy to see that both interpretations are deeply connected, given the relation of the notions of truth-condition (sense) and truth (reference). Finally, two things are worth mentioning. First, considering what we mentioned earlier about Frege's moderate structuralism, it follows that the ontological problem is reducible to the semantic one. Second, given that the epistemological problem is reduced to the problem that HP does not settle the reference of the number names '$Nx : Fx$' (since it does not fix the reference of sentences of the form '$Nx : Fx = q$', which means that the reference of such sentences were not shown to be uniquely determined as either the True or the False), and given that to know how numbers are given to us becomes a matter of determining the reference of the number names, it follows as a direct consequence of Frege's procedure that the epistemological problem is reduced to the semantic one, too.

(d) Frege is celebrated for his claim that concepts must have sharp boundaries, what encouraged him to subscribe to his so-called 'principle of complete determination'. According to this principle, for a predicate to be legitimately used in science, its application should not be ambiguous as in many natural languages (such as 'x is red'). Therefore, Greimann (2003) recalls that if we want to properly introduce the natural language predicate 'x is prime' into the formal language (which is intended to be an ideal one), we must say in what conditions this predicate is true for all objects x given to us, which includes not only numbers, but also the moon, England and Julius Caesar, so to say. As a result, in order to legitimize the use of all predicates of arithmetic in science (such as 'is a natural number', 'is an even number', 'is divisible by 3', 'is the number 1'), it follows that it must be shown whether or not an object a such as Julius Caesar is applicable to them. However, it is worth mentioning that the way this task is carried out by Frege is semantically, i.e. to say in what conditions a predicate F is true for all objects is the same as to fix its reference. This is well stated in the following passage of *Gg* II:

> Ehe ein Wort oder Zeichen seiner Bedeutung nach nicht vollständig erklärt oder sonst bekannt ist, darf es in einer strengen Wissenschaft nicht gebraucht werden, am wenigsten aber dazu, seine eigene Erklärung weiter fortzuführen. (*Gg* II, §57)

Thus, the logical problem is reducible to the semantic problem, too. Thus, if in order to determine the reference of the predicate F we find that Fa is neither true or false for some object a given to us (as we shall see, resulting that the reference of F is undetermined) and it follows that the application of F is vague, having no sharp boundaries at all. But, despite this substantial account of the logical problem, this view has been criticized by Heck (1997) on the ground that there is no indication that Frege subscribed to the principle of complete determination already at the time of writing *Grundlagen*. But Heck's argument is weak, as Greimman (2003) observes that, already in *Gl* §68 and §74, Frege explicitly states his concern that concepts must have sharp boundaries, stressing (before proposing the definition of number as extensions) that his former attempts were not able to provide a concept of number with sharp boundaries:

> Seeing that we cannot by these methods obtain any concept of direction with sharp limits to its application, not therefore, for the same reason, any satisfactory concept of Number either, let us try another way. (*Gl*, §68)

Taking into account all these facts, we confirm that, for the greater part of the Fregean literature, the Julius Caesar problem poses a complex of four problems which are interrelated in the sense that the semantic problem of fixing the reference of the number names is their ultimate root[12].

Before proceeding any further, it is worth mentioning an alternative view of the Julius Ceasar problem by Ruffino (2002) whose purpose is to explain Frege's reason to think that bringing in extensions would be enough to overcome it. We shall keep in mind this view all along this work, mainly in the chapter 4. It goes as follows. In *Über Begriff und Gegenstand*, or *On Concept and Object* (1891), during the famous analysis of Benno

[12] For Greimann the Julius Caesar problem in *Grundlagen* poses primarily the logical problem for Frege. However, since this problem is reducible to the semantic one, this is not a contradiction with our view here.

Kerry's example 'the concept *horse* is easily attained'[13], Frege suggests that in logic there are some objects of a special kind which serve as the representative 'objectification' of concepts. Frege's idea is that sometimes the awkwardness of the language allows us to assert something about a concept, such as in sentences like 'the concept *horse* is easily attained', 'the concept *square root of 4* is realized' or 'there is only one Vienna', which cannot be made without the use of the definite article in the first two cases. Doing so Frege is compelled to awkwardly admit that 'the concept *horse*' is not a concept (for the purpose of not violating his sharp distinction between concepts and objetcs):

> one would expect that the reference of the grammatical subject would be the concept; but the concept as such cannot play this part, in view of its predicative nature; it must first be converted into an object, or, speaking more precisely, represented by an object. (*BuG*, p.46)

According to Ruffino (2002), there is strong textual evidence suggesting that Frege has extensions of concepts in mind, when saying, in this passage, that there are some objects of a special kind that represent concepts. If this is correct, then the expression 'the concept *horse*', as much as 'the extension of the concept *horse*', would have for Frege as the same referent, namely, the extension of the concept horse (Ruffino, 2000). This view is held by Parsons (1984) and Burge (1984) as well. Taking this all into account, Ruffino (2002) argues that it follows from Frege's ontology of logic that extensions are the most fundamental logical objects. In this sense, by identifying numbers with those 'atomic' logical objects, there would be no more dispute whether some number would be equal to a non-logical object such as Julius Caesar. Ultimately, Ruffino claims that the Julius Caesar problem arises in *Grundlagen* insofar as the nature of numbers as logical objects is not uncontroversially established according to his criteria. This view is useful to explain, for example, why in *Grundgesetze*, after numbers being identified with extensions, Frege only shows concern in determining the relation between truth values and value-ranges, but not concrete objects.

2.2.1 Numbers as extensions

Now let us return to the argumentative part of the book. As mentioned before, the greater part of *Grundlagen*, which exceeds three quarters of the book, is spent with criticisms of the view of previous writers. However, it seems that even his own ideas could not avoid his powerful objections. For the first attempt of definition of the concept of number, the inductive definition carried out in *Gl* §55, turned out to be failed (since in order to avoid the Julius Caesar problem it would only serve to define numbers

[13] Kerry's example is addressed to show that no logical rules can be based on linguistic distinctions, so that the Frege's claim that the definite article marks the word's association with a singular term (one of the fundamental pillars of his argument that numbers are objects in *Grundlagen*) is mistaken. Frege's *On Concept and Object* is written as a response to Kerry's critique.

as second-order concepts). Similarly, the second attempt of definition, the contextual definition by HP of *Gl* §63, was defeated by the Julius Caesar problem as well. As a matter of fact, up until this part of the book it seems that the Julius Caesar problem will not only discontinue the investigation of *Grundlagen*, but to undermine the logicist program either. Seeing that he cannot by the former attempts obtain any satisfactory concept of number, Frege decides to try another way. It is then in the beginning of *Gl* §68 that the notion of 'extensions of concepts' appears for the first time in Frege's work, when he defines explicitly the cardinality operator '$Nx : Fx$' as the extension of the concept *stands in one-to-one correspondence with the concept F*:

$$Nx : Fx =_{\text{df}} \text{the extension of the concept } \xi 1-1 F \tag{2.2}$$

But without providing any justification for the importance of bringing in the extensions of concepts, or even for adopting this alternative[14], the notion of extensions actually "comes out of nowhere" at the highest point of the book. More specifically, the extensions of concepts are deliberately introduced without further explanation but a vague footnote remarking "I assume that it is known what an extension of concept is" (*Gl*, §68, n.1).

This passage expresses an underlying uncertainty, which contrasts sharply with the critical, almost arrogant, tone found in the rest of the book (Burge, 1984). Indeed, there are evidences that Frege did fight forcefully against using the notion of the extension of a concept after the publication of *Grundlagen* in 1884. As a historical matter, Burge (1984) tells us that a manuscript from the period immediately after the publication of *Grundlagen* was found having the purpose of defining numbers without using extensions of concepts, according to a list of Frege's *Wissenschaftliche Nachlass*. The list was compiled by Heinrich Scholz. Unfortunately, we are just left with Scholz's very brief summary of the manuscript, once it was among those manuscripts destroyed during an American bombing raid in 1943[15]. Frege's resistance against using extensions is actually suggested already in *Grundgesetze*, where after stressing the importance of introducing a notation for value-ranges into his Begriffsschrift, he admits: "I myself can estimate to some extent the resistance with which my innovations will be met, because I had first to overcome something similar in myself in order to make them" (*Gg*, xi). But regardless of all the struggle against extensions, Frege was somehow compelled to maintain the view on numbers as extensions in *Grundgesetze*, which forced him to provide an explanation about what sort of things extensions actually are.

[14] In an enigmatic closing passage of *Grundlagen*, Frege seems to suggest that there might be other ways to overcome the Julius Caesar problem by providing an equally satisfactory but alternative definition of numbers, "I attach no decisive importance even to bringing in the extensions of concepts at all" (*Gl*, §107). Frege's tone of conviction is actually striking as the passage is continuously denied along *Grundgesetze*, "the introduction of a notation for value-ranges seems to me to be one of the most important supplementations that I have made of my *Begriffsschrift* since my first publication on this subject" (Gg, §9), "I define number itself as the extension of a concept, and extensions of concepts are by my definition value-ranges. Thus we just cannot get on without it." (Gg, Preface).

[15] Cf. Burge 1984, for a detailed discussion

But how the identification of numbers with extensions is supposed to resolve the Julius Caesar problem? Let us consider two ways to see this matter. First, we can take the Julius Caesar problem as the matter of establishing the nature of numbers as logical objects to claim that, since extensions are the most fundamental logic objects for Frege, then numbers would be certainly logical objects. This is the view of Ruffino (2002). Second, we can hold that Frege's most fundamental objection when posing the Julius Caesar problem is the semantic one of determining the reference of number names and observe, such as Heck (1997) and Schirn (2001) do, that identifying numbers with extensions is not a definitive solution to the problem at all, at least not until it is shown that no extension is identical to Julius Caesar (what has encouraged Parsons, Dummett, Wright, Ricketts, Heck, Shirn and other commentators to expect that a conclusive solution for the extensions should be found in *Grundgesetze*).

We give an account of this matter at the chapters 3 and 4. We later claim in the chapter 4 that, considering Frege's way of solving the indeterminacy by the trans-sortal identification and his lack of concern about whether Julius Caesar is a value-range, the first way of seeing the problem, that is, the matter of establishing the nature of numbers as logical objects, is more plausible.

2.2.2 The individual numbers

After defining the cardinality operator '$Nx : Fx$' and making some comments on the legitimacy of his explicit definition, which includes the already mentioned proof sketch of HP, Frege assumes that we can now pass on the definition of the individual numbers (*Gl*, §74) which, as usual, is carried out inductively by defining the number 0 and a successor function. Let us begin with the definition of the number 0, for which we have already indicated the path. Because nothing falls under the concept 'not identical with itself', Frege defines the number 0 as the number which belongs to this concept[16]

$$0 =_{\text{df}} Nx : \text{being an } x \text{ such that } x \neq x \tag{2.4}$$

[16] In order to facilitate the symbolical representation of Frege's definition of the individual numbers which makes extensive use of conceptual terms (names of a concept), we henceforth make use of the so-called 'lambda-notation', which is here helpful to provide us the means of taking a formula, ϕ, to form a corresponding complex predicate 'is such that ϕ' (or 'being an ϕ') which can be seen as a symbolical representation of a conceptual term. For expressing this we introduce the new symbol, λ (often called lambda-abstracts), with the following grammar:

$$\text{if } x \text{ is a variable and } \phi \text{ is a wff then } \lambda x.\phi \text{ is an unary predicate} \tag{λ}$$

We then use the lambda-notation to represent Frege's definition of 0 as follows:

$$0 =_{\text{df}} Nx : \lambda x.(x \neq x) \tag{2.3}$$

which, according to the definition of the cardinality operator by means of extensions, is to say that the number 0 is the extension of the concept *stands in one-to-one correspondence with the concept 'not identical with itself'*. And to justify the definite article 'the' which occurs in "*the* number 0 belonging to the concept F", Frege proceeds to prove 0's uniqueness, which is essentially to prove the statement that the number 0 is the number belonging to every concept in which no object falls[17].

It is important to note that formally Frege could have used for the definition of 0 any other concept under which no object falls. However, having in mind his logicism, Frege deliberately picks out the concept 'not identical with itself' considering that, according to him, the relation of identity can be given in terms of a purely logic vocabulary by Leibniz Law:

> ...I have made a point of choosing one [concept] which can be proved to be such on purely logical grounds; and for this purpose "not identical with itself"is the most convenient that offers, taking for the definition of "identical"the one from Leibniz given above [Leibniz Law (§65)], which is in purely logical terms. (*Gl*, §74)

As we saw, when Frege was trying to complete the Leibnizian definitions of the individual numbers, he had to define the relation of immediate precedence. Therefore, after defining the number 0 and proving its uniqueness, Frege proposes the successor function, defined as a relation in which every two adjacent members of the series of natural numbers stand to each other. Thus he defines the relation m *immediately precedes* n as

m immediately precedes n iff there is a concept F and an object a such that: a falls under F, n is the number belonging to the concept F, and m is the number of the concept object falling under F but not identical with a[18] (2.6)

which, intuitively, can be seen as grasping the fact that m immediately precedes n whenever n is the number belonging to the concept F and m is the number belonging to the concept where there is exactly one object less than F falling into it.

Taking into account the definition of 0 and of the relation of immediate precedence, Frege is able to arrive at the definition of the number 1, which he does by demonstrating that there is something that directly follows in the series of natural numbers after 0 (hence stipulating this object to be 1). Now, recall the relation of immediate precedence. Frege observes that by considering the concept *identical with 0* as the concept F and the object a falling under it as 0, both sentences (a) and (b) hold:

[17] The proof essentially consists in assuming two arbitrary concepts F and G in which no objects fall to find a relation R which is a one-to-one correspondence of F and G, which, by HP determines that the number belonging to F is the same as the number belonging to G, namely, *the* number 0

a) Nx : identical with 0 = Nx : identical with 0;

b) Nx : identical with 0 but not identical with 0 = 0;

then it follows directly by the relation of immediate precedence that the number which belongs to the concept *identical with 0* immediately precedes the number which belongs to the concept *identical with 0 but not identical with 0* in the series of natural numbers. Because the number which belongs to the latter is 0 (which follows from the fact that Frege has proven 0's uniqueness as well), then Frege defines the former as the 1[19]

$$1 =_{\text{df}} Nx : \text{being an } x \text{ such that } x = 0 \tag{2.8}$$

which, again, according to the definition of the cardinality operator by means of extensions, is to say that the number 1 is the extension of the concept *stands in one-to-one correspondence with the concept 'identical with 0'*. (It is worth pointing out, as we remarked before, that this definition does not require the existence of other objects, and therefore does not presuppose the existence of observed facts).

The definition of the remaining numbers goes in a similar way, and 2 is the number belonging to the concept *identical with 0 or 1*, 3 is the number belonging to the concept *identical with 0, 1 or 2* and so on, with Frege defining 'n is a finite number' as equivalent to 'n is a member of the series of natural numbers beginning with 0', and, finally, defining the infinite numbers by identifying Cantor's $\aleph_0$ as the number belonging to the concept *finite number*.

[19] We represent Frege's definition as:

$$1 =_{\text{df}} Nx : \lambda x.(x = 0) \tag{2.7}$$

3 The problem of indeterminacy

3.1 The indeterminacy of value-ranges

In the previous chapter we saw that in order to solve the Julius Caesar problem, Frege chooses to identify numbers with extensions of concepts, in this way making use of the notion of extensions for the first time. Ultimately, his appeal to extensions is encouraged by his incapacity to obtain any satisfactory concept of number by his former attempts, which compels him to try another way. This is where the notion of extensions enters the scene. He defines explicitly his cardinality operator by making use of it, although at that time he had not developed a robust theory of extensions of concepts. Consequently, his results by making use of extensions just made the analytic character of arithmetical propositions no more than probable: for the purpose of ending this dispute once for all and entirely put out of the question the analytical character of the propositions of arithmetic, Frege was convinced that he had to provide a formal account of extensions. With this, he could finally make use of the concept writing to provide proofs of the arithmetical propositions that are deducible solely from purely logical laws, and to make sure that no other type of premise is involved at some point in their proof without our noticing it[1].

In pursuing such a robust theory for extensions, the logicist program had to wait for another nine years after the publication of *Grundlagen*, when Frege's work reached the maturity with his *magnum opus Grundgesetze*. This is the book that exposes the enhancement of the formal system developed *Begriffschrift* in order to vindicate the success of his claim that arithmetic is reducible to logic.

The internal changes found in Frege's formalism from Begriffschrift to Grundgesetze, a system now comprising a total of six axioms plus three rules of inference, are better expressed as follows:

> The primitive signs used in my *Begriffsschrift* occur here also, with one exception. Instead of the three parallel lines I have adopted the ordinary sign of equality, since I have persuaded myself that it has in arithmetic precisely the meaning that I wish to symbolize ... To the old primitive signs two more have now been added: the smooth breathing, for the notation for the value-ranges of a function, and a sign meant to do the

[0] In the present chapter and in the following we shall use Frege's original notation. See the Appendix.

[1] Cf. *Gl*, §90: "This misgiving will not be completely allayed even by the indications I have given of the proof of some of the propositions; it can only be removed by producing a chain of deductions with no link missing, such that no step in it is taken which does not conform to some one of a small number of principles of inference recognized as purely logical.

> work of the definite article of everyday language. The introduction of the value-ranges of functions is a vital advance, thanks to which we gain far greater flexibility. The former derivative signs can now be replaced by other, simpler ones, although the definitions of the many-oneness of a relation, of following in a series, and of a mapping, are essentially the same as those which I gave in part in *Begriffsschrift* and in part in *Grundlagen der Arithmetik*. But the value-ranges are also extremely important in principle; in fact, I define Number itself as the extension of a concept, and extensions of concepts are by my definitions value-ranges. Thus we just cannot get on without them ... The former 'content-stroke' reappears as the 'horizontal' (*Gg*, ix-x)

Particularly, the content-stroke is abandoned in Grundgesetze in order to make room for the distinction between sense and reference, where the acknowledgement of conceptual content of a judgement (*Bs*, §3) gives place to the acknowledgement of its thought (sense) and its truth value (reference).

The most striking internal change of the formalism is provided by the systematical introduction of notion of value-ranges in Grundgesetze[2]. The first appearance of the notion of value-ranges, which is a generalization of the notion of extensions, is actually found in *Funktion und Begriff*, or *Function and Concept* (1891), a paper published separately in order to make room for the main argument of his *magnum opus Grundgesetze*. At this point, the notion of value-ranges appears through the first draft of the law to be later incorporated as the Axiom V of Grundgesetze. Taking this into account, we can compare the role that value-ranges plays for logic with the place that curves take in analytic geometry[3]. As well known from the latter, two curves are said to be the same whenever the values of their corresponding functions universally agree for every argument. Take the functions $y = x^2 - 4x$ and $y = x(x - 4x)$ as example. Because the value of these functions agree for every argument, it follows that the function $x(x-4x)$ has the same *curve* as the function $x^2 - 4x$. Now, considering this fact, Frege believed that this thought could be expressed in genuinely logical terms by stating that the function $x(x - 4x)$ has the same

2 As hastily mentioned in the last chapter, the introduction of the notion of extensions of concepts provides Grundgesetze a common view as a system of logic plus naïve set theory, in the sense that its formalism has no particular concern about how exactly these "sets"should be defined (Jech, 2011; *Ns*, p.269). Although the notion of sets is not explicit in Grundgesetze, this claim is quite in agreement with the fact that value-range names allow us to represent the notion of classes. We make this parallel as follows. Although most axiomatic set theories, e.g. ZFC, define functions as particular case of sets, Frege pursued a, so to say, "reverse path"conception of sets as particular case of functions. Then, Frege conceives concepts particular "boolean functions"that yield either the True or the False (henceforth truth-functions). In this sense, an object a falls under a concept F if and only if it is the case that $F(a)$ yields the True. This immediately led him into the notion of classes as value-ranges of such functions (*BuG*). In an unharmedly sense, this is quite in parallel with the naïve set theoretical notion of transformation of a property $P(x)$ (concept) into a set (value-range of the concept) by means of the axiom of unrestricted comprehension (abstraction). Then, the value-range of a concept F (truth-function) is called 'the extension of the concept F', which is actually the class of objects which fall under the concept F.

3 This comparison nevertheless collapses in light of the fact that Frege regards value-ranges as logical objects. For a more detailed discussion on Frege's conception of logical objects, see Ruffino, 2000; 2002; 2003.

value-range as the function $x^2 - 4x$. This, in his symbolism, could be expressed this way

$$\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \acute{\alpha}(\alpha^2 - 4\alpha) \tag{3.1}$$

where, recall, '$\acute{\epsilon}...(\epsilon)$' is the abstraction operator used to designate the expression 'the value-range of the function ϕ'. Hence, the above proposition means that the function $x(x - 4x)$ has the same value-range as the function $x^2 - 4x$. Similarly, Frege symbolizes

$$\overset{\mathfrak{a}}{\smile}\ \mathfrak{a}(\mathfrak{a} - 4\mathfrak{a}) = (\mathfrak{a}^2 - 4\mathfrak{a}). \tag{3.2}$$

where '$\overset{\mathfrak{a}}{\smile}\ ...\mathfrak{a}$' is the symbolism for the universal quantifier, so that the above proposition means the fact that the values of the functions $x(x - 4x)$ and $x^2 - 4x$ universally agree for every argument.

That being said, the Axiom V was so thought by Frege. More specifically, the Axiom V is the law that states that the value-range of two functions is the same iff it holds universally that those two functions have the same value for the same arguments. Hence, we have the following[4]:

$$\vdash (\acute{\epsilon}f(\epsilon) = \acute{\alpha}g(\alpha)) = \overset{\mathfrak{a}}{\smile}\ f(\mathfrak{a}) = g(\mathfrak{a}) \tag{V}$$

Before proceeding any further to the problem of indeterminacy, it is worth making some remarks about how the abstraction operator, which is ruled by the Axiom V, is supposed to introduce new names in the language.

3.1.1 The formation of abstraction terms

The introduction of the abstraction operator in Grundgesetze inaugurates the device of function abstraction into the system, which, formally speaking, is the idea of transforming function into corresponding 'objects' by means of variable-binding[5]. Notably, the introduction of the value-ranges names by the abstraction operator can be described by the following rule formation

if f stands for a function name and x any variable, then $\acute{x}f(x)$ is a value-range name.

4 The (V) stated here is slightly different from the (V) stated in the introduction. Here we reproduce the law in the exact way Frege conceived it, which means using the identity sign '=' as the most fundamental logical connective, instead of the biconditional '$\longleftrightarrow$'. However, this difference is not substantive, since both sides of the Axiom V refer to truth values, and then, in this context, the identity sign can be seen as functioning as a biconditional as well.

5 This technique is best known for its usage in formal systems such as lambda-calculus or combinatory logic. However, as de Queiroz, de Oliveira and Gabbay (2012) observe, function abstraction and function application (the dual of abstraction) are introduced in a systematical way by Frege already, when he develops techniques for "transforming functions (expressions with free variables) into value-range names (expressions with no free variables) by means of an 'introductory' operator of abstraction"(p. 3). They observe that Frege's abstraction operator is '$\acute{\epsilon}\phi(\epsilon)$' (regarded by Frege as a second-level function name that takes a first level function name f and maps a uniquely determined value-range name $\acute{\epsilon}f(\epsilon)$, see §31) and they identify a corresponding 'elimination' operator of abstraction, which is the operation of application of an argument in a value-range name through the function '$\cap$' of Grundgesetze.

Now one thing needs to be remarked. Although the above rule formation does generate new names, namely, value-range names, it does not guarantee them any semantics. As Frege sees it, they might be even "empty" names: one of his more powerful criticisms while exposing his arguments against the formalists is the objection that the mere stipulation of an operation, or the simple formation of a name, does not provide it a meaning by itself. This critique can be better expressed this way:

> Hankel introduces two sorts of operation, which he calls lytic and thetic, and which he defines by means of certain properties that they are to possess. There is nothing against this, so long as it is only not presupposed that operations of these sorts and objects such as their results would be exist. (*Gl*, §98)

As a result, it follows that the simple formation of value-range names by function abstraction does not provide a meaning by itself to those names. More accurately, it does not determine the reference of the value-range names.

Now, if the abstraction operator does not provide a meaning to them, who should play the role of fixing the reference of the value-range names?

3.1.2 The permutation argument

In the remaining sections of this chapter we now focus on exposing the argumentative part of the *Gg* I §10, the local where Frege addresses the problem of indeterminacy and proposes a solution for it. Although this is a relatively short part of the book, it is perhaps the most studied section of it, considering that it was investigated by many influential scholars such as Parsons (1965), Dummett (1981b), Benacerraf (1981), Wright (1983), Resnik (1986), Moore and Rein (1986, 1987), Schröder-Heister (1987), Ricketts (1997), Heck (1999, 2005), Schirn (2001), Ruffino (2002) and Greimann (2003), who have fairly contributed to its understanding.

In the beginning of this section, Frege realizes that the semantic stipulation of the meaning of the value-range names, introduced informally as the Axiom V at *Gg* I §3, does not fix the reference of those names as well. The complaint goes as follows:

> Although we have laid it down that the combination of signs '($\acute{\epsilon}\Phi(\epsilon) = \acute{\alpha}\Psi(\alpha)$)' has the same denotation as '$\overset{\mathfrak{a}}{\smile}\ \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', this by no means fixes completely the denotation of a term like '$\acute{\epsilon}\Phi(\epsilon)$'. We have only a means of always recognizing a value-range if it is designated by a term like '$\acute{\epsilon}\Phi(\epsilon)$', by which it is already recognizable as a value-range. But we can neither decide, so far, whether an object is a value-range that is not given us as such, and to what a function it may correspond, nor decide in general whether a given value-range has a given property unless we know that this property is connected with a property of the corresponding function. (*Gg* I, §10)

The problem of indeterminacy is therefore the problem that the Axiom V is unable to decide the truth value of sentences of the form '$\acute{\epsilon}f(\epsilon) = q$', and, hence, more accurately, the problem that the axiom is unable to fix the reference of the value-range names.

Immediately after stating the problem of indeterminacy, Frege famously articulates of a very ingenious mathematical argument, now known as the permutation argument[6]. The argument is given to evidentiate the indeterminacy and to technically show the Axiom V's inability to fix the reference of the value-range names. The permutation argument goes as follows:

a) Suppose Δ is an assignment of objects to value-range names in such a way it satisfies the Axiom V;[7]

b) Let h be a nontrivial permutation[8] of all objects in the domain of first-order variables, so that for some value-range a, $h(a) \neq a$;

c) Now consider a new assignment Δ' which is related to Δ as follows: if Δ assigns an object x to a value-range name, then Δ' assigns $h(x)$ as an alternative;

d) Since h is a nontrivial permutation, it follows that at least one value-range name would have both x or $h(x)$ as its referent, without the identity $x = h(x)$ being ensured.

(In other words, the argument states that if there is an assignment that satisfies the Axiom V, then, by a simple permutation, we can find many others, so that there is no uniquely determinate assignment of objects to value-range names that satisfies the Axiom V).

For a practical example, consider a very simple fragment of the language of the theory that consists of only two functions f and g. Suppose that it does not hold universally that f and g have the same value for the same arguments. Now consider an assignment Ω of objects to value-range names, such that a is the object assigned to the value-range name '$\acute{\epsilon}f(\epsilon)$' and b is the object assigned to the value-range name '$\acute{\epsilon}g(\epsilon)$'. (Because by our supposition it does not hold universally that the values of f and g have the same value for the same arguments, we can now say by the Axiom V that $a \neq b$). Now take the permutation argument and consider a new assignment Ω' which is almost the same as Ω except that $h(a)$ is the object assigned to the value-range name '$\acute{\epsilon}f(\epsilon)$' and $h(b)$ is the object assigned to the value-range name '$\acute{\epsilon}g(\epsilon)$', where h is such a nontrivial permutation. (Since Ω' is obtained from Ω by permutation of assignments, it is easy to verify that both are equiconsistent regarding the Axiom V). Since h is nontrivial, there

[6] Since Dummett 1981, p.408.

[7] Note that since the Axiom V is actually inconsistent, then there would be no real assignment of objects to value-range names able to satisfy it.

[8] It is uncontroversial in the literature whether the permutation h which is invoked in the argument has to be one which is expressible in the formal language or not. Dummett (1981) claims that it does not. Moore and Rein (1986, p.378) claim that it does, remarking that this perhaps explains Frege's additional remark "at least if there does exist such a function..."(*Gg* I, §10).

should exist at least one object x for which $x \neq h(x)$ and we cannot assume that neither $a = h(a)$ is *the* object assigned to $\acute{\epsilon}f(\epsilon)$ nor that $b = h(b)$ is *the* object assigned to $\acute{\epsilon}g(\epsilon)$. Since Ω' is obtained from Ω by a nontrivial permutation, it follows that Ω and Ω' will always be different assignments, though both satisfying the Axiom V.

If the Axiom V were able to determine the reference of the value-range names, then for each value-range name a unique referent should be assigned to it[9]. But the argument shows that different assignments of objects to value-range names respect the truth-values of instances of the Axiom V, so that no unique particular assignment of objects to value-range names can be determined.

Before we proceed any further, let's recall that the permutation argument can be also applied in to show that HP does not fix the reference of numerals:

a) Suppose Δ is an assignment of objects to number names in such a way it satisfies HP;

b) Let h be a nontrivial permutation of all objects in the domain of first-order variables, so that for some number a, $h(a) \neq a$;

c) Now consider a new assignment Δ' which is related to Δ as follows: if Δ assigns an object x to a number name, then Δ' assigns $h(x)$ as an alternative;

d) Since h is a nontrivial permutation, it follows that at least one number name would have both x or $h(x)$ as its referent, without the identity $x = h(x)$ being ensured.

As a result, if there exists an assignment that satisfies HP, then, by permutation, there will be others, so that there is no uniquely determinate assignment of objects to number names that satisfies HP.

3.2 The trans-sortal identification

Section 10 can be divided in two distinct parts, with each on them having a different purpose. The aim of the first, occupied by the permutation argument, is to evidentiate the indeterminacy. The goal of the second, pronounced by the trans-sortal identification, is to solve the problem. Frege's way of specifying the unique particular assignment of objects to value-range names is spontaneously stated after immediately articulating the

[9] The way Frege poses the problem of determining the reference of the value-range names, generalized somewhat, suggests that he intends to specify a unique particular interpretation of his theory, namely, the unique assignment of objects to terms that satisfies the axioms of the theory (a modern logician might reasonably ask why Frege is concerned with this, since a formal language can be defined apart from any interpretation of it). Dummett (1981b), however, claims that this is impossible since it could only be made in metalinguistic terms and, as we shall see, the means Frege adopts to specify the unique particular assignment of objects to value-range names is expressible in the object-language. Cf. Dummett, 1981b, p.405 for a detailed discussion.

permutation argument. He states that it is possible to overcome this indeterminacy "by its being determined for every function when it is introduced, what values it takes on for value-ranges as arguments, just as for all other arguments"(*Gg* I, §10). Now, we have to consider every function and determine its values[10]. Up to this point, he recalls that there are only three functions introduced in the theory: the horizontal '$\text{---}\,\xi$', the negation '$\text{-}\!\!\top\!\!\text{-}\,\xi$' and the identity '$\xi = \zeta$'. But Frege shows that if the identity is determined for value-ranges as arguments, the horizontal and the negation will be as well. This is so for the following reason: (i) because $\xi = \xi$ denotes always the True, the horizontal can be expressed through the identity, $\xi = (\xi = \xi)$; (ii) because we can define $\text{-}\!\!\top\!\!\text{-}\,\Delta$ as the negation of $\text{---}\,\Delta$, thus negation can be seen as a function applied to the horizontal. As a result, to determine what values $\xi = \zeta$ assumes for value-ranges as arguments is the same as to determine it for the horizontal and negation.

Taking this into account, Frege justifiably addresses himself to the determination of the values that the identity assumes for value-ranges as arguments. Then, he says:

> Since up to now we have introduced only the truth-values and value-ranges as objects, it can only be a question of whether one of the truth-values can perhaps be a value-range[11]. (*Gg* I, §10)

Nevertheless, the Axiom V does not state whether value-ranges are truth values or not. It is ultimately reduced to comprise a relation about value-ranges with value-ranges (that is, the value-range of two functions is the same iff it holds universally that those two functions have the same value for the same arguments). It says nothing about whether truth values are value-ranges, or, at least, if they do are, what is their corresponding function. As we shall see in the next chapter, the only kind of singular terms expressible in Grundgesetze are the value-range and truth value names. Hence, the difficulty to determine "whether one of the truth-values can perhaps be a value-range"can be reduced to the difficulty of deciding whether any sentence of the form '$\acute{\varepsilon}f(\varepsilon) = q$' holds, where q is not a value-range name[12]. But this is of no help, because every instance of the form '$\acute{\varepsilon}f(\varepsilon) = q$' is actually an undecidable statement in Grundgesetze[13].

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[10] The real justification for this performing this particular procedure will only be clear by *Gg* I §29, where Frege lays down the semantic rules of his system. In particular, for a name of second-level function of one argument $\acute{\epsilon}\phi(\epsilon)$ succeed in referring any of its substitutional instances should be referring as well. See the chapter 4, particularly section 4.1.1 (d).

[11] How far this justification is acceptable, it will be something that we will discuss later on.

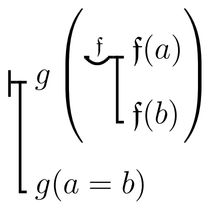
[12] It is interesting to remark how the question of whether a truth value is a value-range, which is primarily ontological, is reduced to the semantic question of whether sentences of the form '$\acute{\varepsilon}f(\varepsilon) = q$' have a reference. This is what Greimann (2003) has pointed out when he, as we stressed in the section 2.2.a, charged Frege of being a moderate structuralist.

[13] A statement is undecidable if it is neither provable nor refutable in a specified deductive system. It should not be misunderstood with a second usage of the term concerning undecidability of theories: A theory, i.e. set of formulas closed under logical consequence, is said to be undecidable if there is no algorithm that calculates, for every formula whether or not it belongs to the theory—though we

Expounding this proof of undecidability would lead us too far from our objective. Instead, we can make some interesting remarks on this direction. For the sake of argument, suppose that some sentence of the form '$\acute{\varepsilon}f(\varepsilon) = q$' is decidable in the system. Without loss of generality, take this sentence to be

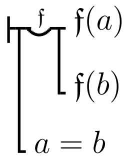
$$\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile} (\mathfrak{a} = \mathfrak{a}). \tag{3.3}$$

Thus there is either a proof or a refutation of this statement. First, suppose that we succeed in constructing a proof of it. Then, we set down a proof in which each line is either (i) an axiom or (ii) follows from earlier lines of the proof by modus ponens[14]. Since the last line of this proof is the very statement itself, which in turn is an identity statement, it would be interesting (but not sufficient) to show that the chief axiom governing the function $\xi = \zeta$ is unable to prove it, namely, the Axiom III:



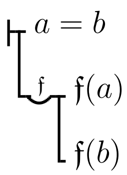
(III)

Although its reading might seem obscure from the start, the Axiom III encompasses all the main principles governing the function $\xi = \zeta$. Some are derived in *Gg* I §50. Among them we find Frege's commitment to the Leibniz Law. First, by instantiating g with the horizontal $—\xi$, we derive the well known principle of the *indiscernibility of identicals*



(IIIa)

Second, by instantiating g with the negation $\neg\xi$ instead, we derive, from contraposition (*Gg* I, §15), its converse, the principle of the *identity of indiscernibles*



(IIIi)

The principle of the identity of indiscernibles will be particularly useful in our intended demonstration. Just observe that it allows us to prove a general identity statement from modus ponens.

can conclude that Grundgesetze is undecidable in this second sense too, since it is partly consisted of first-order logic, which was shown to be undecidable independently by Church (1936) and Turing (1936).

14 For simplicity sake, we are ignoring the other two rules of inference set forth in the book. Cf. *Gg*, §§15-16

Now, let us instantiate a with $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))$ and b with $\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})$. Then we obtain a particular instantiation of the principle, namely

$$\begin{array}{l} \vdash\!\top\; \acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a}) \\ \;\;\llcorner\overset{\mathfrak{f}}{\smile}\!\top\; \mathfrak{f}(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))) \\ \qquad\;\llcorner\; \mathfrak{f}(\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \end{array} \tag{3.4}$$

from which we only need to prove the antecedent

$$\begin{array}{l} \vdash\!\overset{\mathfrak{f}}{\smile}\!\top\; \mathfrak{f}(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))) \\ \qquad\;\llcorner\; \mathfrak{f}(\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \end{array} \tag{3.5}$$

to finish our proof obtaining its consequent by modus ponens. We can do this by second-order universal generalization, assuming the existence of an arbitrary first-level function f and demonstrating that the following holds:

$$\begin{array}{l} \vdash\!\top\; f(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))) \\ \;\;\llcorner\; f(\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \end{array} \tag{3.6}$$

But this is impossible. It suffices to observe that, since Grundgesetze does not preclude the possibility of an impredicative construction (from which Russell's paradox is a famous example), we can construct, say, '$\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \xi$' as a particular first-level function of the language. As a result, to prove that an arbitrary first-level function with the property (3.5) exists, we need to prove that the property holds particularly for $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \xi$, that is, we need to prove that

$$\begin{array}{l} \vdash\!\top\; \acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) \\ \;\;\llcorner\; \acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a}) \end{array} \tag{3.7}$$

holds, what is obviously a *petitio principii*. Hence, we cannot obtain the antecedent, and, for the same reason, prove that $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})$ holds.

Similarly, we can show that, using the Axiom III there is no refutation of $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})$ either. Suppose that we succeed in constructing a disproof of it, i.e. suppose that we construct a proof of $\neg\,(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a}))$. For it, we apply the rule of contraposition in the principle of indiscernibility of identicals to obtain a slightly variant of it, namely:

$$\begin{array}{l} \vdash\!\top\!\!\neg\; (a = b) \\ \;\;\llcorner\!\neg\overset{\mathfrak{f}}{\smile}\!\top\; \mathfrak{f}(a) \\ \qquad\quad\;\llcorner\; \mathfrak{f}(b) \end{array} \tag{IIIa'}$$

Then, we instantiate a with $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))$ and b with $\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})$:

$$\begin{array}{l} \vdash\!\!\top (\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \\ \quad \llcorner\!\overset{\mathfrak{f}}{\smile}\!\top \mathfrak{f}(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))) \\ \qquad\quad \llcorner\, \mathfrak{f}(\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \end{array} \tag{3.8}$$

In order to finish it we need to show that the antecedent holds.

This sentence means that there exists a first-level function f such that:

$$\begin{array}{l} \vdash\!\!\top f(\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon))) \\ \quad \llcorner\, f(\overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})) \end{array} \tag{3.9}$$

and since up to now we have only introduced the horizontal, negation and identity as functions, it is only a matter of whether some of those functions satisfy this property.

However, the values of the horizontal and negation were not yet determined for value-ranges as arguments, they depend on the determination of the identity function. Thus, we are left with the identity function alone. Yet, for the identity function be determined for value-ranges as arguments, we need to decide "whether one of the truth-values can perhaps be a value-range", which is precisely what we were trying to find out. As a result, we cannot obtain this antecedent, and, hence, we cannot complete our disproof of $\acute{\epsilon}(\epsilon(\epsilon - 4\epsilon)) = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a})$ using the Axiom III as well. This is so because this sentence, and, without loss of generality, any sentence of the form '$\acute{\varepsilon}f(\varepsilon) = q$' are neither provable nor refutable. They are undecidable statements in the formalism.

3.2.1 The identification of the truth values

As we saw, Frege's way of solving the indeterminacy consists in determining "for every function when it is introduced, what values it takes on for value-ranges as arguments, just as for all other arguments" (*Gg* I, §10). And the determination of the values that the horizontal and negation assume for value-ranges as arguments was reduced by means of the identity. But this suggestion could not solve the problem, since we have no means to determine "whether one of the truth-values can perhaps be a value-range". The problem is that up to now Grundgesetze is unable to either prove or refute the formalization of such question, which takes the form identity statements of the form '$\acute{\varepsilon}f(\varepsilon) = q$'. Therefore, the axioms of Grundgesetze are insufficient to determine its intended interpretation. Recall that the permutation argument shows us the Axiom V cannot ensure uniqueness of interpretation for value-range names in the system. This being the case, we have no means to determine what (from a very large range of different possible assignments) the referent of a certain term $\acute{\varepsilon}f(\varepsilon)$ is supposed to be. Putting in another way, it could possibily refer to any object of the domain. Since this very result does not rule out the possibility of the reference of a value-range name being the same as q's in any '$\acute{\varepsilon}f(\varepsilon) = q$' statement, the permutation argument also implies a stronger claim: value-ranges *might be* truth values. (In other words, this would not contradict any axiom of the system. Indeed, what the permutation argument shows is that we are powerless to prove that any given object in the domain is *not* a value-range, since we could always construct a counterexample by reductio, considering a particular permutation h').

Thus, the indeterminacy can be curiously overcome by the tecnical advantages of the permutation argument itself. As we saw, the argument shows that if a given assignment satisfies the Axiom V, then there will be others. Frege then demands a solution to the indeterminacy problem by drawing a particular instance of the argument:

a) Suppose Δ is an assignment of objects to value-range names in such a way it satisfies the Axiom V;

b) Let f and g be two arbitrary non-coextensional functions. Consider also that Δ assigns a to $\acute{\varepsilon}f(\varepsilon)$ and b to $\acute{\varepsilon}g(\varepsilon)$;

c) Let h be the following nontrivial permutation of all objects in the domain of first-order variables:

$$h(x) = \begin{cases} \text{True} & \text{if } x = a, \\ a & \text{if } x = \text{True} \\ \text{False} & \text{if } x = b, \\ b & \text{if } x = \text{False} \\ x & \text{otherwise} \end{cases}$$

d) Now consider a new assignment Δ' which is related to Δ as follows: if Δ assigns an object x to a value-range name, then Δ' assigns $h(x)$ as an alternative;

e) It follows that without contradicting the Axiom V, we can identify the value-ranges of two arbitrary non-coextensional functions f and g with the True and False, respectively.

(In other words, the argument states what we remarked already, namely, that without falling into inconsistency with the Axiom V, the two truth values can be identified with the value-ranges of two arbitrary non-coextensional functions[15]).

Note that the chosen functions should not be coextensionals, otherwise we could fall in contradiction with the fact that the True and the False are distinct objects. Besides,

[15] Often known as the identifiability thesis, Cf. Schroeder-Heister, 1987. The legitimacy of this thesis is not univocal, and Schroeder-Heister (1987) himself claims it is mistaken. However, to present the arguments that led Schroeder-Heister to this claim would require us to take a long detour around our subject. On a reply to the criticism of Schroeder-Heister, Cf. Moore, and Rein, 1987, especially p. 52. In this short article, these scholars present a clear defense of the theory of identifiability, arguing against Schroeder-Heister's reading that it is unsustainable.

because it would not be contradictory that value-ranges be truth values, Frege feels free to perform his so-called trans-sortal identification, which is the stipulation that the True and the False are to be identified with particular value-ranges. The identification goes like this:

a) Consider f to be the horizontal function $—\,\xi$ and g to be the 'being the False' function $\xi = \;\neg\overset{\mathfrak{a}}{\smile}\,(a = a)$

b) Let $\acute{\varepsilon}(—\,\varepsilon)$ denote the True and $\acute{\varepsilon}(\varepsilon = \;\neg\overset{\mathfrak{a}}{\smile}\,(a = a))$ denote the False.

Now, because we are currently able to decide in any case if a value-range is a truth value, Frege concludes that "with this we have determined the value-range so far as is here possible"remarking that "as soon as there is a further question of introducing a function that is not completely reducible to known functions already, we can stipulate what value it is to have for value-ranges as arguments", thereby concluding that "this can be regarded as much as a further determination of the value-ranges as of that function"(*Gg* I, §10).

But what Frege did mean by "so much is here possible"is obscure for two reasons. First, is it not obvious why, in order to determine what the reference of the value-range names are, it is sufficient that we stipulate for every function what values it takes on for value-ranges as arguments. In the next chapter, we shall suggest that his argument, though quite vague in *Gg* I §10, is justified by the semantic rules stated in §29. Second, this claim suggests that the system is lacking a further determination of the value-range names, in the sense that we do not know whether other objects, say, Julius Caesar, is a value-range (and if he is, what is his corresponding function).

3.2.2 The proposal of the footnote

Before we proceed any further, some scholars such as Wright (1983) are convinced that Frege attempted to articulate an answer for the second doubt. In fact, section 10 contains a very pregnant footnote, where special attention is certainly deserved. Recall that, up to this point, all that the formalism is able to prove and refute about mixed identity statements is whether value-ranges are truth values or not. This is thanks to the trans-sortal identification which can be seen as performing an identification of truth values with their singletons. This is confirmed by the fact that the identification stipulates nothing but that the True and the False are to be identified with the value-range of the functions 'is identical to the True' and 'is identical to the False'.

For this reason, what is ultimately done by the trans-sortal identification is to stipulate the intuitive idea:

a) Consider f to be the function 'is identical to the True' and g to be the function 'is identical to the False';

b) Let $\acute{\varepsilon}$(ε is identical to the True) denote the True and $\acute{\varepsilon}$(ε is identical to the False) denote the False.

In this footnote, Frege considers the possibility of extending this stipulation, attempting to perform an identification of every object given to us to its singleton, namely, to establish that for every a given to us, a be the value-range of the function 'is identical to a'. Formally, the attempt is to stipulate that the following holds for any object name Δ

$$\Delta = \acute{\varepsilon}(\Delta = \varepsilon). \tag{3.10}$$

so that all objects (including those who are already identified as value-ranges) could be identified with their singletons.

Surprisingly enough, Frege remaks that this attempt would be misleading. He says that by including in the stipulation those objects that are already given to us by value-range names, we would, by reiteration, awkwardly admit "singletons of classes", which do not necessarily respect the Axiom V. To show this, consider the object given by the value-range name '$\acute{\varepsilon}(\neg\, \varepsilon)$'. Now, the intended stipulation, provides us the following identification:

$$\acute{\varepsilon}(\neg\, \varepsilon) = \acute{\alpha}(\acute{\varepsilon}(\neg\, \varepsilon) = \alpha) \tag{3.11}$$

which, in turn, leads us to infer by the Axiom V that the following also holds:

$$\vdash\!\overset{\mathfrak{a}}{\smile}(\neg\, \mathfrak{a}) = (\acute{\varepsilon}(\neg\, \varepsilon) = \mathfrak{a}) \tag{3.12}$$

But it is false, since, by stipulation of the negation function, the lefthand side of the identity, $\overset{\mathfrak{a}}{\smile}\neg\, \mathfrak{a}$, is to hold provided that $\mathfrak{a}$ does not denote the True, whereas its right side, $\overset{\mathfrak{a}}{\smile}(\acute{\varepsilon}(\neg\, \varepsilon) = \mathfrak{a})$, holds whenever '$\mathfrak{a}$' is to denote the False.

Of course, this stipulation only becomes a problem if it is applied to objects already given by value-range names, and we might be led to think that restricting the scope of our stipulation would make it technically plausible. Frege's reaction against this suggestion is is nevertheless negative. Then, he complains:

> the way in which an object is given must not be regarded as an immutable property of it, since the same object can be given in a different way (*Gg* I, §10, n.2).

This objection seems to support Frege's realist view of mathematics, since, considering the doctrine in which value-ranges do exist objectively, it is natural to think that they can be given to us by different guises. For instance, consider the value-range referred by '$\acute{\varepsilon}(-\, \varepsilon)$'. By the Axiom V, we come to know that we can refer to it by different names. Particularly, we can refer to it by value-range names—such as '$\acute{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$', '$\acute{\varepsilon}(\varepsilon = \overset{\mathfrak{a}}{\smile}(\mathfrak{a} = \mathfrak{a}))$' or

‘$\acute{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$’— truth value names—such as ‘$\smile^{\mathfrak{a}} (\mathfrak{a} = \mathfrak{a})$’—or, even other object names q that we can *come to know* as well[16].

In other words, the point is that (except in the case of truth values) we still do not know which objects can come to be value-ranges. Yet, we cannot overcome this difficulty at the present situation, since this further determination of value-ranges is the very problem that our stipulation is supposed to solve.

Yet, suppose that we do want to set forth $\Delta = \acute{\varepsilon}(\Delta = \varepsilon)$ as a restricted stipulation, i.e. for any a non-value-range name Δ, $\Delta = \acute{\varepsilon}(\Delta = \varepsilon)$. Now consider an arbitraty non-value-range name Γ. Since Γ is a object name, but not a value-range name, the stipulation is applicable to it, so that we obtain the identity statement ‘$\Gamma = \acute{\varepsilon}(\Gamma = \varepsilon)'$. As we remarked earlier, the chief laws of identity are ruled by the Axiom III and its theorems, so that we cannot expect that this stipulation will contradict it. Instantiating the principle of indiscernibility of identicals we obtain the following:

$$\vdash \smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\acute{\varepsilon}(\Gamma = \varepsilon)) \\ \Gamma = \acute{\varepsilon}(\Gamma = \varepsilon) \end{array} \tag{3.13}$$

Now, since by our stipulation ‘$\Gamma = \acute{\varepsilon}(\Gamma = \varepsilon)$’ holds, by modus ponens, we have that:

$$\vdash \smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\acute{\varepsilon}(\Gamma = \varepsilon)) \end{array} \tag{3.14}$$

Which is to say that everything that holds of Γ must hold of $\acute{\varepsilon}(\Gamma = \varepsilon)$ as well. Now, consider the function *being identified with its singleton*, namely, ‘$\xi = \acute{\varepsilon}(\xi = \varepsilon)$’. Then, by a simple instantiation of (3.14) we obtain that the following must hold:

$$\vdash \begin{array}{l} \Gamma = \acute{\varepsilon}(\Gamma = \varepsilon) \\ \acute{\varepsilon}(\Gamma = \varepsilon) = \acute{\alpha}(\acute{\varepsilon}(\Gamma = \varepsilon) = \alpha) \end{array} \tag{3.15}$$

and, by a very similar, argument, that the following must hold as well:

$$\vdash \begin{array}{l} \acute{\varepsilon}(\Gamma = \varepsilon) = \acute{\alpha}(\acute{\varepsilon}(\Gamma = \varepsilon) = \alpha) \\ \Gamma = \acute{\varepsilon}(\Gamma = \varepsilon) \end{array} \tag{3.16}$$

But it is easy to see that the restricted stipulation implies that (3.16) should be false. As a result, we have a contradiction caused by (i) the fact that $\Delta = \acute{\varepsilon}(\Delta = \varepsilon)$ is a

[16] This is basically the same objection presented in *Grundlagen* section 67. There, Frege rejects a proposal of treating the way that directions are introduced as an intrinsic property of them: “[i]f, moreover, we were to adopt this way out, we should have to be presupposing that an object can only be given in one single way; for otherwise it would not follow, from the fact that q was not introduced by means of our definition, that it *could* not have been introduced by means of it”.

restricted stipulation that holds only for non-value-range names Δ and (ii) the fact that everything that holds of Δ must hold of $\acute{\varepsilon}(\Delta = \varepsilon)$ as well. This contradiction is found since the stipulation can itself be described as a function that will hold of Δ but not hold of $\acute{\varepsilon}(\Delta = \varepsilon)$. Then, by our hypothesis that $\Delta = \acute{\varepsilon}(\Delta = \varepsilon)$ is a restricted stipulation we found a contradiction. As a result, by reductio, there can be no such restricted stipulation to non-value-range names—otherwise it will contradict the laws of identity.

Interestingly enough, Schirn (2001) seems to defend the plausibility of this restricted stipulation. Schirn questions the fact that the restricted stipulation necessarily would imply the illegitimate consideration of the way an object is given as an essential characteristic of it, improperly assuming that every object Δ not given as a value-range is not, in effect, a value-range[17]. But Schirn's objection seems to overlook the technical result that we showed above, namely, that the admissibility of this stipulation results in a contradiction with the chief laws of identity in the formalism. Yet, against what Schirn would like to admit, this contradiction is precisely motivated by the fact that the stipulation regards the way that an object is given as an immutable characteristic of it: the restricted stipulation identify objects distinguishing them by their names, at the same time. As a result, any reason for admitting a stipulation restricted to non-value-range names seems, therefore, unsustainable.

In the same footnote, Frege considers a second possibility. In order to extend his trans-sortal identification in a consistent way with the Axiom V, he notices that the stipulation $\Delta = \acute{\varepsilon}(\Delta = \varepsilon)$ is a particular case of $\Delta = \acute{\varepsilon}f(\Delta, \varepsilon)$, where $f(\xi, \zeta)$ stands for a binary function. The question, thus, is whether there is another function that can satisfy this property, but without falling into contradiction with the Axiom V.

Up to now we have only the horizontal, negation and identity functions introduced so far. Cleary, $f(\xi, \zeta)$ could be the identity function $\xi = \zeta$, but in this case we know already that the stipulation is contradictory. The horizontal and negation are unary functions, and thus they are out of question. As Frege remarks, the only technically attractive possibility turns out to be the function $\xi \cap \zeta$, defined in *Gg* §34 as:

$$a \cap u = \backslash\acute{\alpha}\left(\begin{array}{l} \neg\overset{\mathfrak{g}}{\smile}\!\top\; \mathfrak{g}(a) = \alpha \\ \qquad\;\; \llcorner\; u = \acute{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right) \qquad \text{(Def. } \cap\text{)}$$

As it turns out, $\xi \cap \zeta$ is the converse of the abstraction-introduction operator, the abstraction-elimination operator[18]. By definition, this is the function enjoying the property of $\Delta \cap \acute{\varepsilon}f(\varepsilon) = f(\Delta)$.

Consider $f(\xi, \zeta)$ to be $\xi \cap \Delta$. Since $\Delta = \acute{\varepsilon}f(\Delta, \varepsilon)$ is the desired property of *f*, we

[17] Cf. Schirn, 2001, n.4

[18] See note 5.

only need to prove that $\Delta = \acute{\varepsilon}(\varepsilon \cap \Delta)$ is to hold. Recall that the possibility of Δ being a value-range name is included: since we cannot restrict the stipulation, we must make sure this case does not fall into contradiction with the Axiom V.

Suppose Δ is value-range name. To keep this in mind let us denote it by '$\acute{\varepsilon}g(\varepsilon)$' instead. Then

$$\xi \cap \Delta = \backslash\acute{\alpha}\left(\smile^{\mathfrak{g}} \begin{array}{l} \mathfrak{g}(\xi) = \alpha \\ \Delta = \acute{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right) \tag{3.17}$$

shall be the same as

$$\xi \cap \acute{\varepsilon}g(\varepsilon) = \backslash\acute{\alpha}\left(\smile^{\mathfrak{g}} \begin{array}{l} \mathfrak{g}(\xi) = \alpha \\ \acute{\varepsilon}g(\varepsilon) = \acute{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right) \tag{3.18}$$

which, in turn, yields $g(\xi)$. Thus, $g(\xi) = \xi \cap \acute{\varepsilon}g(\varepsilon)$. By abstraction on $g(\xi)$ and $\xi \cap \acute{\varepsilon}g(\varepsilon)$, $\acute{\alpha}g(\alpha) = \acute{\alpha}(\alpha \cap \acute{\varepsilon}g(\varepsilon))$ is obtained by the Axiom III. Yet, since Δ stands for '$\acute{\varepsilon}g(\varepsilon)$', then $\Delta = \acute{\varepsilon}(\varepsilon \cap \Delta)$, which is the property we wanted to prove.

What we are left to prove is that this stipulation is consistent with the theory whenever Δ is a non-value-range name. By the definition of '$\backslash\xi$', then, if Δ is a non-value-range name we obtain

$$\xi \cap \Delta = \acute{\alpha}\left(\smile^{\mathfrak{g}} \begin{array}{l} \mathfrak{g}(\xi) = \alpha \\ \Delta = \acute{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right) \tag{3.19}$$

which is to say: $\xi \cap \Delta$ is the same as a mapping from every argument to the False (since there is no function that satisfies the above condition). This being the case, the corresponding extension

$$\acute{\eta}\acute{\alpha}\left(\smile^{\mathfrak{g}} \begin{array}{l} \mathfrak{g}(\xi) = \alpha \\ \Delta = \acute{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right)$$

which in Frege's own terminology corresponds to say that there are no objects that fall under this concept.

Consequently, as Ruffino (2002) observes, it seems that Frege had in mind to identify objects that are not value-ranges with value-ranges of relations that have no instances (a proposal absolutely distinct from its initial idea, namely, to identify each object with its singleton).

Nevertheless, this possibility is from the start not feasible. The function $\xi \cap \zeta$, Frege remarks, is defined through the very notion of value-ranges, so that would incur circularity if we intend to use it to generalize the trans-sortal identification (whose purpose is to determine the reference of the value-range names). As a result, no purpose to further determine the reference of the value-range names in the system was attainable by Frege.

4 Referentiality and determination

4.1 Is Frege's solution enough?

In the beginning of the last chapter we saw how the notion of value-ranges was introduced through the Axiom V to express the notion of extensions in Grundgesetze. This was motivated by the definition of numbers as extensions in *Grundlagen*. Extensions became value-ranges, and hence numbers became value-ranges as well. In the initial passage of *Gg* I §10, however, Frege identified the problem of indeterminacy, stated with the support of the permutation argument, which is seen as the failure of the Axiom V to fix the reference of the value-ranges names. In the end of the last chapter, we saw that, thanks to the trans-sortal identification, Frege was now able to either prove or refute identity statements having, on the one side, value-range names and, on the other side, truth value names and thus he has claimed to have determined the reference of value-range names after all (although the proposal of his footnote is failed).

Over the last few decades, many Fregean scholars have raised doubts in respect to whether Frege's claim is, indeed, appropriate. Since Frege's way of solving the Julius Caesar problem in *Grundlagen* was to define numbers as extensions (and, hence, value-ranges), the determination of the reference of value-range names is expected to fix the reference of numerals as well. Therefore, it is plausible to expect that any complete determination of the reference of the value-range names decide whether or not Julius Caesar is the value-range of some function. Ultimately, because, on the one hand, the proposal of the footnote of generalizing the trans-sortal identification is failed, and, on the other hand, at any point of *Gg* I §10 the presence of all objects (people, mountains, trees) is taken into account in order to completely fix this indeterminacy, many commentators conclude that Frege was unable to solve a persistent version of the Julius Caesar problem, which has now somewhat appeared affecting the value-ranges names (Parsons, 1965; Dummett, 1981b; Wright, 1983; Ricketts, 1997; Heck, 1997, 1999a, 2005; Schirn, 2001).

Dummett (1981) devotes much of a book to expose his position against Frege's claim that the reference of value-ranges names was completely determined. Not surprisingly, he sees a nearly parallel between the difficulties mentioned in *Gg* I §10 and the Julius Caesar problem. The only difference, Dummett points out, is that when asked whether Julius Caesar is a number in *Grundlagen* it is assumed that we want a negative answer, whereas at *Gg* I §10 one cannot assume anything about identity sentences having, on one side, value-range names, and, on the other, names of another kind (because we do not know yet what value-range names are). Dummett also finds strange that on the one hand, Frege makes no specification about the domain of objects of *Grundgesetze* so that the reader is free to understand that variables range over any objects and, secondly, that his trans-sortal identification to determine the reference of value-range names is ultimately restricted to two objects only, the True and the False. Dummett admits that regarding the axioms of the formal theory only value-ranges and truth values are required in the domain. However, he argues, a complete determination of the reference of the value-range names naturally would require us to answer if any value-range is identical to the Moon—of course, if one intends it to be in the domain. In this case, Dummett concludes, it would be irrelevant whether concrete objects are not required by the axioms of the theory or whether they cannot be referred to by their language.

Wright (1983) notes that the suggestion in *Grundlagen* of defining numbers as extensions simply postpones the difficulty for *Grundgesetze*, since identifying numbers with extensions does not solve the Julius Caesar problem for numbers unless the problem is solved for extensions. Wright interprets the difficulty found in *Gg* I §10 as a persistent version of the problem, now affecting the value-ranges. Dummett, of course, also endorses this position. The only motivation for introducing extensions in *Grundgesetze*, observes Wright, is to solve the problem of mixed identities between terms of numbers and other terms not given in this way. Nevertheless, Wright concludes that, ironically, not only the problem is left unsolved, but the theory is affected by the inconsistency of the Axiom V.

Heck (1999) interprets the problem of fixing the value-range names in *Gg* I §10 essentially in the same way as Dummett and Wright. Nevertheless, he acknowledges that *Gg* I §10 is almost impossible to be understood without taking into account the arguments of *Gg* I §31. The latter concerns the semantic properties of the *Grundgesetze*. In a long and detailed account of the arguments of Frege in *Gg* I §10 Heck argues that a proper understanding of this section gives us the answer to the question why, after taking notice of Russell's paradox, Frege did not abandon the Axiom V of his theory and returned to the second attempt of definition of *Grundlagen*, the one that, despite all its technical advantages, could not solve the problem of mixed identities statements between terms numerals and other terms not given in this way. Heck concludes that the trans-sortal identification is not sufficient to eliminate the Julius Caesar problem. As already mentioned, Frege justifies his restriction to the True and the False by observing that "up to now we have introduced only the truth-values and value-ranges as objects" (*Gg* I, §10) but, Heck concludes, what objects were introduced to some extent is irrelevant, what matters is what objects are in the domain.

Now, the Julius Caesar problem for value-ranges naturally arises from the conception of the Julius Caesar problem as mainly the semantic problem of fixing the reference of number names. It points out that the semantic stipulations of system are not sufficient to determine completely the reference of the value-range names (hence number names). On the one hand, the semantic stipulation of the Axiom V does not determine what kind of object we are talking about when we talk about value-ranges. On the other hand, the notion of value-range is treated by Frege as a primitive notion, hence, it would be fair to expect that value-range names refer to value-ranges (Greimann, 2003). But identifying the True and the False with value-ranges of specific functions, the trans-sortal identification, does not seem to ensure this. It does not seem to specify what value-ranges are, and, above all, the referent of value-range names.

In our view, to take the Julius Caesar problem as, mainly, a semantic problem is mistaken. First, this is supported by the fact that we can see the principle of complete determination in such a way that it does not require such complete determination. So, considering the semantics of Grundgesetze, there is no indeterminacy left for the notion of value-ranges. Second, we can see the Julius Caesar problem as the matter of showing the nature of numbers of logical objects, so that there is no Julius Caesar problem for value-ranges. We will address these topics at the end of this chapter. Now, for this purpose, we need to have a better understanding of the semantics of Grundgesetze in advance.

4.1.1 Frege's semantic rules

The semantics of Grundgesetze is usually thought to be built in a special kind of contextualism which Dummett called "the generalized context principle for reference". According to Dummett, it says, roughly, that a name in question has a reference provided that every more complex names of which it is constituent has a reference. Indeed, Dummett's claim is apparently confirmed as we look at the semantic rules that Frege carries out in *Gg* I §29. In this section Frege lays down general conditions for which each of the five logical types of name of his formal language (proper names, first-level function names of one argument, first-level function names of two arguments, second-level function names of one argument and the third-level function names of one argument) succeeds in referring:

> A name of a first-level function of one argument succeeds has a *denotation* (*denotes* something, succeeds in denoting) if the proper name that results from this function-name by its argument-places' being filled by a proper name always has a denotation, if the name substituted denotes something.
>
> A proper name has a *denotation* if the proper name that results from that proper name's filling the argument places of a denoting name of a first-level function of one argument always has a denotation, and if the name of a first-level function of one argument that results from the proper name in question's filling the ξ-argument-places of a denoting name of a first-level function of two arguments always has a denotation, and if the same holds also for the ζ-argument-places.
>
> A name of a first-level function of two arguments has a *denotation* if the proper name that results from this function-name by its

> ξ-argument-places' being filled by a denoting proper name and its ζ-argument-places' being filled by a denoting proper name always has a denotation.
>
> A name of a second-level function of one argument of type 2 has a *denotation* if, from the fact that the name of a first-level function of one argument denotes something, it follows generally that the proper name that results from its being substituted in the argument-places of our [name of a] second-level function has a denotation.
>
> It follows that every name of a first-level function of one argument which, combined with every denoting proper name, forms a denoting proper name, is also such that if combined with any denoting name of a second-level function of one argument of type 2, it forms a denoting [proper] name.
>
> The name "$\overset{\mathfrak{f}}{\smile}\ \mu_{\beta}(f(\beta))$"of a third-level function succeeds in denoting, if, from the fact that a name of a second-level function of one argument of type 2 denotes something, it follows generally that the proper name that results from its being substituted in the argument-place of "$\overset{\mathfrak{f}}{\smile}\ \mu_{\beta}(f(\beta))$"has a denotation. (*Gg* I, §29)

So Frege's semantic rules basically states that the reference of a compound name is yielded by the reference of its components. To put it more clear, such rules could be reformulated as follows:

a) Names of first-level function of one argument:

- '$—\ \xi$' succeeds in referring if '$—\ \Delta$' succeeds in referring, provided that 'Δ' does as well.
- '$-\!\!\!\top\ \xi$' succeeds in referring if '$-\!\!\!\top\ \Delta$' succeeds in referring, provided that 'Δ' does as well.
- '$\backslash\xi$' succeeds in referring if '$\backslash\Delta$' succeeds in referring, provided that 'Δ' does as well.

b) Proper names:

- 'Δ' succeeds in referring if
  - '$—\ \Delta$' succeeds in referring, provided that '$—\ \xi$' does as well.
  - '$-\!\!\!\top\ \Delta$' succeeds in referring, provided that '$-\!\!\!\top\ \xi$' does as well.
  - '$\backslash\Delta$' succeeds in referring, provided that '$\backslash\xi$' does as well.
  - '$\xi = \Delta$' and '$\Delta = \zeta$' succeed in referring, provided that '$\xi = \zeta$' does as well.
  - '$\begin{array}{l}\top\ \Delta\\ \llcorner\ \xi\end{array}$' and '$\begin{array}{l}\top\ \zeta\\ \llcorner\ \Delta\end{array}$' succeed in referring, provided that '$\begin{array}{l}\top\ \zeta\\ \llcorner\ \xi\end{array}$' does as well.

c) Names of first-level function of two arguments:

- '$\xi = \zeta$' succeeds in referring if '$\Delta = \Gamma$' succeeds in referring, provided that 'Δ' and 'Γ' do as well.
- '$\begin{array}{l}\mathrel{\top}\zeta\\ \llcorner\,\xi\end{array}$' succeeds in referring if '$\begin{array}{l}\mathrel{\top}\Gamma\\ \llcorner\,\Delta\end{array}$' succeeds in referring, provided that 'Δ' and 'Γ' do as well.

d) Names of second-level function of one argument:

- '$\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})$' succeeds in referring if
  - '$\overset{\mathfrak{a}}{\smile}\!\!-\,\mathfrak{a}$' succeeds in referring, provided that '$-\,\xi$' does as well.
  - '$\overset{\mathfrak{a}}{\smile}\!\!\mathrel{\top}\,\mathfrak{a}$' succeeds in referring, provided that '$-\!\!\mathrel{\top}\,\xi$' does as well.
  - '$\overset{\mathfrak{a}}{\smile}\,\backslash\mathfrak{a}$' succeeds in referring, provided that '$\backslash\xi$' does as well.
  - '$\overset{\mathfrak{a}}{\smile}\,\xi = \mathfrak{a}$' and '$\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \xi$' succeed in referring, provided that '$\xi = \zeta$' does as well.
  - '$\overset{\mathfrak{a}}{\smile}\begin{array}{l}\mathrel{\top}\mathfrak{a}\\ \llcorner\,\xi\end{array}$' and '$\overset{\mathfrak{a}}{\smile}\begin{array}{l}\mathrel{\top}\zeta\\ \llcorner\,\mathfrak{a}\end{array}$' succeed in referring, provided that '$\begin{array}{l}\mathrel{\top}\zeta\\ \llcorner\,\xi\end{array}$' does as well.
- '$\acute{\epsilon}\phi(\epsilon)$' succeeds in referring if
  - '$\acute{\epsilon}(-\,\epsilon)$' succeeds in referring, provided that '$-\,\xi$' does as well.
  - '$\acute{\epsilon}(-\!\!\mathrel{\top}\,\epsilon)$' succeeds in referring, provided that '$-\!\!\mathrel{\top}\,\xi$' does as well.
  - '$\acute{\epsilon}\backslash\epsilon$' succeeds in referring, provided that '$\backslash\xi$' does as well.
  - '$\acute{\epsilon}(\xi = \epsilon)$' and '$\acute{\epsilon}(\epsilon = \xi)$' succeed in referring, provided that '$\xi = \zeta$' does as well.
  - '$\acute{\epsilon}\left(\begin{array}{l}\mathrel{\top}\epsilon\\ \llcorner\,\xi\end{array}\right)$' and '$\acute{\epsilon}\left(\begin{array}{l}\mathrel{\top}\zeta\\ \llcorner\,\epsilon\end{array}\right)$' succeed in referring, provided that '$\begin{array}{l}\mathrel{\top}\zeta\\ \llcorner\,\xi\end{array}$' does as well.
- Names of third-level function:

  '$\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(f(\beta))$' succeeds in referring if
  - '$\overset{\mathfrak{f}}{\smile}\,\overset{\mathfrak{a}}{\smile}\,(f(\mathfrak{a}))$' succeeds in referring, provided that '$\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})$' does as well.
  - '$\overset{\mathfrak{f}}{\smile}\,\acute{\epsilon}(f(\epsilon))$' succeeds in referring, provided that '$\acute{\epsilon}\phi(\epsilon)$' does as well.

where 'Δ' and 'Γ' stand for any proper names.

First, what are Frege's justifications to assume as sufficient to fix the reference of sentences having, on the one side, value-range names and, on the other side, truth value names, but not sentences involving other names? To answer to this question we need to look deeper at the formal grammar of Grundgesetze. In *Gg* I §30, in a section titled "Two ways to form a name", Frege designs two grammatical rules for name formation in his formal language. The name formation is actually conducted between the less complex names in an inductive way, and Frege's two grammatical rules assures that a new name is well-formed if it is formed by other well-formed names. Interestingly, Frege does this with rules which, as observes Resnik (1986), are totally foreign to contemporary logicians. Instead of constructing '$\forall xy(x = y)$' by forming '$x = y$' first and applying the universal quantifiers '$\forall y$', then '$\forall x$', Frege does this starting with the first-level function name of two arguments '$\xi = \zeta$' and the object name Δ. He combines them to form the first-level function name of one argument '$\Delta = \xi$', and then combines it with the second-level function name '$\overset{\mathfrak{a}}{\smile}\,\phi\mathfrak{a}$' to form '$\overset{\mathfrak{a}}{\smile}\,\Delta = \mathfrak{a}$'. Then he deletes the '$\Delta$' from it, obtaining the new function name '$\overset{\mathfrak{a}}{\smile}\,\xi = \mathfrak{a}$', which is now combined with the second-level function name '$\overset{\mathfrak{d}}{\smile}\,\phi(\mathfrak{d})$' to obtain '$\overset{\mathfrak{d}}{\smile}\overset{\mathfrak{a}}{\smile}\,\mathfrak{d} = \mathfrak{a}$'. The method in which two names are combined to form a new one is called *the first way*, the method in which a new name is formed by well-formed a part of it is called *the second way*. Consequently, there arises:

> [A] a proper name [1] from a proper name and a name of a first-level function of one argument or [2] from a name of first-level function and a name of a second-level function of one argument. or [3] from a name of a second-level function of one argument of type 2 and the name "$\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(f(\beta))$"of a third level function
>
> [B] the name of a first-level function of one argument [1] from a proper name and a name of a fisr-level function of two arguments.
>
> The names so formed may be used in the same way for the formation of further names, and all names arising in this way suceeed in denoting if the primitive simple names do so. (*Gg* I, §30)

It is important to note that the most applicable result of the formal grammar of Grundgesetze, at least for our purposes, is that the only existing object names of the system are the truth-values and value-ranges names. Consider Frege's two rules. Then truth-values names are obtained from (i) a name of first-level function of one argument filling the argument-places of the second-level function name '$\overset{\mathfrak{a}}{\smile}\,(\phi\mathfrak{a})$', from (ii) a name of second-level function of one argument filling the argument-places of the third-level function name '$\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(f(\beta))$' and from (iii) an object name filling the argument-places of the names of first-level function of one argument '$\text{—}\,\xi$' and '$\text{—}\!\!\top\,\xi$' and two object names filling the ξ-argument-places and ζ-argument-places of the names of first-level function of two arguments '$\xi = \zeta$' and '$\begin{array}{l}\top\,\zeta\\ \llcorner\,\xi\end{array}$'. On the other hand, the value-range names are obtained by names of first-level function of one argument filling the argument-places of the second-level function name '$\acute{\epsilon}\phi(\epsilon)$', or abstractor operator. The first-level function name '$\backslash\xi$', by its stipulation, can result either a truth value or a value-range name, depending on the value of its argument ξ[1].

So, an initial attempt to answer, say, Dummett's claim that a complete determination of the reference of the value-range names would require us to answer if any value-range is identical to the Moon (provided that it is intended it to be in the domain) would be to recall that the existing names in the language of Grundgesetze are confined to value-range and truth values names. So, there are no names for such objects like Julius Caesar or the Moon, and Frege apparently assumes that it is sufficient to consider only identity sentences having, on one side, value-range names, and, on the other, truth value names (since there are no names of another kind). In fact, this is the way in which the argument of Moore and Rein goes:

> [We] simply assume that Frege restricted his attention to the True and the False because he was concerned only with questions that could be stated within his formalism. Since his formalism contained proper names only for value-ranges and the two truth-values, questions involving other objects (if there are such objects) cannot be formulated within the system. (Moore and Rein, 1986, p.384, n.9)

and, in a quite identical line of reasoning, Ricketts affirms:

> [The second argument] shows that Frege's introduction of course-of-values names gives rise to the Julius Caesar problem within the begrisschrift, if there are proper names that are not course-of-values names. Sentential names of truth-values do not present the truth-values as courses-of-values. (Ricketts, 1997, p.206)

Although he apparently comes to contradict himself, when, in the following section, he immediately expresses an explicit agreement with Dummett's objection:

> Frege's identification of the truth-values with the course-of-value makes only a small contribution to determining the meanings of course-of-values names. After all, these stipulations do not settle whether Julius Caesar himself is the course-of-values of some function. In general, it appears that the complete determination of meanings for course-of-values names requires stipulations that settle for each object, whether that object is the course-of-values of some function.(Ricketts, 1997, p.206)

Given such considerations, we could suggest (as Moore and Rein really seem to do) that there is no Julius Caesar problem for value-ranges after the trans-sortal identification whenever it is only possible to formalize mixed statements of the form '$\acute{\epsilon} f(\epsilon) = q$' in the language, where either (i) q is a value-range name, or (ii) q is a truth value name.

[1] Recall that the function '$\backslash\xi$' is stipulated as returning Δ if there is a Δ, in which $\xi = \acute{\varepsilon}(\Delta = \varepsilon)$ and returns ξ itself otherwise.

But if this is supposed to answer that we do not need a complete determination, Heck (1999) attempts to show that this would be mistaken. As he stresses it, the point of the indeterminacy is that, as long as objects such as Julius Caesar, or the Moon, are part of the domain of first-order variables, the stipulations do not determine the reference of certain quantified sentences. To put it in his words:

> That Begriffsschrift contains 'proper names only for value-ranges and the two truth-values' simply does not imply that 'questions involving other objects... cannot be formulated within' it. Consider, for example, the sentence: $\forall x \exists F(x = \acute{\epsilon}F(\epsilon))$. This sentence says that every object is a value-range. And if nothing has been said to determine whether Caesar is a value-range, then not enough may have been said to give this sentence a determinate truth-value. (Imagine that we already know that every other object is a value-range.) (Heck, 1999, p.272)

Schirn (2001, p.46) reads it along the same lines as Heck does. According to them, Frege's formalism will fail to fix the reference of value-range names as long as Julius Caesar is in the domain, whether the formal language contains a name for him or not. But which semantic approach are they attributing to Frege? As has often been remarked, however, the semantic rules of Grundgesetze has its own peculiarities, which not necessarily need such truth condition for objectual quantifiers.

Now, in order to show that the Heck–Schirn objection is mistaken, we are compelled to show that Frege's criterion of referentiality for objectual quantifiers is not given in such a way. If it be the case, we can presume that the quantified formulae that their objection points out do not need to bother us, so that, from the fact that the trans-sortal identification fails to determine whether Julius Caesar is a value-range, it does not necessarily follows that the reference of value-range name is not fixed.

4.1.2 The proof of referentiality

So, let us return to Frege's semantic rules. Recall the conditions for a name of each logical type to be said to have a reference, stated in *Gg* I §29. According to Frege, as has often been remarked, the reference of a compound name is yielded by the reference of its components. As a first reaction to it, one might argue that they incur in circularity, so that the matter of each name succeed in referring depends on other names succeed in referring, that, in its turn, depend on the referentiality of other names and so on. So, obviously, the procedure of that section has to assume that the primitive names of the language succeed in referring already, otherwise we would have gone in a circle. Indeed, this is the task of *Gg* I §31, titled "our simple names denotes something".

Futhermore, *Gg* I §31 is the section responsible for Frege's so-called "proof of referentiality", where Frege attempts to run an inductive proof with the foundations he had presented in the previous sections for the basis and inductive cases[2]. Not surprisingly, the semantic tools provided to account the inductive case are the rules laid down in *Gg* I §29. Now, Frege assumes it to be sufficient to provide the basis case by showing that the eight primitive names of the language succeed in referring, presuming that once the reference of each of these names is determined, the reference of the compound names they play a part of immediately follows by the semantic rules from *Gg* I §29. Here are the eight primitive names of Grundgesetze:

a) Names of first-level function of one argument:

$$\text{‘}\!-\!\!\!-\,\xi\text{’},\ \text{‘}\!-\!\!\!\top\,\xi\text{’},\ \text{‘}\backslash\xi\text{’}$$

b) Names of first-level function of two arguments:

$$\text{‘}\xi = \zeta\text{’},\ \text{‘}\begin{array}{l}\top\,\zeta\\ \llcorner\,\xi\end{array}\text{’}$$

c) Names of second-level function of one argument:

$$\text{‘}\smile^{\mathfrak{a}}\,\phi(\mathfrak{a})\text{’},\ \text{‘}\acute{\epsilon}\phi(\epsilon)\text{’}$$

d) Names of third-level function of one argument:

$$\text{‘}\smile^{\mathfrak{f}}\,\mu_{\beta}(f(\beta))\text{’}$$

Interestingly, since Frege's proof is intended to show that each of these names succeed in referring, it will eventually lead to the determination of the reference of the abstraction operator ‘$\acute{\epsilon}\phi(\epsilon)$’, which, by the general conditions laid down by Frege's semantic rules, only has a reference if the proper name—value-range name—that results from a referential name of first-level function of one argument being substituted in its argument-places succeeds in referring. As a result, it is easy to see that Frege's proof of referentiality presupposes, as part of it, a previous proof that every value-range name has a determinate reference.

Frege's proof starts by showing that ‘$-\!\!\!-\,\xi$’ and‘$-\!\!\!\top\,\xi$’ are referential. Taking for granted that, by his own stipulation, the truth value names have a reference (namely, either the True or the False), Frege concludes that what needs to be shown is that those function names yield a referential name when its argument-places are filled by truth value names. However, this follows already from the very stipulation of the functions[3]. In the

[2] Because Frege has an interesting view about consistency, the proof of referentiality of *Gg* I §31 can be also taken to be the theory's consistency proof. According to Frege, in order to a theory T be consistent, it is not enough to be shown that for every sentence φ of the language of T, that either $\vdash \varphi$ or $\vdash (\neg\varphi)$, i.e. the absence of contradictions. For Frege is usually regarded as a realist in mathematics: numerals, relations, functions mean something—namely, logical objects. Since the absence of contradiction regarding a certain term φ does not imply that it means anything, it is also necessary to carry a demonstration that all terms and function terms of language have a reference.

[3] This can only be taken as a partial determination of the function names ‘$-\!\!\!-\,\xi$’ and‘$-\!\!\!\top\,\xi$’, since Frege does not consider, up to this point, the introduction of values-range as object yet. Observe that the introduction of value-ranges we would compel any previous determination of reference of a first-level function name to provide an extra account for the case of value-ranges. But, as previously remarked,

case of the function names ‘$\xi = \zeta$’ and ‘$\begin{array}{l} \top\, \zeta \\ \llcorner\, \xi \end{array}$’, Frege’s argument goes in a similar way. Immediately after the investigation of those functions, it is the time for Frege’s approach of quantifiers. The procedure is less trivial, and before we go too far in the proof, we ought to spend some time examining it in a more meticulous way.

The immediate reason for our further analysis of Frege’s approach to quantifiers is that such investigation would shed light into the Heck–Schirn objection that Frege’s stipulations do not determine the reference of certain quantified sentences. According to Frege, to investigate whether the name ‘$\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$’ of a second-level function refers to something:

> ...we ask whether it follows universally from the fact that the function-name “$\Phi(\xi)$”denotes something, that “$\overset{\mathfrak{a}}{\smile}\, \Phi\mathfrak{a}$”succeeds in denoting. Now “$\Phi(\xi)$”has a denotation if, for every denoting proper name “Δ, “$\Phi(\Delta)$”denotes something. If this is the case, then this denotation either always is the True (wathever “Δ”denotes), or not always. In the first case “$\overset{\mathfrak{a}}{\smile}\, \Phi\mathfrak{a}$”denotes the True, in the second the False. Thus it follows universally from the fact that the substituted function-name “$\Phi(\xi)$”denotes something, that “$\overset{\mathfrak{a}}{\smile}\, \Phi\mathfrak{a}$”denotes something. Consequently the function-name “$\overset{\mathfrak{a}}{\smile}\, \phi\mathfrak{a}$”is to be admitted into the sphere of denoting names. The same follows similarly for “$\overset{\mathfrak{f}}{\smile}\, \mu_{\beta}(f(\beta))$”. (*Gg* I, §31)

Now, at this point, Frege uses a substitutional approach of quantifiers. In the regular (objectual) approach, ‘$\forall xFx$’ means that all objects x are Fs. In the substitutional, however, it states that all substitution instances of F are true (hence the truth of the quantified sentence depends solely on what constants there are available in the language). According to Frege’s approach, the function name ‘$\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$’ is referential if, by filling its argument-places with any function name $\Phi(\xi)$, ‘$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$’ is referential as well. Then he shows that if ‘$\Phi(\Delta)$’ is the True for every argument ‘$\Phi(\Delta)$’, then ‘$\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$’ is the True and False otherwise. In each case, regardless of their logical type, the criterion of referentiality of those names depends exclusively on the reference of their substitutional instances. Hence, the result is clear: ‘$\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$’ is true if and only if all substitution instances of Φ are true. Similar reasoning, though in a different level, shows that the same applies to ‘$\overset{\mathfrak{f}}{\smile}\, \mu_{\beta}(f(\beta))$’.

This suggests that the Heck–Schirn objection overlooks Frege’s actual conception of quantifiers. Heck objects saying that sentences such as ‘$\forall x \exists F(x = \acute{\epsilon}F(\epsilon))$’ says that every object is a value-range. Then, as we saw, he concludes that “if nothing has been said to determine whether Caesar is a value-range, then not enough may have been said to give this sentence a determinate truth-value”(Heck, 1999, p.272). But according to

this would require a precise criteria of identity for value-ranges (and here *Gg* I §10 plays an important role once again), since otherwise we would not be able to account value-ranges “as value-ranges”.

what we just saw about Frege's conception of quantifiers, this sentence actually says that "every object denotable by the language is a value-range". And this is indeed true, and not problematic at all. Given Frege's semantics, the domain of quantification is thus restricted to objects denotable by the formal language, that is, value-ranges and truth values[4].

As Resnik observes, as much as Frege's ontological views preclude the use of concatenation-style grammatical rules, they exclude the Tarski approach to semantics as well[5]. Recall that Frege's semantic conception is that the reference of a compound name is yielded by the reference of its components. Indeed, Frege's approach is much like Tarski's in the sense of being recursive. But to overlook the nuances of Frege's semantic conception is mistaken. Let $\mathfrak{L}$ be a second order language and $\mathfrak{A}$ an $\mathfrak{L}$-structure. Then a Tarski-style rule is defined as follows[6]:

$$\mathfrak{A}(\forall x\alpha) = 1 \text{ iff } U(\alpha[i/x]) = 1, \text{ for every parameter } i \tag{4.1}$$

$$\mathfrak{A}(\exists x\alpha) = 1 \text{ iff } U(\alpha[i/x]) = 1, \text{ for some parameter } i$$

Let us just observe that for the Heck–Schirn argument work, it must necessarily consider a Tarski-style rule for regular (objectual) quantifiers, so that '$\forall xFx$' is true for every assignment of values to the variable x, being false otherwise. So, Tarski's approach to quantifiers is extended to the assignment of truth to open formulae[7], whereas, in the Fregean approach, the assignment of truth to any formulae containing free variables is impossible, as recalls Resnik:

> ...the denotation of the universal quantification, "$\forall x(x = x)$", which is a name, cannot be construed as a function of the denotation of the "$x = x$"; for the latter has a denotation only in the derivative sense of being stipulated to be coreferential with the former.(Resnik, 1986, p.180)

The point is that, as has often been remarked, Frege regards open formulae as function names, so that they are not names of objects, they should not refer to truth values. In this Frege and Tarski differs in a fundamental way. Besides, the Heck–Schirn objection cannot be sustained, since, as we saw, Frege uses a substitutional approach to quantifiers (and not the objectual one, as it suggests).

Now, before we proceed any further, one might argue that the Heck–Schirn objection could be reformulated in substitutional terms. Is this plausible? Well, only

4 Why Frege thought that doing so would be acceptable is a question we attempt to answer at section 4.2.2.

5 Cf. Resnik, 1986, p.180.

6 Tarski (1983) defined it a bit differently in his original formulation from 1931. Instead of substituting constants for variables, Tarski appeals to the notion of satisfiability of a formula (which may or may not include free occurrences of variables) by an infinite sequence of individuals, so that he can define truth (Mortari, 2001).

7 According to the Tarskian approach, we say that $\forall x(Fx)$ is true iff, for all objects u in the domain, Fx is true when x temporarily refers to u.

if there are names for Julius Caesar in the formal language. In fact, the substitutional and objectual approaches differ inasmuch as there is not a name for every object. If every object of the domain is named, like in a one-to-one correlation, then the approaches are essentially the same in the truth they result. But this is not what happens in Grundgesetze (not even in ordinary language). Whenever there is not a name for Julius Caesar in the formal language, there is no indeterminacy concerning quantification after the trans-sortal identification, and Heck–Schirn objection is mistaken.

Frege's account of the abstraction operator '$\acute{\epsilon}\phi(\epsilon)$' is less simple and occupies the most part of the section. The point is, as Frege remarks, with this we are not only introducing a new function name, but, also, we are associating a new proper name for every name of first-level function of one argument. Now, by Frege's semantic rules from *Gg* I §29 we know under what circumstances '$\acute{\epsilon}\phi(\epsilon)$' should succeed in referring. So, according to what has been laid down before, '$\acute{\epsilon}\phi(\epsilon)$' has reference if, for each name of function of one argument Φ, '$\acute{\epsilon}\Phi(\epsilon)$' has a reference. And eventually, the fact that the abstraction operator is referential follows from the fact that every value-range name has a reference as well. Thus, Frege observes:

> To the inquiry whether a value-range name denotes something, we need only subject such value-ranges names as are formed from referential names of first-level functions of one argument. We shall call these for short *fair* value-range names. (*Gg* I, §31)

Now we are driven to investigate if each value-range name (composed by a referential function) is referential. Because value-range names are proper names, we know that, according to the general semantic rule for proper names of *Gg* I §29 (slightly reformulated):

- '$\acute{\epsilon}\Phi(\epsilon)$' succeeds in referring if
  - '$—\, \acute{\epsilon}\Phi(\epsilon)$' succeeds in referring, provided that '$—\, \xi$' does as well.
  - '$—\!\!\!\!\top\, \acute{\epsilon}\Phi(\epsilon)$' succeeds in referring, provided that '$—\!\!\!\!\top\, \xi$' does as well.
  - '$\backslash\acute{\epsilon}\Phi(\epsilon)$' succeeds in referring, provided that '$\backslash\xi$' does as well.
  - '$\xi = \acute{\epsilon}\Phi(\epsilon)$' and '$\acute{\epsilon}\Phi(\epsilon) = \zeta$' succeed in referring, provided that '$\xi = \zeta$' does as well.
  - '$\begin{array}{l} \top\ \acute{\epsilon}\Phi(\epsilon) \\ \llcorner\ \xi \end{array}$' and '$\begin{array}{l} \top\ \zeta \\ \llcorner\ \acute{\epsilon}\Phi(\epsilon) \end{array}$' succeed in referring, provided that '$\begin{array}{l} \top\ \zeta \\ \llcorner\ \xi \end{array}$' does as well.

For which '$\backslash\acute{\epsilon}\Phi(\epsilon)$' will be left out of account at the present moment (we just need to account referential function names, and '$\backslash\xi$' was not shown yet to be referential).

It is important to note that the above procedure can be taken as a determination of the reference of value-range names as much as a further determination of the function names introduced so far. This general rule for value-ranges works as follows: '$\acute{\epsilon}\Phi(\epsilon)$' has a reference if (i) the proper name that results from it filling the argument places of a referential name of a first-level function of one argument always has a reference, and if (ii) the name of a first-level function of one argument that results from '$\acute{\epsilon}\Phi(\epsilon)$''s filling the ξ-argument-places of a referential name of a first-level function of two arguments always has a referencial, and if the same holds also for the ζ-argument-places.

Now consider (ii):

> If we substitute the value-range name "$\acute{\epsilon}\Phi(\epsilon)$" for "$\zeta$" in "$\xi = \zeta$", then the question is thus whether "$\xi = \acute{\epsilon}\Phi(\epsilon)$" is a denoting name of a first-level function of one argument, and to that end it is to be asked in turn whether all proper names denote something that result from out putting in the argument-place either a name of a truth-value or a fair value-range name. (*Gg* I, §31)

In order to investigate whether '$\xi = \acute{\epsilon}\Phi(\epsilon)$' and '$\acute{\epsilon}\Phi(\epsilon) = \zeta$' succeed in referring, Frege recalls that—by his stipulation that '$(\acute{\epsilon}\Phi(\epsilon) = \acute{\alpha}\Psi(\alpha))$' is to have always the same reference as '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', embedded in the Axiom V, and that '$\acute{\varepsilon}(\!-\!\!-\, \varepsilon)$' is to refer to the True, and that '$\acute{\epsilon}(\epsilon = \,\top\!\!\smile^{\mathfrak{a}}\, (a = a))$' is to refer to the False, compassed by his trans-sortal identification—a reference is assured in each case of a proper name of the form '$\Delta = \Gamma$', where 'Δ' and 'Γ' can be fair value-range names or truth value names. Hence, the reference of the identity function '$\xi = \zeta$' is assured and so '$\xi = \acute{\epsilon}\Phi(\epsilon)$' and '$\acute{\epsilon}\Phi(\epsilon) = \zeta$' succeed in referring.

Now, since according to Frege's stipulations the horizontal '$-\!\!-\, \varepsilon$' has to be coextensional with the function '$\xi = (\xi = \xi)$' (which, by the identity function being referential, it always have to yield a referential proper name, particularly, truth value names), then the horizontal function is referential as well. As has often been remarked, the negation '$\top\, \xi$' and the conditional '$\begin{array}{l} \top\, \zeta \\ \llcorner\, \xi \end{array}$' are functions taking the values of the horizontal, then both are referential, too. As a result, fair value-range names are, up to now, referential names. Hence, the abstraction operator '$\acute{\epsilon}\phi(\epsilon)$' is referential as well.

Frege regards very quickly the definite description operator '$\backslash\xi$'. He recalls that, by his stipulation, '$\backslash\Delta$' refers to Γ if there exists an object 'Γ' such that 'Δ' is the name of the value-range '$\acute{\epsilon}(\epsilon = \Gamma)$', and, refers to '$\Delta$'. And he concludes saying that this way a reference is assured for all cases of a proper name of the form '$\backslash\Delta$', and so, for the function name '$\backslash\xi$' as well. Now, in the light of all these considerations, Frege claims to have shown that every name of the system has a reference.

4.2 The principle of complete determination

As has often been remarked, the principle of complete determination says that, in order to define a predicate properly, in such a way that it be legitimate for use in science, one must say under what conditions the predicate will be true of any object at all. However, since Frege's way of solving the problem of indeterminacy by the trans-sortal identification just determine whether value-ranges are truth values (it says nothing about other objects), the claim that *Gg* §10 contains the necessary to completely determine the reference of value-range names can be seen as a violation of this principle (Heck, 1999). This is suggested as follows.

Recall that the demand of complete determination is connected with the view that every concept must have sharp boundaries, suggested already at *Grundlagen* (§68). This worry is best expressed in the following passage:

> Here again we likewise see that the laws of logic presuppose concepts with sharp boundaries, and therefore also complete definitions for names of functions, like the *plus sign*. In vol. i we expressed this as follows: every function-name must have a reference.[8] (*Gg* II, §65)

Therefore, we should expect from the principle of complete determination that in order to determine the reference of predicates such as 'being the value-range of some function'

$$\neg\overset{\mathfrak{f}}{\smile}\neg\ \xi = \acute{\epsilon}\mathfrak{f}(\epsilon) \tag{4.2}$$ [9]

a complete determination of what conditions this predicate will be true of all objects would be needed.

But a complete determination is not what happens: in order to determine the reference of 'being the value-range of some function', Frege would take into account only value-ranges and truth values, so that we would never know if, say, Julius Caesar is a value-range. Obviously, if we take the principle of complete determination in the strict sense, the reference of such predicate (and, in general, the reference of all value-range names) could never be claimed to be fixed at all. Yet, Frege's proof of referentiality from *Gg* I §31 is claimed as correct[10], what implies that, as he sees it, the reference of value-range names is completely determined by his system:

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[8] Translation taken from Geach's *Frege On Definitions I (Vol.II SS56-67)*, In: Translations from the Philosophical Writings of Gottlob Frege, GEACH. P; BLACK, B. (Ed.), Oxford: Basil Blackwell, 1960.

[9] being an ξ such that there exists a function f such that ξ is the value-range of this function

[10] What is at stake is not whether he succeeded or not in showing that all names were referential. With the discovery of the inconsistency of Frege's system, we all know that Frege's proof of referentiality cannot be sound. Also, our point is not whether Frege's proof, regardless of the contradiction, is valid or not. Our question is whether Frege had successfully determined the reference of value-range names. And if he did have, whether he could solve the Julius Caesar problem for value-ranges, too.

> In this way it is shown that our eight primitive names have denotation, and thereby that the same holds good for all names correctly compounded out of these (*Gg* I, §32).

It is easy to see that, at least at first glance, Frege's claim that the reference of the value-range names is completely determined cannot accommodate the principle of complete determination and *vice-versa*.

Is there some way to accommodate those views? Was Frege right in claiming that he has fixed the reference of the value-range names? Is this claim not a violation of his principle of complete determination? To put it in Dummett words:

> ...it seems obscure why the references of value-range terms are fixed by determining the relation of identity between value-ranges and just two objects ... A complete determination of the reference of each such term ought, surely, to settle that question, one way or the other: the fact that the Moon cannot be referred to in the language of the system, or that it need not fail within the domain of a model of the axioms, appears irrelevant, especially if the Moon is meant, on the intended interpretation, to be included in the domain. (Dummett, 1981, p.408)

We claim that, if we look at the principle of complete determination in a different way, it is possible to accommodate it with Frege's claim that *Gg* §10 has determined the reference of value-range names after all. Now we attempt to develop such a view by examining some alternative possibilities.

4.2.1 The all-inclusive domain?

Dummett has suggested that a complete referential determination should consider objects such as Julius Caesar or the Moon, at least if those objects are supposed to be included in the domain. About this Moore and Rein write:

> That Frege's eventual concern in this regard extends only to the truth-values and not to other objects has puzzled many commentators. Was Frege in fact operating with a conception of a domain of objects restricted to just the value-ranges and the two truth-values, contrary to the impression he gives, when introducing his primitive first-level functions, that the domain encompasses *all* objects? (Moore and Rein, 1986, p.384, n.9)

Indeed, this is what Ruffino suggests in his paper "Logical Objects in Frege's Grundgesetze, Section 10"(2002).

In sections IV and V of his paper, Ruffino develops an alternative view regarding Frege's procedure in *Gg* I §10 that "does not allow for any remaining indeterminacy for the notion of extensions". He asks himself why Frege identifies truth values with value-ranges of certain functions in *Gg* I §10, instead of stipulating them to be distinct kind of objects.

Then, he recalls that extensions have a special status for Frege as the most fundamental logical objects[11]. If this is correct, then he observes that the procedure of identifying truth values with value-ranges is a strategy that follows directly from Frege's ontology of logic and from the corresponding epistemology of logical objects: since truth values and value-ranges are both logical objects, but the latter are the most basic ones, it is natural that truth values should be constructed by means of with some of them[12].

With respect to identity statements between value-ranges and concrete objects, Ruffino remarks that most scholars assume that Frege imposes the following two demands on Grundgesetze:

a) The domain of first-order variables is all-inclusive.

b) A definition of a proper name must distinguish its intended referent from any other objects included in the domain of first-order variables, i.e. by definition, we should be able to determine the truth value of all identity statements containing the name being defined.

It is easy to see that both demands imply the Julius Caesar problem in its semantic root, that is, the common assumption that a definition of number should fix the reference of number names, determining whether Julius Caesar or the Moon are numbers or not.

Ruffino suggests that it is doubtful that the problem of indeterminacy could ever be solved if Frege was really attempting to meet both demands. He recalls how this is not a problem in the usual mathematical practice, where the second demand is not made:

> Hardly any definition in mathematics satisfies the second demand. For example, the definition of a particular filter in topology cannot tell whether Julius Caesar is identical with it or not—or, better, it plays no role at all in our knowledge that Julius Caesar is not a filter. (Ruffino, 2002, p.137)

Ultimately, he points out that there is some textual evidence suggesting that Frege did not impose the first demand as well and, despite all impression of being operating with a conception of an all-inclusive domain he gives, he had a conception of domain restricted to just value-ranges and truth values in mind[13].

About this suggestion, Dummett recalls that, nowhere in *Grundgesetze*, Frege attempts a specification of what the first-order variables will range over, so that the reader is left to understand that they will range over all objects whatever[14]. Dummett's claim is strengthened by a notorious passage in *Gg* I §2: "*Objects* stand opposed to functions.

[11] See section 2.2.d.

[12] Cf. Ruffino, 2002, p.136.

[13] In the sense, after the identification of truth values with value-ranges, the domain became exclusively 'logical' giving that it became confined to just value-ranges, the most fundamental kind of logical objects.

[14] Dummett, 1981, p.407-408.

Accordingly I count as *objects* everything that is not a function". Although Ruffino admits this seems to go against his suggestion, he objects:

> But how unrestricted really is the realm of objects that Frege has in mind here? The sentence that immediately follows the above quotation is the following: "for example, numbers, truth-values and the value-ranges that will be introduced bellow". Only logical objects are mentioned here as example of objects. (Ruffino, 2002, p.138)

If this is correct, so that the domain of the system is indeed confined to value-range and truth values, then there is no remaining indeterminacy for the notion of value-ranges arising for both demands, since the first demand is not made by Frege. Moreover, this could explain why Frege's claim that *Gg* I §10 uncontroversially fixes the reference of the value-range names does not violate the principle of complete determination (since only value-ranges and truth values are counted as objects).

Now it is possible to see how the discussion in *Gg* I §10 is supposed to solve the Julius Caesar problem for value-ranges. As remarked earlier, Ruffino's holds an alternative view of the Julius Caesar problem as the matter of reducing the nature of numbers to uncontroversially logical objects. According to this view, the objection in *Gl* §55 and §§62–63 is that the definitions do not unequivocally show that numbers should be regarded as logical objects. In this sense, the definition of *Gl* §68 really says what numbers are, namely, the equivalence classes of equinumerous concepts. The situation is different in *Gg* I §10, where the definition of numbers as extensions is already assumed. If we keep in mind Frege's view of extensions as the most fundamental logical objects for Frege, it follows that:

> ...Once numbers are defined as extensions, there can be no doubt any more that, say, the number 3 is not Julius Caesar, for the latter lacks those crucial features that make extensions paradigmatic cases of logical objects. (Ruffino, 2002, p.140)

Therefore, Ruffino concludes that there is no parallel between the problem of indeterminacy and the Julius Caesar problem: the former is a formal problem involving the semantics of the system, whereas the latter appears in *Grundlagen* as long as the nature of numbers is not uncontroversially established according to the demands of Frege's logicism.

Indeed, the idea of suggesting that Frege was operating with a domain of objects confined to just the value-ranges and the two truth values were also considered by Heck (1999a)[15]. Nevertheless, Heck recalls that scholars have been reluctant to adopt this interpretation, since Frege is celebrated as strongly opposed to other logicians who consider Boolean 'universes of discourse'. Also, Schirn (2001) recalls that Frege explains

[15] Cf. Heck, 1991, p.272–273.

the primitive functions of first-level for any admissible arguments and defines complex particular functions "for all possible objects as arguments"(*Gg* I, §34). In fact, Frege's complaint about universes of discourse can be best understood by taking into account a notorious passage, found in *Gg* II §65, where he seems to admit that it is a mistake to operate with a restricted domain of objects:

> If the sign of addition has been completely defined, then '$\xi + \xi = \zeta$' gives us the name of a relation—the relation of single to double ... Now anybody will answer: 'I forbid anything but numbers to be taken into account at all.' We dealt above with a similar objection; here we may throw light on the matter from other sides. If anybody wants to exclude from consideration all objects that are not numbers, he must first say what he takes 'number' to mean, and then further extension of the term is inadmissible. Such a restriction would have to be incorporated in the definition, which would thus take some such form as: 'If a and b are numbers, then $a+b$ stands for ...' We should have a conditional definition ... By a well-known law of logic, the proposition 'if a is a number and b is a number then $a + b = b + a$' can be transformed into the proposition 'if $a + b$ is not equal to $b + a$, and a is a number, then b is not a number' and here it is impossible to maintain the restriction to the domain of numbers. (*Gg* II, §65)

Therefore, Frege understood that sentences involving restricted quantifications ('Every real number is F') could be transcribed as generalized conditional sentences ('For all x, if x is a real number, then x is F'), so that there is no need to operate with a restricted domain. Furthermore, by making use of the 'universe of discourse', Frege points out that we would make it difficult to analyze the validity or certain inferences, such as the fact that 'Every real number has a non-negative square' entails 'Any number which has a negative square is non-real'. For, in order to represent the latter sentence, we would need to quantify over non-real numbers, which would be impossible if the domain is taken to be confined to real numbers, says Heck.

Although the claim that the domain of the system was restricted to value-ranges and truth values does explain in a reasonable way the question why Frege thought the reference of value-range names was fully determined without violating the principle of complete determination, this claim is not based on a decisive piece of textual evidence (Ruffino, 2002, p.138.). And, although the passage in *Gg* II §65 seems to suggest otherwise, it cannot be uncontroversially taken as a refutation of this view as well. Because nowhere in *Grundgesetze* Frege is transparent about the range of the domain of his system, we conclude that it is more reasonable to seek for another explanation that does not depend on such dubious assumptions.

4.2.2 The criteria of referentiality

It was said before that the criteria of referentiality are commonly assumed to introduce a contextualist notion of reference, which is based on Dummett's interpretation of a generalized context principle for reference. According to Greimann (2003), this view is mistaken. Greimann points out that Dummett's reading goes against the task that Frege ascribes to the criteria, which are explicitly introduced to prescribe the conditions for the legitimate ascription of a reference to a name. With this in mind, let us recall the following passage of *Gg* II §65:

> Here again we likewise see that the laws of logic presuppose concepts with sharp boundaries, and therefore also complete definitions for names of functions, like the *plus sign*. In vol. i we expressed this as follows: every function-name must have a reference. Accordingly all conditional definitions, and any procedure of piecemeal definition, must be rejected. Every symbol must be completely defined at a stroke, so that, as we say, it acquires a reference. (*Gg* II, §65)

This passage strongly suggests that the principle of complete determination is ultimately expressed in the system via Frege's criteria of referentiality (Greimann, 2003, p.273-275).

If this is correct, then the criteria of referentiality are to be regarded as the formal embodiment of the principle of complete determination into the system. However, it is important to note that Frege's criteria do not state that, e.g. '$\Phi(\xi)$' has a reference, if for every *object* a, it follows that '$\Phi(a)$' succeeds in referring. Instead, they say that '$\Phi(\xi)$' has a reference, if for every *name* Δ, '$\Phi(\Delta)$' succeeds in referring. Hence, if the criteria of referentiality are to be seen as expressing the principle of complete determination, it represents it in a quite different way as we would expect. The former extends to names, whereas, the latter, objects. And their difference is substantial, since, obviously, we do not need to have a name for every existing object (or function), neither in natural language nor in a formal language.

In order to explain this unconformity between the ontological based principle and its linguistically based reformulation, Greimann suggests that, as Frege thought it, we could delimit the applicability of a predicate only with respect to objects which are linguistically accessible to us, that is, those objects which are semantically given to us:

> as long as we lack a linguistic means of referring to an object x, we cannot determine what the conditions are for a predicate '$\varphi(\xi)$' to be true of x. Thus, in order to overcome the referential indeterminacy of value-courses terms, Frege confines himself in §10 of *Grundgesetze* Vol. I to fix the truth-conditions of equations of the form '$\acute{\varepsilon}f(\varepsilon)$ =a', where 'a' is a name of a truth-value, because at this stage of the construction of his system only the truth-values have been provided with a linguistic means of referring to them. (Greimann, 2003, p.275)

If this is correct, then Frege's principle of determination can be thought to be confined to objects semantically accessible to us, so that, as we wanted to show, Frege's claim that *Gg* I §10 contains all what is needed to fix the reference of the value-range names does not violate his sense of complete determination after all, simply because the principle is expressed linguistically in the system.

But our task is not finished yet, since this view has to answer at least one objection raised by Dummett (1991). It is obvious that the language of the system is not the only language accessible to us, as he recalls it:

> we start with natural language, and it is natural language, supplemented, indeed, by technical terms Frege has explained in natural language, that serves as the metalanguage in which the interpretation of the system has been set out. In natural language, we can refer to the referent of a value-range term of the formal system, say the term '$\acute{\epsilon}(\epsilon = \epsilon)$', by means of a phrase like 'the referent of "$\acute{\epsilon}(\epsilon = \epsilon)$"in the system of *Grundgesetze*'. (Dummett, 1981, p.411)

According to his complaint, even if the principle of complete determination is to be interpreted linguistically in the system, the reference would still be indeterminate in the metalanguage, in so far we dispose the natural language to formulate sentences—such as 'the referent of some value-range in the system of *Grundgesetze* is Julius Caesar', or 'the Moon is not the referent of a value-range in the system of *Grundgesetze*'—which cannot be said to be true or false.

But at which extent is this objection really applicable? The complaint raised by Dummett observes that there is a referential indeterminacy in the metalanguage. However, one could answer this objection by assuming that Frege was concerned only with questions which could be stated within his formalism (Moore and Rein, 1986), so that he felt he did not have to concern about such indeterminacies simply because Grundgesetze is supposed to contain only proper names for truth values and value-ranges.

4.2.3 Julius Caesar and value-ranges

For the time being, we could show how the principle of complete determination was thought to be expressed in the system, so that we could accommodate Frege's claim of having fixed the notion of value-ranges. The principle was thought linguistically, but this does not explain Frege's silence about objects which are neither value-ranges nor truth values along the book. Why does the formal language contain only proper names for truth values and value-ranges?

Considering the fact that extensions have a special status as logical objects for Frege, there are two reasons to explain why Frege did not mention such objects. They are indicated by Ruffino (2002):

a) Since the purpose of *Grundgesetze* is to show how numbers can be constructed out of logical objects, then it was obvious for Frege that he did not have to consider other objects to compose the ontological basis of the system.

b) As we mentioned before, Frege emphasizes that it is not enough for the consistency of a system that no contradiction is derivable from it so far, that is, what is necessary for the system is a proof that every name has a uniquely determined reference. Therefore, it is plausible that Frege did not think he had to mention other objects because they were not required in order to guarantee consistency.

However, we have already remarked in section 4.2.1. how any dubious assumption about the ontological basis and domain of the system should be avoided.

Yet, we can think of both (a) and (b) in a slightly different way:

a) Considering the purpose of *Grundgesetze* of showing how numbers can be constructed out of logical objects, it was clear for Frege that he did not have to consider other objects to compose the semantic basis of the system.

b) The consistency of the system depends on the proof that every name of the system succeeds in referring. Then there is no need for names of other objects to assure the consistency of the system, insofar as the existing names have a reference.

Taking this all into account, we can explain how Frege thought that names for other objects were not necessary for attaining his purpose of reducing arithmetic to logic. Moreover, the trans-sortal identification would suffice to provide the reference of value-range names a complete determination, without violating the principle of complete determination (which is expressed into the system via the criteria of referentiality). As a result of all this, we do not need to account for Julius Caesar in determining the reference of predicates such as ‘is the value-range of some function’, simply because Julius Caesar is not semantically given to us in the system.

But if we cannot even formulate the question of whether Julius Caesar is a value-range in the formalism, how can we really know that Julius Caesar is not the number of moons of Jupiter? In other words, how can we make sure that there is no more room for the Julius Caesar problem for value-ranges? Let us consider two possibilities.

First, if we embrace the alternative view of the problem suggested by Ruffino (2002), that is, if we see the problem as the matter of establishing without any doubt the nature of numbers as logical objects, then there can be no Julius Caesar problem in Grundgesetze insofar as numbers are treated as extensions of concepts, particular cases of value-ranges, which were thought the most fundamental logical objects for Frege.

Now if we take the problem as mainly the semantic problem of fixing the reference of the number names, then we eventually come to the doubts raised by Parsons (1965), Dummett (1981), Wright (1983), Ricketts (1997), Heck (1997, 1999a, 2005) or Schirn (2001) and their objections that the procedures adopted by Frege do not rule out the possibility of the referent of some value-range be Julius Caesar. The trans-sortal identification adds nothing to this question. Besides, there are not even names for objects such as Julius Caesar in the system, which implies that they are simply disregarded to establish the referentiality and consistency of the system. In pursuing this view, it is at least strange that Frege expresses no special concern on unequivocally showing that the referent of some number name cannot be Julius Caesar. As we remarked already, the simple fact that some number might be Julius Caesar would undoubtedly undermine Frege's attempt to provide a satisfactory reduction of arithmetic to logic. This claim is justified by two facts. First, he stresses in *Grundlagen* that numbers names should refer to one and only one object. If he ultimately admits that some number name might refer to Julius Caesar, then he is violating his own claim. Second, the principle of complete determination (taken strictly) states that in order to grasp the concept of number we must know whether or not Julius Caesar is a number. If he ultimately shows no concern about such determination, then he is violating his own principle (taken strictly) as well.

But, considering our suggestion that Frege's commitment to the principle of complete determination is expressed by the criteria of referentiality of his system, then there is no violation at all. Moreover, since the criteria of referentiality show that value-range names succeed in referring, then there is no violation of the claim that number names should refer to one and only one object, too. Consequently, in light of all these considerations, we see that Frege's lack of concern to solve the Julius Caesar problem for value-ranges can only be reasonably explained by claiming that the Julius Caesar problem does not pose for Frege the semantic problem of fixing the reference of the number names after all (as most scholars take it to be), but just a matter of establishing without any doubt the nature of numbers as logical objects. This is not to say that there is no indeterminacy problem in Grundgesetze. Surely there is a problem. But, against what Parsons, Dummett, Wight and many other scholars would like to admitt, we hoped to have shown that the indeterminacy is completely solved with the trans-sortal identification.

Conclusion

In order to preclude the Julius Caesar problem, the most pregnant philosophical problem found in the logicist project, Frege introduces the notion of extensions of concepts and, subsequently, the notion of value-range in his system, defining each individual number as value-ranges of particular functions. But this leads him to a semantic problem in his formalism, namely, the problem of indeterminacy found in *Gg* I §10. The root of the problem is that Axiom V was after all incapable of fixing the reference of the value-range names, which implies that the reference of some number names is undetermined as well. The problem of indeterminacy is confirmed by the articulation of the permutation argument, that, in addition, plays a role in supporting Frege's way of solving the problem as well, namely, the trans-sortal identification.

Then Frege claims to have fixed the reference of the value-range names, which is confirmed by his proof of referentiality in *Gg* I §29. Yet, because he did not provide a complete determination of the notion of value-ranges (in the sense of deciding whether Julius Caesar is the value-range of some function) Frege's principle of complete determination (according to which, for the usage of a predicate be legitimately accepted in proper science it must be determinate for every object whether it falls under a concept or not) seems to suggest otherwise.

In order to accommodate these facts we attempted to examine some alternative possibilities. First, it might be the case that the domain of first-order variables was confined to value-ranges and truth values. However, this possibility is disagreeably speculative and, as we saw, there is no way to found a conclusive answer in this direction. Second, we showed how Frege's criteria of referentiality can be seen as the expression of the principle of complete determination into the system, concluding that his interest was confined to objects that can be semantically given to us (the criteria of referentiality takes not objects into account, but names). Third, we saw that for attaining his purposes Frege did not need to consider names for objects which are neither value-ranges nor truth values. If this is correct, we can see how Frege determined completely the reference of the value-range names by his trans-sortal identification without violating his principle of complete determination (and without making any assumption about what objects the domain of the system is supposed to range).

If this is all correct, then our final question is why Frege shows no concern in solving the Julius Caesar problem for value-ranges. The Julius Caesar problem is usually seen as a complex of interrelated ontological, epistemiological, semantic and logical problems, of which the most fundamental one seems to be the semantic one, that is, the objection that any acceptable definition of numbers should fix the reference of the number names. This view cannot explain why Frege provides no decisive way to rule out the possibility of some number name refer to Julius Caesar after all, and why he demonstrates no concern about it. A combination of all these considerations points to the fact that the Julius Caesar problem does not pose for Frege a semantic problem, but the matter of showing in an uncontroversial way the nature of numbers as logical objects.

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Appendix A: Frege's Notation

$\acute{\varepsilon}\Phi(\varepsilon)$	Abstraction operator - The value-range of the function Φ
$—\Delta$	The horizontal - Δ denotes the True
$\neg\,\Delta$	Negation - it is not the case that Δ
$\Delta \rightarrow \Gamma$	Material implication - Δ implies Γ
$\Delta \wedge \Gamma$	Conjunction - Δ and Γ
$\Delta \vee \Gamma$	Inclusive Disjunction - Δ or Γ
$\forall\mathfrak{a}\ \Phi(\mathfrak{a})$	Universal quantifier - For all x, $\Phi(x)$
$\exists\mathfrak{a}\ \Phi(\mathfrak{a})$	Existential quantifier - There exists a x such that $\Phi(x)$
$\backslash\Delta$	Definite description function - If there is a Γ such that $\Delta = \acute{\varepsilon}(\Gamma = \varepsilon)$, return Δ; return Δ otherwise