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New Extended Lifetime Distributions

Tese de Doutorado

Recife, 2014.

Ana Carla Percontini da Paixão

NEW EXTENDED LIFETIME DISTRIBUTIONS

Orientador: Prof. Dr. Josenildo dos Santos

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Trabalho apresntado ao programa de Pós-Graduação em Matemática Computacional da Universidade Federal de Pernambuco, como requisito para obtenção do grau de Doutor em Matemática Computacional.

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ANA CARLA PERCONTINI DA PAIXÃO

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A Deus e aos meus amados: pais, filha Isabela e irmã Adriana.

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“Tenho a impressão de ter sido uma criança brincando à beira-mar, divertindo-me em descobrir uma pedrinha mais lisa ou uma concha mais bonita que as outras, enquanto o imenso oceano da verdade continua misterioso diante de meus olhos.”

Isaac Newton

Resumo

Este trabalho está dividido em quatro capítulos independentes. Nos Capítulos 2 e 3 propomos extensões para a distribuição Weibull. A primeira delas, com cinco parâmetros, é uma composição das distribuições beta e Weibull Poisson. Essa nova distribuição tem como submodelos algumas importantes distribuições descritas na literatura e outras ainda não discutidas tais como: bata exponencial Poisson, Weibull Poisson exponencializada, Rayleigh Poisson exponencializada, beta Weibull, Weibull, exponencial, entre outras. Obtemos algumas propriedades matemáticas tais como momentos ordinários e incompletos, estatísticas de ordem e seus momentos e entropia de Rényi. Usamos o método da máxima verossimilhança para obter estimativas dos parâmetros. A potencialidade desse novo modelo é mostrada por meio de um conjunto de dados reais. A segunda extensão, com quatro parâmetros, é uma composição das distribuições Poisson generalizada e Weibull, tendo a Poisson generalizada exponencial, a Rayleigh Poisson, Weibull Poisson e Weibull como alguns de seus sub-modelos. Várias propriedades matemáticas foram investigadas, incluíndo expressões explícitas para os momentos ordinários e incompletos, desvios médios, função quantílica, curvas de Bonferroni e Lorentz, confiabilidade e as entropias de Rényi e Shannon. Estatísticas de ordem e seus momentos são investigados. A estimativa de parâmetros é feita pelo método da máxima verossimilhança e é obtida a matriz de informação obsevada. Uma aplicação a um conjunto de dados reais mostra a utilidade do novo modelo. Nos dois últimos capítulos propomos duas novas classes de distribuições. No Capítulo 4 apresentamos a família G- Binomial Negativa com dois parâmetros extras. Essa nova família inclui como caso especial um modelo bastante popular, a Weibull binomial negativa, discutida por Rodrigues et al.(Advances and Applications in Statistics *22* (2011), 25-55.) Algumas propriedades matemáticas da nova classe são estudadas, incluindo momentos e função geradora. O método de máxima verossimilhança é utilizado para obter estimativas dos parâmetros. A utilidade da nova classe é mostrada através de um exemplo com conjuntos de dados reais. No Capítulo 5 apresentamos a classe Zeta-G com um parâmetro extra e algumas nova distribuições desta classe. Obtemos expressões explícitas para a função quantílica, momentos ordinários e incompletos, dois tipos de entropia, confiabilidade e momentos das estatísticas de ordem. Usamos o método da máxima verossimilhança para estimar os parâmetros e a utilidade da nova classe é exemplificada com um conjunto de dados reais.

Palavras-chave: Distribuição beta. Distribuição Poisson generalizada. Distribuição binomial negativa. Distribuição Weibull Poisson. Distribuição Zeta. Entropia. Máxima verossimilhança.

Abstract

This paper is divided into four independent chapters. In Chapters 2 and 3 we propose extensions to the Weibull distribution. The first one with five parameters is a composition of the beta and the Weibull Poisson distributions. This new distribution has as sub-models some important distributions described in the literature and others that have not been discussed yet, such as: beta exponential Poisson (BEP), exponentiated Weibull Poisson (EWP), exponentiated Rayleigh Poisson (ERP), beta Weibull, Weibull, exponential, among others. We obtain some mathematical properties such as ordinary and incomplete moments, order statistics and their moments and Rényi entropy. We use the method of maximum likelihood to obtain estimates of the parameters. The potential of this new model is shown by a real data set. The second extension, with four parameters, is a composition of generalized Poisson and Weibull distributions having the exponential generalized Poisson, the Rayleigh Poisson, Weibull Poisson and Weibull as some of its sub-models. Several mathematical properties were investigated, including explicit expressions for the ordinary and incomplete moments, mean deviation, Quantile function, Bonferroni and Lorentz curves, reliability and the entropies of Rényi and Shannon. Order statistics and their moments are investigated. The parameter estimation is performed by the method of maximum likelihood and the observed matrix of information is obtained. An application to an actual data set shows the usefulness of the new model. In the last two chapters we propose two new classes of distributions. In Chapter 4 we present the G-negative binomial family with two extra parameters. This new family includes as special case a very popular model, the Weibull negative binomial, discussed by Rodrigues et al. (Advances and Applications Statistics in *22* (2011) , 25-55). Some math properties of the new class are studied, including moments and generating function. The maximum likelihood method is used to obtain parameters estimates. The usefulness of the new class is shown by an example with real data sets. In Chapter 5 we present Zeta-G class with an extra parameter and some new distributions of this class. We obtain explicit expressions for the Quantile function, ordinary and incomplete moments, two types of entropy, reliability and moments of order statistics. We use the method of maximum likelihood to estimate the parameters and the usefulness of the new class is exemplified with a real data set.

Keywords: Beta distribution. Entropy. Generalized Poisson distribution. Maximum likelihood. Negative binomial distribution. Weibull Poisson distribution. Zeta distribution.

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CHAPTER 1

Introduction

This thesis is composed by four independent papers. Two of them introduce new distributions and two others, new families of distributions. So, in this thesis, each of the papers fills a distinct chapter. Therefore, each chapter can be read independently, since each one is self contained. Additionally, we emphasize that each chapter contains a thorough introduction to the presented matter, so this general introduction only shows, quite briefly, the context of each chapter.

In each chapter we are interested in the study of continuous distributions defined on the positive real line. Roughly speaking, any continuous distribution defined on the positive real line can be considered as a lifetime distribution. Obviously, not all such distributions are meaningful for describing an aging (lifetime) phenomenon. The analysis of lifetime data is an important topic in statistical literature, since its applications range from industrial applications to biological studies. We note that in survival analysis, the time to be analyzed refers to the time until the occurrence of any event of interest: diagnosis of a disease, birth, healing, appearance of a tumor, a fault of an equipment or component, etc. Several probabilistic models have proved quite adequate to describe lifetime data. In Chapters 2-5 we present construction methods of continuous distributions used in survival data analysis. Such distributions are characterized by a variety of ways on your rate of failure function.

Chapter 2 introduces a new model obtained by compounding the beta and Weibull Poisson (WP) distributions (Lu and Shi, 2012), called the *beta Weibull Poisson* (BWP) distribution. This distribution has various types of shapes: it can be increasing, decreasing, upside-down bathtub-shaped or unimodal. The WP model is well-motivated for industrial applications and biological studies. As an example, consider the time to relapse of cancer under the first-activation scheme. Suppose that the number, say Z, of carcinogenic cells for an individual left active after an initial treatment follows a truncated Poisson distribution and let W_i be the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i \geq 1$. If $\{W_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (iid) Weibull random variables independent of Z, then the time to relapse of cancer of a susceptible individual can be modeled by the WP distribution. Some mathematical properties are investigated, explicit expressions for the quantile function, Rényi entropy, among several others. We illustrate the potentiality of the new distribution with an application to a real data set.

Chapter 3 presents another extension of the Weibull distribution with four parameters. Taking the baseline distribution as the Weibull model and the distribution of Z as the generalized Poisson, we develop the *Weibull generalized Poisson* (WGP) distribution. This model generalizes the exponential generalized Poisson (EGP) distribution proposed by Gupta et al. (2013) and has several sub-models such as exponential Poisson (EP), Rayleigh generalized Poisson (RGP) and Weibull Poisson (WP) distributions. This new lifetime distribution has strong biological motivation. As an example, consider that the unknown number, say Z, of carcinogenic cells for an individual left active after an initial treatment follows the GP distribution and let Y_i (for $i \geq 1$) be the time spent for the ith carcinogenic cell to produce a detectable cancer mass. If $\{Y_i\}_{i\geq 1}$ is a sequence of iid X random variables independent of Z having the Weibull distribution, then the random variable $X = \text{Min}\{Y_i\}_{i=1}^{Z}$ denoting the cancer recurrence time can be modeled by the WGP distribution. The WGP density function can be written as a linear combination of Weibull density functions. This is one of the main results of this chapter. The usefulness of the new model is illustrated in an application to real data using formal goodness-of-fit tests. By means of a real data application, we prove that the proposed distribution is a very competitive model to the exponentiated Weibull and beta Weibull distributions.

Several new models involving the negative binomial distribution have been proposed and applied in survival analysis. In Chapter 4, we propose a general family of continuous distributions called the *G-negative binomial* (G-NB) family. It includes, as a special case, the *Weibull negative binomial* (WNB) model. This generalization is obtained by increasing the number of parameters compared to the G model. This increase adds more flexibility to the generated distribution. One positive point of the G-NB model is that it includes the G distribution as a sub-model when $s = 1$ and $\beta \to 0$. The G-NB family is well-motivated for industrial applications and biological studies. For example, considers that the failure of a device occurs due to the presence of an unknown number N of initial defects of the same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by X_i the time to the failure of the device due to the ith defect, for $i \geq 1$. If we assume that the X_i's are iid random variables independent of N, which follows a G distribution, then the time to the first failure is appropriately modeled by the G-NB family. For reliability studies, the random variable $X = \text{Min}\{X_i\}_{i=1}^{N}$ can be used in serial systems with identical components, which appear in many industrial applications and biological organisms. An important results is the fact that the G-NB density family is a linear combination of exponentiated-G ("exp-G" for short) density functions.

Finally, in Chapter 5, we propose a new family by compounding any continuous baseline G distribution with the zeta distribution supported on integers $n \geq 1$. By this method, we obtain a new class of distributions, called the *zeta-G*, with an additional shape parameter, whose role is to govern skewness and generate densities with heavier/ligther tails. The cdf of the zeta-G distribution has one representation in terms of polylogarithm function and can be represented by others special functions, for example, using the generalized hypergeometric function, the Lerch transcendent function and the Meijer G-function. We demonstrate that the zeta-G density class is a linear combination of exponentiated-G (“exp-G” for short) density functions. A good characteristic of the zeta-G model is that it includes the G distribution as a special model when $s \to \infty$. This new class extends several widely-known distributions in the literature. So, we present some of its special cases. Its density function will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions, and allow for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. We discuss maximum likelihood estimation and inference on the parameters based on the Cramér-von Mises (CM) and Anderson-Darling (AD) statistics. An example to real data illustrates the importance and potentiality of the new class.

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CHAPTER 2

The beta Weibull Poisson distribution

Resumo

Em Estatística, costuma-se buscar distribuições mais flexíveis. Uma nova distribuição de cinco parâmetros chamada de beta Weibull Poisson é proposta. Ela é obtida através da composição das distribuições Weibull Poisson e beta. Generaliza vários modelos de tempo de vida conhecidos. Nós obtemos algumas propriedades da distribuição proposta, como as funções de sobrevivência e taxa de risco, a função quantílica, momentos ordinários e incompletos, estatísticas de ordem e entropia de Rényi. Estimativas por máxima verossimilhança e inferência para grandes amostras são abordadas. A potencialidade do novo modelo é mostrada por meio de um conjunto de dados reais definido. Na verdade, o modelo proposto pode produzir melhores ajustes do que algumas distribuições conhecidas.

Palavras-chave: Distribuição beta; Distribuição Weibull Poisson; Dados de vida; Função Quantílica; Máxima verossimilhança.

Abstract

In statistics, it is customary to seek more flexible distributions. A new five-parameter distribution called the beta Weibull Poisson is proposed. It is obtained by compounding the Weibull Poisson and beta distributions. It generalizes several known lifetime models. We obtain some properties of the proposed distribution such as the survival and hazard rate functions, quantile function, ordinary and incomplete moments, order statistics and Rényi entropy. Estimation by maximum likelihood and inference for large samples are addressed. The potentiality of the new model is shown by means of a real data set. In fact, the proposed model can produce better fits than some well-known distributions.

Keywords: Beta distribution; Lifetime data; maximum likelihood; Quantile function; Weibull Poisson distribution.

2.1 Introduction

The Weibull distribution is a very popular model in reliability and it has been widely used for analyzing lifetime data. Several new models have been proposed that are either derived from or, in some way, are related to the Weibull distribution. When modelling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modelling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates that are common in reliability and biological studies. An example of the bathtub-shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low level. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The statistics literature is filled with hundreds of continuous univariate distributions. Recent developments focus on new techniques for building meaningful distributions. Several methods of introducing one or more parameters to generate new distributions have been studied in the statistical literature recently. Among these methods, the compounding of some discrete and important lifetime distributions has been in the vanguard of lifetime modeling. So, several families of distributions were proposed by compounding some useful lifetime and truncated discrete distributions.

In recent years, there has been a great interest among statisticians and applied researchers in constructing flexible distributions to furnish better modeling for describing lifetime data. Several authors introduced more flexible distributions to model monotone or unimodal failure rates but they are not useful for modelling bathtub-shaped failure rates. [1] proposed the exponential geometric (EG) distribution to model lifetime data with decreasing failure rate function and [10, 11, 12] defined the generalized exponential (GE) (also called the exponentiated exponential) distribution. The last distribution has only increasing or decreasing failure rate function. Following the key idea of [1], [13] introduced the exponential Poisson (EP) distribution which has a monotone failure rate. [14] proposed a generalization of the Weibull distribution called the beta Weibull (BW) distribution. [3] studied a Weibull geometric (WG) distribution which extends the EG and Weibull distributions. In this paper, we propose a new compounding distribution, called the *beta Weibull Poisson* (BWP) distribution, by compounding the beta and Weibull Poisson (WP) distributions (Lu and Shi, 2012). The failure rate function of the WP distribution has various shapes. In fact, it can be increasing, decreasing, upside-down bathtub-shaped or unimodal.

The proposed generalization stems from a general class of distributions which is defined by the following cumulative distribution function (cdf)

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw, \tag{2.1}$$

where $a > 0$ and $b > 0$ are two additional shape parameters to the parameters of the G-distribution, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $I_{G(x)}(a,b)$ denotes the incomplete beta function ratio evaluated at $G(x)$. The parameters a and b govern both the skewness and kurtosis of the generated distribution.

This class was proposed by [8] and has been widely used ever since. For example, [8] introduced the beta normal (BN) distribution, [16] defined the beta Gumbel (BGu) distribution and [17] proposed the beta Fréchet (BF) distribution. Another example is the beta exponential (BE) model studied by [18].

The probability density function (pdf) corresponding to (2.1) is given by

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1 - G(x)\}^{b-1}, \tag{2.2}$$

where $g(x) = dG(x)/dx$ is the baseline density function.

The paper is organized as follows. In Section 2.2, we define the BWP distribution and highlight some special cases. In Section 2.3.1, we demonstrate that the new density function is a linear combination of WP density functions. The proof is given in Appendix A. Also, we derive the survival and hazard rate functions, moments and moment generating function (mgf), order statistics and their moments and Rényi entropy. Maximum likelihood estimation of the model parameters and the observed information matrix are discussed in Section 2.4. In Section 2.5, we provide an application of the BWP model to the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver. Concluding remarks are given in Section 2.6. Unless otherwise indicated, all results presented in the paper are new and original. It is expected that they could encourage further research of the new model.

2.2 The BWP distribution

We assume that Z has a truncated Poisson distribution with parameter $\lambda > 0$ and probability mass function given by

$$p(z;\lambda) = \mathrm{e}^{-\lambda}\lambda^{z}\,\Gamma^{-1}(z+1)(1-\mathrm{e}^{-\lambda})^{-1},\ z = 1, 2, \ldots,$$

where $\Gamma(p) = \int_0^\infty x^{p-1}\mathrm{e}^{-x}dx$ (for $p > 0$) is the gamma function.

We define $\{W_i\}_{i=1}^{Z}$ to be independent and identically distributed random variable having the Weibull density function defined by

$$\pi(w;\alpha,\beta) = \alpha\beta w^{\alpha-1}\exp(-\beta w^{\alpha}), w > 0,$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

We define $X = min\{W_1, \ldots, W_Z\}$, where the random variables Z and W's are assumed independent. The WP distribution of X has density function given by

$$g(x;\alpha,\beta,\lambda) = \dot{c}\,u\,x^{\alpha-1}\,\mathrm{e}^{\lambda u}, \quad x > 0, \tag{2.3}$$

where $\dot{c} = \dot{c}(\alpha,\beta,\lambda) = \dfrac{\alpha\beta\lambda\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}$ and $u = \mathrm{e}^{-\beta x^{\alpha}}$.

The WP model is well-motivated for industrial applications and biological studies. As a first example, consider the time to relapse of cancer under the first-activation scheme. Suppose that

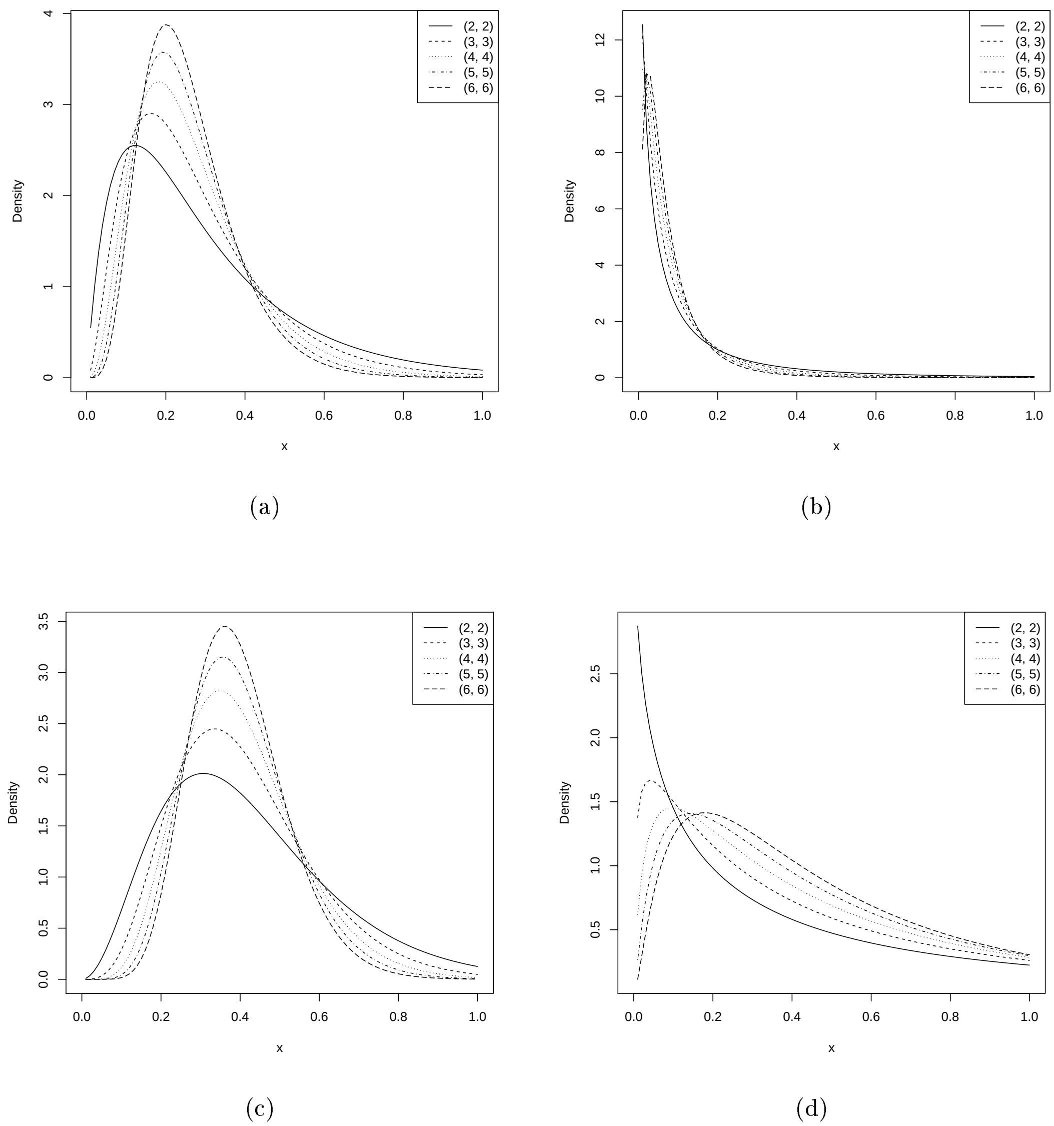

Figure 2.1: Plots of the BWP density function for: (a) $\alpha = 1$, $\beta = 2$ and $\lambda = 1$, (b) $\alpha = 0.5$, $\beta = 2$ and $\lambda = 1$, (c) $\alpha = 1.5$, $\beta = 2$ and $\lambda = 1$, (d) $\alpha = 0.5$, $\beta = 0.5$ and $\lambda = 2$.

the number, say Z, of carcinogenic cells for an individual left active after the initial treatment follows a truncated Poisson distribution and let W_i be the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i \geq 1$. If $\{W_i\}_{i \geq 1}$ is a sequence of independent and identically distributed (iid) Weibull random variables independent of Z, then the time to relapse of cancer of a susceptible individual can be modeled by the WP distribution. Another example considers that the failure of a device occurs due to the presence of an unknown number, say Z, of initial defects of the same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by W_i the time to the failure of the device due to the ith defect, for $i \geq 1$. If we assume that the W_i's are iid Weibull random variables independent of Z, which is a truncated Poisson random variable, then the time to the first failure is appropriately modeled by the WP distribution. For reliability studies, the proposed models for $X = \min\{W_i\}_{i=1}^{Z}$ and $T = \max\{W_i\}_{i=1}^{Z}$ can be used in serial and parallel systems with identical components, which appear in many industrial applications and biological organisms. The first activation scheme may be questioned by certain diseases. Consider that the number Z of latent factors that must all be activated by failure follows a truncated Poisson distribution and assume that W represents the time of resistance to a disease manifestation due to the ith latent factor has the Weibull distribution. In the first-activation scheme, the failure occurs after all Z factors have been activated. So, the WP distribution is able for modeling the time to the failure under last-activation scheme.

The cdf corresponding to (2.3) is

$$G(x) = \frac{\mathrm{e}^{\lambda u} - \mathrm{e}^{\lambda}}{1 - \mathrm{e}^{\lambda}}, \quad x > 0. \tag{2.4}$$

The BWP density function is obtained by inserting (2.3) and (2.4) in equation (2.2). It is given by

$$f(x) = c\, u\, x^{\alpha-1} \mathrm{e}^{\lambda u} (\mathrm{e}^{\lambda} - \mathrm{e}^{\lambda u})^{a-1} (\mathrm{e}^{\lambda u} - 1)^{b-1}, \tag{2.5}$$

where

$$c = \frac{\alpha\beta\lambda\, \mathrm{e}^{-\lambda} (\mathrm{e}^{\lambda} - 1)^{2-a-b}}{B(a,b)(1 - \mathrm{e}^{-\lambda})}.$$

Hereafter, a random variable X having density function (2.5) is denoted by $X \sim \mathrm{BWP}(\alpha, \beta, \lambda, a, b)$.

The cumulative distribution of X is given by

$$F(x) = I_{G(x)}(a,b) = I_{(\mathrm{e}^{\lambda u} - \mathrm{e}^{\lambda})/(1-\mathrm{e}^{\lambda})}(a,b). \tag{2.6}$$

We are motivated to study the BWP distributions because of the wide usage of the Weibull and the fact that the current generalization provides means of its continuous extension to still more complex situations. A second positive point of the current generalization is that the WP distribution is a basic exemplar of the proposed family. A third positive point is the role played by the two beta generator parameters to the WP model. They can add more flexibility in the density function (2.5) by imposing more dispersion in the skewness and kurtosis of X and to control the tail weights.

The beta exponential Poisson (BEP) distribution is obtained from (2.5) when $\alpha = 1$. For $b = 1$, the exponentiated Weibull Poisson (EWP) distribution comes as a special model. In addition, for $\alpha = 1$, we obtain the exponentiated exponential Poisson (EEP) distribution. On the other hand, if $\alpha = 2$, the beta Rayleigh Poisson (BRP) distribution is obtained. In addition, for $b = 1$, it follows the exponentiated Rayleigh Poisson (ERP) distribution. The beta Weibull (BW) distribution comes as the limiting distribution of the BWP distribution when $\lambda \to 0^+$. For $a = b = 1$, equation (2.5) becomes the WP density function. In addition, if $\alpha = 1$, we obtain the exponential Poisson (EP) distribution. The following distributions are new sub-models: the beta Rayleigh Poisson (BRP), exponentiated Weibull Poisson (EWP), beta exponential Poisson (BEP), exponentiated Rayleigh Poisson (ERP), beta Rayleigh (BR), Rayleigh Poisson (RP) and

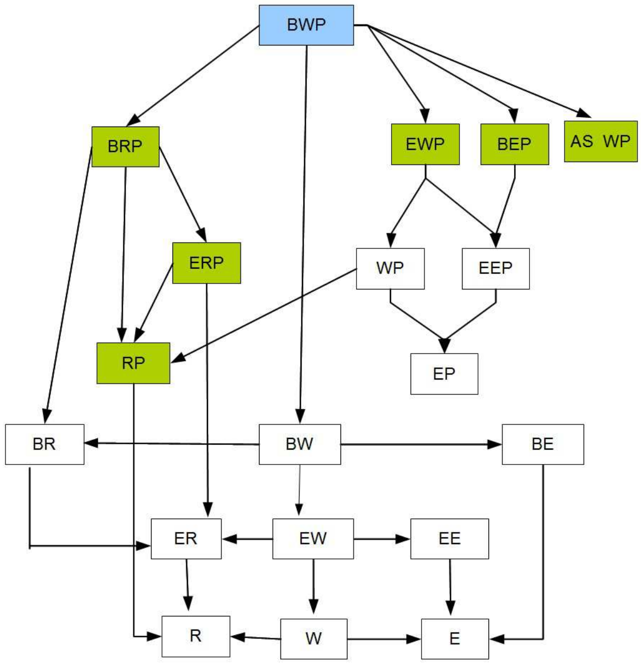

Figure 2.2: Relationships of the BWP sub-models.

arc sine Weibull Poisson (ASWP) distributions (for more details, see Appendix B). Other sub-models are the beta exponential (BE), beta Weibull (BW), beta Rayleigh (BR), exponentiated Rayleigh (ER), exponentiated exponential (EE), exponentiated Weibull (EW), Rayleigh (R), Weibull (W) and exponential (E) distributions. Several special distributions of the BWP model are displayed in Figure 2.2.

2.3 Properties of the new distribution

2.3.1 Density function

We can derive a useful expansion for the BWP density function (see the proof in Appendix A) given by

$$f(x) = \sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\; g(x;\alpha,\beta,\lambda_{r,j}), \tag{2.7}$$

where $\lambda_{r,j} = \lambda(r-j+1) > 0$ and

$$v_{r,j} = \frac{(-1)^j\,(r+1)\,v_r\,\mathrm{e}^{j\lambda}\,(1-\mathrm{e}^{-\lambda_{r,j}})}{(r-j+1)\,\mathrm{e}^{-\lambda_{r,j}}(1-\mathrm{e}^{\lambda})^r(\mathrm{e}^{\lambda}-1)}\binom{r}{j}.$$

Clearly, $\sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j} = 1$. Equation (2.7) reveals that the BWP density function is a linear combination of WP density functions. So, we can obtain some mathematical properties of the BWP distribution directly from those WP properties.

2.3.2 Cumulative function and quantiles

By integrating (2.7), the cdf $F(x)$ becomes

$$F(x) = \sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\ G(x;\alpha,\beta,\lambda_{r,j}). \tag{2.8}$$

Quantile functions are in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. For some baseline distributions with closed-form cdf, it is possible to obtain the quantile function in closed-form. However, for some other distributions, the solution is not possible. The quantile function, say $x = Q(z;\alpha,\beta,\lambda,a,b) = F^{-1}(z;\alpha,\beta,\lambda,a,b)$, of the BWP distribution follows by inverting (2.6) as

$$x = Q(z;\alpha,\beta,\lambda,a,b) = \left\{\log\left(\log[w + \mathrm{e}^{\lambda}(1-w)]^{\frac{1}{\lambda}}\right)^{-\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}, \tag{2.9}$$

where $w = Q_{a,b}(z)$ denotes the beta quantile function with parameters a and b.

Power series methods are at the heart of many aspects of applied mathematics and statistics. We can obtain the moments of the beta G distribution using a power series expansion for the quantile function $x = Q_G(u) = G^{-1}(u)$ of the baseline cdf $G(x)$ with easily computed non-linear recurrence equation for its coefficients.

When the function $Q(u)$ does not have a closed form expression, this function can usually be written in terms of a power series expansion of a transformed variable v, which is usually of the form $v = p(qu-t)^{\rho}$ for p, q, t and ρ known constants.

We can obtain a power series for $Q_{a,b}(z)$ in the Wolfram website given by

$$\begin{aligned} Q_{a,b}(z) &= v + \frac{(b-1)}{(a+1)}v^2 + \frac{(b-1)(a^2+3ba-a+5b-4)}{2(a+1)^2(a+2)}v^3 \\ &+ \frac{v^4(b-1)}{3(a+1)^3(a+2)(a+3)}[a^4+(6b-1)a^3+(b+2)(8b-5)a^2+ \\ &(33b^2-30b+4)a+b(31b-47)+18]+O(v^5), \end{aligned} \tag{2.10}$$

where $v = [azB(a,b)]^{1/a}$ for $a > 0$.

The simulation of the BWP distribution is easy. If W is a random variable having a beta distribution with parameters a and b, then the random variable

$$X = \left\{\log\left(\log[W + \mathrm{e}^{\lambda}(1-W)]^{\frac{1}{\lambda}}\right)^{-\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}$$

follows the BWP distribution.

2.3.3 Survival and hazard rate functions

The BWP survival function is given by

$$S(x;\boldsymbol{\theta}) = 1 - F(x;\boldsymbol{\theta}) = 1 - I_{(\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda})/(1-\mathrm{e}^{\lambda})}(a,b),$$

where $\boldsymbol{\theta} = (\alpha, \beta, \lambda, a, b)$ is the vector of the model parameters. The failure rate function corresponding to (2.5) reduces to

$$h(x;\boldsymbol{\theta}) = \frac{f(x;\boldsymbol{\theta})}{S(x;\boldsymbol{\theta})} = \frac{c\, u\; x^{\alpha-1} \mathrm{e}^{\lambda u} (\mathrm{e}^{\lambda} - \mathrm{e}^{\lambda u})^{a-1} (\mathrm{e}^{\lambda u} - 1)^{b-1}}{\{1 - I_{(\mathrm{e}^{\lambda u} - \mathrm{e}^{\lambda})/(1-\mathrm{e}^{\lambda})}(a, b)\}}.$$

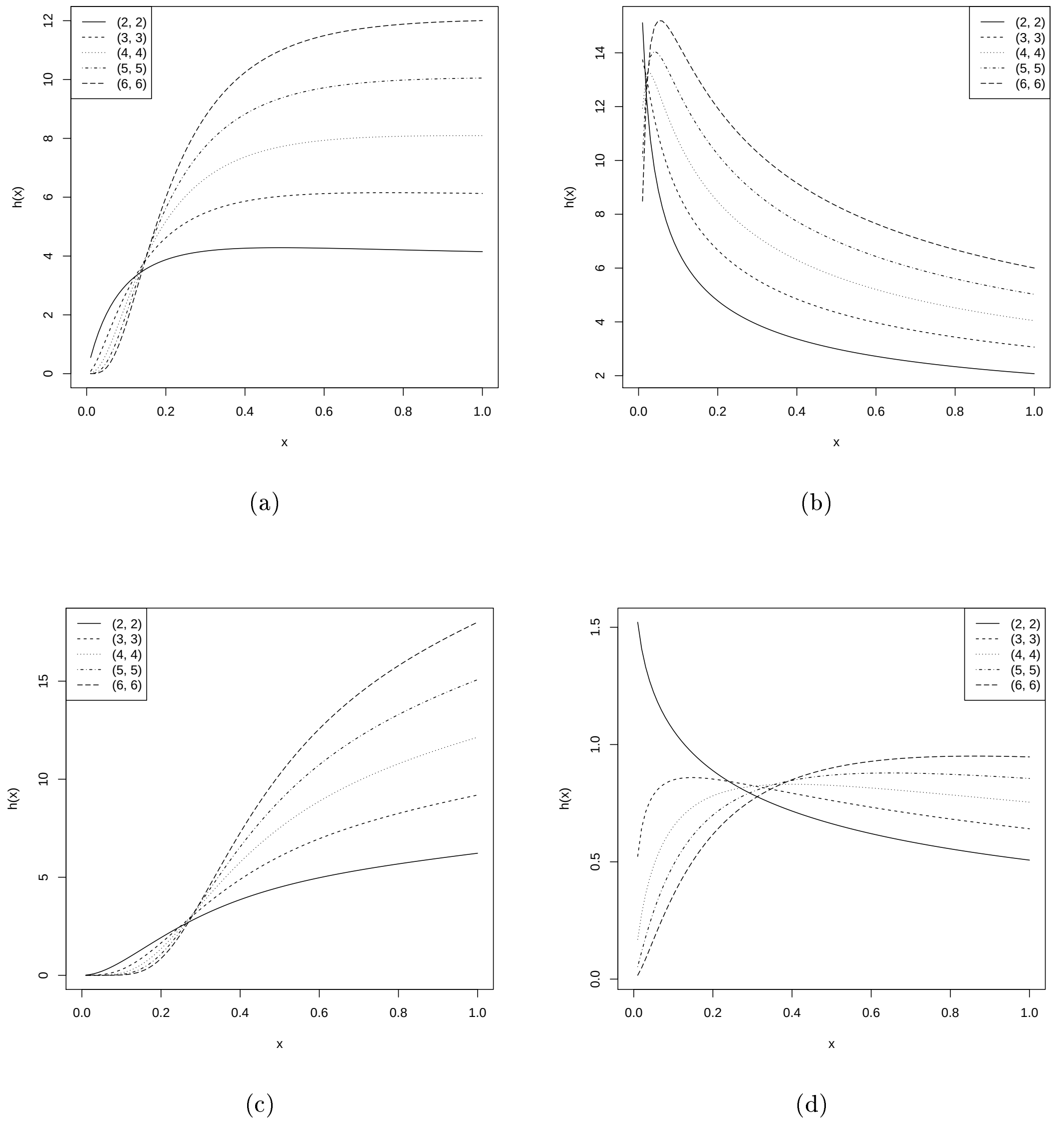

Figure 2.3: Plots of the BWP hazard rate function for (a) $\alpha = 1$, $\beta = 2$ and $\lambda = 1$; (b) $\alpha = 0.5$, $\beta = 2$ and $\lambda = 1$; (c) $\alpha = 1.5$, $\beta = 2$ and $\lambda = 1$; (d) $\alpha = 0.5$, $\beta = 0.5$ and $\lambda = 1$.

2.3.4 Moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

An expression for the mgf of X can be obtained from (2.5) using the WP generating function. Setting $y = \lambda_{r,j} e^{-\beta x^{\alpha}}$ in the definition of the mgf, we can express it as

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\,(e^{\lambda_{r,j}}-1)^{-1} \\ &\times \int_0^{\lambda_{r,j}} \exp\{t(-\beta^{-1}[\log(y)-\log(\lambda_{r,j})])^{1/\alpha}+y\}dy. \end{aligned}$$

Using the power series of the exponential function, after some simplification, we obtain

$$M_X(t) = \sum_{r,m,n=0}^{\infty}\sum_{j=0}^{r} q(r,m,n,j)\,J(\lambda_{r,j},m,n)\,t^n, \tag{2.11}$$

where

$$J(\lambda_{r,j},m,n) = \int_0^{\lambda_{r,j}} y^m\left(-\beta^{-1}[\log(y)-\log(\lambda_{r,j})]\right)^{\frac{n}{\alpha}} dy$$

and

$$q(r,m,n,j) = \frac{v_{r,j}}{(e^{\lambda_{r,j}}-1)\,m!\,n!}.$$

The last integral can be computed using the software Mathematica 8.0. Then,

$$M_X(t) = \sum_{r,m,n=0}^{\infty}\sum_{j=0}^{r} \varpi(r,m,n,j)\,\Gamma\left(\frac{\alpha+n}{\alpha}\right)t^n, \tag{2.12}$$

where

$$\varpi(r,m,n,j) = \beta^{-\frac{n}{\alpha}}\,\lambda_{r,j}^{m+1}\,(1+m)^{-\frac{\alpha+n}{\alpha}}\,q(r,m,n,j).$$

Equation (**??**) can be reduced to

$$M_X(t) = \sum_{n=0}^{\infty}\delta_n\,t^n, \tag{2.13}$$

where $\delta_n = \sum_{m,r=0}^{\infty}\sum_{j=0}^{r}\varpi(r,m,n,j)\,\Gamma\left(\frac{\alpha+n}{\alpha}\right)$, $n = 0, 1, \ldots$

Hence, the *nth* ordinary moment of X, say $\mu'_n = E(X^n)$, is simply given by $\mu'_n = n!\,\delta_n$. Further, the central moments (μ_n) and cumulants (κ_n) of X can be determined as

$$\mu_n = \sum_{s=0}^{n}(-1)^s\binom{n}{s}\mu_1'^s\,\mu'_{n-s} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{s=1}^{n-1}\binom{n-1}{s-1}\kappa_s\,\mu'_{n-s},$$

respectively, where $\kappa_1 = \mu'_1$. Then, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ follow from the second, third and fourth cumulants.

The nth descending factorial moment of X is

$$\mu'_{(n)} = E(X^{(n)}) = E\left[X(X-1) \times \cdots \times (X-n+1)\right] = \sum_{r=0}^{n} s(n,r)\,\mu'_r,$$

where

$$s(n,r) = \frac{1}{r!}\left[\frac{d^r}{dx^r}x^{(n)}\right]_{x=0}$$

is the Stirling number of the first kind which counts the number of ways to permute a list of n items into r cycles. So, we can obtain the factorial moments from the ordinary moments given before.

The incomplete moments of X can be expressed in terms of the incomplete moments of the WP distribution from equation (2.7). We obtain

$$\begin{aligned} m_n(y) &= E(X^n|X<y) = \sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\int_0^y x^n g(x;\alpha,\beta,\lambda_{r,j})dx \\ &= \sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\int_0^y x^n c\,u\,x^{\alpha-1}\mathrm{e}^{\lambda u}dx. \end{aligned} \tag{2.14}$$

Setting $z=\beta x^\alpha$ and integrating by parts, we can write

$$m_n(y) = \frac{\mathrm{e}^{-\lambda}y^n}{1-\mathrm{e}^{\lambda}}\sum_{r=0}^{\infty}\sum_{j=0}^{r} v_{r,j}\left\{n\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\left[\sum_{m=0}^{\infty}\frac{(-1)^m s^m(\beta y^\alpha)^m}{n+m\alpha}\right] - \mathrm{e}^{\lambda \mathrm{e}^{-\beta y^\alpha}}\right\}.$$

The sum in m converges to $(n+m\alpha)^{-1}\mathrm{e}^{-s\beta y^\alpha}$. Then, the nth incomplete moment of X becomes

$$m_n(y) = \sum_{r=0}^{\infty}\sum_{j=0}^{r} p_{r,j}\;y^n\left\{\sum_{s=0}^{\infty}\left[\frac{\lambda^s\mathrm{e}^{-s\beta y^\alpha}}{s!(n+m\alpha)}\right] - \frac{y^n\mathrm{e}^{-\lambda(1-\mathrm{e}^{-\beta y^\alpha})}}{1-\mathrm{e}^{\lambda}}\right\}, \tag{2.15}$$

where $p_{r,j} = \dfrac{n\,v_{r,j}\,\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{\lambda}}$.

We can derive the mean deviations of X about the mean μ'_1 and about the median M in terms of its first incomplete moment. They can be expressed as

$$\delta_1 = 2\big[\mu'_1\,F(\mu'_1) - m_1(\mu'_1)\big] \qquad \text{and} \qquad \delta_2 = \mu'_1 - 2m_1(M), \tag{2.16}$$

where $\mu'_1 = E(X)$ and $m_1(q) = \int_{-\infty}^{q} x\,f(x)\,dx$. The quantity $m_1(q)$ is obtained from (2.15) with $n=1$ and the measures δ_1 and δ_2 in (5.21) are immediately determined from these formulae with $n=1$ by setting $q=\mu_1^{'}$ and $q=M$, respectively. For a positive random variable X, the Bonferroni and Lorenz curves are defined as $B(\pi) = T_1(q)/[\pi\mu'_1]$ and $L(\pi) = T_1(q)/\mu'_1$, respectively, where $q = F^{-1}(\pi) = Q(\pi)$ comes from the quantile function (2.9) for a given probability π.

The formulae derived along the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab. These platforms have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration. The infinity limit in the sums of these expressions can be substituted by a large positive integer such as 20 or 30 for most practical purposes.

2.3.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Moments of order statistics play an important role in quality control and reliability, where some predictors are often based on moments of the order statistics. We derive an explicit expression for the density function of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$ (see Appendix C). For a beta-G model defined from the parent functions $g(x)$ and $G(x)$, $f_{i:n}(x)$ can be expressed as an infinite linear combination of WP density functions

$$f_{i:n}(x) = \sum_{l=0}^{\infty}\sum_{s=0}^{l} \gamma_{i:n}(l,s)\; g(x;\alpha,\beta,\lambda_{l,s}), \tag{2.17}$$

where $\lambda_{l,s} = \lambda(l-s+1)$ and

$$\gamma_{i:n}(l,s) = \sum_{j=0}^{n-i}\sum_{r,k=0}^{\infty} \frac{(-1)^{l+s+j+k}\, \mathrm{e}^{s\lambda} \binom{l}{s}\binom{n-i}{j}\binom{r+a-1}{k}\binom{k+b-1}{l}(1-\mathrm{e}^{-\lambda_{l,s}})\, c_{i+1-j,r}}{(l-s+1)(1-\mathrm{e}^{-\lambda})\,(1-\mathrm{e}^{\lambda})^{l}\, B(a,b)^{i+j} B(i,n-i+1)}.$$

An expression for the mgf of $X_{i:n}$ can be obtained from (2.17) using the WP generating function. Setting $y = \lambda_{l,s}\mathrm{e}^{-\beta x^{\alpha}}$ in the definition of the generating function, we obtain

$$M_{X_{i:n}}(t) \;=\; \sum_{l,m,n=0}^{\infty}\sum_{s=0}^{l} \varpi_i(l,m,n,s)\,\Gamma\left(\frac{\alpha+n}{\alpha}\right) t^{n}, \tag{2.18}$$

where

$$\varpi_i(l,m,n,s) = \frac{\beta^{-\frac{n}{\alpha}}\,\lambda_{l,s}^{m+1}\;(1+m)^{-\frac{\alpha+n}{\alpha}}\gamma_{i:n}(l,s)}{m!\;n!\,(\mathrm{e}^{\lambda_{l,s}}-1)}.$$

Equation (2.18) can be reduced to $M_{X_{i:n}}(t) = \sum_{n=0}^{\infty} \delta_{i:n}\, t^{n}$, where

$$\delta_{i:n} \;=\; \sum_{m,l=0}^{\infty}\sum_{s=0}^{l} \varpi_i(l,m,n,s)\,\Gamma\left(\frac{\alpha+n}{\alpha}\right), n = 0,1,\ldots$$

Hence, the *sth* ordinary moment of $X_{i:n}$ becomes $E(X_{i:n}^{s}) = s!\,\delta_{i:n}$.

2.3.6 Rényi entropy

The entropy of a random variable X with density function $f(x)$ is a measure of the uncertainty variation. The Rényi entropy is defined as

$$I_R(\rho) = (1-\rho)^{-1} \log \left\{ \int f(x)^\rho dx \right\},$$

where $\rho > 0$ and $\rho \neq 1$. If a random variable X has the BWP distribution, we have

$$f(x)^\rho = \left[\frac{g(x;\theta)}{B(a,b)}\right]^\rho G(x)^{(a-1)\rho} \left[1 - G(x)\right]^{(b-1)\rho}. \tag{2.19}$$

By expanding the binomial term, the following expansion holds for any real a,

$$G(x)^{(a-1)\rho+j} = \sum_{r=0}^{\infty} s_r[(a-1)\rho + j]\; G(x)^r,$$

where $s_r[(a-1)\rho+j] = \sum_{i=r}^{\infty} (-1)^{r+i} \binom{(a-1)\rho+j}{j} \binom{i}{r}$. Equation (2.19) can be rewritten as

$$f(x)^\rho = \left[\frac{g(x;\theta)}{B(a,b)}\right]^\rho \sum_{j=0}^{\infty}\sum_{r=0}^{\infty} q_{j,r}\; G(x)^r,$$

where $q_{j,r} = (-1)^j \binom{(b-1)\rho}{j} s_r[(a-1)\rho + j]$.

From equations (2.3) and (2.4), we obtain

$$f(x)^\rho = \left[\frac{c\, u\, x^{\alpha-1} \mathrm{e}^{\lambda u}}{B(a,b)}\right]^\rho \sum_{j=0}^{\infty}\sum_{r=0}^{\infty} q_{j,r} \left(\frac{\mathrm{e}^{\lambda u} - \mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^r.$$

Then,

$$f(x)^\rho = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{t=0}^{r} p_{j,r,t}\, u^\rho\, x^{\alpha(\rho - \frac{\rho}{\alpha})}\, \mathrm{e}^{\lambda(\rho+r-t)u}, \tag{2.20}$$

where $u = \mathrm{e}^{-\beta x^\alpha}$ and

$$p_{j,r,t} = \frac{q_{j,r}(-1)^t \binom{r}{t} \mathrm{e}^{\lambda t} c^\rho}{[B(a,b)]^\rho (1-\mathrm{e}^{\lambda})^r}.$$

Using the power series expansion $\mathrm{e}^{\lambda(\rho+r-t)u} = \sum_{s=0}^{\infty} \dfrac{[\lambda(\rho+r-t)]^s}{s!} \mathrm{e}^{-s\beta x^\alpha}$ in (2.20) and setting $y = \beta s x^\alpha$, the Rényi entropy reduces to

$$I_R(\rho) = (1-\rho)^{-1} \log \left\{ \sum_{j=0}^{\infty} \phi_j(\rho)\, \Gamma\left(\rho + \frac{1-\rho}{\alpha}\right) \right\}, \tag{2.21}$$

where

$$\phi_j(\rho) = \sum_{r,s=0}^{\infty}\sum_{t=0}^{r} \frac{p_{j,r,t}\, \lambda^s (\rho + r - t)^s}{\alpha\, s!\, (\beta s)^{\frac{1-\rho}{\alpha} + \rho}}.$$

2.4 Maximum likelihood estimation

Let $x_1, \ldots, x_n$ be a random sample of size n from the BWP$(a, b, \alpha, \beta, \lambda)$ distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, \alpha, \beta, \lambda)^T$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}) &= n\left[\log(\alpha\beta\lambda) - \lambda - \log[B(a,b)] - \log(1-\mathrm{e}^{-\lambda}) - \log(\mathrm{e}^{\lambda}-1)^{a+b-2}\right] \\ &+(\alpha-1)\sum_{i=1}^{n}\log(x_i) - \beta\sum_{i=1}^{n}x_i^{\alpha} + \lambda\sum_{i=1}^{n}u_i \\ &+(a-1)\sum_{i=1}^{n}\log(^{\lambda}-\mathrm{e}^{\lambda u_i}) + (b-1)\sum_{i=1}^{n}\log(\mathrm{e}^{\lambda u_i}-1), \end{aligned}$$

where $u_i = \exp(-\beta x_i^{\alpha})$ is a transformed observation. The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$\begin{aligned} U_{\alpha}(\boldsymbol{\theta}) &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log(x_i) - \beta\sum_{i=1}^{n}x_i^{\alpha}\log(x_i) - \lambda\beta\sum_{i=1}^{n}u_i x_i^{\alpha}\log(x_i) \\ &+\lambda\beta\sum_{i=1}^{n}u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log(x_i)\left(\frac{1-a}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}} + \frac{b-1}{1-\mathrm{e}^{\lambda u_i}}\right), \\ U_{\beta}(\boldsymbol{\theta}) &= \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{\alpha} - \lambda\sum_{i=1}^{n}u_i x_i^{\alpha} + \lambda\sum_{i=1}^{n}u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i} \\ &\times\left(\frac{1-a}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}} + \frac{b-1}{1-\mathrm{e}^{\lambda u_i}}\right), \end{aligned}$$

$$\begin{aligned} U_{\lambda}(\boldsymbol{\theta}) &= \frac{n}{\lambda} - n + \frac{n\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}} - \frac{n(a+b-2)\mathrm{e}^{\lambda}\log(1-\mathrm{e}^{\lambda})^{a+b-2}}{(1-\mathrm{e}^{\lambda})\log(1-\mathrm{e}^{\lambda})} \\ &+\sum_{i=1}^{n}u_i + (a-1)\sum_{i=1}^{n}\frac{u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}} - (b-1) \\ &\times\sum_{i=1}^{n}\frac{u_i\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}, \\ U_a(\boldsymbol{\theta}) &= -n\left[\psi(a)-\psi(a+b)\right] + n\log(\mathrm{e}^{\lambda}-1)^{a+b-2}\log[\log(1-\mathrm{e}^{\lambda})] \\ &+\sum_{i=1}^{n}\log(\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}), \\ U_b(\boldsymbol{\theta}) &= -n\left[\psi(b)-\psi(a+b)\right] + n\log(\mathrm{e}^{\lambda}-1)^{a+b-2}\log[\log(1-\mathrm{e}^{\lambda})] \\ &+\sum_{i=1}^{n}\log(1-\mathrm{e}^{\lambda u_i}), \end{aligned}$$

where $\psi(\cdot)$ is the digamma function. The maximum likelihood estimates (MLEs) $\widehat{\boldsymbol{\theta}} = (\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\beta}, \widehat{\lambda})^T$ of $\boldsymbol{\theta} = (a, b, \alpha, \beta, \lambda)^T$ are the simultaneous solutions of the non-linear equations: $U_a(\boldsymbol{\theta}) = U_b(\boldsymbol{\theta}) = U_{\alpha}(\boldsymbol{\theta}) = U_{\beta}(\boldsymbol{\theta}) = U_{\lambda}(\boldsymbol{\theta}) = 0$. They can be solved numerically using iterative methods such as a Newton-Raphson type algorithm.

For interval estimation and hypothesis tests on the model parameters, we require the 5×5 observed information matrix $J = J(\boldsymbol{\theta})$ given in Appendix D. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_5(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. In practice, we can replace $I(\boldsymbol{\theta})$ by the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, say $J(\widehat{\boldsymbol{\theta}})$. We can construct approximate confidence regions for the parameters based on the multivariate normal $N_5(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution.

Further, the likelihood ratio (LR) statistic can be used for comparing this distribution with some of its sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some sub-models of the BWP distribution. For example, the test of $H_0 : a = b = 1$ versus $H_1 : H_0$ *is not true* is equivalent to compare the BWP and WP distributions and the LR statistic becomes $w = 2\{l(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) - l(1, 1, \tilde{\alpha}, \tilde{\beta}, \tilde{\lambda})\}$, where $\hat{a}$, $\hat{b}$, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are the MLEs underH_1 and $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\lambda}$ are the estimates under H_0.

2.5 Application

Here, we present an application regarding the BWP model to the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by [2], [4] and [7]. We also fit a five-parameter beta Weibull geometric (BWG) distribution introduced by [6] to make a comparasion with the BWP model. The BWG density function is given by

$$f(x; \boldsymbol{\theta}_1) = \frac{\alpha(1-p)^b \beta^\alpha x^{\alpha-1} \mathrm{e}^{-b(\beta x)^\alpha} (1 - \mathrm{e}^{-(\beta x)^\alpha})^{a-1} (1 - p\mathrm{e}^{-(\beta x)^\alpha})^{-(a+b)}}{B(a,b)},$$

where $\boldsymbol{\theta}_1 = (p, \alpha, \beta, a, b)$ and $x > 0$.

The data are: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0 e 24.5.

In Table 2.1, we list the MLEs of the model parameters and the bias-corrected Akaike information criterion (BAIC), Bayesian information criterion (BIC) and the Hannan-Quinn information criterion (HQIC). We observe that the value of the BAIC criterion is smaller for the BWP distribution as compared with those values of the other models. So, the new distribution seems to be a very competitive model to these data.

The LR test statistic for testing $H_0 : a = b = 1$ against $H_1 : H_0$ *is not true* is $w = 7.08912$ (p-value $= 2.88 \times 10^{-2}$), which is statistically significant. Figure **??** displays the histogram of the data and the plots of the fitted BWP, WP, Weibull and BWG models.

Table 2.1: MLEs of the parameters and BAIC, BIC and HQIC statistics of the BWP, BWG, WP and Weibull models for data of active repair times (hours) for an airborne communication transceiver.

Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	BAIC	BIC	HQIC
BWP	21.969	0.320	0.722	1.439	5.342	207.838	216.981	211.263
	(58.799)	(0.256)	(0.390)	(1.418)	(2.232)			
WP			1.101	0.092	3.522	210.927	216.413	212.982
			(0.120)	(0.052)	(1.917)			
Weibull			0.899	0.334		212.939	216.597	214.309
			(0.096)	(0.075)				
Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	BAIC	BIC	HQIC
BWG	3.269	0.587	1.417	0.212	0.988	208.205	217.348	211.630
	(4.599)	(0.323)	(0.642)	(0.076)	(0.017)			

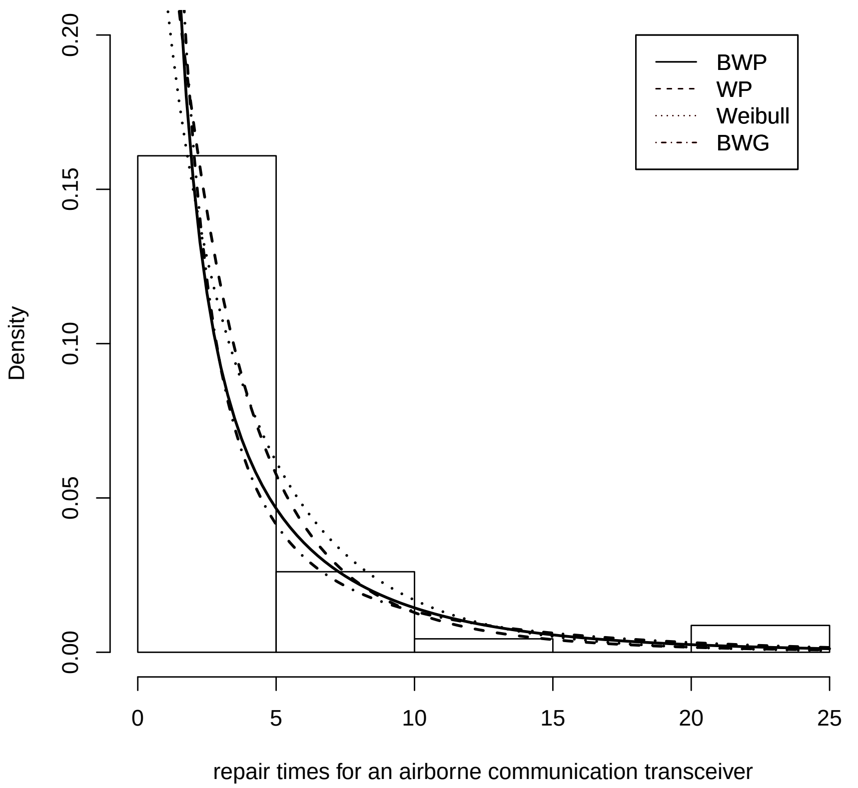

Figure 2.4: The density functions of the fitted BWP, WP, Weibull and BWG distributions.

2.6 Concluding remarks

The Weibull distribution is commonly used to model the lifetime of a system. However, it does not exhibit a bathtub-shaped failure rate function and thus it can not be used to model the complete lifetime of a system. We define a new lifetime model, called the beta Weibull Poisson (BWP) distribution, which extends the Weibull Poisson (WP) distribution proposed by Lu and Shi (2012), whose failure rate function can be increasing, decreasing and upside-down bathtub. The BWP distribution is quite flexible to analyse positive data instead of some other special models. Its density function can be expressed as a mixture of WP densities. We provide a mathematical treatment of the distribution including explicit expressions for the density function, generating function, ordinary and incomplete moments, Rényi entropy, order statistics and their moments. The estimation of the model parameters is approached by the method of maximum likelihood and the observed information matrix is determined. An application to real data reveals that the BWP distribution can provide a better fit than other well-known lifetime models.

Appendix A - The BWP density function

An expansion for the beta-G cumulative function is given by [5] and follows from equation (2.1) as

$$F(x) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} t_r \; G(x)^r, \tag{2.22}$$

where $t_r = \sum_{m=0}^{\infty} w_m \; s_r(a+m)$ for any real a, $w_m = (-1)^m (a+m)^{-1} \binom{b-1}{m}$ and $s_r(a+m) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{a+m}{j} \binom{j}{r}$. Differentiating equation (2.22), we obtain an expansion for the BWP density function

$$f(x) = \sum_{r=0}^{\infty} v_r \, h_{r+1}(x), \tag{2.23}$$

where $v_r = t_{r+1}/B(a,b)$. Note that $h_{r+1}(x) = (r+1) G(x)^r g(x)$ is the density function of the exponentiated G with power parameter $r+1$, say exp-G$(r+1)$, distribution. We can verify that $\sum_{r=0}^{\infty} v_r = 1$. In fact,

$$\sum_{r=0}^{\infty} v_r = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} w_m s_r(a) = 1$$

if and only if

$$\sum_{r=0}^{\infty} \sum_{m=0}^{\infty} w_m \; s_r(a) = B(a,b). \tag{2.24}$$

But

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \sum_{j=0}^{\infty} \binom{b-1}{j} \frac{(-1)^j}{a+j},$$

and, consequently,

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty} w_m\, s_r(a) \quad = \quad \sum_{m=0}^{\infty} \frac{(-1)^m}{(a+m)}\binom{b-1}{m}\sum_{r=0}^{\infty}\sum_{j=r}^{\infty}(-1)^{r+j}\binom{a}{j}\binom{j}{r} = B(a,b)$$

Consider the expressions of $g(x)$ and $G(x)$ from equations (2.3) and (2.4), respectively. Replacing them in (2.23), we obtain an expansion for the BWP density function

$$f(x) = c\, x^{\alpha-1}\, u\, \mathrm{e}^{\lambda u}\sum_{r=0}^{\infty} v_r(r+1)\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^r. \tag{2.25}$$

Hence, from this equation, the BWP density function can be expressed as a linear combination of WP density functions.

Appendix B - Special cases of the BWP distribution

Setting $b=1$ in equation (2.5), we obtain the EWP density function

$$f(x) = c\, u\, x^{\alpha-1}\mathrm{e}^{\lambda u}\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^{a-1}, \quad c = \frac{\alpha\beta\lambda}{B(a,1)(1-\mathrm{e}^{-\lambda})}.$$

Using equation $G(x)^{\alpha} = \sum_{k=0}^{\infty} s_k(\alpha)\, G(x)^k$, we can write

$$\begin{aligned} f(x) &= c\, u\, x^{\alpha-1}\mathrm{e}^{\lambda u}\sum_{k=0}^{\infty} s_k(a-1)\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^k \\ &= c\, u\, x^{\alpha-1}\mathrm{e}^{\lambda u}\sum_{k=0}^{\infty}\frac{s_k(a-1)}{(1-\mathrm{e}^{\lambda})^k}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\mathrm{e}^{\lambda u(k-r)}\mathrm{e}^{\lambda r}. \end{aligned} \tag{2.26}$$

After some algebra, we obtain from (2.26)

$$f(x) = \sum_{k=0}^{\infty}\sum_{r=0}^{k} v_{k,r}\; g(x;\alpha,\beta,\lambda_{k,r}), \tag{2.27}$$

where $\lambda_{k,r} = \lambda(k-r+1)$ and

$$v_{k,r} = \frac{(-1)^r\binom{k}{r}\, B(a,b)\, s_k(a-1)\,\mathrm{e}^{\lambda r}(1-\mathrm{e}^{-\lambda_{k,r}})}{(k-r+1)\, B(a,1)\,(1-\mathrm{e}^{\lambda})^k(1-\mathrm{e}^{-\lambda})}.$$

Equation (2.27) reveals that the density function $f(x)$ is a linear combination of the WP densities.

From equation (2.5) with $a=b=1/2$, we obtain

$$f(x;\boldsymbol{\theta}) = \frac{c_1 x^{\alpha-1} u\mathrm{e}^{\lambda u}}{\pi}\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^{-1/2}\left(1-\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)^{-1/2},$$

where $c_1 = \dfrac{\alpha\beta\lambda\mathrm{e}^{-\lambda}(\mathrm{e}^{\lambda}-1)}{(1-\mathrm{e}^{-\lambda})}$ and $u = \mathrm{e}^{-\beta x^{\alpha}}$. Thus,

$$f(x;\boldsymbol{\theta}) = \frac{c_1 x^{\alpha-1} u \mathrm{e}^{\lambda u}}{\pi\sqrt{\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)\left(\frac{1-\mathrm{e}^{\lambda u}}{1-\mathrm{e}^{\lambda}}\right)}}$$

If λ approaches to 0, then

$$\lim_{\lambda\to 0} f(x;\boldsymbol{\theta}) \quad = \quad \lim_{\lambda\to 0} \frac{c_1 x^{\alpha-1} u \mathrm{e}^{\lambda u}}{\pi\sqrt{\left(\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right)\left(\frac{1-\mathrm{e}^{\lambda u}}{1-\mathrm{e}^{\lambda}}\right)}} = \frac{\alpha\beta x^{\alpha-1} u}{\pi\sqrt{u(1-u)}}$$

So, the BWP distribution reduces as a limiting case to a two-parameter arcsine Weibull-Poisson distribution.

Appendix C - Expansion for the Density Function of the order statistics

The density function $f_{i:n}(x)$ of the ith order statistic, say $X_{i:n}$, for $i = 1, 2, \ldots, n$, from data values $X_1, \ldots, X_n$ having the beta-G distribution can be obtained from (2.2) as

$$f_{i:n}(x) = \frac{g(x)\ G(x)^{a-1}\{1-G(x)\}^{b-1}}{B(a,b)\ B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \qquad (2.28)$$

By application of an equation in Section 0.314 of [9] for a power series raised to any j positive integer

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^j = \sum_{i=0}^{\infty} c_{j,i}\, u^i, \qquad (2.29)$$

where the coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) can be obtained from the recurrence equation

$$c_{j,i} = (i a_0)^{-1} \sum_{m=1}^{i} [m(j+1) - i]\, a_m\, c_{j,i-m}, \qquad (2.30)$$

with $c_{j,0} = a_0^j$. The coefficient $c_{j,i}$ comes from $c_{j,0}, \ldots, c_{j,i-1}$ and then from $a_0, \ldots, a_i$. The coefficients $c_{j,i}$ can be given explicitly in terms of the quantities a_i's, although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

For $a > 0$ real non-integer, we have

$$\begin{aligned} F(x)^{i+j-1} &= \left(\frac{1}{B(a,b)} \sum_{r=0}^{\infty} t_r(a,b) G(x)^r\right)^{i+j-1} \\ &= \left(\frac{1}{B(a,b)}\right)^{i+j-1} \left(\sum_{r=0}^{\infty} t_r G(x)^r\right)^{i+j-1}. \end{aligned}$$

We now use equations (2.29)-(4.33)

$$
\begin{aligned}
f_{i:n}(x) &= \sum_{j=0}^{n-i}(-1)^j \frac{g(x)G(x)^{a-1}[(1-G(x)]^{b-1}}{B(a,b)^{i+j}B(i,n-i+1)}\binom{n-i}{j}\sum_{r=0}^{\infty} c_{i+j-1,r}G(x)^r \\
&= \sum_{j=0}^{n-i}\sum_{r=0}^{\infty}(-1)^j\, c_{i+j-1,r}\binom{n-i}{j}\frac{g(x)[(1-G(x)]^{b-1}G(x)^{r+a-1}}{B(a,b)^{i+j}B(i,n-i+1)},
\end{aligned}
\tag{2.31}
$$

where

$$
c_{i+j-1,r} = (rt_0)^{-1}\sum_{m=1}^{r}((i+j)m-r)t_m c_{i+j-1,r-m}. \tag{2.32}
$$

Equation (2.31) can be written as

$$
f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{r=0}^{\infty}(-1)^j c_{i+j-1,r}\binom{n-i}{j}\frac{g(x)[(1-G(x)]^{b-1}[1-(1-G(x))]^{r+a-1}}{B(a,b)^{i+j}B(i,n-i+1)}.
$$

For any $q>0$ real, we have

$$
G(x)^q = [1-\{1-G(x)\}]^q = \sum_{k=0}^{\infty}(-1)^k\binom{q}{k}[1-G(x)]^k, \tag{2.33}
$$

and then

$$
f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{r=0}^{\infty} g(x)\sum_{k=0}^{\infty}(-1)^k\binom{r+a-1}{k}[1-G(x)]^{k+b-1}.
$$

In the same way, using equation (2.33), it follows that

$$
f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{r,k,l=0}^{\infty}\frac{(-1)^{j+k+l}\binom{n-i}{j}\binom{r+a-1}{k}\binom{k+b-1}{l}c_{i+j-1,r}}{B(a,b)^{i+j}\,B(i,n-i+1)}\,g(x)\,G(x)^l.
$$

Replacing equations (2.3) and (2.4) in the above equation, $f_{i:n}(x)$ can be expressed as an infinite linear combination of WP density functions

$$
\begin{aligned}
f_{i:n}(x) &= \sum_{j=0}^{n-i}\sum_{r,k,l=0}^{\infty}\frac{(-1)^{j+k+l}\binom{n-i}{j}\binom{r+a-1}{k}\binom{k+b-1}{l}c_{i+j-1,r}}{B(a,b)^{i+j}B(i,n-i+1)}\left[\frac{\alpha\beta\lambda \mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\,u\,x^{\alpha-1}\,\mathrm{e}^{\lambda u}\right]\left[\frac{\mathrm{e}^{\lambda u}-\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}\right]^l \\
&= \sum_{j=0}^{n-i}\sum_{r,k,l=0}^{\infty}\frac{(-1)^{j+k+l}\binom{n-i}{j}\binom{r+a-1}{k}\binom{k+b-1}{l}c_{i+j-1,r}}{B(a,b)^{i+j}\,B(i,n-i+1)(1-\mathrm{e}^{\lambda})^l}\left[\frac{\alpha\beta\lambda \mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\,u\,x^{\alpha-1}\,\mathrm{e}^{\lambda u}\right] \\
&\times \sum_{s=0}^{l}(-1)^s\binom{l}{s}(\mathrm{e}^{\lambda u})^{l-s}\mathrm{e}^{s\lambda}.
\end{aligned}
\tag{2.34}
$$

Equation (2.34) reduces to

$$f_{i:n}(x) = \sum_{l=0}^{\infty}\sum_{s=0}^{l} \gamma_{i:n}(l,s)\; g(x;\alpha,\beta,\lambda_{l,s}), \tag{2.35}$$

where $\lambda_{l,s} = \lambda(l-s+1)$ and

$$\gamma_{i:n}(l,s) = \sum_{j=0}^{n-i}\sum_{r,k=0}^{\infty} \frac{(-1)^{l+s+j+k}\,\mathrm{e}^{s\lambda}\binom{l}{s}\binom{n-i}{j}\binom{r+a-1}{k}\binom{k+b-1}{l}(1-\mathrm{e}^{-\lambda_{l,s}})\,c_{i+1-j,r}}{(l-s+1)(1-\mathrm{e}^{-\lambda})(1-\mathrm{e}^{\lambda})^{l}\,B(a,b)^{i+j}\,B(i,n-i+1)}.$$

Appendix D - Information Matrix

Let $u_i = \exp(-\beta x_i^{\alpha})$. The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters $(\alpha,\beta,\lambda,a,b)$ are

$$\begin{aligned} J_{\alpha\alpha} &= -\frac{n}{\alpha^2} - \beta\sum_{i=1}^{n} x_i^{\alpha}\log^2(x_i) - \lambda\beta\sum_{i=1}^{n} u_i x_i^{\alpha}\log^2(x_i) + \lambda\beta^2\sum_{i=1}^{n} x_i^{2\alpha}u_i\log^2(x_i) \\ &\quad +(a-1)\sum_{i=1}^{n}\left[\frac{\lambda\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log^2(x_i)}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right]\psi(x_i) \\ &\quad +(b-1)\sum_{i=1}^{n}\left[\frac{\lambda\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log^2(x_i)}{1-\mathrm{e}^{\lambda u_i}}\right]\varphi(x_i), \end{aligned}$$

$$\begin{aligned} J_{\alpha\beta} = J_{\beta\alpha} &= -\sum_{i=1}^{n} x_i^{\alpha}\log(x_i) - \lambda\sum_{i=1}^{n} u_i x_i^{\alpha}\log(x_i) + \lambda\beta\sum_{i=1}^{n} u_i x_i^{2\alpha}\log(x_i) \\ &\quad +(a-1)\sum_{i=1}^{n}\left[\frac{\lambda u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log(x_i)}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right]\psi(x_i) \\ &\quad +(b-1)\sum_{i=1}^{n}\left[\frac{\lambda u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log(x_i)}{1-\mathrm{e}^{\lambda u_i}}\right]\varphi(x_i), \end{aligned}$$

where

$$\psi(x_i) = \left(-1 + \beta x_i^{\alpha} + \lambda\beta u_i x_i^{\alpha} - \frac{\lambda\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right)$$

and

$$\varphi(x_i) = \left(1 - \beta x_i^{\alpha} - \lambda\beta u_i x_i^{\alpha} - \frac{\lambda\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right).$$

Further,

$$\begin{aligned} J_{\alpha\lambda} = J_{\lambda\alpha} &= -\beta\sum_{i=1}^{n} u_i x_i^{\alpha}\log(x_i) + (a-1)\sum_{i=1}^{n}\rho(x_i)\Bigg[-1-\lambda u_i \\ &\quad +\frac{\lambda(u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda})}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\Bigg] + (b-1)\sum_{i=1}^{n}\phi(x_i)\left(1+\lambda u_i + \frac{\lambda u_i\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right), \end{aligned}$$

$$J_{\alpha a}=J_{a\alpha}=-\sum_{i=1}^{n}\lambda\rho(x_i),\qquad J_{\alpha b}=J_{b\alpha}=\sum_{i=1}^{n}\lambda\phi(x_i),$$

where $\rho(x_i)=\dfrac{\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log(x_i)}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}$ and $\phi(x_i)=\dfrac{\beta u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}\log(x_i)}{1-\mathrm{e}^{\lambda u_i}}$,

$$\begin{aligned}J_{\beta\beta} &= -\frac{n}{\beta^2}+\lambda\sum_{i=1}^{n}u_i x_i^{2\alpha}+(a-1)\sum_{i=1}^{n}\left(\frac{\lambda u_i(x_i^{\alpha})^2\mathrm{e}^{\lambda u_i}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right)\\ &\quad\times\left(1+\lambda u_i-\frac{\lambda u_i\mathrm{e}^{\lambda u_i}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right)+(b-1)\sum_{i=1}^{n}\left(\frac{\lambda u_i x_i^{2\alpha}\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right)\\ &\quad\times\left(-1-\lambda u_i-\frac{\lambda u_i\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right),\end{aligned}$$

$$\begin{aligned}J_{\beta\lambda}=J_{\lambda\beta} &= -\sum_{i=1}^{n}u_i x_i^{\alpha}+(a-1)\sum_{i=1}^{n}\gamma(x_i)\left[-1-\lambda u_i+\frac{\lambda(u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda})}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right]\\ &\quad+(b-1)\sum_{i=1}^{n}\delta(x_i)\left(1+\lambda u_i+\frac{\lambda u_i\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right),\end{aligned}$$

$$J_{\beta a}=J_{a\beta}=-\sum_{i=1}^{n}\lambda\gamma(x_i),\qquad J_{\beta b}=J_{b\beta}=\sum_{i=1}^{n}\lambda\delta(x_i),$$

$\gamma(x_i)=\dfrac{u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}$ and $\delta(x_i)=\dfrac{u_i x_i^{\alpha}\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}$. Furthermore,

$$\begin{aligned}J_{\lambda\lambda} &= -\frac{n}{\lambda^2}+\frac{n\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}+\frac{n\mathrm{e}^{-2\lambda}}{(1-\mathrm{e}^{-\lambda})^2}-\frac{n(a+b-2)\mathrm{e}^{\lambda}\log(1-\mathrm{e}^{\lambda})^{a+b-2}}{1-\mathrm{e}^{\lambda}\log(1-\mathrm{e}^{\lambda})}\\ &\quad\times\left[1-\frac{(a+b-2)\mathrm{e}^{\lambda}}{(1-\mathrm{e}^{\lambda})\log(1-\mathrm{e}^{\lambda})}+\frac{\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{\lambda}}{(1-\mathrm{e}^{\lambda})\log(1-\mathrm{e}^{\lambda})}\right]\\ &\quad+(a-1)\sum_{i=1}^{n}\left[\frac{u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}-\frac{(u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda})^2}{(\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda})^2}\right]\\ &\quad-(b-1)\sum_{i=1}^{n}\left[\frac{u_i^2\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}+\frac{u_i^2(\mathrm{e}^{\lambda u_i})^2}{(1-\mathrm{e}^{\lambda u_i})^2}\right],\end{aligned}$$

$$\begin{aligned}J_{\lambda a}=J_{a\lambda} &= \left[\frac{n\mathrm{e}^{\lambda}\log(1-\mathrm{e}^{\lambda})^{a+b-2}}{(1-\mathrm{e}^{\lambda})\log(1-\mathrm{e}^{\lambda})}\right]\left\{1+(a+b-2)\log[\log(1-\mathrm{e}^{\lambda})]\right\}\\ &\quad+\sum_{i=1}^{n}\left(\frac{u_i\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda u_i}-\mathrm{e}^{\lambda}}\right),\end{aligned}$$

$$\begin{aligned}J_{\lambda b}=J_{b\lambda} &= \left[\frac{n\mathrm{e}^{\lambda}\log(1-\mathrm{e}^{\lambda})^{a+b-2}}{(1-\mathrm{e}^{\lambda})\log(1-\mathrm{e}^{\lambda})}\right]\left\{1+(a+b-2)\log[\log(1-\mathrm{e}^{\lambda})]\right\}\\ &\quad+\sum_{i=1}^{n}\left(\frac{u_i\mathrm{e}^{\lambda u_i}}{1-\mathrm{e}^{\lambda u_i}}\right),\end{aligned}$$

$$
\begin{aligned}
J_{aa} &= -\frac{n\ddot{B}_a(a,b)}{B(a,b)} + \frac{n\left[\dot{B}_a(a,b)\right]^2}{B(a,b)} - n\log(1-\mathrm{e}^{\lambda})^{a+b-2} \\
&\quad \times \log^2[\log(1-\mathrm{e}^{\lambda})],
\end{aligned}
$$

$$
\begin{aligned}
J_{ab} = J_{ba} &= -\frac{n\ddot{B}(a,b)}{B(a,b)} + \frac{n\dot{B}_a(a,b)\dot{B}_b(a,b)}{[B(a,b)]^2} - n\log(1-\mathrm{e}^{\lambda})^{a+b-2} \\
&\quad \times \log^2[\log(1-\mathrm{e}^{\lambda})],
\end{aligned}
$$

$$
\begin{aligned}
J_{bb} &= -\frac{n\ddot{B}_b(a,b)}{B(a,b)} + \frac{n\left[\dot{B}_b(a,b)\right]^2}{[B(a,b)]^2} - n\log(1-\mathrm{e}^{\lambda})^{a+b-2} \\
&\quad \times \log^2[\log(1-\mathrm{e}^{\lambda})],
\end{aligned}
$$

where $\dot{B}_a(a,b) = \dfrac{\partial}{\partial a}B(a,b)$, $\dot{B}_b(a,b) = \dfrac{\partial}{\partial b}B(a,b)$ and $\ddot{B}(a,b) = \dfrac{\partial^2}{\partial b \partial a}B(a,b)$.

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CHAPTER 3

The Weibull Generalized Poisson distribution

Resumo

Em Estatística, costuma-se investigar distribuições mais flexíveis. Uma nova distribuição contínua é estudada pela composição das distribuições Poisson generalizada e Weibull. Considerar a distribuição do tempo de vida de um sistema em série com um número aleatório Z componentes. Tomando a Poisson generalizada para a distribuição de Z, nós definimos a distribuição Weibull Poisson generalizada compondo as duas distribuições. Várias propriedades matemáticas do modelo proposto são investigadas, incluíndo expressões explícitas para os momentos ordinários e incompletos, função geradora, desvios médios, dois tipos de entropias e estatísticas de ordem. Discutimos estimação do modelo de parâmetros por máxima verossimilhança e fornecemos uma aplicação a uma conjunto de dados reais. Esperamos que a proposta de distribuição sirva como um modelo alternativo para outras distribuições para modelar dados reais positivos em muitas áreas.

Palavras-chave: Distribuição Poisson generalizada; distribuição Weibull; matriz de informação; máxima verossimilhança.

Abstract

In statistics, it is customary to seek more flexible distributions. A new continuous distribution is studied by compounding the generalized Poisson and Weibull distributions. We consider the distribution of the lifetime of a series system with a random number Z of components. Taking the generalized Poisson for the distribution of Z, we define the Weibull generalized Poisson distribution by compounding the two distributions. Various mathematical properties of the proposed model are investigated, including explicit expressions for the ordinary and incomplete moments, generating function, mean deviations, two types of entropies and order statistics. We discuss estimation of the model parameters by maximum likelihood and provide an application to a real data set. We hope that the proposed distribution will serve as an alternative model to other distributions for modeling positive real data in many areas.

Keywords: Generalized Poisson distribution; Information matrix; Maximum likelihood; Weibull distribution.

3.1 Introduction

Many distributions lack biological motivation for modeling lifetime data such as cancer recurrence times. Adding new parameters to classical distributions in order to obtain more flexibility has been investigated by several authors in the last twenty years or so.

Let $Y_1, \ldots, Y_Z$ be a random sample of unknown size Z from a distribution with survival function $\overline{G}(t)$, $t > 0$. In realibility analysis, each of the Y_i's denotes the lifetime of a subject (component). For a parallel system, we observe $\max\{Y_1, \ldots, Y_Z\}$, whereas for a series system, we observe $\min\{Y_1, \ldots, Y_Z\}$. In reliability and survival analysis, it is almost impossible to have a fixed sample size because of missing observations. In such cases, the sample size should be considered a random variable.

In this paper, we assume that Z has the *generalized Poisson* (GP) distribution which is an extension of the Poisson distribution with one additional parameter α. Various distributions for Z have been proposed in the literature. Cheng *et al.* (2003), Cooner (2007), Kus (2007) and Karlis (2009) considered the Poisson distribution for Z. Morais and Barreto-Souza (2011) took the power series distribution for Z, which is a more general discrete distribution. We aim to generalize (Gupta *et al.* 2013)'s results who proposed the *exponential generalized Poisson* (EGP) distribution. We compare this distribution with our model in terms of model fitting.

So, we introduce a new four-parameter lifetime distribution with strong biological motivation. As an example, consider that the unknown number, say Z, of carcinogenic cells for an individual left active after an initial treatment follows the GP distribution and let Y_i (for $i \geq 1$) be the time spent for the ith carcinogenic cell to produce a detectable cancer mass. If $\{Y_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (iid) random variables independent of Z having the Weibull distribution, then the random variable X=$\min\{Y_1, \ldots, Y_Z\}$ denoting the cancer recurrence time can be modeled by the *Weibull generalized Poisson* (WGP) distribution.

The rest of the paper is organized as follows. In Section 3.2, we define the new distribution and some special cases. We demonstrate that the WGP density function is a linear combination of Weibull densities and provide explicit expressions for the quantile function, ordinary and incomplete moments, moment generating function (mgf), mean deviations, Shannon entropy, Rényi entropy, reliability and moments of order statistics in Sections 3.3 to 3.6. The estimation of the model parameters using maximum likelihood is discussed in Section 3.7. An application to a real data set is performed in Section 3.8. Finally, some conclusions are addressed in Section 3.9.

3.2 The WGP distribution

Let $Y_1, \ldots, Y_Z$ be a random sample from the Weibull distribution with probability density function (pdf) and survival function given by $g(y; a, b) = a\, b\, y^{a-1} \exp(-b\, y^a)$ (for $y, a, b > 0$) and $\overline{G}(y; a, b) = \exp(-b\, y^a)$, respectively. Let Z be a random variable having a *zero-truncated generalized Poisson* (ZTGP) distribution with probability mass function (pmf)

$$P(z; \lambda, \alpha) = \frac{\lambda(\lambda + \alpha z)^{z-1} \mathrm{e}^{-\lambda - \alpha z}}{(1 - \mathrm{e}^{-\lambda})\Gamma(z+1)},$$

where $z \in \{1, 2, \ldots\}$, $\lambda > 0$, $\max(-1, -\lambda/m) \leq \alpha \leq 1$, $m \geq 4$ is the largest positive integer for which $\lambda + m\alpha > 0$ when $\alpha < 0$ and $\Gamma(\cdot)$ is the gamma function, see Consul and Jain (1973). The ZTGP distribution reduces to the zero-truncated Poisson when $\alpha = 0$. We assume that the random variables Z and the Y_i's are independent. Let $X = \min(Y_1, \ldots, Y_Z)$. Then, $g(y|z; a, b) = a\, b\, z\, y^{a-1} \exp(-b\, z\, y^a)$ is the conditional WGP density function.

[1] expressed the mgf $M_Z(t)$ of the GP distribution in terms of the Lambert W function as

$$M_Z(t) = \exp\left\{-\frac{\lambda}{\alpha}\left[W(-\alpha \mathrm{e}^{-\alpha+t}) + \alpha\right]\right\},$$

where $\alpha \neq 0$ and $W(x)$ is the Lambert W function defined by

$$W(x)\, \mathrm{e}^{W(x)} = x, \tag{3.1}$$

for $x > -\mathrm{e}^{-1}$. For $-\mathrm{e}^{-1} \leq x < 0$, there are two possible values of $W(x)$. We denote the branch satisfying $-1 \leq W(x)$ by $W_0(x)$ and the branch satisfying $W(x) \leq -1$ by $W_{-1}(x)$. $W_0(x)$ is referred to as the principal branch of the Lambert W function. Here, we denote $W_0(x)$ as $W(x)$. The history of this function goes back to J. H. Lambert (1728-1777).

The Lambert W function admits the power series

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!}\, x^n. \tag{3.2}$$

Another elementary property of the Lambert W function is provided by its derivative. After some algebra, we obtain the following formula for the derivative of W

$$W'(x) = \frac{W(x)}{x[1 + W(x)]} = \frac{1}{[1 + W(x)] \exp[W(x)]}, \quad \text{if}\, x \neq 0.$$

Other Lambert W properties have been studied by Corless *et al.* (1996).

Gupta *et al.* (2013) derived the mgf $M_Z^*(t)$ of the ZTGP distribution as

$$\begin{aligned} M_Z^*(t) &= \frac{M_z(t) - P(Z=0)}{1 - P(Z=0)} \\ &= \frac{\exp\left\{-\frac{\lambda}{\alpha}\left[W(-\alpha \mathrm{e}^{-\alpha+t}) + \alpha\right]\right\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}. \end{aligned}$$

Thus, the unconditional WGP survival function reduces to

$$
\begin{aligned}
S(x;\boldsymbol{\theta}) &= \sum_{z=1}^{\infty}[P(Y>x|Z=z)]^{z}P(Z=z) \\
&= \sum_{z=1}^{\infty}(\mathrm{e}^{-bx^{a}})^{z}P(Z=z) = M_{Z}^{*}(-bx^{a}) \\
&= \frac{\exp\left\{-\frac{\lambda}{\alpha}\left[W(-\alpha\,\mathrm{e}^{-\alpha-bx^{a}})+\alpha\right]\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}},
\end{aligned}
$$

where $\boldsymbol{\theta}=(a,b,\lambda,\alpha)$, $x>0$, $a,b>0$, $\lambda>0$ and $0<\alpha<1$. We can verify that the expression $-dS(x)/dx$ becomes a proper density function when has a normalizing constant given by

$$
C=\frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{-\frac{\lambda}{\alpha}W(-\alpha\mathrm{e}^{-\alpha})}-1}.
$$

Thus, the corresponding density function of X reduces to

$$
f(x;\boldsymbol{\theta}) = \frac{\lambda\,a\,b\,x^{a-1}\,\exp\left\{-(\frac{\lambda}{\alpha}+1)W(\psi)-\alpha-b\,x^{a}\right\}}{[1+W(\psi)]\,[\mathrm{e}^{-\frac{\lambda}{\alpha}W(-\alpha\mathrm{e}^{-\alpha})}-1]}, \tag{3.3}
$$

where $\psi(x)=-\alpha\,\mathrm{e}^{-\alpha-bx^{a}}$. We can verify using the Mathematica software that $\int_0^{\infty}f(x;\boldsymbol{\theta})dx=1$, i.e., $f(x;\boldsymbol{\theta})$ is a proper density function with support $\mathbb{R}^{+}$. Hereafter, a random variable X with pdf (3.3) is denoted by $X\sim$ WGP (a,b,λ,α). To the best of our knowledge the density (3.3) is a new result.

The cumulative distribution function (cdf), obtained from the normalized survival function, and the hazard rate function (hrf) of X are given by

$$
F(x;\boldsymbol{\theta}) = \frac{\mathrm{e}^{-\frac{\lambda}{\alpha}W(-\alpha\mathrm{e}^{-\alpha})}-\mathrm{e}^{-\frac{\lambda}{\alpha}[W(\psi)]}}{\mathrm{e}^{-\frac{\lambda}{\alpha}W(-\alpha\mathrm{e}^{-\alpha})}-1} \tag{3.4}
$$

and

$$
h(x;\boldsymbol{\theta}) = \frac{\lambda\,a\,b\,x^{a-1}\,\exp\left\{-\frac{\lambda}{\alpha}\left[W(\psi)-\alpha-b\,x^{a}\right]\right\}}{[1+W(\psi)]\,[\mathrm{e}^{-\frac{\lambda}{\alpha}W(\psi)}-1]},
$$

respectively.

We are motivated to study the WGP distribution because of the importance of the Weibull distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. A second positive point of the current generalization is that the Weibull distribution is a basic exemplar of the new distribution when α and λ tend to zero.

The exponential generalized Poisson (EGP) distribution is obtained from (3.3) when $a=1$. In addition, for $\alpha\to 0$, we have the exponential Poisson (EP) distribution, and letting $\lambda\to 0^{+}$, we obtain the exponential distribution. On the other hand, if $a=2$, the Rayleigh generalized Poisson (RGP) distribution arises as a special case. Further, for $\alpha\to 0$, we have the Rayleigh Poisson (RP) distribution, and adding $\lambda\to 0^{+}$, we obtain the Rayleigh distribution. The Weibull Poisson (WP) distribution comes as the limiting distribution of the WGP distribution when $\alpha\to 0^{+}$. In addition, if $\lambda\to 0^{+}$, we obtain the Weibull (W) distribution.

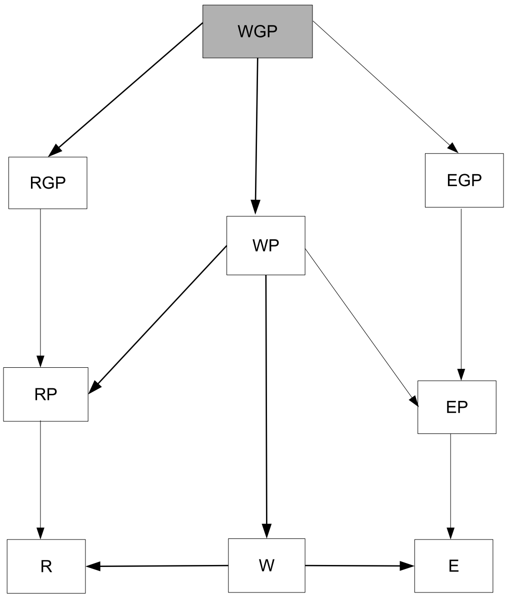

Figure 3.1: Relationships of the WGP sub-models.

3.3 Properties of the new distribution

3.3.1 A useful representation

Henceforth, we use an equation by (Gradshteyn and Ryzhik 2000) for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{3.5}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}, \quad i \geq 1 \tag{3.6}$$

and $c_{n,0} = a_0^n$.

We can derive a useful expansion for the WGP survival function given by

$$S(x;\boldsymbol{\theta}) = \sum_{k,i=0}^{\infty} \omega_{k,i}\, \overline{G}(x; a, b_{k,i}), \tag{3.7}$$

where $b_{k,i} = (k+i+1)\, b > 0$ and

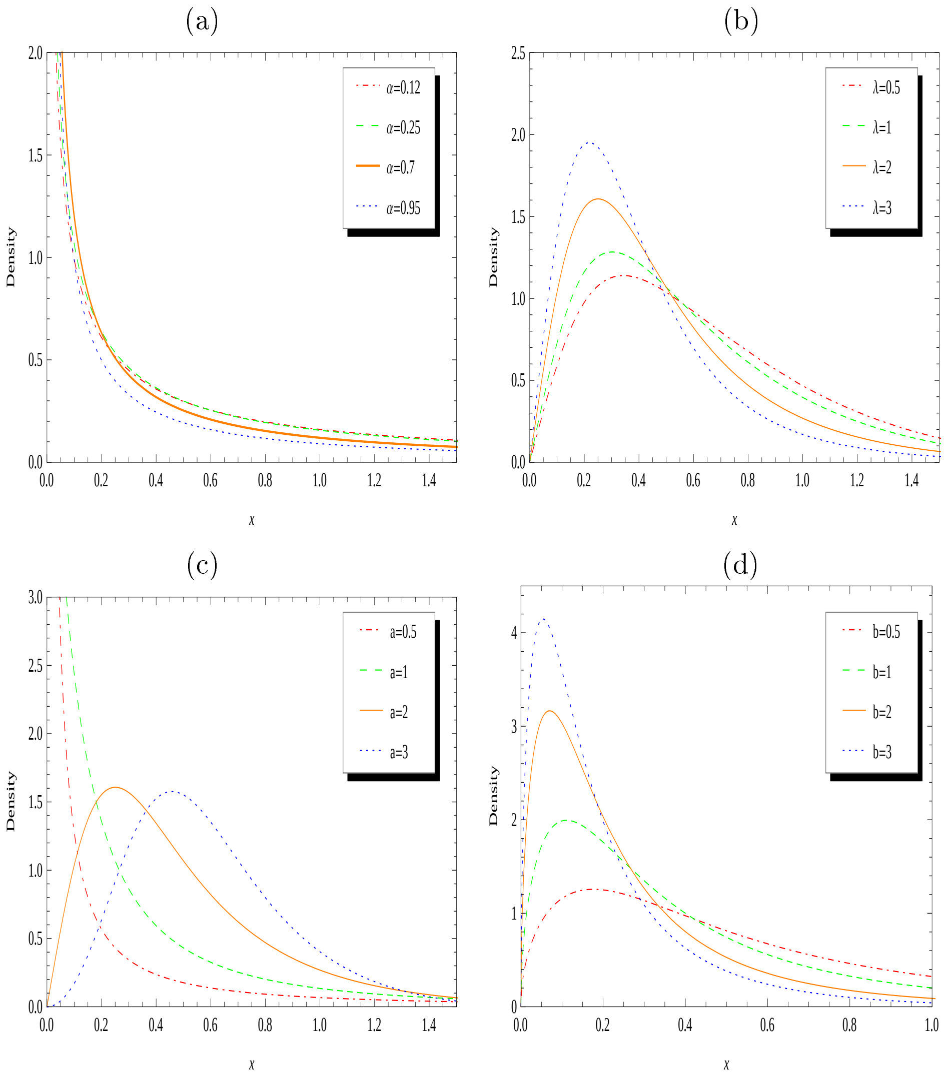

Figure 3.2: Plots of the WGP density function for: (a) $a = 0.5$, $b = 0.5$ and $\lambda = 1$, (b) $a = 2$, $b = 1$ and $\alpha = 0.6$, (c) $b = 1$, $\lambda = 2$ and $\alpha = 0.6$, (d) $a = 1.5$, $\lambda = 2$ and $\alpha = 0.6$.

$$\omega_{k,i} = \frac{(-1)^i \, \lambda^{k+1} \, \alpha^i \, d_{k+1,i} \, \mathrm{e}^{-\alpha(k+i+1)}}{K \, (k+1)!}.$$

The algebraic details that lead to (3.7) and the quantities K and $d_{k+1,i}$ are given in Appendix ??.

Equation (3.7) reveals that the WGP survival function is a linear combination of Weibull survival functions. Clearly,

$$F(x; \boldsymbol{\theta}) = \sum_{k,i=0}^{\infty} \omega_{k,i} \, G(x; a, b_{k,i}). \tag{3.8}$$

The corresponding expansion for the WGP density function becomes

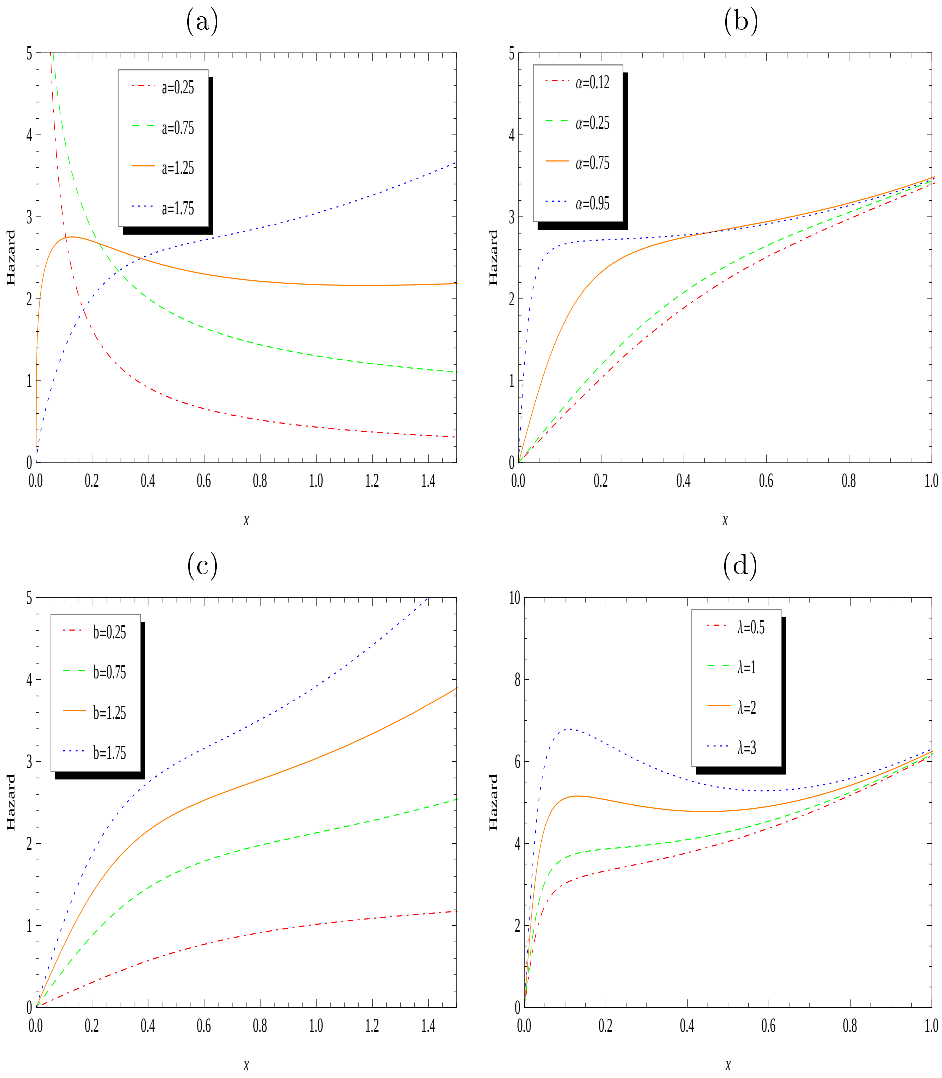

Figure 3.3: Plots of the WGP hrf for: (a) $\lambda = 1$, $\alpha = 0.5$ and $b = 1.5$; (b) $\lambda = 1$, $a = 2$ and $b = 1.5$; (c) $\lambda = 1$, $\alpha = 0.5$ and $a = 2$; (d) $\alpha = 0.9$, $a = 2$ and $b = 3$.

$$f(x;\boldsymbol{\theta}) = -\frac{dS(x)}{dx} = \sum_{k,i=0}^{\infty} \omega_{k,i}\, g(x;a,b_{k,i}). \tag{3.9}$$

Equation (3.9) gives the WGP density function as a linear combination of Weibull density functions. It is the main result of this section. Thus, some mathematical properties of the WGP distribution can be derived directly from those Weibull properties.

3.3.2 Moments

The nth moment of a Weibull random variable $Z_{k,i}$ with scale a and shape $b_{k,i}$ is $E(Z_{k,i}^n) = b_{k,i}^{-n/a}\,\Gamma(n/a+1)$. From equation (3.9), we obtain

$$\mu_n' = E(X^n) = \Gamma\left(\frac{n}{a}+1\right)\sum_{k,i=0}^{\infty}\omega_{k,i}\,b_{k,i}^{-n/a}.$$

The central moments (μ_n) and cumulants (κ_n) of X can be determined as

$$\mu_n = \sum_{s=0}^{n}(-1)^s\binom{n}{s}\mu_1'^s\,\mu_{n-s}' \qquad \text{and} \qquad \kappa_n = \mu_n' - \sum_{s=1}^{n-1}\binom{n-1}{s-1}\kappa_s\,\mu_{n-s}',$$

respectively, where $\kappa_1 = \mu_1'$. Then, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be obtained from the second, third and fourth cumulants. Plots of the skewness and kurtosis of the WGP distribution for some choices of a, b and α as function of λ are displayed in Figure 3.4. We take $a=2$, $b=1$ and $a=1.5$, $b=2$ for the plots of the skewness and kurtosis, respectively. These plots reveal that the shapes of the proposed distribution have strong dependence on the values of α and λ.

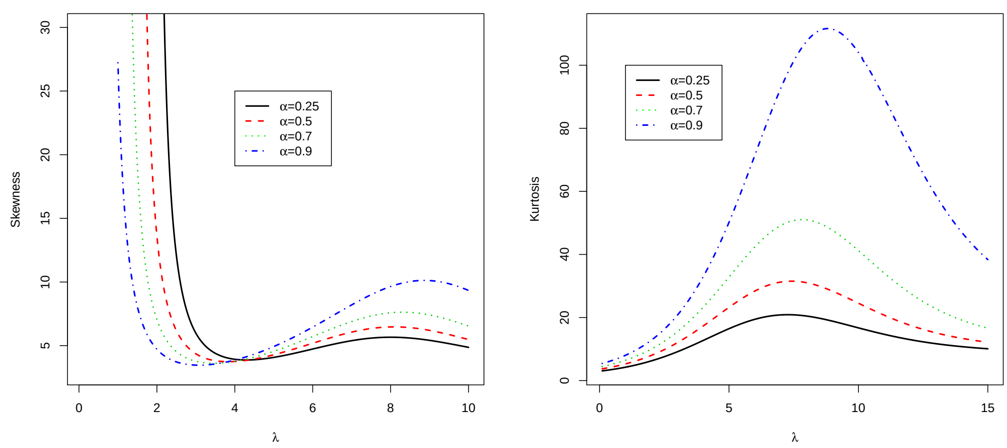

Figure 3.4: Skewness and kurtosis measures of the WGP distribution for some parameter values.

For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves. The incomplete moments of X can be expressed in terms of the incomplete moments of the Weibull distribution from equation (3.9). We can write

$$m_n(y) \;=\; E(X^n|X<y) = \sum_{k,i=0}^{\infty}\omega_{k,i}\int_0^y x^n\,a\,b_{k,i}\,x^{a-1}\mathrm{e}^{-b_{k,i}\,x^a}dx.$$

Using the power series for the exponential function and, after some simplification, we have

$$m_n(y) = \sum_{k,i,j=0}^{\infty} \delta_{k,i,j}\, y^{a(j+1)+n}, \tag{3.10}$$

where

$$\delta_{k,i,j} = \frac{(-1)^j\, a\, \omega_{k,i}\, b_{k,i}^{j+1}}{[a(j+1)+n]\, j!}.$$

The symbolic computational plataforms Maple, Mathematica and Matlab make it possible to automate the formulae derived in this paper since they have currently the ability to deal with analytic recurrence equations and sums of formidable size and complexity. In practical terms, we can substitute ∞ in the sums by a large number such as 30 or 50 for most practical applications. Establishing scripts for the closed-form expressions given throughout the paper can be more accurate computationally than other integral representations which can be prone to rounding off errors among others.

3.3.3 Quantile function

The quantile function of X, say $Q(u;\lambda,\alpha,a,b) = F^{-1}(u;\lambda,\alpha,a,b)$, follows by inverting (3.4) as

$$Q(u) \;=\; \left\{-\frac{\alpha}{b} - \frac{1}{b}\log\left[-\frac{1}{\alpha}\, W^{-1}\left\{-\frac{\alpha}{\lambda}\log\left[1+K(1-u)\right]\right\}\right]\right\}^{1/a}.$$

The Lambert $W(x)$ function is defined as the inverse function of $y\exp(y) = x$ and the solution is given by $y = W(x)$. Then, we can define the inverse function $W^{-1}(y) = x = y\exp(y)$.

We can rewrite the quantile function as

$$Q(u) \;=\; \left\{-\frac{1}{b}\left[\log\left(-\frac{M}{\alpha}\right) + \alpha + M\right]\right\}^{1/a}, \tag{3.11}$$

where

$$M = -\frac{\alpha}{\lambda}\log[1+K(1-u)].$$

Quantiles of interest for X can be obtained from the last equation by substituting appropriate values for u. In particular, the median of X comes when $u = 0.5$.

3.3.4 Generating Function

We provide two representations for the mgf of X, say $M(t) = E(\mathrm{e}^{tX})$. The algebraic developments follow closely the works by (Cheng *et al.* 2003), (Nadarajah and Gupta 2007) and

(Cordeiro *et al.* 2010). We can write $M(t)$ from (3.9) as

$$M(t) = a \sum_{k,i=0}^{\infty} \omega_{k,i}\, b_{k,i}\, L_k(t), \tag{3.12}$$

where

$$L_k(t) = \int_0^\infty x^{a-1} \exp[t\,x - b_{k,i}\, x^a] dx,$$

and $\omega_{k,i}$ and $b_{k,i}$ are defined in Section 3.1.

A first representation for $M(t)$ is based on the Wright generalized hypergeometric function (Wright 1935) defined by

$${}_p\Psi_q \left[\begin{array}{c} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \end{array} ; x \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j\, n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j\, n)} \frac{x^n}{n!}.$$

The Wright function exists if $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$. We have

$$\begin{aligned} L_k(t) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^\infty x^{m+a-1} \exp\left(-b_{k,i}\, x^a\right) dx = \frac{1}{a\, b_{k,i}} \sum_{m=0}^{\infty} \frac{(t/b_{k,i}^a)^m}{m!} \Gamma\left(\frac{m}{a} + 1\right) \\ &= \frac{1}{a\, b_{k,i}}\, {}_1\Psi_0 \left[\begin{array}{c} (1, a^{-1}) \\ - \end{array} ; \frac{t}{b_{k,i}^a} \right] \end{aligned} \tag{3.13}$$

provided that $a > 1$. Combining (3.12) and (3.13), the mgf of X (for $a > 1$) reduces to

$$M(t) = \sum_{k,i=0}^{\infty} \omega_{k,i}\, {}_1\Psi_0 \left[\begin{array}{c} (1, a^{-1}) \\ - \end{array} ; \frac{t}{b_{k,i}^a} \right].$$

A second representation for $M(t)$ follows from the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma\left(b_j + t\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - t\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j + t\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - t\right)} x^{-t} dt,$$

where $i = \sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik 2000, Section 9.3). The Meijer G-function contains many integrals with elementary and special functions. Some of these integrals are given by (Prudnikov *et al.* 1986).

For an arbitrary $g(\cdot)$ function, we can write

$$\exp[-g(x)] = G_{0,1}^{1,0}\left(g(x) \middle| \begin{array}{c} - \\ 0 \end{array} \right)$$

and then

$$L_k(t) = \int_0^\infty \mathrm{e}^{tx}\, x^{a-1}\, G_{0,1}^{1,0}\left(b_{k,i} x^a \middle| \begin{array}{c} - \\ 0 \end{array} \right) dx.$$

We assume that $a = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. Using equation (2.24.1.1) in (Prudnikov *et al.* 1986), we have (for $t < 0$)

$$L_j(t) \;=\; \frac{p^{a-1/2}(-t)^{-a}}{(2\pi)^{(p+q)/2-1}}\; G_{q,p}^{p,q}\left(\frac{(b_{k,i}^a\, a)^q p^p}{(-t)^p q^q} \middle| \begin{array}{c} \frac{1-a}{p}, \frac{2-a}{p}, \ldots, \frac{p-a}{p} \\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q} \end{array} \right).$$

Using (3.12) and the last equation, we obtain (for $t < 0$)

$$M(t) = \frac{a\, p^{a-1/2}\,(-t)^{-a}}{(2\pi)^{(p+q)/2-1}} \sum_{k,i=0}^{\infty} \omega_{k,i}\, b_{k,i}\, G_{q,p}^{p,q}\left(\frac{(b_{k,i}^a\, a)^q p^p}{(-t)^p q^q} \middle| \begin{array}{c} \frac{1-a}{p}, \frac{2-a}{p}, \ldots, \frac{p-a}{p} \\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q} \end{array} \right).$$

Here, the condition $a = p/q$ in the last equation is not very restrictive since every real number can be approximated by a rational number. For irrational a, an approximation of vanishingly small error can be made using increasingly accurate rational approximations of a.

3.3.5 Mean deviations

The mean deviations about the mean ($\delta_1(X) = E(|X - \mu_1'|)$) and about the median ($\delta_2(X) = E(|X - M|)$) of X can be expressed as

$$\delta_1(X) = 2\mu_1'\, F\left(\mu_1'\right) - 2m_1\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2(X) = \mu_1' - 2m_1(M),$$

respectively, where $\mu_1' = E(X)$, $M = Median(X)$ is the median computed from (3.11) with $u = 1/2$, $F(\mu_1')$ is easily calculated from the cdf (3.4) and $m_1(z) = \int_{-\infty}^{z} x f(x) dx$ is the first incomplete moment given by (3.10) with $n = 1$.

The Lorenz and Bonferroni curves are important applications of the mean deviations in fields like economics, reliability, demography insurance and medicine. They are defined for a given probability π by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$ respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is given by (3.11). The Bonferroni and Lorenz curves for the WGP distribution as functions of π are readily calculated from (3.10) for $n = 1$. They are plotted for some parameter values in Figure 3.5.

3.4 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies (Rényi 1961) and (Shannon 1951). The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined by (for $\delta > 0$ and $\delta \neq 1$)

$$I_R(\delta) = \frac{1}{1-\delta} \log\left(\int_0^\infty f^\delta(x) dx \right).$$

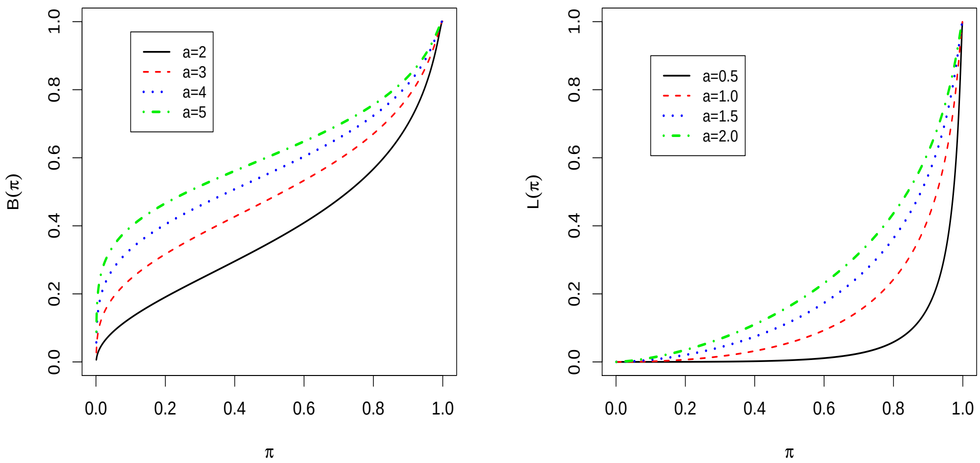

Figure 3.5: Plots of $B(\pi)$ and $L(\pi)$ versus π for the WGP distribution. Here, $b=3$, $\alpha=0.5$ and $\lambda=2$ for $B(\pi)$ and $b=1$, $\alpha=0.5$ and $\lambda=1.5$ for $L(\pi)$.

The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the particular case of the Rényi entropy when δ goes to one.

Here, we derive explicit expressions for the Rényi and Shannon entropies for the WGP distribution. From equation (3.3), we write

$$I_R(\delta) = \frac{1}{1-\delta}\log\left\{\int_0^\infty \left[\frac{L\, x^{\delta(a-1)}\, \mathrm{e}^{-\delta b\, x^a}\, \mathrm{e}^{-\delta(\frac{\lambda}{\alpha}+1)W(\psi)}}{[1+W(\psi)]^\delta}\right] dx\right\},$$

where $L=\left(\frac{\lambda\, a\, b\, \mathrm{e}^{-\alpha}}{K}\right)^\delta$ and the quantity K is defined in Appendix **??**. Using the expansion in Taylor series for the exponential function, we obtain

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}\log\left\{\int_0^\infty L\sum_{j,t=0}^\infty \frac{(-1)^{j+t}\,\delta^{j+t}\,b^t\,(\lambda+\alpha)^j}{\alpha^j\, j!\, t!}\right. \\ &\times \left. W(\psi)^j\,[1+W(\psi)]^{-\delta}\, x^{a(\delta+t)-\delta} dx\right\}. \end{aligned} \quad (3.14)$$

Now, we use the expansion

$$(x+a)^{-k} = \sum_{n=0}^\infty \binom{-k}{n} x^n\, a^{-k-n},$$

where k is any real number, $|x|<a$ and $\binom{-k}{n}$ is an extended binomial coefficient (for any real value of k and positive integer n), given by

$$\binom{-k}{n} = (-1)^n \binom{k+n-1}{n}.$$

Since $|W(\psi)| < 1$, equation (3.14) reduces to

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log\left\{ \int_0^\infty L \sum_{j,t=0}^\infty \frac{(-1)^{j+t}\,\delta^{j+t}\,b^t\,(\lambda+\alpha)^j}{\alpha^j\,j!\,t!} \right. \\ &\times \left. \sum_{n=0}^\infty W(\psi)^{n+j}\,x^{a(\delta+t)-\delta} dx \right\}, \end{aligned}$$

Using equations (3.2), (3.5) and (3.6), we obtain

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log\left\{ L \sum_{j,t,n,i=0}^\infty \frac{(-1)^{j+t}\,\delta^{j+t}\,b^t\,(\lambda+\alpha)^j \binom{-\delta}{n} d_{n+j,i}}{\alpha^j\,j!\,t!} \right. \\ &\times \left. \int_0^\infty \psi^{j+n+i}\,x^{a(\delta+t)-\delta} dx \right\}, \end{aligned}$$

where the constant $d_{n+j,i}$ is defined by $d_{n+j,i} = (i\,q_0)^{-1} \sum_{m=1}^{i} [m(n+j+1)-i]\,q_m\,d_{n+j,i-m}$, with $q_i = (-1)^i\,(i+1)^{i-1}/i!$ (for $i \geq 0$) and $d_{n+j,0} = q_0^{n+j}$.

Replacing the expression of ψ and integrating, the Rényi entropy reduces to

$$I_R(\delta) = \frac{1}{1-\delta} \log\left\{ a^{-1}\,\Gamma\Big(\frac{a(t+\delta)-\delta+1}{a}\Big) \sum_{j,t,n,i=0}^\infty \xi_{j,t,n,i} \right\}, \text{ for } a > 1,$$

where

$$\xi_{j,t,n,i} = \frac{(-1)^{t+n+i}\,L\,\delta^{j+t}\,b^t\,\alpha^{n+i}\,(\lambda+\alpha)^j \binom{-\delta}{n} d_{n+j,i}}{\mathrm{e}^{\alpha(n+j+i)}\,j!\,t!\,[b(j+n+i)]^{\frac{1+a(t+\delta)-\delta}{a}}}.$$

The Shannon entropy can be obtained by limiting $\delta \uparrow 1$ in the last equation. However, it is easier to derive an expression for $I_S(\delta)$ from its definition. We have

$$\begin{aligned} E\{-\log[f(X)]\} &= \alpha + \log(K) - \log(\lambda\,a\,b) - (a-1)E\{\log(X)\} + b\,E(X^a) \\ &+ \Big(\frac{\lambda}{\alpha}+1\Big) E[W(\psi)] + E\{\log[1+W(\psi)]\}, \end{aligned} \quad (3.15)$$

where the quantity K is defined in Appendix A. Here, and for the rest of the section, $\psi = \psi(X)$. The four expectations in (3.15) can be easily evaluated numerically. Using (3.9), they can also be determined as

$$\begin{aligned} E\{\log(X)\} &= \sum_{k,i=0}^\infty \omega_{k,i} \int_0^\infty \log(x)\,g(x;a,b_{k,i}) dx \\ &= -\sum_{k,i=0}^\infty a^{-1}\,\omega_{k,i}\,[\gamma + \log(b_{k,i})], \end{aligned}$$

where $\gamma \simeq 0.577216$ is the Euler's constant. Further,

$$E(X^a) = \sum_{k,i=0}^{\infty} \frac{\omega_{k,i}}{b_{k,i}}.$$

Using the power series expansion (3.2) for the Lambert W function, we have

$$\begin{aligned} E\{W(\psi)\} &= \sum_{k,i=0}^{\infty} \omega_{k,i} \int_0^{\infty} W(\psi)\, g(x; a, b_{k,i}) dx \\ &= -\sum_{k,i=0}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha^n\, n^{n-2}\, \omega_{k,i}\, (k+i+1)\, \mathrm{e}^{-n\alpha}}{(k+i+n+1)\,(n-1)!} \end{aligned}$$

and

$$E\{\log[1 + W(\psi)]\} = \sum_{k,i=0}^{\infty} \omega_{k,i} \int_0^{\infty} \log[1 + W(\psi)]\, g(x; a, b_{k,i}) dx.$$

Using the power series expansion for the logarithm (since $|W(\psi)| \leq 1$), the power series (3.2) for the Lambert W function and equations (3.5) and (3.6), the last equation reduces to

$$E\{\log[1 + W(\psi)]\} = \sum_{k,i,r=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{r-1}\,(k+i+1)\,\omega_{k,i}\, d_{n,r}\, \alpha^{r+n}\, \mathrm{e}^{-\alpha(r+n)}}{n\,(k+i+r+n+1)},$$

where $d_{n,r}$ is defined before for $r > 0$ and $r = 0$.

3.5 Reliability

We derive the reliability, $R = \Pr(X_2 < X_1)$, when $X_1 \sim WGP(\lambda_1, \alpha_1, a, b)$ and $X_2 \sim WGP(\lambda_2, \alpha_2, a, b)$ are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Let f_i and F_i denote the pdf and cdf of X_i, respectively. Based on the representations (3.8) and (3.9), we can write

$$R = \sum_{r,s,u,v=0}^{\infty} \int_0^{\infty} \omega_{r,s}\, \omega_{u,v}\, g(x; a, b_{r,s})\, G(x; a, b_{u,v})\, dx \tag{3.16}$$

where

$$\omega_{r,s} = \frac{(-1)^s\, \lambda_1^{r+1}\, d_{r+1,s}\, \alpha_1^s\, \mathrm{e}^{-\alpha_1(r+s+1)}}{K_1\,(r+1)!}$$

and

$$\omega_{u,v} = \frac{(-1)^v\, \lambda_2^{u+1}\, d_{u+1,v}\, \alpha_2^v\, \mathrm{e}^{-\alpha_2(u+v+1)}}{K_2\,(u+1)!}.$$

Here, $K_1 = \mathrm{e}^{-\frac{\lambda_1}{\alpha_1} W(-\alpha_1 \mathrm{e}^{-\alpha_1})} - 1$ and $K_2 = \mathrm{e}^{-\frac{\lambda_2}{\alpha_2} W(-\alpha_2 \mathrm{e}^{-\alpha_2})} - 1$, $d_{r+1,s}$ and $d_{u+1,v}$ are defined in Appendix A and $d_{r+1,s}$ is defined in Section 3.4.

Using equation (3.21) and after some algebra, we obtain

$$R = \sum_{r,s,u,v=0}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \omega_{r,s}\, \omega_{u,v}\, (u+v+1)^j\, \Gamma(1+j)}{j!\,(r+s+1)^j}.$$

3.6 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Moments of order statistics play an important role in quality control and reliability, where some predictors are often based on moments of the order statistics. We derive an explicit expression for the density function of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$ (see Appendix B). Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the WGP distribution. Let $X_{i:n}$ denote the ith order statistic. From equations (3.3) and (3.4), the pdf of $X_{i:n}$ can be expressed as an infinite linear combination of Weibull density functions

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{t,v,m,r=0}^{\infty} \gamma_{i:n}(j,s,t,v,m,r)\, g(x; a, b_{t,v,m,r}), \tag{3.17}$$

where $b_{t,v,m,r} = b(t+v+m+r+1)$ and

$$\gamma_{i:n}(j,s,t,v,m,r) = \frac{(-1)^{j+s+v}(m+r+1)\,(\lambda s)^t\, \alpha^v\, (K+1)^{j+i-1}\, d_{t,v}\, \omega_{m,r}}{(t+v+m+r+1)\, t!\, K^{j+i-1}\, B(i, n-i+1)} \times \binom{n-i}{j} \binom{j+i-1}{s} \mathrm{e}^{\left\{\frac{\lambda s}{\alpha} W(-\alpha \mathrm{e}^{-\alpha}) - \alpha(v+t)\right\}}.$$

Equation (3.17) reveals that the pdf of $X_{i:n}$ can be represented as a finite mixture of WGP density functions. So, some mathematical properties for $X_{i:n}$ can be obtained from this equation.

For example, the pth moment of the ith WGP order statistic in a sample of size n comes from (3.17) as

$$E(X_{i:n}^p) = \Gamma\left(\frac{p}{a} + 1\right) \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{t,v,m,r=0}^{\infty} \rho_{i:n}(j,s,t,v,m,r), \tag{3.18}$$

where

$$\rho_{i:n}(j,s,t,v,m,r) = \frac{(-1)^{j+s+v}(m+r+1)\,(\lambda s)^t\, \alpha^v\, b^{-p/a}\, (K+1)^{j+i-1}}{t!\, K^{j+i-1}\, B(i, n-i+1)\, (t+v+m+r+1)^{1+p/a}} \times d_{t,v}\, \omega_{m,r} \binom{n-i}{j} \binom{j+i-1}{s} \mathrm{e}^{\left\{\frac{\lambda s}{\alpha} W(-\alpha \mathrm{e}^{-\alpha}) - \alpha(v+t)\right\}}.$$

Alternatively, we obtain another explicit expression for these moments using a result due to (Barakat and Abdelkader 2004) applied to the i.i.d. case

$$E(X_{i:n}^p) = p \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} I_j(p), \tag{3.19}$$

where

$$I_j(p) = \int_0^\infty x^{p-1} \, \{1 - F(x)\}^j \, dx.$$

From equations (3.2), (3.4), (3.5), (3.6) and (3.21) this integral can be reduced to

$$I_j(p) = K^{-j} \sum_{s=0}^{\infty} h_{j,s} \int_0^\infty x^{p-1} \, W(\psi)^{j+s} dx,$$

where the quantity K is defined in Appendix A and the constant $h_{j,s}$ is defined by $h_{j,s} = (s\,e_0)^{-1} \sum_{t=1}^{s} [t(j+1) - s] \, e_t \, h_{j,s-t}$, $e_s = \frac{(-1)^{s+1} \lambda^{s+1}}{\alpha^{s=1} (s+1)!}$ (for $s \geq 0$) and $h_{j,0} = e_0^j$.

Using equations (3.2), (3.5) and (3.6) and after some algebra, we obtain

$$I_j(p) = K^{-j} \sum_{s,t=0}^{\infty} h_{j,s} \, d_{j+s,t} \, (-1)^{j+s+t} \, \alpha^{j+s+t} \, \mathrm{e}^{-\alpha(j+s+t)} \int_0^\infty x^{p-1} \, \mathrm{e}^{-b(j+s+t)x^a} \, dx,$$

where the constant $d_{j+s,t}$ is defined by $d_{j+s,t} = (t\,q_0)^{-1} \sum_{m=1}^{t} [m(j+s+1) - t] \, q_m \, d_{j+s,t-m}$, with $q_t = (-1)^t \, (t+1)^{t-1}/t!$ (for $t \geq 0$) and $d_{j+s,0} = q_0^{j+s}$.

Simple integration yields

$$I_j(p) = a^{-1} \, K^{-j} \, \Gamma\Big(\frac{p}{a}\Big) \sum_{s,t=0}^{\infty} \frac{(-1)^{j+s+t} \, h_{j,s} \, d_{j+s,t} \, \alpha^{j+s+t} \, \mathrm{e}^{-\alpha(j+s+t)}}{[b(j+s+t)]^{p/a}}.$$

From equation (3.19), we obtain

$$\begin{aligned} E(X_{i:n}^p) &= p \, a^{-1} \, K^{-j} \, \Gamma\Big(\frac{p}{a}\Big) \sum_{j=n-i+1}^{n} \sum_{s,t=0}^{\infty} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \\ &\times \frac{(-1)^{j+s+t} \, h_{j,s} \, d_{j+s,t} \, \alpha^{j+s+t} \, \mathrm{e}^{-\alpha(j+s+t)}}{[b(j+s+t)]^{p/a}}. \end{aligned} \tag{3.20}$$

We can compute the moments of the WGP order statistics by two different formulas. Equation (3.18) involves four infinite sums and two finite sums, whereas equation (3.20) is much simpler since it involves only two infinite sums and one finite sum.

3.7 Maximum likelihood estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the WGP distribution from complete samples only. Let $x_1, \ldots, x_n$ be a observed sample of size n from the WGP(a, b, λ, α) distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, \lambda, \alpha)^T$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}) &= n\left[-\lambda - \alpha + \log(N)\right] - b\sum_{i=1}^{n} x_i^a + (a-1)\sum_{i=1}^{n}\log(x_i) \\ &\quad - \left(\frac{\lambda}{\alpha} + 1\right)\sum_{i=1}^{n} W(\psi_i) - \sum_{i=1}^{n}\log\left[1 + W(\psi_i)\right], \end{aligned}$$

where $\psi_i = -\alpha\,\mathrm{e}^{-\alpha - b x_i^a}$ and $N = \frac{\lambda\, a\, b\, (\mathrm{e}^{\lambda}-1)\,(1-\mathrm{e}^{-\lambda})^{-1}}{[\exp\{-\frac{\lambda}{\alpha}W(-\alpha \mathrm{e}^{-\alpha})\}-1]}$. The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$\begin{aligned} U_a(\boldsymbol{\theta}) &= \frac{n}{a} + \sum_{i=1}^{n}\log(x_i) + b\sum_{i=1}^{n}\eta(x_i)\left[\varphi(x_i) - 1\right], \\ U_b(\boldsymbol{\theta}) &= \frac{n}{b} + \sum_{i=1}^{n} x_i^a\left[\varphi(x_i) - 1\right], \\ U_\alpha(\boldsymbol{\theta}) &= -n + \frac{\lambda}{\alpha^2}\sum_{i=1}^{n} W(\psi_i) + \left(\frac{\alpha-1}{\alpha}\right)\sum_{i=1}^{n}\varepsilon(x_i)\left[\frac{\alpha+\lambda}{\alpha} + \frac{1}{1+W(\psi_i)}\right], \\ U_\lambda(\boldsymbol{\theta}) &= n\left[\frac{1-\lambda}{\lambda} - \frac{1}{\mathrm{e}^{\lambda}-1}\right] - \frac{1}{\alpha}\sum_{i=1}^{n} W(\psi_i), \end{aligned}$$

where $\varepsilon(x_i) = \frac{W(\psi_i)}{1+W(\psi_i)}$, $\eta(x_i) = x_i^a\,\log(x_i)$ and

$$\varphi(x_i) = \left[\frac{\varepsilon(x_i)}{1+W(\psi_i)} + \left(1 + \frac{\lambda}{\alpha}\right)\varepsilon(x_i)\right].$$

For interval estimation and hypothesis tests on the model parameters, we require the 4×4 observed information matrix $J = J(\boldsymbol{\theta})$ given in the Appendix C. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_4(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. We can replace $I(\boldsymbol{\theta})$ by the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, say $J(\widehat{\boldsymbol{\theta}})$, to construct approximate confidence intervals for the parameters based on the multivariate normal $N_4(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution.

Further, the likelihood ratio (LR) statistic can be adopted for comparing this distribution with some of its sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models of the WGP distribution. For example, the test of $H_0 : a = 1$ versus $H_1 : H_0$ *is not true* is equivalent to compare the WGP and EGP distributions and the LR statistic becomes

$$w = 2\{\ell(\hat{a}, \hat{b}, \hat{\lambda}, \hat{\alpha}) - \ell(1, \widetilde{b}, \widetilde{\lambda}, \widetilde{\alpha})\},$$

where $\hat{a}$, $\hat{b}$, $\hat{\lambda}$ and $\hat{\alpha}$ are the MLEs under H_1 and $\widetilde{b}$, $\widetilde{\lambda}$ and $\widetilde{\alpha}$ are the estimates under H_0.

3.8 Application

Here, we present an application of the WGP model to the data obtained from (Proschan 1963). The data set denotes the number of successive failures for the air conditioning system of each member in a fleet of 7 Boeing 720 airplanes. The 125 observations refer to aircraft numbers 7910, 7911, 7912, 7913, 7914, 7915 and 7916. We present the fits of the WGP, EP, EGP, EW and BW distributions. The BW distribution has pdf given by

$$f(x; a, b, \alpha, \lambda) = [\alpha\,\lambda/B(a, b)]\, x^{\lambda-1} \exp(-b\,\alpha\, x^{\lambda})[1 - \exp(-\alpha x^{\lambda})]^{a-1},$$

for $x > 0$. The EW model follows if $a = 1$.

Table 3.1 gives the MLEs and corresponding standard errors (SEs), the values of the Cramér-von Mises (CM) and Anderson-Darling (AD) statistics for the current data. In general, the smaller the values of these statistics, the better the fit to the data. To obtain the statistics, one can proceed as follows: (1) compute $v_i = F(x_i; \widehat{\theta})$ and $y_i = \Phi^{-1}(v_i)$, where the x_i's are in ascending order, $\widehat{\theta}$ is an estimate of θ, $\Phi(\cdot)$ is the standard normal cumulative function and $\Phi^{-1}(\cdot)$ denotes its inverse; (2) compute $u_i = \Phi[(y_i - \bar{y})/s_y]$, where $\bar{y}$ is the sample mean of y_i and s_y is the sample standard deviation; (3) compute $\mathrm{CM}^* = \sum_{i=1}^{n}[u_i - (2\,i-1)/2\,n]^2 + 1/(12\,n)$ and $\mathrm{AD}^* = -n - (1/n)\sum_{i=1}^{n}[(2\,i-1)\log(u_i) + (2\,n+1-2\,i)\log(1-u_i)]$, and then $\mathrm{CM} = (1+0.5/n)\mathrm{CM}^*$ and $\mathrm{AD} = (1 + 0.75/n + 2.25/n^2)\mathrm{AD}^*$.

Table 3.1: MLEs, the corresponding SEs (given in parentheses), maximized log-likelihoods, statistics W* and A* for the numbers of successive failures.

Distribution	a	b	λ	α	W*	A*
EP	-	0.0085	1.0371	-	0.0718	0.4592
	-	(0.0015)	(0.5792)	-		
EGP	-	0.0072	0.6213	0.4231	0.0497	0.3282
	-	(0.0019)	(1.2792)	(0.3286)		
WGP	1.3744	0.0006	1.2021	0.8853	**0.0295**	**0.2084**
	(0.1675)	(0.0006)	(0.0159)	(0.1095)		
Distribution	a	b	c	λ		
EW		0.5618	2.3252	0.0338	0.0415	0.2970
		(0.1642)	(1.4154)	(0.0307)		
BW	4.2708	3.4124	0.3932	0.0123	0.0370	0.2805
	(3.0016)	(3.3254)	(0.1266)	(0.0379)		

The figures in Table 3.1 indicate that the fitted WGP distribution to these data is superior than the other fitted models. Further, the QQ-plots of the data for the fitted models are displayed in Figure 3.6. They reveal that the WGP model provides the better fit to these data.

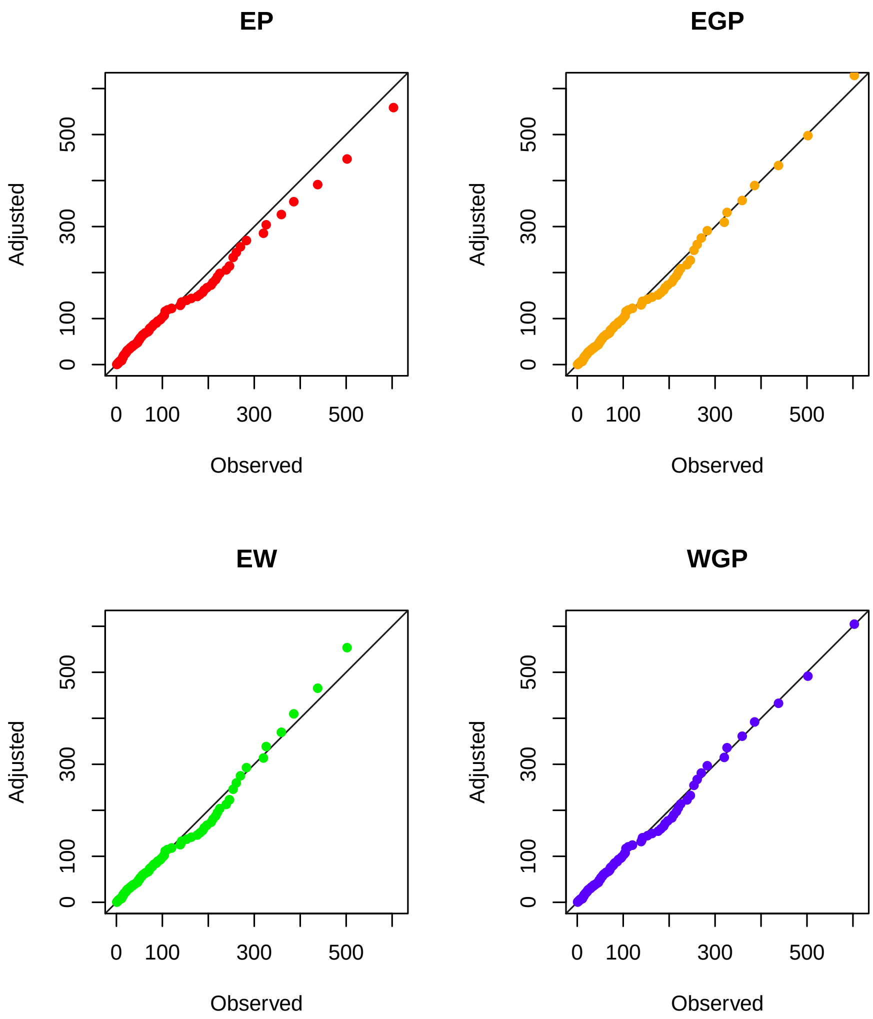

Figure 3.6: QQ-plots to the number of successive failures.

3.9 Concluding remarks

The Weibull distribution is commonly used to model the lifetime of a system. However, it does not exhibit a bathtub-shaped failure rate function and thus it can not be used to model the complete lifetime of a system. We define a new lifetime model, called the Weibull generalized Poisson (WGP) distribution, which extends the exponential generalized Poisson (EGP) distribution proposed by (Gupta *et al.* 2013), whose failure rate function can be increasing, decreasing and upside-down bathtub. The WGP density function can be expressed as a linear combination of Weibull densities, which allows to obtain several of its structural properties. We provide a mathematical treatment of the distribution including explicit expressions for the density function, generating function, ordinary and incomplete moments, Rényi and Shannon entropies, reliability, order statistics and their moments. The parameter estimation is approached by maximum likelihood and the observed information matrix is derived. The usefulness of the new model is illustrated in an application to real data using formal goodness-of-fit tests. By means of a real data application, we show that the proposed distribution is a very competitive model to the exponentiated Weibull and beta Weibull distributions.

Appendix A - The WGP survival function

After some algebra, from equation (3.4) , we can write

$$S(x;\boldsymbol{\theta}) \quad = \quad -\frac{1}{K}\Big\{1-\exp\Big(-\Big[\frac{\lambda}{\alpha}W(\psi)\Big]\Big)\Big\},$$

where $K = \mathrm{e}^{-\frac{\lambda}{\alpha}W(-\alpha \mathrm{e}^{-\alpha})} - 1$.

Using the power series expansion

$$1-\mathrm{e}^{-z} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}z^k}{k!}, \tag{3.21}$$

we obtain

$$S(x;\boldsymbol{\theta}) \quad = \quad -\frac{1}{K}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\lambda^k W(\psi)^k}{\alpha^k k!}.$$

Based on the power series (3.2), we have

$$S(x;\boldsymbol{\theta}) \quad = \quad \sum_{k=1}^{\infty}\frac{(-1)^{k+2}\lambda^k}{K\,\alpha^k k!}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,n^{n-2}}{(n-1)!}\psi^n\right]^k.$$

Setting $i=n-1$ in the last sum gives

$$S(x;\boldsymbol{\theta}) \quad = \quad \sum_{k=1}^{\infty}\frac{(-1)^{k+2}\lambda^k}{K\,\alpha^k k!}\,\psi^k\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}\,(i+1)^{i-1}}{i!}\,\psi^i\right]^k.$$

Applying equations (3.5) and (3.6) in the second sum gives

$$S(x;\boldsymbol{\theta}) \quad = \quad \sum_{k,i=0}^{\infty}\omega_{k,i}\overline{G}(x;a,b(k+i)),$$

where

$$\omega_{k,i} = \frac{(-1)^{i+3}\lambda^{k+1}\,\alpha^i\,\mathrm{e}^{-\alpha(k+1+i)}\,d_{k+1,i}}{K\,(k+1)!},$$

with $d_{k+1,i} = (i\,q_0)^{-1}\sum_{m=1}^{i}[m(k+2)-i]\,q_m\,d_{k+1,i-m}$, $d_{k+1,0} = q_0^{k+1} = 1$ and $q_i = (-1)^i\,(i+1)^{i-1}/i!$ for $i \geq 0$.

Appendix B - Expansion for the density function of the order statistic

The density $f_{i:n}(x)$ of the ith order statistic, for $i = 1, \ldots n$, from iid random variables $X_1, \ldots, X_i$ from the WGP distribution is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}.$$

Substituting (3.4) in this equation, we can write

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \frac{(-1)^j \, (K+1)^{j+i-1} \binom{n-i}{j}}{K^{j+i-1}} \\ &\times \quad \left[1 - \exp\left\{\frac{-\lambda}{\alpha}[W(\psi) - W(-\alpha \mathrm{e}^{-\alpha})]\right\}\right]^{j+i-1}. \end{aligned}$$

Using the binomial theorem and the power series for the exponential function, we have

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \frac{(-1)^{j+s} \, (K+1)^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{s} \, \mathrm{e}^{\frac{\lambda s}{\alpha} W(-\alpha \mathrm{e}^{-\alpha})}}{K^{j+i-1}} \\ &\times \sum_{t=0}^{\infty} \frac{(-1)^t \, (\lambda s)^t}{t! \, \alpha^t} W(\psi)^t. \end{aligned}$$

Substituting expansion (3.2) and using equations (3.5) and (3.6), we obtain

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \frac{(-1)^{j+s} \, (K+1)^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{s} \, \mathrm{e}^{\frac{\lambda s}{\alpha} W(-\alpha \mathrm{e}^{-\alpha})}}{K^{j+i-1}} \\ &\times \quad \sum_{t=0}^{\infty} \frac{(-1)^t \, (\lambda s)^t}{t! \, \alpha^t} \sum_{v=0}^{\infty} d_{t,v} \psi^{t+v}, \end{aligned}$$

where $d_{t,0} = q_0^t$, $q_v = (-1)^v (v+1)^{v-1}/v!$ and $d_{t,v} = (v\, q_0)^{-1} \sum_{m=1}^{v} [m(t+1) - v] \, q_m \, d_{t,v-m}$ (for $v \geq 1$).

Substituting equations of ψ and $f(x)$, the last expression reduces to

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{t,v=0}^{\infty} \frac{(-1)^{j+s+t} \, (\lambda s)^t \, f_{t,v} \, (K+1)^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{s} \, \mathrm{e}^{\frac{\lambda s}{\alpha} W(-\alpha \mathrm{e}^{-\alpha})}}{t! \, \alpha^t \, K^{j+i-1} \, B(i, n-i+1)} \\ &\times \quad (-\alpha \mathrm{e}^{-\alpha - b x^a})^{t+v} \sum_{m,r=0}^{\infty} \omega_{m,r} \, g(x; a, b(m+r+1)) \\ &= \quad \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{t,v,m,r=0}^{\infty} \gamma_{i:n}(j, s, t, v, m, r) \, g(x; a, b_{t,v,m,r}), \end{aligned}$$

where $b_{t,v,m,r} = b(t+v+m+r+1)$ and

$$\begin{aligned}\gamma_{i:n}(j,s,t,v,m,r) &= \frac{(-1)^{j+s+v}(m+r+1)\,(\lambda s)^t\,\alpha^v\,(K+1)^{j+i-1}\,d_{t,v}\,\omega_{m,r}}{(t+v+m+r+1)\,t!\,K^{j+i-1}\,B(i,n-i+1)}\\ &\times \binom{n-i}{j}\binom{j+i-1}{s}\,\mathrm{e}^{\left\{\frac{\lambda s}{\alpha}W(-\alpha \mathrm{e}^{-\alpha})-\alpha(v+t)\right\}}.\end{aligned}$$

Appendix C - Information Matrix

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the model parameters (a, b, λ, α) are given by

$$\begin{aligned}
J_{aa} &= -\frac{n}{a^2} - b\sum_{i=1}^{n} x_i^a\,\log^2(x_i) + \sum_{i=1}^{n}\frac{\delta(x_i)}{1+W(\psi_i)}\Bigg[1+\frac{2b\,x_i^a\,\varepsilon(x_i)}{1+W(\psi_i)}\\
&- \frac{b\,x_i^a}{1+W(\psi_i)}\Bigg] + \left(1+\frac{\lambda}{\alpha}\right)\sum_{i=1}^{n}\delta(x_i)\Big[1+\frac{b\,x_i^a\,\varepsilon(x_i)}{1+W(\psi_i)} - \frac{b\,x_i^a}{1+W(\psi_i)}\Big],\\
J_{ab} = J_{ba} &= -\sum_{i=1}^{n} x_i^a\,\log(x_i) + \sum_{i=1}^{n}\frac{\varphi(x_i)}{1+W(\psi_i)}\Big[1+\frac{2\,b\,x_i^a\,\varepsilon(x_i)}{1+W(\psi_i)} - \frac{b\,x_i^a}{1+W(\psi_i)}\Big]\\
&+ \left(1+\frac{\lambda}{\alpha}\right)\sum_{i=1}^{n}\varphi(x_i)\Big[1+\frac{b\,x_i^a\,\varepsilon(x_i)}{1+W(\psi_i)} - \frac{b\,x_i^a}{1+W(\psi_i)}\Big],\\
J_{a\alpha} = J_{\alpha a} &= -\frac{\lambda\,b}{\alpha^2}\sum_{i=1}^{n}\varphi(x_i) + \frac{b\,(\alpha-1)}{\alpha}\sum_{i=1}^{n}\frac{\varphi(x_i)}{[1+W(\psi_i)]^2}\Big\{2\,[1+W(\psi_i)]^{-3}-1\Big\}\\
&+\Big(\frac{\alpha+\lambda+b(\alpha-1)}{\alpha}\Big)\sum_{i=1}^{n}\frac{\varphi(x_i)}{1+W(\psi_i)}[\varepsilon(x_i)-1],\\
J_{a\lambda} = J_{\lambda a} &= \frac{b}{\alpha}\sum_{i=1}^{n}\varphi(x_i),
\end{aligned}$$

$$J_{bb} = -\frac{n}{b^2} + \sum_{i=1}^{n}\frac{(x_i^a)^2\,\varepsilon(x_i)}{[1+W(\psi_i)]^2}\Big[2\,\varepsilon(x_i)-1\Big] + \left(1+\frac{\lambda}{\alpha}\right)\sum_{i=1}^{n}\frac{(x_i^a)^2\,\varepsilon(x_i)}{1+W(\psi_i)}\Big[\varepsilon(x_i)-1\Big],$$

$$\begin{aligned}
J_{b\alpha} = J_{\alpha b} &= -\frac{\lambda}{\alpha^2}\sum_{i=1}^{n} x_i^a\,\varepsilon(x_i) - \frac{(\alpha-1)}{\alpha}\sum_{i=1}^{n}\frac{x_i^a\,\varepsilon(x_i)}{[1+W(\psi_i)]^2}\Big[2\,\varepsilon(x_i)-1\Big]\\
&+ \left(1+\frac{\lambda}{\alpha}\right)\frac{(\alpha-1)}{\alpha}\sum_{i=1}^{n}\frac{x_i^a\,\varepsilon(x_i)}{1+W(\psi_i)}\Big[\varepsilon(x_i)-1\Big],\\
J_{b\lambda} = J_{\lambda b} &= \alpha^{-1}\sum_{i=1}^{n} x_i^a\,\varepsilon(x_i),
\end{aligned}$$

$$
\begin{aligned}
J_{\alpha\alpha} = J_{\alpha\alpha} &= \frac{n\,\lambda(\alpha + C_2)\,C_2}{\alpha^3\,(1+C_2)\,[\mathrm{e}^{(\lambda/\alpha)C_2}-1]}\left\{\frac{\lambda\,(\alpha + C_2)\,C_2}{\alpha\,(1+C_2)\,[\mathrm{e}^{(\lambda/\alpha)C_2}-1]} + \frac{\lambda\,(\alpha + C_2)\,C_2}{\alpha\,(1+C_2)}\right.\\
&- \left.\frac{\alpha + 3C_2 + 2C_2^2}{(1+C_2)^2}\right\} - \frac{2\lambda}{\alpha^3}\sum_{i=1}^{n}[W(\psi_i) - (\alpha-1)\varepsilon(x_i)] + \frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{\varepsilon(x_i)}{1+W(\psi_i)}\\
&\times \Big[\frac{(\alpha^2-1)(2\,\varepsilon(x_i)-1)}{1+W(\psi_i)} + 1\Big] + \frac{(\lambda+\alpha)}{\alpha^3}\sum_{i=1}^{n}\varepsilon(x_i)\Big[\frac{(\alpha^2-1)(\varepsilon(x_i)-1)}{1+W(\psi_i)} + 1\Big],\\
J_{\alpha\lambda} = J_{\lambda\alpha} &= \frac{1}{\alpha^2}\sum_{i=1}^{n}W(\psi_i) + \frac{(\alpha-1)}{\alpha^2}\sum_{i=1}^{n}\varepsilon(x_i) + \frac{n\,(\alpha + C_2)\,C_2}{\alpha^3\,(1+C_2)\,[\mathrm{e}^{(\lambda/\alpha)C_2}-1]}\\
&\times \Big[\frac{\lambda\,C_2}{\mathrm{e}^{(\lambda/\alpha)C_2}-1} + \alpha - \lambda\,C_2\Big],\\
J_{\lambda\lambda} &= -\frac{n}{\lambda^2} + \frac{n\,C_2^2\,\mathrm{e}^{(\lambda/\alpha)C_2}}{\alpha^2[\mathrm{e}^{(\lambda/\alpha)C_2}-1]^2},
\end{aligned}
$$

where $\varepsilon(x_i)$ is defined in Section 3.7, $\delta(x_i) = b\,x_i^a\,\log(x_i^2)\,\varepsilon(x_i)$ and $\varphi(x_i) = x_i^a\log(x_i)\,\varepsilon(x_i)$.

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CHAPTER 4

The G-Negative Binomial Family: General Properties and Applications

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Resumo

Propomos uma nova classe de distribuições com dois parâmetros positivos adicionais. Alguns casos especiais são apresentados. Obtivemos algumas propriedades matemáticas desta classe, incluíndo expressões explícitas para a função quantil, momentos ordinários e incompletos, função geradora, desvios médios, dois tipos de entropia, confiabilidade, estatísticas de ordem e seus momentos. Discutimos estimativa dos parâmetros do modelo por máxima verossimilhança e fornecemos uma aplicação para um conjunto de dados reais.

Palavras-chave: Distribuição binomial negativa; desvios médios; estimação; momentos.

Abstract

We propose a new class of distributions with two extra positive parameters. Some special cases are presented. We derive some mathematical properties of this class including explicit expressions for the quantile function, ordinary and incomplete moments, generating function, mean deviations, two types of entropy, reliability, order statistics and their moments. We discuss estimation of the model parameters by maximum likelihood and provide an application to a real data set.

Keywords: Estimation; Mean Deviation; Moment; Negative Binomial Distribution.

4.1 Introduction

Providing a wider family of continuous models is always precious for statisticians. The negative binomial distribution has been widely used in mixing procedures of distributions. Several new models have been proposed and applied in survival analysis. Zamani and Ismail [11] introduced the negative binomial-Lindley (NB-L) distribution to model claim counts, one of the most important topics in actuarial theory and practice. Hajebi et al. [16] investigated the exponential negative binomial (ENB) distribution for modeling failure times of a system. Ortega et al. [7] introduced a regression model to predict cure of prostate cancer based on the negative binomial-beta Weibull (NBBW) distribution. Rodrigues et al. [5] pionnered a composition of the truncated negative binomial and Weibull distributions yielding a very popular model to analyze survival data, the so-called Weibull negative binomial (WNB) distribution.

We propose a general family of continuous distributions called the *G-negative binomial* (G-NB) family. It includes as a special case the WNB model. We demonstrate that the G-NB density family is a linear combination of exponentiated-G ("exp-G" for short) density functions.

Let $W_1, \ldots, W_Z$ be a random sample from a random variable having density function $g(\cdot)$, where Z is an unknown positive integer number. We assume that the random variable Z has a zero truncated negative binomial (ZTNB) probability mass function (pmf) with parameters $s > 0$ and $\beta \in (0, 1)$ given by

$$P(z; s, \beta) = \beta^z \binom{s+z-1}{z} [(1-\beta)^{-s} - 1]^{-1}, \quad z \in \mathbb{N}.$$

Here, Z and W are considered to be independent random variables. Let $X = \text{Min}(W_1, \ldots, W_Z)$. Then, the conditional cumulative distribution function (cdf) of X given Z is

$$\begin{aligned} F(x|z) &= 1 - P(X \geq x|z) = 1 - P^z(W_1 \geq x) \\ &= 1 - [1 - P(W_1 \leq x)]^z = 1 - [1 - G(x)]^z. \end{aligned}$$

The unconditional cdf of X becomes

$$F(x) = \sum_{z=0}^{\infty} \beta^z \binom{s+z-1}{z} \{(1-\beta)^{-s} - 1\}^{-1} \{1 - [1 - G(x)]^z\},$$

for $x > 0$. Here, s and β are shape parameters. After some algebra, the cdf of X reduces to

$$F(x) = \frac{(1-\beta)^{-s} - \{1 - \beta[1 - G(x)]\}^{-s}}{[(1-\beta)^{-s} - 1]}. \tag{4.1}$$

The probability density function (pdf) corresponding to (4.1) is given by

$$f(x) = \frac{s\beta}{[(1-\beta)^{-s} - 1]} \, g(x) \, \{1 - \beta[1 - G(x)]\}^{-s-1}. \tag{4.2}$$

This generalization is obtained by increasing the number of parameters compared to the G model, this increase being the price to pay for adding more flexibility to the generated distribution. A first positive point of the G-NB model is that it includes the G distribution as a sub-model when $s = 1$ and $\beta \to 0$. A second one is that it includes as special cases important lifetime models published in recent years. Hereafter, a random variable X following the family (4.2) is denoted by $X \sim$ G-NB(τ, s, β), where τ is the parameter vector associated with G. The survival function and hazard rate function (hrf) of X are given by

$$S(x) = \frac{\{1-\beta[1-G(x)]\}^{-s}-1}{[(1-\beta)^{-s}-1]}$$

and

$$h(x) = \frac{s\,\beta\,g(x)\,\{1-\beta[1-G(x)]\}^{-s-1}}{\{1-\beta[1-G(x)]\}^{-s}-1}, \tag{4.3}$$

respectively. The aim of this paper is to derive some mathematical properties of (4.2) which hold for any baseline continuous G. We obtain explicit expressions for the quantile function, ordinary and incomplete moments, moment generating function (mgf), mean deviations, Bonferroni and Lorenz curves, Shannon entropy, Rényi entropy, reliability and moments of the order statistics.

The G-NB family is well-motivated for industrial applications and biological studies. As a first example, consider that the number, say N, of carcinogenic cells for an individual left active after the initial treatment follows a ZTNB distribution and let X_i be the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i \geq 1$. If $\{X_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (iid) random variables independent of N following the G distribution, then the time to relapse of cancer of a susceptible individual can be modeled by the G-NB family of distributions. Another example considers that the failure of a device occurs due to the presence of an unknown number N of initial defects of the same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by X_i the time to the failure of the device due to the ith defect, for $i \geq 1$. If we assume that the X_i's are iid random variables independent of N, which follows a G distribution, then the time to the first failure is appropriately modeled by the G-NB family. For reliability studies, the random variable $X = \text{Min}\{X_i\}_{i=1}^{N}$ can be used in serial systems with identical components, which appear in many industrial applications and biological organisms.

The rest of the paper is organized as follows. In Section 4.2, we define some new distributions in the G-NB family. A range of mathematical properties of (4.2) is derived in Sections 4.3 to 4.10. The estimation of the model parameters performed by the method of maximum likelihood is presented in Section 4.11. An application to real data is addressed in Section 4.12. Finally, some conclusions are addressed in Section 4.13.

4.2 Special G-NB distributions

The G-NB family of density functions (4.2) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. The new family extends several widely-known distributions in the literature. So, we present some of its special cases. The density function (4.2) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

4.2.1 Normal-negative binomial (NNB) distribution

The NNB distribution is defined from (4.2) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the normal $N(\mu, \sigma^2)$ distribution. Its density function is

$$\begin{aligned} f_{\mathcal{NNB}}(x) &= \frac{s\,\beta\,[(1-\beta)^{-s}-1]^{-1}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \\ &\times \left\{1-\beta\left[\frac{1}{2}-\frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]\right\}^{-s-1}, \end{aligned} \tag{4.4}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, $s > 0$, $\beta \in (0, 1)$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. A random variable with density (4.4) is denoted by $X \sim \mathrm{NNB}(\mu, \sigma^2, s, \beta)$. For $\mu = 0$ and $\sigma = 1$, we obtain the standard NNB distribution. Further, this distribution with $s = 1$ and $\beta \to 0$ tends to the normal distribution. Plots of the NNB density function for selected parameter values are displayed in Figure 4.1.

4.2.2 Gumbel-negative binomial (GuNB) distribution

Consider the Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, where the pdf and cdf (for $x \in \mathbb{R}$) are

$$g(x) = \frac{1}{\sigma}\exp\left\{\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right\}$$

and

$$G(x) = 1 - \exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\},$$

respectively. The mean and variance are equal to $\mu - \gamma\sigma$ and $\pi^2\sigma^2/6$, respectively, where γ is the Euler's constant ($\gamma \approx 0.57722$). Inserting these expressions into (4.2) gives the GuNB density function

$$\begin{aligned} f_{\mathcal{GuNB}}(x) &= \frac{s\,\beta}{\sigma[(1-\beta)^{-s}-1]} \exp\left\{\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right\} \\ &\times \left\{1-\beta\left[\exp\left\{-\exp\left(\frac{\mu-x}{\sigma}\right)\right\}\right]\right\}^{-s-1}, \end{aligned} \tag{4.5}$$

where $x, \mu \in \mathbb{R}$, $s, \sigma > 0$ and $\beta \in (0, 1)$. The Gumbel distribution corresponds to $s = 1$ and $\beta \to 0$. Plots of the density function (4.5) for selected parameter values are displayed in Figure 4.2.

4.2.3 Log-normal-negative binomial (LNNB) distribution

Let $G(x)$ be the log-normal distribution with cdf

$$G(x) = 1 - \Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)$$

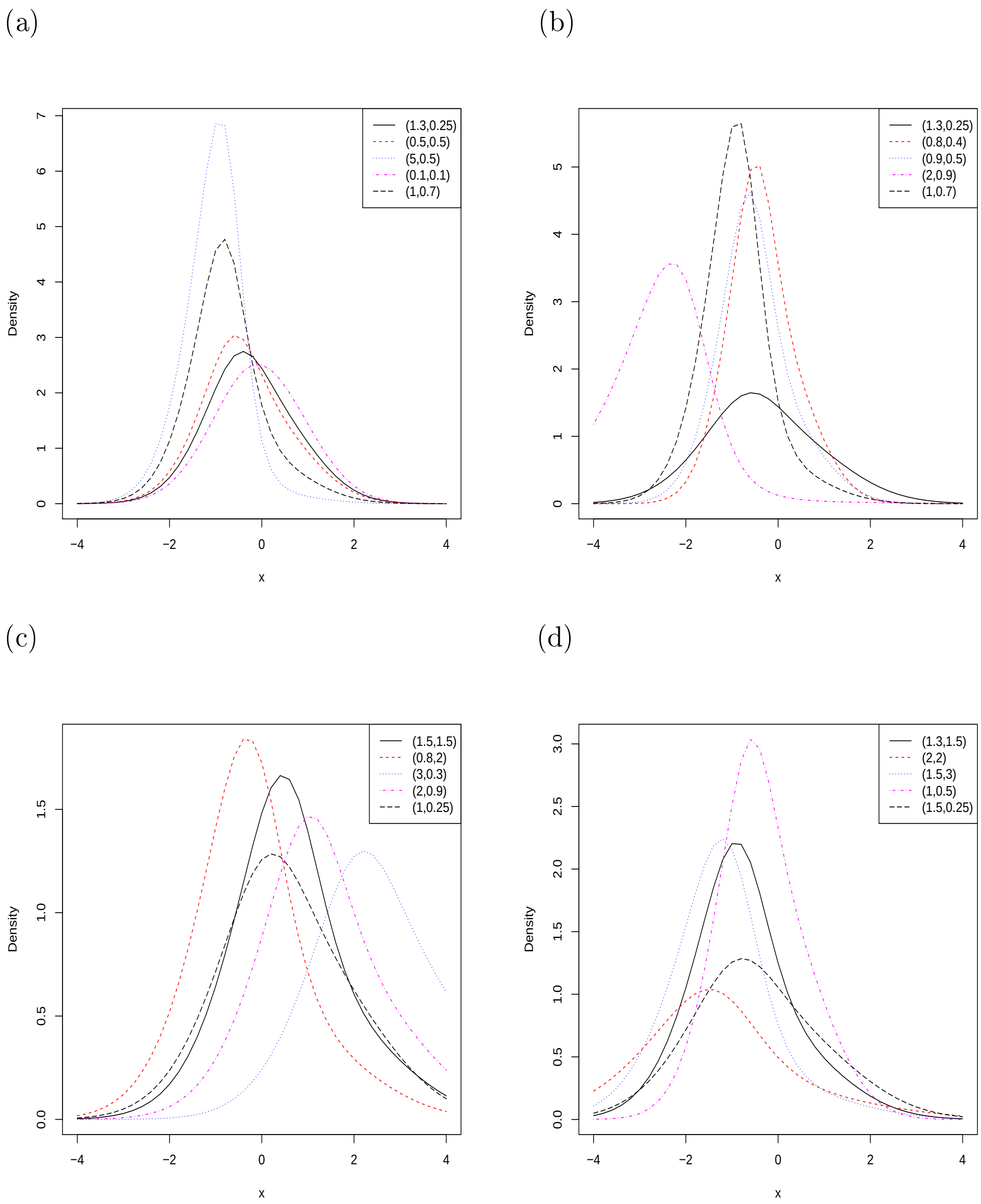


Figure 4.1: The NNB density function for some parameter values: (a) $\mu = 0$ and $\sigma = 1$; (b) $\mu = 0$ and $s = 1.5$; (c) $\sigma = 1$ and $\beta = 0.5$; (d) $\mu = 0$ and $\beta = 0.5$.

for $x > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$. The LNNB density function (for $x > 0$) is given by

$$\begin{aligned} f_{\mathcal{LNNB}}(x) &= \frac{s\,\beta\,[(1-\beta)^{-s}-1]^{-1}}{\sqrt{2\pi}\,\sigma\,x} \exp\left\{-\frac{1}{2}\left[\frac{\log(x)-\mu}{\sigma}\right]^2\right\} \\ &\times \left\{1-\beta\left[\Phi\left(\frac{-\log(x)+\mu}{\sigma}\right)\right]\right\}^{-s-1}. \end{aligned} \tag{4.6}$$

For $s = 1$ and $\beta \to 0$, we obtain the log-normal distribution. Figure 4.3 displays some plots of the LNNB density function for some parameter values.

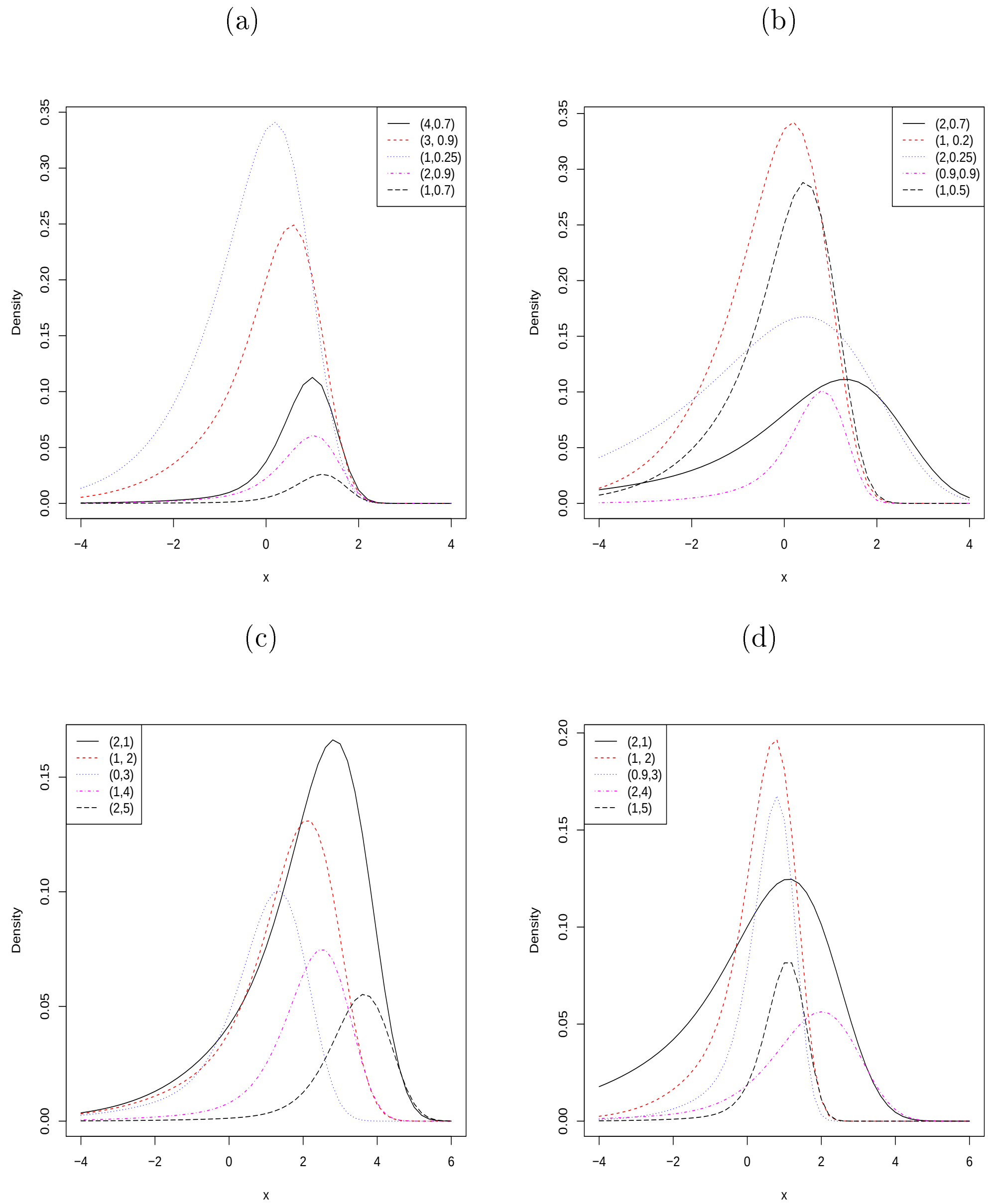


Figure 4.2: The GuNB density function for some parameter values: (a) $\mu = 0$ and $\sigma = 1$; (b) $\mu = 0$ and $s = 1.5$; (c) $\sigma = 1.5$ and $\beta = 0.7$; (d) $\mu = 0$ and $\beta = 0.7$.

4.2.4 Gamma-negative binomial (GaNB) distribution

The gamma cumulative distribution (for $x > 0$) with shape parameter $a > 0$ and scale parameter $b > 0$ is

$$G(x) = \frac{\gamma(a, bx)}{\Gamma(a)}, \tag{4.7}$$

where $\Gamma(a) = \int_0^\infty w^{a-1} \mathrm{e}^{-w} dw$ and $\gamma(a, x) = \int_0^x w^{a-1} \mathrm{e}^{-w} dw$ are the gamma and incomplete gamma functions, respectively. The density function of a random variable X having the GaNB

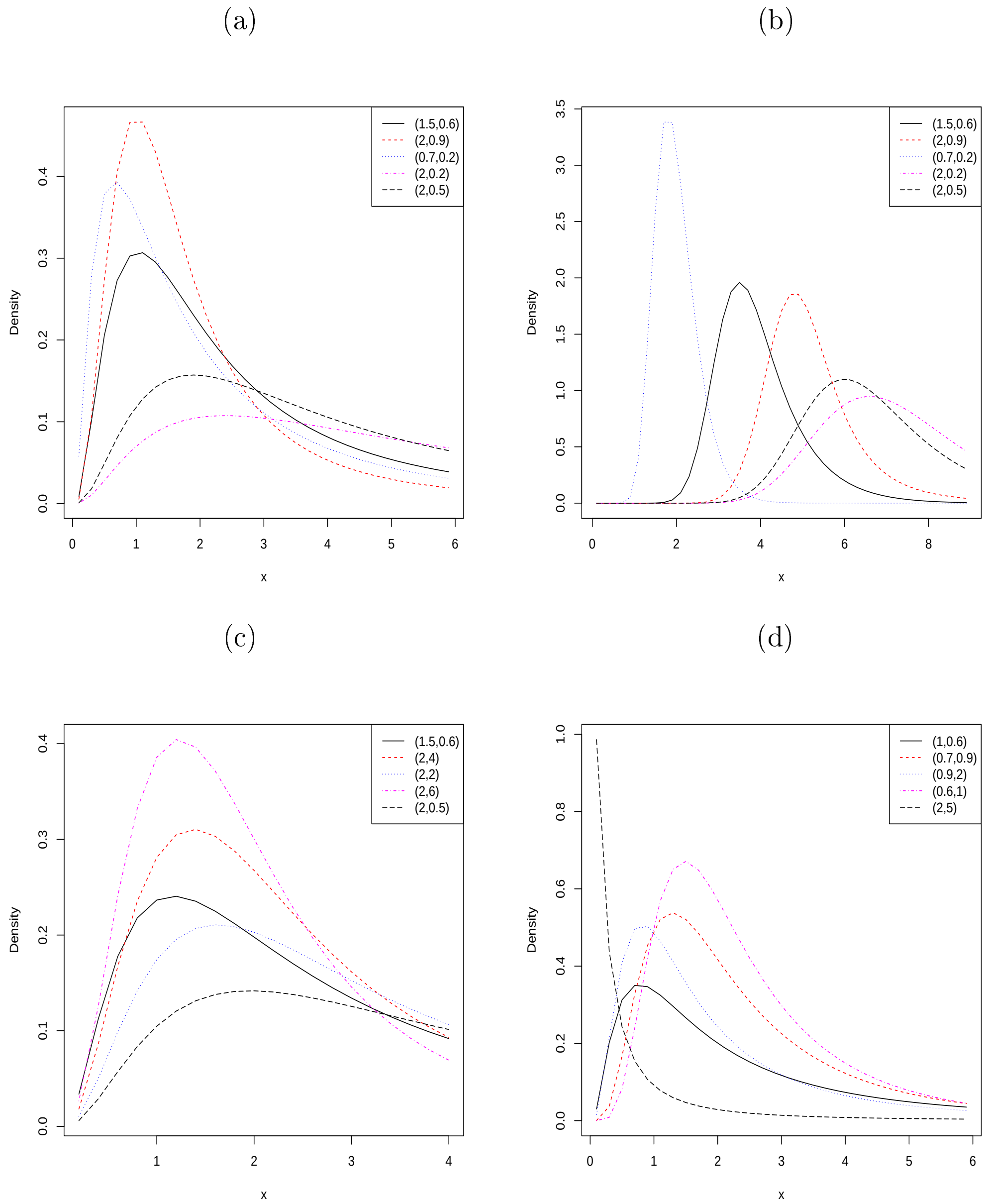


Figure 4.3: The LNNB density function for some parameter values: (a) $s = 1.5$ and $\sigma = 1$; (b) $s = 1.5$ and $\sigma = 0.25$; (c) $\sigma = 1$ and $\beta = 0.6$; (d) $\mu = 1$ and $\beta = 0.5$.

distribution can be expressed as

$$f_{\mathcal{G}a\mathcal{NB}}(x) \quad = \quad \frac{s\,\beta\,b^a\,x^{a-1}\,\mathrm{e}^{-bx}}{[(1-\beta)^{-s}-1]\,\Gamma(a)}\left\{1-\beta\Big[1-\frac{\gamma(a,bx)}{\Gamma(a)}\Big]\right\}^{-s-1}. \tag{4.8}$$

Some plots of the GaNB density function are displayed in Figure 4.4.

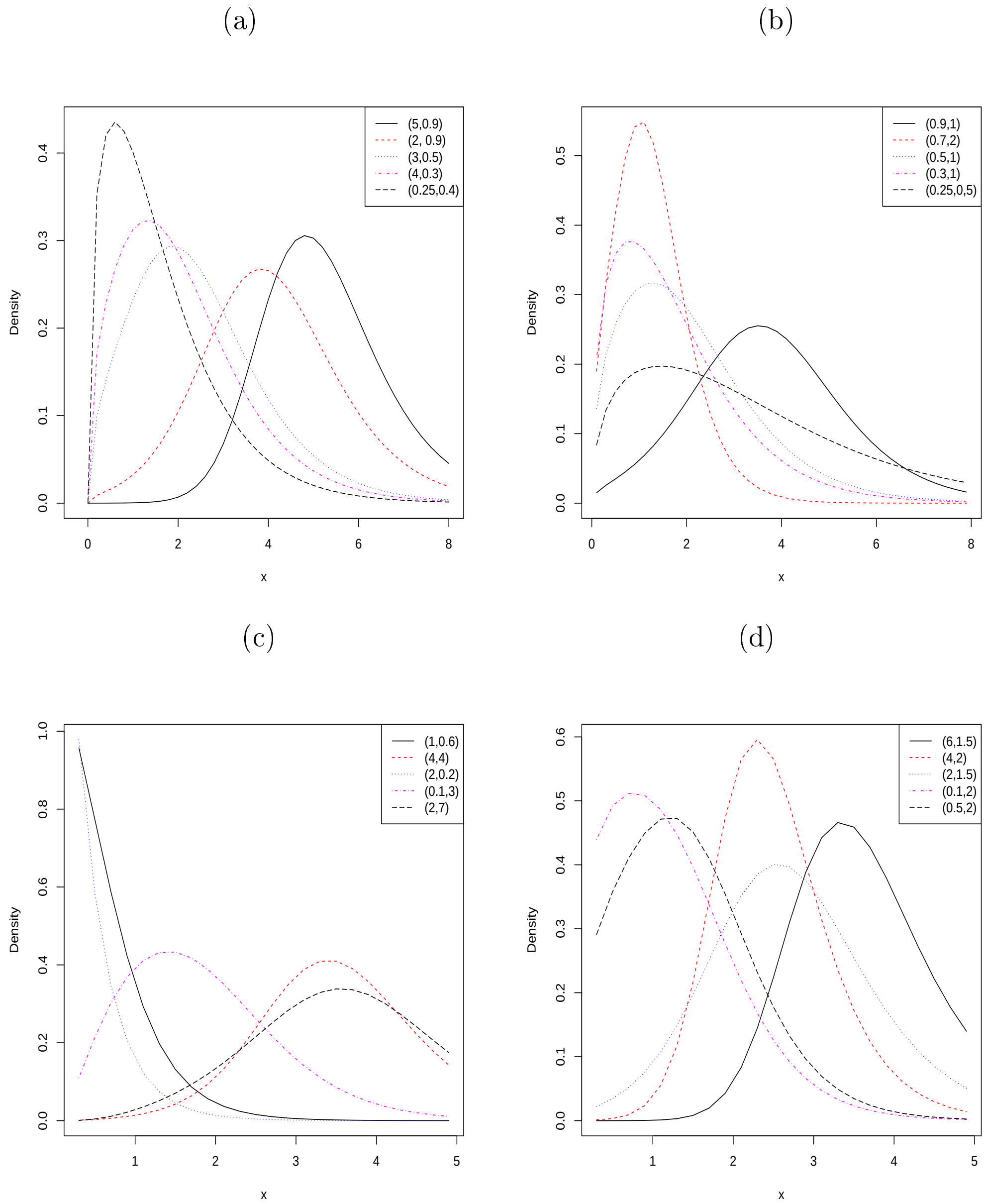


Figure 4.4: The GaNB density function for some parameter values: (a) $a = 1.5$ and $b = 1$; (b) $a = 1.5$ and $s = 1.5$; (c) $b = 2$ and $\beta = 0.7$; (d) $a = 1.5$ and $\beta = 0.9$.

4.2.5 Log-logistic negative binomial (LLNB) distribution

The pdf and cdf of the log-logistic (LL) distribution are (for $x, \alpha, \gamma > 0$)

$$g(x) = \frac{\gamma}{\alpha^\gamma}\, x^{\gamma-1}\left[1 + \left(\frac{x}{\alpha}\right)^\gamma\right]^{-2} \text{and } \ G(x) = 1 - \left[1 + \left(\frac{x}{\alpha}\right)^\gamma\right]^{-1}.$$

Inserting these expressions into (4.2) gives the LLNB density function (for $x > 0$)

$$f_{\mathcal{LLNB}}(x) = \frac{s\,\beta\,\gamma\,x^{\gamma-1}}{\alpha^\gamma\,[(1-\beta)^{-s} - 1]}\left[1 + \left(\frac{x}{\alpha}\right)^\gamma\right]^{-2}\left\{1 - \beta\left[1 + \left(\frac{x}{\alpha}\right)^\gamma\right]^{-1}\right\}^{-s-1}.$$

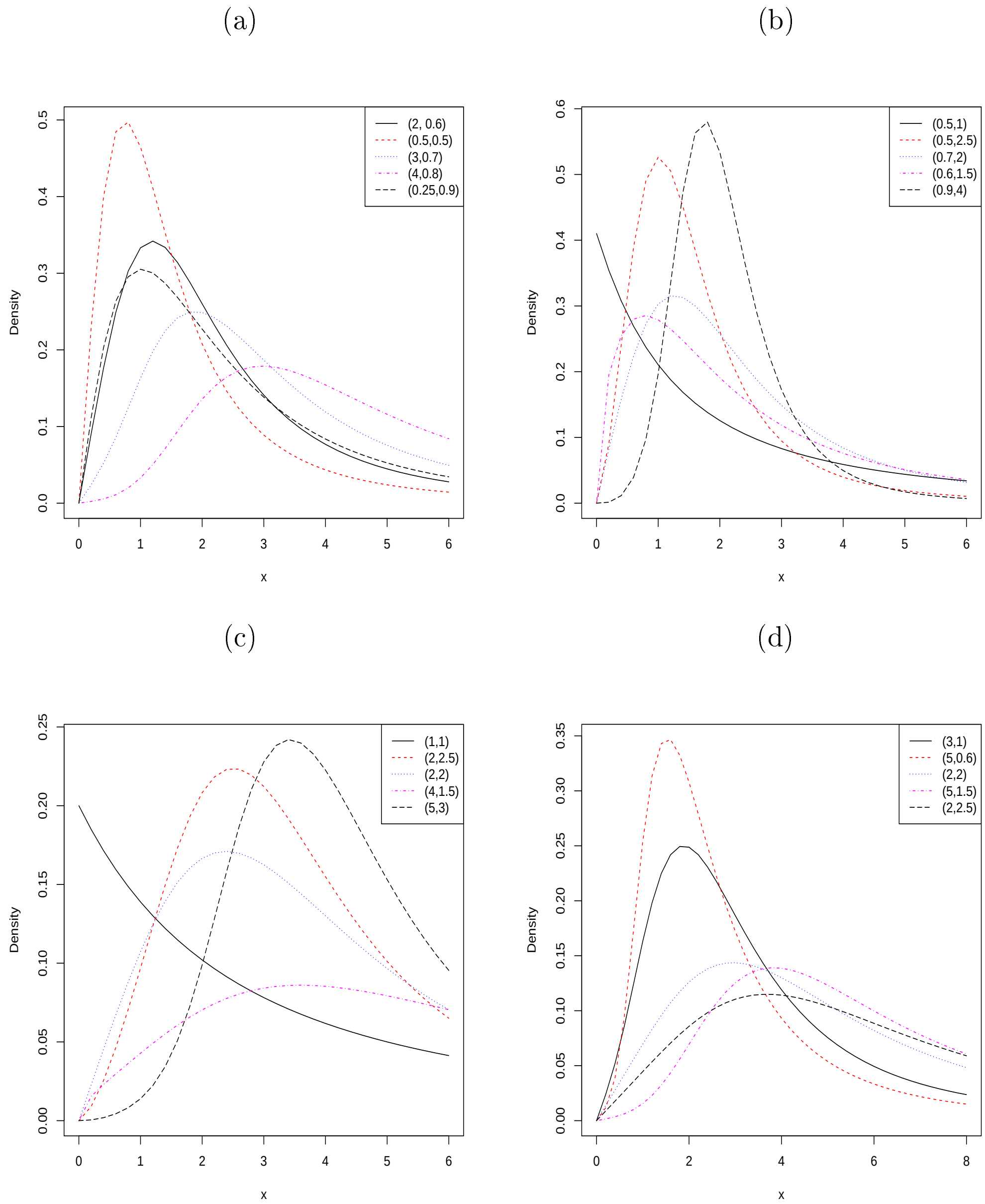


Figure 4.5: The LLNB density function for some parameter values: (a) $\alpha = 1$ and $\gamma = 1.5$; (b) $\alpha = 1.5$ and $s = 1.5$; (c) $\alpha = 1$ and $\beta = 0.5$; (d) $\gamma = 2$ and $\beta = 0.7$.

The LL distribution is obtained for $s = 1$ and $\beta \to 0$. Plots of the LLNB density function for selected parameter values are displayed in Figure 4.5.

4.3 Useful representations

Some useful expansions for (4.1) and (4.2) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exp-G distribution with parameter $j > 0$, say $Y_j \sim$ exp-G$(j+1)$, if its pdf and cdf are

$$h_{j+1}(y) = (j+1)\, G(y)^j\, g(y) \tag{4.9}$$

and

$$H_{j+1}(y) = G(y)^{j+1}, \tag{4.10}$$

respectively. The properties of exponentiated distributions have been studied by several authors in recent years. See Mudholkar and Srivastava (1993) for exponentiated Weibull, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah and Kotz (2006) for exponentiated Fréchet and Nadarajah and Gupta (2007) for exponentiated gamma distributions.

For any real a and $|z| < 1$, we have the power series

$$(1-z)^{-a} = \sum_{k=0}^{\infty} (a)_k \frac{z^k}{k!}, \tag{4.11}$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2)\ldots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. Using expansion (4.11), we can write (4.2) as

$$f(x) = \frac{s\,\beta}{[(1-\beta)^{-s}-1]}\, g(x) \sum_{k=0}^{\infty} \frac{(s+1)_k}{k!}\, \beta^k\, [1-G(x)]^k. \tag{4.12}$$

Expanding the binomial term in Equation (4.2), we can express $f(x)$ as

$$f(x) = \sum_{j=0}^{\infty} \omega_j\, h_{j+1}(x), \tag{4.13}$$

where $h_{j+1}(x)$ denotes the exp-G$(j+1)$ density function and

$$\omega_j = \frac{(-1)^j\, s}{(j+1)[(1-\beta)^{-s}-1]} \sum_{k=j}^{\infty} \frac{(s+1)_k\, \beta^{k+1}}{k!} \binom{k}{j}.$$

We can verify that $\sum_{j=0}^{\infty} \omega_j = 1$. By integrating (4.13), we can express $F(x)$ as

$$F(x) = \sum_{j=0}^{\infty} \omega_j H_{j+1}(x), \tag{4.14}$$

where $H_{j+1}(x)$ denotes the exp-G$(j+1)$ cumulative distribution. So, several mathematical properties of the G-NB family can be obtained by knowing those of the exp-G distribution, see, for example, Mudholkar et al. [9], Nadarajah and Kotz [17], among others.

4.4 Quantile function

Inverting $F(x) = u$ in (4.1), the quantile function of X, say $Q(u)$, for $0 < u < 1$, follows as

$$
\begin{aligned}
Q(u) &= G^{-1}\left\{1 - \frac{1}{\beta}\left[1 - \left([1-\beta]^{-s} - u[(1-\beta)^{-s} - 1]\right)^{-\frac{1}{s}}\right]\right\} \\
&= G^{-1}\left\{1 - \frac{1}{\beta}\left[1 - (1-\beta)(1 - u[1-(1-\beta)^{s}])^{-\frac{1}{s}}\right]\right\}. \qquad (4.15)
\end{aligned}
$$

Using Equation (4.11) in the last equality, we obtain

$$
\begin{aligned}
Q(u) &= G^{-1}\left\{1 - \frac{1}{\beta}\left[1 - (1-\beta)\sum_{j=0}^{\infty}\left(\frac{1}{s}\right)_j \frac{u^j\,[1-(1-\beta)^s]^j}{j!}\right]\right\} \\
&= G^{-1}\left\{\frac{(\beta-1)}{\beta}\left[1 - \sum_{j=0}^{\infty}\left(\frac{1}{s}\right)_j \frac{u^j\,[1-(1-\beta)^s]^j}{j!}\right]\right\} \\
&= G^{-1}\left\{\frac{(\beta-1)}{\beta}\sum_{j=0}^{\infty} a_j\, u^j\right\},
\end{aligned}
$$

where $a_0 = 0$ and $a_j = \left(\frac{1}{s}\right)_j \frac{[1-(1-\beta)^s]^j}{j!}$, $j \geq 1$.

Quantiles of interest can be obtained from (4.15) by substituting appropriate values for u. In particular, the median of X comes when $u = 0.5$.

The motivation for using quantile-based measures is because of the non-existence of classical kurtosis for many generalized distributions. The Bowley's skewness is based on quartiles (Kenney and Keeping, 1962):

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors' kurtosis (Moors, 1984) is based on octiles:

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Plots of the skewness and kurtosis for the GuNB distribution, for some choices of β, σ and μ as function of s, and for some choices of s, σ and μ as function of β are displayed in Figure 4.6. The plots indicate that there is a great flexibility of the skewness and kurtosis curves of this distribution.

4.5 Moments

A first formula for the nth moment of X, say $\mu'_n = E(X^n)$, can be obtained from (4.13) and $Y_j \sim$ exp-G$(j+1)$ as

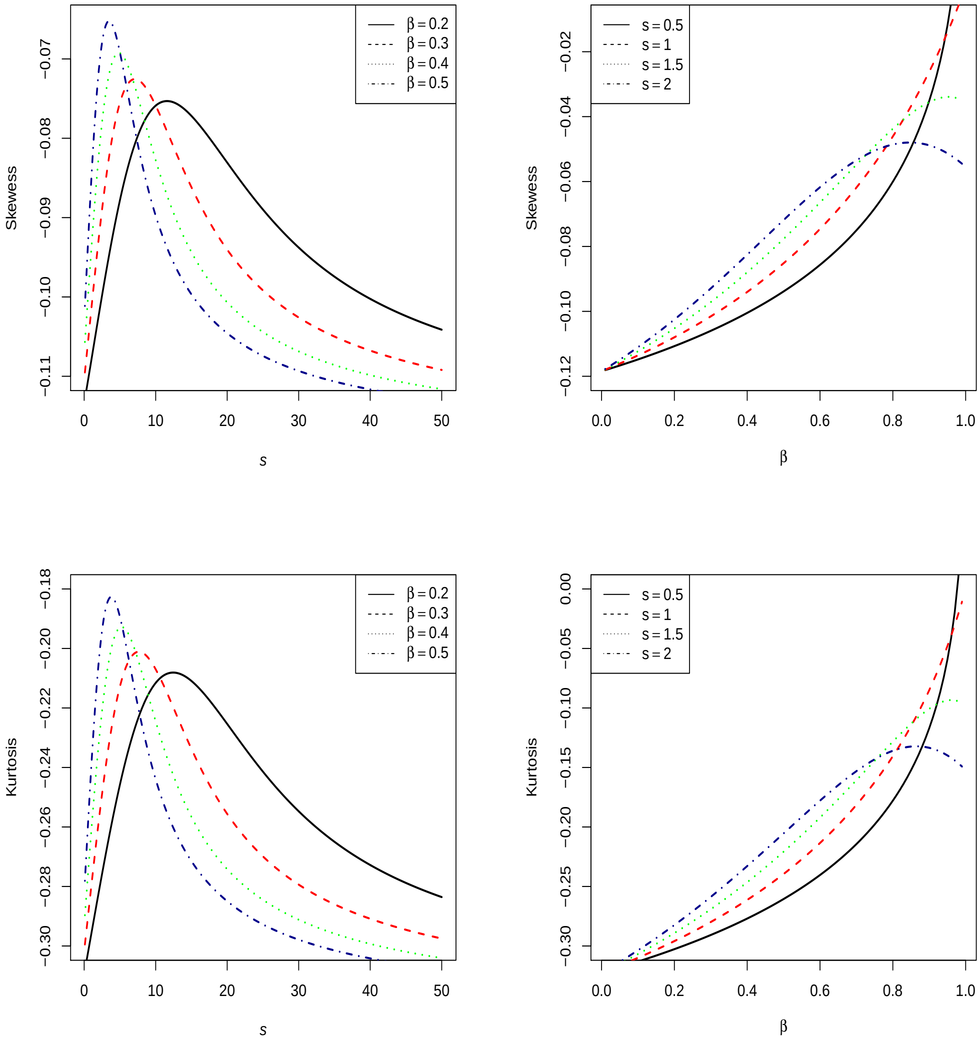


Figure 4.6: Plots of the GuNB skewness and kurtosis as functions of s for selected values of β and as functions of β for selected values of s. Here, $\mu = 0$ and $\sigma = 1$.

$$\mu'_n = \sum_{j=0}^{\infty} \omega_j \, E(Y_j^n). \tag{4.16}$$

Expressions for moments of several exponentiated distributions are given by Nadarajah and Kotz [17], which can be used to obtain $E(X^n)$.

A second formula for μ'_n can be derived from (4.16) in terms of the baseline quantile function $Q_G(x) = G^{-1}(x)$. We obtain

$$\mu'_n = \sum_{j=0}^{\infty} (j+1) \, \omega_j \, \tau_{n,j}, \tag{4.17}$$

where $\tau_{n,j}$ is given by

$$\tau_{n,j} = \int_{-\infty}^{\infty} x^n\, G(x)^j\, g(x)dx = \int_0^1 Q_G(u)^n\, u^j\, du. \tag{4.18}$$

The ordinary moments of several G-NB distributions can be determined directly from equations (4.17) and (4.18). Here, we give three examples. For the standard logistic-negative binomial (LoNB) distribution, where $G(x) = (1+\mathrm{e}^{-x})^{-1}$, using a result from Prudnikov et al. (1986, Section 2.6.13, equation 4), we have (for $t<1$)

$$\mu_n' = \sum_{j=0}^{\infty}(j+1)\,\omega_j\, \left(\frac{\partial}{\partial t}\right)^n\, B(t+j+1, 1-t)\Big|_{t=0},$$

where $B(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1}dt$ is the beta function. The moments of the exponential-negative binomial (ENB) distribution (with parameter $\lambda>0$) are

$$\mu_n' = n!\,\lambda^n \sum_{j,k=0}^{\infty} \frac{(-1)^{n+k}\,(j+1)\,\omega_j}{(k+1)^{n+1}} \binom{j}{k}.$$

For the Pareto-negative binomial (ParNB) distribution, where $G(x) = 1-(1+x)^{-\nu}$, and considering $\nu>1$, we obtain

$$\mu_n' = \sum_{j,k=0}^{\infty} (-1)^{n+k}\,(j+1)\,\omega_j\, B(j, 1-k\nu^{-1}) \binom{n}{k}.$$

For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X can be determined as

$$m_n(y) = E\left(X^n | X<y\right) = \sum_{j=0}^{\infty}(j+1)\,\omega_j \int_0^{G(y)} Q_G(u)^n\, u^j du. \tag{4.19}$$

The last integral can be computed for most baseline G distributions, at least numerically. Equations (4.16) - (4.19) are the main results of this section.

The symbolic computational plataforms Maple, Mathematica and Matlab make it possible to automate the formulae derived in this paper since they have currently the ability to deal with analytic recurrence equations and sums of formidable size and complexity. In practical terms, we can substitute ∞ in the sums by a large number such as 20 or 50 for most practical applications. Establishing scripts for the explicit expressions given throughout the paper can be more accurate computationally than other integral representations which can be prone to rounding off errors among others.

4.6 Generating function

The mgf $M(t) = E(e^{tX})$ of X follows from (4.13) as

$$M(t) = \sum_{j=0}^{\infty} (j+1)\,\omega_j\, M_j(t), \tag{4.20}$$

where $M_j(t)$ is the mgf of Y_j. Hence, $M(t)$ can be immediately determined from the exp-G generating function. Another formula for $M(t)$ can be derived from (4.13) as

$$M(t) = \sum_{j=0}^{\infty} (j+1)\,\omega_j\, \rho_j(t), \tag{4.21}$$

where $\rho_j(t)$ can be determined from $Q_G(u) = G^{-1}(u)$ as

$$\rho_j(t) = \int_{-\infty}^{\infty} e^{tx}\, G(x)^j g(x) dx = \int_0^1 \exp\{t\, Q_G(u)\}\, u^j du. \tag{4.22}$$

We can obtain the mgf's of several G-NB distributions directly from equations (4.21) and (4.22). For example, the mgf's of the LoNB (for $t < 1$), ENB (with parameter λ)(for $t < \lambda^{-1}$) and ParNB (with parameter $\nu > 0$)(for $\nu > 1$) distributions are

$$M(t) = \sum_{j=0}^{\infty} (j+1)\,\omega_j\, B(t+j+1, 1-t),$$

$$M(t) = \sum_{j=0}^{\infty} (j+1)\,\omega_j\, B(j+1, 1-\lambda t),$$

and

$$M(t) = e^{-t} \sum_{j,r=0}^{\infty} \frac{(j+1)\,\omega_j\, B(j+1, 1-r\nu^{-1})}{r!}\, t^r,$$

respectively.

4.7 Mean deviations

The mean deviations about the mean ($\delta_1(X) = E(|X - \mu_1'|)$) and about the median ($\delta_2(X) = E(|X - M|)$) of X can be expressed as

$$\delta_1(X) = 2\mu_1'\, F\left(\mu_1'\right) - 2m_1\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2(X) = \mu_1' - 2m_1(M), \tag{4.23}$$

respectively, where $\mu_1' = E(X)$, $M = Median(X)$ is the median computed from (4.15) with $u = 1/2$, $F(\mu_1')$ is easily calculated from the cdf (4.1) and $m_1(z) = \int_{-\infty}^{z} x f(x) dx$ is the first incomplete moment given by (4.19) with $n = 1$.

In this section, we provide two alternative ways to compute $\delta_1(X)$ and $\delta_2(X)$. A general equation for $m_1(z)$ can be derived from (4.13) as

$$m_1(z) = \sum_{j=0}^{\infty} \omega_j \, R_j(z), \tag{4.24}$$

where

$$R_j(z) = \int_{-\infty}^{z} x \, h_{j+1}(x) dx. \tag{4.25}$$

Equation (4.25) is the basic quantity to compute the mean deviations of the exp-G distribution. Hence, the mean deviations (4.23) depend only on the mean deviations of the exp-G distributions. So, alternative representations for $\delta_1(X)$ and $\delta_2(X)$ are

$$\delta_1(X) = 2\mu_1' F\left(\mu_1'\right) - 2\sum_{j=0}^{\infty} \omega_j \, R_j\left(\mu_1'\right) \quad \text{and} \quad \delta_2(X) = \mu_1' - 2\sum_{j=0}^{\infty} \omega_j \, R_j(M).$$

In a similar manner, the mean deviations of any G-NB distribution can be determined from equation (4.19). Let $T_j(z) = \int_0^{G(z)} Q_G(u) \, u^j du$. For example, the mean deviations of the LoNB, ParNB (with irrational $\nu > 0$) and ENB (with parameter λ) follow, based on the generalized binomial expansion, from the functions

$$T_j(z) = \frac{1}{\Gamma(j)} \sum_{k=0}^{\infty} \frac{(-1)^k \, \Gamma(j+1+k) \; [1 - \exp(-kz)]}{(k+1)!},$$

$$T_j(z) = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^k \, z^{1-r\nu}}{(1-r\nu)} \binom{j+1}{k} \binom{k}{r}$$

and

$$T_j(z) = \lambda^{-1} \sum_{k=0}^{\infty} \frac{(1-j)_{[k]} \; [1 - \exp\left(-k\lambda z\right)]}{(k+1)!},$$

respectively, where $(1-j)_{[k]} = (-1)^k (j-1)(j-2)(j-3) \ldots (j-k)$ is the descending factorial.

Applications of equations (4.24) and (4.25) can be important to obtain Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$ respectively, where $\mu_1' = E(X)$ and

$$q = G^{-1}\left\{1 - \frac{1}{\beta}\left[1 - \left\{(1-\beta)^{-s} - \pi\Big[(1-\beta)^{-s} - 1\Big]\right\}^{-\frac{1}{s}}\right]\right\}$$

is the G-NB quantile function at π (see Section 4.4). For example, the Bonferroni and Lorenz curves for the LLNB distribution (Section 2.5) with parameters $\alpha, \gamma > 0$ are readily calculated from $B(\pi)$ and $L(\pi)$. They are plotted for selected parameter values in Figure 4.7.

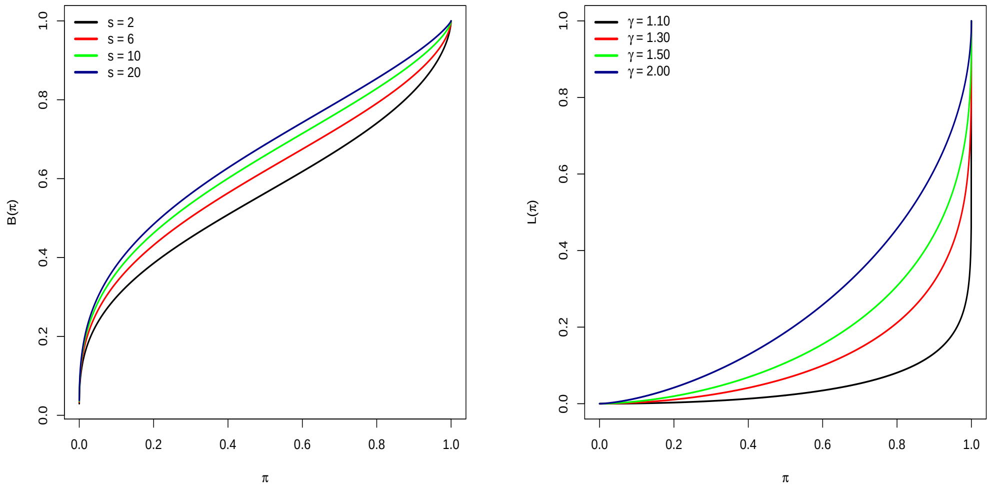


Figure 4.7: Plots of $B(\pi)$ and $L(\pi)$ versus π for the LLNB distribution. Here, $\beta = 0.5$, $\gamma = 3$ and $\alpha = 2$ for $B(\pi)$ and $s = 2$, $\beta = 0.5$ and $\alpha = 1.5$ for $L(\pi)$.

4.8 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies (Shannon, 1951; Rényi, 1961). The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined by (for $\gamma > 0$ and $\gamma \neq 1$)

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_0^\infty f^\gamma(x)dx \right).$$

The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the limit of the Rényi entropy when γ goes to one.

Here, we derive closed-form expressions for the Rényi and Shannon entropies when X is a G-NB random variable. From Equations (4.2) and (4.11), we obtain

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma}\left\{ \log(K) + \log\left[\int_0^\infty g(x)^\gamma \left(1-\beta[1-G(x)]\right)^{-\gamma(s+1)} dx \right] \right\} \\ &= \frac{1}{1-\gamma}\left\{ \log(K) + \log\left[\int_0^\infty \left\{ g(x)^\gamma \sum_{k=0}^\infty (-1)^k \binom{-\gamma(s+1)}{k} \right.\right.\right. \\ &\quad \times \left.\left.\left. \beta^k \left[1-G(x)\right]^k \right\} dx \right] \right\}, \end{aligned}$$

where $K = (s\beta)^\gamma / [(1-\beta)^{-s} - 1]^\gamma$. By expanding the binomial term, we obtain

$$
\begin{aligned}
I_R(\gamma) &= \frac{1}{1-\gamma}\Bigg\{\log(K)+\log\left(\int_0^\infty\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^k\beta^k\binom{k}{j}\binom{-\gamma(s+1)}{k}\right.\right.\\
&\times \left.\left. g(x)^\gamma\, G(x)^j\right]dx\right)\Bigg\}.
\end{aligned}
$$

The above sum converges to $g(x)^\gamma\{1-[1+G(x)]\beta\}^{-\gamma(s+1)}$. Then, the Rényi entropy reduces to

$$
\begin{aligned}
I_R(\gamma) &= \frac{\gamma}{1-\gamma}\left\{\log(s\beta)-\log[(1-\beta)^{-s}-1]\right\}\\
&+ \frac{1}{1-\gamma}\log\left(\int_0^\infty g(x)^\gamma\left\{1-[1+G(x)]\,\beta\right\}^{-\gamma(s+1)}dx\right).
\end{aligned}
\tag{4.26}
$$

The last integral depends only on the baseline G distribution.

The Shannon entropy can be obtained by limiting $\gamma\uparrow 1$ in (4.26). However, it is easier to derive its expression from the definition. We have

$$
\begin{aligned}
E\{-\log[f(X)]\} &= -\log(s\beta)+\log[(1-\beta)^{-s}-1]-E\{\log[g(X)]\}\\
&\quad -E\{-\log[1-\beta(1-G(X))]^{(s+2)-1}\}.
\end{aligned}
$$

For any real $a>0$, the following formula given by (http:// functions.wolfram.com/ ElementaryFunctions/ Log/ 06/ 01/ 04/ 03/) holds

$$
\{-\log\,[1-G(x)]\}^{a-1}=(a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}J_{j,k}(a)\,G(x)^{a+k-1},
\tag{4.27}
$$

where

$$
J_{j,k}(a)=\frac{(-1)^{j+k}\,p_{j,k}}{(a-1-j)}\binom{k}{j}
$$

and the constants $p_{j,k}$ can be calculated recursively by

$$
p_{j,k}=k^{-1}\sum_{m=1}^{\infty}\frac{(-1)^m\,[m(j+1)-k]}{(m+1)}\,p_{j,k-m}
\tag{4.28}
$$

for $k=1,2,\ldots$ and $p_{j,0}=1$.

Then, using expansion (4.27), we obtain

$$
\begin{aligned}
E\left\{-\log[f(X)]\right\} &= -\log(s\beta)+\log[(1-\beta)^{-s}-1]-E\{\log[g(X)]\}\\
&\quad -\sum_{k,i=0}^{\infty}\sum_{j=0}^{k}q_{k,j,i}\,E\left[G(X)^i\right],
\end{aligned}
\tag{4.29}
$$

where

$$q_{k,j,i} = \frac{(-1)^{j+k}\,(s+1)\,\beta^{s+k+1}\,p_{j,k}}{(s+1-j)}\binom{k}{j}\binom{k-s-1}{k}.$$

The two expectations in (4.29) can be easily evaluated numerically for given $G(\cdot)$ and $g(\cdot)$. Using (4.13), they can also be represented as

$$E\left[G(X)^i\right] = \sum_{j=0}^{\infty}(j+1)\,\omega_j\int_0^{\infty}G(x)^{i+j}\,g(x)dx = \sum_{j=0}^{\infty}\frac{(j+1)\,\omega_j}{(i+j+1)}\,G(x)^{i+j+1},$$

and

$$E\left\{\log[g(X)]\right\} = \sum_{j=0}^{\infty}(j+1)\,\omega_j\int_0^{\infty}\log[g(x)]\,G(x)^j g(x)dx,$$

respectively. The last equation can also be expressed in terms of the baseline quantile function as

$$E\left\{\log[g(X)]\right\} = \sum_{j=0}^{\infty}(j+1)\,\omega_j\int_0^{1}\log\left\{g\left[Q_G(u)\right]\right\}\,u^j du.$$

The last integral can be calculated for most baseline distributions using a power series expansion for $Q_G(u)$.

4.9 Reliability

We derive the reliability, $R = \Pr(X_2 < X_1)$, when $X_1 \sim G_1$-NB and $X_2 \sim G_2$-NB are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Let f_i and F_i denote the pdf and cdf of X_i, respectively. By using the representations (4.13) and (4.14), we can write

$$R = \sum_{n,m=0}^{\infty} p_{nm}\int_0^{\infty}H_{m+1}(x)\,h_{n+1}(x)dx = \sum_{n,m=0}^{\infty} p_{nm}\,R_{nm}, \tag{4.30}$$

where

$$p_{nm} = \frac{(-1)^{n+m}\,s_1 s_2\,v_n(s_1,\beta_1)\,v_m(s_2,\beta_2)}{(n+1)(m+1)[(1-\beta_1)^{-s_1}-1]\,[(1-\beta_2)^{-s_2}-1]},$$

where $v_j(s_i,\beta_i)$ (for $i=1,2$ and $j=n,m$) is given by

$$v_j(s_i,\beta_i) = \sum_{k=j}^{\infty}\frac{\beta_i^{k+1}\,(s_i+1)_k}{k!}\binom{k}{j}$$

and $R_{nm} = \Pr(Y_m < Y_n)$ is the reliability between the independent random variables $Y_n \sim$ exp-G$(n+1)$ and $Y_m \sim$ exp-G$(m+1)$. Hence, the reliability of the G-NB random variables is a linear combination of those for exp-G random variables. For example, we derive the reliability when X_1 and X_2 have independent WNB distributions with the same shape parameter b, namely, WNB(a_1, b, s_1, β_1) and WNB(a_2, b, s_2, β_2). The reliability obtained from Equation (4.30) is

$$\begin{aligned} R &= \sum_{n,m=0}^{\infty} (n+1)\, p_{nm} \int_0^{\infty} g(x; a_1, b)\, G(x; a_1, b)^n\, G(x; a_2, b)^{m+1} dx \\ &= \sum_{n,m=0}^{\infty} (n+1)\, p_{nm} \int_0^{\infty} [a_1 b\, x^{b-1} \mathrm{e}^{-a_1 x^b}]\, [1-\mathrm{e}^{-a_1 x^b}]^n\, [1-\mathrm{e}^{-a_2 x^b}]^{m+1} dx. \end{aligned}$$

By application of the binomial expansion, we obtain

$$\begin{aligned} R &= \sum_{n,m=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{m+1} (-1)^{k+l}\, (n+1)\, a_1 b\, p_{nm} \binom{n}{l} \binom{m+1}{k} \\ &\times \int_0^{\infty} x^{b-1} \mathrm{e}^{-[a_1(1+l)+a_2 k]\, x^b} dx. \end{aligned} \tag{4.31}$$

Calculating the last integral, we can write the reliability $R = Pr(X_2 < X_1)$ as

$$R = \sum_{n,m=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{m+1} \frac{(-1)^{k+l}\, b\, a_1\, (n+1)\, p_{nm}}{a_1\, b(1+l) + a_2\, b\, k} \binom{n}{l} \binom{m+1}{k}.$$

4.10 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the G-NB distribution. Let $X_{i:n}$ denote the ith order statistic. From equations (4.13) and (4.14), the pdf of $X_{i:n}$ is

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{r=0}^{\infty} \omega_r\, (r+1)\, G(x)^r\, g(x)\right] \\ &\times \left[\sum_{k=0}^{\infty} \omega_k\, G(x)^{k+1}\right]^{j+i-1}. \end{aligned}$$

Here, we use an equation by Gradshteyn and Ryzhik (2000, Section 0.314) for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{4.32}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m\,(n+1) - i]\, a_m\, c_{n,i-m}, \tag{4.33}$$

and $c_{n,0} = a_0^n$. The coefficient $c_{n,i}$ can be determined from $c_{n,0}, \ldots, c_{n,i-1}$ and then from the quantities $a_0, \ldots, a_i$. In fact, $c_{n,i}$ can be given explicitly in terms of the coefficients $a_i's$, although it is not necessary for programming numerically the expansions in any algebraic or numerical software.

Using Equations (4.32) and (4.33), we can write

$$\left[G(x)\sum_{k=0}^{\infty}\omega_k\,G(x)^k\right]^{j+i-1} = \sum_{k=0}^{\infty}\gamma_{j+i-1,k}\;G(x)^{k+j+i-1},$$

where $\gamma_{j+i-1,0} = \omega_0^{j+i-1}$ and $\gamma_{j+i-1,k} = (k\,\omega_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]\,\omega_m\,\gamma_{j+i-1,k-m}$.

Hence,

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty}\sum_{j=0}^{n-i} m_{r,k,j}\,h_{r+k+j+i}(x), \tag{4.34}$$

where

$$m_{r,k,j} = \frac{(-1)^j\,i\,(r+1)\,\omega_r\,\gamma_{j+i-1,k}}{(r+k+j+i)}\binom{n}{i}\binom{n-i}{j}.$$

Equation (4.34) reveals that the pdf of the G-NB order statistics is a triple linear combination of exp-G density functions. So, several mathematical quantities of the G-NB order statistics such as the ordinary, incomplete and factorial moments, mgf and mean deviations can be obtained from those quantities of the exp-G distributions. Clearly, the cdf of $X_{i:n}$ can be expressed as

$$F_{i:n}(x) = \sum_{r,k=0}^{\infty}\sum_{j=0}^{n-i} m_{r,k,j}\,H_{r+k+j+i}(x).$$

For example, from equation (5.27), the moments of the G-NB order statistics can be expressed directly in terms of the exp-G moments as

$$E(X_{i:n}^s) = \sum_{r,k=0}^{\infty}\sum_{j=0}^{n-i} m_{r,k,j}\int_0^{\infty} x^s\,h_{r+k+j+i}(x)dx. \tag{4.35}$$

Using equation (4.35), the moments of the WNB order statistics can be written directly in terms of the exp-Weibull moments (with parameters $r+k+j+i>0$, $a>0$ and $b>0$) as

$$E(X_{i:n}^s) = a^{-\frac{s}{b}}\sum_{r,k=0}^{\infty}\sum_{j=0}^{n-i} p_{r,k,j}\,\Gamma\left(\frac{b+s}{b}\right), \tag{4.36}$$

where

$$\begin{aligned} p_{r,k,j} &= (-1)^j\,i\,(r+1)\,\omega_r\,\gamma_{j+i-1,k}\binom{n}{i}\binom{n-i}{j} \\ &\times \sum_{t=0}^{r+k+j+i-1}(-1)^t\binom{r+k+j+i-1}{t}(1+t)^{-\frac{b+s}{b}}. \end{aligned}$$

Alternatively, we obtain another expression for these moments using a result due to Barakat and Abdelkader [12] applied to the independent and identically distributed (i.i.d.) case, subject to existence,

$$E(X_{i:n}^s) = s \sum_{t=n-i+1}^{n} (-1)^{t-n+i-1} \binom{t-1}{n-i} \binom{n}{t} I_t(s), \tag{4.37}$$

where $I_t(s)$ denotes the integral

$$I_t(s) = \int_{-\infty}^{\infty} x^{s-1} [1 - F(x)]^t dx$$

Using the binomial expansion and interchanging terms, the lest integral becomes

$$\begin{aligned} I_t(s) &= \sum_{m=0}^{t} (-1)^m \binom{t}{m} \int_{-\infty}^{\infty} x^{s-1} F(x)^m dx \\ &= \sum_{m=0}^{t} (-1)^m \binom{t}{m} \int_{-\infty}^{\infty} x^{s-1} \left[\sum_{j=0}^{\infty} \omega_j H_{j+1}(x) \right]^m dx \end{aligned}$$

Using Equation (4.32), we obtain

$$\begin{aligned} I_t(s) &= \sum_{m=0}^{t} (-1)^m \binom{t}{m} \int_0^{\infty} x^{s-1} \sum_{j=0}^{\infty} c_{m,j} \, H_{j+1}(x) dx \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^{t} (-1)^m \binom{t}{m} c_{m,j} \int_0^{\infty} x^{s-1} G(x)^{j+1} dx, \end{aligned}$$

where $c_{m,0} = \omega_0^m$ and $c_{m,j} = (j\,\omega_0)^{-1} \sum_{n=1}^{j} [n\,(m+1) - j]\,\omega_n\, c_{m,j-n}$.

Inserting the expression for $I_t(s)$ in equation (4.37) yields

$$\begin{aligned} E(X_{i:n}^s) &= s \sum_{t=n-i+1}^{n} \sum_{m=0}^{t} \sum_{j=0}^{\infty} c_{m,j} \, (-1)^{t-n+i+m-1} \binom{t-1}{n-i} \binom{n}{t} \binom{t}{m} \\ &\times \int_0^{\infty} x^{s-1} \, G(x)^{j+1} dx. \end{aligned}$$

The last integral can be computed for most baseline distributions.

4.11 Estimation

We calculate the maximum likelihood estimates (MLEs) of the parameters of the G-NB distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the G-NB(s,β,$\boldsymbol{\tau}$) distribution, where $\boldsymbol{\tau}$ is a $p \times 1$ vector of unknown parameters in the baseline distribution $G(x;\boldsymbol{\tau})$. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (s, \beta, \boldsymbol{\tau}^T)^T$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}) &= n\{\log(s\beta) - \log[(1-\beta)^{-s} - 1]\} + \sum_{i=1}^{n} \log[g(x_i;\boldsymbol{\tau})] \\ &- (s+1)\sum_{i=1}^{n} \log\{1 - \beta[1 - G(x_i;\boldsymbol{\tau})]\}. \end{aligned} \tag{4.38}$$

The log-likelihood can be maximized by using well established routines like nlm or optimize in the R statistical package or by solving the nonlinear likelihood equations obtained by differentiating (4.38). The components of the score vector $U(\boldsymbol{\theta})$ are

$$\begin{aligned} U_s(\boldsymbol{\theta}) &= \frac{n}{s} + \frac{n(1-\beta)^{-s}\log(1-\beta)}{(1-\beta)^{-s}-1} - \sum_{i=1}^{n} \log\left\{1 - \beta[1 - G\left(x_i;\boldsymbol{\tau}\right)]\right\}, \\ U_\beta(\boldsymbol{\theta}) &= \frac{n}{\beta} - \frac{n\,s\,(1-\beta)^{-s}}{(1-\beta)[(1-\beta)^{-s}-1]} - (s+1)\sum_{i=1}^{n} \frac{G\left(x_i;\boldsymbol{\tau}\right) - 1}{1 - \beta[1 - G\left(x_i;\boldsymbol{\tau}\right)]}, \\ U_{\tau_j}(\boldsymbol{\theta}) &= \sum_{i=0}^{n} \frac{\partial g(x_i;\boldsymbol{\tau})/\partial\tau_j}{g(x_i;\boldsymbol{\tau})} - (s+1)\sum_{i=1}^{n} \frac{\beta\left[\partial G\left(x_i;\boldsymbol{\tau}\right)/\partial\tau_j\right]}{1 - \beta[1 - G\left(x_i;\boldsymbol{\tau}\right)]}, \end{aligned}$$

for $j = 1, \ldots p$.

For interval estimation and hypothesis tests on the model parameters, we require the $(p+2)\times(p+2)$ observed information matrix $J = J(\boldsymbol{\theta})$ given in the Appendix. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_{p+2}(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. In practice, we can replace $I(\boldsymbol{\theta})$ by the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, say $J(\widehat{\boldsymbol{\theta}})$. We can construct approximate confidence intervals for the parameters based on the multivariate normal $N_{p+2}(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution.

Further, the likelihood ratio (LR) statistic can be used for comparing this distribution with some of its sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models of the G-NB distribution. For example, the test of $H_0 : s = 1 \,\text{and}\, \beta \to 0$ versus $H_1 : H_0$ *is not true* is equivalent to compare the G-NB and G distributions and the LR statistic becomes

$$w = 2\{\ell(\hat{\boldsymbol{\tau}}, \hat{s}, \hat{\beta}) - \ell(\widetilde{\boldsymbol{\tau}}, 1, 0)\},$$

where $\hat{\boldsymbol{\tau}}$, $\hat{s}$ and $\hat{\beta}$ are the MLEs under H_1 and $\widetilde{\boldsymbol{\tau}}$ and $\widetilde{\beta}$ are the estimates under H_0.

4.12 Application

In this section, we fit the Fréchet negative binomial (FNB) distribution to a real data set. In order to estimate the parameters of this special model, we adopt the maximum likelihood method (as discussed in Section 11) with all computations performed using the subroutine NLMixed of the SAS software. The data set obtained from Murthy et al. (2004) consist of the failure times of 20 mechanical components. The data are: 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, 0.485.

We compare the fit of the FNB distribution with three alternative models not belonging to the G-NB family:

- the *beta Fréchet* (BF) distribution (see Barreto-Souza et al., 2011) with pdf (for $x > 0$):

$$f(x;\sigma,\lambda,a,b) = \frac{\lambda\,\sigma^{\lambda}\,x^{-(\lambda+1)}}{B(a,b)}\exp[-a(\sigma/x)^{\lambda}]\{1-\exp[-(\sigma/x)^{\lambda}]\}^{b-1},$$

  where $\lambda > 0, \sigma > 0, a > 0$ and $b > 0$, and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$;

- the *beta Weibull* (BW) distribution (see Famoye et al., 2005) with pdf (for $x > 0$):

$$f(x;\alpha,\gamma,a,b) = \frac{\alpha\,\gamma}{B(a,b)}\,x^{\gamma-1}\exp(-b\,\alpha\,x^{\gamma})[1-\exp(-\alpha x^{\gamma})]^{a-1},$$

  where $\alpha > 0, \gamma > 0, a > 0$ and $b > 0$; and

- the *Marshall-Olkin Weibull* (MOW) distribution (see Marshall and Olkin, 1997) with pdf (for $x > 0$):

$$f(x;\alpha,\gamma,\delta) = \frac{\delta\,\gamma\,\alpha\,x^{\gamma-1}\exp(-\alpha x^{\gamma})}{[1-(1-\delta)\exp(-\alpha x^{\gamma})]^2},$$

  where $\alpha > 0, \gamma > 0$ and $\delta > 0$.

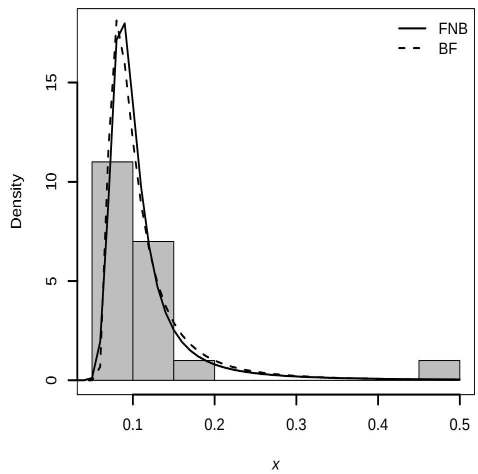
Table 4.1: MLEs, the corresponding SEs (given in parentheses), maximized log-likelihoods, CM and AD statistics and the p-values for the failure time data.

Distribution	Estimates	$\ell(\widehat{\boldsymbol{\theta}})$	CM	p-Value	AD	p-Value
FNB(σ,λ,β,s)	0.1906, 1.8463, 0.9811, 0.6264	39.25	0.0425	0.6365	0.2723	0.6703
	(0.2985, 1.9279, 0.1494, 1.2789)					
BF(σ,λ,a,b)	0.0745, 7.7243, 1.1875, 0.3540	39.20	0.0446	0.5977	0.3233	0.5261
	(0.0466, 5.7291, 4.4002, 0.3595)					
BW(α,γ,a,b)	1.3011, 0.0973, 636.50, 344.03	33.30	0.1761	0.0109	1.2506	0.0030
	(0.0454, 0.0152, 340.23, 188.76)					
MOW(α,γ,δ)	7.0694, 5.0872, 0.0001	36.20	0.0892	0.1577	0.6697	0.0804
	(5.9804, 0.9628, 0.0001)					

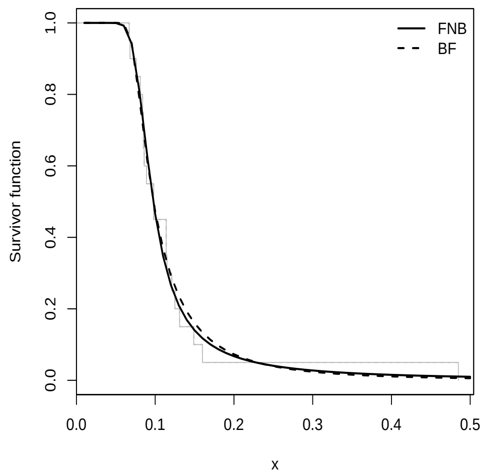
Table 4.1 gives the MLEs and corresponding standard errors (SEs), maximized log-likelihoods, the values of the Cramér-von Mises (CM) and Anderson-Darling (AD) statistics and the p-values for the current data. In general, the smaller the values of these statistics, the better the fit to the data. To obtain the statistics, one can proceed as follows: (1) compute $v_i = F(x_i;\widehat{\theta})$ and

$y_i = \Phi^{-1}(v_i)$, where the x_i's are in ascending order, $\widehat{\theta}$ is an estimate of θ, $\Phi(\cdot)$ is the standard normal cumulative function and $\Phi^{-1}(\cdot)$ denotes its inverse; (2) compute $u_i = \Phi[(y_i - \bar{y})/s_y]$, where $\bar{y}$ is the sample mean of y_i and s_y is the sample variance; (3) compute $\mathrm{CM}^* = \sum_{i=1}^{n}[u_i - (2\,i-1)/2\,n]^2 + 1/(12\,n)$ and $\mathrm{AD}^* = -n - (1/n)\sum_{i=1}^{n}[(2\,i-1)\log(u_i) + (2\,n+1-2\,i)\log(1-u_i)]$, and then $\mathrm{CM} = (1+0.5/n)\mathrm{CM}^*$ and $\mathrm{AD} = (1+0.75/n+2.25/n^2)\mathrm{AD}^*$.

Thus, according to these formal tests, the FNB model fits better to these data than the other models. This evidence can also be noted in Figure 4.8, where we can check that the FNB model captures the behavior of the data.



(a)



(b)

Figure 4.8: Plots of the estimated density (a) and estimated survivor function (b) for the failure time data.

4.13 Concluding remarks

For the first time, we propose a family of generalized negative binomial (G-NB) distributions. The G-NB family extends several common distributions such as the normal, Weibull, gamma, log-logistic and Gumbel distributions. In fact, for each distribution G, we can define the corresponding G-NB distribution using a simple equation. We demonstrate that some mathematical properties of the G-NB distribution can be readily obtained from those of the exponentiated-G distribution. Explicit expressions for the ordinary and incomplete moments, generating function, mean deviations, Bonferroni and Lorenz curves, Rényi and Shannon entropies, reliability and order statistics are derived for any G-NB distribution. We discuss maximum likelihood estimation and inference on the parameters based on Cramér-von Mises (CM) and Anderson-Darling (AD) statistics. An example to real data illustrates the importance and potentiality of the new family.

Appendix A - Information Matrix

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the model parameters (s, β, τ) are given by

$$
\begin{aligned}
J_{ss} &= -\frac{n}{s^2} - \frac{n(1-\beta)^{-s}\log(1-\beta)^2}{[(1-\beta)^{-s}-1]^2}, \\
J_{s\beta} &= \frac{1}{[(1-\beta)^{s}-1]^3}\{n\log(1-\beta)[s(1-\beta)^{s-1}\log(1-\beta) + s(1-\beta)^{2s-1} \\
&\quad \times \log(1-\beta) + 2(1-\beta)^{s-1} - 2(1-\beta)^{2s-1}]\}, \\
J_{s\tau_j} &= -\sum_{i=1}^{n} \frac{\beta\dot{\Phi}}{1-\beta(1-\Phi)}, \\
J_{\beta\beta} &= -\frac{n}{\beta^2} + \frac{ns^2(1-\beta)^{s-4}}{[(1-\beta)^{s}-1]^2}\Big\{2 + (1+s)(\beta^2-2\beta) - \frac{1}{(1-\beta)^{s-2}}\Big\} \\
&\quad +(s+1)\sum_{i=1}^{n}\frac{(\Phi-1)^2}{[1-\beta(1-\Phi)]^2}, \\
J_{\tau_j\tau_j} &= \sum_{i=1}^{n}\Big[\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2}\Big] - (s+1)\sum_{i=1}^{n}\Big\{\frac{\beta\ddot{\Phi}}{1-\beta(1-\Phi)} - \frac{\beta^2\dot{\Phi}^2}{[1-\beta(1-\Phi)]^2}\Big\},
\end{aligned}
$$

where $\phi = g(x_i;\tau)$,$\Phi = G(x_i;\tau)$, $\dot{\phi} = \frac{\partial g(x_i;\tau)}{\partial \tau_j}$, $\dot{\Phi} = \frac{\partial G(x_i;\tau)}{\partial \tau_j}$, $\ddot{\phi} = \frac{\partial^2 g(x_i;\tau)}{\partial \tau_j^2}$ and $\ddot{\Phi} = \frac{\partial^2 G(x_i;\tau)}{\partial \tau_j^2}$.

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CHAPTER 5

The Zeta-G Class: General Properties and Application

Resumo

Propomos uma nova classe de distribuições com um parâmetro de forma adicional. Alguns casos especiais são apresentados. Obtivemos algumas propriedades matemáticas desta classe, incluíndo expressões explícitas para a função quantílica, momentos ordinários e incompletos, função geradora, desvios médios, dois tipos de entropia, confiabilidade, estatísticas de ordem e seus momentos. Discutimos estimativa dos parâmetros do modelo por máxima verossimilhança e fornecemos uma aplicação para um conjunto de dados reais.

Palavras-chave: Distribuição Zeta; desvios médios; estatísticas de ordem; função geradora; momentos.

Abstract

We propose a new class of distributions with one extra shape parameter. Some special cases are presented. We derive some mathematical properties of this class including explicit expressions for the quantile function, ordinary and incomplete moments, generating function, mean deviations, two types of entropy, reliability, order statistics and their moments. We discuss estimation of the model parameters by maximum likelihood and provide an application to a real data set.

Keywords: Generating function; Mean deviation; Moment; Order Statistic; Zeta distribution.

5.1 Introduction

Recently, new distributions have been proposed by compounding any continuous baseline G distribution with a discrete distribution supported on integers $n \geq 1$. By this method, we can obtain a new class of distributions with additional parameters whose role is to govern skewness and generate densities with heavier/ligther tails. These parameters are sought as a manner to furnish a more flexible distribution for modeling the hazard rate function (hrf). Another important method for generating continuous distributions was proposed by Alzaatreh *et al.* (2013). Accordingly, several new distributions have been appeared, such as the extended Weibull distribution Cordeiro and Lemonte (2013) that includes the Weibull distribution as a special case and gives more flexibility to model various types of data.

We propose a general class of continuous distributions called the *Zeta-G class* with an additional shape parameter. The Zeta-G can generate new distributions from specified baseline distributions. We demonstrate that the Zeta-G density class is a linear combination of exponentiated-G ("exp-G" for short) density functions.

Let $W_1, \ldots, W_Z$ be a random sample from a continuous cumulative distribution function (cdf) $G(\cdot)$ with positive support, where Z is an unknown positive integer number. We assume that the random variable Z has a zeta probability mass function (pmf)

$$P(z; s) = \frac{z^{-s}}{\zeta(s)}, \quad z \in \{1, 2, \ldots\},\ s \in (1, \infty),$$

where $\zeta(s)$ is the Riemann zeta function (which is undefined for $s = 1$). Let Z and W be independent random variables and $X = min(W_1, \ldots, W_Z)$. Then, the conditional cdf of X given Z is

$$\begin{aligned} F(x|z) &= 1 - P(X \geq x|z) = 1 - P^z(W_1 \geq x) \\ &= 1 - [1 - P(W_1 \leq x)]^z = 1 - [1 - G(x)]^z. \end{aligned}$$

The unconditional cdf of X becomes

$$F(x; s) = \sum_{z=1}^{\infty} \frac{z^{-s}}{\zeta(s)} \{1 - [1 - G(x)]^z\},$$

for $x > 0,\ s > 1$ and $z \in \{1, 2, \ldots\}$. Here, s is a shape parameter. After some algebra, the cdf of X reduces to

$$F(x; s) = \frac{\zeta(s) - \mathrm{Li_s}[1 - G(x)]}{\zeta(s)}, \tag{5.1}$$

where $\mathrm{Li_s}(x)$ is the polylogarithm function [Abramowitz and Stegun] defined by the power series

$$\mathrm{Li_s}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \tag{5.2}$$

with $|z| < 1$. The model defined by (5.1) is called the *Zeta-G* distribution

The polylogarithm function can be represented by more general functions, for example, using the generalized hypergeometric function, the Lerch transcendent function and the Meijer G-function (can be found in wolfram website[1]).

[1]http://functions.wolfram.com/10.08.26.0008.01 - Accessed 13/06/2013.

We provide two motivations for the Zeta-G class of distributions. First, suppose the failure of a device occurs due to the presence of an unknown number Z of initial defects of same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by W_i the time to the failure of the device due to the ith defect, for $i \geq 1$. Under the assumptions that the W_i's are iid random variables with cdf $G(x)$ independent of Z, where Z has a Zeta distribution, equation (5.1) is appropriate for modeling the time to the first failure. Secondly, suppose that an individual in the population is susceptible to a certain type of cancer. Let Z be the number of carcinogenic cells for that individual left active after the initial treatment and denote by W_i the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i \geq 1$. Under the assumptions that $\{W_i\}_{i \geq 1}$ is a sequence of iid random variables independent of Z having the cdf $G(x)$, where Z has a Zeta distribution, the time to relapse of cancer of a susceptible individual is defined by $X = \min\{W_i\}_{i=1}^{Z}$, which follows (5.1).

The probability density function (pdf) corresponding to (5.1) is given by

$$f(x) = \frac{\mathrm{Li}_{s-1}[1 - G(x)]\, g(x)}{\zeta(s)\,[1 - G(x)]}, \tag{5.3}$$

where $g(x) = dG(x)/dx$. We can verify using Mathematica that $\int_0^\infty f(x)dx = 1,\ \forall\, s > 1$.

This generalization is obtained by increasing the number of parameters of the G model by one, this increase being the price to pay for adding more flexibility to the generated distribution. A positive point of the Zeta-G model is that it includes the G distribution as a special model when $s \to \infty$. Hereafter, a random variable X having the density (5.3) is denoted by $X \sim$ Zeta-G(τ, s), where τ is the parameter vector associated with G. The survival function and hazard rate function (hrf) of X are given by

$$S(x) = \frac{\mathrm{Li}_s[1 - G(x)]}{\zeta(s)}$$

and

$$h(x) = \frac{g(x)\,\mathrm{Li}_{s-1}[1 - G(x)]}{[1 - G(x)]\,\mathrm{Li}_s[1 - G(x)]},$$

respectively. The aim of this paper is to derive some mathematical properties of (5.3) which hold for any continuous G distribution.

Throughout the paper we use an expansion in Taylor series for x^λ, where λ is any real number, given by

$$x^\lambda = \sum_{k=0}^{\infty} (\lambda)_k \frac{(x-1)^k}{k!} = \sum_{i=0}^{\infty} f_i\, x^i, \tag{5.4}$$

where $f_i = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}(\lambda)_k}{k!}\binom{k}{i}$ and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ is the descending factorial. Further, we use an equation by [4, Section 0.314] for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{5.5}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation

$$c_{n,i} = (i\,a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m} \tag{5.6}$$

and $c_{n,0} = a_0^n$.

The rest of the paper is organized as follows. In Section 5.2, we present some new distributions in the Zeta-G class. We obtain explicit expressions for the quantile function, ordinary and incomplete moments, moment generating function (mgf), mean deviations, Shannon entropy, Rényi entropy, reliability and moments of the order statistics in Sections 5.2 to 5.10. The estimation of the model parameters using the method of maximum likelihood is presented in Section 5.11. An application to a real data set is performed in Section 5.12. Finally, some conclusions are addressed in Section 5.13.

5.2 Special Zeta-G distributions

The Zeta-G class of density functions (5.3) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. This new class extends several widely-known distributions in the literature. So, we present some of its special cases. The density function (5.3) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

5.2.1 Zeta-Weibull (ZW) distribution

If $G(x)$ is the Weibull cdf with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$, say $G(x) = 1 - \exp(-\beta x^{\alpha})$, the pdf (for $x > 0$) and cdf of the ZW distribution reduce to

$$f_{\mathcal{ZW}}(x) = \frac{\alpha\,\beta\,x^{\alpha-1}\,\mathrm{Li}_{s-1}[\mathrm{e}^{-\beta x^{\alpha}}]}{\zeta(s)} \quad \text{and} \quad F_{\mathcal{ZW}}(x) = \frac{\zeta(s) - \mathrm{Li}_{s}[\mathrm{e}^{-\beta x^{\alpha}}]}{\zeta(s)}.$$

Figure 5.1 displays some possible shapes of the ZW density function.

5.2.2 Zeta-Kumaraswamy (ZKw) distribution

Consider the Kumaraswamy distribution with pdf and cdf in the forms [for $x \in (0,1)$ and a, $b > 0$] $g(x) = a\,b\,x^{a-1}$ and $G(x) = 1 - (1 - x^a)^b$, respectively. This distribution, introduced by [6], was investigated by [5]. The ZKw distribution, for $x \in (0,1)$, has pdf and cdf given by

$$f_{\mathcal{ZK}w}(x) \;=\; \frac{a\,b\,x^{a-1}\,\mathrm{Li}_{s-1}[(1-x^a)^b]}{\zeta(s)\,(1-x^a)^b} \tag{5.7}$$

and

$$F_{\mathcal{ZK}w}(x) \;=\; \frac{\zeta(s) - \mathrm{Li}_{s}[(1-x^a)^b]}{\zeta(s)},$$

respectively, where a and b are shape parameters. Plots of (5.7) for selected parameter values are displayed in Figure 5.2.

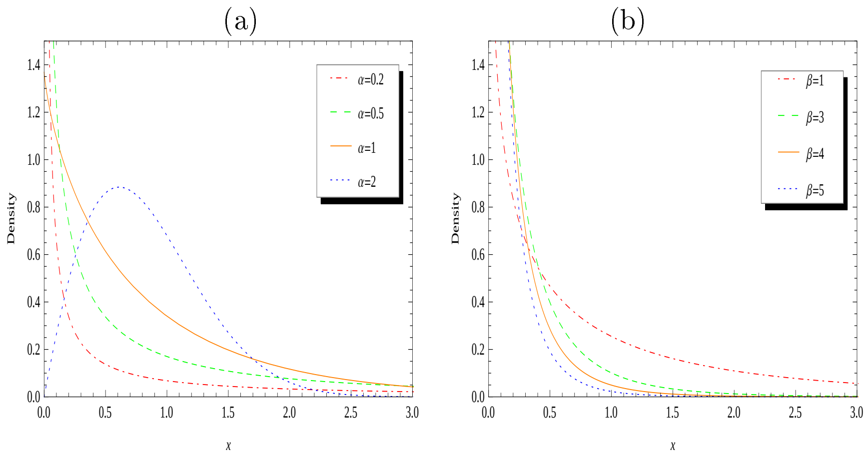

Figure 5.1: The ZW density function for some parameter values: (a) $s = 3$ and $\beta = 1$; (b) $s = 5$ and $\alpha = 0.7$.

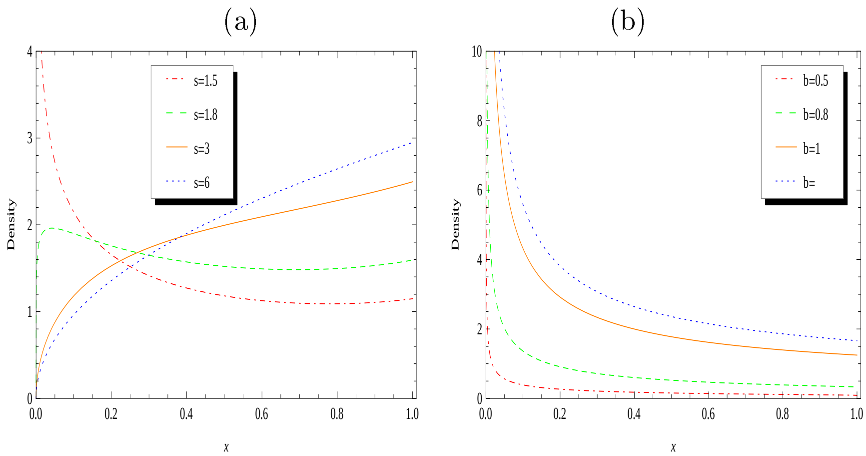

Figure 5.2: The ZKw density function for some parameter values: (a) $a = 1.5$ and $b = 2$; (b) $s = 3$ and $a = 0.5$.

5.2.3 Zeta-Fréchet (ZFr) distribution

Consider the Fréchet distribution (for $x, \sigma, \lambda > 0$) with cdf and pdf given by $G(x) = \exp\{-(\sigma/x)^{\lambda}\}$ and $g(x) = \lambda\, \sigma^{\lambda}\, x^{-\lambda-1} \exp\{-(\sigma/x)^{\lambda}\}$, respectively.

The ZFr distribution, for $x > 0$, has pdf and cdf given by

$$f_{\mathcal{ZF}r}(x) \quad = \quad \frac{\lambda\, \sigma^{\lambda}\, x^{-\lambda-1}\, \mathrm{Li}_{s-1}[1 - \exp\{-(\frac{\sigma}{x})^{\lambda}\}]}{\zeta(s)\, [\exp\{(\frac{\sigma}{x})^{\lambda}\} - 1]} \tag{5.8}$$

and

$$F_{\mathcal{ZF}r}(x) \quad = \quad \frac{\zeta(s) - \mathrm{Li}_s[1 - \exp\{(\frac{\sigma}{x})^{\lambda}\}]}{\zeta(s)},$$

respectively, where $\sigma > 0$ is scale parameter and $\lambda > 0$ is a shape parameter. Plots of (5.8) for selected parameter values are given in Figure 5.3.

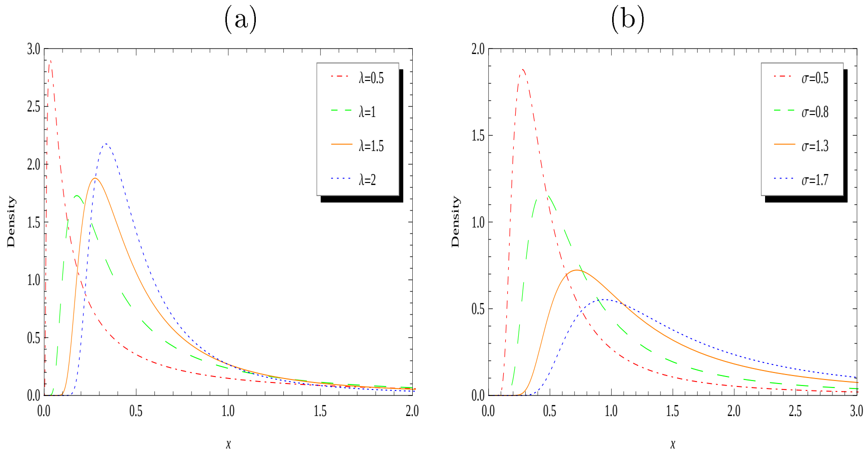

Figure 5.3: The ZFr density function for some parameter values: (a) $s = 2$ and $\sigma = 0.5$; (b) $s = 2$ and $\lambda = 1.5$; (c) $\lambda = 1.5$ and $\sigma = 1$.

5.2.4 Zeta-Exponentiated Pareto (ZEPa) distribution

The pdf and cdf of the exponentiated Pareto distribution are (for $\theta, \gamma, k > 0$) $g(x) = \gamma\, k\, \theta^k\, x^{-k-1}[1 - (\theta/x)^k]^{\gamma-1}$ and $G(x) = [1 - (\theta/x)^k]^\gamma$, respectively.

The ZEPa distribution, for $x \geq \theta$, has pdf and cdf given by

$$f_{\mathcal{ZEP}a}(x) \quad = \quad \frac{\gamma\, k\theta^k\, x^{-k-1}\, \{1 - (\frac{\theta}{x})^k\}^{\gamma-1}\, \mathrm{Li}_{s-1}[(1 - \{1 - (\frac{\theta}{x})^k\}^\gamma]}{\zeta(s)\, [1 - \{1 - (\frac{\theta}{x})^k\}^\gamma]} \tag{5.9}$$

and

$$F_{\mathcal{ZEP}a}(x) \quad = \quad \frac{\zeta(s) - \mathrm{Li}_s[1 - \{1 - (\frac{\theta}{x})^k\}^\gamma]}{\zeta(s)},$$

respectively. Some plots of the ZEPa density function are displayed in Figure 5.4.

5.3 Useful representations

Some useful expansions for (5.1) and (5.3) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G ("exp-G" for short) distribution with parameter $r > 0$, say $Y_r \sim$ exp-G(r), if its pdf and cdf are

$$h_r(x) = r\, G(x)^{r-1}\, g(x) \quad \text{and} \quad H_r(x) = G(x)^r,$$

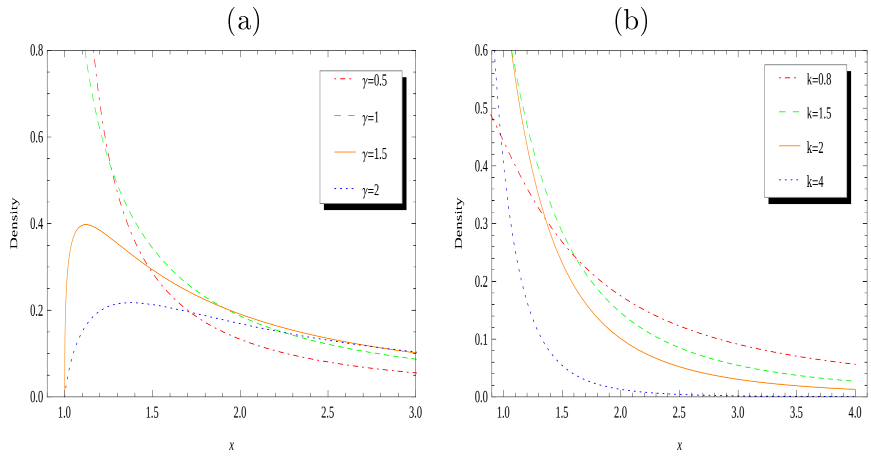

Figure 5.4: The ZEPa density function for some parameter values: (a) $s = 2$, $\theta = 1$ and $k = 0.5$; (b) $s = 3$, $\theta = 0.5$ and $\gamma = 2$.

respectively. The properties of exponentiated distributions have been studied by several authors in recent years. See [9] for exponentiated Weibull, [7] for exponentiated exponential, [11] for exponentiated Fréchet and [10] for exponentiated gamma distributions.

Using expansion (5.2), we can write (5.3) as

$$f(x) = \frac{g(x)}{\zeta(s)} \sum_{k=1}^{\infty} \frac{1}{k^{s-1}} \left[1 - G(x)\right]^{k-1}.$$

Expanding the binomial term in this equation, we can express $f(x)$ as

$$f(x) = \sum_{r=1}^{\infty} \omega_r \, h_r(x), \tag{5.10}$$

where $h_r(x)$ denotes the exp-G(r) density function and

$$\omega_r = \frac{(-1)^r}{r \, \zeta(s)} \sum_{k=r}^{\infty} k^{1-s} \binom{k-1}{r-1}.$$

We prove using Mathematica that $\sum_{r=1}^{\infty} \omega_r = 1$. By integrating (5.10), we can express $F(x)$ as

$$F(x) = \sum_{r=1}^{\infty} \omega_r H_r(x), \tag{5.11}$$

where $H_r(x)$ denotes the exp-G(r) cdf. So, several mathematical properties of the Zeta-G class can be obtained by knowing those of the exp-G distribution, see, for example, [9], [11], among others.

5.4 Quantile function

The quantile function is defined by $x = Q(u;s) = F^{-1}(u;s)$, where $F(x;s)$ follows (5.1) and (5.2) by

$$F(x;s) = 1 - \sum_{j=0}^{k} c_j\, G(x)^j,$$

where $c_j = \sum_{k=1}^{\infty} \frac{(-1)^j \binom{k}{j}}{\zeta(s)\, k^s}$. We shall use the Lagrange theorem [8, p. 88] to obtain the expansion for the quantile function. We can rewrite $w = F(x;s)$ as

$$w = F(x;s) = w_0 - \sum_{j=1}^{k} c_j\, z^j, \qquad F'(x) = -c_1 \neq 0, \tag{5.12}$$

where $w_0 = 1 - c_0$ and $z = G(x)$. The quantile function of the Zeta-G is given by $x = G^{-1}(z)$. First, by inverting (5.12), the inverse function $z = F^{-1}(w;s)$ can be written as a power series around zero

$$z = \sum_{n=1}^{k} g_n\, (w - w_0)^n, \tag{5.13}$$

where $g_n = (1/n!)\mathrm{d}^{n-1}\Psi(x)^n/\mathrm{d}x^{n-1}|_{z=0}$ and $\Psi(z) = \frac{z}{[F(z)-w_0]} = -\frac{1}{\sum_{j=0}^{k} c_{j+1} z^j}$.

We can obtain the inverse of the power series $\sum_{j=0}^{k} c_{j+1} z^j$ using Equation (0.313) from [4]. We have

$$\Psi(z) = -\frac{1}{c_1}\sum_{j=0}^{k} d_j z^j,$$

where d_j can be calculated recursively from the quantities c_j by $d_0 = 1$ and $d_j = -c_1^{-1}\sum_{i=1}^{j} d_{j-i}\, c_{i+1}$ $(j \geq 1)$.

We can obtain $\Psi(z)^n$ using (5.5). Then,

$$\Psi(z)^n = \left(-\frac{1}{c_1}\sum_{j=0}^{k} d_j\, z^j\right)^n = \frac{(-1)^n}{c_1^n}\sum_{j=0}^{k} f_{j,n}\, z^j, \quad n \geq 1,$$

where the coefficients $f_{j,n}$ (for $j = 1, 2, \ldots$) can be determined from the recurrence relation

$$f_{j,n} = j^{-1}\sum_{m=1}^{j} [m\,(n+1) - j]\, d_m\, f_{j-m,n}, \tag{5.14}$$

and $f_{0,n} = d_0^n = 1$. The coefficient $f_{j,n}$ can be calculated from the quantities $f_{0,n}, \ldots, f_{j-1,n}$ and therefore from $d_0, \ldots, d_j$, although it is not necessary for programming numerically our expansions in any algebraic or numerical software. The power series with the first $(n+1)$ terms can be expressed as

$$\Psi(z)^n = \frac{(-1)^n}{c_1^n}\left(f_{0,n} + f_{1,n}\, z + \cdots + f_{n-1,n}\, z^{n-1} + f_{n,n}\, z^n + \cdots\right)$$

The derivative of order $(n-1)$ is given by

$$\frac{d^{n-1}}{dz^{n-1}}\{[\Psi(z)]^n\}\Big|_{z=0} = \frac{(-1)^n (n-1)!\, f_{n-1,n}}{c_1^n},$$

and then

$$g_n = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\{[\Psi(z)]^n\}\Big|_{z=0} = \frac{(-1)^n f_{n-1,n}}{n\, c_1^n}.$$

The inverse function (5.13) can be written as

$$z = \sum_{n=1}^{k} \frac{(-1)^n f_{n-1,n}}{n\, c_1^n}\,(w-w_0)^n$$

and, therefore, the quantile function $x = Q(w;s)$ reduces to

$$x = Q(w;s) = G^{-1}\left\{\sum_{n=1}^{k} \frac{(-1)^n f_{n-1,n}}{n\, c_1^n}\,(w-w_0)^n\right\},$$

where the coefficients $f_{j,n}$ are calculated from (5.14).

5.5 Moments

A first formula for the nth moment of X, say $\mu'_n = E(X^n)$, can be obtained from (5.10) and $Y_r \sim$ exp-G(r) as

$$\mu'_n = \sum_{r=1}^{\infty} \omega_r\, E(Y_r^n). \tag{5.15}$$

Expressions for moments of several exponentiated distributions are given by [11], which can be used to obtain $E(X^n)$. We now provide an application of (5.15) by taking the baseline Weibull introduced in Section 5.2.1. The pdf of the exp-Weibull distribution with power parameter r is given by $h_r(x) = r\,\alpha\,\beta\, x^{\alpha-1}\,\mathrm{e}^{-\beta x^\alpha}\,(1-\mathrm{e}^{-\beta x^\alpha})^{r-1}$. The nth moment of the ZW distribution becomes

$$\mu'_n = \beta^{-n/\alpha}\,\Gamma\left(\frac{n}{\alpha}+1\right)\sum_{k=0}^{\infty}\sum_{r=1}^{\infty}(-1)^k\, r\,\omega_r\,(1+k)^{-\frac{n}{\alpha}-1}\binom{r-1}{k}.$$

Plots of skewness and kurtosis for the ZW distribution for some choices of α and β as function of s are displayed in Figure 5.5.

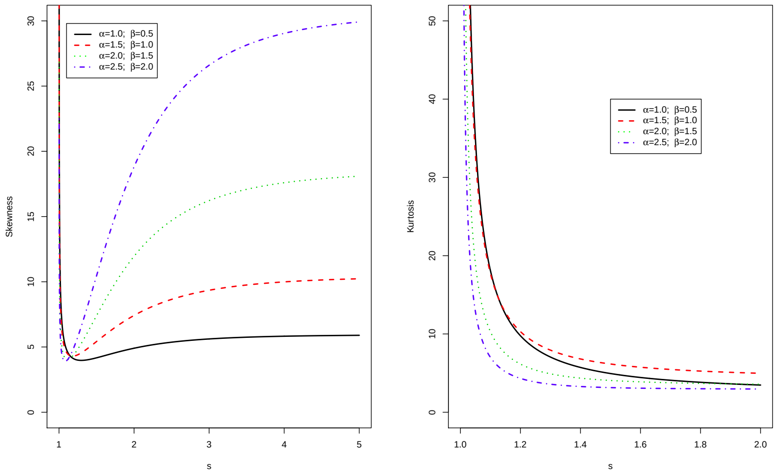

Figure 5.5: Skewness and kurtosis measures of the ZW distribution for some parameter values.

A second formula for μ'_n can be derived from (5.15) in terms of the baseline quantile function $Q_G(x) = G^{-1}(x)$. We can write

$$\mu'_n = \sum_{r=1}^{\infty} r\,\omega_r\,\tau_{n,r}, \tag{5.16}$$

where $\tau_{n,r}$ can be obtained from

$$\tau_{n,r} = \int_{-\infty}^{\infty} x^n\,G(x)^{r-1} g(x) dx = \int_0^1 Q_G(u)^n\,u^{r-1} du. \tag{5.17}$$

The ordinary moments of several Zeta-G distributions can be determined directly from Equations (5.16) and (5.17).

Here, we give two examples. First, the moments of the ZKw distribution (Section 5.2.2) are given by

$$\mu'_n = \sum_{r=1}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\,r\,\omega_r\,\Gamma\left(\frac{n}{a}+1\right)}{k!\,\Gamma\left(\frac{n}{a}+1-k\right)}\,B\left(r+1, \frac{k}{b}+1\right),$$

where $B(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1} dt$ is the beta function. The moments of the ZFr distribution (Section 5.2.3) are given by

$$\mu' n = \sigma^n \sum_{r=1}^{\infty} r\,\omega_r \sum_{i,t=0}^{\infty} h_i\,d_{i,t}\,B(r, t+2),$$

where we define from Equations (5.4), (5.5) and (5.6) $h_i = \sum_{j=i}^{\infty} \frac{(-1)^{j-i}(-n/\lambda)_j}{j!}\binom{j}{i}$. Here, $(-n/\lambda)_j = (-n/\lambda)\ldots(-n/\lambda - j + 1)$ is the descending factorial, $d_{i,t} = (t\,a_0)^{-1}\sum_{m=1}^{i}[m(i+1)-t]\,a_m\,d_{i,t-m}$ for $t \geq 1$ and $d_{i,0} = a_0^i$ with $a_t = (-1)^{t+2}/(t+1)$.

The moments of the ZEPa distribution (Section 5.2.4) are given by

$$\mu'_n = \gamma\,\theta^n \sum_{r=1}^{\infty} r\,\omega_r \sum_{j=0}^{\infty} \frac{[\frac{n}{k}]_j}{j!\,(j + r\,\gamma)},$$

where $[\frac{n}{k}]_j = (\frac{n}{k})(\frac{n}{k}+1)\ldots(\frac{n}{k}+j-1)$ is the ascending factorial.

For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X can be determined from (5.10) as

$$m_n(y) = E\left(X^n | X < y\right) = \sum_{r=1}^{\infty} r\,\omega_r \int_0^{G(y)} Q_G(u)^n\, u^{r-1} du. \tag{5.18}$$

The last integral can be computed for most baseline G distributions, at least numerically.

The symbolic computational software Maple, Mathematica, Matlab make it possible to automate the formulae derived in this paper since they have currently the ability to deal with analytic recurrence equations and sums of formidable size and complexity. In practical terms, we can substitute ∞ in the sums by a large number such as 20 or 50 for most practical applications. Establishing scripts for the explicit expressions given throughout the paper can be more accurate computationally than other integral representations which can be prone to rounding off errors among others. Equations (5.15) - (5.18) are the main results of this section.

5.6 Generating function

The mgf $M(t) = E(\mathrm{e}^{tX})$ of X follows from (5.10) as

$$M(t) = \sum_{r=1}^{\infty} \omega_r\, M_r(t),$$

where $M_r(t)$ is the mgf of Y_r. Hence, $M(t)$ can be immediately determined from the generating function of the exp-G distribution. Another formula for $M(t)$ follows from (5.10) as

$$M(t) = \sum_{r=1}^{\infty} r\,\omega_r\,\gamma_r(t), \tag{5.19}$$

where $\gamma_r(t)$ can be determined from the baseline quantile function $Q_G(u) = G^{-1}(u)$ as

$$\gamma_r(t) = \int_0^{\infty} \mathrm{e}^{tx}\, G(x)^{r-1} g(x) dx = \int_0^1 \exp\{t\, Q_G(u)\}\, u^{r-1} du. \tag{5.20}$$

We can obtain the mgf's of several Zeta-G distributions directly from equations (5.19) and (5.20). For example, the mgf's of the Zeta-Exponential (with parameter λ and for $t < \lambda^{-1}$) and Zeta-Standard Logistic (for $t < 1$) distributions can be expressed as

$$M(t) = \sum_{r=1}^{\infty} r\,\omega_r\, B(r, 1-\lambda\, t) \qquad \text{and} \qquad M(t) = \sum_{r=1}^{\infty} r\,\omega_r\, B(t+r, 1-t),$$

respectively.

5.7 Mean deviations

The mean deviations about the mean ($\delta_1(X) = E(|X - \mu_1'|)$) and about the median ($\delta_2(X) = E(|X - M|)$) of X are given by

$$\delta_1(X) = 2\mu_1'\, F\left(\mu_1'\right) - 2m_1\,\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2(X) = \mu_1' - 2m_1(M), \tag{5.21}$$

respectively, where $\mu_1' = E(X)$, $M = Median(X)$ is the median, $F(\mu_1')$ is easily calculated from the cdf (5.1) and $m_1(z) = \int_{-\infty}^{z} x f(x) dx$ is the first incomplete moment given by (5.18) with $n = 1$.

In this section, we provide two alternative ways to compute $\delta_1(X)$ and $\delta_2(X)$. A general equation for $m_1(z)$ can be derived from (5.10) as

$$m_1(z) = \sum_{r=1}^{\infty} \omega_r\, S_r(z),$$

where

$$S_r(z) = \int_{-\infty}^{z} x\, h_r(x) dx. \tag{5.22}$$

Equation (5.22) is the basic quantity to compute the mean deviations of the exp-G distributions. Hence, the mean deviations in (5.21) depend only on the mean deviations of the exp-G distribution. So, alternative representations for $\delta_1(X)$ and $\delta_2(X)$ are

$$\delta_1(X) = 2\mu_1' F\left(\mu_1'\right) - 2\sum_{r=1}^{\infty} \omega_1\, S_r\left(\mu_1'\right) \quad \text{and} \quad \delta_2(X) = \mu_1' - 2\sum_{r=1}^{\infty} \omega_r\, S_r(M).$$

In a similar manner, the mean deviations of the Zeta-G distribution can be determined from Equation (5.18) with $n = 1$ and letting $T_r(z) = \int_0^{G(z)} Q_G(u)\, u^{r-1} du$. For example, the mean deviations of Zeta-Logistic (ZL), Zeta-Pareto (ZPa) (with parameter $\nu > 0$) and ZE (with parameter λ) distributions are calculated using the generalized binomial expansion from the following functions

$$T_r(z) = \frac{1}{\Gamma(r-1)} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(r+k)\, \{1 - \exp(-kz)\}}{(k+1)!},$$

$$T_r(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^k\, z^{1-j\nu}}{(1-j\nu)} \binom{r}{k}\binom{k}{j}$$

and

$$T_r(z) = \lambda^{-1} \sum_{k=0}^{\infty} \frac{(2-r)_k\, \{1-\exp(-k\lambda z)\}}{(k+1)!},$$

respectively, where $(2-r)_k = (2-r)(1-r)\ldots(2-r-k)$ is the descending factorial.

5.8 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies [14, 13]. The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_0^{\infty} f^{\gamma}(x)dx\right)$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $I_S = E\{-\log[f(X)]\}$. It is a limit case of the Rényi entropy when $\gamma \uparrow 1$.

Here, we derive closed-form expressions for the Rényi and Shannon entropies when X has a Zeta-G distribution. From equation (5.3), we obtain

$$I_R(\gamma) \quad = \quad \frac{1}{1-\gamma} \log\left[\int_0^{\infty} \frac{g(x)^{\gamma}\,[1-G(x)]^{-\gamma}\{\mathrm{Li}_{s-1}[1-G(x)]\}^{\gamma}}{\zeta(s)^{\gamma}} dx\right].$$

For any real a and $|z| < 1$, we have the power series

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)\, k!} z^k, \tag{5.23}$$

From Equations (5.2) and (5.23), we write

$$\begin{aligned} I_R(\gamma) \quad &= \quad \frac{1}{1-\gamma} \log\left\{\zeta(s)^{-\gamma} \int_0^{\infty} g(x)^{\gamma} \sum_{j=0}^{\infty} \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)\, j!}\, G(x)^j \right. \\ &\times \quad \left. \left[\sum_{k=1}^{\infty} \frac{[1-G(x)]^k}{k^{(s-1)}}\right]^{\gamma} dx\right\}. \end{aligned}$$

Applying (5.4) in the last equation, we obtain

$$
\begin{aligned}
I_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\zeta(s)^{-\gamma}\int_0^\infty g(x)^\gamma \sum_{j=0}^\infty \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)\, j!}\, G(x)^j\right.\\
&\times \left.\sum_{i=0}^\infty g_i\left(\sum_{k=1}^\infty \frac{[1-G(x)]^k}{k^{(s-1)}}\right)^i dx\right\}.
\end{aligned}
$$

Now, applying (5.5) and (5.6) in the last sum, we can write

$$
\begin{aligned}
I_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\zeta(s)^{-\gamma}\int_0^\infty g(x)^\gamma \sum_{j=0}^\infty \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)\, j!}\, G(x)^j\right.\\
&\times \left.\sum_{i,k=0}^\infty \widetilde{h}_i\, e_{i,k} \sum_{v=0}^{k+i}(-1)^{k+i}\binom{k+i}{v} G(x)^v dx\right\},
\end{aligned}
$$

where $e_{i,k} = (k\, a_0)^{-1}\sum_{m=1}^{k}[m(i+1)-k]\, a_m\, e_{i,k-m}$ for $k \geq 1$, $e_{i,0} = a_0^i$ with $a_k = (k+1)^{1-s}$, $\widetilde{h}_i = \sum_{t=i}^\infty \frac{(-1)^{t-i}(\gamma)_t}{t!}\binom{t}{i}$ and $(\gamma)_t$ is the descending factorial.

Then, the Rényi entropy reduces to

$$
\begin{aligned}
I_R(\gamma) &= \frac{\gamma}{\gamma-1}\log[\zeta(s)] + \frac{1}{1-\gamma}\log\left\{\sum_{j,i,k=0}^\infty \sum_{v=0}^{k+i} \frac{(-1)^{k+i}\,\widetilde{h}_i\, e_{i,k}\binom{k+i}{v}\Gamma(\gamma+j)}{\Gamma(\gamma)\, j!}\right.\\
&\times \left.\int_0^\infty g(x)^\gamma\, G(x)^{j+v} dx\right\}.
\end{aligned}
$$

The Shannon entropy can be obtained by limiting $\gamma \uparrow 1$ in the last equation. However, it is easier to derive an expression for I_S from its definition

$$
\begin{aligned}
I_S &= \log[\zeta(s)] + E\{-\log(\mathrm{Li}_{s-1}[1-G(X)])\} - E\{\log[g(X)]\}\\
&+ E\{\log[1-G(X)]\}. \qquad (5.24)
\end{aligned}
$$

The three expectations in (5.24) can be easily determined numerically given $G(\cdot)$ and $g(\cdot)$. From Equations (5.2) and (5.10), we obtain

$$
\begin{aligned}
E\{-\log(\mathrm{Li}_{s-1}[1-G(X)])\} &= \sum_{r=1}^\infty \sum_{k,m=0}^\infty \sum_{j=0}^{k+1} \frac{(-1)^j\, r\, \omega_r\, c_{j,m}}{k+1}\binom{k+1}{j}\\
&\times\ B(r, m+j+1),
\end{aligned}
$$

where $c_{j,m} = (m\, a_0)^{-1}\sum_{n=1}^{m}[n(j+1)-m]a_n c_{j,m-n}$ for $m \geq 1$, $c_{j,0} = a_0^j$ with $a_m = (m+1)^{1-s}$ (for $m \geq 0$)

$$
E\{\log[G(X)]\} = \sum_{r=1}^\infty r\,\omega_r \int_0^1 u^{r-1}\log(u)\, du = -\sum_{r=1}^\infty \frac{\omega_r}{r}
$$

and

$$E\{\log[g(X)]\} = \sum_{r=1}^{\infty} r\,\omega_r \int_0^{\infty} \log[g(x)]\, G(x)^{r-1}\, g(x)\, dx$$

respectively. The last of these equations can be expressed in terms of the baseline quantile function, $Q_G(u)$, as

$$E\{\log[g(X)]\} = \sum_{r=1}^{\infty} r\,\omega_r \int_0^{1} \log\{g[Q_G(u)]\}\, u^{r-1}\, du.$$

5.9 Reliability

We derive the reliability, $R = \Pr(X_2 < X_1)$, when $X_1 \sim$ Zeta-G(τ, s_1) and $X_2 \sim$ Zeta-G(τ, s_2) are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Let f_i and F_i denote the pdf and cdf of X_i, respectively. By using the representations (5.10) and (5.11), we can write

$$R = \sum_{n,m=1}^{\infty} \omega_{n,m} \int_0^{\infty} H_m(x)\, h_n(x) dx = \sum_{n,m=1}^{\infty} \omega_{n,m}\, R_{nm}, \tag{5.25}$$

where $R_{nm} = \Pr(Y_m < Y_m)$ is the reliability between the independent random variables $Y_n \sim$ exp-G(n) and $Y_m \sim$ exp-G(m). Here,

$$\omega_{n,m} = \frac{(-1)^{n+m-2}\, v_n(s_1)\, v_m(s_2)}{n\, m},$$

and $v_j(s_i)$ (for $j = n, m$ and $i = 1, 2$) is given by

$$v_j(s_i) = \sum_{k=j}^{\infty} \frac{k^{1-s_i}}{\zeta(s_i)} \binom{k-1}{j-1}.$$

Hence, the reliability of Zeta-G random variables is a linear combination of those for exp-G random variables. For example, we derive the reliability when X_1 and X_2 have independent Zeta-Weibull distributions with the same shape parameter β, namely Zeta-Weibull(α_1, β, s_1) and Zeta-Weibull(α_2, β, s_2). The reliability obtained from Equation (5.25) is

$$R = \sum_{n,m=1}^{\infty} n\,\omega_{n,m} \int_0^{\infty} (\alpha_1 \beta\, x^{\beta-1} \mathrm{e}^{-\alpha_1 x^{\beta}})\, (1 - \mathrm{e}^{-\alpha_1 x^{\beta}})^{n-1}\, (1 - \mathrm{e}^{-\alpha_2 x^{\beta}})^{m} dx.$$

By application of the binomial expansion, we obtain

$$\begin{aligned} R &= \sum_{n,m=1}^{\infty} \sum_{k=0}^{n-1} \sum_{l=0}^{m} (-1)^{k+l}\, n\, \alpha_1 \beta\, \omega_{n,m} \binom{n-1}{k} \binom{m}{l} \\ &\times \int_0^{\infty} x^{\beta-1}\, \mathrm{e}^{-[\alpha_1(1+k)+\alpha_2 l] x^{\beta}} dx. \end{aligned} \tag{5.26}$$

Finally, we have

$$R = \sum_{n,m=1}^{\infty} \sum_{k=0}^{n-1} \sum_{l=0}^{m} \frac{(-1)^{k+l}\, n\, \alpha_1 \beta\, \omega_{n,m}}{[\alpha_1\, \beta(1+k) + \alpha_2\, \beta\, l]} \binom{n-1}{k} \binom{m}{l}.$$

5.10 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Zeta-G distribution. Let $X_{i:n}$ denote the ith order statistic. From Equations (5.10) and (5.11), the pdf of $X_{i:n}$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{k=1}^{\infty} k\, \omega_k\, g(x)\, G(x)^{k-1} \right] \\ &\times \left[\sum_{r=1}^{\infty} \omega_r\, G(x)^r \right]^{j+i-1}. \end{aligned}$$

Using (5.5), (5.6) and setting $r = t+1$, we write

$$\left[G(x) \sum_{t=0}^{\infty} \omega_{t+1}\, G(x)^t \right]^{j+i-1} = \sum_{t=0}^{\infty} \gamma_{j+i-1,t}\; G(x)^{t+j+i-1},$$

where $\gamma_{j+i-1,t} = (t\,\omega_1)^{-1} \sum_{m=1}^{t} [m(j+i) - t]\, \omega_{m+1}\, \gamma_{j+i-1,t-m}$ and $\gamma_{j+i-1,0} = \omega_1^{j+i-1}$.

Hence,

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k,t=0}^{\infty} q_{j,k,t}\, h_{k+t+j+i}(x), \tag{5.27}$$

where

$$q_{j,k,t} = \frac{(-1)^j i\,(k+1)\,\omega_{k+1}\,\gamma_{j+i-1,t}}{(k+t+j+i)} \binom{n}{i} \binom{n-i}{j}.$$

Equation (5.27) reveals that the pdf of the Zeta-G order statistics is a linear combination of exp-G density functions. So, several mathematical quantities of the Zeta-G order statistics such as the ordinary, incomplete and factorial moments, mgf and mean deviations can be obtained from those quantities of the exp-G distributions. Clearly, the cdf of $X_{i:n}$ can be expressed as

$$F_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k,t=0}^{\infty} q_{j,k,t}\, H_{k+t+j+i}(x).$$

For example, from Equation (5.27), the moments of the Zeta-G order statistics can be written directly in terms of the exp-G moments as

$$E(X_{i:n}^s) = \sum_{j=0}^{n-i} \sum_{k,t=0}^{\infty} q_{j,k,t} \int_0^{\infty} x^s \, h_{k+t+j+i}(x)dx. \tag{5.28}$$

As a simple example of Equation (5.28), the moments of the ZW order statistics can be written directly in terms of the exp-Weibull moments, with baseline parameters $\alpha > 0$ and $\beta > 0$, as

$$\begin{aligned} E(X_{i:n}^s) &= \beta^{-\frac{s}{\alpha}} \, \Gamma\left(\frac{\alpha+s}{\alpha}\right) \sum_{j=0}^{n-i} \sum_{k,t,r=0}^{\infty} (-1)^r \, (k+t+j+i) \, (1+r)^{s/\alpha-1} \\ & \times q_{j,k,t} \binom{k+t+j+i-1}{r}. \end{aligned}$$

5.11 Estimation

We calculate the maximum likelihood estimates (MLEs) of the parameters of the Zeta-G distribution from complete samples only. Let $x_1, \ldots, x_n$ be a observed sample of size n from the Zeta-G(s,$\boldsymbol{\tau}$) distribution, where $\boldsymbol{\tau}$ is a $p \times 1$ vector of unknown parameters in the baseline distribution $G(x; \boldsymbol{\tau})$. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (s, \boldsymbol{\tau}^T)^T$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}) &= -n \, \log\{\zeta(s)\} - \sum_{i=1}^{n} \log\{1 - G(x_i; \boldsymbol{\tau})\} + \sum_{i=1}^{n} \log\{g(x_i; \boldsymbol{\tau})\} \\ &+ \sum_{i=1}^{n} \log\{\mathrm{Li_{s-1}}[1 - G(x_i; \boldsymbol{\tau})]\}. \end{aligned} \tag{5.29}$$

The log-likelihood can be maximized by using well established routines like nlm or optimize in the R statistical package or by solving the nonlinear likelihood equations obtained by differentiating (5.29). The components of the score vector $U(\boldsymbol{\theta})$ are

$$\begin{aligned} U_s(\boldsymbol{\theta}) &= \sum_{i=1}^{n} \frac{\frac{\partial}{\partial s}\mathrm{Li_{s-1}}\left[1 - G\left(x_i, \boldsymbol{\tau}\right)\right]}{\mathrm{Li_{s-1}}\left[1 - G\left(x_i, \boldsymbol{\tau}\right)\right]} - \frac{n\zeta\left(1, s\right)}{\zeta\left(s\right)}, \\ U_{\boldsymbol{\tau}_j}(\boldsymbol{\theta}) &= -\sum_{i=1}^{n} \frac{\mathrm{Li_{s-2}}\left[1 - G\left(x_i, \boldsymbol{\tau}\right)\right] \frac{\partial}{\partial \boldsymbol{\tau}_j} G\left(x_i, \boldsymbol{\tau}\right)}{\left[1 - G\left(x_i, \boldsymbol{\tau}\right)\right] \mathrm{Li_{s-1}}\left[1 - G\left(x_i, \boldsymbol{\tau}\right)\right]} + \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \boldsymbol{\tau}_j} g\left(x_i, \boldsymbol{\tau}\right)}{g\left(x_i, \boldsymbol{\tau}\right)} \\ &+ \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \boldsymbol{\tau}_j} G\left(x_i, \boldsymbol{\tau}\right)}{1 - G\left(x_i, \boldsymbol{\tau}\right)}, \end{aligned}$$

for $j = 1, \ldots p$ and $\zeta\left(1, s\right) = \frac{d}{ds}\zeta(s)$.

For interval estimation and hypothesis tests on the model parameters, we require the $(p+1) \times (p+1)$ observed information matrix $J = J(\boldsymbol{\theta})$ given in the Appendix. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the

Table 5.1: MLEs, the corresponding SEs (given in parentheses), maximized log-likelihoods, statistics CM, AD and the p-values for successive failure data.

Distribution	Estimates	CM	AD
ZW(s, α, β)	4.1020, 0.9320, 0.0145	0.1276	0.7973
	(0.0009, 0.0488, 0.0037)		
ZFr(s, σ, λ)	14.8759, 0.7360, 25.9844	0.7115	4.5757
	(95.3446, 0.0346, 2.5720)		
ZBXII(s, c, k)	10.7162, 13.2045, 0.0194	0.9549	6.1318
	(8.2407, 0.0029, 0.0013)		
ZLo(s, α, λ)	24.7604, 6.0747, 474.6617	0.0822	0.5404
	(0.0087, 3.1962, 288.3710)		

asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_{p+1}(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. In practice, we can replace $I(\boldsymbol{\theta})$ by the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, say $J(\widehat{\boldsymbol{\theta}})$, to construct approximate confidence intervals for the parameters based on the multivariate normal $N_{p+1}(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution.

5.12 Application

In this section, we fit the Zeta-Weibull (ZW), Zeta-Fréchet (ZFr), Zeta-Burr XII (ZBII) and Zeta-Lomax (ZLo) distributions to a real data set. In order to estimate the parameters of these specials models, we adopt the maximum likelihood method (as discussed in Section 5.11) with all computations done using the script **bbmle** of the R software (version 3.0.0). The data set obtained from [12] consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes reported with 213 observations.

Table 5.1 gives the MLEs and corresponding standard erros (SEs) and the values of the Cramér-von Mises (CM) and Anderson-Darling (AD) statistics for the current data. In general, the smaller the values of these statistics, the better the fit to the data. To obtain the statistics, one can proceed as follows: (i) compute $v_i = F(x_i; \widehat{\theta})$ and $y_i = \Phi^{-1}(v_i)$, where the x_i's are in ascending order, $\widehat{\theta}$ is an estimate of θ, $\Phi(\cdot)$ is the standard normal cumulative function and $\Phi^{-1}(\cdot)$ denotes its inverse; (ii) compute $u_i = \Phi[(y_i - \bar{y})/s_y]$, where $\bar{y}$ is the sample mean of y_i and s_y is the sample standard deviation; (iii) compute $\text{CM}^* = \sum_{i=1}^{n}[u_i - (2\,i-1)/2\,n]^2 + 1/(12\,n)$ and $\text{AD}^* = -n - (1/n)\sum_{i=1}^{n}[(2\,i-1)\log(u_i) + (2\,n+1-2\,i)\log(1-u_i)]$, and then $\text{CM} = (1+0.5/n)\text{CM}^*$ and $\text{AD} = (1 + 0.75/n + 2.25/n^2)\text{AD}^*$.

Thus, according to these formal tests, the ZLo model yields a better fit to these data than the other models. This evidence can also be noted in Figure 5.6, where we can check that the ZLo model captures the behavior of the data.

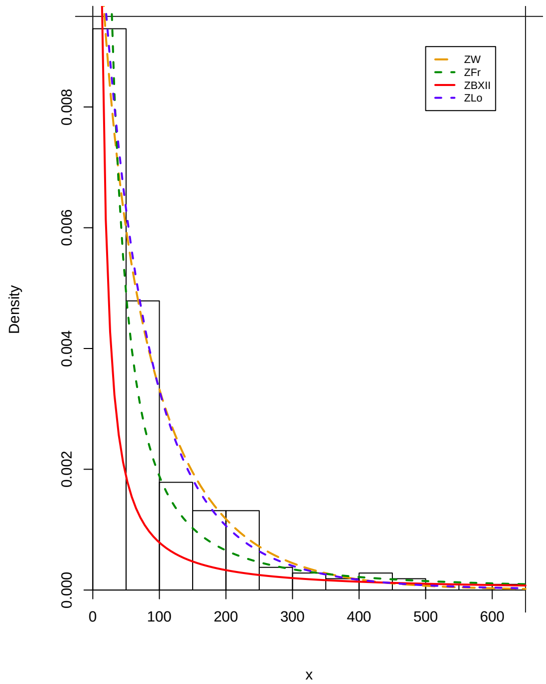

Figure 5.6: Histogram of the data and fitted ZW, ZFr, ZBXII and ZLo density functions to successive failure data.

5.13 Concluding remarks

We propose a general class of continuous distributions called the Zeta-G class. It extends several common distributions such as the Weibull, Kumaraswamy, Fréchet, Burr XII, Lomax and exponentiated Pareto distributions. In fact, for each distribution G, we can define the Zeta-G generator using a simple equation. We demonstrate that some mathematical properties of the Zeta-G distribution can be readily obtained from those of the exponentiated-G distribution. The ordinary and incomplete moments, the generating function and the mean deviations of the Zeta-G class can be expressed explicitly in terms of the baseline quantile function. We discuss maximum likelihood estimation and inference on the parameters based on the Cramér-von Mises (CM) and Anderson-Darling (AD) statistics. An example to real data illustrates the importance and potentiality of the new class.

Appendix: Information Matrix

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the model parameters $(s, \boldsymbol{\tau})$ of the Zeta-G class are given by

$$
\begin{aligned}
J_{ss} &= \sum_{i=1}^{n}\left\{\frac{\frac{\partial^2}{\partial s^2}\mathrm{Li_{s-1}}\left[1-\Phi\right]}{\mathrm{Li_{s-1}}\left[1-\Phi\right]}-\frac{\left\{\frac{\partial}{\partial s}\mathrm{Li_{s-1}}\left[1-\Phi\right]\right\}^2}{\left\{\mathrm{Li_{s-1}}\left[1-\Phi\right]\right\}^2}\right\}-\frac{n\,\zeta(s)''}{\zeta\left(s\right)}+\frac{n\left[\zeta(s)'\right]^2}{\left[\zeta\left(s\right)\right]^2},\\
J_{s\boldsymbol{\tau}_j} &= -\sum_{i=1}^{n}\left\{\frac{\dot{\Phi}\,\frac{\partial}{\partial s}\mathrm{Li_{s-2}}\left[1-\Phi\right]}{\left[1-\Phi\right]\ \mathrm{Li_{s-1}}\left[1-\Phi\right]}+\frac{\dot{\Phi}\,\frac{\partial}{\partial s}\,\mathrm{Li_{s-1}}\left[1-\Phi\right]\ \mathrm{Li_{s-2}}\left[1-\Phi\right]}{\left\{\mathrm{Li_{s-1}}\left[1-\Phi\right]\right\}^2\left[1-\Phi\right]}\right\},\\
J_{\boldsymbol{\tau}_j\boldsymbol{\tau}_j} &= \sum_{i=1}^{n}\left\{\frac{\mathrm{Li_{s-3}}\left[1-\Phi\right]\dot{\Phi}^2}{\left[1-\Phi\right]^2\mathrm{Li_{s-1}}\left[1-\Phi\right]}-\frac{\mathrm{Li_{s-2}}\left[1-\Phi\right]\dot{\Phi}^2}{\left[1-\Phi\right]^2\mathrm{Li_{s-1}}\left[1-\Phi\right]}\right.\\
&\quad\left.-\frac{\mathrm{Li_{s-2}}\left[1-\Phi\right]\ddot{\Phi}}{\left[1-\Phi\right]\mathrm{Li_{s-1}}\left[1-\Phi\right]}-\frac{\left\{\mathrm{Li_{s-2}}\left[1-\Phi\right]\right\}^2\dot{\Phi}^2}{\left[1-\Phi\right]^2\left\{\mathrm{Li_{s-1}}\left[1-\Phi\right]\right\}^2}\right\}\\
&\quad+\sum_{i=1}^{n}\left\{\frac{\ddot{\phi}}{\phi}-\frac{\dot{\phi}^2}{\phi^2}\right\}+\sum_{i=1}^{n}\left\{\frac{\ddot{\Phi}}{1-\Phi}+\frac{\dot{\Phi}^2}{\left[1-\Phi\right]^2}\right\},
\end{aligned}
$$

where $\phi = g(x_i;\boldsymbol{\tau})$,$\Phi = G(x_i;\boldsymbol{\tau})$, $\dot{\phi} = \frac{\partial g(x_i;\boldsymbol{\tau})}{\partial \tau_j}$, $\dot{\Phi} = \frac{\partial G(x_i;\boldsymbol{\tau})}{\partial \tau_j}$, $\ddot{\phi} = \frac{\partial^2 g(x_i;\boldsymbol{\tau})}{\partial \tau_j^2}$ and $\ddot{\Phi} = \frac{\partial^2 G(x_i;\boldsymbol{\tau})}{\partial \tau_j^2}$.

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