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BRUNA GABRIELLY DE MORAES ARAÚJO

QUANTUM LOGIC OPERATIONS WITH CONTINUOS VARIABLES IN A SINGLE TRAPPED ION

Recife
2015

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Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestra em Física.

Orientador:
Prof. Dr. Alessandro de Sousa Villar
Universidade Federal de Pernambuco

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BANCA EXAMINADORA

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Prof. Dr. Alessandro de Sousa Villar
Orientador
Universidade Federal de Pernambuco

_____

Prof. Dr. Daniel Felinto Pires Barbosa
Examinador Interno
Universidade Federal de Pernambuco

_____

Prof. Dr. Gustavo Silva Wiederhecker
Examinador Externo
Universidade Estadual de Campinas

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Ao Pedro Cruz com amor.

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"A viagem não acaba nunca. Só os viajantes acabam. E mesmo estes podem prolongar-se em memória, em lembrança, em narrativa. Quando o visitante sentou na areia da praia e disse:
'Não há mais o que ver', saiba que não era assim. O fim de uma viagem é apenas o começo de outra. É preciso ver o que não foi visto, ver outra vez o que se viu já, ver na primavera o que se vira no verão, ver de dia o que se viu de noite, com o sol onde primeiramente a chuva caía, ver a seara verde, o fruto maduro, a pedra que mudou de lugar, a sombra que aqui não estava. É preciso voltar aos passos que foram dados, para repetir e para traçar caminhos novos ao lado deles. É preciso recomeçar a viagem. Sempre. O viajante volta já." (José Saramago)

"Uma nova Roma, lavada em sangue negro e sangue índio, destinada a criar uma esplêndida civilização, mestiça e tropical, mais alegre, porque mais sofrida, e melhor, porque assentada na mais bela província da Terra."
(O povo brasileiro - Darcy Ribeiro)

"Só há duas opções nesta vida: se resignar ou se indignar. E eu não vou me resignar nunca." (Darcy Ribeiro)

RESUMO

Nesta dissertação, tratamos da computação quântica em variáveis contínuas empregando a armadilha de íons como plataforma física. A proposta central deste trabalho consiste na sistematização de uma *toolbox* com portas lógicas gaussianas a partir da manipulação coerente dos modos de vibração de um íon aprisionado. Através da irradiação de feixes de lasers clássicos monocromáticos e bicromáticos em um íon confinado, propomos portas lógicas gaussianas análogas às operações de ótica linear e não linear. Relacionamos essas portas lógicas a operações já bem conhecidas do caso discreto, tais como transformada de Fourier, gates CNOT e CPHASE. A execução de cada uma dessas operações lógicas é selecionada pela frequência do laser de interação e pelo parâmetro de Lamb-Dicke. Reunindo todas as operações obtidas, gaussianas e não gaussianas, mostramos ser possível com nosso sistema simular hamiltonianas descritas por um polinômio em cada coordenada do espaço de fase, permitindo com isso a realização de dinâmicas hamiltonianas polinomiais nesse espaço.

Palavras-chave: *Computação quântica. Variáveis contínuas. Portas lógicas gaussianas. Portas lógicas quânticas. Modos de vibração. Armadilha de íons.*

ABSTRACT

In this monography, we propose the realization of quantum computation over continuous variables using the ion trap as physical platform. The central idea of our work is to provide a toolbox of Gaussian logic gates from the coherent manipulation of the vibrational modes of a trapped ion. By irradiating monochromatic and bichromatic classical laser beams in a confined ion, we propose gaussian logic gates similar to the operations of linear and nonlinear optics. We connect these gates with operations already employed in the discrete case, such as Fourier, CNOT and CPHASE gates. The execution of each of these logical operations is selected by the frequency of the interacting laser and the Lamb-Dicke parameters. Bringing together all the proposed operations, Gaussian and non-Gaussian, we show the simulation of Hamiltonians with polynomial expansion in the phase space coordinates.

Keywords: *Quantum computation. Continuous variables. Gaussian gates. Quantum gates. Vibrational modes. Ion trap.*

RESUMÉ

Dans cette thèse, nous traitons de l'informatique quantique à variables continues, en utilisant le piège à ions comme plate-forme physique. L'objectif central de ce travail est de construire une boite à outils avec des portes logiques gaussiennes à travers de la manipulation cohérente des modes de vibration d'un ion piégé. En irradiant un ion confiné par des faisceaux lasers classiques monochromatiques et dichromatique , nous proposons des opérations quantiques gaussiennes similaires aux opérateurs d'optique linéaire et non linéaire . Nous relions ces opérateurs des portes quantiques déjà bien connus pour le cas discret, comme des transformées de Fourier, par des portes quantiques CNOT et CPHASE. L'exécution de chacune de ces opérations logiques est sélectionnée par fréquence de l'interaction laser et par les paramètres de Lamb-Dicke. En réunissant toutes les opérations pertinentes, gaussiennes et non gaussiennes, nous avons montré par simulation que l'hamiltonien est décrit par un polynôme en chaque coordonnée de l'espace des phases, ce qui permet d'effectuer l'hamiltonien dynamique, c'est à dire l'expression polynomiale dans cet espace.

Mots-clés: *Information quantique. Variables continues. Portes gaussiennes. Portes quantiques. Modes de vibration. Ion piège.*

Contents

Chapter 1
Introduction

> "If you decide you don't have to get A's, you can learn an enormous amount in college."
>
> Isidor Isaac Rabi

1.1 Quantum information and computation over continuous variables

The first steps in developing quantum information theory were given in the 1970s. Let us begin by citing some of the notorious names that contributed to it. In 1970, Stephen Wiesner invented "Conjugate coding", an important tool for quantum cryptography [1]. In 1973, Alexander Holevo showed that n qubits cannot be used to communicate more information that n bits, a result known today as the "Holevo bound"[2]. Shortly after that, Charles Bennett showed that computation can be realized reversibly [3]. In 1975, R. P. Poplovskii argued for the impracticality of realizing quantum simulations with classical computers[4]. In 1973, the mathematical physicist Roman Ingarden showed in a seminal work that the information theory devised by Shannon cannot be generalized directly to the quantum case; at the same time, he also showed that it is possible to construct a quantum information theory that is more general than Shannon's theory for the formalism of quantum mechanics in open systems[5].

In 1982, Richard Feynman considered which kind of universal computer could simulate quantum systems. He conjectured that this problem would not find efficient solution in a classical computer and started to devise a new type of computer that would be capable of doing so: a quantum computer[6]. In 1994, a key discovery by Peter Shor showed that a quantum computer would have the capability to efficiently perform tasks believed to be unfeasible with classical computers [7]. He showed that a quantum computer could factor a large number in polyno-

mial time, a result that contrasts with the current belief that classical Turing machines must take exponential resources to solve the factorization problem. Two years after that, Seth Lloyd developed more on the idea that the simulation of quantum systems could be implemented in a quantum computer [8].

Quantum information is a recent branch of physics. One of its goals is to contribute for the development of quantum theory in a very fundamental level, by uniting physics and information. Let us not forget that much of computation as well as information theories are based in physical concepts, according to Feynmann and Landauer [9, 10]. Exploring fundamental concepts of quantum information, we may cite the two prominent applications of *quantum computation*, believed to achieve exponential speedup over classical computation for certain tasks, and *quantum key distribution*, which may bring unconditional security to communications.

There are two general ways in which to represent quantum information in physical systems, either through continuous or discrete observables. For each type of observable spectrum, the Hilbert space is exploited differently.

The general motivations to work with continuous observables, or *continuous variables* (CV), are the high measurement efficiency and unconditionality of implementation. In optical systems, uncondicionality is difficult to obtain in the case of discrete variables (i.e. qubit-based implementations). However, as we will show, continuous variables can also be implemented in trapped ion systems, in which case they offer a novel way to interpret the quantum state of those systems.

Considering the set of quantum operations capable of acting on continuous variables, they may be classified as Gaussian and non-Gaussian. Gaussian operations are based on interactions which are quadratic on the phase space observables (e.g. position and momentum). Examples of these include squeezing and the beam-splitter transformation, among others. Despite the advent of several theoretical proposals, experimental quantum computation over **CV** has made little progress in recent years [11].

Gaussian operations are special in the sense that they map Gaussian states into Gaussian states. However, Non-Gaussian states are required in more advanced quantum communications. The measurement unit of quantum information in the CV regime is the 'qunat', in analogy to classical conventional terminology for continuous computers, 1 nat $= \log_2 e$ bits. We note that both experimentalists and theorists are particularly interested in the study of entanglement entropy in quantum optics to connection with condensed matter, for example, by the use of

conformal field theory to describe coherent states and compute the entanglement entropy of some systems [12].

In general, the field of quantum information has been growing in importance, boosted primarily by technological applications of presumably great impact once available. In fact, the research community has been dedicated in the last years to increase the degree of control over quantum systems. There is a great interest in exploring more fundamental aspects of quantum mechanics using concepts borrowed from quantum information theory, in particular to better understand the building blocks of physical reality [13, 14, 15, 16].

1.1.1 Outline of the thesis

In the first chapter, we highlight certain results in the literature considering topics that permeate our area of study and are important for the research frontier between quantum information and other branches of physics.

In chapter two, we study the quantum logic gates developed to coherently manipulate trapped ions. We give a brief description of the computation procedure involved in the coherent control of trapped ions. We describe the physical system composed of trapped ions in interaction with an external laser field. We consider in detail the interaction Hamiltonian driving the two-level system associated with each ion as well as the collective vibrational modes. We describe the two best known examples of collective quantum gates in this context: the Cirac-Zoller and Mølmer-Sørensen gate.

In chapter three, we present how to realize quantum operations over continuous variables, with the development of analytic calculations to obtain the general expression for the ion-laser interaction Hamiltonian in the ion trap with two vibrational modes. These analytical calculations were carried out in an analogous manner to the general expression for a single-mode Hamiltonian of the quantum harmonic oscillator. From these general expressions, we determine the quantum operations realized by laser light with a convenient set of frequencies. We obtain Gaussian operations, analogous to the linear quantum optics operations of displacement, phase rotation and beam splitter, as well as non-linear ones, such as squeezing of one or two modes. We develop a toolbox of gates to manipulate the position and momentum of trapped ions. Finally, we investigate non-Gaussian operations and how they contribute to achieve universal computation over continuous variables. As we show, the ion trap processor is a system that naturally supports Gaussian and non-Gaussian operations.

1.1.2 Quantized fields and canonical operators of bosonic systems

Continuous variables (**CV**) systems are defined as those associated with observables of continuous spectra. The Hilbert space employed to describe the quantum state of one such system has infinite dimension. One example of a quantized system with CV spectrum is the electromagnetic field, described in terms of quadrature observables [17]. To avoid conceptua difficulties, the field is quantized considering discrete modes, as e.g. in an optical cavity [18].

The set of operators describing a **CV** system are represented by N bosonic modes, from N quantum harmonic oscillators of the Hilbert space $\mathscr{I} = \otimes_{k=1}^{N} I_k$, where each $I_{k's}$ describes a single mode. In the case of the electromagnetic field, the Hamiltonian can be written as

$$\hat{H} = \sum_{k=1}^{N} \hat{H}_k, \qquad \hat{H} = \hbar\omega_k(\hat{a}_k^{\dagger}\hat{a}_k + \frac{1}{2}), \tag{1.1}$$

where each term in the Hamiltonian of the k^{th} mode of the field represents a single harmonic oscillator. Here the ladder operators change the number of excitations (photons) in the k^{th} mode. They follow the commutation relations

$$\left[\hat{a}_i, \hat{a}_j^{\dagger}\right] = \delta_{ij}, [\hat{a}_k, \hat{a}_l] = \left[\hat{a}_i{}^{\dagger}, \hat{a}_j^{\dagger}\right] = 0. \tag{1.2}$$

We can construct a group that gathers all ladder operators in a single vector, due to the properties of the symplectic space, in the form

$$\left[\hat{b}_i, \hat{b}_j\right] = \Omega_{ij}, \;\; (i, j = 1, ..., 2N), \tag{1.3}$$

where Ω_{ij} is a generic element of N-modes in the symplectic form of $2N \times 2N$,

$$\Omega = \otimes_{k=1}^{N} \omega, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.4}$$

The Hilbert space of this system has infinite dimension and can be spanned by a Fock basis $\{|n\rangle\}_{n=0}^{\infty}$, composed of eigenstates of the number operators $\hat{n} = \hat{a}^{\dagger}\hat{a}$, which are compatible with the Hamiltonian. In particular we have the well known relations

$$\hat{a}\,|0\rangle = 0, \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \;\text{ for }\; (n \geqslant 1), \tag{1.5}$$

$$\hat{a}^{\dagger}\,|n+1\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{for}\,(n \geqslant 0). \tag{1.6}$$

Besides bosonic field operators, the system can also be described through field quadratures as $\{\hat{q}_k, \hat{p}_k\}_{k=1}^{N}$, by defining the vector

$$\hat{x} \equiv (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^T. \tag{1.7}$$

The amplitude and phase quadrature observables are respectively given by

$$\hat{q}_k \equiv \hat{a}_k + \hat{a}_k^\dagger \;\; \hat{p}_k \equiv i(\hat{a}_k^\dagger - \hat{a}_k), \tag{1.8}$$

$$\left[\hat{x}_i, \hat{x_j}\right] = 2i\Omega_{ij}. \tag{1.9}$$

The quadrature operators are observables with continuous spectra. Indeed, the two operators have eigenstates that are incompatible, denoted by $\{|q\rangle\}_{q\varepsilon\mathbb{R}}$ and $\{|p\rangle\}_{p\varepsilon\mathbb{R}}$, two basis connected by the Fourier transform, as we can see below

$$|q\rangle = \frac{1}{2\pi}\int dp \exp^{-ipq/2}|p\rangle, \tag{1.10}$$

$$|p\rangle = \frac{1}{2\pi}\int dq \exp^{-ipq/2}|q\rangle. \tag{1.11}$$

In general, for N-modes we have the relations

$$\hat{x}^T|x\rangle = x^T|x\rangle \;\; ; |x\rangle \equiv (|x_1\rangle, \;\; \ldots, |x_{2N}\rangle)^T, \tag{1.12}$$

with $x\,\varepsilon\,\mathbb{R}^{2N}$, the quadratures observables. They will be used to define continuous variables in the phase space.

1.1.3 Phase space representation

Although the quadrature observables obey the same algebra as the classical position and momentum of a quantum harmonic oscillator (provided we substitute the Poisson brackets by the commutator), the classical construction of a phase space in quantum mechanics is not straightforward. The problem is that the quadrature observables do not commute. Hence the meaning of a point described in phase space by coordinates given by concomitant eigenvalues of p and q has to be carefully considered. We can take guidance in the classical limit of quantum mechanics.

In particular, given a quantum state ρ, we look for a distribution in phase space over the canonical variables q and p that reproduces the classical behavior. Defining a distribution function ρ_Q in phase space, it must have the following properties:

$$\int \rho_Q(q,p)dp = < q|\rho|q >, \tag{1.13}$$

$$\int \rho_Q(q,p)dq = < p|\rho|p >, \tag{1.14}$$

$$\rho_Q(q,p) \geqslant 0. \tag{1.15}$$

In fact, there are infinite functions $\rho_Q(q,p)$ which satisfy these 3 equations, and none of them present a clear physical meaning to lead us in our path. In 1971, Wigner proved that $\rho_Q(q,p)$ could not satisfy the conditions 1.13, 1.14 and 1.15. The bilinearity condition in the distribution function is attractive but lacks physical motivation [19, 20, 21]. However, this condition has been generalized, and the condition of bilinearity was replaced by the "mixture condition". This property requires that the distribution in phase space should depend only on the operator, and not on the particular basis in which it is represented, as a mixture of any set of pure states. To circumvent these negative results, two approaches have been proposed. The first is the Wigner function that satisfies the mixture properties 1.13, 1.14, but not 1.15. The consequence is that the Wigner function cannot be interpreted as a formal probability density in phase space, since it may show negative values. The second is the Husimi distribution, that we can interpret probabilistically, since it is always positive, but as a downside does not provide clear cut evidence of quantum behavior. Here, we adopt the Wigner approach. We consider $\hat{\rho}:\ H^{\otimes N} \to H^{\otimes N}$, with density space denoted by $D(H^{\otimes N})$. The density operator finds a representation equivalent to the quasi-probability defined over the real symplectic space. With this, we can introduce the Weyl operator, as follows:

$$D(\xi) \equiv \exp(i\hat{x}^T \Omega \xi), \tag{1.16}$$

where $\xi \in \Re^{2N}$, so $\hat{\rho}$ is equivalent to a characteristic function $\chi(\xi)$, that is defined in relation to Weyl operator

$$\chi(\xi) = tr(\hat{\rho} D(\xi)), \tag{1.17}$$

Its Fourier Transform is the Wigner function

$$W(x) = \int_{\Re^{2N}} \frac{d^{2N}\xi}{(2\pi)^{2N}} \exp(-i\hat{x}^T \Omega \xi) X(\xi). \tag{1.18}$$

In the equation above, the continuous variables $x \in \Re^{2N}$ are the eigenvalues of the quadrature operator $\hat{x}$. These variables generate a real symplectic space $\mathbf{K} \equiv (\Re^{\mathbf{2N}}, \blacksquare)$ which is called phase space. The quantum state $\hat{\rho}$ of bosonic systems of N-modes is equivalent to the Wigner function $W(x)$ defined over the phase space $\mathbf{K}$.

The Wigner function is particularly suitable to define a "quantum distribution in phase space", to describe the effects of the observables of quantum theory and statistical physics which have a probability distribution with classical analog. So we need to compute the measurement quantities as mean values and variance of quadratures. For quantum states with no

classical analog, the Wigner function presents negative values [22]. In these cases, $W(x,p)$ is similar to a probability distribution only in the sense that it is normalizable as

$$\int w(\alpha)d^2\alpha = 1, \tag{1.19}$$

being $d^2\alpha = d(Re\alpha)d(Im\alpha) = dxdp$. This equation would furnish the mean value of any operator $\hat{A}$ in the quantum state ρ if the average value could be calculated as

$$\langle\hat{A}\rangle = tr(\hat{\rho}\hat{A}) = \int w(x)A(\alpha)d^2\alpha. \tag{1.20}$$

The marginal distribution for p, $\langle p|\,\rho\,|p\rangle$, is obtained by a change of variables, for example $x-y=u$ and $x+y=v$, by using the Fourier transform and the identity $\int \exp(4iyp)dp = \frac{\pi}{2}\delta(y)$. The normalization of the Wigner function follows from $tr(\hat{\rho}) = 1$. For any symmetric operator, we can define the Weyl operator

$$tr[\hat{\rho}S(\hat{x}^n\hat{p}^m)] = \int w(x,p)x^n p^m dxdp, \tag{1.21}$$

where $\hat{S}(\hat{x}^n\hat{p}^m)$ is the symmetrization operator, e.g. $\hat{S}(\hat{x}^2\hat{p}) = (\hat{x}^2\hat{p}+\hat{x}\hat{p}\hat{x}+\hat{p}\hat{x}^2)/3$ corresponds to $\hat{x}^2\hat{p}$ [23]. However, the quantities that characterize the Wigner function (W or χ) are moments of a quantum statistical distribution. A first moment that is also called the mean value is described by

$$\bar{x} \equiv <\hat{x}> = tr(\hat{x}\rho), \tag{1.22}$$

and the second moment or the covariance matrix V can be defined by arbitrary elements, as

$$V_{ij} = \frac{1}{2} < \{\delta\hat{x}_i, \delta\hat{x}_j\} >, \tag{1.23}$$

where $\delta\hat{x}_i = \hat{x}_i - \langle\hat{x}_i\rangle$ and $\{,\}$ is the anti-commutator. This matrix $2N\times 2N$, is real and symplectic, and satisfies the uncertainty relation in the form

$$V + i\Omega \geq 0. \tag{1.24}$$

Since $V > 0$, we have

$$V(\hat{q}_k)V(\hat{p}_k) \geq 1. \tag{1.25}$$

We can generalize these relations to define all Gaussian states as a bosonic state in which the Wigner function and its characteristic function obey, respectively, the following expressions

$$\chi(\xi) = \exp - \left(\frac{1}{2}\xi^T\left(\Omega V\Omega^T\right)\xi - i\left(\Omega\bar{x}\right)^T\xi\right) \tag{1.26}$$

$$W(x) = \frac{\exp - \left(\frac{1}{2}\left(x-\bar{x}\right)^T V^{-1}\left(x-\bar{x}\right)\right)}{(2\pi)^N\sqrt{\det V}} \tag{1.27}$$

1.2 Symplectic geometry in a nutshell

The structure of the Hamiltonian dynamics can always be seen from a geometric point of view. We will need to work with differentiable manifolds for which the Hamiltonian dynamics is bound in a symplectic structure, i.e. one that has a closed and non degenerated differential 2-form. The main appeal of the symplectic structure is that it is entirely geometric, and for this reason we will briefly study its vectorial space. The geometry brought by bilinear forms is quite rich when compared with symmetric tensors. We can bring the intuition we have from the Euclidean vector space to the symplectic vector space [24]. For 2-forms, the metric creates invariant subsets and quasi-complex structures. We will use the Euclidean set with dimension $\Re^{\mathbf{2N}}$ (local theorems and real manifolds). In this section, we base our approach in Refs. [25, 26, 27].

1.2.1 Symplectic vector space

We consider an open set U in $\Re^{2N}$ to invoke the definition below:

Definition: A symplectic structure is a real, non-degenerate and asymmetric bilinear application defined over the vectors in the U set. Given η and ξ, we have that

$$[\xi,\eta] = -[\eta,\xi], \tag{1.28}$$

together with the condition of non-degenerescence for all η different from zero, in such a way that

$$[\xi,\eta] = 0 \tag{1.29}$$

implies $\eta = 0$. Let the components of a vector v be given by $\{p_1,...,p_n,q_1,...,q_n\}$. We can define a 2-form as

$$\omega = dp_1 \wedge dq_1 + ...dp_n \wedge dq_n \tag{1.30}$$

and impose the relation $[\xi,\eta] = \omega(\xi,\eta)$. This is a non-degenerate form and the determinant of its component is non null. The vector space $\Re^{2N}$ is parametrized by the canonical coordinates **p** and **q**, and therefore we get a symplectic structure. Geometrically, this is the sum of parallelogram areas in the (p_i,q_i) plane. Two vectors are called non orthogonal if their inner product is null. This notion of orthogonality is important when we want to build a geometric structure with the symplectic form like planes, spheres and related. In analogy with Riemannian

geometry, we define an orthogonal complement. As the symplectic application is bilinear, the orthogonal complement will also be a vectorial space.

Definition: The symplectic complement of W contained in U is the set

$$W^{\omega} = \{\eta \, \varepsilon \, U | \, \omega(\xi, \eta) = 0, \, \xi \, \varepsilon W\}. \tag{1.31}$$

The symplectic complement is necessarily transverse to ξ, and so W can be labeled according to the following conditions [28]:

- Isotropic, when $W \subset W^{\omega}$.
- Coisotropic, when $W^{\omega} \subset W$.
- Symplectic, when $W^{\omega} \cap W = \{0\}$.
- Lagrangian, when $W = W^{\omega}$.

This means W will be isotropic iff ω is null in W. For each physical system, we can explore one of these conditions. Throughout this text, we will only use the third condition.

Lemma: For any subspace $W \subset U$ the following definitions are valid

$$\dim W + \dim W^{\omega} = \dim V, \tag{1.32}$$

and

$$W = W^{\omega}. \tag{1.33}$$

This lemma is very important for several reasons. One of them is that the isomorphism between the symplectic vector space and its dual is built. This connection is physically important because all dynamic structures of the Hamiltonians is based on this. The lemma helps us to build an intuition about the geometry of the symplectic spaces. For physical systems, we can interpret this geometric intuition through the concept of the Hamiltonian flux that permeates the whole of classical mechanics. In short, there are always two non-orthogonal vectors. If we fix η, the dimension of its complement is $2n-1$. This is employed to define the symplectic basis.

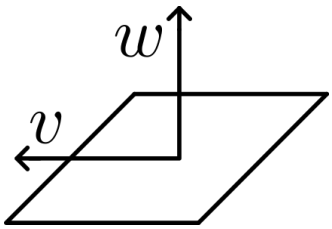

Figure 1.1: Orthogonal complement of **v** and **w**.

Definition: A symplectic basis is a set of $2n$ vectors, $\mathbf{e}_{pa}$ and $\mathbf{e}_{qa}$, for which $a = 1,...,n$. So

$$[\mathbf{e}_{pa}, \mathbf{e}_{pa}] = \left[\mathbf{e}_{qa}, \mathbf{e}_{qa}\right] = 0 \tag{1.34}$$

with

$$\left[\mathbf{e}_{pa}, \mathbf{e}_{qa}\right] = 1 \tag{1.35}$$

We have that all the basis vectors are orthogonal to all the others except one. The next result is the most important about symplectic spaces that we will show. The theorem shows us that we should only think in terms of linear transformations applied to the basis.

Theorem: All symplectic structures allow a symplectic basis, and any non null vector can be taken as basis element.

1.2.2 Symplectic group

Orthogonal transformations preserve the Euclidean structure. Analogously, symplectic transformations are defined as those which preserve the symplectic structure.

Definition: The linear transformation $S: \Re^{2N} \to \Re^{2N}$ of the symplectic space in itself is called symplectic if it satisfies

$$[\mathbf{S}\xi, \mathbf{S}\eta] = -[\xi, \eta] \tag{1.36}$$

for all vectors in $\Re^{2N}$. The set of all the symplectic transformations in $\Re^{2N}$ is the symplectic group, represented by $S_p(2n)$. Being $\mathbf{S}$ a linear transformation, the composition of two symplectic transformations will also be symplectic. For the structure displayed in Eq. 1.36, we say that $\mathbf{S}$ should preserve the differential form ω, which is the sum of the parallelogram area in the planes defined by the symplectic vectors. Thus we have a volume element in $\Re^{2N}$ to all exterior power of ω, which implies $\det \mathbf{S} = \mathbf{1}$. So we have

$$vol = \underbrace{\omega \wedge ... \wedge \omega}_{n \text{ times}}. \tag{1.37}$$

Theorem: A linear transformation $\mathbf{S}: \Re^{2N} \to \Re^{2N}$ is symplectic iff it takes a symplectic basis to another symplectic basis. For a symplectic basis $\{\mathbf{e}_{pa}, \mathbf{e}_{qb}\}$, the component of ω is

$$\{\mathbf{e}_{pa}, \mathbf{e}_{qb}\} = \omega_{ab}, \tag{1.38}$$

satisfying

$$\mathbf{S}_c^a \omega_{ab} \mathbf{S}_d^b = \omega_{cd}, \tag{1.39}$$

by the action of this transformation, we can write the equations more easily in matrix form using Eq. 1.30 for $n = 1$. Following the same procedure of Ref.[29], Eq.1.39 can be represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.40}$$

The matrix coefficients must obey $\mathbf{detS} = \mathbf{1}$ and there are no fixed parameters. In $\Re^{2N}$ the symplectic form is exactly the volume element, which implies the following equation

$$S_p(2, \mathbb{R}) = S_L(2, \mathbb{R}). \tag{1.41}$$

For higher dimensions, in general, $S_p(2n)$ is contained in $S_L(2n)$ as proper subset.

1.2.3 Local symplectic invariants of Gaussian states

Now that we saw the basic definitions about the phase space and the symplectic group, let us deepen our understanding of the role of Gaussian states and operations in quantum information theory. We have that a class of transformations known in analytical mechanics makes the connection between unitary transformations U and its counterpart at the phase-space, this is called the symplectic transformation. For Hamiltonian equations of motion, we have an arbitrary vector $\mathbf{X} = (q_1, p_1, \ldots, q_n, p_n)^T$ that has the form in the symplectic formalism

$$\dot{\mathbf{X}} = \sigma_{ij} \partial_j H. \tag{1.42}$$

The transformation of coordinates ($\mathbf{X} \to S\mathbf{X}$) leaves the form of equation invariant iff $S\sigma S^T = \sigma$. Thus, we can start to define the symplectic group.

Definition: The symplectic group $S_p(2, \mathbb{R})$ is the set of $2n \times 2n$ real matrices S that satisfy

$$S\sigma S^T = \sigma. \tag{1.43}$$

The symplectic group $S_p(2,\mathbb{R})$ has dimension $n(2n+1)$ and its elements are called canonical or symplectic transformations. In quantum mechanics, the elements of $S_p(2,\mathbb{R})$ are preserved due to the relations of commutation, and all modes transformation are generated by linear and bilinear symplectic interactions. The relation between the unitary and symplectic transformations is a consequence the *Stone - von Neumann's Theorem*: every transformation S in the $2n$ dimensional of phase space $\rho = \mathbb{R}^{2n}$ has its counterpart at the Hilbert space level I via an unitary transformation U[30].

In Tab. 1.1, the main differences between the description of physical states in the Hilbert space and in the phase space are summarized.

Property	Hilbert space H	Phase space Γ
dimension	∞	$2N$
structure	$\otimes$	$\otimes$
description	ρ	$\mathbf{d}, \sigma$
bona fide	$\rho \geq 0$	$\sigma + i\Omega \geq 0$
Unitary operations	$\hat{U}\|\hat{U}^\dagger\hat{U} = \hat{I}$ $\rho \to \hat{U}\rho\hat{U}^\dagger$	$\mathbf{S}\|\mathbf{S}\Omega\mathbf{S}^{\mathrm{T}} = \Omega$ $\mathbf{d} \to \mathbf{S}\mathbf{d}$, $\sigma \to \mathbf{S}\sigma\mathbf{S}^{\mathrm{T}}$
spectra	$\hat{U}\rho\hat{U}^\dagger = \underset{0\leq\lambda_j\leq 1}{diag}\ \{\lambda_j\}_{j=1}^{\infty}$	$\mathbf{S}^{\mathrm{T}}\sigma\mathbf{S} = \underset{0\leq\nu_k\leq 1}{\mathrm{diag}}\ \{(\nu_k,\nu_k)\}_{k=1}^{N}$
pure states	$\lambda_i = 1$, $\lambda_{j\neq i} = 0$	$\nu_k = 1$, $\forall k = 1,...,N$
purity	$tr\rho^2 = \sum_j \lambda_j^2$	$1/\sqrt{\det\sigma} = \prod_k \nu_k^{-1}$

Table 1.1: Comparison between Hilbert space and phase space, in the case of N mode Gaussian states. We have that $\mathbf{d}$ is the first moment and σ is the second moment. Table reproduced from Ref.[18].

The quasi-classical states (coherent states) of the field can be obtained through the displacement operator of the vacuum state, as we will describe in the next section. This operator shifts the vacuum to another point in the phase space. We know that the covariance matrix is the same for the vacuum: $V_{ij} = \frac{1}{2}\delta_{ij}$. Thus, all displacement operators are complete, so every operator O on H can be written as

$$\mathscr{O} = \int_{C^n} \frac{d^{2n}\lambda}{\pi} tr[\mathscr{O}\mathscr{D}(\lambda)]D^\dagger,(\lambda) \tag{1.44}$$

as derived by Glauber. The characteristic function of operator $\mathscr{O}$ is

$$\chi[\mathscr{O}](\lambda) = tr[\mathscr{O}D(\lambda)]. \tag{1.45}$$

Definition: A state ρ of a canonical system with $2n$ degrees of freedom is Gaussian iff its characteristic function is Gaussian, i.e.

$$\chi[\rho](\lambda) = \chi[\rho](0)e^{-\lambda^T \Sigma \lambda + d^T \lambda}, \tag{1.46}$$

when Σ is a matrix 2n x 2n and $d\ \varepsilon\ \mathbb{R}^{2n}$. From the definition above, we can characterize the Gaussian states by its first- and second-order momenta. The operators $\mathscr{O}$ and the covariance matrix are given by

$$d = \sigma \mathrm{tr}[\mathscr{O}\rho], \tag{1.47}$$

$$\sum = \sigma^T V \sigma. \tag{1.48}$$

The quantum correlation is given by the second-order momenta. These criteria are quite important for the study of entanglement and separability of Gaussian states. Then, we need to invoke an important theorem, that is quite used in Bogoliubov's transformation for diagonalization of quadratic Hamiltonians:

Williamson's Theorem: Given a $V\,\varepsilon\, M(2n,\mathbb{R})$ satisfy $V^\dagger = V$ and $V > 0$, there is a symplectic transformation $S\,\varepsilon\, S_p(2n,\mathbb{R})$ and the diagonal matrix $D\,\varepsilon\, M(n,\mathbb{R})$ positive definite which can be represented in the following way [31]:

$$V = S^T \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} S. \tag{1.49}$$

Matrices S and D are unique, apart from permutations of elements D. The eigenvalues of D are called symplectic eigenvalues. Thus, we can generate a Gaussian state ρ through the unitary transformation U_S associated with the symplectic matrix S that is given for

$$\rho = U_S \Xi U_S^\dagger, \tag{1.50}$$

where Ξ is a thermal state, a consequence of the Williamson's Theorem. Let us consider now the particular case of two-mode Gaussian states, represented by the vector of quadratures in phase space $\mathscr{O}_{\mathscr{A}\mathscr{B}} = (X_A, P_A, X_B, P_B)^T$. We can write the covariance matrix in blocks of 2×2 matrices as

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{1.51}$$

where A, B, and C are 2×2 matrices that contain the information about the local states A and B and the correlation C between the two subsystems. Local invariants with respect to $S_p(2,\mathbb{R}) \otimes S_p(2,\mathbb{R})$ are derived considering the action of the generic local sympletic transformations $S_A \ominus S_B$ and $S_A,\ S_B\ \varepsilon\ S_p(2,\mathbb{R})$ over V, as shown below

$$A \to S_A A S_A^{\dagger} B \to S_B B S_B^{\dagger} C \to S_A C S_B^{\dagger}. \tag{1.52}$$

The determinant of all blocks will not change under the action of symplectic transformation $S_p(2,\mathbb{R}) \otimes S_p(2,\mathbb{R}) \subset S_p(4,\mathbb{R})$, since $detA, detB, detC, detV$ are symplectic invariants. The theorem allows us to diagonalize the matrices A and B suitable of S_A and S_B, so that we have

$$S_A A S_A^{\dagger} = D_A = a\mathbb{1}_2, \tag{1.53}$$

$$S_B B S_B^{\dagger} = D_B = b\mathbb{1}_2. \tag{1.54}$$

Lastly, we make a final simplification of Eq. 1.40 by noting that matrices C and C^T are real matrices 2×2 that admit diagonalization by a proper choice of orthogonal matrix $\mathscr{O}_{\mathscr{A}\mathscr{B}}$, so

$$V = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix}. \tag{1.55}$$

The covariance matrix is then in the final form, with three symplectic invariants that have as feature: $\det A = a^2$, $\det B = b^2$ and $\det C = cd$ [32, 30]. A practical form of obtaining eigenvalues of the covariance matrix is given by

$$\sqrt{2}d_{\pm} = \sqrt{\sum[V] \pm \sqrt{(\sum[V])^2 - 4detV}}, \tag{1.56}$$

with $\sum[V] = \det A + \det B + 2\det C$. We have that $d_{\pm}$ are eigenvalues of D that follows the Williamson's form of Eq. 1.55 being simplified to

$$V = S^T \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix} \oplus \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix} S. \tag{1.57}$$

1.3 Bogoliubov Transformation

In a few words, the Bogoliubov transformation is a change of basis from a set of modal solutions to another basis that preserve the commutation relations (or anti-commutation) of field operators. In general, is it a unitary transformation from one field modal basis to another.

1.3.1 Linear unitary Bogoliubov orthogonal (LUBO)- stationary scenarios

In the context of Gaussian transformations, we can make linear transformations for an arbitrary quadrature **M**, where

$$\mathbf{M} : x \rightarrow \mathbf{M}x = x'. \tag{1.58}$$

For this transformation to be physical, it must preserve the commutation relations of Eqs.1.2 and 1.9. This restriction can be described as:

$$M\Omega M^T = \Omega. \tag{1.59}$$

Since Ω is the symplectic form, this condition implies that the elements of **M** belong to the real symplectic group $S_p(2N, \Re)$. We can construct an unitary operator or diagonalize the Hamiltonian using a linear transformation by determining the unitary $\mathbf{U_{2N\times 2N}}$ such that $U^\dagger xU = \mathbf{Mx}$. Then we obtain the transformation of the quantum state as $\rho \rightarrow U\rho U^\dagger$.

These unitary operators are refereed to as *linear unitary Bogoliubov operations* (**LUBO**). The evolution of the quantum states proceeds via $U \sim e^{iH}$, where H is an unitary Gaussian operator. These operators are generators of quadratic Hamiltonians in the quadrature operators, that is, they generate the symplectic group. In its simplest case, quadratic functions do not have second order terms, as in the case of operations that only use linear optics. However, if we want to add non linearities to the Hamiltonian we will obtain mode mixing. The second order terms of the Hamiltonian are generated by a mixture of ladder operators. For these terms, the use of LUBO becomes necessary[33].

1.3.2 Diagonalization of quadratic Hamiltonian

A quadratic Hamiltonian (or bilinear) can be always diagonalized, meaning that it is possible to reduce it to a linear combination of number operators. A quadratic Hamiltonian is implemented for non-interacting systems, and also in the context of a mean field approximation. We can try to verify if the number conservation is obeyed by a quadratic Hamiltonian. We find that even with a change of basis the number operator $\hat{n}$ is conserved. A quadratic Hamiltonian is written as

$$H = \sum_{ij} H_{ij} a_i^\dagger a_j, \tag{1.60}$$

where H_{ij} are real numbers, and

$$a_l^\dagger = \sum_i a_i^\dagger U_{il}. \tag{1.61}$$

Inverting this relation, we have

$$\sum a_l^\dagger \left(u^\dagger\right)_{lj} = a_j, \tag{1.62}$$

Considering the Hermitian conjugate, we have

$$\sum_l U_{jl} a_l = a_j \tag{1.63}$$

Replacing the ladder operators by Bogoliubov coefficients, we find

$$H = \sum_{lm} \alpha_l^\dagger \left(U^\dagger H U\right)_{lm} \alpha_m = \sum_n \varepsilon_n \alpha_n^\dagger \alpha_n = \sum_n \varepsilon_n \hat{n}_n \tag{1.64}$$

Thus demonstrating the conservation of the number of particles.

1.3.3 Bosonic example

A number of physically important systems, when treated approximately, give rise to bilinear Hamiltonians involving two modes. An example is the Hamiltonian of trapped ion-laser interaction with two modes of ionic vibration. This Hamiltonian can be diagonalized by Bogoliubov transformations which mix the creation and annihilation operators, but always preserve the commutation relations.

Now, we can consider a simple application of the *Williamson Theorem* seen in Sec. **??**. Additionally, we illustrate the **LUBO** transformation applied in the last section as described by the ransformation of Gaussian operators. We start with the description of our bilinear Hamiltonian,

$$H = \varepsilon \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}\right) + \lambda \left(\hat{a}^\dagger \hat{b}^\dagger + \hat{b}\hat{a}\right). \tag{1.65}$$

We consider the transformation

$$\hat{a}^\dagger = u\hat{d}_1^\dagger + v\hat{d}_2, \tag{1.66}$$

$$\hat{b}^\dagger = u\hat{d}_2^\dagger + v\hat{d}_1, \tag{1.67}$$

to which we impose the commutation relations,

$$\left[\hat{a}^\dagger, \hat{b}^\dagger\right] = 0. \tag{1.68}$$

We then find the condition

$$\left[\hat{a}, \hat{a}^\dagger\right] = u^2 \left[\hat{d}_1, \hat{d}_1^\dagger\right] - v^2 \left[\hat{d}_2, \hat{d}_2^\dagger\right] = 1, \tag{1.69}$$

and thus the requirement that $u^2 - v^2 = 1$. The Bogoliubov transformation can be also parametrized by $u = \cosh\theta$ and $v = \sinh\theta$. Introducing the matrix notation as enunciated in the *Williamson Theorem*, we have

$$H = \frac{1}{2}\begin{pmatrix} \hat{a}^\dagger\hat{b} & \hat{b}^\dagger\hat{a} \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda & 0 & 0 \\ \lambda & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & \lambda \\ 0 & 0 & \lambda & \varepsilon \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \\ \hat{b} \\ \hat{a}^\dagger \end{pmatrix} - \varepsilon. \tag{1.70}$$

Where for bosons we need to use the commutators and replace in Eq. 1.70. Selecting the block (2×2) as below,

$$\begin{pmatrix} \hat{a}^\dagger & \hat{b} \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & \varepsilon \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix}, \tag{1.71}$$

we describe the Bogoliubov transformation in matrix form as

$$\begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \hat{d_1} \\ \hat{d_2}^\dagger \end{pmatrix}. \tag{1.72}$$

Replacing the ladder operators by the transformation given in Eqs. 1.66 and 1.67, we have

$$\begin{pmatrix} \hat{d_1} & d_2 \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & \varepsilon \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \hat{d_1} \\ \hat{d_2} \end{pmatrix}. \tag{1.73}$$

Detailing the calculation, we have

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & \varepsilon \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix} = \begin{pmatrix} \varepsilon\left[u^2+v^2\right] + 2\lambda uv & 2\varepsilon uv + \lambda\left[u^2+v^2\right] \\ 2\varepsilon uv + \lambda\left[u^2+v^2\right] & \varepsilon\left[u^2+v^2\right] + 2\lambda uv \end{pmatrix}, \tag{1.74}$$

by using the relations $u^2 + v^2 = \cosh 2\theta$ and $2uv = \sinh 2\theta$. Thus, since $\tanh 2\theta = -\lambda/\varepsilon$, we find

$$H = \hat{\varepsilon}\left(\hat{d_1}\hat{d_1} + \hat{d_2^\dagger}\hat{d_2}\right) - \varepsilon + \bar{\varepsilon}. \tag{1.75}$$

We note that for $\varepsilon > \lambda$, we have that H is a Hamiltonian with a normal mode of oscillation in unstable equilibrium.

1.4 Examples of Gaussian states

The study of entanglement in the CV regime finds many difficulties in systems of infinite dimension, but a large simplification is possible by restricting it to the set of "Gaussian states".

This set is relevant because it is what we get most often in the classical limit, and appears multiple times in mean-field theories. This states are probably the movasive members of this class. Gaussian states form the most important class of states of the harmonic oscillator, since thermal and coherent states are both of this type [34]. They have the special property of being kept as Gaussian under the action of linear optical devices such as beam splitters and phase shifters. They can also be employed for teleportation and quantum error correction [35].

1.4.1 Vacuum and thermal states

Thermal states are a subclass of Gaussian states. Bosonic thermal states can be defined as those which maximizes the von Neumann entropy, given by

$$S = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho}), \tag{1.76}$$

for fixed mean energy. We have $tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}) = \bar{n}$, where $\bar{n} \geq 0$ is the mean number of phonons in the bosonic mode. The representation of a thermal state in terms of number states is given by

$$\hat{\rho}^{th}(\bar{n}) = \sum_{n=0}^{+\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} |n\rangle \langle n| . \tag{1.77}$$

Their Wigner function is Gaussian, with null quadrature average and covariance matrix given by $V = (2\bar{n}+1)I$ where $\mathbb{1}_{2\times 2}$.

1.4.2 Displacement and coherent states

Coherent states are well known in quantum optics, due to the facts that they describe the statistical properties of a laser light source and are invariant when interacting with linear optical elements. They are used in others branches of physics for their properties. The coherent states $|\alpha\rangle$ are defined as eigenstates of ladder operators,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle . \tag{1.78}$$

Expanding this relation in the Fock basis, $\{|n\rangle\}$(with $n \,\varepsilon\, \mathbb{N}_0$), in the both sides of Eq. 1.78, we have a recurrence relation whose solution is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n\,\varepsilon\,\mathbb{N}_0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{1.79}$$

Using the BCH relation, we obtain that coherent states can be understood as displacements of the oscillator ground state,

$$|\alpha\rangle = D(\alpha)|\alpha\rangle = e^{\alpha a^{\dagger} - \alpha * a}|0\rangle , \tag{1.80}$$

where $a \to a+\alpha$, and $D(\alpha)$ is not related to the Weyl's operator. This operation is equivalent to Bogoliubov's transformation corresponding to a displacement $a \to a+\alpha$ or a displacement in the quadrature $\mathbf{x} \to \mathbf{x}+(\mathbf{p},\mathbf{q})^{\mathbf{T}}$.

Below we list the main properties of the displacement operator and coherent states.

- In the Heisenberg representation, a displacement α of the annihilation operator

$$D^{\dagger}(\alpha)aD(\alpha) = a+\alpha. \tag{1.81}$$

- The concatenated action of two displacement operators displaces the vacuum as

$$D(\alpha)D(\beta) = e^{iIm(\alpha\beta*)}D(\alpha+\beta). \tag{1.82}$$

- The coherent states are not orthogonal

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha*\beta} \tag{1.83}$$

- They make an over-complete basis of the quantum oscillator Hilbert space

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n\,\varepsilon\,\mathbb{N}_0} |n\rangle\langle n| = 1. \tag{1.84}$$

1.4.3 Phase rotation

Of the most important operations in linear optics, the phase rotation is perhaps the Gaussian unitary operator which is more physically intuitive. This operation can be realized in this context with a highly refractive medium. This gives rise to a difference of phase when compared to fixed local harmonic oscillator. Mathematically, the phase rotation is generated by the Hamiltonian $H = 2\theta a^{\dagger}a$ which generates the Gaussian operator

$$R(\theta) = \exp(-i\theta a^{\dagger}a). \tag{1.85}$$

Using a **LUBO**, this transformation maps $a \to \exp i\theta' a$ for a single mode. Thus, this mode transforms in phase space as

$$\mathbf{x} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mathbf{x}. \tag{1.86}$$

1.4.4 Beam splitter

The beam splitter optical element is quite useful in quantum optics. Its Hamiltonian contains the terms of form $\hat{a}^\dagger\hat{b}$ and $\hat{a}\hat{b^\dagger}$. We note it promotes the coherent combination between the modes $\hat{a}$ and $\hat{b}$, giving rise to a Gaussian unitary represented by

$$B(\theta) \rightarrow \exp(\theta(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)), \tag{1.87}$$

like a **LUBO**. The ladder operators of two modes are submitted to a linear combination by the expression

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \rightarrow \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.88}$$

Considering the four quadrature operators ($\mathbf{x} = (q_1, p_1, q_2, p_2)$) this relation becomes

$$\mathbf{x} \rightarrow \begin{pmatrix} \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \\ -\cos\theta & 0 & \sin\theta & 0 \\ 0 & -\cos\theta & 0 & \sin\theta \end{pmatrix} \mathbf{x} = \begin{pmatrix} \sin\theta\mathbb{I} & \cos\theta\mathbb{I} \\ -\cos\theta\mathbb{I} & \sin\theta\mathbb{I} \end{pmatrix} \mathbf{x}. \tag{1.89}$$

1.4.5 Single-mode squeezed state

As seen previously, Gaussian operators originate from operations of linear optics. A notorious example of entanglement in **CV** systems arise in the form of squeezed states [36].

Definition: *Squeezing* is a process from which the variance of one quadrature in a particular direction in phase space is decreased and, as a consequence, its conjugated direction is increased with respect to the vacuum noise. We describe in the following two types of squeezing transformations, acting on one mode or two modes [23].

An example of a physical phenomenon capable of producing single mode squeezing is degenerate parametric down conversion. This is a process whereby photons with given energy pass through a non-linear crystal and are converted into two photons with equal energy and momentum. Since pairs of photons are generated, the interaction Hamiltonian must contain two creation operators for a single mode (e. g. $\hat{a}^{\dagger^2}$) and its Hermitian conjugate. We assume that the input pump is a classical field, e.g.,$\alpha_{pump} = ke^{i\theta}$, where k and θ are real numbers representing the field strength and phase. So the Hamiltonian is given by

$$H = ik(\hat{a}^{\dagger^2}e^{i\theta} - \hat{a}^2e^{-i\theta}). \tag{1.90}$$

The evolution operator associated with the Hamiltonian above represents a single-mode squeezing operator, given by

$$S(r) = \exp\left(\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)\right), \tag{1.91}$$

where $r = kt$ is the squeezing parameter. In the Heisenberg picture, we obtain a **LUBO** in the form

$$\hat{a} \to \hat{a}\cosh r - \hat{a}^\dagger \sinh r, \tag{1.92}$$

whereby the quadrature observables become

$$\begin{pmatrix} p' \\ q' \end{pmatrix} \to \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{1.93}$$

1.4.6 Two-mode squeezed states

The two-mode squeezer is implemented optically through a non-degenerate optical amplifier. Being that the main difference with respect to the single mode case is the interaction between distinct modes, the generalization to two modes results in the Hamiltonian

$$H = 2i\kappa(\hat{a}^\dagger \hat{b}^\dagger e^{i\theta} - \hat{a}\hat{b}e^{i\theta}), \tag{1.94}$$

where the Gaussian operator corresponding to the two-mode squeezer is

$$S_2(r) = \exp(\frac{r}{2}(\hat{a}\hat{b} - \hat{a}^\dagger \hat{b}^\dagger)). \tag{1.95}$$

This unitary acts over the quadratures $\mathbf{x} = (q_a, p_a, q_b, p_b)$ as

$$\mathbf{x} \to \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix} = \begin{pmatrix} \cosh r\mathbb{I} & \sinh r\sigma_z \\ \sinh r\sigma_z & \cosh r\mathbb{I} \end{pmatrix} \mathbf{x}, \tag{1.96}$$

where $\sigma_z = diag(1,-1)$ is the Pauli matrix and $\mathbb{I}$ is a identity matrix, both with dimension (2×2). The **LUBO** associated with the two-mode squeezing transformation affects the field operators as

$$\hat{a} \to \hat{a}\cosh r + \hat{b}^\dagger \sinh r, \tag{1.97}$$

$$\hat{b} \to \hat{b}\cosh r + \hat{a}^\dagger \sinh r. \tag{1.98}$$

As the two-mode squeezing transforms a phase space composed of four quadrature observables, two of them become squeezed and their conjugates are anti-squeezed. For example, if

squeezing takes place in the directions of the quadratures $q_a - q_b$ and $p_a + p_b$, the anti-squeezed quadratures are $q_a - p_b$ and $p_a - q_b$. Applying $S_2(r)$ to a pair of vacuum modes, we obtain the EPR state as an important class of Gaussian states,

$$S_2(r)\,|0\rangle = |r\rangle_{EPR} = \sqrt{1-\tanh^2 r}\sum_{n=0}^{\infty}(-\tanh r)^n\,|n\rangle_a\,|n\rangle_b\,. \tag{1.99}$$

Considering $\bar{\mathbf{x}} = 0$ (mean value), we have

$$\mathbf{V}_{EPR} = \begin{pmatrix} \cosh 2r\mathbb{I} & \sinh 2r\sigma_{\mathbf{z}} \\ \sinh 2r\sigma_{\mathbf{z}} & \cosh 2r\mathbb{I} \end{pmatrix}. \tag{1.100}$$

These states correspond to the Einstein-Poldoski-Rosen (EPR) state in the limit of infinite amount of squeezing, producing two-mode entanglement.

Chapter 2
Quantum logic using trapped ions

> "Science knows no country, because knowledge belongs to humanity, and is the torch which illuminates the world."
>
> Louis Pasteur

2.1 Engineering quantum gates

What is a quantum computer? Before answering this question, we need to consider that a classical computer is a machine that executes logical operations over binary numbers, represented by 0s or 1s. In quantum logic, the basic unit of information becomes instead the quantum bit, also called qubit – a two-level quantum system that can assume any superposition of the logical basis states $|0\rangle$ and $|1\rangle$. The usual physical realization of a quantum computer consists of several qubits in which the elementary logical operations, the quantum gates, are performed in sequence [37].

The high degree of experimental control achieved with trapped ions makes them very attractive candidates for the implementation of quantum logic. This idea was initially proposed by Ignacio Cirac and Peter Zoller in 1995 [38], where a string of trapped ions would be cooled down to near the motional ground state to perform quantum gate operations. The ions would then be manipulated with lasers to perform quantum gate operations. In addition to quantum computation, the CZ (Cirac-Zoller) proposal finds important applications in spectroscopy and time keeping [39].

One important concept in quantum computation is that of *entanglement*. The entangled states are based on the onset of strong correlations between two or more quantum systems. Entanglement means that the quantum state of the system cannot be split into a product quantum

states describing subsets of the system without losing information. For instance, a well known entangled state of two qubits is the Bell singlet state

$$\Phi = \frac{|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\sqrt{2}}. \tag{2.1}$$

An entangled state has the counter-intuitive property that measurements realized on one of the particles seem to bring to collapse the state of the other particle – instantaneously. Although information cannot be transmitted by this apparent collapse (i.e. local measurements on one particle cannot be used to infer whether local measurements have been performed on the other particle), there are consequences of this kind of effect to the interpretation of quantum theory with respect to its non-locality and/or non-realism.

In a seminal paper from 1935, Einstein, Poldosky, and Rosen (EPR) [36] considered a thought experiment in which two non-commuting observables could be alledgedly measured using such kind of quantum states, hence concluding that quantum mechanics should be considered incomplete. To determine from experiments the compatibility of quantum mechanics with factual data, Bell developed an inequality capable of distinguishing between the predictions of classical-like theories (local/realist hidden variables theories) and those of quantum mechanics. This inequality and a class of others can be violated by observables described as in quantum mechanics, but will be obeyed by the so-called "hidden variables" theories, a concept that was originated in the EPR paper. This fact unleashed a big interest in the fundamentals of entangled states and quantum non-locality in a wide variety of physical systems such as trapped ions and single photons [40, 41, 42].

The possibilities to implement quantum logic and produce entangled states with trapped ions make this system a very interesting experimental platform for investigations in quantum information. This is a field in rapid progression where important proof-of-principle experiments have been realized, especially in the research group of David Wineland in NIST (Boulder, USA), and in the group of Rainer Blatt (Innsbruck, Austria) with the first demonstrations of gates with two qubits [43], the realization of simple quantum algorithms, the generation of entangled states of many particles [40, 44], and the violation of fundamental inequalities [45]. There are tremendous possibilities within this field, which several groups around the world are working hard to explore [46].

2.1.1 Why ‘qubit’?

[1]

Twenty years ago was a great year for quantum computation: as we already know, 1995 was the year when Cirac and Zoller proposed their quantum computation platform using the ion trap and Wineland’s group implemented it for the first time. Simultaneously, quantum error correction codes were invented by Shor and Steane, entanglement concentration and purification protocols were described by Bennett, besides many other developments made this year so surprising for the quantum world.

Twenty years ago nobody spoke the word ‘qubit’ instead of ‘two-level quantum system’. The advent of this new word brought, besides a somewhat large amount of saved time and breath, a powerful message: a quantum state can be used to encode a new kind of information.

Ben Schumacher, in his article *‘Quantum Coding’* [47], was the first to introduce the word ‘qubit’. In the acknowledgments of this article, the author talks about the origin of the word:

“The term ‘qubit’, was coined in jest during one of the author’s many intriguing and valuable conversations with W. K. Wootters, and become the initial impetus for this work”

Recently, the theoretical physicist Paul Ginsparg, inspired in the post of the blog *Quantum Frontiers* signed by the theoretical physicist John Preskill, did a plot of the word ‘qubit’ in papers posted on the arXiv, reproduced in Fig. 2.1 This discussion was strongly based in the

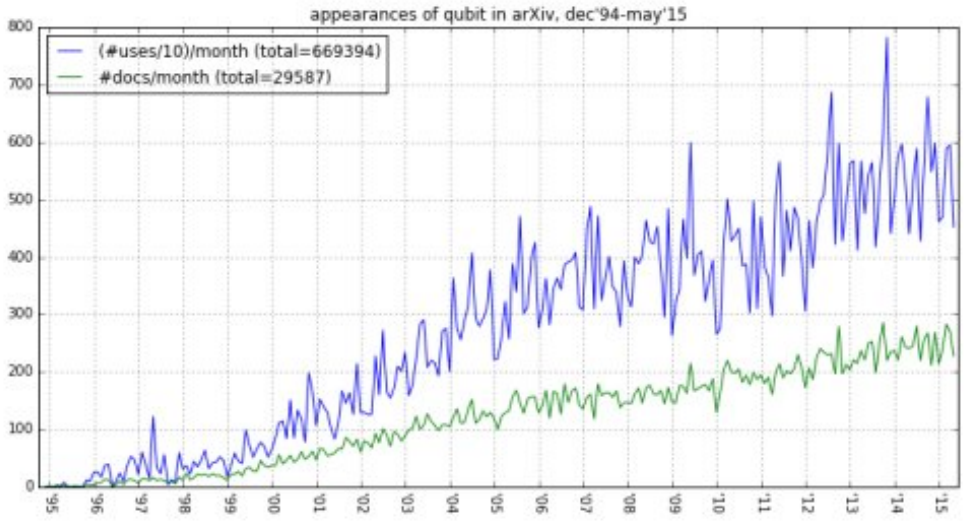

Figure 2.1: The figure, we have: i) uses of qubit in arxiv (divided by 10) per month, and a total of documents per month:an impressive 669394 total in 29587 documents. Credits to Paul Ginsparg.

[1] David Mermin coined the word ’qbit’, but sadly the term that has been mostly used is the ‘qubit’.

blog Quantum Frontiers. Happy birthday, young "qubit"!

2.1.2 Single-qubit gates

Every quantum algorithm can be broken into a sequence of single-qubit operations and two-qubit entangling operations, such as e.g. a conditional phase gate, a controlled NOT gate or a SWAP gate [48].

In the ion trap, operations with a single qubit are the easiest ones to implement in order to achieve universal computation. In fact, the Rabi oscillation between the two energy levels of the qubit implement a single qubit rotation. We may describe the effect of resonant radiation through the gate $R^C(\theta,\phi)$ acting over state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ [37] as

$$R^C(\theta,\phi) = \exp\left(\frac{i\theta}{2}\left(e^{i\varphi}\sigma_+ + e^{-i\varphi}\sigma_-\right)\right) = I\cos\left(\frac{\theta}{2}\right) + i\left(\sigma_x\cos\varphi - \sigma_y\sin\varphi\right)\sin\frac{\theta}{2} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & ie^{i\varphi}\sin\frac{\theta}{2} \\ ie^{-i\varphi}\sin\frac{\theta}{2} & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2.2}$$

These operations acting on a single qubit can be better visualized in the Bloch sphere, where they perform rotations of the spin vector. Any linear combination of Pauli matrix operations can be implemented by a sequence of laser pulses with the appropriate relative phases.

2.1.3 Two-qubit gates

The CNOT gate involves two qubits and thus acts on the superposition

$$|\psi\rangle = \alpha|1\rangle|1\rangle + \beta|1\rangle|0\rangle + \gamma|0\rangle|1\rangle + \delta|0\rangle|0\rangle \qquad \alpha,\beta,,\gamma,\delta\,\varepsilon\,\mathbb{C} \tag{2.3}$$

The truth table 2.1 of the CNOT gate is the result of its operation on the vectors of the logical basis, given by tensor products of the 'control qubit' and the 'target qubit', respectively. An

Input		**Output**	
Control	Target	Control	Target
1	1	1	1
1	0	1	0
0	1	0	0
0	0	0	1

Table 2.1: Truth table for the CNOT gate.

alternative realization of a conditional operation relies on the controlled phase gate, which adds

a π phase to one of the basis states. The CNOT-gate and the Controlled-Z gate are equivalent up to a local rotation of the target qubit, as we can see by the action of the C-Z gate in the rotated basis $|0\rangle$,$|1\rangle$

$$\underbrace{\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}}_{CNOT} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R(\frac{\pi}{2}),target\,qubit} \underbrace{\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix}}_{Controlled-Z} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R(-\frac{\pi}{2}),target\,qubit} \tag{2.4}$$

Thus the CNOT gate acting on the logical basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ is equivalent to the transformation produced by the C-Z gate on the alternative logical basis $\{|0\rangle, |1\rangle\}$, since the two basis differ only by local qubit rotations. Quantum gates and the preparation of entangled states are related issues, since with the ability to make the CNOT gate, one is also able to prepare entangled states. For instance, we find that

$$CNOT\left\{\left[\cos\left(\frac{\theta}{2}\right)|-\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|+\rangle\right]\otimes|-\rangle\right\} = \cos\left(\frac{\theta}{2}\right)|-\rangle|-\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|+\rangle|+\rangle, \tag{2.5}$$

is an entangled state for a suitably chosen angle θ. The preparation of these states for trapped ions is discussed in the next sections. To achieve a set of universal gates, we must obtain complementary operations through one and two qubit operations. These operations are the most important for quantum computation and allows for the generation of entanglement among qubits. Through the combined operations of one qubit or more qubits, we can implement any important operation [49]. In the next sections, we will describe briefly the Cirac-Zoller gate and the Mølmer - Sørensen gate as examples of two-qubit gates.

2.2 Quantum information in the ion trap

Trapped ions can be seen as the experimental realization of an ideal system composed of 'atoms in a box' [50]. In this physical system, collective modes of vibration of an ensemble of ions are kept confined in space by a strong electromagnetic field which behaves like a virtual box. The collective modes of the ionic oscillators, together with the electronic transition between two internal energy levels of each ion (the qubit), can be manipulated by an external source of coherent radiation, usually laser light.

The method employed to manipulate trapped ions is conceptually similar to methods used in cavity quantum electrodynamics (**CQED**) mainly because there are many similarities in the equations of motion. While the qubit is represented in both types of systems by internal energy levels of an atom, the modes of **CQED** stand as quantized electromagnetic fields confined to an optical cavity, whereas trapped ions offer their vibrational modes as **CV** degrees of freedom. On the conceptual level ,the **CQED** Hamiltonian is more fundamental than the one of a trapped ion, since in the atomic case the modes result from a quadratic approximation of the confining potential.

The spin-spring interaction was well studied by Haroche [50], and in the ion trap it is more subtle and less direct than in the **CQED**. On the other hand, in ion trapped systems we have a large variety of effects and a wealth of details with respect to the possible coupling that can always be improved with changes in the parameters of the laser, which in **CQED** can not occur. In particular, since energy conservation is not required in the interaction between qubit and vibrational modes in the case of trapped ions (given that the excitation energy is provided by an external laser source), anti-Jaynes-Cummings coupling is also available in this system, as opposed to **CQED** systems [50].

Ions can be trapped for long times, much longer than the time needed for coherent quantum manipulation and read-out of the quantum state – this is the case of qubit states that have a long lifetime, making this system more suitable for quantum information applications. In summary, the experiments with **CQED** are conceptually simpler, however the ion traps are more promising for applications. Fig. 2.2 schematizes the ion trap system constructed in the research group of Prof. Rainer Blatt (IQOQI / University of Innsbruck).

The system of single ion trapping is one of the most successful experimental implementations of quantum computation. It still stands as one of the few quantum systems for which several basic protocols of coherent manipulation have been demonstrated and scalability does not seem unfeasible (yet) [52]. The usual design utilizes a Paul trap, a set of electrodes subject to an oscillatory external electric potential. The time-average potential associated with the secular motion of the ions provides confinement of charged particles in the three directions of space [43].

However, when it comes to the implementation of quantum information with continuous variables, the quantized electromagnetic field continues to be the physical system of choice for most experimentalists. In this case, the optical amplitude and phase quadratures are the

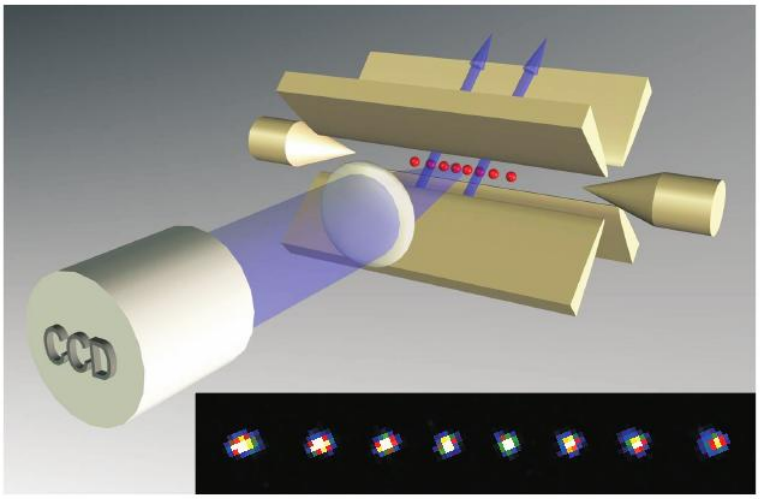

Figure 2.2: Scheme of a linear ion trap employing an ion string of Ca^+. A single laser addresses the ions individually and manipulates their quantum states. Credit: IQOQI / University of Innsbruck. Ref.[51].

usual **CV** degrees of freedom. Although light is suitable for the experimental realization of **CV** quantum information, usually only Gaussian states and operations (explored in the previous chapter) are available in the laboratory [53]. Those alone are not enough, nevertheless, to implement universal quantum computation in **CV** [54], since non-Gaussian resources (such as 'macroscopic cat' states) are needed to achieve quantum speed-up in many quantum information processes. Those types of operations are very challenging to achieve with light, and probably for this reason research in **CV** quantum computation (**CVQC**) is less developed than with discrete variables.

We believe trapped ions can furnish a new impetus to **CVQC**. By making use of the position and momentum (i.e. the wavefunction) of a single trapped ion as the **CV** degrees of freedom, **CVQC** could be investigated in theory and experiment taking this system as basic physical implementation. Due to the special characteristics of the trapped ions, it is possible to strongly couple vibrational modes through the mediation of the ionic qubit, by employing pulses of classical light for the coherent manipulation, such that the non-linear interactions and non-Gaussian operations interactions are possibly not a major problem to implement. The higher order non-linearity that gives rise to non-Gaussian operations can be realized by compound light pulses, creating and destroying single phonons at will. Pioneering experiments employing related ideas have produced motional cat states of a single ion, a class of states applicable in quantum computation [55, 56].

Experimental studies on this topic can help understand the practical differences between qubits and **CV** in the implementation of quantum information processing. **CV** systems typ-

ically have Hilbert spaces of higher dimension than discrete systems, allowing for a larger configuration space within current technological constraints. A better understanding of these topics is as interesting experimentally as from a theoretical point of view, and could help in the development of a more practical model for quantum computing. We believe pioneering work in **CVQC** could be performed with the ion trap [23].

2.3 Two-level model

We consider an atom with two levels where $|g\rangle$, the ground state, is connected to $|e\rangle$, the excited state, and this connection is given by a dipole with the angular frequency ω_{eg} represented in the figure. This system is equivalent to a spin-1/2 system immersed in a magnetic field along the **z** axis. We can do a correspondence according to the notation used in quantum information $|e\rangle \rightarrow |0\rangle$, and $|g\rangle \rightarrow |1\rangle$, with eigenvalues $+1$ and -1, respectively, as shown in figure 2.3. The

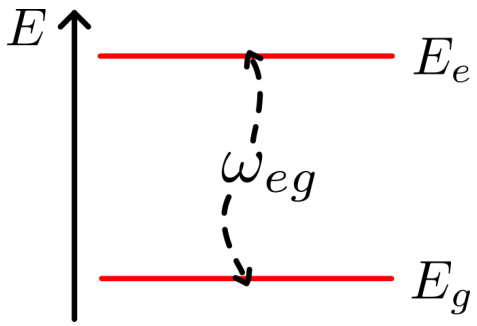

Figure 2.3: Two-level atom

Hamiltonian operator of the atom represented by this two-level system is given by

$$H_A = E_e |e\rangle \langle e| + E_g |g\rangle \langle g| . \tag{2.6}$$

In 2-D space, 4 linear operators are possible, which are generated through the basis $|e\rangle$, $|g\rangle$. Then

$$\mathbb{1} = |e\rangle \langle e| + |g\rangle \langle g| , \tag{2.7}$$

$$\sigma_z = |e\rangle \langle e| - |g\rangle \langle g| , \tag{2.8}$$

$$\sigma^+ = |e\rangle \langle g| , \tag{2.9}$$

$$\sigma^- = |g\rangle \langle e| , \tag{2.10}$$

from which comes the algebra

$$\sigma_z^2 = \mathbb{1}, \quad \sigma^+\sigma^- - \sigma^-\sigma^+ = \sigma_z, \tag{2.11}$$

$$\left(\sigma^{-}\right)^{2} = |g\rangle\langle g|, \quad \left(\sigma^{+}\right)^{2} = |e\rangle\langle e|. \tag{2.12}$$

Let us introduce the atomic raising and lowering operators $\sigma_{\pm}$,

$$\sigma_{\pm} = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right). \tag{2.13}$$

In terms of the spin eigenstates along the z axis, these operators are

$$\sigma^{+} = |0\rangle\langle 1|, \qquad \sigma^{-} = |1\rangle\langle 0|, \tag{2.14}$$

and

$$\sigma^{+}|1\rangle = |0\rangle, \quad \sigma^{-}|0\rangle = |1\rangle, \quad \sigma^{+}|0\rangle = |0\rangle, \quad \sigma^{-}|1\rangle = 0. \tag{2.15}$$

If we separate the term $\left(E_e + E_g\right)/2\mathbb{1}$ in the Hamiltonian 2.6, we can rewrite the Hamiltonian and obtain, for a two-level system, the following equation

$$H_A = \frac{1}{2}\hbar\omega_{eg}\sigma_{\mathbf{z}}. \tag{2.16}$$

The transition frequency between levels can be written

$$\omega_{eg} = \frac{1}{\hbar}\left(E_e - E_g\right). \tag{2.17}$$

We can describe the commutation relations between the above operators as

$$\left[\sigma^{+}, \sigma^{-}\right] = \sigma_z, \qquad \left[\sigma^{+}, \sigma_z\right] = -2\sigma^{+}, \qquad \left[\sigma^{-}, \sigma_z\right] = 2\sigma^{-}, \tag{2.18}$$

and complement these with the Pauli's operators

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \qquad [\sigma_y, \sigma_z] = 2i\sigma_x, \qquad [\sigma_z, \sigma_x] = 2i\sigma_y. \tag{2.19}$$

The vector state in 2-D can be represented as vectors in $\mathbb{C}^2$ according to the rule

$$|\psi\rangle = c_g|g\rangle + c_e|e\rangle \rightarrow \begin{pmatrix} c_e \\ c_g \end{pmatrix}. \tag{2.20}$$

The operations are represented by matrices

$$\sigma^{+} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^{-} \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{1} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.21}$$

2.3.1 Two-level atom interacting with a classical electric field

The main aspects of atom-light interaction can be understood by considering a simple model in which the electronic cloud and nucleus form an electric dipole that can be excited by the electric field of an external source of coherent light. We add to the free two-level Hamiltonian an interaction term,

$$H = H_a + H_r, \tag{2.22}$$

with

$$H_r = -\overrightarrow{D} \cdot \overrightarrow{E_r}, \tag{2.23}$$

where we describe the atom-light coupling. The atomic dipole operator reads as $\overrightarrow{D} = q\overrightarrow{R}$, where q is the free charge in the dipole and R is the position of the electron. The classical electric field depends on time,

$$\overrightarrow{E_r} = i\varepsilon_r \left[\overrightarrow{\varepsilon_r} e^{-i\omega_r t} e^{-i\varphi} - \overrightarrow{\varepsilon_r^*} e^{i\omega_r t} e^{i\varphi} \right], \tag{2.24}$$

In this expression, ε_r is the real amplitude of the classical electric field, ω_r is the optical frequency, $\vec{\varepsilon}_r$ is a complex unit vector describing the polarization, and ϕ is its phase. The dipole operator has odd parity, $\overrightarrow{D}$ its diagonal matrix elements must be null, and the operator is described by the matrices $\sigma_\pm$ according to

$$\overrightarrow{D} = d\left(\overrightarrow{\varepsilon_a}\sigma_- + \overrightarrow{\varepsilon_a^*}\sigma_+ \right), \tag{2.25}$$

where the notation below was introduced,

$$q \langle g | \overrightarrow{R} | e \rangle = d\overrightarrow{\varepsilon_a}. \tag{2.26}$$

Note that d is the matrix elements of the atomic dipole operators and $\overrightarrow{\varepsilon_a}$ is a unit vector that describes the atomic polarization.

2.4 Motion of the ion

Before starting the next chapter, let us briefly describe some qualitative aspects of the ion motion. A moving atom emits or absorbs light with frequency ω different from its transition frequency ω_0 due to the Doppler effect,

$$\omega = \omega_0 + \mathbf{k} \cdot \mathbf{v} \tag{2.27}$$

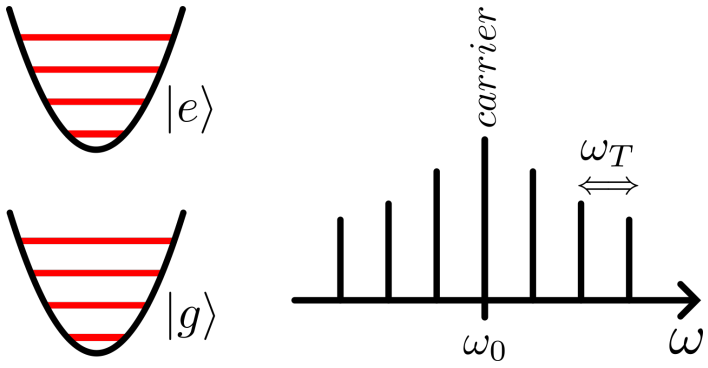

Figure 2.4: Energy levels of the trapped atom (ground and excited) and its resulting spectrum. ω_T is the frequency of oscillation of the confining potential.

where **k** is the wave vector of the emitted/absorbed light and **v** is the atomic speed vector. For free atoms, the speed can be arbitrary, following the Boltzmann distribution for the vapour temperature. The optical spectrum of atomic ensembles is broadened and/or displaced according to their motion in space.

In trapped ions, the energy of the motion is quantized. Since the ion motion can be approximated by the motion of a set of harmonic oscillators, the motional Doppler shift creates sidebands at specific frequencies displaced from the carrier frequency of atomic emission. Fig. 2.4 illustrates a set of sidebands, showing that the resulting emission spectrum contains a large number of resonant lines. The main idea of this chapter is to present some tools to explore the motion of the ion, i.e., its continuous variables (CV) position and momentum, as a quantum system amenable to coherent manipulation with the aim of performing quantum information processing.

The interaction of an ion with a resonant external laser can be understood in terms of a simple two-level system driven by an external coupling. Each level corresponds to an internal electronic level of the ion. While the ion interacts with the laser, it is bounded to move in an area of space restricted by the confining potential of a Paul trap. As the ion is cooled to have low kinetic energy, its motion can be treated as that of a quantum oscillator. The coupling between the electronic levels and the vibrational degree of freedom takes place due to the absorption and emission of photons and the resulting recoil of the ion.

We can start the physical discussion with the classical Hamiltonian of an ion with mass m confined by an harmonic 1-D potential, as follows

$$H_{vib} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{2.28}$$

where ω is the angular frequency of the vibrational motion, x is the displacement of the ion rel-

ative to its equilibrium position and p is its momentum. The motion of the ion can be quantized introducing the canonical operators $\hat{x}$ and $\hat{p}$ that can be written in terms of ladder operators. Thus, the Hamiltonian of Eq. 2.28 is quantized in the following way

$$H_{vib} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{2.29}$$

with eigenstates given by the Fock basis $|n\rangle$ corresponding to a well defined number n of excitations (phonons). The complete free Hamiltonian of the ion involves the qubit and the motional degree of freedom, yielding

$$\hat{H}_0 = \hat{H}_{int} + \hat{H}_{vib} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^{\dagger}\hat{a}. \tag{2.30}$$

The dipole interaction of this system with an external light source was already discussed in previous sections.

2.4.1 Interaction Hamiltonian

For the simple case of a single trapped ion, we begin by employing the Hamiltonian of a two-level atom interacting with an external laser field [46]. To include the fact that the atom is confined in space, we consider the electric field of the laser at the quantum-mechanical position of the ion, described by the operator $x = a + a^{\dagger}$. The interaction Hamiltonian then reads as

$$H = -i\hbar\Omega\sigma_{+}e^{-i(\Delta t)}\exp\left(i\eta\left(ae^{-i\omega t} + a^{\dagger}e^{i\omega t}\right)\right) + h.c., \tag{2.31}$$

where a and $a^{\dagger}$ are respectively the annihilation and creation operators of vibrational excitations (phonons) of the harmonic oscillator associated with the trapped atom and Δ is the qubit-laser detuning. The Rabi frequency Ω depends on the laser electric field amplitude as an experimentally controllable parameter. The ϕ denotes the phase of the laser, ω denotes the secular frequency of the ions in confined movement and $\eta = k_z z_0$ is the Lamb-Dicke parameter. This parameter reflects the strength of laser-motion coupling. It depends on k_z, the projection of the laser-field wave vector along the direction z of the atomic oscillation, and $z_0 = \sqrt{\frac{\hbar}{2m\omega}}$ is the typical spatial extension of the ground state wave function of the ion in the harmonic oscillator.

Only some terms of the Hamiltonian in Eq. 2.31 contribute resonantly to the ionic transitions. The rotating wave approximation (RWA) considers only terms oscillating with small frequencies in comparison with the typical time scale of the quantum evolution. In addition, the Lamb-Dicke approximation can also be employed when the oscillator wave function is much

smaller in extension than the radiation wavelength $\left(\eta\sqrt{\left\langle (a+a^{\dagger})^2\right\rangle} << 1\right)$. We obtain the simplified expression

$$H \backsimeq -i\hbar\Omega\left\{\sigma_+ e^{-i(\Delta t-\varphi)} + \sigma_- e^{i(\Delta t-\varphi)} + i\eta\left(\sigma_+ e^{-i(\Delta t-\varphi)} - \sigma_- e^{i(\Delta t-\varphi)}\right)\left(ae^{-i\omega t} + a^{\dagger}e^{i\omega t}\right)\right\}. \tag{2.32}$$

The basis for quantum logical operations is now formed by a combination of motional and atomic qubit, i.e., by the states: $|g,0\rangle$,$|g,1\rangle$,$|e,0\rangle$,$|e,1\rangle$ as depicted in Fig. 2.5. In the Lamb-Dicke approximation, there are three important transitions to consider: red sideband,carrier and blue sideband transitions. Each of them corresponds to a value of laser detuning $\Delta = +\omega, 0, -\omega$.

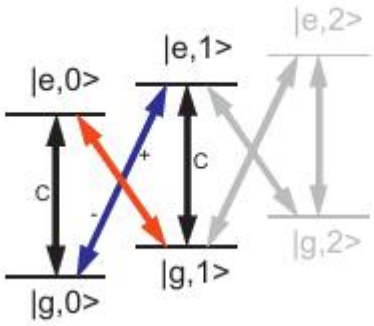

Figure 2.5: A single ion with carrier (C) and sideband $\pm$ transitions. Reproduced from Ref. [51].

Each of the Hamiltonians that follow are derived from Eq. 2.32 by neglecting off-resonant terms, giving rise to the following quantum dynamics:

i)**Carrier frequency excitation:** For $\Delta = 0$, the carrier excitation induces transitions in the qubit without affecting the motion. It is described by the Hamiltonian

$$H_c \simeq -i\hbar\frac{\Omega_l}{2}\left(e^{-i\varphi}\sigma_+ - e^{i\varphi}\sigma_-\right). \tag{2.33}$$

This Hamiltonian couples the basis states $|g,n\rangle$ and $|e,n\rangle$. All the quantum gates acting on one qubit may be obtained by tuning the laser phase ϕ, and the time of interaction of atom-laser. The speed of the gate is tuned by the laser intensity through the Rabi frequency Ω. When the Lamb-Dicke parameter is not small, other terms of the expansion should be considered. Since the resonant terms of the Hamiltonian are diagonal in the Fock basis, it couples $|g,n\rangle$ and $|e,n\rangle$, although with Rabi frequencies depend on the phonon number. This effect becomes relevant only in the case the interaction acts over very long times.

ii)**Red sideband excitation:** If the driving laser is tuned to $\Delta = -\omega$, we find the following approximation for the resonant Hamiltonian:

$$H_r \simeq -i\hbar\frac{\Omega_l\eta}{2}\left(a\sigma_+ - a^\dagger\sigma_-\right). \tag{2.34}$$

This Hamiltonian is essentially the Jaynes-Cummings term of CQED. The red sideband couple the states $|g,n\rangle$ and $|e,n-1\rangle$, inducing an excitation of the qubit and the simultaneous absorption of a phonon in the motional state. Since it realizes a dynamics of the qubit conditioned on the number of phonons, it produces coupling between the internal and external degrees of freedom of the ion. The Rabi frequency in this case strongly depends on the phonon number, as

$$\Omega_{n,n-1} = \sqrt{n}\eta\Omega \tag{2.35}$$

iii)**Blue sideband excitation:** When the external laser assumes the detuning $\Delta = +\omega$, we can describe it through the approximate equation:

$$H_{b;1} \simeq -i\hbar\frac{\Omega_l\eta}{2}\left(a^\dagger\sigma_+ - a\sigma_-\right) \tag{2.36}$$

The blue sideband interaction implements an anti-Jaynes-Cumming model, which induces in the qubit a transition between the states $|g\rangle$ and $|e\rangle$ while emitting a phonon in the vibrational mode. This process cannot be resonant in **CQED**, for example, but in our system we can realize it, since in the ion trap the driving laser provides the extra energy required to realize the transition. This interaction can also be used to entangle the qubit and the motional mode of the ion. The Rabi frequency is given by

$$\Omega_{n,n+1} = \sqrt{n+1}\eta\Omega \tag{2.37}$$

where n describe the number of motional quanta (phonons).

2.5 Motion of ion strings

In the CZ proposal, the interaction between ionic qubits is obtained using the motional degree of freedom as a quantum bus[2] for the distribution of quantum information between pairs of ions [60]. The simplest method to couple a single ion with a vibrational mode is schematized in Fig. 2.6.

The laser field is made resonant with a blue-sideband transition. The following unitary operator is realized. This operation is important it is one of the gates that we obtained in the

[2]This is a device which can be used to store or transfer information between independent qubits. This concept was first demonstrated by some researches, as we can see these references: [57, 58, 59].

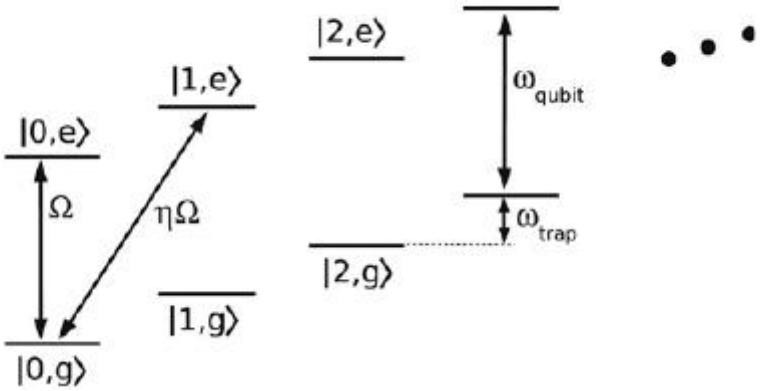

Figure 2.6: Energy level of a single trapped ion, being $|g\rangle$a ground level and $|e\rangle$ an excited level in a Paul trap. The symbols ω_{qubit} and ω_{trap}. Figure borrowed from reference [61]

Gaussian toolbox.

$$R^{+}(\theta,\phi) = \exp(i\frac{\theta}{2}(e^{i\phi}\sigma^{+}a^{\dagger} + e^{-i\phi}\sigma^{-}a)). \tag{2.38}$$

The parameter θ depends on the strength and duration of the applied pulse, and ϕ describes the relative phase between optical field and the atomic polarization. Qubit and motional state change simultaneously. For experiments with quantum logic, the case of multiple ions is also of interest. Because of the Coulomb coupling between the ions movement takes place in terms of the normal modes of a string of ions [62, 63].

Although we do not consider the treatment of multiples ions in detail, we take some space to describe a bit the normal modes of vibration. For three ions, the normal modes are, in increasing order of frequency, the center-of-mass, breathing (or stretch) and an additional axial mode, where we denote the center-of-mass mode by the frequency ω_1, the breathing mode by ω_2 and axial mode by ω_3, as in Fig. 2.7.

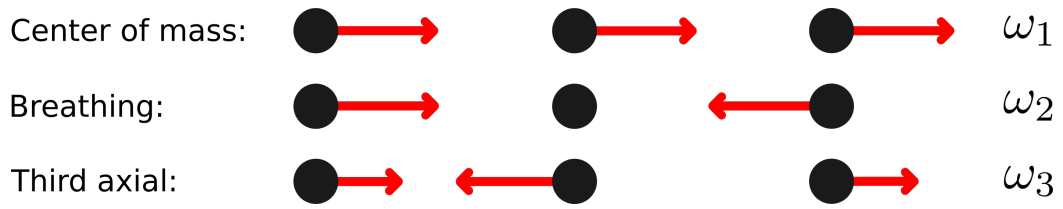

Figure 2.7: Normal modes of the three-ion string along the axial direction with frequency ω_i

2.6 Quantum gates using trapped ions

There is a considerable number of proposals to implement quantum gates using a string of cold trapped ions [62, 63, 64]. In the next sections, we discus the Cirac-Zoller proposal in

some detail, as demonstrated by the Innsbruck group [43]. Another relevant proposal is the Mølmer-Sørensen gate [65, 66], which, unlike the Cirac-Zoller gate, does not require cooling the oscillator to the ground state, nor the addressing of individuals ions. This proposal was successfully implemented by the NIST group using two or four ions, and subsequently by the Innsbruck group to produce a two-qubit entangled Bell state with high fidelity [67].

2.6.1 Cirac-Zoller gate

In this proposal, a string of ions is confined in the effective harmonic potential of a linear Paul trap as schematized in Fig. 2.8

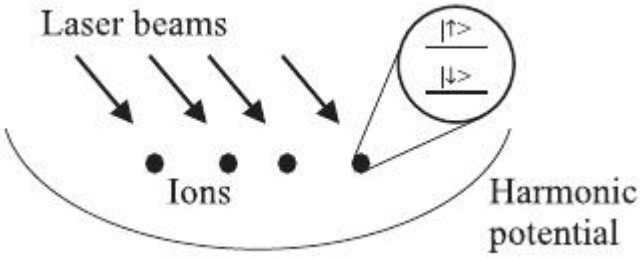

Figure 2.8: Basic scheme of the Cirac-Zoller proposal. A chain of ions trapped in a harmonic potential is radiated by laser beams. Internal states define the qubit states $|0\rangle$, $|1\rangle$. Reproduced from Ref. [68].

The ions are cooled to the ground state of oscillation, and can be addressed individually by a laser beam with spectral resolution better than the motional oscillation frequency in the trap. The Lamb-Dicke limit is supposed to be a valid approximation, and hence the most relevant transitions for achieving gate operations are the carrier red- and blue-sideband transitions (RSB and BSB, respectively), which were described in detail in the previous section. A simple example of how two ions could effectively interact mediated by the vibrational mode is provided below, where a two qubit Bell state is produced by a sequence of sideband laser pulses.

$$|-\rangle\,|-\rangle\,|0\rangle$$

$$\Downarrow$$

$$\text{BSB } \pi/2 \text{ -pulse , Ion 2}$$

$$|-\rangle\,[|-\rangle\,|0\rangle - i\,|+\rangle\,|1\rangle]$$

$$\Downarrow$$

RSB π- pulse, Ion 1

$$[|-\rangle|-\rangle - |+\rangle|+\rangle]|0\rangle \tag{2.39}$$

Firstly, we apply a BSB $\frac{\pi}{2}$ pulse on ion 2. A simple example of how two ions could be made to effectively interact mediated by the vibrational mode is provided below, where a two qubit Bell state is produced by a sequence of sideband laser pulses, giving rise to entanglement between ion 2 and the motional mode. Subsequently, an RSB π-pulse on the ion 1 transforms only the $|-\rangle\ |+\rangle\ |1\rangle$ ket, such that the entangled state is mapped into the internal state of the ions. The excitation and de-excitation of the motional state provide a kind of communication channel between the qubits. It is essential in this procedure to have individual addressing and ions cooled to the motional ground state.

Although RSB and BSB transitions are sufficient to perform entangling operations between qubits, as we have seen in the previous example, there is a more systematic way of realizing conditional operations between two qubits, by employing the CNOT or C-phase gate. The central importance of the Cirac-Zoller proposal is in fact offering a means to realize the C-phase gate in a viable physical system [38]. The C-phase gate between a pair of ions works as follows: initially, the quantum state of the first ion is transferred to the quantum state of the motional mode, the 'quantum information bus', so that the information is made collectively available to all ions. Subsequently, the conditional gate is performed between the motional mode and a second ion of choice. The final step requires mapping back the motional state to the first ion.

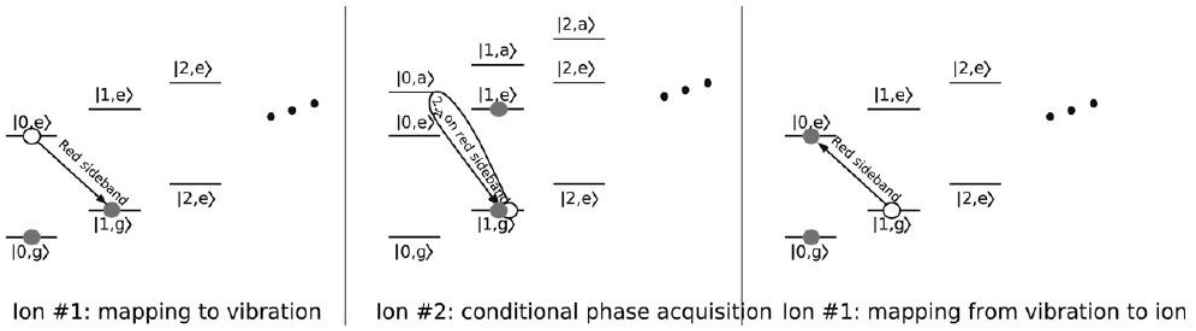

Figure 2.9: Representation of the three steps that perform a phase gate between two ions with an electronic ground state $|g\rangle$, an excited state $|e\rangle$ and an ancilla state $|a\rangle$ (an extra state). The illustration was taken from reference [61].

The steps employed to realize the Cirac-Zoller phase gate, as illustrated in Fig. 2.9, are: a RSB pulse of area π addresses the first ion to move all the population from state $|0,e\rangle$ to $|1,g\rangle$,

while for the state $|0,g\rangle$ the transfer is not performed. This step maps the quantum information from qubit to motion, rendering the first ion in the separable state $|g\rangle$. The inverse step is realized after the gate is realized. Hence the crucial step is the intermediate one, whereby the basis state $|0,g\rangle$ involving the second ion is delayed by a π phase. In the original CZ proposal, that is realized by employing an alternative intermediate electronic state that couples only to the $|g\rangle$ state of the second ion. The actual experimental realization utilizes as intermediate state the Fock state $|2\rangle$ of the motional mode, achieving the same result.

The applied Rabi transitions induced by sideband gates are restricted to have a time duration **T**, since the sideband needs to be resolved and must be limited in coupling strength so that off-resonant excitation of the carrier transition is avoided. For instance, for a $\frac{\pi}{2}$ sideband pulse, the following restriction holds

$$T = \frac{\pi}{\eta\Omega} >> \frac{\pi}{\eta\omega_z}, \tag{2.40}$$

where η is the Lamb-Dicke parameter, and Ω is of Rabi frequency for the free ion. For a single qubit gate, there is no need to excite the motional state, we can implement a carrier pulse for which the duration should obey for $\frac{\pi}{2}$ pulses the relation $T >> \frac{\pi}{\omega_z}$. Thus we can conclude that a single ion gates is not only easier to implement, they are much faster than the two-qubit gate.

2.6.2 Mølmer-Sørensen gate

The CZ gate strict requirement of ground state cooling can be relaxed in a second class of two-qubit gates proposed by Molmer and Sorensen in 1999 [66]. The MS gate employs bichromatic radiation, i.e., a laser with two frequency components, to displace the ion wave function in phase space in a manner which is dependent on the qubit quantum state. The laser effectively applies a state-dependent force on the ion, coherently displacing it in one internal state is $|g\rangle$ and vice-versa. By applying radiation with frequency slightly detuned from the oscillating frequency, the motional wavefunction returns to its initial state some time after the interaction begins. However, the area enclosed in phase space during the interaction imposes a relative phase delay between the $|g\rangle$ and $|e\rangle$ qubit states, similarly to a Berry phase [69, 70].

To better understand the MS gate, we suppose again the Lamb-Dicke limit, $\eta\sqrt{n+1} << 1$, detuning $0 < \delta < \omega_z$ and a weak field such that $\eta\Omega << \omega_z - \delta$. Although one-photon transitions are not allowed, however, two-photon transition of the type $|-\rangle|-\rangle|n\rangle - |+\rangle|+\rangle|n\rangle$ are resonant if they happen via the intermediate states $|-\rangle|+\rangle|m\rangle$ and $|+\rangle|-\rangle|m\rangle$, where ($m = n-1, n, n+1$). Considering the transitions path via intermediate states in a second-order per-

turbation theory, we find that the Rabi frequency for the two-photon transition is [66]

$$\tilde{\Omega} = -\frac{2\omega_z(\eta\Omega)^2}{\omega_z^2 - \delta^2}. \tag{2.41}$$

We also have $|-\rangle|+\rangle|n\rangle - |+\rangle|-\rangle|n\rangle$ two-photon transitions via the intermediate states $|-\rangle|-\rangle|m\rangle$ and $|+\rangle|+\rangle|m\rangle$ ($m = n-1, n, n+1$) are allowed, and the corresponding two-photon Rabi frequency is $-\tilde{\Omega}$. A bichromatic laser pulse induces the transfer of population between two states of the computational basis using two other states as non-resonant intermediates. Since the two-photon Rabi frequency is n-independent, the dynamics is insensitive to heating during the gate operations and ground state cooling is not required; in addition, both ions are irradiated by the same laser, rendering individual addressing not necessary.

The proposal has been refined to the case where $\eta\Omega = \omega_z - \delta$, under the assumptions $\Omega^2 << \omega_z^2$ and $\eta^2 << 1$ [69]. In this case, the gate operations are faster, however, and the fact that the motional state is excited during the gate operations means that only at specific points in time can the motional mode return to its initial state, making the scheme sensitive to heating during the gate operation. Fig. 2.10 schematizes the quantum transitions involved in the gate operation

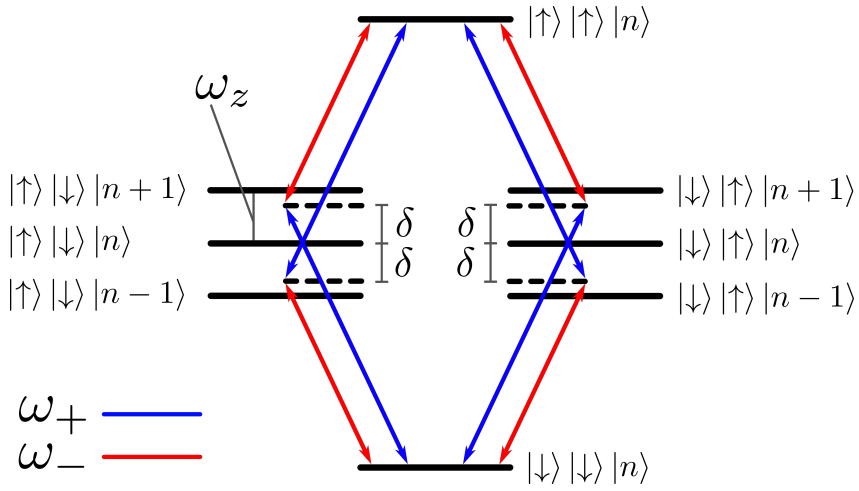

Figure 2.10: Energy-level diagram of two trapped ions by the principle of the Sørensen and Mølmer gate. The bus mode can be initially populated with n phonons. Two laser beams tuned close to the blue and red sidebands drive the system through the dashed virtual energy levels.

For interactions with the appropriate time, the MS gate performs the following transforma-

tion of the basis states:

$$\begin{aligned} |ee\rangle &\rightarrow (|ee\rangle + i|gg\rangle)/\sqrt{2} \\ |eg\rangle &\rightarrow (|eg\rangle + i|ge\rangle)/\sqrt{2} \\ |ge\rangle &\rightarrow (|ge\rangle + i|eg\rangle)/\sqrt{2} \\ |gg\rangle &\rightarrow (|gg\rangle + i|ee\rangle)/\sqrt{2} \end{aligned} \tag{2.42}$$

To obtain a universal set of gates, we introduce a new basis $|\pm\rangle = (|e\rangle_i \pm |g\rangle_i)/\sqrt{2}$, with $|\pm\rangle|\pm\rangle$ described in Eq. 2.42. The MS gate transforms them according to $|++\rangle \rightarrow |++\rangle$, $|+-\rangle \rightarrow i|+-\rangle$,$|-+\rangle \rightarrow i|-+\rangle$, $|--\rangle \rightarrow |--\rangle$, where a global phase $e^{-i\pi/4}$ has been omitted. Thus the MS gate realizes the two-qubit conditional phase gate understood to be an essential resource to achieve universality in the quantum manipulation of discrete variables.

Chapter 3
CV quantum computation using the motion of the ion

> "When we look at reality through the equations of physics, we find that they describe patterns and regularities. But to me, mathematics is more than a window on the outside world: I argue that our physical world not only is described by mathematics, but that it is mathematics: a mathematical structure, to be precise."
>
> Max Tegmark

The main idea of this chapter consists in increasing the size of the Hilbert space through the manipulation of the vibrational modes of trapped ions. We consider the simplest case of using only a single trapped ion and take as a starting point its three vibrational modes. In this proposed paradigm, the qubit is used as mediator of the interaction between the pairs of vibrational modes, i.e., the qubit is our 'quantum bus'. This proposal is in a sense an inversion of the 'CZ paradigm' of quantum computing, which employs one or more motional modes of the quantum harmonic oscillator to mediate the interaction between the qubit pairs, as we have seen previously. This situation is represented in Fig. 3.1

From this different viewpoint, we develop a CV quantum computing toolbox to manipulate each of the single vibrational modes and make them interact in pairs. The quantum gates we propose are realized by the coherent manipulation of the vibrational modes of the ion through bichromatic beams of light. Furthermore, with the elements of our toolbox, it is shown that we can obtain gates that are already known from discrete quantum computing, such as the Fourier transform, the CNOT and CPHASE gates. Besides, we can also obtain generalized Pauli ma-

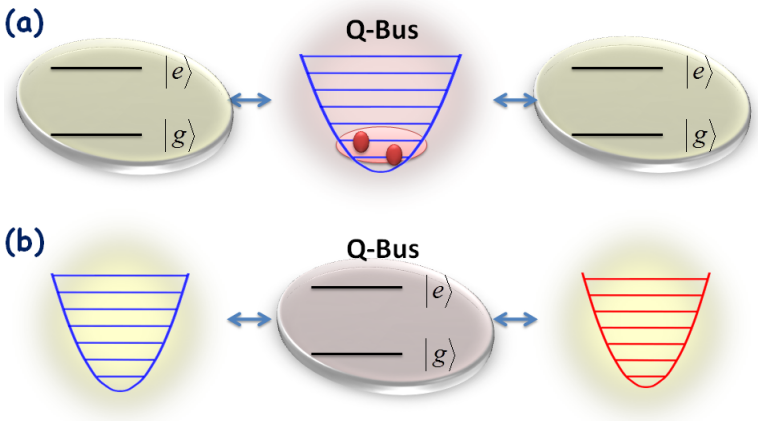

Figure 3.1: Comparison of two proposals of CVQC using an ion trap. (a) The CZ paradigm, where the 'quantum bus' consists of the motional modes. (b) Our proposal, where the 'quantum bus' is composed of the qubits, i.e., our proposal is the inverse of the CZ paradigm. The size of the Hilbert space in our proposal is larger than it is in the CZ paradigm.

trices, which are used to explore quantum error correction codes in the stabilizer formalism. Gathering Non-Gaussian and Gaussian gates, we also show that it is possible to generate dynamics through the commutators, as described by Lloyd [8].

3.1 Interaction Hamiltonian for two vibrational modes

We can start describing our proposal. It is built from the implementation of an ion trap processor: a single ion offers the qubit and its modes of vibration. The manipulation of the quantum state of trapped ions is realized by exciting their internal and external degrees of freedom, as detailed in Ch. 2, employing a laser resonant to specific transitions. The total Hamiltonian of the system can be written as

$$H = H_0 + H_{int}. \tag{3.1}$$

The H_0 term describes the dynamics before the laser interaction, and is given by

$$H_0 = \hbar\omega(a^\dagger a + b^\dagger b) + \frac{\hbar\omega_{eg}\sigma_z}{2}. \tag{3.2}$$

Where the operators $\hat{a}$ and $\hat{b}$ are the vibrational modes of quantum harmonic oscillator. Thus, the Hamiltonian of interaction is

$$H_{int} = \frac{\hbar\Omega}{2} e^{-i\omega_l t} e^{ikzcos\theta} e^{-i\phi}\sigma^+ + h.c. \tag{3.3}$$

where Ω is the Rabi frequency, ω_l is the laser frequency, θ is the angle between the wave vector $\mathbf{k}$ of the laser and the z direction of oscillation of the ion in two dimensions, and ϕ is the phase of the laser field.

It is convenient to consider the above Hamiltonian in the interaction picture. To this end, we need to manipulate the term associated with the ion motion $e^{ikz\cos\theta}$. Using the fact that our system is described by a quantum harmonic oscillator, we write

$$e^{ikzcos\theta} = e^{ik[z_0(a+a^\dagger)+z_1(b+b^\dagger)]} \tag{3.4}$$

where z_0 and z_1 are the amplitudes of oscillation

$$z_0 = \sqrt{\frac{\hbar}{2m\omega_0}} \qquad \text{and} \qquad z_1 = \sqrt{\frac{\hbar}{2m\omega_1}}. \tag{3.5}$$

Using the identity $e^{(A+B)} = e^A e^B e^{-[A,B]/2}$, we have

$$e^{i[\eta_a(a+a^\dagger)+\eta_1(b+b^\dagger)]} = e^{i\eta_a(a+a^\dagger)} e^{i\eta_b(b+b^\dagger)}. \tag{3.6}$$

The definition of the exponential yields

$$e^{i[\eta_a(a+a^\dagger)+\eta_b(b+b^\dagger)]} = \left(\sum_{n=0}^{\infty}\frac{(i\eta_a a^\dagger)^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{(i\eta_a a)^m}{m!}\right)\left(\sum_{p=0}^{\infty}\frac{(i\eta_b b^\dagger)^p}{p!}\right)\left(\sum_{k=0}^{\infty}\frac{(i\eta_b a^\dagger)^k}{k!}\right) \tag{3.7}$$

$$= \sum_{n=0,m=0}^{\infty}(i\eta_a)^{n+m}\frac{(a^\dagger)^n(a)^m}{n!m!}\sum_{p=0,k=0}^{\infty}(i\eta_a)^{p+k}\frac{(b^\dagger)^p(b)^k}{p!k!}. \tag{3.8}$$

For $\eta_a = \eta_b = \eta$, it then follows

$$e^{i\eta(a+a^\dagger+b+b^\dagger)} = \left(\sum_{n=0,m=0,p=0,k=0}^{\infty}(i\eta)^{n+m+p+k}\frac{(a^\dagger)^n(a)^m(b^\dagger)^p(b)^k}{n!m!p!k!}\right) \tag{3.9}$$

As for the one-dimensional case, we need to factor the sums with the goal of sorting out the distinct sidebands. We can break this double sum in three parts:

- $n > m$, with $n = m + q'$ and $p > k$, with $p = k + q''$;
- $n = m$ and $p = k$;
- $n < m$, with $m = n + q'$ and $p < k$, with $p = k + q''$;

For $n > m$ and $p > q$

$$\left(\sum_{q'>0}\sum_{m=0}^{\infty}(-1)^m(i\eta)^{q'}\eta^{2m}\frac{(a^\dagger)^{m+q'}(a)^n}{m!(m+q')!}\right)\left(\sum_{q''>0}\sum_{k=0}^{\infty}(-1)^k(i\eta)^{q''}\eta^{2k}\frac{(b^\dagger)^{k+q''}(b)^k}{k!(k+q'')!}\right) \tag{3.10}$$

By defining

$$f_{q'}(a^\dagger,a) = \sum_{m=0}^{\infty}\eta^{2m}\frac{(a^\dagger)^m(a)^m}{m!(m+q')!} \tag{3.11}$$

$$f_{q''}(b,b^\dagger) = \sum_{k=0}^{\infty} \frac{(-1)^k (i\eta)^{2k} (b^\dagger)^k (b)^k}{k!(k+q'')!} \tag{3.12}$$

We come to

$$\sum_{q'>0;q''>0} (i\eta)^{q'+q''} (a^\dagger)^{q'} (b^\dagger)^{q''} f_{q'}(a^\dagger,a) f_{q''}(b,b^\dagger) \tag{3.13}$$

For $n=m$ and $p=k$, we have

$$\sum_{n=0;p=0}^{\infty} (-1)^{n+k} (\eta)^{2n+2p} \frac{(a^\dagger a)^n (b^\dagger b)^p}{(n!)^2 (p!)^2} = f_0(a^\dagger,a) f_0(b,b^\dagger) \equiv f_{00}(a^\dagger,a;b,b^\dagger) \tag{3.14}$$

In the case $n<m$ and $p<k$, we obtain

$$\left(\sum_{q'>0} \sum_{n=0}^{\infty} (-1)^n \frac{(i\eta)^{q'} \eta^{2n} (a^\dagger)^n (a)^{n+q}}{n!(n+q)!} \right) \left(\sum_{q''>0} \sum_{p=0}^{\infty} (-1)^p \frac{(i\eta)^{q''} \eta^{2p} (b^\dagger)^p (b)^{p+q''}}{p!(p+q'')!} \right) \Rightarrow$$

$$\Rightarrow \sum_{q'>0;q''>0} (i\eta)^{q'+q''} (a)^{q'} (b)^{q''} f_{q'}(a^\dagger,a) f_{q''}(b,b^\dagger) \tag{3.15}$$

Then,

$$\begin{aligned} e^{ikzcos\theta} &= (f_{00}(a^\dagger,a;b,b^\dagger) + \sum_{q'>0;q''>0} (i\eta)^{q'+q''} (a)^{q'} (b)^{q''} f_{q'}(a^\dagger,a) f_{q''}(b,b^\dagger) + \\ &+ \sum_{q'>0;q''>0} (i\eta)^{q'+q''} (a^\dagger)^{q'} (b^\dagger)^{q''} f_{q'}(a^\dagger,a) f_{q''}(b,b^\dagger)) \end{aligned} \tag{3.16}$$

Here, we can assume that at first $f_{q'}(a^\dagger,a) \neq f_{q'}(a^\dagger a) = f_q(N_a)$, where $N_a = a^\dagger a$ is the number operator. Let's now check if $f_q(a^\dagger,a) = f_q(N_a)$. We see that $(a^\dagger)^m a^m = (a^\dagger)^{m-1} N_a a^{m-1}$, for the commutator $[N_a, a^m]$, so we have the expression below

$$[N,a^m] = [a^\dagger a, a^m] = a^\dagger [a,a^m] + [a^\dagger,a^m] a$$

$$[a^\dagger,a^m]a = (a[a^\dagger,a^{m-1}] + [a^\dagger,a]a^{m-1})a$$

$$\vdots \tag{3.17}$$

$$\Rightarrow a^{m-1}[a^\dagger,a] - (m-1)a^{m-1}a = (-ma^{m-1})a = -ma^m \tag{3.18}$$

Which means we have

$$\Rightarrow [N_a, a^q] = -qa^q \tag{3.19}$$

Similarly, for $N_b = b^\dagger b$ it follows

$$\Rightarrow [N_b, b^q] = -qb^q \tag{3.20}$$

And in another way, we have

$$([N_a,a^q])^\dagger = (Na^q - a^q N)^\dagger = -[N_a,(a^\dagger)^q] \tag{3.21}$$

then,

$$[N_a,(a^\dagger)^q] = q(a^\dagger)^q \tag{3.22}$$

and,

$$[N_b,(b^\dagger)^q] = q(b^\dagger)^q \tag{3.23}$$

Returning to the term of $(a^\dagger)^m a^m$

$$(a^\dagger)^m a^m = a^\dagger)^{m-1} N a^{m-1} = (a^\dagger)^{m-1}[a^{m-1}N-(m-1)a^{m-1}] = (a^\dagger)^{m-1}a^{m-1}[N-(m-1)] \Rightarrow$$

$$\Rightarrow (a^\dagger)^{m-2}.a^{m-2}[N-(m-1)].[N-(m-1)] = (a^\dagger)^{m-2}.a^{m-2}[N-(m-1)].[N-(m-1)]$$

$$\vdots$$

$$\Rightarrow (a^\dagger)a[N-(m-(m-1))][N-(m-(m-2))]...[N-2][N-1] \tag{3.24}$$

As a result, we have

$$\Rightarrow (a^\dagger)^m a^m = \prod_{j'=1}^{m} (N_a-(m-j')) = \prod_{j'=0}^{m-1} (N_a-j') \tag{3.25}$$

Analogously for the $\hat{b}$ mode, we have

$$(b^\dagger)^k b^k = \prod_{j''=1}^{k} (N_b-(k-j'')) = \prod_{j'=0}^{k-1} (N_b-j'') \tag{3.26}$$

As the dependences in $f_{q'}(a^\dagger,a)$ and $f_{q''}(b^\dagger,b)$ are exactly in $(a^\dagger)^m a^m$ and $(b^\dagger)^k b^k$, we have

$$f_{q'}(a^\dagger,a) = f_{q'}(N_a) = \sum_{m=0}^{\infty} (-1)^m \eta^{2m} \prod_{j'=0}^{m-1} \frac{(N_a-j')}{m!(m+q')!} \tag{3.27}$$

and

$$f_{q''}(b^\dagger,b) = f_{q''}(N_b) = \sum_{k=0}^{\infty} (-1)^k \eta^{2k} \prod_{j''=0}^{k-1} \frac{(N_b-j'')}{m!(k+q'')!}. \tag{3.28}$$

Now we can finally move on to the interaction picture. Considering the operators in the Heisenberg picture with respect to the free Hamiltonian (H_0), we have the interaction Hamiltonian

$$H_{int}^{'} = U_0 H_{int} U_0, \tag{3.29}$$

where

$$U_0 = e^{-iH_0 t/\hbar} = e^{-i\omega N t} e^{-i\omega_{eg}\sigma_z t/2}, \qquad N = N_a + N_b. \tag{3.30}$$

The operator N acts in the space of number states and the operator σ_z acts in the space of qubits $|e\rangle$, $|g\rangle$, and so that N and σ_z commute. As U_0 and H_{int} factor in different terms with dependence on the spaces of number of excitations and qubits, we can treat these operator parts separately and make the product between each part, as below

- Space of number states

$$e^{i\omega Nt}e^{i\eta(a^\dagger+a+b^\dagger+b)}e^{-i\omega Nt} \Rightarrow$$

$$+\sum_{q'>0;q''>0}(i\eta)^{q'+q''}(a^\dagger)^{q'}(b^\dagger)^{q''}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)e^{-i\omega(N_a+N_b)t}. \tag{3.31}$$

We next use the fact that these are polynomial functions of $N=N_a+N_b$ and thus commute with $e^{\pm i\omega Nt}$. In this way, we need only to consider the effect of the evolution operator over a^q and $(a^\dagger)^q$. First we expand the term $a^q e^{-i\omega N_a t}$,

$$a^q e^{-i\omega N_a t}=\sum_{m=0}^{\infty}\frac{(-i\omega)^m}{m!}a^q(N_a)^m. \tag{3.32}$$

And it follows that

$$\begin{aligned}
a^q(N_a)^m=a^qN(N)^{m-1} &= (Na^q+qa^q)N^{m-1}=(N+q)a^qN^{m-1}\Rightarrow\\
\Rightarrow(N+q)(Na^q+qa^q)N^{m-2} &= (N+q)(N+q)a^qN^{m-2}=(N+q)^2a^qN^{m-2}\\
&\vdots\\
a^qe^{-i\omega N_at}=\sum_{m=0}^{\infty}\frac{(-i\omega)^m}{m!}(N+q)^ma^q &= e^{-i\omega(N_a+q)t}a^q.
\end{aligned} \tag{3.33}$$

Now we need to work out the term $e^{i\omega N_a t}(a^\dagger)^q$. We have

$$\begin{aligned}
e^{i\omega N_at}(a^\dagger)^q &= \sum_{m=0}^{\infty}\frac{(-i\omega)^m}{m!}(N_a)^m(a^\dagger)^q\\
N_a^m(a^\dagger)^q=N_a^{m-1}N_a(a^\dagger)^q &= N_a^{m-1}((a^\dagger)^qN_a+q(a^\dagger)^q)=N_a^{m-1}(a^\dagger)^q(N_a+q)\Rightarrow\\
\Rightarrow N_a^{m-2}((a^\dagger)^qN_a+q(a^\dagger)^q)(N_a+q) &= N_a^{m-2}(a^\dagger)^q(N_a+q)(N_a+q)=N_a^{m-2}(a^\dagger)^q(N_a+q)^2\\
&\vdots\\
(N_a)^m(a^\dagger)^q &= (a^\dagger)^q(N_a+q)^m
\end{aligned} \tag{3.34}$$

So,

$$\begin{aligned}
e^{i\omega N_at}(a^\dagger)^q=\sum_{m=0}^{\infty}\frac{(i\omega)^m}{m!}(a^\dagger)^q(N+q)^m &\equiv (a^\dagger)^qe^{i\omega(N_a+q)t}\\
(b^{q'})e^{-i\omega N_bt} &= e^{-i\omega(N_b+q')t}b^{q'}\\
e^{i\omega N_bt}(b^\dagger)^{q'} &= (b^\dagger)^{q'}e^{i\omega(N_b+q)t}.
\end{aligned} \tag{3.35}$$

Using the results of Eq. 3.33 and 3.35, we obtain for Eq. 3.31 the expression

$$\begin{aligned}
&e^{i\omega(N_a+N_b)t}e^{i\eta(a^\dagger+a+b^\dagger+b)}e^{-i\omega(N_a+N_b)t}=\\
\Rightarrow&\left(e^{i\omega(N_a)t}e^{i\eta(a^\dagger+a)}e^{-i\omega(N_a)t}\right).\left(e^{i\omega(N_b)t}e^{i\eta(b^\dagger+b)}e^{-i\omega(N_b)t}\right).
\end{aligned} \tag{3.36}$$

Hence the motional term of the Hamiltonian above in the interaction picture reads as

$$\begin{aligned}\Rightarrow\ & [f_{00}(a^\dagger,a;b,b^\dagger)+\sum_{q'>0;q''>0}(i\eta)^{q'+q''}e^{-i(q'+q'')\omega t}(a)^{q'}(b)^{q''}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)+\\ +\ & \sum_{q'>0;q''>0}(i\eta)^{q'+q''}(a^\dagger)^{q'}(b^\dagger)^{q''}e^{i(q'+q'')\omega t}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)]\end{aligned}\tag{3.37}$$

- For the qubit space $|g\rangle$ and $|e\rangle$ we can find

$$e^{i\omega\sigma_z t/2}\sigma^+e^{-i\omega\sigma_z t/2}=\sum_{m=0;n=0}^{\infty}\left(\frac{i\omega_{eg}t}{2}\right)^m\left(\frac{-i\omega_{eg}t}{2}\right)^n\frac{(\sigma_z)^m\sigma^+(\sigma_z)^n}{m!n!},\tag{3.38}$$

where $(\sigma_z)^{2m}=1$, $(\sigma_z)^{2m+1}=\sigma_z$, $\sigma_z\sigma^+=\sigma^+$, and $\sigma^+\sigma_z=-\sigma^+$. We then separate the two sums in parts with n/m being $even/odd$,

$$\begin{aligned}&\sum_{m=0;n=0}^{\infty}\left(\frac{i\omega_{eg}t}{2}\right)^m\left(\frac{-i\omega_{eg}t}{2}\right)^n\frac{(\sigma_z)^m\sigma^+(\sigma_z)^n}{m!n!}\Rightarrow\\ \Rightarrow&\left(\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(\frac{i\omega_{eg}t}{2}\right)^{2m}+\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{i\omega_{eg}t}{2}\right)^{2m+1}\sigma_z\right).\sigma^+.\\ &.\left(\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{-i\omega_{eg}t}{2}\right)^{2n}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{-i\omega_{eg}t}{2}\right)^{2n+1}\sigma_z\right).\end{aligned}\tag{3.39}$$

Identifying the sums with higher order terms in $\cos(\omega_{eg}t/2)$ and terms with $\pm i\sin(\omega_{eg}t/2)$, we have

$$\begin{aligned}\Rightarrow\ & \left(\cos^2(\omega_{eg}t/2)-\sin^2(\omega_{eg}t/2)\right)\sigma^+-i\sin(\omega_{eg}t/2)\cos(\omega_{eg}t/2)\sigma^+\sigma_z+\\ +\ & i\sin(\omega_{eg}t/2)\cos(\omega_{eg}t/2)\sigma_z\sigma^+\\ \Rightarrow\ & \cos(\omega_{eg}t)\sigma^++i\sin(\omega_{eg}t)\sigma^+=e^{i\omega_{eg}t}\sigma^+\end{aligned}\tag{3.40}$$

Thus, as conclusion of our results, we have a single sum of the Hamiltonian of interaction in the interaction picture for two vibrational modes of the trap. The general expression is given by

$$\begin{aligned}H'_{int}\ =\ & \frac{-i\hbar\Omega e^{-i\phi}}{2}[f_{00}(a^\dagger,a;b,b^\dagger)e^{i(\omega_{eg}-\omega_l)t}+\\ +\ & \sum_{q'>0;q''>0}(i\eta)^{q'+q''}e^{i(\omega_{eg}-\omega_l-(q'+q'')\omega)t}(a)^{q'}(b)^{q''}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)+\\ +\ & \sum_{q'>0;q''>0}(i\eta)^{q'+q''}e^{i(\omega_{eg}-\omega_l-(q'+q'')\omega)t}(a)^{q'}(b)^{q''}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)+\\ +\ & \sum_{q'>0;q''>0}(i\eta)^{q'+q''}(a^\dagger)^{q'}(b^\dagger)^{q''}e^{i(\omega_{eg}-\omega_l+(q'+q'')\omega)t}f_{q'}(a^\dagger,a)f_{q''}(b,b^\dagger)]\sigma^++h\end{aligned}\tag{3.41}$$

3.2 CV Gaussian quantum gates over two motional modes

Due to the properties of saturation with absorption of a single quantum, the qubit promotes a way to add and subtract phonons to motional modes. It also prevents the motional state from reaching a fast increase of the number of excitations. We can make interesting combinations of Hamiltonians using terms that are resonant and considering bichromatic beams to drive the quantum dynamics [71]

Let us now explore the Hamiltonian of one and two modes separately. We will construct a toolbox of Gaussian operations by tuning the laser radiation frequency. We also show that non-Gaussian operations are available by employing the Hamiltonian for the blue and red sideband transitions.

3.2.1 Single-mode Gaussian gate

The Gaussian operations are better visualized in the phase space of the Wigner function used to describe the CV quantum system. We write Eq. 3.41 for the two mode ion-laser Hamiltonian in a restricted form to initially consider operations acting on only one mode,

$$\begin{aligned} H'_{int_{1-mode}} &= \left(\frac{-i\hbar e^{-\eta^2/2}e^{-i\phi}}{2}\right)\cdot\Big[f_0(N)e^{-i\delta t}\sigma^+ + \\ &+ \sum_{q>0}(i\eta)^q e^{-i(\delta+q\omega)t} f_q(N)a^q\sigma^+ + \\ &+ \sum_{q>0}(i\eta)^q e^{-i(\delta-q\omega)t} a^{\dagger q} f_q(N)\sigma^+\Big] \end{aligned} \tag{3.42}$$

where $\delta = \omega_l - \omega_{eg}$ is the detuning frequency and the function $f_{q'}(a^\dagger, a)$ is defined by Eq. 3.27. Expanding Eq. 3.42 explicitly, and considering first that we will focus on Gaussian operations, for $\mathbf{q} = 0, 1, 2$ and η up to second order, we have

$$\begin{aligned} H_{int_{1-mode}} &= \left(-\frac{i\hbar\Omega e^{-i\phi}}{2}\right)\Bigg(e^{-i\delta t}\left(1-\frac{\eta^2}{2}(aa^\dagger + a^\dagger a)+\cdots\right)+ \\ &+ e^{-i(\delta+\omega)t}\left(i\eta - i\frac{\eta^3}{3}a^2a^\dagger - i\frac{\eta^3}{6}a^\dagger a^2+\cdots\right)+ \\ &+ e^{-i(\delta-\omega)t}\left(i\eta - i\frac{\eta^3}{3}a^{\dagger 2}a - i\frac{\eta^3}{6}aa^{\dagger 2}+\cdots\right) + e^{-i(\delta+2\omega)t}\left(\frac{-\eta^2a^2}{2}+\cdots\right)+ \\ &+ e^{-i(\delta-2\omega)t}\left(\frac{-\eta^2a^{\dagger 2}}{2}+\cdots\right) + e^{-i(\delta+3\omega)t}\left(\frac{-i\eta^3a^3}{6}+\cdots\right)+ \\ &+ e^{-i(\delta-3\omega)t}\left(\frac{-i\eta^3a^{\dagger 3}}{6}+\cdots\right) + e^{-i(\delta+4\omega)t}\left(\frac{\eta^4a^4}{5!}+\cdots\right)+ \\ &+ e^{-i(\delta-4\omega)t}\left(\frac{\eta^4a^{\dagger 4}}{5!}+\cdots\right)+\cdots\Bigg)\sigma^+ + h.c.. \end{aligned} \tag{3.43}$$

The fact that this Hamiltonian has so many different frequencies allows us to choose a single resonant term through the laser detuning δ. We consider a single and well defined laser frequency and we use the expansion of the Hamiltonian in powers of η. Then, up to the second-order approximation, we have five resonant frequencies to choose from. The evolution operators corresponding to the resonant part of the Hamiltonian are

$$U(t) = \exp\left(\frac{-iH_{int}t}{\hbar}\right). \tag{3.44}$$

The first-order term in η is resonant for $\omega_l = \omega_{eg}$ and corresponds to a qubit rotation in the Bloch sphere.

$$\hat{H}^{(0)} = \left(\frac{i\hbar\Omega}{2}\right)\left(e^{i\phi}\hat{\sigma}_+ + e^{-i\phi}\hat{\sigma}_-\right). \tag{3.45}$$

The Hamiltonian induces a qubit transition without changing the motional state of the ion. Now, we will describe the operations obtained and, at the end of this subsection, present a summary of the Gaussian gates. Initially, we consider the Rotation operation in the phase space, that is the Fourier transform. The Fourier gate introduces a number dependent phase shift in the form $|n_s\rangle \to e^{in\theta}|n_s\rangle$, where n_s is the number of phonons in the mode $\hat{s}$. The Fourier transform gate is realized by frequency $\omega_l = \omega_{eg}$, i.e. by a monochromatic laser. The Hamiltonian is given by

$$H_{int(1)} = \left(\frac{i\Omega\hbar e^{i\varphi}}{2}\right)\left(-\frac{\eta^2 a^\dagger a}{2}\right)\sigma_+ \tag{3.46}$$

that is the Fourier gate $\hat{F} = \exp^{-i\theta\hat{a}^\dagger\hat{a}}$ or a rotation in the phase space, where the θ is the rotation.

We consider now the displacement operator that realizes a mode displacement $\hat{s}$. Using a bichromatic beam with composite frequency given by $\delta_1 = -\delta_2 = \delta = \omega_s$, we have

$$\hat{H}_s^{(1)} = i\hbar\Omega\hat{\sigma}_+\left(\tilde{\alpha}\hat{s}^\dagger - \tilde{\alpha}^*\hat{s}^\dagger\right). \tag{3.47}$$

The realization of a single-mode displacement operation requires the qubit state to be initialized in an eigenstate of the operator $\hat{\sigma}_+$. Then applying the evolution operator from this Hamiltonian, we obtain the $\hat{D}(\alpha)$ of mode $\hat{s}$.

Another operation available to us is the squeezing operation. This gate is obtained by a bichromatic beam with detuning $\delta_1 = -\delta_2 = \delta = 2\omega_s$ that produce the Hamiltonian

$$\hat{H}_s^{(2)} = i\hbar\Omega\hat{\sigma}_+\left(\tilde{\xi}\hat{s}^{\dagger 2} - \tilde{\xi}^*\hat{s}^{\dagger 2}\right), \tag{3.48}$$

where ξ is the squeezing parameter. Acting with this Hamiltonian on the motional state, we have the evolution operation given by $\hat{S}(\xi)$.

The Gaussian operations are realized with the terms of the Hamiltonian stemming from the second-order expansion in η. However, these terms involve not only the creation of quanta in the vibrational modes, but also the excitation of the qubit, a feature that will produce undesirable entanglement between them. We would like the qubit to be an expectator of the interaction, with the sole role of mediating the coupling among the motional modes. A bichromatic laser as shown in Tab. 3.1 solves this problem by factoring the qubit operators from the motional ones. Next we analyze each of the possibilities of Gaussian operations.

Hamiltonian	Frequency	Operator
$H_{int(1)} = \left(\frac{i\Omega\hbar e^{i\varphi}}{2}\right)\left\{1 - \frac{\eta^2 aa^\dagger}{2} - \frac{\eta^2 a^\dagger a}{2}\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg}$	$\hat{R}\left(\frac{\Omega t e^{i\varphi}\eta^2}{4}\right)$
$H_{int(2)} = \left(\frac{-\Omega\hbar e^{i\varphi}}{2}\right)\{\eta a\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} - \omega$	-
$H_{int(3)} = \left(\frac{-\Omega\hbar e^{i\varphi}}{2}\right)\left\{\eta a^\dagger\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} + \omega$	-
$H_{int(4)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{4}\right)\left\{\eta^2 a^2\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} - 2\omega$	-
$H_{int(5)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{4}\right)\left\{\eta^2 a^{2\dagger}\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} + 2\omega$	-
$H'_{total} = H_{int(3)} + e^{i\theta}H_{int(2)}$	$\omega_{l'}; \omega_{l''}$	$\hat{D}\left(\frac{i\Omega t e^{i\varsigma}\eta}{2}\right)$
$H''_{total} = H_{int(5)} + e^{i\theta}H_{int(4)}$	$\omega_{l'}; \omega_{l''}$	$\hat{S}\left(\frac{\Omega t e^{i\varsigma}\eta^2}{2}\right)$

Table 3.1: Gaussian operations achieved through the application of a bichromatic beam on the Hamiltonian with one external degree of freedom.

3.2.2 Two-mode Gaussian gates

Let us now consider a Hamiltonian of the ion-laser interaction considering two vibrational modes. Using the rotating-wave approximation (RWA) for two modes, we expand the general expression given in equation 3.41, in an analogous way to the expansion of one mode. Thus we have

$$\begin{aligned} H'_{int_{2-mode}} &= \left(\frac{-i\hbar\Omega e^{-i\phi}}{2}\right)e^{\eta^2}e^{i\omega_{eg}t}\Big(1 + i\eta(ae^{-i\omega_a t} + a^\dagger e^{i\omega_a t} + be^{-i\omega_b t} + b^\dagger e^{i\omega_b t}) + \\ &+ \frac{-\eta^2}{2}(ae^{-i\omega_a t} + a^\dagger e^{i\omega_a t} + be^{-i\omega_b t} + b^\dagger e^{i\omega_b t})^2 + \\ &+ \frac{-i\eta^3}{6}(ae^{-i\omega_a t} + a^\dagger e^{i\omega_a t} + be^{-i\omega_b t} + b^\dagger e^{i\omega_b t})^3 + \cdots\Big)e^{-i\omega_l}\sigma^+ + h.c. \end{aligned} \quad (3.49)$$

Right now, we will not consider the third term of the expansion, because we will explore it in the non-linear analysis in a later section. Simplifying the expression, we have

$$\begin{aligned} H'_{int_{2-mode}} &= \left(\frac{-i\hbar\Omega e^{-i\phi}}{2}\right)\left(e^{-i\delta t}\left(1-\frac{\eta^2}{2}\left(aa^\dagger + a^\dagger a + bb^\dagger + b^\dagger b\right)\right)+\right. \\ &+ e^{-i(\delta+\omega_a)t}(i\eta a) + e^{-i(\delta-\omega_a)t}(i\eta a^\dagger) + e^{-i(\delta+\omega_b)t}(i\eta b) + \\ &+ e^{-i(\delta-\omega_b)t}(i\eta b^\dagger) + e^{-i(\delta+2\omega_a)t}\left(\frac{-\eta^2 a^2}{2}\right) + \\ &+ e^{-i(\delta+2\omega_b)t}\left(\frac{-\eta^2 b^2}{2}\right) + e^{-i(\delta-2\omega_a)t}\left(\frac{-\eta^2 a^{\dagger 2}}{2}\right) + \\ &+ e^{-i(\delta+(\omega_a+\omega_b))t}\left(\frac{-\eta^2 ab}{2}\right) + e^{-i(\delta-(\omega_b-\omega_a))t}\left(\frac{-\eta^2 ab^\dagger}{2}\right) + \\ &+ \left. e^{-i(\delta-(\omega_a-\omega_b))t}\left(\frac{-\eta^2 a^\dagger b}{2}\right) + \cdots\right)\sigma^+ + h.c. \end{aligned} \tag{3.50}$$

In this Hamiltonian, we have a pair of phonons that is created and destroyed in each mode. The term of n^{th} order in η will coherently distribute n phonons between the two modes, although the coupling strength decreases with increasing powers of η^n. The action of these Hamiltonians can be understood in the Fock basis of the motional modes by describing the wave function in the form of quantized excitations. We consider the Lamb-Dicke parameter up to powers of η^2.

As we saw previously, Gaussian operations acting on two vibrational modes produce transformations in linear combinations of modes. The bichromatic field can modify the Eq. 3.50 to produce two types of dynamics. The first Hamiltonian that we start to describe is obtained by choosing the bichromatic radiation $\delta_1 = -\delta_2 = \delta = \omega_a - \omega_b$, for $\omega_a > \omega_b$. The resulting Hamiltonian is

$$\hat{H}^{(2)}_{ab} = \hbar\Omega\hat{\sigma}_+\left(e^{-i\varphi}\hat{a}^\dagger\hat{b} + e^{-i\varphi}\hat{a}\hat{b}^\dagger\right), \tag{3.51}$$

as a result the beam splitter operator $\hat{B}$. In the case of two-mode quantum state is initially separable, the beam splitter dynamics entangles the modes.

The second type of quantum gate involving two modes is obtained from choosing the detuning $\delta_1 = -\delta_2 = \delta = \omega_a + \omega_b$. The resonant term generates the Hamiltonian

$$\hat{H}^{(2)}_{ab} = \hbar\Omega\hat{\sigma}_+\left(e^{-i\varphi}\hat{a}\hat{b} + e^{-i\varphi}\hat{a}^\dagger\hat{b}^\dagger\right), \tag{3.52}$$

then as resulted we have the two-mode squeezing operator using the propagator 3.44. This produces an EPR-type entanglement between the motional modes when acting over the oscillator ground state.

We summarize all these operations in Tab. 3.2. As we can see, we obtain Gaussian operators

Hamiltonian	Frequency (2 mode)	Operator
$H_{int(1)} = \left(\frac{i\Omega\hbar e^{i\varphi}}{2}\right)\left\{1-\frac{\eta^2}{2}\left(aa^\dagger+a^\dagger a\right) -\frac{\eta^2}{2}(bb^\dagger+b^\dagger b\right\}\sigma_+ + h.c$	$\omega_l = \omega_{eg}$	$\hat{R}_{\hat{a}}\left(\frac{\Omega t e^{i\varsigma}\eta^2}{4}\right)$ $\hat{R}_{\hat{b}}\left(\frac{\Omega t e^{i\varsigma}\eta^2}{4}\right)$
$H_{int(10)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{4}\right)\left\{\frac{\eta^2}{2}a^\dagger b^\dagger\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} + (\omega_a + \omega_b)$	-
$H_{int(11)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{4}\right)\left\{\frac{\eta^2}{2}ab\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} - (\omega_a + \omega_b)$	-
$H_{int(12)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{2}\right)\left\{\frac{\eta^2}{2}ab^\dagger\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} - (\omega_a - \omega_b)$	-
$H_{int(13)} = \left(\frac{-i\Omega\hbar e^{i\varphi}}{2}\right)\left\{\frac{\eta^2}{2}a^\dagger b\right\}\sigma_+ + h.c.$	$\omega_l = \omega_{eg} + (\omega_a - \omega_b)$	-
$H_{total} = H_{13} + e^{i\theta}H_{12}$	$\omega_{l'}; \omega_{l''}$	$\hat{B}\left(\frac{\Omega t e^{i\varsigma}\eta^2}{4}\right)$
$H_{total} = H_{10} + e^{i\theta}H_{11}$	$\omega_{l'}; \omega_{l''}$	$\hat{S}_{\hat{a},\hat{b}}\left(\frac{\Omega t e^{i\varsigma}\eta^2}{2}\right)$

Table 3.2: Gaussian operations achieved through the application of bichromatic lasers on Hamiltonians with two vibrational modes.

like the beam-splitter $\hat{B}\left(\Omega t e^{i\varsigma}\eta^2/4\right)$, two-mode squeezeer $\hat{S}_{\hat{a},\hat{b}}\left(\Omega t e^{i\varsigma}\eta^2/2\right)$, and rotations in the individual modes $\hat{R}_{\hat{a}}\left(\Omega t e^{i\varsigma}\eta^2/4\right)$, $\hat{R}_{\hat{b}}\left(\Omega t e^{i\varsigma}\eta^2/4\right)$.

3.2.3 Non Gaussian and Gaussian operations

To produce an arbitrary CV quantum state, we need at least one of the operations to be non-Gaussian. We have also seen that the operations of creation and annihilation of phonons of one-mode through **BSH** and **RSH**, and the linear coupling η_s, make these terms stronger than others in the interaction. An alternative to this is to perform the non-Gaussian operations using resonant laser fields. Non Gaussian operations can be done with bichromatic light. In this way, the number of phonons continues to increase, changing with the Wigner function due to the non-Gaussian terms. In the case of two-modes, the cross function of the Hamiltonian $\hat{H}_{ab}$ allows the simultaneous creation or annihilation of phonons in each motional mode, using light of a single frequency.

3.2.4 Generalization of Gaussian operations

Quantum Mechanics gives us aspects of information processing that could not be realized using the classical laws of information theory. For instance, it may be possible to perform an algorithm efficiently on a quantum computer that cannot be realized in this way in a classical computer. The Gottesman-Knill theorem [72] for qubits gives us a tool for the evaluation of classical complexity of a given process. Firstly, the states of some quantum algorithm begin with a computational basis and employ only a class of gates restricted to: Hadamard, Phase,

Controlled-NOT and Pauli gates. With this, we can simulate in an efficient way the quantum computation on a classical computer [37]. We can analyse an extension of the Gottesman-Knill theorem to try to understand which CVQC algorithms may be efficiently simulated by a classical computer. During this section we develop a set of sufficient conditions for CVQC that can be simulated in a classically efficient way. We will not give a complete proof in here, but we use the stabilizer formalism in terms of their generators. Moreover, we can also connect these elements to the idea of encoding the quantum state in CV from our Gaussian and Non-Gaussian gates.

We will start our discussion about quantum logic from the Gaussian operations of one and two-modes represented in CV. Thus, converting our 'objects' to the phase space, we have:

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) \to \exp\left(ip\hat{X} - ix\hat{P}\right) \tag{3.53}$$

$$\hat{S}(\xi) = \exp\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2}\right) \to \left[(\xi-\xi^*)\left(\hat{X}^2-\hat{P}^2\right) + i(\xi+\xi^*)\left(\hat{X}\hat{P}+\hat{P}\hat{X}\right)\right] \tag{3.54}$$

$$\hat{R}(\theta) = \exp\left(i\theta\hat{a}^\dagger\hat{a}\right) \to \exp\left(i\theta\left(\hat{X}^2+\hat{P}^2-1\right)\right) \tag{3.55}$$

$$\hat{B}(\theta') = \exp\left(i\theta'\left(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger\right)\right) \to \exp\left(2i\theta'\left(\hat{X}_1\hat{P}_2 - \hat{P}_1\hat{X}_2\right)\right) \tag{3.56}$$

$$\hat{S}_{\hat{a},\hat{b}}(\zeta) = \exp\left(\frac{\zeta}{2}\left(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger\right)\right) \to \exp\left[2(\xi-\xi^*)\left(\hat{X}_1\hat{X}_2 - \hat{P}_1\hat{P}_2\right) + 2(\xi+\xi^*)\left(\hat{X}_1\hat{P}_2 - \hat{P}_1\hat{X}_2\right)\right] \tag{3.57}$$

We can now direct our attention to the operations in $\hat{x}$ and $\hat{p}$. For linear and quadratic Hamiltonians we can explore one- and two-modes Gaussian gates. We then start by generalizing the Pauli operator **X** in CV as a displacement operator, and so we have

$$X(x) = \exp\left(-ix\hat{P}\right) \tag{3.58}$$

where $\hat{P}$ is the momentum operator. This adds a logic number to the state of the computational basis, as follows

$$X|0\rangle \equiv |0\oplus 1\rangle \equiv |1\rangle, \quad \hat{X}\left(x'\right)\left|x'\right\rangle = \left|x+x'\right\rangle. \tag{3.59}$$

Another important construction is the generalization of the Pauli operator **Z**, which can be represented as

$$Z(p) = \exp\left(ip\hat{X}\right) \tag{3.60}$$

This operator adds a phase to the computational states basis, as

$$\hat{Z}(p)|x\rangle = e^{2ixp}|x\rangle. \tag{3.61}$$

The generalizations of the Pauli operators are elements of the Heisenberg-Weyl group, which are fundamental objects in quantum error correction, specifically in the stabilizer formalism [73]. Furthermore, we can also interpret these elements of the group as displacement operators. We then have

$$X(q) = D\left(\hat{X}\right), \quad Z(p) = D\left(i\hat{P}\right). \tag{3.62}$$

In other words, $X(q)$ performs a displacement along the real axis and $Z(p)$ does it along the complex axis of the phase space. Both operators can be obtained by bichromatic beams $\delta_1 = -\delta_2 = \delta = \omega_s$ of the Hamiltonian $\hat{H}_s^{(1)}$. These operators are noncommutative, and act as follows

$$\hat{X}(q)\hat{Z}(p) = e^{-2ixp}\hat{Z}(p)\hat{X}(q). \tag{3.63}$$

We also have that the rotation operator given in Eq. 3.55, or the operator part $H_{int(1)}$ of Tab. 3.2, is analogous to the unitary operator that represents a Fourier transform in CV, as we can check by

$$\mathscr{F} = \exp\left(i\frac{\pi}{4}\left(\hat{X}^2 + \hat{P}^2\right)\right) \propto \exp\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \tag{3.64}$$

We can verify the action of $\mathscr{F}$ over the Heisenberg-Weyl group and the canonical operators through a mathematical manipulation using the BCH expansion and the identity $UA^nU^\dagger = UAU^\dagger UA\ldots AU^\dagger = \left(UAU^\dagger\right)^n$. It follows that

$$\mathscr{F}X(q)\mathscr{F}^\dagger = Z(q) \tag{3.65}$$

$$\mathscr{F}\hat{P}\mathscr{F}^\dagger = -\hat{X} \tag{3.66}$$

$$\mathscr{F}Z(p)\mathscr{F}^\dagger = X^{-1}(p) = X(-p). \tag{3.67}$$

Comparing it with the discrete case, these transformations are equivalent to the Hadamard gate described by H, which also acts as a discrete Fourier transform. Then

$$HZH = X \qquad ,HXH = Z \tag{3.68}$$

Following the order of relevant operations, we introduce the phase gate as follows

$$\Phi\left(\tilde{\theta}\right) = \exp\left(-i\tilde{\theta}\frac{\hat{X}^2}{2}\right). \tag{3.69}$$

As we can see in Eqs. 3.54 and 3.55, this gate can be generated by the combination of a single-mode squeezer and a rotation of quadratures. Its action on the Pauli operators in Eq. 3.58 and 3.60 is given by

$$\Phi\left(\tilde{\theta}\right)X(q)\Phi^{-1}\left(\tilde{\theta}\right) = X(q)Z\left(\tilde{\theta}q\right)e^{i\tilde{\theta}'\hat{q}^2} \tag{3.70}$$

$$\Phi\left(\tilde{\theta}\right) Z\left(p\right) \Phi^{-1}\left(\tilde{\theta}\right) = Z\left(p\right) \tag{3.71}$$

or simply:

$\Phi\left(\tilde{\theta}\right)$:

$$X\left(q\right) \rightarrow e^{i\tilde{\theta}'\hat{q}^2} X\left(q\right) Z\left(\tilde{\theta}q\right), \tag{3.72}$$

$$Z\left(p\right) \rightarrow Z\left(p\right). \tag{3.73}$$

This gate may suggest another phase gate that is generated by a momentum quadrature, as given below

$$\Phi_p\left(\tilde{\theta}\right) = \exp\left(i\tilde{\theta}\frac{\hat{P}}{2}\right). \tag{3.74}$$

This operator is obtained through the Fourier transform from operator $\Phi(\tilde{\theta})$ as we can see below

$$\Phi_p\left(\tilde{\theta}\right) = \mathscr{F}\Phi\left(\tilde{\theta}\right)\mathscr{F}^{\dagger}, \tag{3.75}$$

which may be implemented in the ion trap. This can also be achieved from the rotation operator.

We will now move on to discuss what can be obtained from two-mode Gaussian gates. We know from the discrete case that the **CX** or the **CNOT** gates are two of the entangling gates that can accomplish a bit-flip **X** to the 'target qubit' that is controlled by the 'control qubit'. In the continuous variables version, we have

$$CX_{ij} = \exp\left(-i\hat{X}_i \otimes \hat{P}_j\right). \tag{3.76}$$

The expression above can also be read as (control)$\otimes$(target). This operator displaces the position of the momentum of the control mode, and is similar to the beam splitter gate or to two-mode gates depending on the squeezer parameter as we can see in Eqs. 3.56 and 3.57. The action of this gate on the Pauli operators follows as

$$X_1\left(q\right) \rightarrow X_1\left(q\right) X_2\left(q\right), \qquad Z_1\left(p\right) \rightarrow Z_1\left(p\right), \tag{3.77}$$

$$X_2\left(q\right) \rightarrow X_2\left(q\right), \quad Z_2^{-1}\left(p\right) Z_1\left(p\right), \tag{3.78}$$

or, $\hat{CX}_{ij}$:

$$\begin{aligned} X_i\left(q\right) \otimes \mathbb{I}_j &\rightarrow X_i\left(q\right) \otimes X_j\left(q\right), \\ Z_i\left(p\right) \otimes \mathbb{I}_j &\rightarrow Z_i\left(p\right) \otimes \mathbb{I}_j, \\ \mathbb{I}_i \otimes X_j\left(q\right) &\rightarrow \mathbb{I}_i \otimes X_j\left(q\right), \\ \mathbb{I}_i \otimes Z_j\left(p\right) &\rightarrow Z_i^{-1}\left(p\right) \otimes Z_j\left(p\right) \end{aligned} \tag{3.79}$$

Another quite known operation in discrete quantum computation is the controlled-Z gate, also called CPHASE, denoted by C-Z, which applies a phase-flip Z in the qubit target that is controlled through another qubit. For the continuous case, we have the following equation

$$CZ_{ij} = \exp\left(i\hat{X}_i \otimes \hat{X}_j\right) \tag{3.80}$$

which also acts on (control)$\otimes$(target). This is a similar gate to the two-mode squeezer operation as we can see in Eq. 3.57. This CZ_{ij} gate produce a transformation associated with Pauli operations. We have

$$X_1(q) \to X_1(q) Z_2(p) \tag{3.81}$$

Both conditional operations, CX_{ij} and CZ_{ij}, can be obtained via tetrachromatic laser, with frequencies: $\delta_1 = -\delta_2 = \omega_s - \omega_{s'}$ and $\delta_3 = -\delta_4 = \omega_s + \omega_{s'}$. In addition to the fundamentals of quantum logic, these gates have been shown to have the form of the Pauli group of the stabilizer formalism used in quantum error correction. Thus, besides implementing a quantum logic with our 'Gaussian and Non Gaussian objects' for the ion trap processor, we can also take another step in quantum information processing: encoding of our quantum states.

3.2.5 Dynamic via commutators

In this section, we investigate how to generate the quantum dynamics via commutators of the basic gates explored above, the aim to obtain the power of the quadrature operators and show that it is possible to generate a polynomial Hamiltonian in the phase space, as proposed in Ref. [8]. Using elements of our toolbox of Gaussian and non Gaussian operations we reproduced the Lloyd proposal.

From the Gottesman-Knill theorem we obtain a large class of operations that can keep track of the stabilizer. Thus, for a quantum computer based on continuous variables offering a speed-up in its information processing, we need to include non Gaussian operations. We will start this section with quadratic interaction Hamiltonians. We know that the commutator of two quadratic functions is also a quadratic function of the quadrature operator, as we write below

$$\left[\hat{X}^2, \hat{P}^2\right] = i\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right), \qquad \left[\hat{X}^2, \hat{P}\hat{X}\right] = i\hat{X}^2 \tag{3.82}$$

When we determine the evolution operator in the Heisenberg picture, we solve the commutators repeatedly via the expansion **BCH**. However, the elements of the commutator do not increase the polynomial order of the quadrature operators, and thus we need operators having higher

polynomial order in position and momentum, as we can see in the following equation

$$H_K = \left(\hat{X}^2 + \hat{P}^2\right)^2 \propto \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)^2 \tag{3.83}$$

In this way, using the same commutators, we can generate the following polynomials having the minimum possible degree in position and momentum required to perform universal computation. These are

$$\left[H_K, \hat{X}\right] = -i\left(\hat{X}^2\hat{P} + \hat{P}\hat{X}^2 + 2\hat{P}^3\right), \tag{3.84}$$

$$\left[H_K, \hat{P}\right] = -i\left(\hat{P}^2\hat{X} + \hat{X}\hat{P}^2 + 2\hat{X}^3\right), \tag{3.85}$$

$$\left[H_K, \hat{X}\hat{P} + \hat{P}\hat{X}\right] = 4i\left(\hat{X}^4 - \hat{P}^4\right). \tag{3.86}$$

As we can see, we have polynomials of third and fourth order. This means we can explore the nonlinear terms of the expansions in one- and two-modes of our interaction Hamiltonian to generate polynomials with mixed Gaussian elements to verify universal computation and the effectiveness of these generated objects.

We can explore the non Gaussian terms in more depth, making use of quantum simulations as in Ref. [74]. However, we will now expand the terms corresponding to η^3 of a two-mode Hamiltonian as in Eq. 3.49, which are non Gaussian terms but have a reasonable coupling strength compared to higher-order terms. With the Hamiltonian of these terms we can generate commutators with canonical variables and obtain a reasonable polynomial to simulate quantum computation over continuous variables. In the next tables, we have Hamiltonians in the phase space whose commutators can be employed to generate new kinds of dynamics. We can always build the polynomial of order k, where k has degree of at least 3, as the Hamiltonians in Tabs. 3.3,3.4. Then for $\left[p^3, p^k x^l\right] = ip^{k+2}x^{l-1}$, and $\left[x^3, p^k x^n\right] = ip^{k-1}x^{l+2}$, where we generate a monomial via these commutators of order $k+1$ from monomial of order k.

3.2.6 Computation

Computation with continuous variables requires Hamiltonians with higher polynomial order in the canonical variables $\hat{X}$ and $\hat{P}$. Hamiltonians of this type generate a set of universal gates which allows computation in an efficient way. Mathematically, the reason why Hamiltonian produces linear transformations of the canonical position and momentum operators is due to fact that the commutator $[\hat{X}, \hat{P}]$ is a constant, and that $[H(\hat{X}, \hat{P}), \hat{X}]$ can create higher order position and momentum polynomials if $H(\hat{X}, \hat{P})$ is quadratic in its variables. To create higher order functions of $\hat{X}$ and $\hat{P}$, we need Hamiltonians of at least cubic order in their canonical variables.

Frequency	$\hat{H}(\hat{x}_1,\hat{x}_2;\hat{p}_1,\hat{p}_2)$
$\omega_{l1}=\omega_{eg}-\omega_a$	$\frac{-1}{2}p_1^3-p_1p_2^2+\frac{1}{2}ip_1^2x_1+ip_2^2x_1+\frac{1}{2}ix_1^3-\frac{1}{2}p_1x_1^2-p_1x_2^2+ix_1x_2^2$
$\omega_{l2}=\omega_{eg}+\omega_a$	$\frac{1}{2}p_1^3+p_1p_2^2+\frac{1}{2}ip_1^2x_1+ip_2^2x_1+\frac{1}{2}p_1x_1^2+\frac{1}{2}ix_1^3+p_1x_2^2+ix_1x_2^2$
$\omega_{l3}=\omega_{eg}-3\omega_a$	$\frac{1}{6}p_1^3-\frac{1}{2}ip_1^2x_1-\frac{1}{2}p_1x_1^2+\frac{1}{6}ix_1^3$
$\omega_{l4}=\omega_{eg}+3\omega_a$	$-\frac{1}{6}p_1^3-\frac{1}{2}ip_1^2x_1+\frac{1}{2}p_1x_1^2+\frac{1}{6}ix_1^3$
$\omega_{l5}=\omega_{eg}-\omega_b$	$-p_1^2p_2-\frac{1}{2}p_2^3+ip_1^2x_2+i\frac{1}{2}p_2^2x_2-p_2x_1^2+ix_1^2x_2-\frac{1}{2}p_2x_2^2+\frac{i}{2}x_2^3$
$\omega_{l6}=\omega_{eg}+\omega_b$	$p_1^2p_2+\frac{1}{2}p_2^3+ip_1^2x_2+i\frac{1}{2}p_2^2x_2+p_2x_1^2+ix_1^2x_2+\frac{1}{2}p_2x_2^2+\frac{i}{2}x_2^3$
$\omega_{l7}=\omega_{eg}-3\omega_b$	$\frac{1}{6}p_2^3-\frac{i}{2}p_2^2x_2-\frac{1}{2}p_2x_2^2+\frac{i}{6}x_2^3$
$\omega_{l8}=\omega_{eg}+3\omega_b$	$-\frac{1}{6}p_2^3-\frac{i}{2}p_2^2x_2+\frac{1}{2}p_2x_2^2+\frac{i}{6}x_2^3$
$\omega_{l9}=\omega_{eg}+(\omega_a+2\omega_b)$	$-\frac{1}{2}p_1^2p_2-\frac{i}{2}p_2^2x_1-ip_1p_2x_2+p_2x_1x_2+\frac{1}{2}p_1x_2^2+\frac{i}{2}x_1x_2^2$
$\omega_{l10}=\omega_{eg}+(\omega_a-2\omega_b)$	$-\frac{1}{2}p_1^2p_2-\frac{i}{2}p_2^2x_1+ip_1p_2x_2-p_2x_1x_2+\frac{1}{2}p_1x_2^2+\frac{i}{2}x_1x_2^2$
$\omega_{l11}=\omega_{eg}+(2\omega_a+\omega_b)$	$-\frac{1}{2}p_1^2p_2-ip_1p_2x_1+\frac{1}{2}p_2x_1^2-\frac{i}{2}p_1^2x_2+p_1x_1x_2+\frac{i}{2}x_1^2x_2$
$\omega_{l12}=\omega_{eg}+(2\omega_a-\omega_b)$	$\frac{1}{2}p_1^2p_2+ip_1p_2x_1-\frac{1}{2}p_2x_1^2-\frac{i}{2}p_1^2x_2+p_1x_1x_2+\frac{i}{2}x_1^2x_2$
$\omega_{l13}=\omega_{eg}-(2\omega_a-\omega_b)$	$-\frac{1}{2}p_1^2p_2+ip_1p_2x_1+\frac{1}{2}p_2x_1^2-\frac{i}{2}p_1^2x_2-p_1x_1x_2+\frac{i}{2}x_1^2x_2$
$\omega_{l14}=\omega_{eg}-(2\omega_a+\omega_b)$	$\frac{1}{2}p_1^2p_2-ip_1p_2x_1-\frac{1}{2}p_2x_1^2-\frac{i}{2}p_1^2x_2-p_1x_1x_2+\frac{i}{2}x_1^2x_2$
$\omega_{l15}=\omega_{eg}-(\omega_a-2\omega_b)$	$\frac{1}{2}p_1p_2^2-\frac{i}{2}p_2^2x_1+ip_1p_2x_2-\frac{1}{2}p_1x_2^2+\frac{i}{2}x_1x_2^2+p_2x_1x_2$
$\omega_{l16}=\omega_{eg}-(\omega_a+2\omega_b)$	$\frac{1}{2}p_1p_2^2-\frac{i}{2}p_2^2x_1-ip_1p_2x_2-\frac{1}{2}p_1x_2^2+\frac{i}{2}x_1^2x_2-p_2x_1x_2$

Table 3.3: Two-modes non-Gaussian gates over continuous variables using a monochromatic laser, using terms of the η^3 order. We have 16 frequencies to select a polynomial Hamiltonian of higher order in $\hat{x}$ and/or $\hat{p}$, which is a good situation to generate some interesting commutators for the simulation of quantum computation over continuous variables.

Repeated commutators generate an ever higher order polynomial in position and momentum, as required. Thus, in a general form, we can perform a quantum computation as follows

$$|X_1,...,X_n\rangle_x \rightarrow \sum_k c_k |X_1,...,X_n;f(X_1,...,X_n)\rangle_x \tag{3.87}$$

where $|c_k|^2$ is the probability. An evolution operator U_F can be generated for a large polynomial F described by multiple variables over CV,

$$U_F=\exp\left[-iF\left(\hat{X}_1,\ldots,\hat{X}_N;\hat{P}_1,\ldots,\hat{P}_N\right)\right] \tag{3.88}$$

This unitary transformation can be decomposed in a linear sequence, quadratic and higher order interactions. The biggest difficulty is to find the 'F' polynomials of interest to quantum computation, and this is a current topic of research.

Now, we have all the ingredients for our **CVQC** scheme: we have the single-mode Gaussian gates, such as the Heisenberg-Weyl gates, the two-modes entangling gates such as the CX and the CZ gates, besides the non Gaussian gates that are based on a Kerr nonlinearity that forms a set of universal gates for quantum computing.

Frequency	$\hat{H}(\hat{x}_1,\hat{x}_2;\hat{p}_1,\hat{p}_2)$
$\omega_{l1}+\omega_{l2}$	$ip_1^2x_1+2ip_2^2x_1+ix_1^3+2ix_1x_2^2$
$\omega_{l3}+\omega_{l4}$	$-ip_1^2x_1+\frac{1}{3}ix_1^3$
$\omega_{l5}+\omega_{l6}$	$2ip_1^2x_2+ip_2^2x_2+2ix_1^2x_2+ix_2^3$
$\omega_{l7}+\omega_{l8}$	$-ip_2^2x_2+\frac{i}{3}x_2^3$
$\omega_{l9}+\omega_{l10}$	$-p_1^2p_2-ip_2^2x_1+p_1x_2^2+ix_1x_2^2$
$\omega_{l11}+\omega_{l12}$	$-ip_1^2x_2+2p_1x_1x_2+ix_1^2x_2$
$\omega_{l13}+\omega_{l14}$	$-ip_1^2x_2-2p_1x_1x_2+ix_1^2x_2$
$\omega_{l15}+\omega_{l16}$	$p_1^2p_2-ip_2^2x_1-p_1x_2^2+ix_1x_2^2$

Table 3.4: Two-modes non-Gaussian gates over continuous variables using bichromatic lasers, using terms of the η^3 order. The polynomial Hamiltonian is realized via the composition of frequencies.

However, putting all gates together in a practical quantum computer is quite challenging, due to the many squeezers and noise of the Kerr nonlinearities. One way to try to work around this issue would be if we could put off the use of the Kerr nonlinearities to the final step, just before the measurement of the modes[75].

3.2.7 Schwinger map

To construct the algebra of quantum angular momentum, operators associated with two harmonic oscillators can be used to build the raising and lowering operators according to the map $\hat{J}_+ = \hat{a}^\dagger\hat{b}$ and $\hat{J}_- = \hat{a}\hat{b}^\dagger$. The z-component of the angular momentum vector is expressed by the operator $\hat{J}_z = \left(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}\right)/2$. Each eigenvalue of the total angular momentum squared $\hat{J}^2$ corresponds to the total number $N = n_a + n_b$ of phonons accessible in the two modes. The eigenstates of $\hat{J}^2$ can be represented by an angular momentum vector over a sphere whose poles are the eigenstates $|N\rangle$ and $|-N\rangle$ of $\hat{J}_z$.

When possible, the two-mode quantum operations described before can be conceived as the manipulation of such angular momentum vector. Although most frequently the two-mode motional quantum state will not obey $n_a+n_b=N$ in a superposition of Fock states, the Schwinger map can be useful to treat the beam splitter operation, because it conserves the total number of excitations in the two modes.

If we describe our Gaussian gates in a conformal field theory in two dimensions we can also associate to it an algebra of quantum angular momentum using the Virasoro algebra.

3.3 CVQC Toolbox of a single trapped ion

Now, we have obtained a set of basic operations through the ion-laser interaction Hamiltonian for our system of a single ion confined in a Paul trap. With this in hand, we can control the external modes of this quantum oscillator choosing certain frequencies and relative phases in the addition of the Hamiltonians, i.e. with distinct frequencies we have distinct Gaussian operations which are our quantum gates for a system over continuous variables. We can generalize these operations of one and two modes of the trap to obtain a toolbox with these gates. Fig. 3.2 shows that adding the Hamiltonians with distinct frequencies allows us to obtain Gaussian operations.

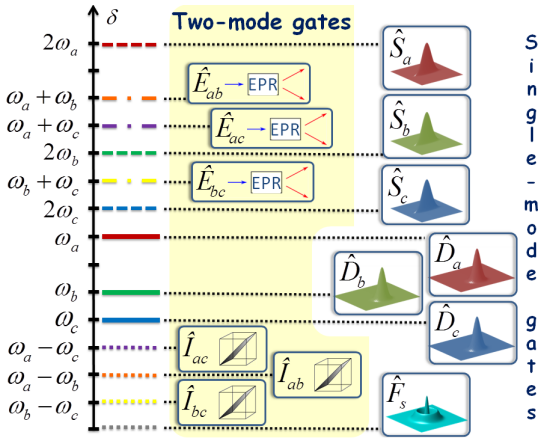

Figure 3.2: The toolbox of Gaussian operations is obtained by a manipulation of vibrational states of a single trapped ion. The desired quantum gate is selected choosing the radiation frequency. All Gaussian gates require bichromatic radiation with $\delta_1 = -\delta_2 = \delta$, with the exception of the Fourier transform. Possible values of the detuning frequency correspond to any of the vibrational frequencies ω_s (continuous lines), double of those frequencies (dashed lines), subtraction (dotted lines) or sum (dashed dot lines) of pairs of frequencies. The corresponding quantum operations are displacement $\hat{D}_s$, squeezer $\hat{S}_s$, beam splitter $\hat{I}ss'$, and two-mode squeezer $\hat{E}_{ss'}$, respectively, where $s, s' \varepsilon \{a, b, c\}$.

In Fig. 3.2 we have depicted the Gaussian operations together with the bichromatic laser detuning required for its performance. The obtained single-mode quantum gates are: displacement $\hat{D}_s$, squeezer $\hat{S}_s$, and Fourier gate $\hat{F}_s$. Considering the correspondence between gates and frequencies, we have

$$\begin{aligned} \delta_1 = -\delta_2 = \omega_s : &\quad \hat{D}_s, \\ \delta_1 = -\delta_2 = 2\omega_s : &\quad \hat{S}_s, \\ \delta = 0 : &\quad \hat{F}_s. \end{aligned} \tag{3.89}$$

Considering now two-mode operations, we have as quantum gates: the beam splitter $\hat{I}_{ss'}$, the two-mode squeezer $\hat{E}_{ss'}$, and two of the conditional operations $\hat{C}^{(1)}_{ss'}$, $\hat{C}^{(2)}_{ss'}$. Thus, these operations

require the frequencies

$$\begin{aligned}
&\delta_1 = -\delta_2 = \omega_s - \omega_{s'} : && \hat{I}_{ss'}, \\
&\delta_1 = -\delta_2 = \omega_s + \omega_{s'} : && \hat{E}_{ss'}, \\
&\begin{cases} \delta_1 = -\delta_2 = \omega_s - \omega_{s'} \\ \delta_3 = -\delta_4 = \omega_s + \omega_{s'} \end{cases} : && \hat{C}^{(1)}_{ss'},\ \hat{C}^{(2)}_{ss'}.
\end{aligned} \tag{3.90}$$

In summary, one of the quantum gates makes use of a monochromatic beam (Fourier gate $\hat{F}_s$), four require bichromatic radiation (displacement $\hat{D}_s$, squeezer $\hat{S}_s$, beam splitter $\hat{I}_{ss'}$, and two-mode squeezer $\hat{E}_{ss'}$), and two use tetrachromatic radiation (the conditional operations $\hat{C}^{(1)}_{ss'}$, $\hat{C}^{(2)}_{ss'}$).

The different operations are selected by the laser detuning frequency δ in such a way as to keep the off-resonance terms of the Hamiltonian small. Our architecture requires frequencies ω_s, $\omega_{s'}$, $2\omega_{s'}$, $2\omega_s$, $\omega_s - \omega_{s'}$ and $\omega_s + \omega_{s'}$ to be well defined and sufficiently separated. In general, 12 frequencies are available. The Lamb-Dicke parameter can be used to control some of the quantum gates, for instance the Fourier transform $\hat{F}_s$. With this control, it is possible to decrease undesired interactions of the modes if we employ radiation in a suitable direction.

In our work, we considered a single trapped ion modeled as a quantum harmonic oscillator with three independent modes. These vibrational modes are dropped in the 'CZ paradigm'. However, they were explored in our construction to increase the available Hilbert space and the size of the quantum system.

The size of the Hilbert space associated with each mode is limited by the trap potential, which makes the distance between the Fock states' energy levels dependent of the phonon numbers. Therefore, we can estimate the performance of the ion trap motional modes. Considering electrode distances of the order of 1 mm we can have approximately 100 phonons according to [76]. Thus, the available Hilbert space in the vibrational modes would be $D \approx (100)^3 \sim 10^6$. The necessary number of qubits to produce a Hilbert space of the same dimension in the CZ paradigm would be $N \approx \log 10^6 / \log 2 \approx 20$.

In conclusion, in this M.Sc. thesis we propose an alternative route to help mitigate the scalability problem in the ion trap processor. The degrees of freedom of a single trapped ion give us a configuration space with a size restricted by the non-linearities of the trap potential. For instance, if we consider a vibrational frequency $\omega_s \approx 10^6$ Hz, the ion wave function occupies a physical volume $V \approx 10^3$ nm. Thus, this implies that the linear dimension of the ion wave function would be given by $l = \sqrt{N+1}x_s$ and the physical volume by $V \approx (N+1)^{\frac{3}{2}} V_0$, where $V_o = x_a x_b x_c$ is the volume of the ground state wavefunction. The ratio between the size of the

Hilbert space and the volume occupied by a single ion is $\approx N^{32}/V_0$ for $N >> 1$. Furthermore, we can ensure that switching to CVQC would increase the available Hilbert space in the current ion trap processor making it better than the previously explored limits and giving it a small increase in complexity.

Chapter 4
Conclusion and outlook

> "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others."
>
> Letter to Bolyai, 1808, by Carl Friedrich Gauss.

In this M.Sc. thesis, we developed a toolbox of quantum gates to manipulate the vibrational modes of trapped ions. We have provided the conceptual tools to realize one- and two-mode Gaussian quantum operations as well as non-Gaussian single-mode operations.

Our trap has a single atom that is irradiated with a classical laser field, tuned to different frequencies. The most interesting part is that we obtain distinct gates just by the control of the frequency of the laser, i.e. by manipulating the external degrees of freedom of our system and coherently manipulating the ion.

We showed that with this toolbox, we can realize operations of quantum computation over continuous variables, such as the Fourier transform, the CNOT and CPHASE gates. Bringing together the Gaussian and non-Gaussian terms, we show that it is possible to simulate quantum computation over **CV** from commutators that generate polynomials with a degree higher than

three, in momentum or position operators. We explored more deeply the behavior of non-Gaussian terms for one and two modes of the interaction Hamiltonian. We showed, that our gates can also be written as generalized Pauli matrices, and so we obtain Weyl-Heisenberg operators that are objects of the Pauli group used in the stabilizer formalism. Thus, with these elements of the Pauli group, we can construct codes to encode our system.

We could still try to further develop the theoretical developments to construct quantum error correction codes over **CV** in quantum topology making use of the AdS/CQE theory [77].

We could also try to connect our tools with the protocols of relativistic quantum information. We know that there are some papers showing gates over **CV** for **CQED** systems, which are used to explore some aspects of the interface between quantum information and relativity [78]. The problems of relativistic quantum information have been gaining enormous interest due to fact that we could in principle realize simulations of relativistic effects via controlled systems, in particular in a single trapped ion [79, 80, 81].

It is also conceivable to simulate the Dirac's equation in N dimensions by controlling the motional degree of freedom. In particular, we may consider our ion in the Paul trap as being in an anisotropic and/or isotropic space and explore the Hamiltonians that were investigated in this work. With this, we could try to make the necessary adjustments needed to get the Dirac equation, which would allow us to analyze important physical effects, such as the 'Zitterbewegung' and Klein's paradox, in different situations. The implementation of these quantum simulations using the Dirac dynamic in a single trapped ion should initially use the electronic internal states coupled to two motional degrees of freedom.

Being able to simulate a Hamiltonian in the Dirac dynamics, we could still study the increase of the ion effective mass starting by gradually increasing the coupling strength to try to understand its relation with the spontaneous symmetry breaking mechanism of the Higgs field [79].

Another idea that we could explore is the complexification of the Fock's space in the formulation of path integrals. To do this we would need to use a conformal field theory in 2-D as a tool. We might use our Gaussian toolbox to describe our objects by the formalism of vertex operators for coherent states. Thus, we may make the operator primary expansion (OPE's) between the vertex operators (Gaussian operators) and the vertex operator and tensor momentum-energy for bosons and fermions. Using the Verma vertex through an expansion in (OPE's), we could analyse the Gaussian parameters and connect them with the cosmological constant expansion

of the universe. Besides, we could try to generate the central charge 'c', a very important theme to study the entanglement entropy of systems with finite dimension. Thus, we could compute the entropy of our system in distinct geometries [82, 83].

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