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A PROPOSAL TO QUANTIFY QUANTUM NON-LOCALITY

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A PROPOSAL TO QUANTIFY QUANTUM NON-LOCALITY

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A PROPOSAL TO QUANTIFY QUANTUM NON-LOCALITY

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Presidente e Orientador

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ABSTRACT

Bell functions are known by the central role played on the characterization of non-locality in quantum theory. They are often used in the quantification of the non-locality strength for specific quantum systems by calculating their maximum among all possible states and detector's configurations. However, even if two quantum states present different "non-local density configurations"(States with different contributions of detector configurations that generate non-locality), if these display the same value for the maximum of Bell function, then they are considered equally non-local. Making use of this criteria, Acín et al. (Phys. Rev. A 65, 052325, 2002) found that for *qunits* (Joint states of d-dimensional quantum systems), the maximally entangled state does not match with the maximally non-local state, this is known as an anomaly of quantum non-locality (Méthot & Scarani; Quant. Inf. Comput. 7, 157, 2008).

In order to solve the anomaly problem, in this dissertation it is proposed a non-locality strength measure in which the whole contributions of detector's configurations that give rise to non-locality are taken into account. Such a measure is proportional to the Bell function integration over the violation region on the space of the parameters that characterize the detector's configuration (Usually, relative angles between orientations of detectors).

The non-locality strength was calculated for several two and three-level bipartite systems, with and without a contribution of white noise to the whole state of the system, by using three kinds of Bell inequalities: Bell inequality in its original version, CHSH inequality and GCLMP inequality (Acín et al. 2002). In all the cases, it was observed agreement between maximally entangled states and maximally non-local ones, thus solving the problem of anomaly of non-locality.

Keywords: Quantum Non-Locality. QuNit Systems. Non-Locality Anomaly.

RESUMO

As funções de Bell são conhecidas pelo papel central desempenhado na caracterização da não localidade da teoria quântica. Usualmente são empregadas na quantificação do grau de não localidade de sistemas quânticos específicos através do calculo do seu valor máximo entre todos os possíveis estados e configurações associadas aos detectores. No entanto, embora dois estados quânticos tenham diferentes "densidades de configuração não local"(Estados com diferentes contribuições de configurações associadas aos detectores que geram não localidade), se eles possuem o mesmo valor do máximo da função de Bell, então são considerados igualmente não locais. Usando este critério, Acín et al. (Phys. Rev. A 65, 052325, 2002) encontraram que para *qunits* (Estados de sistemas quânticos conjuntos d-dimensionais), o estado maximamente emaranhado não corresponde ao estado maximamente não local, fato que é considerado como uma anomalia da não localidade da teoria quântica (Méthot & Scarani; Quant. Inf. Comput. 7, 157, 2008).

A fim de resolver o problema da anomalia, nesta dissertação é proposta uma medida do grau de não localidade, na qual são tomadas em conta todas as contribuições de configurações dos detectores que geram não localidade do estado. Tal medida é proporcional à integral da função de Bell na região de violação, no espaço dos parâmetros que caracterizam as configurações (Em geral ângulos relativos entre orientações dos detectores).

Foi calculado o grau de não localidade de vários sistemas de dois e três níveis, com e sem uma contribuição de ruído quântico ao estado, usando três tipos de desigualdades de Bell: A desigualdade na versão original, a desigualdade CHSH e a desigualdade GCLMP (Acín et al. 2002). Em todos os casos estudados foi observada concondância entre o estado maximamente emaranhado e o estado maximamente não local, resolvendo assim o problema da anomalia da não localidade.

Palavras-chave: Não-Localidade Quântica. Sistemas QuNit. Anomalia de Não Localidade.

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CHAPTER 1

INTRODUCTION

Entanglement is the concept behind some of the most bemusing features of quantum mechanics. In spite of this, it alone may be seen as a purely mathematical concept: the failure of a vector in a large Hilbert space $\mathcal{H}$ to be represented as a single tensor product of vectors in smaller spaces $\mathcal{H}_i$, that together form $\mathcal{H} = \underset{i}{\otimes}\mathcal{H}_i$. As a result of intense research in the field, it is presently clear that there is no unique way to quantify entanglement. In this sense its manifestations or effects, as for example, non-locality, are more concrete (or even meaningful) from a physical perspective. Put differently, entanglement happens in Hilbert spaces while non-locality manifests itself in our ordinary (3+1)D-space. It seems, therefore, that the pursuit of measures to quantify non-locality is a task worth of some effort. There is little dispute, if any, on the fact that entanglement and non-locality are different resources. This consideration, however, has been pushed to a point that may not correspond to the actual state of affairs, as it is suggested in this dissertation.

Although different quantities, there are two desirable properties that candidates to quantifiers of entanglement and non-locality should have in common. First, (i) maximally entangled and maximally non-local states should coincide, and in addition, (ii) as we can attach a number to a state $|\Psi\rangle$ to estimate its degree of entanglement, given a particular experiment one should be able to refer to the amount of non-locality embodied by $|\Psi\rangle$ regarding that experiment with no direct mention to particular settings ("tracing out" settings).

In an essay in honor of Abner Shimony, Nicolas Gisin give a list of open questions related to Bell inequalities [2]. The one most directly related to the present work is "Why are almost all known Bell inequalities for more than 2 outcomes maximally violated by states that are not maximally entangled?". This means that, by adopting the maximum of Bell functions as quantifiers of non-locality, condition (i) in the previous paragraph is not fulfilled. This problem has been often mentioned and studied in the literature, in some cases, it has been refereed to as an "anomaly"[3].

In this dissertation we propose a new way to quantify non-locality that accounts for properties (i) and (ii), thus, eliminating the alleged anomaly in the non-locality of two entangled three-level systems.

In **chapter 2** it is given a brief review about the development of the concept of non-locality in quantum mechanics from the EPR paradox, passing through Bell's theorem and inequalities to recent experimental confirmations.

Chapter 3 is devoted to present some progresses concerning the problem of non-locality in high-dimensional bipartite systems, the Bell-CGLMP inequality and the non-locality anomaly as one of its consequences.

In **chapter 4** our proposal to quantify non-locality is presented and some results on bipartite spin-$\frac{1}{2}$ entangled systems are discussed.

In **chapter 5** the main result is exposed. It is shown that, by applying our quantifiers (eqs. 4.1 and 4.2), no conflict between maxima of entanglement and non-locality appears. This seems to indicate a quite plausible solution to the so-called non-locality anomaly.

Finally, in **chapter 6** conclusions and future perspectives of the present dissertation are given.

CHAPTER 2

EPR PARADOX AND BELL INEQUALITIES

2.1 THE EPR ARGUMENT

In 1935 Albert Einstein, Boris Podolsky and Nathan Rosen [4] (hereafter EPR) published an elegant set of arguments which led them to conclude that quantum theory is incomplete.

The EPR's discussion begins with two requirements which, according to them, must be satisfied by a physical theory:

- **Correctness**, referring to the agreement between predictions of a theory and results of experiments (In the domain of validity of the theory). And

- **Completeness**, in the words of EPR: "*Every element of physical reality must have a counterpart in the physical theory*".

  Then, a definition of what they mean by element of physical reality is given, which amounts to a quantity that is not changed by a measurement and whose outcome can be predicted with certainty (with probability equal to unity).

Finally, the principle of **locality** is assumed: *There is no action-at-a-distance in nature* [5].

2.1.1 The Bohm Version

In the EPR paper, a treatment using operators of position and momentum is carried out, but a more convenient way to illustrate the EPR points of view is to employ a set of operators with a discrete-finite set of eigenvalues, such as the spin angular momentum operators, as argued by David Bohm and Yakir Aharonov[1][6].

The system considered by Bohm and Aharonov consists of a stationary spin-zero particle decaying in two entangled spin-$\frac{1}{2}$ particles[2], each one in turn moving towards two opposite observers, say Alice and Bob as usual. Assuming that the total angular momentum of the system is conserved and that there is no spin-orbit interaction, the quantum state of the system $|\Psi\rangle$ may be written as:

$$|\Psi\rangle = |\Phi\rangle \otimes |\chi\rangle,$$

where $|\Phi\rangle$ is the orbital (symmetric) part and $|\chi\rangle$ is the spin (antisymmetric) part of the wave function. The latter is known as singlet state and is given by:

$$|\chi\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_1 \otimes |-\rangle_2 - |-\rangle_1 \otimes |+\rangle_2 \right],$$

[1]Furthermore it represents the first experimental proposal to test the validity of the EPR hypothesis.
[2]For instance a pion π^0 decaying into an electron e^- and a positron e^+.

here, $|\pm\rangle_k$ expresses an eigenvector of the k particle spin operator in the z direction $\boldsymbol{S}_z^{(k)}$ ($\boldsymbol{\sigma}_z^{(k)}$), with associated eigenvalue $\pm\frac{\hbar}{2}$ (± 1).

For simplicity it can be written as:

$$|\chi\rangle = \frac{1}{\sqrt{2}} \left[| +-\rangle - | -+\rangle \right]. \tag{2.1}$$

Moreover the singlet state is spherically symmetric, then it is invariant under the action of a rotation $\boldsymbol{D}_{\hat{n}}$ to an arbitrary direction $\hat{n}$:

$$|\chi\rangle \to |\chi\rangle_{\hat{n}} = \boldsymbol{D}_{\hat{n}}|\chi\rangle = \frac{1}{\sqrt{2}} \left[| +-\rangle_{\hat{n}} - | -+\rangle_{\hat{n}} \right],$$

thus, if Alice carries out a measurement of $\boldsymbol{S}_{\hat{n}}^{(1)}$, its outcome will always be opposite to that obtained by Bob when he measures $\boldsymbol{S}_{\hat{n}}^{(2)}$.

Given that Alice, e. g., can switch the orientation of the detectors at any moment before the measurement on the particles[3] (for instance $\boldsymbol{S}_{\hat{z}}^{(1)} \to \boldsymbol{S}_{\hat{x}}^{(1)}$) and since no matter her choice, the corresponding quantity for Bob must be determined, in the sense of EPR, it is possible to attribute elements of physical reality to the operators $\boldsymbol{S}_{\hat{z}}^{(2)}$ and $\boldsymbol{S}_{\hat{x}}^{(2)}$ simultaneously.

However quantum theory predicts that observables associated with operators which do not commute cannot be known simultaneously (in our case $\left[\boldsymbol{S}_{\hat{x}}^{(2)}, \boldsymbol{S}_{\hat{z}}^{(2)}\right] \neq 0$), or in other words cannot have elements of physical reality simultaneously.

This contradiction led EPR to conclude that quantum theory does not provide a complete description of nature and opened the question of the possibility of construction of a more general and complete theory.

2.2 BELL THEOREM

In analogy to the relation between statistical mechanics and thermodynamics, many of the attempts made to find a generalization of quantum theory were based on *hidden variables* descriptions in which, besides the standard quantum mechanical state, there should exist a set of additional unknown variables [7, 8]. These hidden variables were supposed to contain the whole information about measurements and outcomes in order to eliminate the apparent action at a distance emerged in EPR's paper. Furthermore, quantum theory must arise as an average of the general theory over all possible states. A very pedagogical example and more details about hidden variables can be found in [9].

In order to deal with the problem of the completeness of quantum theory, John Bell in 1964, using a hidden variable scheme, proposed a set of constraints that must be satisfied to ensure locality of quantum theory. These constraints are known today as *Bell's inequalities* [10].

In his work, Bell considers a system consisting of two spin$-\frac{1}{2}$ particles in the singlet state, moving apart in opposite directions towards two separated observers Alice and

[3]This time may be arbitrarily small in order to ensure that there will be no transport of information about the orientation shift between the apparatus and the other particle.

Bob, each one equipped with a device to measure the spin component associated to the particles $\boldsymbol{S}_{\hat{a}}^{(1)}$ and $\boldsymbol{S}_{\hat{b}}^{(2)}$, when their devices are orientated in the directions $\hat{a}$ and $\hat{b}$ respectively.

A condition of *determinism* is implicitly introduced within a parameter λ that characterizes the set of hidden variables[4] in such a way that the knowledge of this parameter determines the outcome of any measurement performed by Alice or Bob. Without loss of generality, consider λ as a continuous parameter within a space of allowed states Λ. On this, a distribution function $\rho(\lambda)$ may be defined to specify the probability of the state of the system to lie on the interval $[\lambda, \lambda + d\lambda]$: $d\rho = \rho(\lambda)d\lambda$, and is normalized:

$$\int_{\Lambda} \rho(\lambda)d\lambda = \int_{\Lambda} d\rho = 1.$$

According to this description, the result C of a measurement of any observable, in addition to the dependence on variables related with internal settings of the detector α, must depend on the value of the parameter λ, then the mean value E_C of the observable is given by:

$$E_C(\alpha) = \int_{\Lambda} C(\alpha, \lambda)\rho(\lambda)d\lambda.$$

Similarly, in the case of normalized spin operators (or Pauli matrices), $A(\hat{a}, \lambda) = \pm 1$ and $B(\hat{b}, \lambda) = \pm 1$:

$$E_A(\hat{a}) = \int_{\Lambda} \rho(\lambda)A(\hat{a}, \lambda)d\lambda, \tag{2.2}$$

and:

$$E_B(\hat{b}) = \int_{\Lambda} \rho(\lambda)B(\hat{b}, \lambda)d\lambda. \tag{2.3}$$

A second condition takes into account the independence of the results of the measurement on each part of the system, or *locality*. Following this implication, the result of a measurement of the component of spin in an arbitrary direction by Alice $A(\hat{a}, \lambda)$ depends only on the orientation of her Stern-Gerlach device $\hat{a}$ and on the value of the hidden variables λ, and is completely independent on the orientation of the Bob's apparatus $\hat{b}$. As in the case of only one observer (Equations 2.2 and 2.3), the mean value of a joint measurement carried out by Alice and Bob is given by:

$$E(\hat{a}, \hat{b}) = \int_{\Lambda} \rho(\lambda)A(\hat{a}, \lambda)B(\hat{b}, \lambda)d\lambda.$$

$E(\hat{a}, \hat{b})$ is also known as correlation between measurements on the particles 1 and 2 when the detectors associated with Alice and Bob are oriented in the directions $\hat{a}$ and $\hat{b}$, respectively.

If Alice and Bob perform a measurement on a system in the singlet state and their detectors are orientated in the same direction (i.e. $\hat{a}$), it is said that the spin components are perfectly anti-correlated:

$$A(\hat{a}, \lambda) = -B(\hat{a}, \lambda).$$

[4]This set of hidden variables may be composed by only one variable or as many as required by the theory and can be discrete or continuous.

Using the fact that $A(\hat{a},\lambda) = \pm 1$ and the normalization condition for the distribution function of hidden variables $\rho(\lambda)$, the correlation takes the form:

$$E(\hat{a},\hat{b}) = \int_\Lambda \rho(\lambda) A(\hat{a},\lambda) B(\hat{b},\lambda) d\lambda = -\int_\Lambda \rho(\lambda) A^2(\hat{a},\lambda) d\lambda,$$

$$E(\hat{a},\hat{b}) = -\int_\Lambda \rho(\lambda) d\lambda,$$

$$E(\hat{a},\hat{b}) = -1.$$

Now suppose that Alice fixes the orientation of her measuring device on the direction $\hat{a}$ while Bob has two choices $\hat{b}$ or $\hat{b}'$, then:

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') = \int_\Lambda \rho(\lambda) \left[A(\hat{a},\lambda)B(\hat{b},\lambda) - A(\hat{a},\lambda)B(\hat{b}',\lambda)\right] d\lambda,$$

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') = -\int_\Lambda \rho(\lambda) \left[A(\hat{a},\lambda)A(\hat{b},\lambda) - A(\hat{a},\lambda)A(\hat{b}',\lambda)\right] d\lambda,$$

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') = -\int_\Lambda \rho(\lambda) A(\hat{a},\lambda)A(\hat{b},\lambda) \left[1 - A(\hat{b},\lambda)A(\hat{b}',\lambda)\right] d\lambda,$$

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') = \int_\Lambda \rho(\lambda) A(\hat{a},\lambda)B(\hat{b},\lambda) \left[1 + A(\hat{b},\lambda)B(\hat{b}',\lambda)\right] d\lambda.$$

Here it has been used the fact that $A^{-1}(\hat{n},\lambda) = A(\hat{n},\lambda)$.

Two well behaved functions $f(x)$ and $g(x)$ in D, satisfy the following inequality:

$$\left|\int_D f(x)\ g(x) dx\right| \leq \int_D |f(x)\ g(x)|\, dx,$$

then:

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| = \left|\int_\Lambda \rho(\lambda) A(\hat{a},\lambda)B(\hat{b},\lambda) \left[1 + A(\hat{b},\lambda)B(\hat{b}',\lambda)\right] d\lambda\right|,$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq \int_\Lambda \left|\rho(\lambda) A(\hat{a},\lambda)B(\hat{b},\lambda) \left[1 + A(\hat{b},\lambda)B(\hat{b}',\lambda)\right]\right| d\lambda,$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq \int_\Lambda \left|A(\hat{a},\lambda)B(\hat{b},\lambda)\right| \left|\rho(\lambda) \left[1 + A(\hat{b},\lambda)B(\hat{b}',\lambda)\right]\right| d\lambda,$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq \int_\Lambda \left|\rho(\lambda) \left[1 + A(\hat{b},\lambda)B(\hat{b}',\lambda)\right]\right| d\lambda,$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq 1 + E(\hat{b},\hat{b}'). \tag{2.4}$$

This is the Bell Inequality in its original form, or first Bell inequality and must be satisfied by any set of correlations resulting from a local hidden variables theory.

2.2.1 Correlations Predicted by Quantum Theory

In order to test whether quantum theory predictions agree with Bell's inequalities, it is necessary to calculate quantum correlations between measurements performed by Alice and Bob when their detectors are orientated in the directions $\hat{a}$ and $\hat{b}$, respectively. It is equivalent to calculate $\langle \vec{\boldsymbol{\sigma}} \cdot \hat{a} \otimes \vec{\boldsymbol{\sigma}} \cdot \hat{b} \rangle_{\boldsymbol{\rho}}$, where $\vec{\boldsymbol{\sigma}}$ is a vector whose components are the Pauli matrices and $\boldsymbol{\rho}$ is the density operator of the system.

In the case of one-particle system, any operator $\boldsymbol{A}$ can be written in the Dirac notation as:

$$\boldsymbol{A} = \sum_{m,n} A_{mn} \, |m\rangle\langle n|,$$

where A_{mn} are the matrix elements of the operator $\boldsymbol{A}$ in the basis $\{|m\rangle\}$.

In the two-particle case:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \sum_{m,n,\tilde{m},\tilde{n}} A_{m,n} B_{\tilde{m},\tilde{n}} |m\rangle\langle n| \otimes |\tilde{m}\rangle\langle \tilde{n}|,$$

more compactly:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \sum_{m,n,\tilde{m},\tilde{n}} A_{m,n} B_{\tilde{m},\tilde{n}} |m, \tilde{m}\rangle\langle n, \tilde{n}|.$$

Similarly, an arbitrary two party density operator can be expressed as:

$$\boldsymbol{\rho} = \sum_{j,k,\tilde{j},\tilde{k}} \rho_{j,k,\tilde{j},\tilde{k}} |j, k\rangle\langle \tilde{j}, \tilde{k}|.$$

The mean value of $\boldsymbol{A} \otimes \boldsymbol{B}$ on a system characterized by $\boldsymbol{\rho}$ is given by:

$$\langle \boldsymbol{A} \otimes \boldsymbol{B} \rangle_{\boldsymbol{\rho}} = \mathrm{Tr}\left(\boldsymbol{\rho} \boldsymbol{A} \otimes \boldsymbol{B} \right),$$

$$\langle \boldsymbol{A} \otimes \boldsymbol{B} \rangle_{\boldsymbol{\rho}} = \sum_{l,\tilde{l}} \langle l, \tilde{l} | \boldsymbol{\rho} \boldsymbol{A} \otimes \boldsymbol{B} | l, \tilde{l} \rangle,$$

$$\langle \boldsymbol{A} \otimes \boldsymbol{B} \rangle_{\boldsymbol{\rho}} = \sum_{j,k,\tilde{j},\tilde{k},l,\tilde{l},m,n,\tilde{m},\tilde{n}} \rho_{j,k,\tilde{j},\tilde{k}} A_{m,n} B_{\tilde{m},\tilde{n}} \langle l, \tilde{l} | j, k \rangle \langle \tilde{j}, \tilde{k} | m, \tilde{m} \rangle \langle n, \tilde{n} | l, \tilde{l} \rangle,$$

$$\langle \boldsymbol{A} \otimes \boldsymbol{B} \rangle_{\boldsymbol{\rho}} = \sum_{j,k,\tilde{j},\tilde{k},l,\tilde{l},m,n,\tilde{m},\tilde{n}} \rho_{j,k,\tilde{j},\tilde{k}} A_{m,n} B_{\tilde{m},\tilde{n}} \; \delta_{l,j} \delta_{\tilde{l},k} \delta_{\tilde{j},m} \delta_{\tilde{k},\tilde{m}} \delta_{n,l} \delta_{\tilde{n},\tilde{l}} \;,$$

$$\langle \boldsymbol{A} \otimes \boldsymbol{B} \rangle_{\boldsymbol{\rho}} = \sum_{l,\tilde{l},m,\tilde{m}} \rho_{l,\tilde{l},m,\tilde{m}} A_{m,l} B_{\tilde{m},\tilde{l}} \;.$$

Applying this result, the quantum correlation between measurements of spin, performed by Alice in the direction $\hat{a}$ and Bob in the direction $\hat{b}$, $E(\hat{a}, \hat{b})$ takes the form:

$$E(\hat{a}, \hat{b}) = \langle \vec{\boldsymbol{\sigma}} \cdot \hat{a} \otimes \vec{\boldsymbol{\sigma}} \cdot \hat{b} \rangle_{\boldsymbol{\rho}} = \sum_{l,\tilde{l},m,\tilde{m}} \rho_{l,\tilde{l},m,\tilde{m}} \sigma^{a}_{m,l} \sigma^{b}_{\tilde{m},\tilde{l}} \;, \tag{2.5}$$

where $\sigma^n_{m,l}$ are the matrix elements of the projection of the vector of Pauli matrices $\vec{\boldsymbol{\sigma}}$ in the direction $\hat{n}$, in the basis $\{|+\rangle, |-\rangle\}$:

$$\vec{\boldsymbol{\sigma}} \cdot \hat{n} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}. \tag{2.6}$$

The density operator of a system composed by two spin-$\frac{1}{2}$ particles in the singlet state in the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ is given by:

$$\boldsymbol{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituting, the quantum correlation takes the form:

$$E(\hat{a}, \hat{b}) = \rho_{+-+-}\sigma^a_{++}\sigma^b_{--} + \rho_{-+-+}\sigma^a_{--}\sigma^b_{++} + \rho_{+--+}\sigma^a_{-+}\sigma^b_{+-} + \rho_{-++-}\sigma^a_{+-}\sigma^b_{-+} \ ,$$

then:

$$E(\hat{a}, \hat{b}) = -a_x b_x - a_y b_y - a_z b_z = -\hat{a} \cdot \hat{b},$$
$$E(\hat{a}, \hat{b}) = -\cos\theta_{ab}, \tag{2.7}$$

where θ_{ab} is the angle subtended by the unitary vectors $\hat{a}$ and $\hat{b}$.

Thus, quantum correlations for a system on the singlet state depend only on the relative orientations of the involved detectors. For a particular choice of coplanar orientations in which the unitary vector $\hat{a}$ forms an angle of $\frac{\pi}{3}$ with $\hat{b}$ and $\frac{2\pi}{3}$ with $\hat{b}'$, Bell's inequality becomes:

$$\left| -\frac{1}{2} - \frac{1}{2} \right| \leq 1 - \frac{1}{2} \qquad \text{or} \qquad 1 \leq \frac{1}{2}!$$

Thus, quantum correlations do not always satisfy Bell's inequality.

This result constitutes the proof of a first version of Bell's theorem [5]: No *deterministic* hidden-variables theory that admits *perfect anti-correlations*, satisfying a *locality* condition can agree with all predictions by quantum mechanics concerning the spins of a pair of spin-$\frac{1}{2}$ particles in the singlet state.

Although some stages in the construction of Bell theorem are grounded on idealized conditions[5], it constitutes an experimental scheme to follow in order to decide between quantum theory and local hidden-variables theories to deal with the EPR paradox.

2.3 CHSH INEQUALITY

In a seminal work, John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt [11](Herafter CHSH), keeping assumptions of locality and determinism and dispensing with the condition of perfect correlations, obtained a more general version of Bell's inequality. Here, the original proof of the inequality will not be shown. Instead, it is presented a version due to John Bell [12]:

[5] The perfect anti-correlation requirement implies the existence of detectors with an efficiency of 100% and zero attenuation.

2.4 SECOND BELL INEQUALITY

As in the first version, the system is composed by two entangled spin-$\frac{1}{2}$ particles traveling away one from each other towards two observers Alice and Bob, each of them equipped with measuring devices that are capable of measuring spin's component along the directions $\hat{a}$ and $\hat{b}$, respectively.

The state of measuring devices must be taken in consideration, since it may affect the correlations. Thus, values of outcomes A and B are averaged over hypothetical variables related to the measurement instruments, so the averaged outcomes $\overline{A}$ and $\overline{B}$ are no longer binary quantities, but continuous between -1 and 1. It is important to note that the possibility of no detection ($\overline{A} = 0$ or $\overline{B} = 0$) is also allowed.

Following the notation used in the first Bell inequality, correlations may be written as:

$$E(\hat{a}, \hat{b}) = \int_{\Lambda} \rho(\lambda) \overline{A}(\hat{a}, \lambda) \overline{B}(\hat{b}, \lambda) d\lambda.$$

The locality condition is again taken into account, while this is not the case for determinism because of the randomness implicit in $\overline{A}$ and $\overline{B}$.

Let $\hat{a}'$ and $\hat{b}'$ be alternative orientations of the Alice and Bob detectors respectively, then:

$$E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') = \int_{\Lambda} \rho(\lambda) \left[\overline{A}(\hat{a}, \lambda) \overline{B}(\hat{b}, \lambda) - \overline{A}(\hat{a}, \lambda) \overline{B}(\hat{b}', \lambda) \right] d\lambda \tag{2.8}$$

For simplicity, let:

$$A_a \equiv \overline{A}(\hat{a}, \lambda) \qquad \text{and} \qquad B_b \equiv \overline{B}(\hat{b}, \lambda).$$

Adding and subtracting $\pm A_a B_b A_{a'} B_{b'}$ in equation 2.8 and taking the absolute value:

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') \right| = \left| \int_{\Lambda} \rho(\lambda) \left\{ A_a B_b \left[1 \pm A_{a'} B_{b'} \right] - A_a B_{b'} \left[1 \pm A_{a'} B_b \right] \right\} d\lambda \right|.$$

Using the property $|a - b| \leq |a + b| \leq |a| + |b|$:

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') \right| \leq \left| \int_{\Lambda} \rho(\lambda) A_a B_b \left[1 \pm A_{a'} B_{b'} \right] d\lambda \right| + \left| \int_{\Lambda} \rho(\lambda) A_a B_{b'} \left[1 \pm A_{a'} B_b \right] d\lambda \right|.$$

As:

$$\left| \int f(x) dx \right| \leq \int |f(x)| \, dx,$$

$$\rho(\lambda) \geq 0,$$

and:

$$|A_a B_b| \leq 1,$$

then:

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') \right| \leq \int_{\Lambda} \rho(\lambda) \left[1 \pm A_{a'} B_{b'} \right] d\lambda + \int_{\Lambda} \rho(\lambda) \left[1 \pm A_{a'} B_b \right] d\lambda,$$

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') \right| \leq 1 \pm E(\hat{a}', \hat{b}') + 1 \pm E(\hat{a}', \hat{b}),$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq 2\left|E(\hat{a}',\hat{b}') + E(\hat{a}',\hat{b})\right|,$$

or more symmetrically:

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| + \left|E(\hat{a}',\hat{b}') + E(\hat{a}',\hat{b})\right| \leq 2. \tag{2.9}$$

This is the second Bell inequality and is equivalent to the expression derived independently by CHSH [6].

Using absolute value properties on equation 2.9:

$$-2 \leq E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') + E(\hat{a}',\hat{b}') + E(\hat{a}',\hat{b}) \leq 2. \tag{2.10}$$

Due to its irreducibility, it is more common to find texts in the literature referring to equation 2.10 (instead of 2.9) as the Bell-CHSH inequality. This in turn may be written as:

$$-2 \leq S(\hat{a},\hat{b},\hat{a}',\hat{b}') \leq 2,$$

where $S(\hat{a},\hat{b},\hat{a}',\hat{b}')$ is known as Bell's parameter or Bell-CHSH function.

A simple test of compatibility between quantum predictions and CHSH inequality may be carried out by restricting the possible orientations of measuring devices $(\hat{a},\hat{a}')$ and $(\hat{b},\hat{b}')$ to a plane, placed in a set of configurations such that every pair of unit vectors $(\hat{a}_i,\hat{b}_j)$ are separated by an angle φ, except for the corresponding to $(\hat{a},\hat{b}')$, which form an angle of 3φ. Since measurements are performed on a system consisting of two entangled two-level particles in the singlet state, then correlations given by equation 2.7 can be used. Substituting, the Bell-CHSH function takes the form:

$$S(\varphi) = \cos(3\varphi) - 3\cos(\varphi). \tag{2.11}$$

A plot of the Bell-CHSH function for this arrangement is shown in figure 2.1, it presents two regions where Bell-CHSH inequality is violated and each of these has associated two angles in which maximal violation is reached[7].

Given that certain measurement configurations on a particular entangled state lead to violation of Bell-CHSH inequality and in order to exclude perfect anti-correlations and determinism conditions present in its former version, then Bell's theorem may be reformulated as [13]:

"No physical theory of local hidden variables can ever reproduce all of the predictions of quantum mechanics."

[6]Note that doing $\hat{a}' = \hat{b}'$ and allowing perfect anti-correlation ($E(\hat{b}',\hat{b}')=-1$), the second Bell inequality (equation 2.9) becomes:

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| + 1 - E(\hat{a}',\hat{b}) \leq 2,$$

$$\left|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')\right| \leq +1 + E(\hat{b}',\hat{b}),$$

that is the first Bell inequality, as expected.

[7]It can be easily shown that the extrema of Bell-CHSH function are $\pm 2\sqrt{2}$ for any of the following angle's configurations: $\varphi* = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

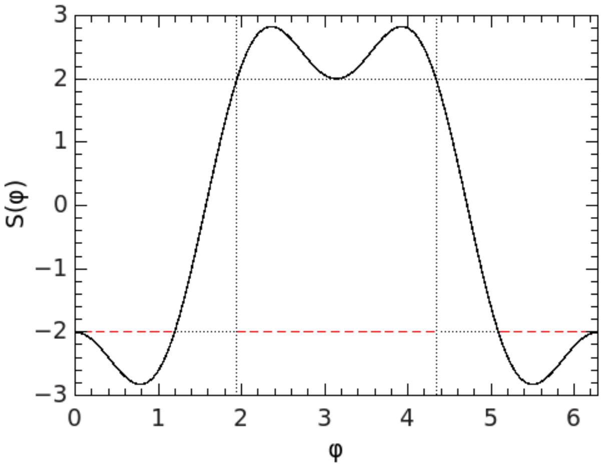

Figure 2.1 *(Black solid)* Bell-CHSH function for a system composed by two two-level particles in the singlet state and a coplanar set of orientations of Alice and Bob's detectors $(\hat{a}, \hat{a}')$ and $(\hat{b}, \hat{b}')$ respectively, separated (in the next order: $\hat{a} \rightarrow \hat{b} \rightarrow \hat{a}' \rightarrow \hat{b}'$) by an angle φ. *(Red dashed)* Regions of violation of CHSH inequality.

At that point only an experiment could give the final verdict between hidden variables theories and quantum mechanics in the search for a solution to the EPR paradox. However, the experiment proposed by CHSH requires a two-channel detection scheme and was carried out only in 1982, since at the time suitable devices still had not been developed. Because of this, several alternative proposals based on single channel detection schemes to detect polarization states of photons were formulated[8].These inequalities instead of relate correlations (like CHSH does) use the number of detection counts in each one and both detectors.

Clauser and Horne [14] elaborated an inequality that includes inherently stochastic theories, in which if it is used the same device configuration that was used in [11], then CHSH inequality is obtained.

Freedman and Clauser [15] proposed an inequality with correlations of linear polarization of photons in an atomic cascade of calcium. Their predicted quantum correlations violate such inequality for sufficiently small detector solid angles and highly efficient polarizers. Finally, they found experimentally a strong agreement between measured and quantum predictions.

These are known as "first generation experiments"[16] which even showing great compatibility between results and quantum predictions, were not taken seriously because in the construction of the inequalities they had to make several assumptions that were far from ideal. One of their suppositions was that the detected particles were part of a significant sample of emerging particles from the source, which was a very strong assumption.

The next generation of experiments is represented by the work of Alain Aspect and

[8]In this approach each arm has only one no event-ready detector, so the possible outcomes are either count or no-count.

colleagues [17] in which experiments following a two channel scheme to test CHSH inequality are developed. This generation of experiments again cannot be considered conclusive because some features in their experimental arrangements are not disposed in such a way that wrong conclusions from the results can be avoided: First, it is necessary to ensure that information about the choice of orientation done by one of the observers cannot in any way reach the other observer (traveling at a velocity equal or less than speed of light), before the latter one has chosen his own orientation[9]. Any experiment of this kind that satisfy such a condition is said to be free from the *Locality Loophole*. Second, in most of the experiments using photons, the proportion of pairs of particles detected to emitted is extremely low, making the experiment non-viable statistically. In order to avoid the so called *Detection Loophole*, either the detection efficiencies must improve or experiments with material particles, like in the Bohm version of EPR paradox should be carried out [8].

More recently, "third generation experiments" have been devoted to close definitely the loopholes. In 1998 a group at Innsbruck led by Anton Zeilinger [18] performed an experiment that closed the loophole of locality completely by putting two observers separated far enough ($\sim 400\ m$) and using polarizers able to change their orientations randomly as quickly as needed to ensure causal independence. Then, in 2001 the first experiment closing detection loophole was carried out using massive entangled pairs of $^9Be^+$ ions [19]. Each of the third generation experiments confirm the validity of quantum theory closing one of the loopholes independently, but fail to close both simultaneously.

Recent experiments have succeeded to close the detection loophole using photons instead of massive particles [20, 21], indicating that it will not take too long until results of Bell test experiments appear closing locality and detection loopholes simultaneously and definitely confirming the non-local character of quantum theory.

[9]It is equivalent to say that events in which orientations are chosen by observers must be causally disconnected.

CHAPTER 3

CGLMP INEQUALITY AND NON-LOCALITY ANOMALY

3.1 BELL MULTIPORT BEAM SPLITTERS

First progresses in the study of quantum non-locality used measurement schemes based on Stern-Gerlach devices. Nevertheless, such approaches are too limited because these only work properly for systems that are described by two-dimensional Hilbert spaces (i.e. spin one-half particles) due to the fact that this kind of measurements are characterized by two parameters only, thus the whole space is not exploited completely in higher dimensional systems [1].

In order to study high dimensional systems, Anton Zeilinger and collaborators developed a scheme based on devices known as *multiport beam splitters*, which are arrangements of simple optical elements, like mirrors and beam splitters [22], able to represent discrete finite-dimensional unitary operators [23]. This scheme may be used to obtain pairs of photons in N-dimensional entangled states (analogous to entangled states of pairs of spin-N particles) which obey quantum correlations in the same fashion as spin-$\frac{1}{2}$ particles do within the framework of Stern-Gerlach measurements [24, 25].

As it can be seen in figure 3.1, the experiment is divided in three stages. A brief description of each one of them is given in subsections 3.1.1 to 3.1.3:

3.1.1 Generation of Photon Pairs

The generation of pairs of photons is given by the process of spontaneous parametric down-conversion [26, 27]. In this process a photon passing through a non-linear crystal gives rise to two photons. If $\vec{k}_i$ denotes the wave vector and ω_i the frequency associated to the i-th photon, conservation of linear momentum and energy respectively may be written as:

$$\vec{k}_0 = \vec{k}_1 + \vec{k}_2, \tag{3.1}$$

$$\omega_0 = \omega_1 + \omega_2, \tag{3.2}$$

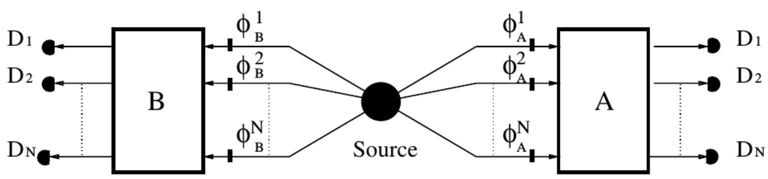

Figure 3.1 Experimental arrangement of Bell multiport beam splitters. Figure reproduced from [1].

where the subscript 0 denotes the decaying photon, while 1 and 2, correspond to the produced photons. An additional condition is that the production of both photons must be simultaneous.

If equations 3.1 and 3.2 are satisfied, a pair of photons with frequencies $\omega_1 = \omega_2 = \frac{\omega_0}{2}$ are generated in the symmetrical case. Schematically, the system can be seen as 3 photons lying over the surface of a standard cone in the following manner: The decaying photon on the vertex and the two produced in points diametrically opposite over the circumference formed by the base of the cone. Thus, it is possible to locate a plate arranged with two pinholes such that only pairs of photons created simultaneously are able to cross. The quantity of pairs of diametrically disposed pinholes over the plate may be increased as much as required, in the present case, up to the dimension of interest N.

At this point, a set of N unpaired photons is channeled and directed to an observer (say Alice) and the remaining N unpaired photons is also channeled and sent to another observer, Bob. Thus, the state of the system may be written as:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} |m\rangle_A \otimes |m\rangle_B, \tag{3.3}$$

where $|m\rangle_A$ represents a photon into the channel m corresponding to Alice, and with $|m\rangle_B$ being the analogous state to Bob.

It is important to note that this state is formally equivalent to a maximally entangled state of two particles of spin-N.

If, somehow, the source of pairs of entangled photons is configured in such a way that the feeding of each one of the channels is not uniform, then the state have to be written as:

$$|\Psi\rangle = \sum_{m=0}^{N-1} \alpha_m |m\rangle_A \otimes |m\rangle_B.$$

For convenience, the variable corresponding to the channel number m has been changed to range from 0 to $N-1$.

3.1.2 Phase Shift

In order to introduce macroscopic parameters (like orientations of detectors in Stern-Gerlach experiments), local adaptive phase shifters are employed over all the channels belonging to Alice and Bob. Let the phase shifts corresponding to the m-th channel in the devices of Alice and Bob be ϕ_a^m and φ_b^m respectively. Thus the state of a photon passing through a phase shifter is transformed as follows:

$$|m\rangle_A \to e^{i\phi_a^m} |m\rangle_A,$$

for a photon traveling to Alice, and

$$|m\rangle_B \to e^{i\varphi_b^m} |m\rangle_B,$$

for a photon traveling to Bob.

The state of the whole system is then transformed into[10]:

$$|\Psi\rangle \rightarrow |\Psi'\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i(\phi_a^m + \varphi_b^m)} |m\rangle_A \otimes |m\rangle_B,$$

for the maximally entangled case, and:

$$|\Psi\rangle \rightarrow |\Psi'\rangle = \sum_{m=0}^{N-1} \alpha_m e^{i(\phi_a^m + \varphi_b^m)} |m\rangle_A \otimes |m\rangle_B, \tag{3·4}$$

for the general case.

3.1.3 Multiport Beam Splitters

A multiport beam splitter is an optical device consisting of $2N$ channels (N input and N output) able to represent any unitary discrete operation on a finite dimension by successive application of unitary operations on two dimensional subsystems performed by arrays of ordinary beam splitters [23, 28, 29]. A special case of multiports known as *symmetric unbiased multiport beam splitters* is a very good alternative to carry out generalized *Stern-Gerlach* type measurements on high dimensional systems due to its main feature: Regardless of the gate that a photon enters, it has equal probability of leaving from any of the N output channels[11] [25].

The action of a symmetric beam splitter on a photon entering via channel m is given by a N-dimensional discrete Fourier transform:

$$\boldsymbol{U}_N |m\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}mk} |k\rangle,$$

thus, the operator associated to the transformation has the form:

$$\boldsymbol{U}_N = \frac{1}{\sqrt{N}} \sum_{k,m=0}^{N-1} e^{i\frac{2\pi}{N}mk} |k\rangle\langle m|.$$

As it may be seen, the operator $\boldsymbol{U}_N$ connects any input state $|m\rangle$ to all the output states $|k\rangle$ with the same probability $(1/N)$, as required.

Putting together the action of symmetric multiport beam splitters on Alice and Bob photons:

$$\boldsymbol{U}_N^A \otimes \boldsymbol{U}_N^B = \frac{1}{N} \sum_{l,n=0}^{N-1} \sum_{k,m=0}^{N-1} e^{i\frac{2\pi}{N}mk} e^{i\frac{2\pi}{N}nl} |k\rangle_A \langle m|_A \otimes |l\rangle_B \langle n|_B.$$

[10] Actually there is no need to introduce $2N$ phases, given that the phase is relative, only $2(N-1)$ suffice.

[11] Note that for the case $N = 2$, symmetric unbiased multiport beam splitters are reduced to conventional 50-50 beam splitters.

Simplifying notation as $|a\ b\rangle \equiv |a\rangle_A \otimes |b\rangle_B$, then:

$$\boldsymbol{U}_N^A \otimes \boldsymbol{U}_N^B = \frac{1}{N} \sum_{k,l,m,n=0}^{N-1} e^{i\frac{2\pi}{N}(mk+nl)} |k\ l\rangle\langle m\ n|.$$

The state of a pair of entangled photons after passing on phase shifters and symmetric multiport beam splitters $|\Psi''\rangle$ is given by:

$$|\Psi''\rangle = \boldsymbol{U}_N^A \otimes \boldsymbol{U}_N^B |\Psi''\rangle,$$

$$|\Psi''\rangle = \frac{1}{N} \sum_{j,k,l,m,n=0}^{N-1} \alpha_j e^{i(\phi_a^j + \varphi_b^j)} e^{i\frac{2\pi}{N}(mk+nl)} \langle m\ n|j\ j\rangle |k\ l\rangle,$$

$$|\Psi''\rangle = \frac{1}{N} \sum_{j,k,l=0}^{N-1} \alpha_j e^{i(\phi_a^j + \varphi_b^j)} e^{i\frac{2\pi}{N}j(k+l)} |k\ l\rangle.$$

Thus, quantum mechanical joint probability $P_{ab}^{QM}(k,l)$ of Alice and Bob detect a pair of photons in the output channels k and l respectively, given a set of phase configurations $\{\phi_a^j, \varphi_b^j\}$, may be calculated as:

$$P_{ab}^{QM}(k,l) = \mathrm{tr}\left(\boldsymbol{\Pi}_k \otimes \boldsymbol{\Pi}_l\ |\Psi''\rangle\langle\Psi''|\right),$$

where, $\boldsymbol{\Pi}_k$ and $\boldsymbol{\Pi}_l$ are projection operators on the subspaces associated to the first and second particle respectively. Substituting $|\Psi''\rangle$, and after some alculations, the expression for joint probability takes the form:

$$P_{ab}^{QM}(k,l) = \left| \frac{1}{N} \sum_{j=0}^{N-1} \alpha_j \exp\left[i \left\{ \phi_a^j + \varphi_b^j + \frac{2\pi}{N} j(k+l) \right\} \right] \right|^2,$$

$$P_{ab}^{QM}(k,l) = \frac{1}{N^2} \sum_{j,j'=0}^{N-1} \alpha_j \alpha_{j'}^* \exp\left[i \left\{ \beta_{ab}^j(k,l) - \beta_{ab}^{j'}(k,l) \right\} \right],$$

with $\beta_{ab}^j(k,l) \equiv \phi_a^j + \varphi_b^j + \frac{2\pi}{N} j(k+l)$.

$$P_{ab}^{QM}(k,l) = \frac{1}{N^2} \sum_{j=0}^{N-1} |\alpha_j|^2 + \frac{1}{N^2} \sum_{j \neq j'}^{N-1} \alpha_j \alpha_{j'}^* \exp\left[i \left\{ \beta_{ab}^j(k,l) - \beta_{ab}^{j'}(k,l) \right\} \right],$$

$$P_{ab}^{QM}(k,l) = \frac{1}{N^2} + \frac{2}{N^2} \sum_{j>j'}^{N-1} \Re(\alpha_j \alpha_{j'}^*) \cos\left(\beta_{ab}^j(k,l) - \beta_{ab}^{j'}(k,l) \right),$$

or:

$$P_{ab}^{QM}(k,l) = \frac{1}{N^2} + \frac{2}{N^2} \sum_{j>j'}^{N-1} \Re(\alpha_j \alpha_{j'}^*) \cos \Delta\beta_{ab}^{j,j'}(k,l), \tag{3.5}$$

where $\Delta\beta_{ab}^{j,j'}(k,l) \equiv \beta_{ab}^{j}(k,l) - \beta_{ab}^{j'}(k,l) = \phi_a^j + \varphi_b^j - \phi_a^{j'} - \varphi_b^{j'} + \frac{2\pi}{N}(j-j')(k+l)$.

It is important to note that the numerical value $k+l$ is taken modulo N.

It can be shown that for certain configurations of phase shifters $\{\phi_a^{j*}, \varphi_b^{j*}\}$ in equation 3.5, the positive detection of a photon in a given channel belonging to one of the observers determines with a probability equal to unity the channel whereby the other observer will execute the detection of the corresponding photon, leading to perfect correlations as in the case of spin one half pairs of entangled particles. For details, see [25].

Other proposed experiments [30] and realizations of entangled systems in $N=3$ or *Qutrits* [31, 32] and $N=4$ or *Ququarts* [33] have been recently carried out, representing significant progresses in the field of quantum information.

3.2 PREVIOUS RESULTS

As an extension to the study of pairs of entangled qubits, an arbitrary increase of the dimensionality of each subsystem showed that maximal violation of Bell inequalities as a measurement of non-locality diminishes with N, which was in concordance with the old vision of quasi-classical mechanical behavior of large numbers in quantum mechanics [1]. However, those studies were conducted applying Stern-Gerlach type measurements and as was already said, these kind of measurements are not optimal for high dimensional systems. Later results showed that non-locality survives even if $N \to \infty$. This ambiguity led Dagomir Kaszlikowski, Marek Żukowski and collaborators to work in this problem [1]. They consider a maximally entangled state of two quNits $|\Psi_{max}^N\rangle$ (eq. 3.3) with an additional contribution of white noise characterized by the noise fraction parameter F_N. The density operator of the whole system is given by:

$$\boldsymbol{\rho}_N(F_N) = F_N \boldsymbol{\rho}_{noise} + (1-F_N)|\Psi_{max}^N\rangle\langle\Psi_{max}^N|, \tag{3.6}$$

where the component of white noise is proportional to the unity operator $\boldsymbol{\rho}_{noise} = \boldsymbol{I}/N^2$.

In order to quantify the strength of non-locality, they calculate the maximum amount of noise F_N^{max} for which it is still impossible to describe the system with a local realistic theory (i.e. the maximum amount of noise necessary to ensure no violation of a Bell inequality.). They consider two observers (Alice and Bob as usual) able to perform one out of two measurements of local non-degenerate observables $(A_1, A_2; B_1, B_2)$ following a Bell multiport beam splitters scheme, on a pair of entangled quNits whose state is described by $\boldsymbol{\rho}_N(F_N)$. They find a system of $4N^2$ equations relating quantum joint probabilities and hypothetical probabilities to be satisfied by a local hidden variable theory. Applying numerical methods of linear optimization they found values of F_N^{max}, showing an increasing behavior $F_2^{max} < F_3^{max} < ... < F_9^{max}$, with a tendency to reach a constant value for $N \to \infty$.

In a subsequent work [34], it is carried out an extension up to $N=16$, confirming previous results and additionally finding sets of local detectors settings (values of the phase shifts) that maximize violation of local realism for N-dimensional maximally entangled states.

The work done so far only could be numerical due to the fact that analytical expressions available to date were valid only for two-level systems. Based on Clauser and Horne inequality, Dagomir Kaszlikowski and collaborators [35] derived a set of inequalities that joint probabilities in a local realistic theory must satisfy for entangled pairs of qutrits. This inequality succeeded to reproduce previous numerical results, however it was limited to three-level systems. Surprisingly, when Kaszlikowski and collaborators submitted their paper, a report showing a generalization to arbitrary dimension had been submitted three days before. The latter is known today as Bell-CGLMP inequality and is briefly exposed in the next section.

3.3 CGLMP INEQUALITY

Based on a set of constraints that joint probabilities in local variable theories must satisfy, Daniel Collins, Nicolas Gisin, Noah Linden, Serge Massar, and Sandu Popescu (Hereafter CGLMP) developed inequalities for arbitrary N dimensional bipartite systems [36].

The scenario in which the inequality is constructed is the same used in previous works: A system composed by two parties, two observers and two possible local measurements for each one. The Bell function I_N has the form:

$$I_N \equiv \sum_{k=0}^{[N/2]-1} \left(1 - \frac{2k}{N-1}\right) \Big\{ \big[P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)\big] - \big[P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)\big] \Big\}, \quad (3.7)$$

where $[x]$ indicates the integer part of the variable x and $P(A_a = B_b + k)$ is the probability that the outcomes A_a and B_b differ by k (modulo N):

$$P(A_a = B_b + k) \equiv \sum_{j=0}^{N-1} P(A_a = j, B_b = j + k \mod N). \quad (3.8)$$

It may be easily shown that the maximum value attainable by I_N for a theory based in local variables is 2 and for theory based in non-local variables is 4, then Bell-CGLMP inequality takes the form:

$$I_N \leq 2. \quad (3.9)$$

By substituting equation 3.8 into the Bell function I_N, the inequality becomes:

$$I_N = \sum_{k=0}^{[N/2]-1} \left(1 - \frac{2k}{N-1}\right) \sum_{j=0}^{N-1} \Big\{ \big[P_{11}(j, j+k) + P_{21}(j+k+1, j) + P_{22}(j, j+k) + P_{12}(j+k, j)\big] - \big[P_{11}(j, j-k-1) + P_{21}(j-k, j) + P_{22}(j, j-k-1) + P_{12}(j-k-1, j)\big] \Big\} \leq 2, \quad (3.10)$$

where subscripts a and b in $P_{ab}(j, k)$ represent the possible choices of measurement in Alice and Bob devices respectively.

The most effective way for Alice and Bob to perform measurements is using Bell multiport beam splitters. Note that when quantum joint probabilities (eq. 3.5) are inserted into the inequality, the eight terms $1/N^2$ cancel out. Because of this, and assuming that the amplitudes of the initial state α_j are real, it is possible to define *quantum pseudo-probabilities* $\widetilde{P}_{AB}^{QM}(k,l)$:

$$\widetilde{P}_{ab}^{QM}(k,l) \equiv \frac{2}{N^2}\sum_{m>n}^{N-1} \alpha_m\alpha_n \cos\Delta\beta_{ab}^{mn}(k,l).$$

Bell-CGLMP inequality takes the form:

$$I_N = \sum_{k=0}^{[N/2]-1}\sum_{j=0}^{N-1}\left(1-\frac{2k}{N-1}\right)\Big\{\widetilde{P}_{11}(j,j+k)-\widetilde{P}_{11}(j,j-k-1)+\widetilde{P}_{21}(j+k+1,j)-\widetilde{P}_{21}(j-k,j)$$
$$+\widetilde{P}_{22}(j,j+k)-\widetilde{P}_{22}(j,j-k-1)+\widetilde{P}_{12}(j+k,j)-\widetilde{P}_{12}(j-k-1,j)\Big\} \leq 2. \quad (3.11)$$

Substituting quantum pseudo-probabilities into the inequality:

$$I_N = \frac{2}{N^2}\sum_{k=0}^{[N/2]-1}\sum_{\substack{j=0\\m>n}}^{N-1}\alpha_m\alpha_n\left(1-\frac{2k}{N-1}\right)\Big\{\cos\Delta\beta_{11}^{mn}(j,j+k)-\cos\Delta\beta_{11}^{mn}(j,j-k-1)$$
$$+\cos\Delta\beta_{21}^{mn}(j+k+1,j)-\cos\Delta\beta_{21}^{mn}(j-k,j)+\cos\Delta\beta_{22}^{mn}(j,j+k)$$
$$-\cos\Delta\beta_{22}^{mn}(j,j-k-1)+\cos\Delta\beta_{12}^{mn}(j+k,j)-\cos\Delta\beta_{12}^{mn}(j-k-1,j)\Big\} \leq 2. \quad (3.12)$$

Previous numerical results in [34] yield the following set of phases that maximize Bell-CGLMP function I_N for a maximally entangled state [12]:

$$\phi_1^j = 0, \qquad \phi_2^j = \frac{\pi}{N}j, \qquad \varphi_1^j = \frac{\pi}{2N}j, \qquad \varphi_2^j = -\frac{\pi}{2N}j,$$

for $j = 0, ..., N-1$. Table 3.1 summarizes the calculated values of maximal violation of Bell-CGLMP inequality for a maximally entangled state up to $N = 10$.

It is straightforward to show that addition of white noise to the state as in equation 3.6 results in a change in joint probabilities $P_{ab}^{\prime QM}(k,l)$ and consequently in Bell-CGLMP function I'_N as follows:

$$P_{ab}^{\prime QM}(k,l) = (1-F_N)P_{ab}^{QM}(k,l) + \frac{F_N}{N^2},$$

$$I'_N = (1-F_N)I_N,$$

then, the critical amount of noise is given by:

$$F_N^* = \frac{I_N^{max}-2}{I_N^{max}}.$$

[12]A brief summary of Von Neumann entropy as quantifier of entanglement and maximally N-dimensional entangled bipartitie states is given in appendix A.

Table 3.1 Maximal violation of Bell-CGLMP inequality for a maximally entangled state and corresponding critical amount of white noise for $N \leq 10$.

Dimension N	Maximal Violation I_N^{max}	Critical Noise F_N^*
2	2.82842712	0.29289321
3	2.87293405	0.30384757
4	2.89624321	0.30945026
5	2.91054480	0.31284342
6	2.92020360	0.31511624
7	2.92716094	0.31674409
8	2.93240960	0.31796704
9	2.93650953	0.31891928
10	2.93980029	0.31968167

Table 3.1 shows calculated critical amount of noise F_N^* for $N \leq 10$ which are in accordance with previous numerical calculations [34], as expected. As can be seen, the critical amount of noise increases slightly with N, thus the greater the dimension, the harder the system can be described by local realistic theory. From this, [1] conclude that systems composed by two entangled quNits violate local realism more than pairs of entangled qubits.

3.4 NON-LOCALITY ANOMALY

Following the results presented above, it may be inferred a relation of correspondence between entanglement and non-locality (i.e. the more entangled the system, the larger its resistance to noise). Nevertheless, Antonio Acín and collaborators found that this is no longer true if the idea of resistance to noise as a measure of non-locality continues to be accepted [37]. By using analytical and numerical optimization methods they reached the conclusion that the state that maximally violates the Bell-CGLMP inequality is not the maximally entangled one. Specifically for the case of two qutrits, they show that the state which maximizes the Bell-CGLMP function under the optimal set of phases presented previously is given by:

$$|\Psi_3^{mv}\rangle = \frac{1}{\sqrt{\gamma_{13}^2 + 2}} \left(|00\rangle + \gamma_{13}|11\rangle + |22\rangle\right), \tag{3.13}$$

where $\gamma_{13} = \left(\sqrt{11} - \sqrt{3}\right)/2 \simeq 0.7923$ and the maximum value attained by the Bell-CGLMP function: $I_3^{max}\left(|\Psi_3^{mv}\rangle\right) = 1 + \sqrt{11/3} \simeq 2.9149$. Furthermore they find numerically values of maximal violation of Bell-CGLMP inequalities I_N^{max} up to $N = 8$, but do not give any kind of insight about the quantum state involved $|\Psi_N^{mv}\rangle$. Via numerical calculations we obtained approximate values of amplitude coefficients γ_j for a maximal violation state $|\Psi_N^{mv}\rangle$ given by:

$$|\Psi_N^{mv}\rangle = \frac{1}{\sqrt{\sum \gamma_{jN}^2}} \sum_{j=0}^{N-1} \gamma_{jN} |jj\rangle. \tag{3.14}$$

The values of γ_j and maximal violation of Bell-CGLMP inequality up to $N = 6$ are summarized in table 3.2.

Table 3.2 Approximate values of amplitude coefficients γ_{jN} that lead to maximal violation of Bell-CGLMP inequality and maximum value attained by the Bell-CGLMP function $I_N^{max}\left(|\Psi_N^{mv}\rangle\right)$ up to $N \leq 6$.

N	γ_{0N}	γ_{1N}	γ_{2N}	γ_{3N}	γ_{4N}	γ_{5N}	$I_N^{max}\left(\|\Psi_N^{mv}\rangle\right)$
3	1.0000	0.7923	1.0000	-	-	-	2.9148542
4	1.0000	0.7394	0.7394	1.0000	-	-	2.9726983
5	1.0000	0.7189	0.6605	0.7189	1.0000	-	3.0157105
6	1.0000	0.7094	0.6256	0.6256	0.7094	1.0000	3.0497004

Acín and collaborators end their report suggesting that resistance to noise is not a suitable measure of non-locality and propose some alternatives.

André Methot and Valerio Scarani describe the previous results as an anomaly of quantum non-locality [3], and propose an explanation to it, in their words: *non-locality and entanglement are not only different concepts, but are really quantitatively different resources.* We will show that, although this statement is true, it is far too strong.

The next two chapters devoted an attempt to reconcile the concepts of quantum entanglement and quantum non-locality by a proposal of quantifier for the latter.

CHAPTER 4

A PROPOSAL TO QUANTIFY QUANTUM NON-LOCALITY

4.1 THE PROPOSAL

We argue that the anomaly of non-locality arises due to the fact that the maximum of the Bell function was adopted as criterion to quantify non-locality. From this point of view, two states exhibiting the same maximal violation of a given Bell inequality are equally non-local even if the amount of configurations of detectors that lead to non-locality are different.

As an illustration, Bell functions associated to the CHSH inequality are plotted in figure 4.1, corresponding to two different states under a given measurement scheme characterized by a parameter θ. The violation of local realism by the Bell-CHSH function of the state II may be reached by a range of parameters wider than that corresponding to the state I. In principle one could argue that the extent to which locality is violated by state II is stronger than the violation for state I, nevertheless following the current conception, the fact that the Bell function I attains a higher maximum value makes it the most non-local.

In order to take into account the whole set of configurations leading to violations of a given Bell inequality, instead of the single setting for maximal violation, we developed an approach to quantify how non-local is a given state under measurements characterized by a Bell inequality.

Defining $R(\theta, \alpha)$ as the region in which a given tight Bell inequality is violated and $d\Omega(\theta)$ as a differential element of hyper-volume in the space of the parameters associated to the measurement devices $\{\theta\}$, we propose $V(\alpha)$ and $A(\alpha)$ as quantifiers of the strength of non-locality of the state involved (which may be characterized by a set of parameters $\{\alpha\}$), under given measurement conditions contained in a Bell function $S(\theta, \alpha)$:

$$V(\alpha) \equiv \int_{R(\theta,\alpha)} d\Omega(\theta), \tag{4.1}$$

$$A(\alpha) \equiv \int_{R(\theta,\alpha)} d\Omega(\theta) S(\theta, \alpha). \tag{4.2}$$

Whereas the value of $V(\alpha)$ is equal to the portion of the space of configurations $\{\theta\}$ that leads to violation of the Bell inequality, the quantity $A(\alpha)$ in addition to such a hyper-volume takes into account by how much the inequality is being violated[13].

[13]Notice that it is also reasonable to make a substitution $S(\theta, \alpha) \to S'(\theta, \alpha) = \frac{S(\theta,\alpha)-b}{S_{max}(\theta,\alpha)-b}$ in equation 4.2, where b is the locality boundary, to ensure that the weight function lie between 0 and 1. For the Bell-CHSH function, $b = 2$ and $S'(\theta, \alpha) = \frac{S(\theta,\alpha)-2}{2(\sqrt{2}-1)}$.

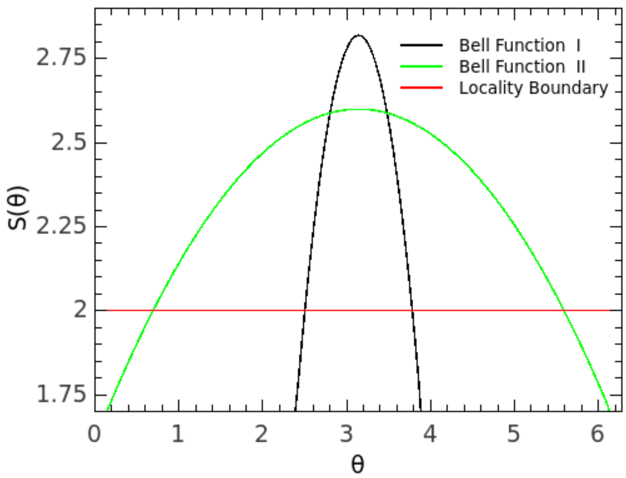

Figure 4.1 Pictorial representation of Bell-CHSH functions showing different maxima of violation of local realism for different detectors configurations. Depending on the interpretation, a state is more non-local than the other one. Following our criteria although the state I presents a higher maximum, the state II is the most non-local.

4.2 TWO SPIN-1/2 PARTICLES - FIRST BELL INEQUALITY

Consider the following state for a system composed by two spin-1/2 particles:

$$|\Psi\rangle = \alpha|+-\rangle + \sqrt{1-\alpha^2}e^{i\psi}|-+\rangle, \tag{4.3}$$

where $\alpha \in [0,1]$ and $\psi \in [0, 2\pi]$.

In the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ the associated density operator is given by:

$$\boldsymbol{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_\alpha & \rho_\delta & 0 \\ 0 & \rho_\gamma & \rho_\beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.4}$$

with $\rho_\alpha \equiv \alpha^2$, $\rho_\beta \equiv 1-\alpha^2$, $\rho_\delta \equiv \alpha\sqrt{1-\alpha^2}e^{i\psi}$ and $\rho_\gamma \equiv \alpha\sqrt{1-\alpha^2}e^{-i\psi}$.

Any matrix element ρ_{ijkl} of the density operator $\boldsymbol{\rho}$ may be written as:

$$\rho_{ijkl} = \delta_{j,i+1}\delta_{l,k+1}\left[\delta_{i,+}\left(\rho_\alpha\delta_{k,+} + \rho_\delta\delta_{k,-}\right) + \delta_{i,-}\left(\rho_\gamma\delta_{k,+} + \rho_\beta\delta_{k,-}\right)\right], \tag{4.5}$$

where additions in the subindex are taken modulo 2.

Then, quantum correlations between Alice and Bob measurements of spin in arbitrary directions $\hat{a}$ and $\hat{b}$ respectively, (eq. 2.5) take the form:

$$E(\hat{a},\hat{b}) = \sum_{ijkl} \sigma^a_{k,i}\sigma^b_{l,j}\delta_{j,i+1}\delta_{l,k+1}\left[\delta_{i,+}\left(\rho_\alpha\delta_{k,+} + \rho_\delta\delta_{k,-}\right) + \delta_{i,-}\left(\rho_\gamma\delta_{k,+} + \rho_\beta\delta_{k,-}\right)\right]$$

$$E(\hat{a},\hat{b}) = \sum_k \sigma^a_{k,+}\sigma^b_{k+1,-}(\rho_\alpha\delta_{k,+} + \rho_\delta\delta_{k,-}) + \sum_k \sigma^a_{k,+}\sigma^b_{k+1,-}(\rho_\gamma\delta_{k,+} + \rho_\beta\delta_{k,-})$$

$$E(\hat{a},\hat{b}) = \rho_\alpha \sigma^a_{++}\sigma^b_{--} + \rho_\delta \sigma^a_{-+}\sigma^b_{+-} + \rho_\gamma \sigma^a_{+-}\sigma^b_{-+} + \rho_\beta \sigma^a_{--}\sigma^b_{++} \tag{4.6}$$

Now, let Alice and Bob perform measurements of spin projections in the directions $\hat{a}$, $\hat{b}$ and $\hat{c}$:

$$\hat{a} = [\sin\theta_a, 0, \cos\theta_a], \qquad \hat{b} = [\sin\theta_b \cos\varphi_b, \sin\theta_b \sin\varphi_b, \cos\theta_b],$$
$$\hat{c} = [\sin\theta_c \cos\varphi_c, \sin\theta_c \sin\varphi_c, \cos\theta_c],$$

thus, by substituting into the general equation for Pauli operators (eq. 2.6):

$$\vec{\boldsymbol{\sigma}} \cdot \hat{a} = \begin{pmatrix} \cos\theta_a & \sin\theta_a \\ \sin\theta_a & -\cos\theta_a \end{pmatrix},$$

and

$$\vec{\boldsymbol{\sigma}} \cdot \hat{n} = \begin{pmatrix} \cos\theta_n & \sin\theta_n e^{-i\varphi_n} \\ \sin\theta_n e^{-i\varphi_n} & -\cos\theta_n \end{pmatrix},$$

where $\hat{n}$ stands for $\hat{b}$ or $\hat{c}$. For this set of orientations, calculations of quantum correlations give:

$$E(\hat{a},\hat{b}) = -\cos\theta_a \cos\theta_b (\rho_\alpha + \rho_\beta) + \sin\theta_a \sin\theta_b (\rho_\delta e^{-i\varphi_b} + \rho_\gamma e^{i\varphi_b}),$$
$$E(\hat{a},\hat{b}) = -\cos\theta_a \cos\theta_b + 2\alpha\sqrt{1-\alpha^2}\sin\theta_a \sin\theta_b \cos(\varphi_b + \psi). \tag{4.7}$$

Analogously, for the correlation between $\hat{a}$ and $\hat{c}$:

$$E(\hat{a},\hat{c}) = -\cos\theta_a \cos\theta_c + 2\alpha\sqrt{1-\alpha^2}\sin\theta_a \sin\theta_c \cos(\varphi_c + \psi). \tag{4.8}$$

And:

$$E(\hat{b},\hat{c}) = -\cos\theta_b \cos\theta_c (\rho_\alpha + \rho_\beta) + \sin\theta_b \sin\theta_c \left(\rho_\delta e^{i(\varphi_b - \varphi_c)} + \rho_\gamma e^{-i(\varphi_b - \varphi_c)}\right),$$

$$E(\hat{b},\hat{c}) = -\cos\theta_b \cos\theta_c + 2\alpha\sqrt{1-\alpha^2}\sin\theta_b \sin\theta_c \cos(\varphi_b - \varphi_c - \psi). \tag{4.9}$$

To illustrate the need of a Bell function associated to a tight inequality, we employ the first Bell inequality (which is not tight). Reordering terms in the first Bell inequality (eq. 2.4), the first Bell function $S(\hat{a},\hat{b},\hat{c})$ reads:

$$S(\hat{a},\hat{b},\hat{c}) \equiv \left| E(\hat{a},\hat{b}) - E(\hat{a},\hat{c}) \right| - E(\hat{b},\hat{c}),$$

or:

$$\begin{aligned} S(\hat{a},\hat{b},\hat{c},\alpha,\psi) = \Big| & 2\alpha\sqrt{1-\alpha^2}\sin\theta_a \big(\sin\theta_b \cos(\varphi_b + \psi) - \sin\theta_c \cos(\varphi_c + \psi)\big) \\ & + \cos\theta_a \big(\cos\theta_c - \cos\theta_b\big)\Big| + \cos\theta_b \cos\theta_c \\ & - 2\alpha\sqrt{1-\alpha^2}\sin\theta_b \sin\theta_c \cos(\varphi_b - \varphi_c - \psi) \le 1. \end{aligned} \tag{4.10}$$

Now we proceed to calculate quantum non-locality strengths for this system. For $V(\alpha,\psi)$ we have:

$$V(\alpha,\psi) = \int_0^\pi d\theta_a \int_0^\pi d\theta_b \sin\theta_b \int_0^\pi d\theta_c \sin\theta_c \int_0^{2\pi} d\varphi_b \int_0^{2\pi} d\varphi_c f(\theta_a,\theta_b,\theta_c,\varphi_b,\varphi_c,\alpha,\psi),$$

where the function $f(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi)$ ensures that only configurations that lead to violation of the Bell inequality are being taken into account:

$$f(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) \equiv \begin{cases} 1, & \text{if } S(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) > 1 \\ 0, & \text{if } S(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) \leq 1. \end{cases}$$

The expression for $A(\alpha, \psi)$ is given by:

$$A(\alpha, \psi) = \int_0^{\pi} d\theta_a \int_0^{\pi} d\theta_b \sin\theta_b \int_0^{\pi} d\theta_c \sin\theta_c \int_0^{2\pi} d\varphi_b \int_0^{2\pi} d\varphi_c g(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi)$$

where the function $g(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi)$ contains implicitly information about the region of violation of the first Bell inequality, and is defined as:

$$g(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) \equiv \begin{cases} S(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi), & \text{if } S(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) > 1 \\ 0, & \text{if } S(\theta_a, \theta_b, \theta_c, \varphi_b, \varphi_c, \alpha, \psi) \leq 1. \end{cases}$$

Normalized results of numerical[14] evaluations of the integrals $V(\alpha, \psi)$ and $A(\alpha, \psi)$ for different values of phases ψ are shown in figure 4.2.

Note that independently of the value of the phase ψ, both measures for normalized strength of non-locality $V'(\alpha, \psi)$ and $A'(\alpha, \psi)$ reach maximum values for $\alpha = 1/\sqrt{2}$ (i.e. the singlet state) and vanish for separable states $\alpha = 0$ and $\alpha = 1$, as expected.

For values of the amplitude α different from the mentioned above, V' and A' present dependence on the phase. This feature is due to the non-tightness of the first Bell inequality[15]. According to Masanes [39], measures of non-locality which use non-tight Bell inequalities are not reliable.

[14] Integrals involving a few number of variables (~ 8) were solved using standard quadrature methods [38].

[15] As will be shown in a later section, a treatment performed using Bell-CHSH inequality (the only tight Bell inequality for the bipartite case with binary inputs and outputs [2]), shows that even if the Bell function presents dependence on a phase, the quantities V and A do not.

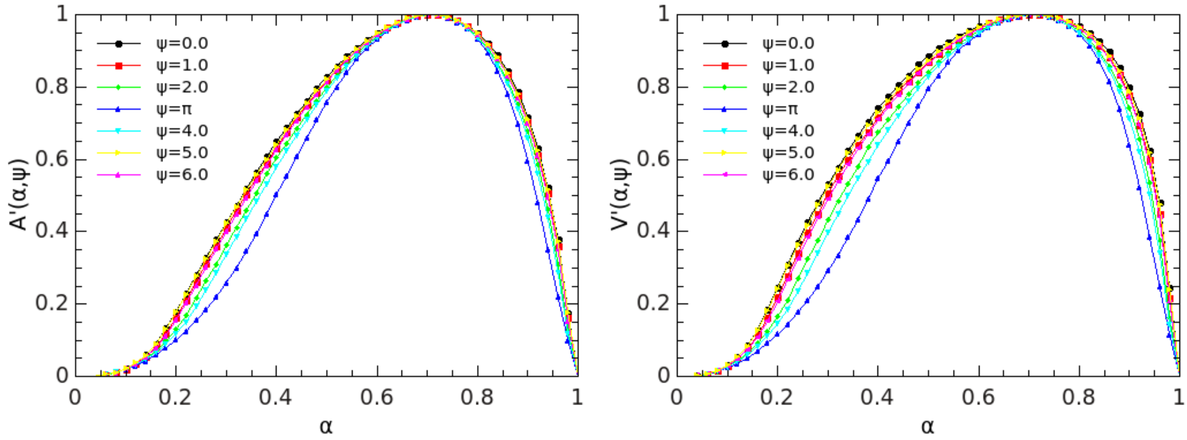

Figure 4.2 Normalized quantum non-locality strengths $A'(\alpha,\psi)$ and $V'(\alpha,\psi)$ for a two-level bipartite system under first Bell inequality measurement conditions.

4.3 TWO SPIN-1/2 PARTICLES + WHITE NOISE - FIRST BELL INEQUALITY

For the sake of completeness we study the effect of noise still in the case of Bell first inequality. Let consider the state studied in the former section with an additional contribution of white noise:

$$\boldsymbol{\rho}_F = \boldsymbol{\rho}_{noise} + (1-F)|\Psi\rangle\langle\Psi|, \tag{4.11}$$

with $\boldsymbol{\rho}_{noise} = \frac{F}{4}\boldsymbol{I}$.

In matrix form, the density operator may be written as:

$$\boldsymbol{\rho}_F = \begin{pmatrix} \rho_\alpha & 0 & 0 & 0 \\ 0 & \rho_\delta & \rho_\beta^* & 0 \\ 0 & \rho_\beta & \rho_\gamma & 0 \\ 0 & 0 & 0 & \rho_\alpha \end{pmatrix}, \tag{4.12}$$

where $\rho_\alpha \equiv F/4$, $\rho_\beta \equiv (1-F)\alpha\sqrt{1-\alpha^2}e^{i\psi}$, $\rho_\gamma \equiv F/4 + (1-F)(1-\alpha^2)$ and $\rho_\delta \equiv F/4 + (1-F)\alpha^2$.

A matrix element ρ_{ijkl} of the density operator $\boldsymbol{\rho}_F$ is given by:

$$\rho_{ijkl} = \rho_\alpha \delta_{i,j}\delta_{j,k}\delta_{k,l}\left(\delta_{i,+} + \delta_{i,-}\right) + \delta_{j,i+1}\delta_{i,k}\delta_{l,j}\left(\rho_\delta\delta_{i,+} + \rho_\gamma\delta_{i,-}\right) + \delta_{j,i+1}\delta_{i,l}\delta_{k,j}\left(\rho_\beta^*\delta_{i,+} + \rho_\beta\delta_{i,-}\right). \tag{4.13}$$

Additions in the subindex are taken modulo 2.

By substituting matrix elements ρ_{ijkl} into equation 2.5, quantum correlations become:

$$E(\hat{a},\hat{b}) = \rho_\alpha\sigma^a_{++}\sigma^b_{++} + \rho_\alpha\sigma^a_{--}\sigma^b_{--} + \rho_\delta\sigma^a_{++}\sigma^b_{--} + \rho_\gamma\sigma^a_{--}\sigma^b_{++} + \rho_\beta^*\sigma^a_{-+}\sigma^b_{+-} + \rho_\beta\sigma^a_{+-}\sigma^b_{-+} \tag{4.14}$$

For the set of orientations given in the last section $(\hat{a}, \hat{b}, \hat{c})$ quantum correlations take the form:

$$E(\hat{a},\hat{b}) = \cos\theta_a \cos\theta_b \left(2\rho_\alpha - \rho_\delta - \rho_\gamma\right) + 2\sin\theta_a \sin\theta_b \Re\left(\rho_\beta e^{i\varphi_b}\right)$$

$$E(\hat{a},\hat{b}) = (1-F)\left[-\cos\theta_a\cos\theta_b + 2\alpha\sqrt{1-\alpha^2}\sin\theta_a\sin\theta_b\cos(\varphi_b+\psi)\right]. \quad (4.15)$$

Similarly for the correlation between $\hat{a}$ and $\hat{c}$:

$$E(\hat{a},\hat{c}) = (1-F)\left[-\cos\theta_a\cos\theta_c + 2\alpha\sqrt{1-\alpha^2}\sin\theta_a\sin\theta_c\cos(\varphi_c+\psi)\right]. \quad (4.16)$$

And:

$$E(\hat{b},\hat{c}) = \cos\theta_b \cos\theta_c \left(2\rho_\alpha - \rho_\delta - \rho_\gamma\right) + 2\sin\theta_b \sin\theta_c \Re\left(\rho_\beta e^{i(\varphi_c-\varphi_b)}\right),$$

$$E(\hat{b},\hat{c}) = (1-F)\left[-\cos\theta_b\cos\theta_c + 2\alpha\sqrt{1-\alpha^2}\sin\theta_b\sin\theta_c\cos(\psi+\varphi_c-\varphi_b)\right]. \quad (4.17)$$

In conclusion, quantum correlations for this case are the same as for the noiseless situation, except for a constant $(1-F)$.

Bell first inequality becomes:

$$\begin{aligned} S(\hat{a},\hat{b},\hat{c},\alpha,\psi) = (1-F)\Bigg[\Big|&2\alpha\sqrt{1-\alpha^2}\sin\theta_a\big(\sin\theta_b\cos(\varphi_b+\psi) - \sin\theta_c\cos(\varphi_c+\psi)\big) \\ &+ \cos\theta_a\big(\cos\theta_c - \cos\theta_b\big)\Big| + \cos\theta_b\cos\theta_c \\ &- 2\alpha\sqrt{1-\alpha^2}\sin\theta_b\sin\theta_c\cos(\varphi_b-\varphi_c-\psi)\Bigg] \leq 1. \quad (4.18)\end{aligned}$$

Following the same numerical procedures carried out in the preceding section, we calculated quantum non-locality strengths $V(\alpha,\psi,F)$ and $A(\alpha,\psi,F)$. In addition we performed calculations of entanglement of formation (see appendix B). Some results of quantum non-locality strength, entropy of formation and maximum of Bell function are presented in figure 4.3. Once again, except for the concurrence, there is an evident dependence on the phase ψ by the quantifiers of non-locality and it is possible to recognize a concordance between maximally entangled and the most non-local state (i.e. $\alpha = 1/\sqrt{2} \approx 0.707$). We pay special attention to the dependence of the maximum of the Bell function, since it disables us to work on a characterization in terms of critical noise.

4.4 TWO SPIN-1/2 PARTICLES - CHSH INEQUALITY

From now on we will focus on tight inequalities. In this section we study the case of two spin-$\frac{1}{2}$ particles described by the family of pure entangled states introduced previously (eq. 4.3), under a CHSH-Bell inequality measurement scheme in which each observer

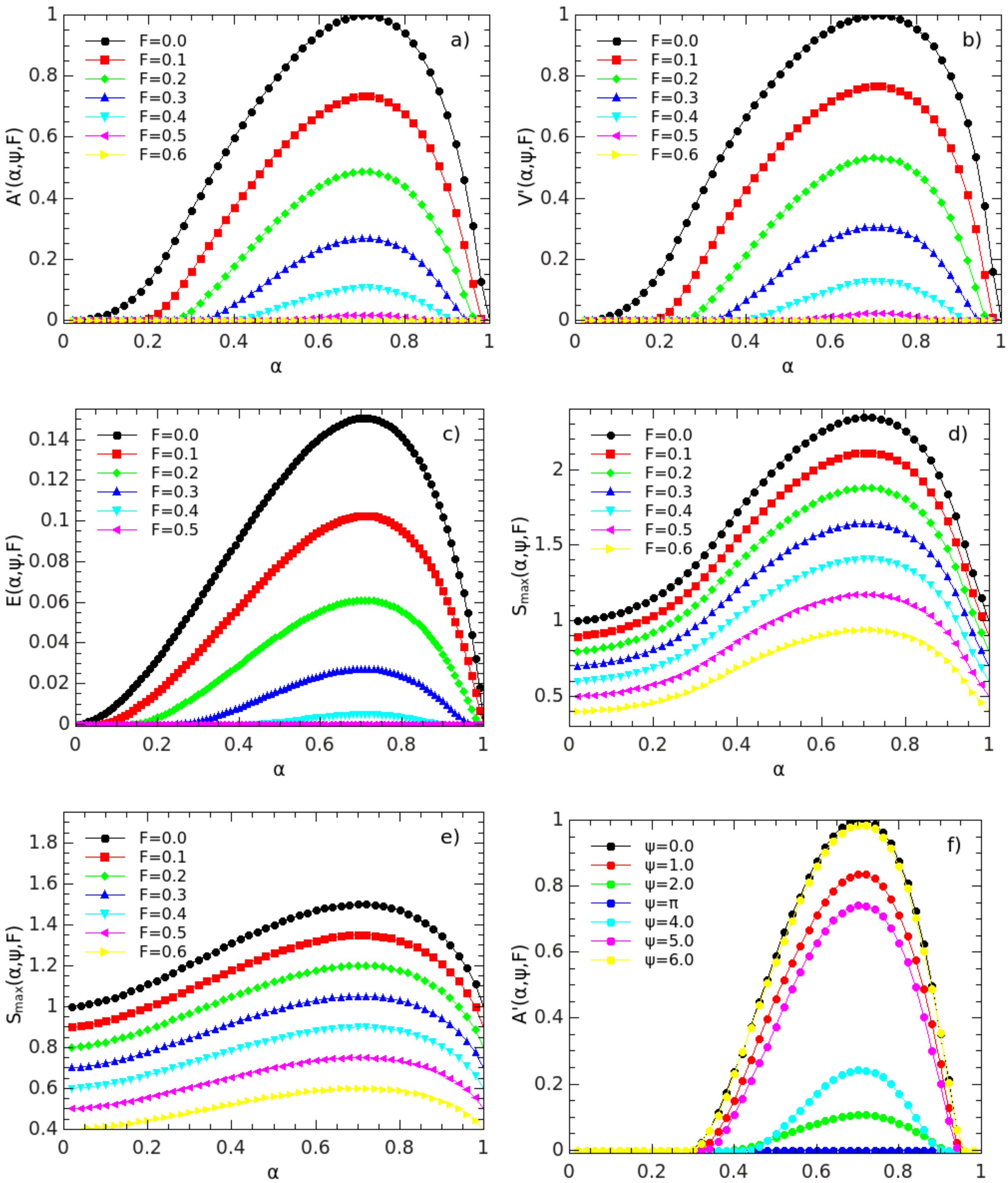

Figure 4.3 Some results from a two-level bipartite system with a contribution of white noise (eq. 4.11) under first Bell inequality measurement conditions. **a)** and **b)**: Normalized quantum non-locality strengths $A'(\alpha, \psi)$ and $V'(\alpha, \psi)$ for a phase $\psi = 2.0$ **c)** Entanglement of formation for several values of noise fraction F. **d)** and **e)** Maximum of Bell function for phase values $\psi = 2.0$ and $\psi = \pi$ in function of the fraction of noise F. **f)** Normalized quantum non-locality strength $A'(\alpha, \psi)$ for $F = 0.6$ and several values of phase ψ.

is able to choose one out of two possible orientations of detectors to measure the spin component of his/her associated particle. We consider the following directions[16]:

$$\hat{a} = [\sin\theta_a, 0, \cos\theta_a], \qquad \hat{b} = [\sin\theta_b\cos\varphi_b, \sin\theta_b\sin\varphi_b, \cos\theta_b],$$

$$\hat{a}' = [\sin\theta_{a'}\cos\varphi_{a'}, \sin\theta_{a'}\sin\varphi_{a'}, \cos\theta_{a'}], \qquad \hat{b}' = [\sin\theta_{b'}\cos\varphi_{b'}, \sin\theta_{b'}\sin\varphi_{b'}, \cos\theta_{b'}].$$

Calculations of quantum correlations for orientations as above have been previously carried out, by substitution on Bell-CHSH inequality, it becomes[17]:

$$\begin{aligned} \big|S(\hat{a},\hat{b},\hat{a}',\hat{b}',\alpha,\psi)\big| = \Big| & \cos\theta_a\big(\cos\theta_{b'} - \cos\theta_b\big) - \cos\theta_{a'}\big(\cos\theta_b + \cos\theta_{b'}\big) \\ & + 2\alpha\sqrt{1-\alpha^2}\Big\{\sin\theta_a\big[\sin\theta_b\cos(\varphi_b+\psi) - \sin\theta_{b'}\cos(\varphi_{b'}+\psi)\big] \\ & + \sin\theta_{a'}\big[\sin\theta_b\cos(\varphi_{a'}-\varphi_b-\psi) + \sin\theta_{b'}\cos(\varphi_{a'}-\varphi_{b'}-\psi)\big]\Big\}\Big| \le 2. \end{aligned} \quad (4.19)$$

A differential element of hyper-volume in the space of parameters $\{\theta\}$ is given by:

$$d\Omega = \sin\theta_{a'}\sin\theta_b\sin\theta_{b'}d\theta_a d\theta_{a'} d\theta_b d\theta_{b'} d\varphi_{a'} d\varphi_b d\varphi_{b'}.$$

Numerical integration was carried out to obtain non-locality strengths $A(\alpha,\psi)$ and $V(\alpha,\psi)$. The integration code was further exploited to calculate the maximum of the Bell-CHSH function $S_{max}(\alpha,\psi)$ over the space of parameters $\{\theta\}$.

We found no dependence of quantum non-locality quantifiers on the phase ψ. In figure 4.4 a plot of $A(\alpha,\psi)$ is shown for seven different phase values, all of them lying on the same curve.

Results of normalized quantum non-locality measurements $V'(\alpha,\psi)$, $A'(\alpha,\psi)$, maximum of Bell-CHSH function $S'_{max}(\alpha,\psi)$ and entropy of entanglement $E'(\alpha,\psi)$ (see appendix A) as function of the amplitude α are plotted in figure 4.5. It can be seen that for any of the three measures of non-locality, the maximally non-local state corresponds to the maximally entangled one ($\alpha = 1/\sqrt{2}$).

4.5 TWO SPIN-1/2 PARTICLES + WHITE NOISE - CHSH INEQUALITY

We have seen that quantum correlations for systems under white-noise influence differ from noiseless ones by a constant $(1-F)$, where $0 \le F \le 1$. Since Bell functions are proportional to correlations, then inequalities have the same response to noise. CHSH

[16]Note that we could have set one of the orientations (say $\hat{a}$) fixed on the positive part of the $\hat{z}$ axis and equivalently a second orientation (say $\hat{b}$) would no longer need the specification of an azimuthal angle, thus having 5 instead of 7 free angle variables to integrate. However we preferred to maintain the generality and allow an extra freedom to $\hat{a}$.

[17]Since the Bell-CHSH inequality may be violated when S attains a value greater than 2 or less than -2 and in order to consider positive and negative contributions to A on the same foot, the integration is performed over $|S|$ instead of S.

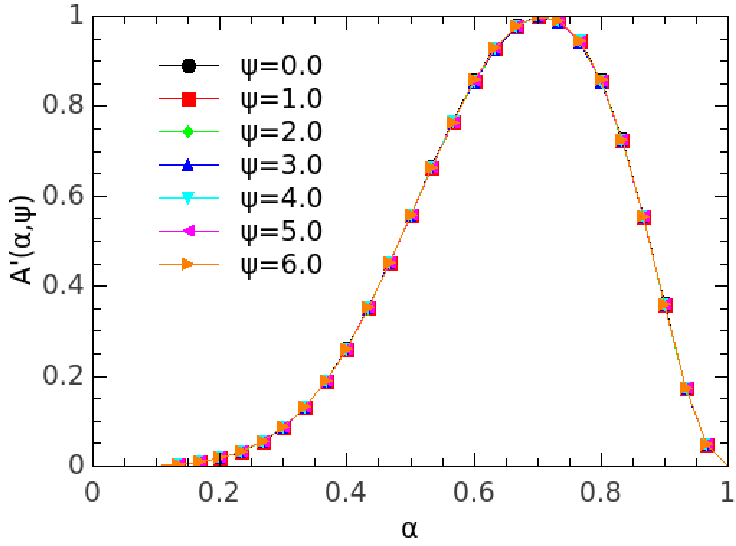

Figure 4.4 Normalized quantum non-locality strength $A(\alpha, \psi)$ for a two-level bipartite system in a state given by equation 4.3, under CHSH inequality measurement conditions. It can be noticed the independence of the phase ψ on the quantum non-locality strength A.

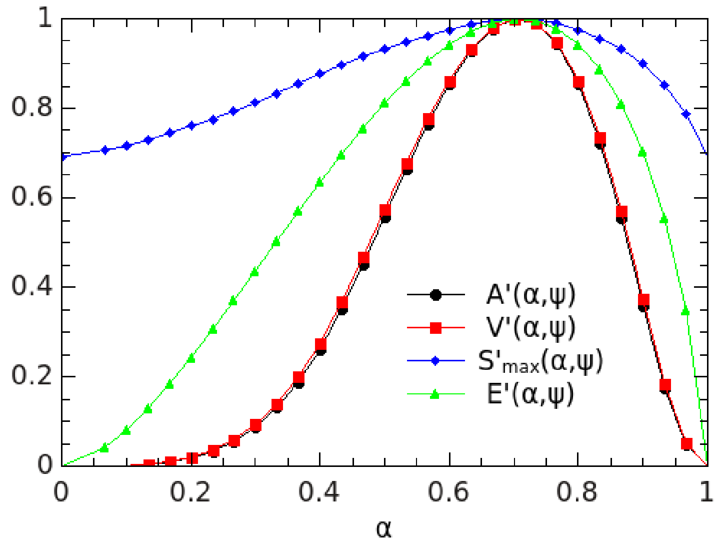

Figure 4.5 Normalized quantum non-locality strengths $A'(\alpha, \psi)$, $V'(\alpha, \psi)$, maximum of Bell-CHSH function $S'_{max}(\alpha, \psi)$ and entropy of entanglement $E'(\alpha, \psi)$ for a two-level entangled bipartite system in a state given by eq. 4.3, under CHSH inequality measurement conditions.

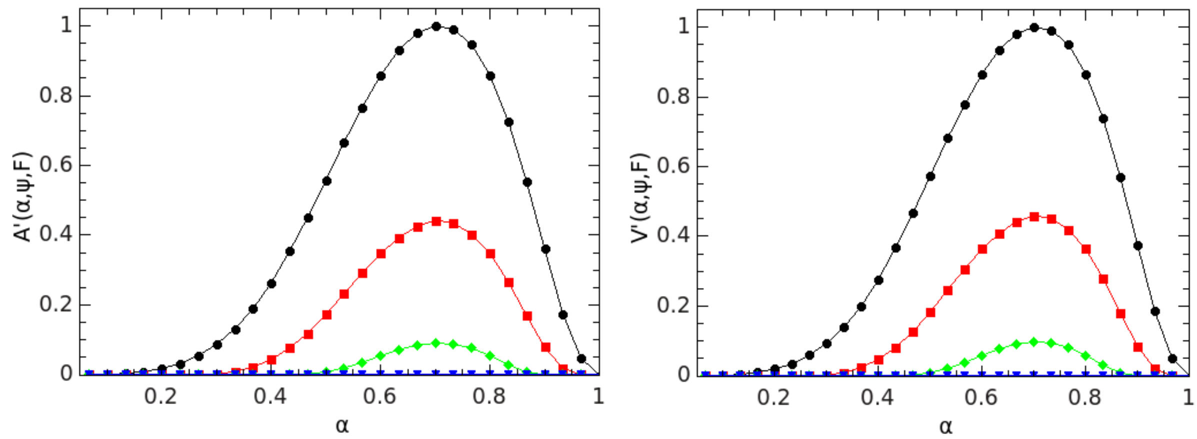

Figure 4.6 Normalized quantum non-locality strength $A'(\alpha, \psi, F)$ and $V'(\alpha, \psi, F)$ for a two-level bipartite system in a generalized singlet state plus a contribution of white noise, under CHSH inequality measurement conditions, for several values of noise fraction F.

inequality for the case treated in the preceding section plus a component of noise becomes:

$$\begin{aligned}\left|S(\hat{a},\hat{b},\hat{a}',\hat{b}',\alpha,\psi,F)\right| = (1-F)\Big|\cos\theta_a\big(\cos\theta_{b'}-\cos\theta_b\big) - \cos\theta_{a'}\big(\cos\theta_b+\cos\theta_{b'}\big)\\ + 2\alpha\sqrt{1-\alpha^2}\Big\{\sin\theta_a\big[\sin\theta_b\cos(\varphi_b+\psi) - \sin\theta_{b'}\cos(\varphi_{b'}+\psi)\big]\\ + \sin\theta_{a'}\big[\sin\theta_b\cos(\varphi_{a'}-\varphi_b-\psi) + \sin\theta_{b'}\cos(\varphi_{a'}-\varphi_{b'}-\psi)\big]\Big\}\Big| \leq 2. \quad (4.20)\end{aligned}$$

Using numerical methods, quantum non-locality strengths $A(\alpha, \psi, F)$ and $V(\alpha, \psi, F)$ were obtained for several values of noise fraction F. Plots of these are shown in figure 4.6. As in the former case it was found that the maximally non-local states correspond to the maximally entangled ones for values of F smaller than the critical noise fraction $F_3^* \approx 2.2928$ (see table 3.1). For values of F larger than F_3^* we found that quantum non-locality strengths A and V vanished i.e. no violation of local realism, as expected.

So far, definitions 4.1 and 4.2, although consistent, did not bring any genuinely new information. However, this will be the case for two three-level entangled systems.

CHAPTER 5

NON-LOCALITY STRENGTH FOR ENTANGLED QUTRITS - CGLMP INEQUALITY

5.1 BELL-CGLMP FUNCTION FOR QUTRITS

We have already applied our proposal of non-locality measures for a variety of cases involving bipartite two-level systems or *qubits* in entangled states, getting familiar results consigned in chapter 4. Henceforth, the discussion will be focused on bipartite three-level systems or *qutrits* in entangled states, under a measurement scheme given by a Bell-CGLMP inequality[18], using Bell multiport beam splitters.

The Bell-CGLMP function I_N has been previously introduced for arbitrary dimensionality in section 3.3. Particularly for $N = 3$, we have:

$$
\begin{aligned}
I_3 = \frac{2}{9} \sum_{\substack{j=0 \\ m>n}}^{2} \alpha_m \alpha_n \Big\{ & \cos \Delta\beta_{11}^{mn}(j, j+k) - \cos \Delta\beta_{11}^{mn}(j, j-k-1) \\
& + \cos \Delta\beta_{21}^{mn}(j+k+1, j) - \cos \Delta\beta_{21}^{mn}(j-k, j) + \cos \Delta\beta_{22}^{mn}(j, j+k) \\
& - \cos \Delta\beta_{22}^{mn}(j, j-k-1) + \cos \Delta\beta_{12}^{mn}(j+k, j) - \cos \Delta\beta_{12}^{mn}(j-k-1, j) \Big\},
\end{aligned} \tag{5.1}
$$

with $\Delta\beta_{ab}^{j,j'}(k,l) = \phi_a^j + \varphi_b^j - \phi_a^{j'} - \varphi_b^{j'} + \frac{2\pi}{3}(j-j')(k+l)$, for a qutrit initially prepared in the following state:

$$
|\Psi\rangle = \sum_{m=0}^{2} \alpha_m |mm\rangle.
$$

Note that the space of parameters associated to the measurements is 12-dimensional[19]. Thus, in order to calculate the strength of non-locality (either A or V), it is necessary to carry out integration of a twelve variables function within a non-trivial boundary. To deal with this, we implemented an algorithm based on Monte Carlo techniques [38] which improved calculations based on numerical integration methods used in the preceding chapter.

The differential element of hyper-volume $d\Omega$ is given by:

$$
d\Omega = \prod_{a,b,j,j'} d\phi_a^j d\varphi_b^{j'},
$$

where $a, b = 1, 2$ and the phase variables may take values between 0 and 2π.

[18] The fact that Bell-CGLMP inequality is tight enables us to use it to quantify non-locality [39].

[19] The set of parameters that define measurements is composed by 12 tunable phases: $\{\phi_1^j, \phi_2^j; \varphi_1^j, \varphi_2^j\}$ (for $j = 0, 1, 2$).

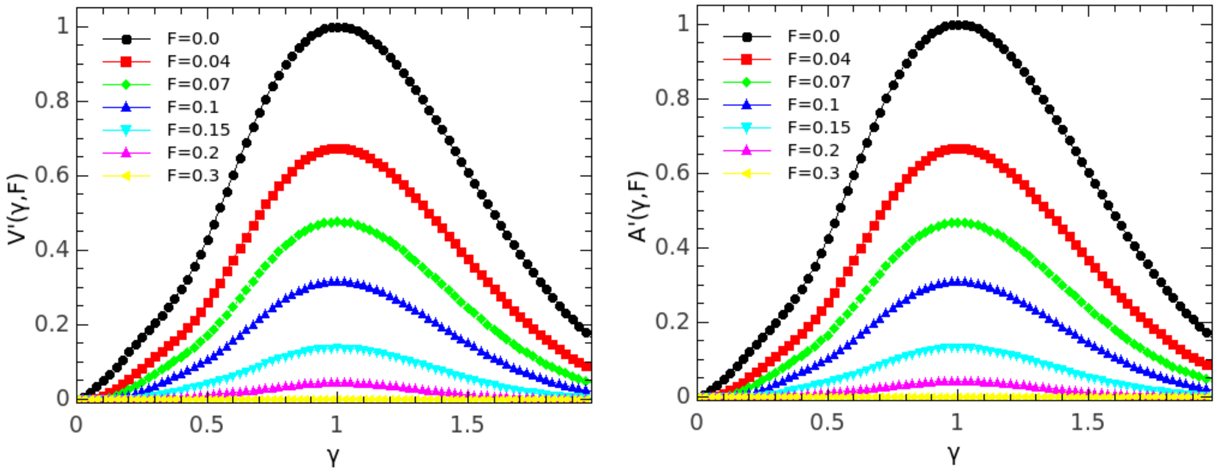

Figure 5.1 Normalized quantum non-locality strength $V(\gamma, F)$ and $A(\gamma, F)$ for a qutrit in an entangled state under the influence of white noise characterized by a noise fraction F.

5.2 SOLVING THE PROBLEM OF NON-LOCALITY ANOMALY

Following the integration procedure carried out in section 4.2, we calculated non-locality strengths $V(\gamma)$ and $A(\gamma)$ for a class of pure states that includes the one that leads to the non-locality anomaly (eq. 3.13):

$$|\Psi\rangle = \frac{1}{\sqrt{\gamma^2+2}}\big(|00\rangle + \gamma|11\rangle + |22\rangle\big). \tag{5.2}$$

Furthermore, a contribution of white noise characterized by a noise fraction F was added to the density operator of the entangled qutrit:

$$\boldsymbol{\rho}_F = \boldsymbol{\rho}_{noise} + (1-F)|\Psi\rangle\langle\Psi| = \frac{F}{9}\boldsymbol{I} + (1-F)|\Psi\rangle\langle\Psi|.$$

Results of normalized non-locality strength $V'(\gamma, F)$ and $A'(\gamma, F)$ are shown in figure 5.1. A remarkable observed feature is the location of the maximum of non-locality strength, as instead of finding this value for $\gamma = \gamma_{13}$ (which in a previous work led to the non-locality anomaly [37, 3]), it was found that the state with larger extent of violation of local reality corresponds to the most entangled one, i.e. $\gamma = 1$, showing that such an anomaly comes from the inappropriate use of the maximum of the Bell function as quantifier for non-locality.

A decreasing behavior with the increase of F is observed up to the critical noise. For noise fraction values higher than the critical one, the non-locality strength vanishes as expected.

To get some intuition about the result, we have plotted in figure 5.2 normalized results of non-locality strength $A'(\gamma)$ for the noiseless case ($F = 0$), normalized Von-Neumann entropy $E'(\gamma)$, as a measure of entanglement of the system and normalized maximum of the Bell-CGLMP function $I'_{3,max}(\gamma)$, which is a currently accepted measure of non-locality

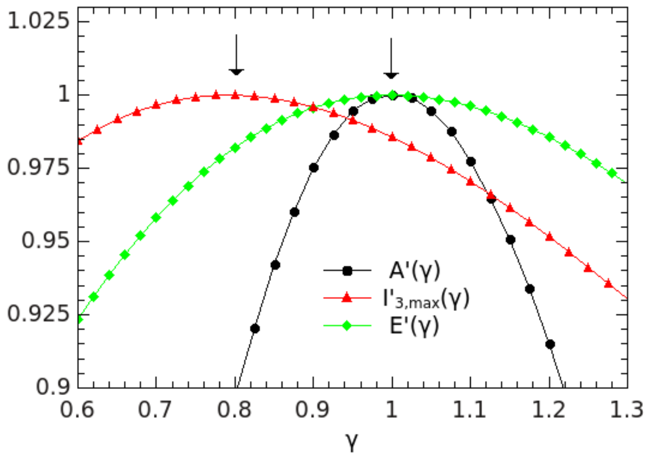

Figure 5.2 Normalized quantum non-locality and entanglement measures. The correspondence between maximum of entanglement E' and our proposed measure of non-locality A' solves the non-locality anomaly, which is also showed through the curve corresponding to the maximum of the Bell-CGLMP function. The function V' presents a behavior very close to that of A' and for this reason it is not plotted here.

strength. Following our interpretation, the agreement between maximally entangled and maximal non-local states is undoubtedly revealed, closing the apparent paradox involving the entanglement and non-locality for the qutrit case.

CHAPTER 6

CONCLUSION AND FUTURE PERSPECTIVES

Based on a scheme that takes into account the whole space of possible settings contributing to violations in tight Bell inequalities, we developed an approach to quantify the strength of non-locality for quantum systems under given measurement conditions related to the specific inequality under study.

The application of our proposal of non-locality measure to bipartite three-level systems under measurement conditions given by a Bell-CGLMP inequality led us to close an apparent inconsistency between quantum non-locality and entanglement that persists in the literature since 2002. In fact, we showed that, contrary to the present belief, the maxima of entanglement and non-locality do coincide also in this case.

The ideas developed in the present dissertation are general enough to allow a conceptually direct (though computationally harder) extension to convey systems with dimensionality higher than three. This is our next step in the study of quantifiers of non-locality.

A precise quantitative understanding of entangled systems described by Hilbert spaces with more than two dimensions represents an important progress in the development of applications in the field of quantum information such as quantum key distributions, that rely on unambiguously non-local states

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APPENDIX A

MAXIMALLY N-DIMENSIONAL ENTANGLED STATES

A.1 VON NEUMANN ENTROPY

There are several ways to measure quantum entanglement. Due to its simplicity, the entropy of entanglement is one of the most known. It is calculated using the Von Neumann entropy of the density operator of the system:

$$E(\boldsymbol{\rho}) = S_i(\boldsymbol{\rho}) = S(\boldsymbol{\rho}_i) = -\mathrm{tr}\left(\boldsymbol{\rho}_i \log(\boldsymbol{\rho}_i)\right),$$

where $\boldsymbol{\rho}_i$ is the partial trace of the density operator on one of the parts of the system. For instance, for a bipartite system, $\boldsymbol{\rho}_1$ can be obtained as follows:

$$\boldsymbol{\rho}_1 = \sum_j \langle \mu_{2,j}|\boldsymbol{\rho}|\mu_{2,j}\rangle,$$

For the case of a bipartite system described by a general entangled state:

$$|\Psi\rangle = \sum_{m=0}^{N-1} \alpha_m |m\rangle_A \otimes |m\rangle_B,$$

its density operator is given by:

$$\boldsymbol{\rho} = |\Psi\rangle\langle\Psi| = \sum_{m,n=0}^{N-1} \alpha_m \alpha_n^* |mm\rangle\langle nn|,$$

thus, the partial trace on any of the parties takes the form:

$$\boldsymbol{\rho}_i = \sum_{j=0}^{N-1} |\alpha_j|^2 |j\rangle\langle j|.$$

Using the fact that $\log \boldsymbol{\rho}_i = \sum_{j=0}^{N-1} \log |\alpha_j|^2 |j\rangle\langle j|$,

$$E(\boldsymbol{\rho}) = -\sum_{k=0}^{N-1} \langle k| \left(\sum_{l=0}^{N-1} |\alpha_l|^2 |l\rangle\langle l| \sum_{m=0}^{N-1} \log |\alpha_m|^2 |m\rangle\langle m| \right) |k\rangle,$$

then:

$$E(\boldsymbol{\rho}) = -\sum_{m=0}^{N-1} |\alpha_m|^2 \log |\alpha_m|^2. \tag{A.1}$$

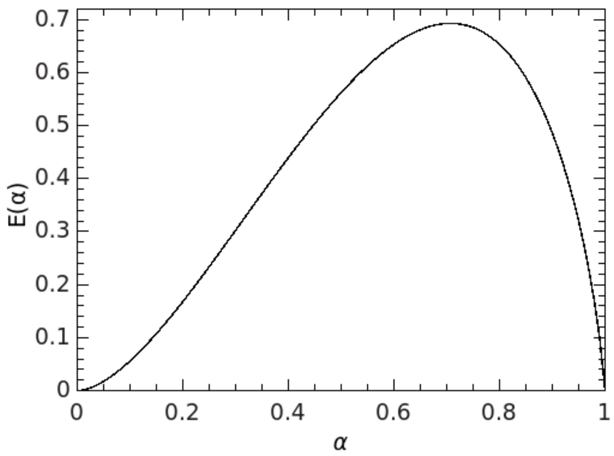

Figure A.1 Entropy of entanglement for a two-level bipartite system.

A.2 BIPARTITE TWO-LEVEL SYSTEM

Recall the bipartite two-level state used in chapter 4:

$$|\phi\rangle = \alpha|00\rangle + \sqrt{1-\alpha^2}e^{i\varphi}|11\rangle,$$

where $\alpha \in [0,1]$ and $\varphi \in [0,2\pi]$.

In this case the entropy of entanglement (eq. A.1) takes the form:

$$E(\alpha) = -\left[\alpha^2 \log \alpha^2 + (1-\alpha^2)\log(1-\alpha^2)\right].$$

In figure A.1 is plotted the entropy of entanglement in function of the amplitude α. Note that it reaches a maximum $E(\alpha^*) = \log 2 \simeq 0.69314$ for $\alpha^* = 1/\sqrt{2} \simeq 0.7071$ (i.e. the Bell state $|\Phi^+\rangle$). Thus the Bell state $|\Phi^+\rangle$ corresponds to the maximally entangled state for a two-level bipartite system[1]:

$$|\phi_{max}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

It is important to note that the entropy of entanglement is independent of phase φ in the state.

A.3 BIPARTITE THREE-LEVEL SYSTEM

For the three-level case, let examine the state studied by Acín and collaborators in [37]:

$$|\Psi\rangle = \frac{1}{\sqrt{2+\gamma^2}}\left(|00\rangle + \gamma|11\rangle + |22\rangle\right). \qquad \text{(A.2)}$$

[1] Actually, any element of the Bell basis is a maximally entangled state for a two-level bipartite system.

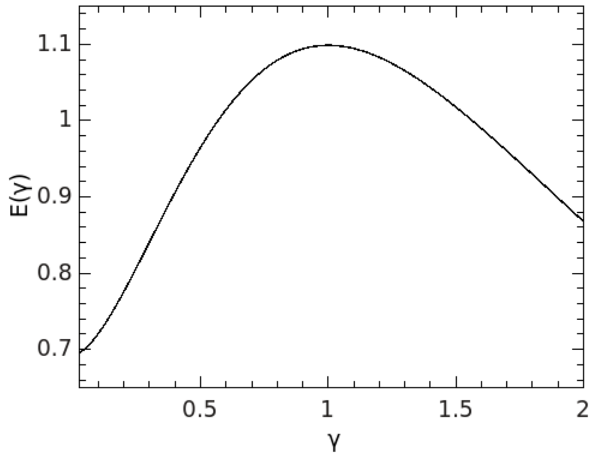

Figure A.2 Entropy of entanglement for a three-level bipartite system.

The entropy of entanglement for this state is:

$$E(\gamma) = \frac{-1}{2+\gamma^2}\left[2\log\left(\frac{1}{2+\gamma^2}\right) + \gamma^2 \log\left(\frac{\gamma^2}{2+\gamma^2}\right)\right].$$

Figure A.2 shows a plot of the entropy of entanglement for a bipartite three-level system described by the state given in equation A.2 in function of the coefficient of amplitude γ. In this case, it reaches a maximum $E(\gamma^*) = \log 3 \simeq 1.09861$ for $\gamma^* = 1$. As in the former case, the maximally entangled state for a three-level bipartite system corresponds to:

$$|\Phi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right).$$

It is important to highlight that the value of the entropy of entanglement for the state that maximally violates the CGLMP inequality is lower than for a maximally entangled state: $E(\gamma = 0.7923) \approx 1.077 < \log 3$.

It is straightforward to show that the maximally entangled N-dimensional bipartite state is given by:

$$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}|mm\rangle, \tag{A.3}$$

with an entropy of entanglement value of $\log N$.

APPENDIX B

ENTANGLEMENT OF FORMATION

B.1 DEFINITION

A suitable measure of the amount of resources necessary to construct an entangled mixed state of a two-level bipartite system that is described by a density operator $\boldsymbol{\rho}$ is the entanglement of formation, defined as [40]:

$$E(\boldsymbol{\rho}) = \mathcal{E}(C(\boldsymbol{\rho})), \tag{B.1}$$

where

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right),$$
$$h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$$

and $C(\boldsymbol{\rho})$ is the concurrence, given by:

$$C(\boldsymbol{\rho}) = \max\{0, 2\lambda_m - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{B.2}$$

here λ_i is the square root of the i-th eigenvalue of the non-hermitian matrix $\boldsymbol{\rho}\tilde{\boldsymbol{\rho}}$ and $\lambda_m = \max\{\lambda_i\}$, where $\tilde{\boldsymbol{\rho}}$ is the *spin-flippped* state:

$$\tilde{\boldsymbol{\rho}} = (\sigma_y \otimes \sigma_y)\boldsymbol{\rho}^*(\sigma_y \otimes \sigma_y)$$

B.2 CALCULATION OF ENTANGLEMENT OF FORMATION, SECTION 4.3

Given an arbitrary density operator $\boldsymbol{\rho}$:

$$\boldsymbol{\rho} = \sum_{ijkl} \rho_{ijkl}|ij\rangle\langle kl|, \tag{B.3}$$

the *spin-flippped* state can be written as:

$$\tilde{\boldsymbol{\rho}} = (\sigma_y \otimes \sigma_y)\boldsymbol{\rho}^*(\sigma_y \otimes \sigma_y) = \sum_{ijkl}(-1)^{i+j+k+l \pmod 2}\rho^*_{i-1,j-1,k+1,l+1}|ij\rangle\langle kl|. \tag{B.4}$$

For the density operator in equation 4.12, we have:

$$\tilde{\boldsymbol{\rho}} = \begin{pmatrix} \rho^*_\alpha & 0 & 0 & 0 \\ 0 & \rho^*_\gamma & \rho^*_\beta & 0 \\ 0 & \rho_\beta & \rho^*_\delta & 0 \\ 0 & 0 & 0 & \rho^*_\alpha \end{pmatrix}, \tag{B.5}$$

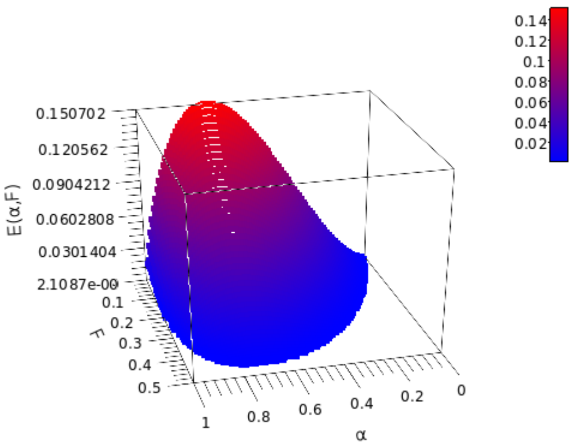

Figure B.1 Entropy of formation for an entangled state of a two-level bipartite system with a contribution of white noise (equation 4.12) in function of the state parameter α and the noise fraction F.

thus:

$$\boldsymbol{\rho}\tilde{\boldsymbol{\rho}} = \begin{pmatrix} |\rho_\alpha|^2 & 0 & 0 & 0 \\ 0 & |\rho_\beta|^2 + \rho_\gamma^*\rho_\delta & \rho_\beta^*(\rho_\delta + \rho_\delta^*) & 0 \\ 0 & \rho_\beta(\rho_\alpha + \rho_\alpha^*) & |\rho_\beta|^2 + \rho_\delta^*\rho_\gamma & 0 \\ 0 & 0 & 0 & |\rho_\alpha|^2 \end{pmatrix}. \tag{B.6}$$

After some calculations, the square roots of the eigenvalues of $\boldsymbol{\rho}\tilde{\boldsymbol{\rho}}$ are:

$$\lambda_1 = \lambda_2 = \frac{F}{4}$$

$$\lambda_3 = \sqrt{\frac{F}{4} + 2(1-F)^2\alpha^2(1-\alpha^2) - \frac{3}{16}F^2 + \frac{1-F}{4}\alpha\sqrt{1-\alpha^2}\sqrt{4F + 16(1-F)^2\alpha^2(1-\alpha^2) - 3F^2}}$$

$$\lambda_4 = \sqrt{\frac{F}{4} + 2(1-F)^2\alpha^2(1-\alpha^2) - \frac{3}{16}F^2 - \frac{1-F}{4}\alpha\sqrt{1-\alpha^2}\sqrt{4F + 16(1-F)^2\alpha^2(1-\alpha^2) - 3F^2}}$$

From these, the calculation of entanglement of formation is straightforward. Results are plotted in figure B.2.

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