Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
Programa de Pós-Graduação em Estatística

Olivia Lizeth Leal Alturo

Nonnested Hypothesis Testing Inference in Regression Models for Rates and Proportions

Recife
2017

Olivia Lizeth Leal Alturo

Nonnested Hypothesis Testing Inference in Regression Models for Rates and Proportions

Dissertação submetida como requerimento parcial para obtenção do grau de Mestre pelo Programa de Pós-graduação em Estatística da Universidade Federal de Pernambuco.

Orientador: Prof. Francisco Cribari Neto

Recife
2017

Catalogação na fonte
Bibliotecária Monick Raquel Silvestre da S. Portes, CRB4-1217

A792n Arturo, Olivia Lizeth Leal
Nonnested hypothesis testing inference in regression models for rates and proportions / Olivia Lizeth Leal Arturo. – 2017.
99 f.: il., fig., tab.

Orientador: Francisco Cribari Neto.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CCEN, Estatística, Recife, 2017.
Inclui referências e apêndice.

1. Análise de regressão. 2. Regressão beta. I. Cribari Neto, Francisco (orientador). II. Título.

519.536 CDD (23. ed.) UFPE- MEI 2017-63

OLIVIA LIZETH LEAL ALTURO

NONNESTED HYPOTHESIS TESTING INFERENCE IN REGRESSION MODELS FOR RATES AND PROPORTIONS

Dissertação apresentada ao Programa de Pós-Graduação em Estatística da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Estatística.

Aprovada em: 16 de fevereiro de 2017.

BANCA EXAMINADORA

Prof. Francisco Cribari Neto
UFPE

Prof. Raydonal Ospina Martínez
UFPE

Prof.ª Verônica Maria Cadena Lima
UFBA

Acknowledgement

Firstly, I would like to express my sincere gratitude to my advisor Professor Francisco Cribari Neto, for his safe orientation (guidance) which is proof of the great professional, professor and researcher that he is, for his disposition, patience and intelligence unique to clarify every question that came to me during the time of preparation of this dissertation and especially for the great motivation and contribution to writing this manuscript in English. I am immensely grateful to have worked on your guidance.

To my parents, who always instilled my love for the study. To my dear Mon Rosa, for the love that every day gives me, by confronting the technological world to be in contact with me for almost two years, and so, not feel we were far from each other. To my beloved Dad Ruben, for his exigency, his discipline, his affection and for always being there supporting me in each new adventure. To my beautiful princesses, my nieces Sara Valentina and Mariana Ximena, my everything, my motor to be every day a better person. Thank you, my princesses, for always being available for a video call. To my dear brothers Paulo and Carolina, for their example, trust, love and always be aware of the little sister, either at 1 meter or 4,570 kilometers away, love you so much.

To Joelton Ivson, for being my biggest company here in Recife, for always loving me, for helping me day by day to improve my Portuguese, for being always available and with a right attitude to get out of the routine life of an international student of postgraduate in Brazil. Infinitely grateful to fate for having met you, my dear filmmaker and photographer.

To my classmates, Alejandro Arias, Rodney Fonseca, Jucelino Matos, Vinicius Scher and Yuri Alves, for many and many hours of study shared, thanks to which it was possible to approve each of the disciplines we attended. I would like to thank, especially, Rodney and Jucelino, because they were my private teachers, who were always available to clarify the thousands of questions that came to me in the first year. I am sure that in the future I will have the privilege to say that I was a classmate in the masters of these two great professionals and researchers in statistics.

To my favorite doctorate students and next doctors in statistics Renata Rojas and Fernando Peña, I am immensely grateful to you for your beautiful friendship, for all their acts of kindness with which I will always be indebted, for the hours of study, for the coffees, anyway, for so many good moments shared and memories that would never be erased from my mind.

To my best friends Erika Huesa, Javier López, Carol Ladino and Jennifer Duarte, that despite the distance and the obligations that we each have daily we never lost contact and were always there in trying times.

I thank full to professors of the department of statistics at UFPE, especially prof. Gauss Cordeiro, prof. Alex Dias, prof. Abraão Nascimento and prof. Audrey Cysneiros for all the academic knowledge transmitted.

I would also like to thank the members of the Examination Board prof. Verônica Lima and prof. Raydonal Ospinal for your comments and suggestions.

To Valéria Bittencourt, secretary of the postgraduate in statistics, for the competence, affection and attention.

Finally, I would like to thank CAPES for the financial support.

Education is the most powerful weapon which you can use to change the world.

Nelson Mandela

Educação não transforma o mundo. Educação muda pessoas. Pessoas transformam o mundo.

Paulo Freire

Abstract

There are several different regression models that can be used with rates, proportions and other continuous responses that assume values in the standard unit interval, $(0, 1)$. When only one class of models is considered, model selection can be based on standard hypothesis testing inference. In this dissertation, we develop tests that can be used when the practitioner has at his/her disposal more than one plausible model, the competing models are nonnested and possibly belong to different classes of models. The competing models can differ in the regressors they use, in the link functions and even in the response distribution. The finite sample performances of the proposed tests are numerically evaluated. We evaluate both the null and nonnull behavior of the tests using Monte Carlo simulations. The results show that the tests can be quite useful for selecting the best regression model when the response assumes values in the standard unit interval.

Keywords: Beta regression. Johnson S_B regression. Gradient test. Likelihood ratio test. Nonnested hypothesis test. Score test.

Resumo

Existem diferentes modelos de regressão que podem ser usados para modelar taxas, proporções e outras variáveis respostas que assumem valores no intervalo unitário padrão, (0,1). Quando só uma classe de modelos de regressão é considerada, a seleção do modelos pode ser baseada nos testes de hipóteses usuais. O objetivo da presente dissertação é apresentar e avaliar numericamente os desempenhos em amostras finitas de testes que podem ser usados quando há dois ou mais modelos que são plausíveis, são não-encaixados e pertencem a classes de modelos de regressão distintas. Os modelos competidores podem diferir nos regressores que utilizam, nas funções de ligação e/ou na distribuição assumida para a variável resposta. Através de simulações de Monte Carlo nós estimamos as taxas de rejeição nulas e não-nulas dos testes sob diversos cenários. Avaliamos também o desempenho de um procedimento de seleção de modelos. Os resultados mostram que os testes podem ser bastante úteis na escolha do melhor modelo de regressão quando a variável resposta assume valores no intervalo unitário padrão.

Palavras-chave: Regressão beta. Regressão Johnson S_B. Teste gradiente. Teste da razão de verossimilhanças. Testes de hipóteses não-encaixadas. Teste escore.

List of Figures

List of Tables

Contents

Chapter 1

Introduction

Resumo

A presente dissertação se ocupa da realização de inferências em modelagem de respostas que assumem valores no intervalo $(0, 1)$ de forma contínua. O foco reside na escolha de um entre vários modelos não-encaixados. A avaliação dos desempenhos dos testes foi realizada através de simulações de Monte Carlo das taxas de rejeição nulas e não nulas, considerando diferentes cenarios em que os modelos competidores são ditos não-encaixados.

1.1 Introduction

In many empirical applications, the interest lies in modeling behavior of a given variable of interest (dependent variable, response) by taking into account its dependence on other variables (independent variables, covariates, regressors). Such a modeling strategy is known as regression analysis. The most commonly used class of models for continuous responses that assume values in the standard unit interval, $(0, 1)$, is the beta regression model proposed by Ferrari and Cribari-Neto (2004), where the response is assumed to follow the beta law.

Alternative classes of regression models for rates and proportions can also be used, some of which have only recently been introduced in the literature. For instance, the class of simplex regression models whose underlying assumption is that the variable of interest is simplex-distributed, was proposed by Barndorff-Nielsen and Jørgensen (1991). Simplex regression belongs to a wider class of models known as dispersion models; see Jørgensen (1997). Song and Tan (2000) and Song, Qiu and Tan (2004) considered a longitudinal version of the simplex regression model. Practitioners can also use the class of unit gamma models, as proposed by Mousa, El-Sheikh and Abdel-Fattah (2016) and Abdel-Fattah (forthcoming), in which it is assumed that the dependent variable follows a unit gamma distribution (Grassia, 1977). Finally, the modeling process can be based on the class of regression models proposed by Lemonte and Bazán (2016) which is based on the assumption that the response follows the Johnson S_B distribution (Johnson, 1949). Unlike the previous models, here it is assumed that the distribution median (not mean) is affected by regressors. In all models, parameter estimation can be carried out by maximum likelihood.

When only a single class of models is considered, hypothesis testing inference can be performed using the standard testing strategies. For instance, one can test restrictions on the parameters that index the model using the likelihood ratio test or an alternative test. Such tests are useful when searching for the best fitting model. Our interest in this paper lies in the situation in which the statistician has two or more models that are plausible candidates for the statistical analysis he/she wants to carry out but the models belong to different classes of models.

Cox (1961) and Cox (1962) started the literature on nonnested hypothesis testing inference. According to Pesaran and Weeks (2001), such literature can be viewed of consisting of three different branches: the modified log-likelihood ratio procedure or the Cox test, the comprehensive models approach developed by Atkinson (1970) and later employed by Davidson and MacKinnon (1981), and the encompassing procedure which was initially considered by Deaton (1982) and Dastoor (1983) and was further developed by Gourieroux, Monfort and Trognon (1983) and Mizon and Richard (1986).

The J test was proposed by Davidson and MacKinnon (1981) and later extended by Hagemann (2012), who introduced the MJ test. Both tests were developed for the classic linear regression model. Cribari-Neto and Lucena (2015) proposed the use of J-like and MJ-like tests in the class of beta regressions, i.e., when all candidate models are beta regressions. More recently, Cribari-Neto and Lucena (2017) pursued a similar approach for nonnested models that belong to the class of Generalized Additive Models for Location, Scale and Shape (GAMLSS) models. We extend their results to cover the situation in which the different nonnested models under consideration belong to different classes of regression models. We show how such testing inference can be carried out. We also extend their results by implementing the testing inference not only via the likelihood ratio test, but also on the basis of the score and gradient tests. We numerically evaluate the tests null and nonnull finite sample performances using Monte Carlo simulations. The score test is shown to outperform the competition in small samples.

1.2 Organization of dissertation

This dissertation consists of six chapters, including this introduction. In Chapter 2 we present the classes of beta, simplex, Johnson S_B and unit gamma of regression models. Such models can be used for modeling variates that assume values in the standard unit interval. We outline point estimation of the parameters by maximum likelihood. The J and MJ tests for nonnested hypothesis tests in the linear regression model and their adaptation for classes of regression models that can be used with rates and proportions are present in Chapter 3.

In Chapter 4 we present the results of several Monte Carlo simulations that were performed to evaluate the finite sample performances of J and MJ tests. We report the tests null and nonnull rejection rate for several scenarios in which the competing models differ in their regression structures and there are at least two classes of regression models. We also evaluate the finite sample performance of the MJ model selection procedure. In Chapter 5 we present and discuss two empirical applications. Finally, some concluding remarks are offered in Chapter 6.

1.3 Computing platform

The simulations were carried out using the `Ox` matrix programming language version 7.1 for Windows and Linux operating systems. The `Ox` programming language was developed by Jurgen Doorking and it is freely available for academic usage at `http://www.doornik.com`. For more information on `Ox`, see Doornik (2009). The graphics were produced using the `R` statistical computing environment version 3.2.1 for Windows. `R` is a free software environment for statistical computing and graphics. For details, see Core and Team (2013). The typesetting environment chosen was LaTeX (Lamport, 1986). LaTeX is a document preparation system developed based on TeX, which was developed by Donald Knuth. It is freely available.

Chapter 2

Regression models for rates and proportions

Resumo

Neste capítulo apresentamos quatro classes diferentes de modelos de regressão para respostas continuamente distribuídas no intervalo $(0,1)$, a saber: o modelo de regressão beta proposto por Ferrari e Cribari-Neto (2004), o modelo de regressão simplex introduzido como parte dos modelos de dispersão em Jørgensen (1997), o modelo de regressão Johnson S_B desenvolvido por Lemonte e Bazán (2016) e o modelo de regressão gama unitário introduzido por Mousa, El-Sheikh e Abdel-Fattah (2016) e Abdel-Fattah (forthcoming). Para cada um deles indicamos sua função de densidade de probabilidade e apresentamos gráficos das densidades para distintos valores dos parâmetros, fornecemos a função de log-verossimilhança para estimação dos parâmetros de regressão, o vetor escore e a matriz de informação de Fisher ou a matriz de informação observada.

2.1 Introduction

In what follows we shall briefly present four different classes of regression models that can be used to model responses that assume values in the standard unit interval, namely: the beta, simplex, Johnson S_B and unit gamma regression models.

2.2 The beta regression model

Ferrari and Cribari-Neto (2004) proposed the class of beta regression models by using a new beta parameterization. The beta density in its standard parameterization is

$$f(y;p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 < y < 1,$$

where $p > 0$, $q > 0$ and $\Gamma(\cdot)$ is the gamma function. Let $\mu = p/(p+q)$ and $\phi = p+q$; thus, $p = \mu\phi$ and $q = (1-\mu)\phi$. The beta density in its new parameterization is

$$f(y;\mu,\phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\,\Gamma((1-\mu)\phi)} y^{\mu\phi-1}(1-y)^{(1-\mu)\phi-1}, \quad 0 < y < 1, \tag{2.1}$$

where $0 < \mu < 1$ and $\phi > 0$. Here, $\mathbb{E}(y) = \mu$ and $\mathrm{Var}(y) = [\mu(1-\mu)]/(1+\phi)$. Thus, μ is the distribution mean and ϕ can be interpreted as a precision parameter, since, for fixed μ, the larger the value of ϕ, the smaller the variance of y.

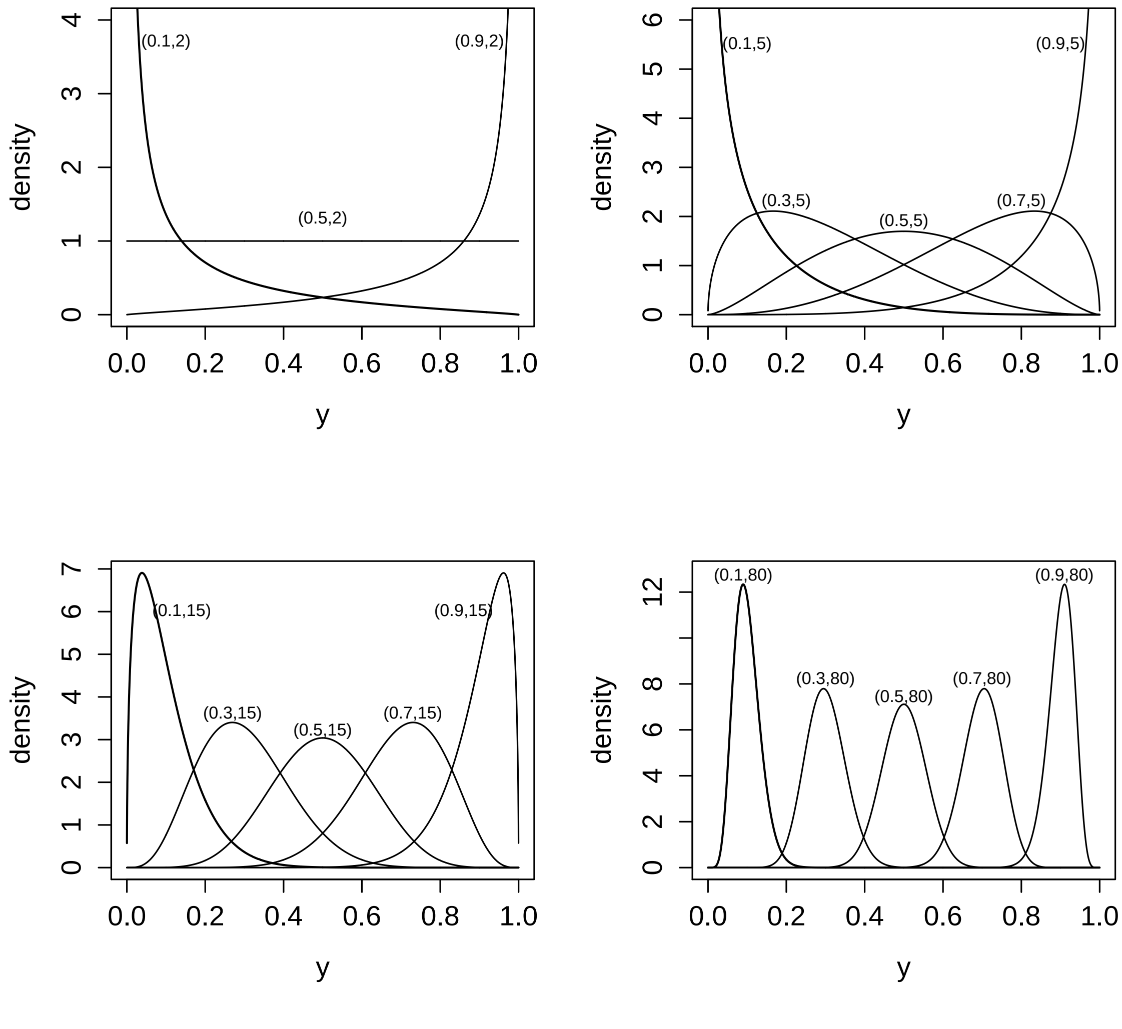

Figure 2.1: The beta density function for some parameter values of μ and ϕ.

Figure 2.1 shows different beta densities indexed by (μ, ϕ). Notice that the densities can display quite different shapes depending on the values of the two parameters. In the upper panels, the density displays 'J shapes' and inverted 'J shapes' and, for $\mu = 0.5$ and $\phi = 2$, reduces to the standard uniform distribution. When $\mu = 0.5$ the density is symmetric, when $\mu < 0.5$ it is right skewed and with $\mu > 0.5$ is left skewed. Finally, as we noticed above, for fixed μ the dispersion decreases when ϕ increases.

Let $y_1, \ldots, y_n$ be independent sample random variables such that each y_t, $t = 1, \ldots, n$, follows the density in (2.1), denoted as $y_t \sim \mathfrak{B}(\mu_t, \phi)$. Here, y_t has mean μ_t and precision ϕ. In the regression model proposed by Ferrari and Cribari-Neto (2004), μ_t is related to a linear predictor as

$$g(\mu_t) = \sum_{i=1}^{k} x_{ti}\beta_i = \mathbf{x}_t^\top \boldsymbol{\beta} = \eta_t,$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)^\top$ is a vector of unknown regression parameters ($\boldsymbol{\beta} \in \mathrm{I\!R}^k$), $\mathbf{x}_t^\top = (x_{t1}, \ldots, x_{tk})$ is a vector of k observed covariates ($k < n$) which are assumed fixed and known, η_t is a linear predictor and $g(\cdot)$ is a strictly increasing and twice-differentiable link function that maps $(0, 1)$ onto $\mathrm{I\!R}$. Some commonly used link function are: logit $g(\mu) = \log(\mu/1 - \mu)$, probit $g(\mu) = \Phi^{-1}(\mu)$, where Φ^{-1} is the standard normal quantile function, log-log $g(\mu) = -\log[-\log(\mu)]$, complementary log-log $g(\mu) = \log[-\log(1 - \mu)]$ and Cauchy $g(\mu) = \tan[\pi(\mu - 0.5)]$; see Atkinson (1985, Chapter 7).

Simas, Barreto-Souza and Rocha (2010) extended the above beta regression model to allow for nonconstant precision. Their model is commonly known as the varying dispersion beta regression model. It contains two submodels, one for the mean parameter and another for the precision parameter. The latter can be expressed as

$$h(\phi_t) = \sum_{j=1}^{l} z_{tj}\gamma_j = \mathbf{z}_t^\top \boldsymbol{\gamma} = \vartheta_t,$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_l)^\top$ is a vector of unknown regression ($\boldsymbol{\gamma} \in \mathrm{I\!R}^l$), $\mathbf{z}_t^\top = (z_{t1}, \ldots, z_{tl})$ is a vector of l observed covariates ($l < (n - k)$) which are assumed fixed and known, ϑ_t is a linear predictor and $h(\cdot)$ is a strictly increasing and twice-differentiable link function that maps $(0, \infty)$ onto $\mathrm{I\!R}$. The most commonly used precision link functions are: log $h(\phi) = \log(\phi)$, square root $h(\phi) = \sqrt{\phi}$ and identity $h(\phi) = \phi$.

Parameter estimation can be carried out by maximum likelihood. The log-likelihood function for a sample of n independent beta responses is

$$\ell(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{t=1}^{n} \ell_t(\mu_t, \phi_t),$$

where

$$\ell_t(\mu_t, \phi_t) = \log\Gamma(\phi_t) - \log\Gamma(\mu_t, \phi_t) - \log\Gamma[(1 - \mu_t)\phi_t] + (\mu_t\phi_t - 1)\log y_t \\ + [(1 - \mu_t)\phi_t - 1]\log(1 - y_t).$$

Let $y_t^* = \log\{y_t/(1 - y_t)\}$ and $y_t^\star = \log(1 - y_t)$. Their first two moments are $\mu_t^* = \mathrm{I\!E}(y_t^*) = \psi(\mu_t\phi_t) - \psi[(1 - \mu_t)\phi_t]$, $v_t^* = \mathrm{Var}(y_t^*) = \psi'(\mu_t\phi_t) + \psi'[(1 - \mu_t)\phi_t]$, $\mu_t^\star = \mathrm{I\!E}(y_t^\star) = \psi[(1 - \mu_t)\phi_t] - \psi(\phi_t)$, $v_t^\star = \mathrm{Var}(y_t^\star) = \psi'[(1 - \mu_t)\phi_t] - \psi'(\phi_t)$ and $c_t = \mathrm{Cov}(y_t^*, y_t^\star) = -\psi'[(1 - \mu_t)\phi]$, where $\psi(\cdot)$ is the digamma function.

The score function is obtained by differentiating the log-likelihood function with respect to the model parameters. It can be expressed as

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{\boldsymbol{\beta}}(\boldsymbol{\beta},\boldsymbol{\gamma}) \\ \mathbf{U}_{\boldsymbol{\gamma}}(\boldsymbol{\beta},\boldsymbol{\gamma}) \end{pmatrix},$$

where

$$\mathbf{U}_{\boldsymbol{\beta}}(\boldsymbol{\beta},\boldsymbol{\gamma}) = X^\top \Phi T(y^* - \mu^*)$$

and

$$\mathbf{U}_{\boldsymbol{\gamma}}(\boldsymbol{\beta},\boldsymbol{\gamma}) = Z^\top H[\mathcal{M}(y^* - \mu^*) + (y^\star - \mu^\star)].$$

Here, X is an $n \times k$ matrix whose tth row is $\mathbf{x}_t^\top = (x_{t1}, \ldots, x_{tk})$ and Z is an $n \times l$ matrix whose tth row is $\mathbf{z}_t^\top = (z_{t1}, \ldots, z_{tl})$. Additionally, T, H, Φ and $\mathcal{M}$ are diagonal matrices defined as $T = \mathrm{diag}\{1/g'(\mu_t)\}$, $H = \mathrm{diag}\{1/h'(\phi_t)\}$, $\Phi = \mathrm{diag}\{\phi_t\}$ and $\mathcal{M} = \mathrm{diag}\{\mu_t\}$. Finally, $y^* = (y_1^*, \ldots, y_n^*)^\top$, $y^\star = (y_1^\star, \ldots, y_n^\star)^\top$, $\mu^* = (\mu_1^*, \ldots, \mu_n^*)^\top$ and $\mu^\star = (\mu_1^\star, \ldots, \mu_n^\star)^\top$.

Fisher's information matrix is given by

$$K = K(\boldsymbol{\beta},\boldsymbol{\gamma}) = \begin{pmatrix} K_{\boldsymbol{\beta\beta}} & K_{\boldsymbol{\beta\gamma}} \\ K_{\boldsymbol{\gamma\beta}} & K_{\boldsymbol{\gamma\gamma}} \end{pmatrix},$$

with

$$\begin{aligned} K_{\boldsymbol{\beta\beta}} &= X^\top \Phi T V^* T \Phi X, \\ K_{\boldsymbol{\beta\gamma}} &= K_{\boldsymbol{\gamma\beta}}^\top = X^\top \Phi T(\mathcal{M}V^* + C)HZ, \\ K_{\boldsymbol{\gamma\gamma}} &= Z^\top H(\mathcal{M}^2 V^* + 2\mathcal{M}C + V^\star)HZ, \end{aligned}$$

where $C = \mathrm{diag}\{c_t\}$, $V^* = \mathrm{diag}\{v_t^*\}$ and $V^\star = \mathrm{diag}\{v_t^\star\}$. It is noteworthy that the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are not orthogonal, unlike the corresponding parameters in the class of generalized linear models.

Under standard regularity conditions and when the sample size is large, the maximum likelihood estimators are approximately normally distributed, i.e.,

$$\begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{pmatrix} \sim N_{k+l}\left(\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, K^{-1} \right),$$

where K^{-1} is the inverse of Fisher's information matrix.

2.3 The simplex regression model

The simplex regression model is based on assumption that the response is simplex-distributed. The simplex distribution was introduced by Barndorff-Nielsen and Jørgensen (1991). Its regression model belongs to the wider class of dispersion models proposed by

Jørgensen (1997) which also extends the class of generalized linear models developed by Nelder and Wedderburn (1972).

A random variable follows the simplex distribution with mean $\mu \in (0,1)$ and dispersion parameter $\lambda > 0$ if its probability density function is given by

$$f(y;\mu,\lambda) = \left\{\frac{\lambda}{2\pi[y(1-y)]^3}\right\}^{1/2} \exp\left\{-\frac{\lambda}{2}d(y;\mu)\right\},$$

where

$$d(y;\mu) = \frac{(y-\mu)^2}{y(1-y)\mu^2(1-\mu)^2} \tag{2.2}$$

is the deviance function. The simplex distribution is denoted by $S^-(\mu,\lambda)$.

The variance of y is

$$\mathrm{Var}(y) = \mu(1-\mu) - \sqrt{\frac{\lambda}{2}} \exp\left\{\frac{\lambda}{\mu^2(1-\mu)^2}\right\} \Gamma\left\{\frac{1}{2}, \frac{\lambda}{2\mu^2(1-\mu^2)}\right\},$$

where $\Gamma(a,b)$ is the incomplete gamma function, i.e., $\Gamma(a,b) = \int_b^\infty t^{a-1}e^{-t}dt$.

Figure 2.2 shows different simplex densities. Notice that, unlike the beta density, it can be bimodal, as shows in the left upper panel. The simplex density is asymmetric when $\mu \neq 0.5$ and symmetric when $\mu = 0.5$. For small values of λ, it is asymmetric and for larger values of λ it is more symmetric.

Let $y_1,\ldots,y_n$ be independent random variables such that $y_t \sim S^-(\mu_t,\lambda)$, $t = 1,\ldots,n$. In the constant dispersion simplex regression model

$$g(\mu_t) = \sum_{i=1}^{k} x_{ti}\beta_i = \mathbf{x}_t^\top \boldsymbol{\beta} = \eta_t,$$

where $g(\cdot)$ is a strictly increasing and twice-differentiable link function that maps $(0,1)$ onto $\mathrm{I\!R}$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_k)^\top$ is a vector of unknown regression parameters, $\mathbf{x}_t' = (x_{t1},\ldots,x_{tk})$ are values of k covariates, which are fixed and known, and η_t is a linear predictor.

Song, Qiu and Tan (2004) introduced the class of varying dispersion simplex regressions, where the dispersion parameter λ_t is modeled as

$$h(\lambda_t) = \sum_{j=1}^{l} z_{tj}\gamma_j = \mathbf{z}_t^\top \boldsymbol{\gamma} = \vartheta_t.$$

Here, $h(\cdot)$ is a strictly increasing and twice-differentiable link function that maps $(0,\infty)$ onto $\mathrm{I\!R}$, $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_l)^\top$ is a vector of unknown regression parameters ($\boldsymbol{\gamma} \in \mathrm{I\!R}^l$), $\mathbf{z}_t^\top = (z_{t1},\ldots,z_{tl})$ is a vector of observations on l covariates ($l < (n-k)$) which are assumed to be fixed and known, ϑ_t is a linear predictor.

The log-likelihood function based on a sample of n independent observations can be

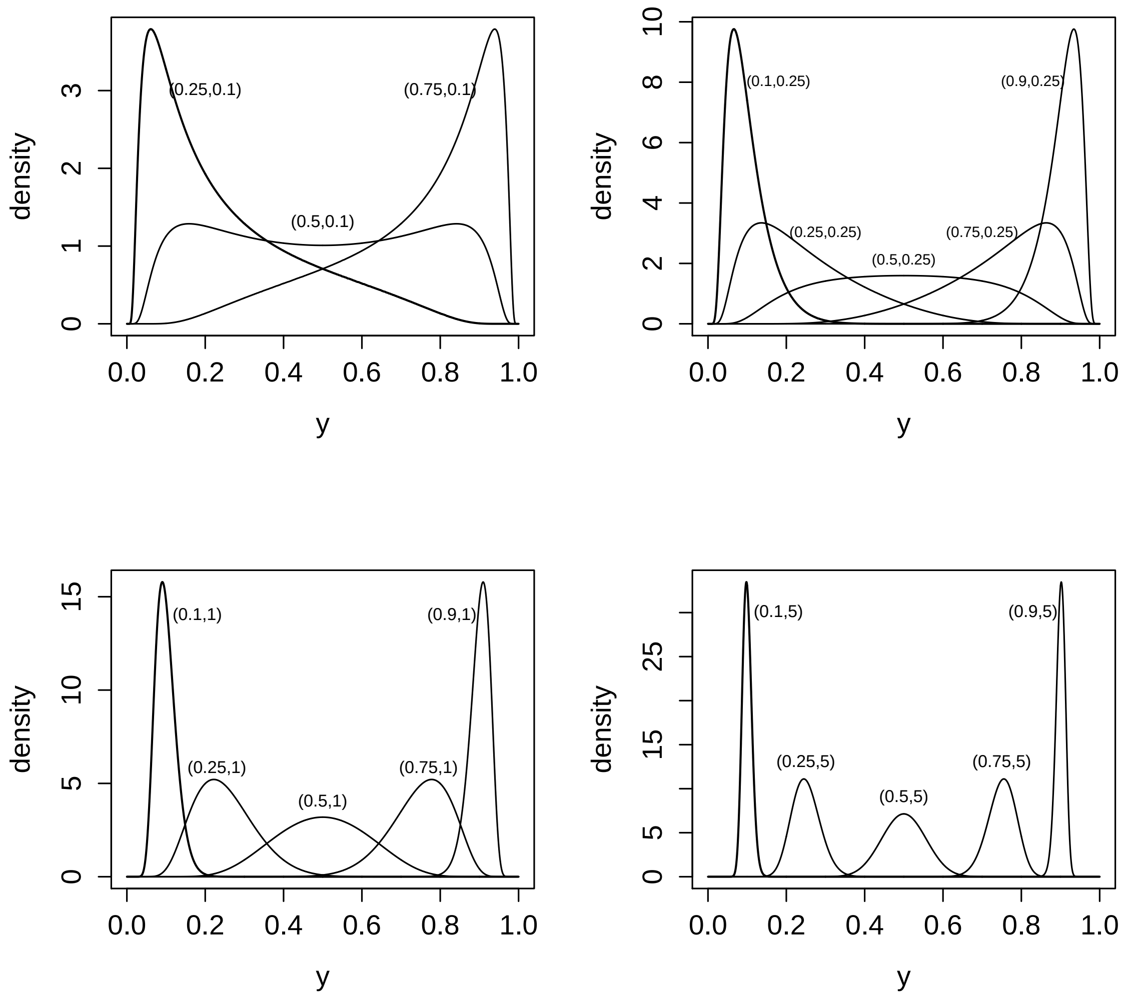

Figure 2.2: The simplex density function for some parameter values of μ and λ.

expressed as

$$\ell(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{t=1}^{n} \ell_t(\mu_t, \lambda_t),$$

where

$$\ell_t(\mu_t, \lambda_t) = \frac{1}{2}\log(\lambda_t) - \frac{1}{2}\log(2\pi) - \frac{3}{2}\log[y_t(1-y_t)] - \frac{\lambda_t}{2} d(y_t; \mu_t),$$

with $d(y_t; \mu_t)$ as defined in (2.2).

The score function with respect to the parameters that index the mean submodel is

$$\mathbf{U}_{\boldsymbol{\beta}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = X^{\top} \Lambda T C (\mathbf{y} - \boldsymbol{\mu}),$$

where X is an $n \times k$ matrix whose tth row is $\mathbf{x}_t^{\top}$, $T = \operatorname{diag}\{1/g'(\mu_t)\}$, $\Lambda = \operatorname{diag}\{\lambda_t\}$ and $C = \operatorname{diag}\{c_t\}$ with

$$c_t = \frac{1}{\mu_t(1-\mu_t)} \left\{ d(y_t; \mu_t) + \frac{1}{\mu_t^2(1-\mu_t)^2} \right\}.$$

The score function with respect to the parameters that index the dispersion submodel is

$$\mathbf{U}_{\gamma}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = Z^{\top} H \mathbf{a},$$

where Z is an $n \times l$ matrix whose tth row is $\mathbf{z}_t^{\top}$, $H = \mathrm{diag}\{1/h'(\lambda_t)\}$ and $\mathbf{a} = (a_1, \ldots, a_n)^{\top}$, with

$$a_t = \frac{1}{2\lambda_t} - \frac{d(y_t; \mu_t)}{2}.$$

Fisher's information matrix can be written in matrix form as

$$K = K(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \begin{pmatrix} K_{\boldsymbol{\beta}\boldsymbol{\beta}} & 0 \\ 0 & K_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \end{pmatrix},$$

where $K_{\boldsymbol{\beta}\boldsymbol{\beta}} = X^{\top} \Lambda W X$ and $K_{\boldsymbol{\gamma}\boldsymbol{\gamma}} = Z^{\top} V Z$, W and V being diagonal matrices whose tth diagonal elements are, respectively,

$$w_t = \left\{ \frac{3}{\lambda[\mu_t(1-\mu_t)]} + \frac{1}{\mu_t^3(1-\mu_t^3)} \right\} \frac{1}{[g'(\mu_t)]^2} \quad \text{and} \quad v_t = \frac{1}{2\lambda_t [h'(\lambda_t)]^2}.$$

It is noteworthy that, unlike the beta regression model, the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are orthogonal.

Under standard regularity conditions and when the sample size is large, the maximum likelihood estimators are approximately normally distributed, i.e.,

$$\begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{pmatrix} \sim N_{k+l} \left(\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, K^{-1} \right).$$

2.4 The Johnson S_B regression model

Johnson (1949) considered a system of transformations of the normal distribution that includes the S_B class of distributions, which has bounded support, as a particular case. Let $X \sim N(0, 1)$ and $Y = t^{-1}[(X - \iota)/\delta]$, i.e., $X = \iota + \delta t(y)$, for some appropriate function $t(\cdot)$ and parameters $\iota \in \mathbb{R}$ and $\delta > 0$. The Johnson S_B distribution, which has support in the standard unit interval, is obtained by using $t(y) = \log[y/(1-y)]$.

Lemonte and Bazán (2016) generalized the Johnson S_B distribution since instead of assuming that $X \sim N(0, 1)$, the authors considered $X \sim S(0, 1, a)$, i.e., X is taken to follow a distribution that belongs to the symmetric family of distributions for some function $a(\cdot)$, which is know as the density generator of symmetric distributions; for details, see Fang and Anderson (1990). Some distributions that belong to the symmetric family distributions are normal, Cauchy, Student-t, generalized Student-t, among others.

When $a(u) = (2\pi)^{-1/2} e^{-u/2}$, the generalized Johnson S_B distribution reduces to the Johnson S_B distribution. We shall focus on such a probability law. The random variable y is said to be Johnson S_B-distributed, denoted by $y \sim JS(\iota, \delta)$, if its probability density function is given by

$$f(y;\iota,\delta) = \frac{\delta a\{[\iota + \delta t(y)]^2\}}{y(1-y)} = \frac{\delta\phi[\iota + \delta t(y)]}{y(1-y)}, \tag{2.3}$$

where $\phi(\cdot)$ denotes the standard normal probability density function, $y \in (0,1)$, $t(y) = \log[y/(1-y)]$, $\delta > 0$, $\iota \in \mathbb{R}$. Lemonte and Bazán (2016) showed that the nth Johnson S_B moment is given by

$$\mu_n^{'} = \int_{\mathbb{R}} w(u)^n a(u^2) du,$$

where $w(u) = w(u;\iota,\delta) = e^{(u-\iota)/\delta}/[1 + e^{(u-\iota)/\delta}]$ and $a(\cdot)$ is the density generator described above. Thus, there is no closed-form expression for the distribution moments which must be computed numerically.

The Johnson S_B median can be expressed in closed-form as $\xi = (1 + e^{\iota/\delta})^{-1}$. It then follows that

$$\iota = \delta \log\left(\frac{1-\xi}{\xi}\right) = -\delta t(\xi), \tag{2.4}$$

where $t(\cdot)$ is as defined previously. Thus, by plugging (2.4) into the probability density function given in (2.3) we obtain

$$f(y;\xi,\delta) = \frac{\delta\phi(\delta[t(y) - t(\xi)])}{y(1-y)}, \tag{2.5}$$

where $0 < \xi < 1$ is a location parameter and $\delta > 0$ is a dispersion parameter. Using such a parameterization it is possible to define a regression model.

Figure 2.3 shows different Johnson S_B densities. Like the simplex density, it can be bimodal (left upper panel). It assumes different shapes depending on the values of ξ and δ. Like the beta and simplex densities, it can be symmetric ($\xi = 0.5$) or asymmetric ($\xi \neq 0.5$).

Let $y_1, \ldots, y_n$ be independent random variable such that y_t, for $t = 1, \ldots, n$, is distributed according to the density function given in (2.5) with median ξ_t and dispersion parameter δ_t, that is, $y_t \sim JS(\xi_t, \delta_t)$. In the Johnson S_B regression model,

$$g(\xi_t) = \eta_t = \sum_{i=1}^{k} x_{ti}\boldsymbol{\beta} \text{ and } h(\delta_t) = \vartheta_t = \sum_{j=1}^{l} z_{tj}\boldsymbol{\gamma},$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)^\top$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_l)^\top$ are vectors of unknown regression parameters which are assumed to be functionally independent, $\boldsymbol{\beta} \in \mathbb{R}^k$ and $\boldsymbol{\gamma} \in \mathbb{R}^l$ with $k + l < n$, η_t and ϑ_t are linear predictors, $\mathbf{x}_t^\top = (x_{t1}, \ldots, x_{tk})$ and $\mathbf{z}_t^\top = (z_{t1}, \ldots, z_{tl})$ are observations on k and l covariates, respectively, $g(\cdot)$ is the median link function which maps $(0,1)$ onto $\mathbb{R}$ and $h(\cdot)$ is the dispersion link function maps $(0,\infty)$ onto $\mathbb{R}$. The two

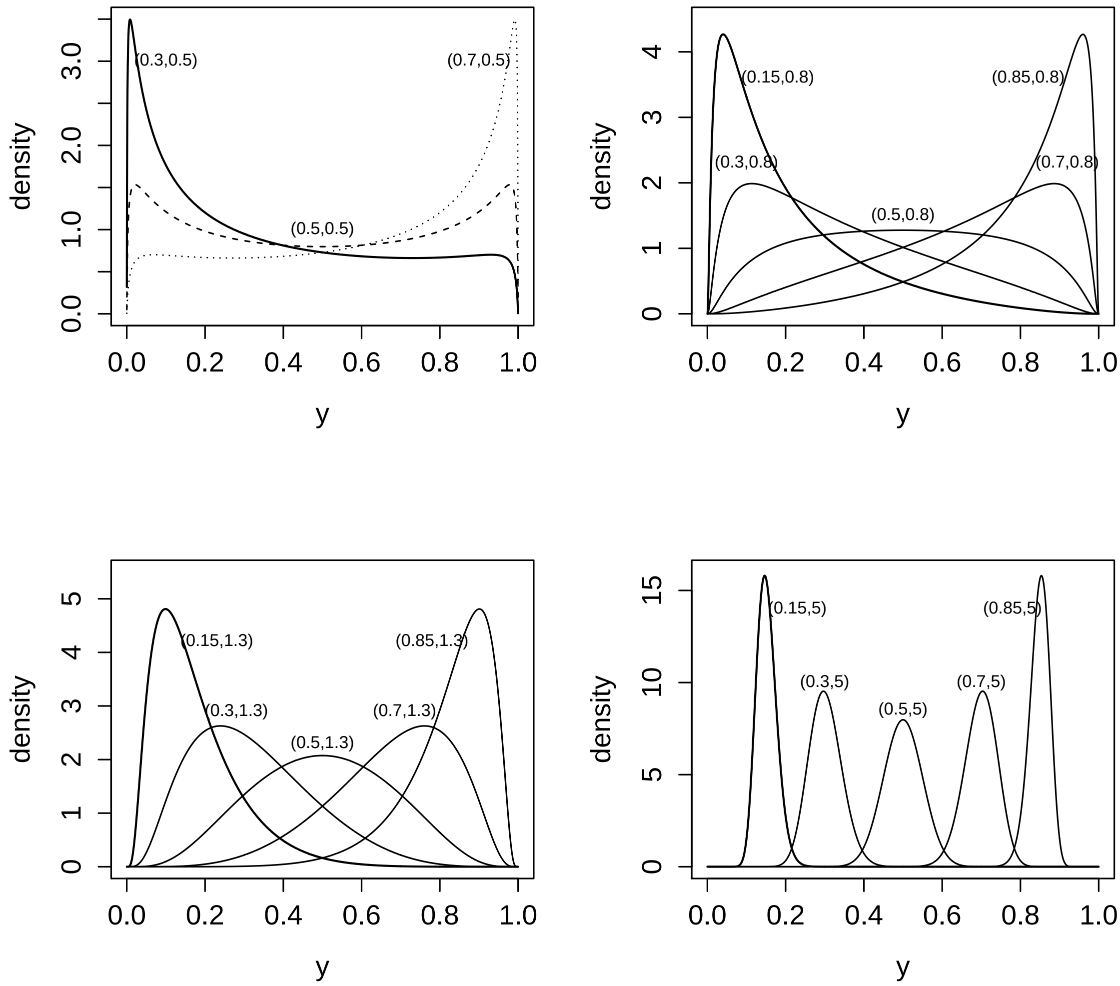

Figure 2.3: The Johnson S_B density function for some parameter values of ξ and δ.

link functions are strictly increasing and twice differentiable.

Based on a sample of n independent observations, the Johnson S_B regression model log-likelihood function is

$$\ell(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{t=1}^{n} \ell_t(\xi_t, \delta_t),$$

where

$$\ell_t(\xi_t, \delta_t) = \log(\delta_t) + \log[g(\{\delta_t[t(y_t) - t(\xi_t)]\}^2)] - \log[y_t(1 - y_t)],$$

$\xi_t = g^{-1}(\eta_t)$ and $\delta_t = h^{-1}(\vartheta_t)$ being functions of $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$.

The score function components are

$$\mathbf{U}_{\boldsymbol{\beta}}(\boldsymbol{\beta}, \boldsymbol{\tau}) = X^\top \Delta W T \boldsymbol{\xi}^* \text{ and } \mathbf{U}_{\boldsymbol{\tau}}(\boldsymbol{\beta}, \boldsymbol{\tau}) = Z^\top H \boldsymbol{\delta}^*,$$

where $T = \operatorname{diag}\{1/g'(\xi_t)\}$, $H = \operatorname{diag}\{1/h'(\delta_t)\}$, $W = \operatorname{diag}\left\{-\frac{1}{2}w_t\right\}$, $\Delta = \operatorname{diag}\{\delta_t\}$,

$\boldsymbol{\xi}^* = (\xi_1^*, \ldots, \xi_n^*)^\top$, $\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_n^*)^\top$, $\xi_t^* = -2/[\xi_t(1-\xi_t)]$, $\delta_t^* = (1-w_t^2)/\delta_t$ and $w_t = \delta_t[t(y_t) - t(\xi_t)]$.

As in the previous regression models, let $K(\boldsymbol{\beta}, \boldsymbol{\gamma})$ the Fisher information matrix for $(\boldsymbol{\beta}, \boldsymbol{\gamma})$. Under standard regularity conditions and when n is large, the maximum likelihood estimators are approximately normally distributed, i.e.,

$$\begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{pmatrix} \sim N_{k+l}\left(\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, K(\boldsymbol{\beta}, \boldsymbol{\gamma})^{-1} \right)$$

approximately.

Since there is no closed-form expression for $K(\boldsymbol{\beta}, \boldsymbol{\gamma})$, interval estimation and hypothesis testing inference are typically based on the observed information evaluated at $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$. The observed information matrix is

$$J(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \begin{pmatrix} J_{\boldsymbol{\beta\beta}} & J_{\boldsymbol{\beta\gamma}} \\ J_{\boldsymbol{\beta\gamma}}^\top & J_{\boldsymbol{\gamma\gamma}} \end{pmatrix}.$$

It can be shown that $J_{\boldsymbol{\beta\beta}} = X^\top W_1 T X$, $J_{\boldsymbol{\beta\gamma}} = X^\top W_2 T H Z$ and $J_{\boldsymbol{\gamma\gamma}} = Z^\top W_3 H Z$, where $W_1 = \mathrm{diag}\{w_t^{(1)}\}$, $W_2 = \mathrm{diag}\{w_t^{(2)}\}$ and $W_3 = \mathrm{diag}\{w_t^{(3)}\}$ with

$$\begin{aligned} w_t^{(1)} &= -\frac{\delta_t^2}{[\xi_t(1-\xi_t)]^2[g'(\xi_t)]}\left[\frac{(1-2\xi_t)w_t}{2\delta_t} + 1\right] - \frac{\delta_t w_t}{\xi_t(1-\xi_t)}\frac{g''(\xi_t)}{[g'(\xi_t)]^2}, \\ w_t^{(2)} &= \frac{2w_t}{\xi_t(1-\xi_t)}, \\ w_t^{(3)} &= -\frac{1}{h'(\delta_t)}\left(\frac{1+w_t^2}{\delta_t^2}\right) - \left(\frac{1-w_t^2}{\delta_t}\right)\frac{h''(\delta_t)}{[h'(\delta_t)]^2}. \end{aligned}$$

2.5 The unit gamma regression model

The last regression model we shall present is the unit gamma model which was recently proposed by Mousa, El-Sheikh and Abdel-Fattah (2016) and Abdel-Fattah (forthcoming). The underlying assumption is that the variable of interest follows the unit gamma distribution introduced by Grassia (1977) which is a logarithmic transformation of the gamma distribution with support limited to the standard unit interval. The gamma probability density function is

$$f(u; \alpha, \tau) = \frac{\alpha^\tau}{\Gamma(\tau)} e^{-\alpha u} u^{(\tau-1)}, \tag{2.6}$$

$\alpha, \tau > 0$ and $0 < u < \infty$. Let $y = \log(1/u)$. It then follows that the probability density function given in (2.6) becomes

$$f(y; \alpha, \tau) = \frac{\alpha^\tau}{\Gamma(\tau)} y^{\alpha-1} \left[\log\left(\frac{1}{y}\right)\right]^{(\tau-1)}, \tag{2.7}$$

where $\alpha, \tau > 0$ and $0 < y < 1$. The distribution mean and variance are given, respectively, by

$$\mathbb{E}(y) = \left[\frac{\alpha}{\alpha+1}\right]^{\tau} \text{ and } \mathrm{Var}(y) = \left\{\left[\frac{\alpha}{\alpha+2}\right]^{\tau} - \left[\frac{\alpha}{\alpha+1}\right]^{2\tau}\right\}.$$

Since regression models are typically constructed for modeling the population mean and possibly the precision or dispersion, both Mousa, El-Sheikh and Abdel-Fattah (2016) and Abdel-Fattah (forthcoming) considered a new parameterization for the density in (2.7). Let $\alpha = \mu^{1/\tau}/(1-\mu^{1/\tau})$. It then follows that the unit gamma mean and variance are given by

$$\mathbb{E}(y) = \left[\frac{\alpha}{\alpha+1}\right]^{\tau} = \mu \text{ and } \mathrm{Var}(y) = \left\{\left[\frac{\alpha}{\alpha+2}\right]^{\tau} - \left[\frac{\alpha}{\alpha+1}\right]^{2\tau}\right\} = \mu\left\{\left[\frac{1}{\left(2-\mu^{\frac{1}{\tau}}\right)^{\tau}}\right] - \mu\right\},$$

respectively. Notice that the variance is a function of the mean. It is also a function of a precision parameter, τ: for fixed μ, the variance decreases as the value of τ increases. The unit gamma probability density function then becomes

$$f(y;\mu,\tau) = \frac{\left[\frac{\mu^{1/\tau}}{1-\mu^{1/\tau}}\right]^{\tau}}{\Gamma(\tau)} y^{\frac{\mu^{1/\tau}}{1-\mu^{1/\tau}}-1}\left[\log\left(\frac{1}{y}\right)\right]^{\tau-1},$$

$0 < \mu, y < 1$ and $\tau > 0$.

Figure 2.4 shows different unit gamma densities. It is noteworthy the J, inverted J and U shapes in the upper panels. The density is symmetric when $\mu = 0.5$ and asymmetric when $\mu \neq 0.5$. We note that, for large values of τ and regardless value of μ, the density becomes more symmetric.

Mousa, El-Sheikh and Abdel-Fattah (2016) and Abdel-Fattah (forthcoming) introduced the fixed and variable dispersion unit gamma regression models. In what follows we shall focus on the latter. Let $y_1, \ldots, y_n$ be independent variates such that $y_t \sim UG(\mu_t, \tau_t)$. The mean μ_t and precision τ_t parameters can be modeled using two submodels, as in the previous models. The regression structure is

$$g(\mu_t) = \sum_{i=1}^{k} x_{ti}\beta_i = \mathbf{x}_t^\top\boldsymbol{\beta} = \eta_t \text{ and } h(\tau_t) = \sum_{j=1}^{l} z_{tj}\gamma_j = \mathbf{z}_t^\top\boldsymbol{\gamma} = \vartheta_t,$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are regression parameters, $\mathbf{x}_t^\top$ and $\mathbf{z}_t^\top$ the tth rows of the corresponding matrices of explanatory variables, $g(\cdot)$ and $h(\cdot)$ strictly increasing and twice-differentiable link functions, and η_t and ϑ_t are linear predictors for μ_t and τ_t, respectively.

The unit gamma regression log-likelihood function is

$$\ell(\boldsymbol{\beta},\boldsymbol{\gamma}) = \sum_{t=1}^{n} \ell_t(\mu_t, \tau_t),$$

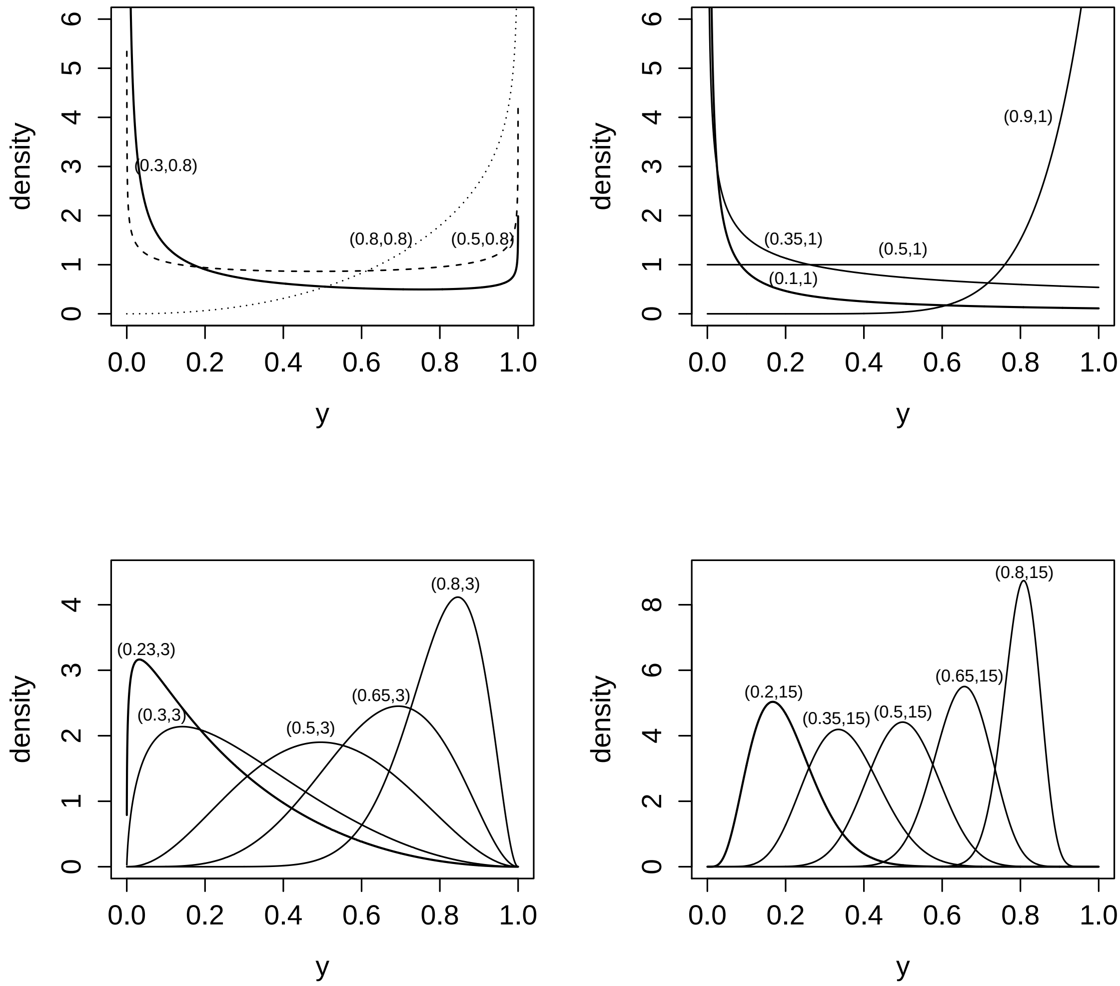

Figure 2.4: The unit gamma density function for some parameter values of μ and τ.

where

$$\ell_t(\mu_t, \tau_t) = \tau_t \log(d_t) - \log\Gamma(\tau_t) + (d_t - 1)\log(y_t) + (\tau_t - 1)\log\left[-\log y_t\right],$$

$d_t = \mu_t^{1/\tau_t}/(1 - \mu^{1/\tau_t})$, $\mu_t = g^{-1}(\eta_t)$ and $\tau_t = h^{-1}(\vartheta_t)$.

The components of the score function corresponding to $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are

$$\mathbf{U}_{\boldsymbol{\beta}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = X^\top T\mathbf{p} \text{ and } \mathbf{U}_{\boldsymbol{\gamma}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = Z^\top H\mathbf{q},$$

where X, Z, T and H diagonal matrices (previously defined) and $\mathbf{p} = (p_1, \ldots, p_n)^\top$ and

$\mathbf{q} = (q_1, \ldots, q_n)^\top$, whose tth elements are

$$
\begin{aligned}
p_t &= \frac{d_t}{\mu_t^{1/\tau_t+1}} \left[1 + \frac{1}{\tau_t} d_t \log y_t\right], \\
q_t &= \log[-\log(y_t)] - \left[\frac{1}{\tau_t} d_t \log(\mu_t)\right] \left[1 + \frac{d_t \log(y_t)}{\tau_t \mu_t^{1/\tau_t}}\right] - \log\left[\frac{\mu_t^{1/\tau_t}}{d_t}\right] - \psi(\tau_t).
\end{aligned}
$$

Fisher's information matrix can be expressed as

$$
K = K(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \begin{pmatrix} K_{\boldsymbol{\beta}\boldsymbol{\beta}} & K_{\boldsymbol{\beta}\boldsymbol{\gamma}} \\ K_{\boldsymbol{\gamma}\boldsymbol{\beta}} & K_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \end{pmatrix} = \begin{pmatrix} X^\top \mathrm{W}_{\boldsymbol{\beta}\boldsymbol{\beta}} X & X^\top \mathrm{W}_{\boldsymbol{\beta}\boldsymbol{\gamma}} Z \\ Z^\top \mathrm{W}_{\boldsymbol{\gamma}\boldsymbol{\beta}} X & Z^\top \mathrm{W}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} Z, \end{pmatrix},
$$

where

$$
\mathrm{W}_{\boldsymbol{\beta}\boldsymbol{\beta}} = \mathrm{diag}\left\{\frac{\tilde{p}_t^2}{\tau_t}\left[\frac{1}{g'(\mu_t)}\right]^2\right\}, \ \mathrm{W}_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \mathrm{diag}\left\{\frac{\tilde{q}_t}{\tau_t}\left[\frac{\tilde{r}_t}{\tau_t} + 1\right]\left[\frac{1}{g'(\mu_t)}\right]\left[\frac{1}{h'(\tau_t)}\right]\right\}, \ \mathrm{W}_{\boldsymbol{\gamma}\boldsymbol{\beta}} = \mathrm{W}_{\boldsymbol{\beta}\boldsymbol{\gamma}}^\top,
$$

$$
\mathrm{W}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} = \mathrm{diag}\left\{\left[\frac{2\tilde{r}_t}{\tau_t^2} + \frac{\tilde{r}_t^2}{\tau_t^3} + \psi'(\tau_t)\right]\left[\frac{1}{h'(\tau_t)}\right]^2\right\}, \ \tilde{p}_t = \frac{d_t}{\mu_t^{\frac{1}{\tau_t}+1}}, \ \tilde{q}_t = -\tilde{p}_t \text{ and } \tilde{r}_t = \frac{d_t \log(\mu_t)}{\mu_t^{\frac{1}{\tau_t}}}.
$$

As in the beta and Johnson S_B regression models, the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are not orthogonal.

Under standard regularity conditions and when the sample size is large, the maximum likelihood estimators are approximately normally distributed, i.e.,

$$
\begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{pmatrix} \sim N_{k+l}\left(\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, K^{-1}\right)
$$

approximately.

Chapter 3

Nonnested hypothesis tests for models used with rates and proportions

Resumo

Neste capítulo introduzimos, na Seção 3.2, o teste J proposto por Davidson e MacKinson (1981) e na Seção 3.3, o teste MJ desenvolvido por Hagemann (2012) para o modelo de regressão linear clássico. Tais testes podem ser usados com modelos não encaixados. Na Seção 3.4 apresentamos a adoptação desses testes para modelos de regressão para dados contínuos limitados ao intervalo $(0, 1)$, sendo este o objetivo da presente dissertação.

3.1 Introduction

The J and MJ tests were originally proposed by Davidson and MacKinnon (1981) and Hagemann (2012), respectively, for testing nonnested hypotheses in the classical linear regression model. They were later extended to beta regression models by Cribari-Neto and Lucena (2015) and to the class of Generalized Additive Models for Location, Scale and Shape (GAMLSS) models by Cribari-Neto and Lucena (2017). The tests proved useful for distinguishing between different beta regressions and GAMLSS that are not nested. All competing models, however, are taken to belong to the same class of regression models. In what follows we shall pursue a different investigation, namely: we shall investigate the usefulness of nonnested hypothesis testing inference for distinguishing between models that belong to different classes of regressions models. Since the some of the competing models may belong to different classes of models they are not nested in a wider, encompassing general class. It is thus important to evaluate the usefulness of the nonnested hypothesis tests in this more challenging setup. It is noteworthy that it is not uncommon for practitioners to have at their disposal a few plausible models that belong to different classes of models. For instance, when modeling rates and proportions statisticians may consider beta and simplex regression models. They may end up with one or two models from each class that seem to provide good fits. How to distinguish between them? This is our motivation in this dissertation.

3.2 J test

The J test was introduced by Davidson and MacKinnon (1981) for the classical linear regression model. It is based on the construction of an artificial that encompasses all competing nonnested models. Suppose there are M ($M \geq 2$) nonnested regression models that can be used to model the behavior of $\mathbf{y} = (y_1, \ldots, y_n)^\top$. Let $\mathcal{M} = \{1, \ldots, M\}$ and let m^* denote the true model. We are then interested in testing the null hypothesis $\mathrm{H}_0 : m = m^*$ versus $\mathrm{H}_A : m \neq m^*$ for a given model $m \in \mathcal{M}$. To that end, consider the following encompassing model:

$$y = \left(1 - \sum_{l \in \mathcal{M} \setminus \{m\}} \alpha_l\right) X_m \boldsymbol{\beta}_m + \sum_{l \in \mathcal{M} \setminus \{m\}} \alpha_l X_l \boldsymbol{\beta}_l + u, \tag{3.1}$$

where α_l, $l = 1, \ldots, M$, are additional parameters, X_m is the $n \times k_m$ matrix of explanatory variables of the mth model, $\boldsymbol{\beta}_l$ is a vector of k_m unknown parameters and $\mathbf{u}$ is a vector of random errors. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_l$ in (3.1) may not be identifiable, Davidson and MacKinnon (1981) proposed to replaced $\boldsymbol{\beta}_l$ ($l \neq m$) by the corresponding least squares estimates. We thus have

$$y = \left(1 - \sum_{l \in \mathcal{M} \setminus \{m\}} \alpha_l\right) X_m \boldsymbol{\beta}_m + \sum_{l \in \mathcal{M} \setminus \{m\}} \alpha_l X_l \hat{\boldsymbol{\beta}}_l + u, \tag{3.2}$$

which can be estimated by ordinary least squares. One can then test $\mathrm{H}_0 : \alpha_m = 0$ against a two-sided alternative.

Let

$$\boldsymbol{\upsilon}_m = n^{-\frac{1}{2}} \left(\mathbf{y}^\top P_l M_m \mathbf{y}\right)_{l \in \mathcal{M} \setminus \{m\}},$$
$$\hat{\Sigma}_m = n^{-1} \left(\mathbf{y}^\top P_l M_m \hat{\Omega}_m M_m P_{l'} \mathbf{y}\right)_{l,l' \in \mathcal{M} \setminus \{m\}},$$

where $P_m = X_m \left(X_m^\top X_m\right)^{-1} X_m^\top$ and $M_m = I_n - P_m$ are the usual projection matrices and $\hat{\Omega}_m = \mathrm{diag}\{\hat{u}_{t,m}^2\}$ is a diagonal matrix of square residuals defined by $\hat{u}_{t,m} = y_t - \mathbf{x}_{t,m}^\top \hat{\boldsymbol{\beta}}_m$. The J test statistic for testing the validity of model m is

$$J = \boldsymbol{\upsilon}_m^\top \hat{\Sigma}_m^{-1} \boldsymbol{\upsilon}_m. \tag{3.3}$$

Under null hypothesis, J is asymptotically distributed as χ^2_{M-1}. The null hypothesis is rejected at nominal level α if J exceeds the $1-\alpha$ upper χ^2_{M-1} quantile. When that happens the model fit is considerably improved upon inclusion of additional information obtained from the alternative models, and model m is rejected. The J test must be carried out for each candidate model. The final inference may be the acceptance of one of the models, more than one model or none model.

3.3 MJ test

The MJ test of Hagemann (2012) considers a different null hypothesis. The interest lies in testing $\mathrm{H}_0 : m^* \in \mathcal{M}$ against $\mathrm{H}_A : m^* \notin \mathcal{M}$. That is, one tests whether the true model is one of the candidate models. Under the null hypothesis, it is; under the alternative hypothesis, in contrast, all models listed in $\mathcal{M}$ are incorrect. Under the null hypothesis, J_{m^*} is asymptotically distributed as χ^2_{M-1}. In contrast, all remaining test statistics, i.e., J_m for $m \in \mathcal{M}\backslash\{m^*\}$, are such that $\lim_{n\to\infty} \mathbb{P}(J_m > b) = 1 \,\forall\, b \in \mathbb{R}$. Notice that the test statistic that corresponds to the correct model has a well defined asymptotic null distribution whereas all other test statistics diverge to ∞. As a consequence, the natural candidate for true model when the null hypothesis is not rejected is that associated to the minimal J test statistic. The MJ test can thus be carried out as follows:

1. For each $m \in \mathcal{M}$, fit the regression model in (3.2) and compute the J statistic given in (3.3).
2. Obtain the MJ test statistic as $MJ = \min\{J_1, \ldots, J_M\}$, where $J_1, \ldots, J_M$ are the M J test statistics.
3. Reject the null hypothesis H_0 at significance level α if $MJ > \chi^2_{M-1,1-\alpha}$, where $\chi^2_{M-1,1-\alpha}$ is the $1-\alpha$ upper χ^2_{M-1} quantile.

Failure to reject the null hypothesis when performing the MJ test indicates that one of the candidate models is the true model. In that case, one is left with the following question: Which model is the true model? Such a question can be answered by considering an MJ-based model selection strategy. Since, under the null hypothesis, the minimal J test statistic is the only J statistic to have a well defined asymptotic distribution, the model that corresponds to it can be taken as the true model. Indeed, Hagemann (2012) showed that such a model selection strategy will consistently choose the true model whenever the correct model is in the set of candidate models, that is, $\lim_{n\to\infty} \mathbb{P}\left(\hat{m} = m^* | MJ \leq \chi^2_{M-1,1-\alpha}\right) = 1$, where $\hat{m}$ indicates the selected model.

3.4 Nonnested hypothesis tests in models for rates and proportions.

Nonnested hypothesis are arguably more useful in regression models for responses that are rates or proportions than in the classical linear. This is so because, as seen in previous section, regression models for variables that assume values in the standard unit interval involve extra layers of complexity. For instance, such models typically include two submodels: one for the mean and another one for the precision/dispersion. Each model in turn includes a link function, which is a common source of nonnestedness. Cribari-Neto and Lucena (2015) considered J and MJ testing inference for beta regressions. In what follows, we shall explain how such testing inference can be performed and we shall also take one step further and consider nonnested testing inference for responses that assume values in the standard unit interval when some of the competing models belong to different classes of regression models.

J and MJ tests can be performed in more elaborated regression models as outlined by Cribari-Neto and Lucena (2015), who considered nonnested hypothesis testing inference in beta regressions. A similar approach can be used for the other models described Chapter 2. As before, suppose there are M competing regression models for a response that assumes values in the standard unit interval:

$$\begin{aligned} \mathrm{H}_l : g_l(\mathfrak{l}) &= X_l\boldsymbol{\beta}_l = \eta_l \\ h_l(\mathfrak{p}) &= Z_l\boldsymbol{\gamma}_l = \vartheta_l, \end{aligned}$$

$l = 1, \dots, M$, where g_l and h_l are strictly increasing and twice-differentiable link functions for the location and precision submodels, respectively, $\mathfrak{l}$ and $\mathfrak{p}$ are location and precision vectors for the lth regression model, X_l and Z_l are matrices of explanatory variables, $\boldsymbol{\beta}_l$ and $\boldsymbol{\gamma}_l$ are unknown parameter vectors, and η_l and ϑ_l are linear predictors for the location and precision submodels, respectively. It is important to note that nonnested models can differ in explanatory variables and/or link functions.

The J test of the model specified in H_l can be performed as follows. At the outset, the parameters that index the competing models are estimated by maximum likelihood. One thus obtain $\hat{\mathfrak{l}}_m$ and $\hat{\mathfrak{p}}_m$ $(m = 1, \dots, M, m \neq l)$. Such fitted values are then used to augment the model under scrutiny, i.e., they are included in the model specified in H_l as additional covariates. Finally, we test the exclusion of the added testing variables (i.e., the added regressors). To that end, one of the following test statistics can be used:

$$J_{S_l} = \tilde{\mathbf{U}}^\top \tilde{K}^{-1} \tilde{\mathbf{U}}, \tag{3.4}$$

$$J_{G_l} = \tilde{\mathbf{U}}^\top \left[(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) - (\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}) \right], \tag{3.5}$$

$$J_{LR_l} = 2\{\ell(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) - \ell(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}})\}. \tag{3.6}$$

They are, respectively, the score, gradient and likelihood ratio test statistics. Here, hats and tildes indicate, respectively, evaluation at the unrestricted (augmented model) and restricted (Model H_l) maximum likelihood estimates of $(\boldsymbol{\beta}, \boldsymbol{\gamma})$. Let $M_{L(l)}$ and $M_{P(l)}$ denote the number of location and precision submodels that differ from the corresponding submodel of the lth model (the model specified in H_l). Notice that the augmented model is obtained by including $M_{L(l)}$ additional explanatory variables in the location submodel and $M_{P(l)}$ extra regressors in the precision submodel. We reject the model specified in H_l at significance level α if $J_l > \chi^2_{M_{L(l)}+M_{P(l)},1-\alpha}$.

As noted earlier, our interest lies in the case where the competing models may belong to different classes of regression models. For instance, the practitioner may have at his/her disposal a beta, a simplex and an unit gamma regression that seem to fit the data well. When all candidate models differ in at least one of the submodels, the above procedure is to be used. It is possible, nonetheless, that the candidate models belong to different classes of models but use the same regression structures for the two submodels (mean and precision/dispersion), i.e., they only differ in the assumption made on the response distribution. When that is the case, we propose augmenting the two submodels of each model with testing variables obtained from the other competing models and then test their exclusion. The test critical value is thus obtained from the $\chi^2_{2(M-1)}$ distribution.

The MJ test can be used to test the null hypothesis that one of the candidate models is the true model. As noted earlier, the MJ test statistic is the minimal J test statistic. Thus,

$$\begin{aligned} MJ_S &= \min\{J_{S_1}, \ldots, J_{S_M}\}, \\ MJ_G &= \min\{J_{G_1}, \ldots, J_{G_M}\}, \\ MJ_{LR} &= \min\{J_{LR_1}, \ldots, J_{LR_M}\}, \end{aligned}$$

where J_{S_l}, J_{G_l} and J_{LR_l} are defined in (3.4), (3.5) and (3.6), respectively. The null hypothesis is rejected if $MJ > \chi^2_{\omega,1-\alpha}$, where ω equals the number of degrees of freedom in the J test associated with the smallest J statistic. When the null hypothesis is not rejected, MJ-based model selection can be carried out: the model that corresponds to the smallest J test statistic is selected.

All nonnested hypothesis testing inferences described above were designed to select a particular model from a set of candidate models when all candidate models belong to the same class of models (e.g., they are all linear regressions or they are all beta regressions). In contrast, our interest lies in the situation where one is modeling a response that assumes values in the standard unit interval and has at his/her disposal plausible candidate models that are not only nonnested, but belong to different classes of models.

Chapter 4

Numerical results

Resumo

Neste capítulo apresentamos os resultados numéricos da avaliação dos desempenhos em amostras finitas dos testes J e MJ apresentados na Seção 3.4 através de simulações de Monte Carlo. Foram calculadas as taxas de rejeição nula e não nula dos testes, considerando $M = 2$ e $M = 3$ modelos concorrentes. Os modelos concorrentes diferem na distribuição assumida pela variável resposta (beta, simplex, gama unitária e Johnson S_B), como também nos regressores usados e nas funções de ligação no submodelo da locação e/ou no submodelo da precisão/dispersão. Também, são apresentados os resultados do desempenho do procedimento de seleção de modelos.

4.1 Introduction

In what follows we shall present the results of several Monte Carlo simulations that were performed to evaluate the finite sample performances of J and MJ tests based on the score (S), gradient (G) and likelihood ratio (LR) test statistics. We consider situations where $M = 2$ (there are two competing models) and $M = 3$ (there are three competing models). The interest lies in modeling responses that assume values in the standard unit interval. The nonnested models may differ in the covariates and/or in the link functions they employ. They may also belong to different classes of models. That is, the competing models are not restricted to a given class of models.

We consider several different situations in our numerical evaluations. They are determined by the classes of regression models under evaluation ('case') and by the models' link functions and regressors ('scenario'). Table 4.1 presents the different cases and scenarios used in our simulations.

We performed 10,000 Monte Carlo replications for each data generating process. Each competing model was taken to be the true model in a set of simulations. Hence, when $M = 2$ ($M = 3$) two (three) sets of simulations were performed, each based on 10,000 replications. The sample sizes are $n = 50, 100$ and 250 for scenarios 1 through 6. For scenario

Table 4.1: Cases and scenarios used in the numerical evaluations.

CASES			SCENARIOS	
Two models ($M = 2$)	I	beta vs. simplex	1.	Different regressors in the location submodels.
	II	unit gamma vs. beta	2.	Different regressors in the precision/dispersion submodels.
	III	simplex vs. unit gamma	3.	Different regressors in the location and precision/dispersion submodels.
	IV	beta vs. Johnson S_B	4.	Different link functions in the location submodels.
	V	Johnson S_B vs. simplex	5.	Different link functions in the precision/dispersion submodels.
Three models ($M = 3$)	VI	beta vs. simplex vs. unit gamma	6.	Different link functions in location and precision/dispersion submodels.
	VII	beta vs. simplex vs. Johnson S_B	7.	Only different in the distribution.

7, we consider $n = 100, 250$ and 400. The covariates values used in location and precision (or dispersion) submodels were obtained as random standard uniform draws and were kept fixed through the experiment. We randomly generated 50 values all covariates, and then replicated such values (twice, five and eight times) to obtain larger samples. This was done so that the heterogeneity strength, which is measured by $\lambda_v = [\max \mathrm{Var}(y_t)]/[\min \mathrm{Var}(y_t)]$, would remain constant for all sample sizes. All log-likelihood maximizations were carried out using the BFGS quasi-Newton method with analytical first derivatives and all tests were performed at the $\alpha = 1\%$, 5% and 10% significance levels.

For cases I, II, III and VI, we shall report simulation results on the finite sample performances of J and MJ tests based on score, gradient and likelihood ratio statistics; for cases IV, V and VII, we shall only report results obtained using the gradient and likelihood ratio test statistics, since in Johnson S_B regression models we oftentimes obtained negative values for the score test statistic even when $n = 250$. Recall that since there is no closed form expression for the expected information matrix in that class of models the score test uses the observed information. As noted by Morgan, Palmer and Ridout (2007), the score statistic may assume negative values when it is based on the observed information. That happens when performing score testing inference in Johnson S_B regression models.

The tests critical values were obtained from the test statistics limiting null distribution, which is χ^2_c. For scenarios 1, 2, 4 and 5, $c = 1$ when $M = 2$ and $c = 2$ when $M = 3$. In scenario 7, the competing models only differ in the response distribution. In that case, we augment both submodels and test the exclusion of the added variables. Therefore, for scenarios 3, 6 and 7, $c = 2$ when $M = 2$ and $c = 4$ when $M = 3$.

4.2 Models differ in their regressors

Table 4.2 contains the tests null rejection rates (%) for scenario 1 when $M = 2$ (two competing models), the models being

$$\mathrm{H}_1 : \log\left(\frac{\mathfrak{l}_{1t}}{1-\mathfrak{l}_{1t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \qquad \mathrm{H}_2 : \log\left(\frac{\mathfrak{l}_{2t}}{1-\mathfrak{l}_{2t}}\right) = \beta_0 + \beta_3 x_{t1} + \beta_4 x_{t3}$$

$$\log(\mathfrak{p}_{1t}) = \gamma_0 + \gamma_1 x_{t1} \qquad \log(\mathfrak{p}_{2t}) = \gamma_2 + \gamma_3 x_{t1},$$

$t = 1, \ldots, n$. The two competing models only differ in the location submodel: H_1 uses x_2 whereas H_2 uses x_3. The parameter values for the location submodels are $\beta_0 = 1.1$, $\beta_1 = -2.9$, $\beta_2 = 0.7 = \beta_4$ and $\beta_3 = -2.8$. It follows that $\mathfrak{l}_{1t}$ and $\mathfrak{l}_{2t}$ assume values in $(0.18, 0.81)$. The values of γ_l, $l = 0, \ldots, 3$, were chosen so that, for both models, $\lambda_v \approx 55$ for all sample sizes.

The figures in Table 4.2 show that the size distortions decrease as the sample size increases. Overall, for all sample sizes in cases I, II and III the score-based J and MJ tests are less size-distorted than the corresponding tests based on alternative test statistics. The gradient test is the second best performer; it slightly outperforms the likelihood ratio test. In cases IV and V the gradient-based test was also slightly superior to the likelihood ratio test. For instance, in case II, when the true data generating process is unit gamma, $n = 50$ and $\alpha = 5\%$, the score-based J and MJ null rejection rates are equal to 5.89% whereas the corresponding figure for the likelihood ratio test is 7.17%.

Table 4.3 contains the tests null rejection rates when there are three competing models for scenario 1. The first two models are as before and the third model is

$$\mathrm{H}_3 : \log\left(\frac{\mathfrak{l}_{3t}}{1 - \mathfrak{l}_{3t}}\right) = \beta_0 + \beta_5 x_{t1} + \beta_6 x_{t4}$$
$$\log(\mathfrak{p}_{3t}) = \gamma_4 + \gamma_5 x_{t1},$$

$t = 1, \ldots, n$. The parameter values in Models H_1 and H_2 are as before and, for Model H_3, $\beta_5 = -2.7$ and $\beta_6 = 0.5$ and $\mathfrak{l}_{3t} \in (0.18, 0.81)$. The size distortions of the tests based on the LR statistic are considerably larger when there are three competing models than when $M = 2$. That is, the likelihood ratio tests are noticeably less accurate when $M = 3$. For instance, in case VI, when the data generating process is unit gamma, $n = 50$ and $\alpha = 5\%$ (10%), the LR-based MJ null rejection rate is 10.14% (17.28%). All tests become more accurate as the sample size increases.

Tables 4.4 and 4.5 contain the tests null rejection rates for scenario 2, situation in which the competing models differ in the covariates used in the precision/dispersion submodels. The models are

$$\mathrm{H}_1 : \log\left(\frac{\mathfrak{l}_{1t}}{1 - \mathfrak{l}_{1t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}$$
$$\log(\mathfrak{p}_{1t}) = \gamma_0 + \gamma_1 x_{t1}$$

$$\mathrm{H}_2 : \log\left(\frac{\mathfrak{l}_{2t}}{1 - \mathfrak{l}_{2t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}$$
$$\log(\mathfrak{p}_{2t}) = \gamma_2 + \gamma_3 x_{t2},$$

$$\mathrm{H}_3 : \log\left(\frac{\mathfrak{l}_{3t}}{1 - \mathfrak{l}_{3t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}$$
$$\log(\mathfrak{p}_{3t}) = \gamma_4 + \gamma_5 x_{t3},$$

$t = 1, \ldots, n$. When $M = 2$ the competing models are H_1 and H_2; when $M = 3$, all three models are considered. The parameters values for the location submodels are $\beta_0 = 1.1$, $\beta_1 = -2.9$ and $\beta_3 = 0.7$. The parameter values in the precision submodels were selected so that $\lambda_v \approx 55$ in all three models. For that scenario, the performances of the J and MJ tests are quite similar for $M = 2$ (two competing models) and $M = 3$ (three competing models). The size distortions are larger than when the models differ in the specification of the location submodel. We also note that the score-based tests are quite conservative, especially when the sample size is small. In contrast, the gradient and likelihood ratio tests overreject the null hypothesis; they are noticeably liberal when n is small.

We shall now consider the situation in which both submodels differ in the regressors they include (scenario 3). More precisely, the models at hand are as follows:

$$\mathrm{H}_1 : \log\left(\frac{\mathfrak{l}_{1t}}{1-\mathfrak{l}_{1t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \qquad \mathrm{H}_2 : \log\left(\frac{\mathfrak{l}_{2t}}{1-\mathfrak{l}_{2t}}\right) = \beta_0 + \beta_3 x_{t1} + \beta_4 x_{t3}$$

$$\log(\mathfrak{p}_{1t}) = \gamma_0 + \gamma_1 x_{t2} \qquad \log(\mathfrak{p}_{2t}) = \gamma_2 + \gamma_3 x_{t3},$$

$$\mathrm{H}_3 : \log\left(\frac{\mathfrak{l}_{3t}}{1-\mathfrak{l}_{3t}}\right) = \beta_0 + \beta_5 x_{t1} + \beta_6 x_{t4}$$

$$\log(\mathfrak{p}_{3t}) = \gamma_4 + \gamma_5 x_{t4},$$

$t = 1, \ldots, n$. The parameter values are as before.

The tests null rejection rates are presented in Tables 4.6 ($M = 2$) and 4.7 ($M = 3$). When $M = 2$, for cases I to III and for all sample sizes the score-based J and MJ tests outperform the corresponding tests based on the alternative testing criteria (likelihood ratio and gradient). For instance, under case III, $n = 50$ and $\alpha = 5\%$ the score-based tests null rejection rate equals 5.02%, which is very close to 5% (nominal level). The gradient tests outperform the likelihood ratio tests under cases IV and V.

When there are three competing models at hand (case VI), the score-based J and MJ tests under beta and unit gamma data generating mechanisms were conservative when $n = 50$, especially the MJ test. For instance, under the beta (unit gamma) law, the MJ-score null rejection rate at the 5% nominal level was 2.80% (3.47%). Under case VII, the gradient and likelihood ratio implementations of the J and MJ tests are considerably liberal when the sample size is small ($n = 50$). Their size distortions are substantially smaller in larger samples.

Table 4.2: Null rejection rates (%), two competing models that differ in the location submodels regressors.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
		True Model: beta									**true model: simplex**								
I	J_S	1.38	1.07	0.80	6.57	5.34	4.52	12.33	11.08	9.56	1.02	1.03	1.00	5.78	5.44	5.24	11.34	10.96	10.55
	MJ_S	1.38	1.07	0.80	6.56	5.34	4.52	12.32	11.08	9.56	0.97	1.03	1.00	5.71	5.44	5.24	11.31	10.96	10.55
	J_G	1.48	1.10	0.83	6.83	5.48	4.58	12.78	11.29	9.58	1.15	1.04	1.03	6.24	5.54	5.27	11.88	11.26	10.68
	MJ_G	1.48	1.10	0.83	6.82	5.48	4.58	12.77	11.29	9.58	1.10	1.04	1.03	6.17	5.54	5.27	11.85	11.26	10.68
	J_{LR}	2.11	1.30	0.86	7.90	6.01	4.74	13.67	11.85	9.73	1.74	1.22	1.11	6.95	5.93	5.44	12.65	11.62	10.86
	MJ_{LR}	2.11	1.30	0.86	7.89	6.01	4.74	13.66	11.85	9.73	1.68	1.22	1.11	6.89	5.93	5.44	12.62	11.62	10.86
		true model: unit gamma									**true model: beta**								
II	J_S	1.05	1.00	0.97	5.89	5.32	5.20	11.68	10.93	10.15	1.31	0.96	0.85	6.49	5.51	4.78	12.56	10.41	9.97
	MJ_S	1.05	1.00	0.97	5.89	5.32	5.20	11.68	10.93	10.15	1.29	0.96	0.85	6.46	5.51	4.78	12.53	10.41	9.97
	J_G	1.09	1.04	0.98	6.07	5.43	5.21	12.26	11.10	10.27	1.38	1.02	0.84	6.82	5.63	4.80	13.13	10.57	10.05
	MJ_G	1.09	1.04	0.98	6.07	5.43	5.21	12.26	11.10	10.27	1.36	1.02	0.84	6.81	5.63	4.80	13.10	10.57	10.05
	J_{LR}	1.72	1.32	1.06	7.17	5.87	5.34	13.27	11.57	10.44	2.10	1.31	0.92	7.97	5.99	4.97	13.97	10.94	10.17
	MJ_{LR}	1.72	1.32	1.06	7.17	5.87	5.34	13.27	11.57	10.44	2.09	1.31	0.92	7.95	5.99	4.97	13.94	10.94	10.17
		true model: simplex									**true model: unit gamma**								
III	J_S	1.04	0.99	1.06	6.05	5.59	4.96	11.66	10.61	9.81	0.97	0.86	0.93	5.43	4.79	5.02	10.58	10.03	10.31
	MJ_S	1.04	0.99	1.06	6.05	5.59	4.96	11.66	10.61	9.81	0.97	0.86	0.93	5.43	4.79	5.02	10.58	10.03	10.31
	J_G	1.10	1.01	1.12	6.36	5.76	4.98	12.28	10.82	9.87	0.99	0.83	0.94	5.73	4.90	5.08	11.22	10.29	10.42
	MJ_G	1.10	1.01	1.12	6.36	5.76	4.98	12.28	10.82	9.87	0.99	0.83	0.94	5.73	4.90	5.08	11.22	10.29	10.42
	J_{LR}	1.88	1.33	1.21	7.38	6.21	5.17	13.32	11.19	9.96	1.57	1.09	0.97	6.59	5.44	5.19	12.14	10.66	10.64
	MJ_{LR}	1.88	1.33	1.21	7.38	6.21	5.17	13.32	11.19	9.96	1.57	1.09	0.97	6.59	5.44	5.19	12.14	10.66	10.64
		true model: beta									**true model: Johnson S_B**								
IV	J_G	1.21	0.94	0.93	6.38	4.83	4.51	11.91	9.78	9.56	0.76	1.05	1.04	5.48	5.18	5.39	10.91	10.53	10.74
	MJ_G	1.20	0.94	0.93	6.37	4.83	4.51	11.91	9.78	9.56	0.76	1.05	1.04	5.48	5.18	5.39	10.91	10.53	10.74
	J_{LR}	1.86	1.09	1.00	7.21	5.22	4.63	12.83	10.11	9.68	1.28	1.40	1.13	6.35	5.55	5.58	11.92	10.93	10.89
	MJ_{LR}	1.86	1.09	1.00	7.21	5.22	4.63	12.83	10.11	9.68	1.28	1.40	1.13	6.35	5.55	5.58	11.92	10.93	10.89
		true model: Johnson S_B									**true model: simplex**								
V	J_G	1.17	1.08	1.08	6.20	5.44	5.00	12.16	11.00	10.05	1.14	1.04	1.15	5.89	5.60	5.30	11.91	11.08	10.21
	MJ_G	1.17	1.08	1.08	6.20	5.44	5.00	12.16	11.00	10.05	1.14	1.04	1.15	5.89	5.60	5.30	11.91	11.08	10.21
	J_{LR}	1.76	1.40	1.14	7.28	5.91	5.23	13.28	11.43	10.23	1.74	1.30	1.25	6.87	5.93	5.42	12.74	11.59	10.41
	MJ_{LR}	1.76	1.40	1.14	7.28	5.91	5.23	13.28	11.43	10.23	1.74	1.30	1.25	6.87	5.93	5.42	12.74	11.59	10.41

Table 4.3: Null rejection rates (%), three competing models that differ in the location submodels regressors.

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
	J_S	1.18	1.19	1.11	7.10	5.95	5.30	13.66	12.16	10.84
	MJ_S	1.03	1.19	1.11	6.79	5.94	5.30	13.32	12.15	10.84
	J_G	1.37	1.29	1.13	7.61	6.24	5.47	14.40	12.38	11.01
	MJ_G	1.23	1.29	1.13	7.33	6.24	5.47	14.09	12.37	11.01
	J_{LR}	2.61	1.71	1.30	9.87	7.10	5.83	16.65	13.32	11.29
	MJ_{LR}	2.47	1.71	1.30	9.54	7.10	5.83	16.37	13.31	11.29
		true model: simplex								
	J_S	1.01	0.88	1.01	6.19	5.61	5.36	12.65	11.39	10.48
	MJ_S	1.01	0.88	1.01	6.19	5.61	5.36	12.65	11.39	10.48
VI	J_G	1.21	0.98	1.05	6.91	5.86	5.46	13.69	11.79	10.61
	MJ_G	1.21	0.98	1.05	6.91	5.86	5.46	13.69	11.79	10.61
	J_{LR}	2.50	1.57	1.24	9.20	6.75	5.77	16.24	12.87	10.96
	MJ_{LR}	2.50	1.57	1.24	9.20	6.75	5.77	16.24	12.87	10.96
		true model: unit gamma								
	J_S	1.41	1.26	1.01	7.37	6.22	5.39	13.91	12.20	10.51
	MJ_S	1.41	1.26	1.01	7.33	6.22	5.39	13.88	12.20	10.51
	J_G	1.45	1.36	1.00	7.84	6.33	5.48	14.67	12.57	10.62
	MJ_G	1.45	1.36	1.00	7.81	6.33	5.48	14.64	12.57	10.62
	J_{LR}	2.77	1.68	1.14	10.17	7.19	5.80	17.31	13.70	11.02
	MJ_{LR}	2.75	1.68	1.14	10.14	7.19	5.80	17.28	13.70	11.02

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
	J_G	1.24	1.20	1.21	7.37	6.00	5.54	14.60	11.83	10.81
	MJ_G	1.10	1.17	1.21	6.98	5.97	5.54	14.23	11.83	10.81
	J_{LR}	2.52	1.57	1.33	9.66	6.88	5.73	16.70	12.86	11.26
	MJ_{LR}	2.30	1.55	1.33	9.26	6.85	5.73	16.37	12.86	11.26
		true model: simplex								
VII	J_G	1.26	1.37	1.09	7.20	5.95	5.33	13.89	12.10	10.66
	MJ_G	1.26	1.37	1.09	7.20	5.95	5.33	13.89	12.10	10.66
	J_{LR}	2.91	1.86	1.19	9.49	6.83	5.59	16.87	13.11	11.13
	MJ_{LR}	2.91	1.86	1.19	9.49	6.83	5.59	16.87	13.11	11.13
		true model: Johnson S_B								
	J_G	1.24	1.06	1.01	6.86	6.50	5.09	13.61	12.25	10.20
	MJ_G	1.24	1.06	1.01	6.86	6.50	5.09	13.61	12.25	10.20
	J_{LR}	2.68	1.57	1.15	9.39	7.52	5.32	16.16	13.22	10.47
	MJ_{LR}	2.68	1.57	1.15	9.39	7.52	5.32	16.16	13.22	10.47

Table 4.4: Null rejection rates (%), two competing models that differ in the precision/dispersion submodels regressors.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta									**true model: simplex**								
I	J_S	0.26	0.47	0.77	2.73	4.05	4.36	7.40	9.08	9.68	0.24	0.59	0.73	3.17	4.47	4.50	8.01	9.32	9.28
	MJ_S	0.25	0.47	0.77	2.73	4.05	4.36	7.40	9.08	9.68	0.14	0.59	0.73	2.82	4.47	4.50	7.58	9.31	9.28
	J_G	1.64	1.28	0.99	7.17	6.01	5.30	12.82	11.44	10.47	1.55	1.17	1.04	6.72	5.91	5.04	13.16	11.39	9.98
	MJ_G	1.63	1.28	0.99	7.17	6.01	5.30	12.82	11.44	10.47	1.29	1.17	1.04	6.37	5.91	5.04	12.86	11.39	9.98
	J_{LR}	2.26	1.50	1.07	7.96	6.43	5.49	13.53	11.96	10.69	1.92	1.35	1.10	7.38	6.24	5.20	13.84	11.76	10.09
	MJ_{LR}	2.25	1.50	1.07	7.96	6.43	5.49	13.53	11.96	10.69	1.68	1.35	1.10	7.05	6.24	5.20	13.58	11.76	10.09
		true model: unit gamma									**true model: beta**								
II	J_S	0.30	0.52	0.65	2.67	3.67	4.57	7.13	8.60	10.10	0.25	0.66	0.82	2.97	4.21	4.67	7.85	9.13	9.47
	MJ_S	0.18	0.52	0.65	2.59	3.67	4.57	7.04	8.60	10.10	0.07	0.66	0.82	2.66	4.21	4.67	7.60	9.13	9.47
	J_G	1.81	1.29	1.02	7.50	5.61	5.30	13.81	11.20	11.35	1.54	1.24	1.10	6.96	6.02	5.44	12.91	11.32	10.38
	MJ_G	1.69	1.29	1.02	7.42	5.61	5.30	13.74	11.20	11.35	1.45	1.23	1.10	6.83	6.02	5.44	12.81	11.32	10.38
	J_{LR}	2.49	1.48	1.12	8.34	5.90	5.46	14.51	11.48	11.42	2.02	1.46	1.17	7.68	6.34	5.61	13.64	11.51	10.43
	MJ_{LR}	2.35	1.48	1.12	8.26	5.90	5.46	14.44	11.48	11.42	1.94	1.45	1.17	7.53	6.34	5.61	13.54	11.51	10.43
		true model: simplex									**true model: unit gamma**								
III	J_S	0.07	0.41	0.63	2.35	3.74	4.69	7.10	8.66	9.34	0.30	0.63	0.93	3.19	3.91	4.70	8.19	8.99	9.76
	MJ_S	0.06	0.41	0.63	2.33	3.74	4.69	7.06	8.66	9.34	0.10	0.63	0.93	2.80	3.91	4.70	7.87	8.99	9.76
	J_G	1.48	1.32	0.90	6.81	5.68	5.22	13.06	11.34	10.16	1.36	1.14	1.17	6.45	5.34	5.34	12.22	10.75	10.39
	MJ_G	1.47	1.32	0.90	6.81	5.68	5.22	13.06	11.34	10.16	1.15	1.14	1.17	6.24	5.34	5.34	12.05	10.75	10.39
	J_{LR}	1.94	1.54	0.97	7.59	6.08	5.38	13.95	11.71	10.34	1.93	1.29	1.26	7.04	5.71	5.49	12.76	11.11	10.42
	MJ_{LR}	1.91	1.54	0.97	7.59	6.08	5.38	13.95	11.71	10.34	1.74	1.29	1.26	6.84	5.71	5.49	12.59	11.11	10.42
		true model: beta									**true model: Johnson S_B**								
IV	J_G	1.51	1.06	0.88	7.34	5.92	5.52	13.24	11.36	10.64	1.45	1.35	1.23	6.40	5.97	5.12	12.28	11.40	10.00
	MJ_G	1.50	1.06	0.88	7.34	5.92	5.52	13.24	11.36	10.64	1.35	1.35	1.23	6.35	5.97	5.12	12.20	11.40	10.00
	J_{LR}	1.88	1.22	0.90	7.85	6.04	5.67	13.64	11.61	10.77	1.93	1.53	1.30	7.11	6.28	5.20	13.03	11.84	10.17
	MJ_{LR}	1.85	1.22	0.90	7.85	6.04	5.67	13.63	11.61	10.77	1.83	1.53	1.30	7.03	6.28	5.20	12.95	11.84	10.17
		true model: Johnson S_B									**true model: simplex**								
V	J_G	1.78	1.22	1.15	7.38	6.27	5.51	13.49	12.05	11.00	1.64	1.07	0.96	7.12	5.31	5.23	13.27	10.88	10.21
	MJ_G	1.78	1.22	1.15	7.37	6.27	5.51	13.47	12.05	11.00	1.41	1.07	0.96	6.79	5.31	5.23	13.00	10.88	10.21
	J_{LR}	2.39	1.57	1.24	8.09	6.75	5.69	14.18	12.53	11.18	1.92	1.21	1.03	7.58	5.50	5.34	13.70	11.07	10.29
	MJ_{LR}	2.39	1.57	1.24	8.09	6.75	5.69	14.16	12.53	11.18	1.71	1.21	1.03	7.33	5.50	5.34	13.45	11.07	10.29

Table 4.5: Null rejection rates (%), three competing models that differ in the precision/dispersion submodels regressors.

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
	J_S	0.33	0.52	0.81	2.51	3.56	4.37	6.28	7.57	9.16
	MJ_S	0.23	0.52	0.81	2.46	3.56	4.37	6.24	7.57	9.16
	J_G	2.12	1.34	1.14	8.19	5.74	5.42	14.77	11.23	10.72
	MJ_G	2.09	1.34	1.14	8.18	5.74	5.42	14.77	11.23	10.72
	J_{LR}	2.98	1.61	1.33	9.55	6.46	5.59	16.20	11.99	11.06
	MJ_{LR}	2.96	1.61	1.33	9.55	6.46	5.59	16.20	11.99	11.06
		true model: simplex								
	J_S	0.25	0.48	0.80	2.56	3.82	4.78	6.84	8.37	9.48
	MJ_S	0.00	0.46	0.80	1.37	3.73	4.78	5.45	8.23	9.48
VI	J_G	2.01	1.34	1.20	8.45	6.33	5.69	14.92	12.33	11.01
	MJ_G	1.36	1.29	1.20	7.31	6.20	5.69	13.55	12.25	11.01
	J_{LR}	2.97	1.59	1.38	9.49	6.83	5.88	16.29	12.93	11.27
	MJ_{LR}	2.18	1.53	1.38	8.24	6.71	5.88	14.99	12.85	11.27
		true model: unit gamma								
	J_S	0.28	0.71	0.80	2.87	3.91	4.83	7.38	9.32	9.86
	MJ_S	0.06	0.71	0.80	2.19	3.91	4.83	6.64	9.32	9.86
	J_G	2.12	1.48	1.13	8.39	6.47	5.85	14.81	12.59	11.10
	MJ_G	1.78	1.48	1.13	7.96	6.47	5.85	14.48	12.59	11.10
	J_{LR}	2.88	1.73	1.23	9.52	7.03	6.10	15.96	13.21	11.31
	MJ_{LR}	2.53	1.72	1.23	9.15	7.03	6.10	15.64	13.21	11.31

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
	J_G	1.58	1.10	1.08	7.70	5.84	5.36	14.59	11.71	10.32
	MJ_G	1.55	1.10	1.08	7.67	5.84	5.36	14.55	11.71	10.32
	J_{LR}	2.37	1.42	1.19	9.06	6.33	5.60	16.03	12.21	10.60
	MJ_{LR}	2.33	1.42	1.19	9.02	6.33	5.60	15.99	12.21	10.60
		true model: simplex								
VII	J_G	2.06	1.52	1.03	8.64	6.41	5.17	15.09	12.39	10.52
	MJ_G	1.39	1.45	1.03	7.35	6.34	5.17	13.70	12.32	10.52
	J_{LR}	2.66	1.65	1.07	9.60	6.82	5.28	16.16	12.87	10.58
	MJ_{LR}	1.92	1.58	1.07	8.36	6.77	5.28	14.78	12.81	10.58
		true model: Johnson S_B								
	J_G	2.22	1.17	1.10	8.56	6.38	5.53	15.55	11.89	10.96
	MJ_G	2.07	1.17	1.10	8.38	6.38	5.53	15.34	11.89	10.96
	J_{LR}	2.98	1.50	1.19	9.97	6.77	5.70	16.96	12.54	11.29
	MJ_{LR}	2.79	1.50	1.19	9.78	6.77	5.70	16.75	12.54	11.29

Table 4.6: Null rejection rates (%), two competing models differ in both submodels, which include different regressors.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta									**true model: simplex**								
I	J_S	0.60	0.79	0.94	4.51	4.83	5.01	10.11	9.90	9.66	0.91	1.01	0.85	4.76	4.95	5.00	9.88	10.12	10.11
	MJ_S	0.43	0.76	0.94	3.99	4.83	5.01	9.57	9.90	9.66	0.91	1.01	0.85	4.76	4.95	5.00	9.88	10.12	10.11
	J_G	1.84	1.28	1.09	8.28	6.15	5.68	15.69	12.16	10.51	1.58	1.06	1.03	7.08	6.07	5.53	13.17	11.81	10.41
	MJ_G	1.78	1.28	1.09	8.19	6.15	5.68	15.60	12.16	10.51	1.56	1.06	1.03	7.08	6.07	5.53	13.16	11.81	10.41
	J_{LR}	3.07	1.71	1.24	10.33	6.94	5.87	17.63	12.90	10.79	2.38	1.34	1.13	8.53	6.60	5.75	14.57	12.57	10.68
	MJ_{LR}	3.01	1.71	1.24	10.26	6.94	5.87	17.55	12.90	10.79	2.38	1.34	1.13	8.53	6.60	5.75	14.57	12.57	10.68
		true model: unit gamma									**true model: beta**								
II	J_S	0.77	0.83	0.92	4.55	4.84	4.74	9.64	9.84	9.79	0.68	0.91	0.92	4.54	4.81	4.78	9.57	9.57	9.63
	MJ_S	0.69	0.83	0.92	4.40	4.84	4.74	9.50	9.84	9.79	0.49	0.89	0.92	4.11	4.81	4.78	9.03	9.57	9.63
	J_G	1.90	1.20	1.18	7.90	6.14	5.31	14.44	11.76	10.43	1.50	1.17	1.09	7.34	6.05	5.31	13.75	11.56	10.37
	MJ_G	1.90	1.20	1.18	7.90	6.14	5.31	14.43	11.76	10.43	1.43	1.17	1.09	7.30	6.05	5.31	13.72	11.56	10.37
	J_{LR}	2.89	1.61	1.28	9.36	6.84	5.51	16.02	12.74	10.65	2.38	1.50	1.18	8.96	6.65	5.52	15.27	12.32	10.64
	MJ_{LR}	2.89	1.61	1.28	9.34	6.84	5.51	16.00	12.74	10.65	2.34	1.50	1.18	8.93	6.65	5.52	15.25	12.32	10.64
		true model: simplex									**true model: unit gamma**								
III	J_S	1.10	0.88	0.85	5.02	5.42	4.98	10.28	10.74	10.05	1.08	0.98	0.97	5.40	5.09	4.86	10.43	10.02	9.81
	MJ_S	1.10	0.88	0.85	5.02	5.42	4.98	10.28	10.74	10.05	1.08	0.98	0.97	5.39	5.09	4.86	10.41	10.02	9.81
	J_G	1.51	1.07	0.98	6.68	6.11	5.37	13.14	11.91	10.77	1.42	1.15	1.04	6.67	5.75	5.05	12.23	10.93	10.14
	MJ_G	1.51	1.07	0.98	6.68	6.11	5.37	13.14	11.91	10.77	1.42	1.15	1.04	6.66	5.75	5.05	12.22	10.93	10.14
	J_{LR}	2.07	1.29	1.05	7.90	6.64	5.52	14.28	12.46	11.01	2.15	1.36	1.09	7.72	6.21	5.18	13.34	11.41	10.34
	MJ_{LR}	2.07	1.29	1.05	7.90	6.64	5.52	14.28	12.46	11.01	2.14	1.36	1.09	7.71	6.21	5.18	13.33	11.41	10.34
		true model: beta									**true model: Johnson S_B**								
IV	J_G	1.54	1.30	1.05	7.87	6.07	5.78	14.95	11.92	10.42	1.72	1.14	1.03	7.29	5.79	5.59	13.52	11.66	10.80
	MJ_G	1.49	1.30	1.05	7.77	6.07	5.78	14.87	11.92	10.42	1.72	1.14	1.03	7.29	5.79	5.59	13.52	11.66	10.80
	J_{LR}	2.61	1.65	1.19	9.47	6.81	5.95	16.65	12.71	10.75	2.49	1.48	1.13	8.97	6.36	5.74	15.31	12.39	11.04
	MJ_{LR}	2.55	1.65	1.19	9.40	6.81	5.95	16.58	12.71	10.75	2.49	1.48	1.13	8.97	6.36	5.74	15.31	12.39	11.04
		true model: Johnson S_B									**true model: simplex**								
V	J_G	1.82	1.15	1.04	7.86	5.89	5.22	14.31	11.47	10.57	1.54	1.03	1.05	6.71	5.57	5.41	13.29	11.11	11.03
	MJ_G	1.81	1.15	1.04	7.85	5.89	5.22	14.30	11.47	10.57	1.53	1.03	1.05	6.70	5.57	5.41	13.23	11.11	11.03
	J_{LR}	2.64	1.48	1.15	9.52	6.40	5.43	15.93	12.30	10.91	2.20	1.33	1.13	8.14	6.03	5.68	14.69	11.62	11.22
	MJ_{LR}	2.64	1.48	1.15	9.52	6.40	5.43	15.93	12.30	10.91	2.20	1.33	1.13	8.14	6.03	5.68	14.69	11.62	11.22

Table 4.7: Null rejection rates (%), three competing models differ in both submodels, which include different regressors.

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VI	J_S	0.77	0.90	1.11	4.18	4.51	5.17	9.37	9.84	10.55
	MJ_S	0.28	0.86	1.11	2.80	4.44	5.17	7.35	9.75	10.55
	J_G	2.15	1.32	1.33	9.15	6.09	5.80	16.55	12.52	11.57
	MJ_G	1.88	1.32	1.33	8.84	6.09	5.80	16.22	12.51	11.57
	J_{LR}	4.23	1.96	1.51	12.57	7.64	6.39	20.83	14.28	12.21
	MJ_{LR}	3.99	1.96	1.51	12.27	7.64	6.39	20.53	14.27	12.21
		true model: simplex								
	J_S	0.82	0.95	0.97	4.77	4.90	4.89	10.58	10.05	10.50
	MJ_S	0.82	0.95	0.97	4.77	4.90	4.89	10.58	10.05	10.50
	J_G	1.70	1.31	1.18	8.31	6.23	5.64	15.73	12.29	11.38
	MJ_G	1.68	1.31	1.18	8.27	6.23	5.64	15.67	12.29	11.38
	J_{LR}	3.47	1.76	1.36	11.74	7.19	6.16	19.41	13.49	11.91
	MJ_{LR}	3.47	1.76	1.36	11.74	7.19	6.16	19.41	13.49	11.91
		true model: unit gamma								
	J_S	0.61	0.73	1.12	4.01	4.44	5.13	8.67	9.31	10.09
	MJ_S	0.43	0.72	1.12	3.47	4.44	5.13	8.01	9.31	10.09
	J_G	1.90	1.30	1.38	8.36	6.35	5.84	15.15	12.21	11.06
	MJ_G	1.73	1.29	1.38	8.06	6.35	5.84	14.84	12.21	11.06
	J_{LR}	3.69	1.89	1.63	11.36	7.78	6.38	18.47	13.72	11.59
	MJ_{LR}	3.52	1.88	1.63	11.04	7.78	6.38	18.12	13.72	11.59

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VII	J_G	2.06	1.37	1.13	9.23	6.71	5.45	17.07	12.73	11.04
	MJ_G	1.84	1.37	1.13	9.03	6.71	5.45	16.88	12.73	11.04
	J_{LR}	4.27	2.22	1.34	13.55	8.04	5.96	21.52	14.69	11.62
	MJ_{LR}	4.19	2.22	1.34	13.47	8.04	5.96	21.39	14.69	11.62
		true model: simplex								
	J_G	1.67	1.30	0.97	8.45	6.38	5.31	16.11	12.34	10.95
	MJ_G	1.66	1.30	0.97	8.44	6.38	5.31	16.07	12.34	10.95
	J_{LR}	3.20	1.81	1.12	11.65	7.54	5.79	19.80	13.84	11.48
	MJ_{LR}	3.20	1.81	1.12	11.65	7.54	5.79	19.80	13.84	11.48
		true model: Johnson S_B								
	J_G	1.40	1.04	0.96	7.08	6.13	5.22	14.04	12.71	10.69
	MJ_G	1.40	1.04	0.96	7.08	6.13	5.22	14.04	12.71	10.69
	J_{LR}	2.93	1.66	1.22	10.28	7.50	5.70	17.79	14.40	11.19
	MJ_{LR}	2.93	1.66	1.22	10.28	7.50	5.70	17.79	14.40	11.19

4.3 Models differ in their link functions

We shall now consider scenarios 4, 5 and 6, where the regression models differ in the link functions. At the outset, we consider

$$\begin{aligned} \mathrm{H}_1 : g_1(\mathfrak{l}_{1t}) &= \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \\ \log(\mathfrak{p}_{1t}) &= \gamma_0 + \gamma_1 x_{t1} \end{aligned} \qquad \begin{aligned} \mathrm{H}_2 : g_2(\mathfrak{l}_{2t}) &= \beta_3 + \beta_4 x_{t1} + \beta_5 x_{t2} \\ \log(\mathfrak{p}_{2t}) &= \gamma_2 + \gamma_3 x_{t1}, \end{aligned}$$

$$\begin{aligned} \mathrm{H}_3 : g_3(\mathfrak{l}_{3t}) &= \beta_6 + \beta_7 x_{t1} + \beta_8 x_{t2} \\ \log(\mathfrak{p}_{3t}) &= \gamma_4 + \gamma_5 x_{t1}, \end{aligned}$$

where $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ are link functions. Table 4.8 contains the results for $M = 2$, the competing models being H_1 and H_2. Under case I, $g_1(\cdot)$ is log-log, $g_2(\cdot)$ is Cauchy, and $\beta_0 = 1.2$, $\beta_1 = 2.0$, $\beta_2 = 0.6$, $\beta_3 = 1.1$, $\beta_4 = -2.9$ and $\beta_5 = 0.7$. For cases II through V, $g_1(\cdot)$ is log-log, $g_2(\cdot)$ is complementary log-log, the true parameters values for the latter model being $\beta_3 = 0.9$, $\beta_4 = -1.5$ and $\beta_5 = -1.1$. In all cases, $\mathfrak{l}_{lt} \in (0.18, 0.81)$ for $l = 1, 2$ and the γ's values were selected so that $\lambda_v \approx 55$.

It is noteworthy that, unlike the J test, the MJ test is considerably undersized under case I when the true model is beta, especially when the sample size is not large. For instance, when $n = 100$ and $\alpha = 5\%$, the MJ null rejection rates are 2.28% (score), 2.13% (gradient) and 2.37% (likelihood ratio). Overall, the three testing criteria delivered similar inferences and the tests proved to be reliable, especially when the sample size is not small.

Table 4.9 contains the tests null rejection rates when there are three competing models. The location link functions $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ are log-log, complementary log-log and Cauchy for case VI, and Cauchy, log-log and complementary log-log for case VII. The location true parameter values under case VI are $\beta_0 = 1.1$, $\beta_1 = 2.9$, $\beta_2 = 0.7$, $\beta_3 = 1.2$, $\beta_4 = -2.0$, $\beta_5 = 0.6$, $\beta_6 = 0.9$, $\beta_7 = -1.5$ and $\beta_8 = -1.1$, and under case VII are $\beta_0 = 1.2$, $\beta_1 = -2.0$, $\beta_2 = 0.6$, $\beta_3 = 0.9$, $\beta_4 = -1.5$, $\beta_5 = -1.1$, $\beta_6 = 1.1$, $\beta_7 = -2.8$ and $\beta_8 = 0.7$ so that $\mathfrak{l}_{lt} \in (0.18, 0.81)$ for $l = 1, 2, 3$. Again, the parameter values in the precision submodel were selected so that $\lambda_v \approx 55$. The MJ test is undersized when the sample size is small. For instance, in case VI when the model is unit gamma, for $n = 100$ and $\alpha = 10\%$ the null rejection rate of the MJ score (gradient) [likelihood ratio] is 6.71% (7.08%) [7.52%]. When the J test is oversized it tends to be more so when based on the likelihood ratio testing criterion. Overall, the tests display good finite sample performances.

We shall now focus on models that differ in the link functions used in the precision submodels. The models are

$$\mathrm{H}_1 \;:\; g\left(\mathfrak{l}_{1t}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \qquad \mathrm{H}_2 \;:\; g\left(\mathfrak{l}_{2t}\right) = \beta_3 + \beta_4 x_{t1} + \beta_5 x_{t2}$$
$$h_1\left(\mathfrak{p}_{1t}\right) = \gamma_0 + \gamma_1 x_{t1} \qquad h_2\left(\mathfrak{p}_{2t}\right) = \gamma_2 + \gamma_3 x_{t1},$$

where $g(\cdot)$ is the link function for the location submodel, and $h_1(\cdot)$ and $h_2(\cdot)$ are the link functions for the precision submodels. For cases I, II and V the location and precision link functions of model H_1 (H_2) are log-log and square root (log-log and log). In case III, the corresponding link functions are log-log and log (log-log and square root) for model H_1 (H_2). Finally, in case IV, the location and precision link functions for model H_1 (H_2) are complementary log-log and square root (complementary log-log and log), respectively. For this scenario we do not present simulation results for three competing models, since there are only three well-know link functions for the precision submodels and the numerical evaluation would be similar to that when the competing models only differ in the response distribution and such results will be presented later.

The tests null rejection rates are reported in Table 4.10. In some cases, the MJ test is considerably undersized, especially when the sample size is small. The score implementation of the MJ test tends to be more conservative than the gradient and likelihood ratio tests. For instance, in case III when the true model is simplex, for $n = 100$ and $\alpha = 1\%$ the MJ null rejection rates are 0.32% (score), 1.01% (gradient) and 1.24% (likelihood ratio). The tests performances improve as the sample increases. For testing inference in small samples it is recommended to use bootstrap resampling, as described in Cribari-Neto and Lucena (2015).

In scenario 6 we consider competing models that differ in both link functions:

$$\mathrm{H}_1 \;:\; g_1\left(\mathfrak{l}_{1t}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \qquad \mathrm{H}_2 \;:\; g_2\left(\mathfrak{l}_{2t}\right) = \beta_3 + \beta_4 x_{t1} + \beta_5 x_{t2}$$
$$h_1\left(\mathfrak{p}_{1t}\right) = \gamma_0 + \gamma_1 x_{t1} \qquad h_1\left(\mathfrak{p}_{2t}\right) = \gamma_2 + \gamma_3 x_{t1},$$

$$\mathrm{H}_3 \;:\; g_3\left(\mathfrak{l}_{3t}\right) = \beta_6 + \beta_7 x_{t1} + \beta_8 x_{t2}$$
$$h_1\left(\mathfrak{p}_{3t}\right) = \gamma_4 + \gamma_5 x_{t1},$$

where $g_l(\cdot)$ and $h_l(\cdot)$, $l = 1, 2, 3$, are the link functions, $t = 1, \ldots, n$. As before, models H_1 and H_2 were used when only two competing models were considered. Table 4.11 contains the tests null rejection rates for distinguishing between two models. In case I, model H_1 uses the log-log and square root link functions whereas model H_2 uses the Cauchy and log link functions, the true parameter values being $\beta_0 = 1.2$, $\beta_1 = -2.0$, $\beta_2 = 0.6$, $\beta_3 = 1.1$, $\beta_4 = -2.2$ and $\beta_5 = 0.7$. Cases II through V only differ from I in the location link function used in H_2, which is complementary log-log. The true parameter values are $\beta_3 = 0.9$, $\beta_4 = -1.5$ and $\beta_5 = -1.1$.

The figures in Table 4.11 show that in case I the tests tend to be conservative, especially when the sample size is small. In cases II and III, the score tests tend to be less size-distorted than the gradient and likelihood ratio tests. For instance, in case III when the true model is simplex, $n = 50$ and $\alpha = 5\%$, the null rejection rates of the J and MJ score (gradient) [likelihood ratio] tests are 4.50% (7.74%) [8.68%].

Table 4.12 contains the tests null rejection rates for selecting a model out of three models. In case VI the location and precision link functions used are: logit and square root (H_1), log-log and log (H_2), and complementary log-log and log (H_3). The true parameter values are $\beta_0 = 0.9$, $\beta_1 = -1.5$, $\beta_2 = -1.1$, $\beta_3 = 1.2$, $\beta_4 = -2.0$, $\beta_5 = 0.6$, $\beta_6 = 1.1$, $\beta_7 = -2.8$ and $\beta_8 = 0.7$. In case VII, the link functions and the true parameter values are: for model H_1, complementary log-log and square root with $\beta_0 = 1.1$, $\beta_1 = -2.9$ and $\beta_2 = 0.7$; for model H_2, log-log and log with $\beta_3 = 1.2$, $\beta_4 = -2.0$ and $\beta_5 = 0.6$; finally, for model H_3, logit and log with $\beta_6 = 0.9$, $\beta_7 = -1.5$ and $\beta_8 = -1.1$. Under case VI, the likelihood ratio tests are considerably oversized in small samples. For instance when $n = 50$ and $\alpha = 10\%$ the null rejection rate for the J (MJ) tests are 16.65% (16.37%) , 16.24% (16.24%) and 17.31% (17.28%) when the true model was beta, simplex and unit gamma, respectively. The score implementations of the J and MJ tests are the least size-distorted. Under case VII, the gradient tests are more reliable than the likelihood ratio counterparts in small samples. For instance, when the true data generating process is simplex, $n = 100$ and $\alpha = 10\%$, the gradient (likelihood ratio) J and MJ null rejection rates are 11.20% and 11.18% (13.74% and 13.74%), respectively.

Table 4.8: Null rejection rates (%), two competing models that differ in the location submodel, which use different link functions.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta									**true model: simplex**								
I	J_S	1.01	1.10	1.05	5.87	5.56	5.32	12.44	11.21	10.84	0.84	0.89	1.10	5.25	5.01	4.95	10.96	10.19	9.89
	MJ_S	0.12	0.23	0.52	1.78	2.28	3.75	4.79	5.59	8.44	0.47	0.73	1.10	3.43	4.69	4.95	8.50	9.89	9.89
	J_G	1.12	1.12	1.05	6.23	5.70	5.37	12.90	11.45	10.91	0.89	0.91	1.11	5.78	5.24	4.99	11.52	10.27	9.96
	MJ_G	0.13	0.19	0.51	1.70	2.13	3.68	4.70	5.56	8.37	0.66	0.89	1.11	5.17	5.21	4.99	10.96	10.26	9.96
	J_{LR}	1.91	1.40	1.12	7.46	6.16	5.52	14.05	11.98	11.07	1.38	1.15	1.22	6.44	5.67	5.10	12.31	10.68	10.12
	MJ_{LR}	0.25	0.31	0.54	2.09	2.37	3.79	5.20	5.88	8.54	1.38	1.15	1.22	6.44	5.67	5.10	12.31	10.68	10.12
		true model: unit gamma									**true model: beta**								
II	J_S	1.29	1.17	0.94	6.06	5.54	5.37	11.59	11.18	10.31	1.12	1.00	1.18	5.15	5.17	5.40	10.37	10.44	10.57
	MJ_S	1.28	1.17	0.94	6.05	5.54	5.37	11.59	11.18	10.31	0.52	0.90	1.18	3.98	5.14	5.40	9.31	10.43	10.57
	J_G	1.47	1.27	1.00	6.48	5.68	5.41	12.25	11.47	10.47	1.27	1.02	1.22	5.62	5.41	5.44	11.19	10.75	10.69
	MJ_G	1.47	1.27	1.00	6.48	5.68	5.41	12.25	11.47	10.47	0.79	0.92	1.22	4.67	5.40	5.44	10.23	10.74	10.69
	J_{LR}	2.02	1.42	1.06	7.35	6.02	5.55	12.95	11.79	10.59	1.48	1.20	1.26	6.04	5.69	5.57	11.70	11.00	10.78
	MJ_{LR}	2.02	1.42	1.06	7.35	6.02	5.55	12.95	11.79	10.59	0.89	1.11	1.26	5.12	5.68	5.57	10.72	10.99	10.78
		true model: simplex									**true model: unit gamma**								
III	J_S	1.13	1.02	1.10	5.51	5.47	4.75	11.03	11.24	10.34	0.95	1.23	1.15	5.28	5.44	5.12	10.56	10.65	10.13
	MJ_S	1.13	1.02	1.10	5.51	5.47	4.75	11.03	11.24	10.34	0.93	1.23	1.15	5.28	5.44	5.12	10.55	10.65	10.13
	J_G	1.32	1.07	1.12	6.07	5.69	4.90	11.96	11.52	10.48	1.09	1.19	1.16	5.56	5.62	5.20	11.03	10.91	10.15
	MJ_G	1.32	1.07	1.12	6.07	5.69	4.90	11.96	11.52	10.48	1.08	1.19	1.16	5.56	5.62	5.20	11.02	10.91	10.15
	J_{LR}	1.75	1.25	1.16	6.84	6.02	5.05	12.49	11.80	10.58	1.39	1.32	1.23	6.09	5.85	5.28	11.67	11.22	10.29
	MJ_{LR}	1.75	1.25	1.16	6.84	6.02	5.05	12.49	11.80	10.58	1.38	1.32	1.23	6.09	5.85	5.28	11.66	11.22	10.29
		true model: beta									**true model: Johnson S_B**								
IV	J_G	1.22	1.18	0.99	6.55	5.88	5.48	12.38	11.21	10.87	1.25	0.97	0.97	5.86	5.16	5.15	11.32	11.07	10.40
	MJ_G	0.62	1.00	0.99	5.05	5.67	5.48	10.82	11.06	10.87	1.24	0.97	0.97	5.85	5.16	5.15	11.31	11.07	10.40
	J_{LR}	1.78	1.34	1.10	7.31	6.21	5.60	13.11	11.56	10.96	1.51	1.09	1.01	6.43	5.35	5.21	11.84	11.38	10.51
	MJ_{LR}	0.93	1.17	1.10	5.76	5.99	5.60	11.43	11.40	10.96	1.50	1.09	1.01	6.42	5.35	5.21	11.83	11.38	10.51
		true model: Johnson S_B									**true model: simplex**								
V	J_G	1.49	1.14	1.13	6.39	5.87	5.37	12.20	11.43	10.20	0.97	0.95	1.03	5.42	5.02	5.53	11.30	10.41	10.76
	MJ_G	1.49	1.14	1.13	6.39	5.87	5.37	12.20	11.43	10.20	0.95	0.95	1.03	5.42	5.02	5.53	11.30	10.41	10.76
	J_{LR}	1.93	1.32	1.21	7.14	6.27	5.46	12.79	11.67	10.24	1.23	1.08	1.11	6.01	5.28	5.64	11.79	10.77	10.87
	MJ_{LR}	1.93	1.32	1.21	7.14	6.27	5.46	12.79	11.67	10.24	1.22	1.08	1.11	6.01	5.28	5.64	11.79	10.77	10.87

Table 4.9: Null rejection rates (%), three competing models that differ in the location submodel, which use different link functions.

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VI	J_S	0.80	1.02	0.91	5.19	5.45	5.02	10.60	10.43	10.39
	MJ_S	0.34	0.53	0.73	2.46	3.24	4.18	5.83	6.93	8.73
	J_G	1.12	1.08	0.98	5.66	5.69	5.31	11.41	11.02	10.54
	MJ_G	0.23	0.35	0.68	2.19	2.90	4.18	5.56	6.58	8.65
	J_{LR}	1.84	1.44	1.04	7.33	6.35	5.48	12.86	11.69	10.93
	MJ_{LR}	0.81	0.71	0.81	3.73	3.63	4.49	7.30	7.62	9.13
		true model: simplex								
	J_S	0.96	1.00	0.97	5.65	5.12	5.59	11.26	9.98	10.58
	MJ_S	0.87	1.00	0.97	5.53	5.12	5.59	11.18	9.98	10.58
	J_G	1.03	1.02	0.99	5.97	5.26	5.63	11.82	10.23	10.65
	MJ_G	0.95	1.02	0.99	5.91	5.26	5.63	11.79	10.23	10.65
	J_{LR}	1.86	1.52	1.12	7.50	5.84	5.93	13.74	11.04	10.99
	MJ_{LR}	1.77	1.52	1.12	7.46	5.84	5.93	13.72	11.04	10.99
		true model: unit gamma								
	J_S	0.93	0.87	0.97	5.43	4.88	5.35	10.74	10.53	10.73
	MJ_S	0.24	0.33	0.69	2.26	2.81	3.89	5.57	6.71	8.23
	J_G	1.03	0.90	0.94	5.74	5.08	5.43	11.17	10.79	10.87
	MJ_G	0.35	0.41	0.68	2.73	3.02	3.96	6.14	7.08	8.38
	J_{LR}	1.97	1.14	1.14	7.27	5.86	5.78	13.24	11.49	11.08
	MJ_{LR}	0.73	0.54	0.84	3.79	3.65	4.26	7.71	7.52	8.56

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VII										
	J_G	0.77	0.97	0.98	5.53	5.47	4.89	11.17	10.64	10.04
	MJ_G	0.33	0.50	0.60	2.81	3.36	3.83	6.05	6.74	8.85
	J_{LR}	1.63	1.36	1.07	6.96	6.17	5.18	12.94	11.29	10.27
	MJ_{LR}	0.64	0.72	0.66	3.62	3.75	4.05	7.28	7.25	9.08
		true model: simplex								
	J_G	1.00	1.03	0.97	5.75	5.91	4.99	12.15	11.34	9.96
	MJ_G	0.65	0.84	0.97	4.28	5.37	4.99	9.83	10.92	9.96
	J_{LR}	1.96	1.38	1.15	7.44	6.64	5.17	14.12	11.97	10.22
	MJ_{LR}	1.45	1.14	1.15	5.70	6.05	5.17	11.62	11.58	10.22
		true model: Johnson S_B								
	J_G	1.12	0.93	1.11	5.98	5.21	5.19	12.19	10.43	10.16
	MJ_G	1.05	0.93	1.11	5.95	5.21	5.19	12.18	10.43	10.16
	J_{LR}	1.98	1.15	1.23	7.67	5.88	5.42	14.15	11.25	10.56
	MJ_{LR}	1.91	1.15	1.23	7.64	5.88	5.42	14.15	11.25	10.56

Table 4.10: Null rejection rates (%), two models that differ in the precision/dispersion submodel, which use different link functions.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta									**true model: simplex**								
I	J_S	0.52	0.78	1.04	2.22	3.47	4.31	5.99	7.66	9.40	0.54	0.85	0.99	4.34	4.84	4.59	9.66	10.17	9.72
	MJ_S	0.51	0.75	1.03	1.66	3.03	4.31	3.33	7.13	9.40	0.20	0.37	0.40	2.96	3.00	2.77	6.90	6.77	7.32
	J_G	5.19	2.39	1.60	12.51	7.95	6.35	19.30	13.67	11.91	2.31	1.52	1.22	9.06	7.05	5.32	15.92	12.89	10.82
	MJ_G	1.44	1.66	1.59	5.88	7.15	6.35	11.85	13.15	11.91	2.02	1.12	0.68	7.53	5.27	3.68	12.94	9.36	8.95
	J_{LR}	3.48	1.80	1.39	10.83	7.31	6.07	17.62	12.89	11.64	2.91	1.70	1.30	9.89	7.46	5.38	16.57	13.41	10.92
	MJ_{LR}	0.94	1.20	1.38	4.87	6.52	6.07	10.61	12.33	11.64	2.65	1.33	0.79	8.45	5.78	3.76	13.74	10.01	9.03
		true model: unit gamma									**true model: beta**								
II	J_S	0.54	0.52	0.91	2.87	4.02	4.63	7.85	8.74	9.55	0.47	0.79	0.92	4.38	4.59	5.04	10.04	9.61	10.07
	MJ_S	0.53	0.46	0.65	1.81	2.05	2.92	3.43	3.81	7.02	0.13	0.39	0.52	2.77	2.97	4.55	7.17	6.83	9.79
	J_G	3.36	1.97	1.35	10.39	6.90	5.68	17.31	13.15	11.10	2.26	1.51	1.16	9.28	6.42	5.59	16.24	11.87	10.98
	MJ_G	0.22	0.21	0.43	1.55	1.34	2.87	3.63	3.44	7.99	2.00	1.11	0.84	7.69	4.88	5.30	13.21	9.25	10.79
	J_{LR}	2.48	1.58	1.27	9.16	6.57	5.53	16.49	12.80	10.98	2.93	1.65	1.27	10.06	6.73	5.73	16.89	12.08	11.09
	MJ_{LR}	0.32	0.25	0.46	1.77	1.52	2.86	3.94	3.56	7.84	2.66	1.31	0.95	8.55	5.21	5.42	13.86	9.55	10.89
		true model: simplex									**true model: unit gamma**								
III	J_S	0.56	0.61	1.12	4.41	4.63	5.16	9.66	10.01	10.12	0.45	0.81	0.88	3.43	4.30	4.62	8.14	9.04	9.50
	MJ_S	0.28	0.32	1.01	3.27	3.37	5.14	7.39	8.64	10.12	0.39	0.61	0.56	1.65	2.07	3.14	3.12	4.03	8.12
	J_G	2.64	1.26	1.45	8.92	6.72	5.80	16.07	12.91	11.19	2.83	1.89	1.29	9.31	6.51	5.44	15.65	12.18	10.63
	MJ_G	2.33	1.01	1.40	7.69	5.82	5.78	13.84	12.03	11.19	0.26	0.38	0.44	1.47	1.63	3.49	3.59	3.86	9.15
	J_{LR}	3.17	1.49	1.51	9.72	7.15	5.95	16.92	13.17	11.24	2.27	1.75	1.26	8.57	6.33	5.30	14.97	11.88	10.56
	MJ_{LR}	2.87	1.24	1.45	8.50	6.20	5.93	14.67	12.25	11.24	0.32	0.43	0.44	1.66	1.69	3.30	3.72	3.96	9.04
		true model: beta									**true model: Johnson S_B**								
IV	J_G	2.59	1.55	1.10	8.28	6.31	5.39	14.12	11.72	10.67	2.17	1.42	1.18	8.18	5.95	5.67	14.81	11.90	11.10
	MJ_G	0.40	0.29	0.36	2.22	1.75	2.12	4.16	3.77	5.94	1.78	0.99	0.82	6.38	4.17	3.73	11.34	8.27	7.66
	J_{LR}	2.10	1.35	1.08	7.70	6.13	5.20	13.47	11.44	10.57	2.63	1.58	1.25	8.95	6.33	5.78	15.41	12.23	11.17
	MJ_{LR}	0.46	0.33	0.40	2.37	1.82	2.10	4.47	3.88	5.88	2.18	1.19	0.90	7.17	4.63	3.90	12.05	8.68	7.81
		true model: Johnson S_B									**true model: simplex**								
V	J_G	4.36	1.93	1.26	12.30	7.90	6.43	19.36	13.69	11.84	2.66	1.41	1.28	9.17	6.45	5.53	15.88	11.98	10.69
	MJ_G	0.35	0.29	0.29	1.59	1.41	3.47	3.26	3.01	9.59	2.66	1.38	0.94	9.08	5.72	4.53	15.51	10.18	9.78
	J_{LR}	4.31	1.95	1.25	12.28	7.96	6.46	19.43	13.73	11.89	3.23	1.66	1.31	9.95	6.77	5.61	16.57	12.46	10.86
	MJ_{LR}	0.37	0.27	0.27	1.72	1.37	3.43	3.39	2.97	9.62	2.95	1.32	0.95	8.71	5.19	3.89	14.07	9.35	7.13

Table 4.11: Null rejection rates (%), two models that differ in the link functions of both submodels.

CASE		$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 1\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
I		**true model: beta**									**true model: simplex**								
	J_S	0.68	0.89	0.82	3.90	4.38	4.79	8.57	9.35	10.07	0.72	0.81	0.79	4.09	4.45	4.71	8.98	9.47	9.79
	MJ_S	0.60	0.82	0.82	3.10	3.93	4.79	6.46	8.71	10.07	0.24	0.44	0.52	2.01	2.71	4.03	4.59	6.15	8.90
	J_G	3.25	1.84	1.18	10.79	7.34	6.06	17.89	13.41	11.65	1.79	1.19	1.07	7.71	5.95	5.24	14.34	11.70	10.62
	MJ_G	0.69	1.08	1.18	4.69	5.82	6.06	10.23	11.88	11.65	0.68	0.70	0.75	4.17	3.60	4.59	8.28	8.02	9.68
	J_{LR}	3.41	1.91	1.20	11.40	7.61	6.16	18.52	13.65	11.81	2.90	1.65	1.21	9.55	6.72	5.57	16.46	12.67	11.04
	MJ_{LR}	1.19	1.19	1.20	5.44	6.11	6.16	11.09	12.07	11.81	1.29	0.94	0.82	5.13	4.00	4.86	9.74	8.57	10.08
II		**true model: unit gamma**									**true model: beta**								
	J_S	0.88	0.75	1.10	4.17	4.14	4.69	9.20	9.39	9.64	0.74	0.73	1.00	4.22	4.27	4.98	9.09	9.64	9.63
	MJ_S	0.88	0.75	1.10	4.17	4.14	4.69	9.20	9.39	9.64	0.33	0.68	1.00	3.06	4.21	4.98	7.71	9.59	9.63
	J_G	2.11	1.22	1.28	7.91	6.17	5.39	14.66	11.82	10.78	1.72	1.03	1.28	7.64	5.81	5.34	14.08	11.79	10.45
	MJ_G	2.11	1.22	1.28	7.91	6.17	5.39	14.66	11.82	10.78	1.38	1.00	1.28	6.83	5.79	5.34	13.38	11.76	10.45
	J_{LR}	2.47	1.28	1.29	8.50	6.29	5.51	15.33	12.40	10.93	2.73	1.37	1.33	9.10	6.35	5.55	15.39	12.50	10.74
	MJ_{LR}	2.47	1.28	1.29	8.50	6.29	5.51	15.33	12.40	10.93	2.34	1.35	1.33	8.43	6.33	5.55	14.77	12.48	10.74
III		**true model: simplex**									**true model: unit gamma**								
	J_S	0.75	0.79	1.08	4.50	4.77	4.98	9.81	10.23	10.43	0.69	0.83	1.23	4.53	4.89	4.99	9.54	10.11	9.71
	MJ_S	0.75	0.79	1.08	4.50	4.77	4.98	9.81	10.23	10.43	0.69	0.83	1.23	4.52	4.89	4.99	9.53	10.11	9.71
	J_G	1.90	1.29	1.25	7.74	6.30	5.66	14.53	12.20	11.11	1.65	1.14	1.30	7.62	6.13	5.42	14.05	11.91	10.46
	MJ_G	1.90	1.29	1.25	7.74	6.30	5.66	14.53	12.20	11.11	1.64	1.14	1.30	7.60	6.13	5.42	14.05	11.91	10.46
	J_{LR}	2.18	1.38	1.32	8.68	6.55	5.76	15.15	12.58	11.17	2.52	1.46	1.43	8.94	6.83	5.63	15.55	12.69	10.78
	MJ_{LR}	2.18	1.38	1.32	8.68	6.55	5.76	15.15	12.58	11.17	2.51	1.46	1.43	8.92	6.83	5.63	15.55	12.69	10.78
IV		**true model: beta**									**true model: Johnson S_B**								
	J_G	2.25	1.52	1.19	8.85	6.76	5.63	15.75	12.63	11.26	1.61	1.36	0.90	7.58	6.11	5.60	13.91	11.61	11.10
	MJ_G	1.77	1.44	1.19	7.65	6.67	5.63	14.45	12.59	11.26	0.75	1.18	0.90	5.74	5.92	5.60	11.87	11.48	11.10
	J_{LR}	2.56	1.69	1.23	9.72	7.09	5.74	16.58	12.95	11.51	2.61	1.68	1.09	9.07	6.86	5.88	15.57	12.36	11.48
	MJ_{LR}	1.91	1.61	1.23	8.35	7.00	5.74	15.14	12.90	11.51	1.60	1.49	1.09	7.36	6.69	5.88	13.86	12.23	11.48
V		**true model: Johnson S_B**									**true model: simplex**								
	J_G	2.53	1.50	1.13	10.08	7.14	5.70	17.36	13.26	11.49	1.46	0.99	1.02	7.65	5.99	5.44	14.10	11.36	11.23
	MJ_G	2.32	1.50	1.13	9.81	7.14	5.70	17.08	13.26	11.49	1.44	0.99	1.02	7.62	5.99	5.44	14.08	11.36	11.23
	J_{LR}	3.78	1.91	1.34	11.87	7.96	6.05	19.27	14.17	11.85	2.47	1.30	1.20	9.12	6.57	5.73	15.85	12.01	11.42
	MJ_{LR}	3.50	1.91	1.34	11.55	7.96	6.05	18.98	14.17	11.85	2.47	1.30	1.20	9.12	6.57	5.73	15.85	12.01	11.42

Table 4.12: Null rejection rates (%), three competing models that differ in the link functions of both submodel.

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VI	J_S	1.18	1.19	1.11	7.10	5.95	5.30	13.66	12.16	10.84
	MJ_S	1.03	1.19	1.11	6.79	5.94	5.30	13.32	12.15	10.84
	J_G	1.37	1.29	1.13	7.61	6.24	5.47	14.40	12.38	11.01
	MJ_G	1.23	1.29	1.13	7.33	6.24	5.47	14.09	12.37	11.01
	J_{LR}	2.61	1.71	1.30	9.87	7.10	5.83	16.65	13.32	11.29
	MJ_{LR}	2.47	1.71	1.30	9.54	7.10	5.83	16.37	13.31	11.29
		true model: simplex								
	J_S	1.01	0.88	1.01	6.19	5.61	5.36	12.65	11.39	10.48
	MJ_S	1.01	0.88	1.01	6.19	5.61	5.36	12.65	11.39	10.48
	J_G	1.21	0.98	1.05	6.91	5.86	5.46	13.69	11.79	10.61
	MJ_G	1.21	0.98	1.05	6.91	5.86	5.46	13.69	11.79	10.61
	J_{LR}	2.50	1.57	1.24	9.20	6.75	5.77	16.24	12.87	10.96
	MJ_{LR}	2.50	1.57	1.24	9.20	6.75	5.77	16.24	12.87	10.96
		true model: unit gamma								
	J_S	1.41	1.26	1.01	7.37	6.22	5.39	13.91	12.20	10.51
	MJ_S	1.41	1.26	1.01	7.33	6.22	5.39	13.88	12.20	10.51
	J_G	1.45	1.36	1.00	7.84	6.33	5.48	14.67	12.57	10.62
	MJ_G	1.45	1.36	1.00	7.81	6.33	5.48	14.64	12.57	10.62
	J_{LR}	2.77	1.68	1.14	10.17	7.19	5.80	17.31	13.70	11.02
	MJ_{LR}	2.75	1.68	1.14	10.14	7.19	5.80	17.28	13.70	11.02

CASE	n	$\alpha = 1\%$ 50	100	250	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta								
VII	J_G	3.83	1.88	1.11	10.76	7.37	5.95	18.47	13.58	11.09
	MJ_G	1.60	1.30	1.08	6.43	5.81	5.93	12.60	11.85	11.07
	J_{LR}	3.35	1.93	1.15	11.60	7.57	5.95	19.27	14.05	11.32
	MJ_{LR}	2.18	1.55	1.13	8.46	6.23	5.93	14.77	12.46	11.31
		true model: simplex								
	J_G	2.28	0.96	1.06	8.30	5.59	5.07	15.71	11.20	10.51
	MJ_G	1.38	0.94	1.06	6.35	5.56	5.07	13.26	11.18	10.51
	J_{LR}	5.02	1.90	1.36	14.20	7.45	5.74	21.36	13.74	11.29
	MJ_{LR}	4.70	1.90	1.36	13.76	7.45	5.74	21.00	13.74	11.29
		true model: Johnson S_B								
	J_G	2.38	1.32	1.14	9.15	6.19	5.34	16.46	12.02	10.56
	MJ_G	2.19	1.32	1.14	8.86	6.19	5.34	16.21	12.02	10.56
	J_{LR}	5.46	2.29	1.36	14.92	8.19	6.09	22.71	14.64	11.26
	MJ_{LR}	5.30	2.29	1.36	14.74	8.19	6.09	22.54	14.64	11.26

4.4 Models only differ in their response distribution

In the last, scenario 7, we consider competing models which only differ in the response distributions. The models are

$$\mathrm{H}_1 \; : \; \log\left(\frac{\mathfrak{l}_{1t}}{1-\mathfrak{l}_{1t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \qquad \mathrm{H}_2 \; : \; \log\left(\frac{\mathfrak{l}_{2t}}{1-\mathfrak{l}_{2t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}$$
$$\log(\mathfrak{p}_{1t}) = \gamma_0 + \gamma_1 x_{t1} \qquad \log(\mathfrak{p}_{3t}) = \gamma_2 + \gamma_3 x_{t1},$$

$$\mathrm{H}_3 \; : \; \log\left(\frac{\mathfrak{l}_{3t}}{1-\mathfrak{l}_{3t}}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}$$
$$\log(\mathfrak{p}_{3t}) = \gamma_4 + \gamma_5 x_{t1},$$

$t = 1, \ldots, n$. The parameter values for the location submodels are $\beta_0 = 1.1$, $\beta_1 = -2.9$ and $\beta_2 = 0.7$ which cause $\mathfrak{l}_{lt}$ to assume values in $(0.18, 0.81)$ for $l = 1, 2, 3$. The γ's values were chosen so that $\lambda_v \approx 55$ when M $= 2$ and $\lambda_v \approx 25$ when M $= 3$.

The tests null rejection rates are presented in Table 4.13 for $M = 2$ competing models and in Table 4.14 for $M = 3$ competing models. Here, we evaluate the tests finite samples performance for samples sizes $n = 100$, 250 and 400. We consider larger samples since there is a more challenging situation.

The score-based tests are quite conservative, especially the MJ test when the sample size is small. For instance, in case I (III) when the true model is simplex, $n = 100$ and $\alpha = 1\%$ the MJ null rejection rates based on the score, gradient and likelihood statistics are 0.20%, 0.76% and 1.50% (0.19%, 0.73% and 1.70%), respectively. The size distortions become smaller as the sample size grows for all criteria. On the other hand, the likelihood ratio implementation of the J test is considerably liberal for $n = 100$ and three competing models. For instance, under the unit gamma (Johnson S_B) law, the J_{LR} null rejection rate at the 10% nominal level was 20.76% (21.61%); for $n = 400$ the corresponding rejection rate was 15.49% (16.13%), these are closer to the nominal level, as expected. With both $M = 2$ and $M = 3$ competing models, the gradient J and MJ tests tend to be less size-distorted than the likelihood ratio tests.

Table 4.13: Null rejection rates (%), two competing models that differ only in the distribution.

CASE	n	$\alpha = 1\%$ 100	250	400	$\alpha = 5\%$ 100	250	400	$\alpha = 10\%$ 100	250	400	$\alpha = 1\%$ 100	250	400	$\alpha = 5\%$ 100	250	400	$\alpha = 10\%$ 100	250	400
		true model: beta									**true model: simplex**								
I	J_S	0.86	1.22	0.98	4.26	5.14	4.94	9.07	10.47	9.86	0.77	0.86	0.94	4.46	5.08	4.80	9.36	9.88	9.99
	MJ_S	0.78	1.03	0.83	3.35	3.94	4.07	6.68	8.21	8.67	0.20	0.47	0.65	2.41	3.77	3.82	6.12	7.94	8.71
	J_G	1.28	1.28	1.10	5.94	5.99	5.30	12.13	11.34	10.55	1.20	1.12	1.09	6.16	5.76	5.21	11.96	11.04	10.68
	MJ_G	0.65	0.81	0.73	3.15	3.93	4.03	7.14	8.16	9.11	0.76	0.86	0.89	4.52	4.77	4.45	9.17	9.51	9.61
	J_{LR}	2.10	1.54	1.32	7.69	6.60	5.75	13.95	12.02	10.98	1.89	1.36	1.29	7.63	6.16	5.44	13.78	11.68	10.93
	MJ_{LR}	0.91	0.83	0.80	3.72	4.19	4.24	7.66	8.52	9.45	1.50	1.18	1.13	6.04	5.30	4.76	11.19	10.26	9.97
		true model: unit gamma									**true model: beta**								
II	J_S	0.97	1.10	1.03	4.73	4.96	4.85	9.62	9.82	10.02	0.97	0.98	0.96	4.36	4.93	4.86	8.76	9.68	9.97
	MJ_S	0.44	0.47	0.36	3.06	2.97	2.72	7.07	6.73	6.80	0.92	0.86	0.74	3.82	3.80	3.55	7.19	7.19	7.09
	J_G	1.39	1.09	1.01	6.39	5.58	5.13	12.38	10.82	10.69	1.28	1.16	1.03	6.13	5.47	5.20	11.89	10.80	10.45
	MJ_G	1.05	0.68	0.53	5.06	3.93	3.45	10.11	8.10	7.84	0.81	0.66	0.59	4.02	3.44	3.30	8.17	7.09	6.79
	J_{LR}	2.44	1.29	1.15	8.10	6.15	5.53	14.45	11.58	11.13	2.25	1.53	1.23	7.90	6.02	5.62	13.95	11.48	10.90
	MJ_{LR}	2.13	0.99	0.73	7.01	4.71	3.96	12.48	9.02	8.58	1.38	0.73	0.58	5.02	3.53	3.28	9.17	7.11	6.76
		true model: simplex									**true model: unit gamma**								
III	J_S	0.75	0.99	0.97	4.22	4.99	4.97	9.43	10.48	10.00	0.70	1.00	0.90	4.21	4.81	4.78	9.18	9.81	9.62
	MJ_S	0.19	0.45	0.51	2.35	2.87	3.24	6.06	7.16	7.29	0.62	0.84	0.69	3.29	3.57	3.54	6.63	6.99	6.92
	J_G	1.08	1.23	1.11	6.27	5.81	5.55	12.82	11.36	10.73	1.27	1.10	0.99	6.07	5.50	5.34	12.51	10.93	10.22
	MJ_G	0.73	0.81	0.76	4.89	4.18	4.16	10.13	8.58	8.47	0.60	0.70	0.60	3.52	3.49	3.38	7.31	6.94	6.77
	J_{LR}	1.99	1.47	1.32	8.10	6.32	5.90	14.92	11.97	11.16	2.17	1.27	1.13	7.80	6.15	5.68	14.57	11.71	10.50
	MJ_{LR}	1.70	1.15	1.03	6.98	5.00	4.70	12.70	9.46	9.12	1.03	0.69	0.62	4.18	3.59	3.42	8.21	6.98	6.64
		true model: beta									**true model: Johnson S_B**								
IV	J_G	1.29	1.22	1.14	5.97	5.87	5.39	12.01	11.22	10.49	1.35	0.98	1.05	6.38	5.22	5.71	12.38	10.58	10.86
	MJ_G	0.75	0.72	0.74	3.59	3.61	3.22	7.46	7.14	6.71	0.89	0.66	0.62	4.95	3.75	4.21	9.83	7.72	8.18
	J_{LR}	1.93	1.39	1.26	7.37	6.25	5.73	13.38	11.84	10.78	2.30	1.16	1.24	7.88	5.73	6.01	14.32	11.18	11.58
	MJ_{LR}	1.07	0.69	0.75	4.22	3.71	3.26	8.21	7.16	6.71	1.89	0.90	0.86	6.60	4.41	4.71	12.13	8.64	9.07
		true model: Johnson S_B									**true model: simplex**								
V	J_G	1.16	1.05	1.08	6.31	5.79	5.69	12.78	11.07	10.85	1.08	1.11	1.05	5.99	5.36	5.07	12.23	11.00	10.41
	MJ_G	0.72	0.62	0.55	3.63	3.44	3.47	7.57	6.89	6.88	0.70	0.71	0.69	4.51	3.94	3.56	9.61	8.23	7.79
	J_{LR}	1.87	1.30	1.18	8.34	6.32	6.06	14.84	11.69	11.18	1.64	1.27	1.15	7.56	5.78	5.29	13.79	11.62	10.70
	MJ_{LR}	0.99	0.64	0.55	4.38	3.42	3.41	8.60	6.86	6.85	1.35	0.94	0.87	6.21	4.51	3.95	11.55	9.03	8.26

Table 4.14: Null rejection rates (%), three competing models that differ only in the distribution.

CASE	n	$\alpha = 1\%$ 100	250	400	$\alpha = 5\%$ 100	250	400	$\alpha = 10\%$ 100	250	400
		true model: beta								
	J_S	0.92	1.03	1.10	5.07	5.06	4.69	10.46	10.16	9.61
	MJ_S	0.83	0.83	0.94	4.03	4.20	3.84	8.43	8.35	7.76
	J_G	1.54	1.12	1.10	7.40	5.76	5.30	14.14	11.41	10.22
	MJ_G	0.99	0.78	0.78	5.17	4.29	3.98	10.19	8.71	7.73
	J_{LR}	2.77	1.46	1.26	10.08	6.42	5.76	17.09	12.31	10.71
	MJ_{LR}	1.71	0.95	0.86	6.88	4.59	4.11	12.24	9.11	7.97
		true model: simplex								
	J_S	1.28	1.43	1.36	6.14	6.55	6.70	12.06	12.86	12.63
	MJ_S	0.49	0.78	0.82	3.48	4.64	4.98	7.97	9.93	10.07
VI	J_G	1.63	1.50	1.58	8.01	7.24	6.67	15.76	13.92	13.15
	MJ_G	1.08	1.09	1.19	6.24	5.85	5.54	12.77	11.73	11.14
	J_{LR}	2.91	1.90	1.77	10.82	8.10	7.11	18.81	14.93	13.78
	MJ_{LR}	2.39	1.58	1.41	9.12	6.87	6.14	16.26	12.90	11.95
		true model: unit gamma								
	J_S	1.47	1.82	1.64	7.28	7.62	7.35	14.02	14.55	14.05
	MJ_S	0.55	0.72	0.66	3.48	3.95	4.16	7.72	8.62	8.64
	J_G	1.80	1.92	1.61	9.42	8.51	7.94	17.07	15.66	14.67
	MJ_G	0.89	0.92	0.78	5.11	4.81	4.76	10.35	9.76	9.50
	J_{LR}	3.30	2.35	1.83	11.78	9.51	8.40	20.76	16.66	15.49
	MJ_{LR}	1.70	1.26	0.93	6.88	5.54	5.23	12.92	10.44	10.08

CASE	n	$\alpha = 1\%$ 100	250	400	$\alpha = 5\%$ 100	250	400	$\alpha = 10\%$ 100	250	400
		true model: beta								
	J_G	1.40	1.18	1.31	7.32	6.12	5.70	14.14	12.01	11.65
	MJ_G	0.88	0.85	0.99	5.00	4.83	4.85	10.44	10.15	10.44
	J_{LR}	2.57	1.53	1.46	9.84	6.78	6.19	17.44	12.82	12.14
	MJ_{LR}	1.64	1.06	1.09	6.82	5.32	5.20	12.73	10.71	10.79
		true model: simplex								
VII	J_G	1.59	0.96	0.85	7.68	4.92	4.73	14.28	10.25	9.78
	MJ_G	1.00	0.74	0.63	6.11	3.98	3.85	11.59	8.58	8.54
	J_{LR}	2.75	1.28	0.96	9.98	5.51	5.17	17.10	11.30	10.15
	MJ_{LR}	2.20	1.06	0.78	8.63	4.75	4.39	14.94	9.83	9.07
		true model: Johnson S_B								
	J_G	1.90	1.74	1.75	9.59	7.91	7.98	18.24	15.63	15.44
	MJ_G	0.59	0.43	0.46	3.56	2.64	3.26	8.03	6.00	7.18
	J_{LR}	3.35	2.12	2.02	12.58	9.16	8.56	21.61	16.74	16.13
	MJ_{LR}	1.23	0.59	0.58	5.40	3.09	3.53	10.52	6.85	7.59

4.5 Model selection using the MJ statistic

The MJ test statistic can be used as a model selection criterion whenever the null hypothesis that one of the candidate models is the true model is not rejected (Section 3.3). In that case, we select the model that corresponds to the smallest J test statistic. We have numerically evaluated the effectiveness of such a model selection strategy. In Tables 4.15, 4.16 and 4.17 we report the frequencies (%) of the correct model selection based on the score, gradient and likelihood ratio MJ statistics, respectively, when two competing models are considered for scenarios 1 through 6. Table 4.18 contains results when three competing models are considered for scenarios 1 thought 6. Finally, Tables 4.19 and 4.20 contain the results for scenario 7 with $M = 2$ and $M = 3$, respectively. The results are for nominal levels $\alpha = 5\%$ and 10%.

The tests based on the three statistics deliver similarly regardless of whether there are two or three models in scrutiny. When two competing models are considered, the frequency of MJ correct model selections in scenarios 1 through 3 (in which the competing models differ in the explanatory variables used in the location submodel, in the precision submodel, and in both submodels) were very close to 100%, even for the smallest sample size ($n = 50$). In scenario 4, the frequency of correct model selection was also close to 100%, except for case I when the true data generating process was beta (for instance, the frequencies of correct model selection based on the MJ likelihood ratio criterion at $\alpha = 5\%$ are 64.97% for $n = 50$ and 76.60% for $n = 100$). In scenario 5, where the competing models only differ in the link functions used in the precision submodels, the frequencies of correct model selection for cases II, IV and V when the true data generating process were unit gamma, beta and Johnson S_B for both significance levels were smaller than 65% for $n = 50$. However, when $n = 250$ model selection proved to be quite accurate. Under scenario 6, the frequencies of correct model selection were similar to those of the first scenario, except for case I, where the frequency of correct model selection fell below 85% for samples of 50 observations. The weakest performance took place in last scenario (7), where the competing models only differ in the response distributions. This is, we note, a challenging situation, since the models only differ in the response distribution, the regressors and link functions being the same. In cases II and V, at least 100 observations are required for the frequency of correct model selection to be above 50%.

Finally, when three competing models were considered, MJ model selection proved reliable when the competing models differ in the specification of their regression structures (scenarios 1 through 6), since, even in small samples ($n = 50$) such frequencies are very closer to 100% for both significance levels. On the other hand, when the true data generating process was Johnson S_B (case VI) and unit gamma (case VII) larger ($n > 250$) sample sizes are required for the frequency of correct model selection to exceed 50%.

Table 4.15: Frequencies (%) of correct model selection using the MJ_S statistic as a model selection criterion (given that the null hypothesis is not rejected at nominal level α) with two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	Sce. 1	99.99	100.00	100.00	99.99	100.00	100.00	99.90	100.00	100.00	99.95	100.00	100.00
	Sce. 2	99.99	100.00	100.00	99.99	100.00	100.00	98.93	99.99	100.00	99.17	99.99	100.00
	Sce. 3	98.60	99.98	100.00	98.92	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	66.11	77.39	92.29	65.90	77.27	92.49	94.43	99.33	100.00	94.81	99.55	100.00
	Sce. 5	91.19	99.20	100.00	91.13	99.36	100.00	85.71	89.93	96.66	85.13	89.53	97.14
	Sce. 6	88.29	98.20	100.00	88.23	98.31	100.00	73.03	86.32	97.82	72.93	86.45	98.12
		true model: unit gamma						**true model: beta**					
II	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	99.89	100.00	100.00	99.93	100.00	100.00
	Sce. 2	99.74	100.00	100.00	99.84	100.00	100.00	99.24	100.00	100.00	99.48	100.00	100.00
	Sce. 3	99.60	100.00	100.00	99.70	100.00	100.00	98.62	100.00	100.00	98.86	100.00	100.00
	Sce. 4	99.99	100.00	100.00	100.00	100.00	100.00	96.94	99.93	100.00	97.47	99.97	100.00
	Sce. 5	63.85	80.25	95.95	63.42	79.97	96.28	89.08	94.16	99.36	88.80	94.10	99.65
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	97.26	99.91	100.00	97.79	99.94	100.00
		true model: simplex						**true model: unit gamma**					
III	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.92	100.00	100.00	99.94	100.00	100.00	98.95	100.00	100.00	99.27	100.00	100.00
	Sce. 3	100.00	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	99.96	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	99.96	100.00	100.00
	Sce. 5	93.69	97.43	99.98	93.51	98.02	100.00	68.34	85.15	97.75	67.98	84.89	98.29
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	99.97	100.00	100.00	99.98	100.00	100.00

Table 4.16: Frequencies (%) of correct model selection using the MJ_G statistic as a model selection criterion (given that the null hypothesis is not rejected at nominal level α) with two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	Sce. 1	99.99	100.00	100.00	99.99	100.00	100.00	99.90	100.00	100.00	99.95	100.00	100.00
	Sce. 2	100.00	100.00	100.00	100.00	100.00	100.00	99.33	100.00	100.00	99.51	100.00	100.00
	Sce. 3	99.84	100.00	100.00	99.86	100.00	100.00	99.88	100.00	100.00	99.87	100.00	100.00
	Sce. 4	65.22	76.60	91.95	65.11	76.65	92.12	98.64	99.97	100.00	98.82	99.99	100.00
	Sce. 5	88.72	99.02	100.00	89.26	99.37	100.00	85.82	90.49	97.33	84.96	90.06	97.89
	Sce. 6	84.31	97.27	100.00	84.63	97.67	100.00	73.20	87.03	97.83	72.76	87.07	98.15
		true model: unit gamma						**true model: beta**					
II	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	99.92	100.00	100.00	99.93	100.00	100.00
	Sce. 2	99.88	100.00	100.00	99.91	100.00	100.00	99.71	100.00	100.00	99.82	100.00	100.00
	Sce. 3	99.99	100.00	100.00	99.99	100.00	100.00	99.89	100.00	100.00	99.92	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	97.41	99.96	100.00	97.75	99.97	100.00
	Sce. 5	62.38	78.85	95.21	62.18	78.59	95.90	90.11	95.11	99.62	89.56	95.07	99.76
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	98.35	99.96	100.00	98.72	99.97	100.00
		true model: simplex						**true model: unit gamma**					
III	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.99	100.00	100.00	99.99	100.00	100.00	99.53	100.00	100.00	99.66	100.00	100.00
	Sce. 3	100.00	100.00	100.00	100.00	100.00	100.00	99.99	100.00	100.00	99.99	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	99.98	100.00	100.00	99.98	100.00	100.00
	Sce. 5	94.91	98.48	99.98	94.72	98.86	100.00	67.44	83.70	97.47	67.20	83.53	98.28
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	99.96	100.00	100.00	99.99	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	Sce. 1	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.97	100.00	100.00	99.98	100.00	100.00	99.83	100.00	100.00	99.85	100.00	100.00
	Sce. 3	99.84	100.00	100.00	99.86	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	95.32	99.53	100.00	95.76	99.69	100.00	99.99	100.00	100.00	99.99	100.00	100.00
	Sce. 5	62.77	77.71	92.93	62.60	77.36	93.28	80.51	86.48	94.18	79.61	85.87	94.21
	Sce. 6	96.89	99.89	100.00	97.36	99.95	100.00	94.36	99.72	100.00	95.11	99.83	100.00
		true model: Johnson S_B						**true Model: simplex**					
V	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.96	100.00	100.00	99.97	100.00	100.00	99.22	100.00	100.00	99.40	100.00	100.00
	Sce. 3	99.99	100.00	100.00	99.99	100.00	100.00	99.69	100.00	100.00	99.68	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 5	62.07	80.68	96.49	61.87	80.40	97.50	91.90	94.48	98.77	91.36	94.49	98.98
	Sce. 6	99.30	100.00	100.00	99.47	100.00	100.00	99.97	100.00	100.00	99.98	100.00	100.00

Table 4.17: Frequencies (%) of correct model selection using the MJ_{LR} statistic as a model selection criterion (given that the null hypothesis is not rejected at nominal level α) with two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	Sce. 1	99.99	100.00	100.00	99.99	100.00	100.00	99.91	100.00	100.00	99.95	100.00	100.00
	Sce. 2	100.00	100.00	100.00	100.00	100.00	100.00	99.40	100.00	100.00	99.55	100.00	100.00
	Sce. 3	99.88	100.00	100.00	99.89	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	64.97	76.60	91.94	64.95	76.67	92.18	99.98	100.00	100.00	99.98	100.00	100.00
	Sce. 5	88.93	99.05	100.00	89.39	99.33	100.00	85.60	90.38	97.30	84.73	89.93	97.86
	Sce. 6	84.22	97.23	100.00	84.59	97.60	100.00	75.52	86.74	97.85	75.26	86.89	98.14
		true model: unit gamma						**true model: beta**					
II	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	99.92	100.00	100.00	99.93	100.00	100.00
	Sce. 2	99.88	100.00	100.00	99.91	100.00	100.00	99.74	100.00	100.00	99.83	100.00	100.00
	Sce. 3	99.98	100.00	100.00	99.98	100.00	100.00	99.95	100.00	100.00	99.96	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	97.43	99.96	100.00	97.75	99.97	100.00
	Sce. 5	62.59	78.88	95.23	62.34	78.61	95.84	89.84	95.04	99.60	89.31	94.99	99.75
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	98.53	99.96	100.00	98.79	99.98	100.00
		true model: simplex						**true model: unit gamma**					
III	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.99	100.00	100.00	99.99	100.00	100.00	99.56	100.00	100.00	99.66	100.00	100.00
	Sce. 3	100.00	100.00	100.00	100.00	100.00	100.00	99.98	100.00	100.00	99.99	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	99.98	100.00	100.00	99.98	100.00	100.00
	Sce. 5	94.73	98.37	99.98	94.53	98.80	100.00	67.58	83.68	97.39	67.35	83.50	98.23
	Sce. 6	100.00	100.00	100.00	100.00	100.00	100.00	99.97	100.00	100.00	99.99	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.98	100.00	100.00	99.98	100.00	100.00	99.83	100.00	100.00	99.85	100.00	100.00
	Sce. 3	99.90	100.00	100.00	99.90	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	95.25	99.53	100.00	95.70	99.67	100.00	99.99	100.00	100.00	99.99	100.00	100.00
	Sce. 5	62.92	77.71	92.91	62.73	77.36	93.25	80.22	86.40	94.17	79.37	85.80	94.15
	Sce. 6	96.65	99.89	100.00	97.20	99.94	100.00	95.59	99.73	100.00	96.22	99.84	100.00
		true model: Johnson S_B						**true Model: simplex**					
V	Sce. 1	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	99.97	100.00	100.00	99.97	100.00	100.00	99.32	100.00	100.00	99.43	100.00	100.00
	Sce. 3	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 5	61.99	80.47	96.42	61.79	80.18	97.48	79.58	83.59	91.17	78.45	82.85	90.86
	Sce. 6	99.25	100.00	100.00	99.46	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.18: Frequencies (%) of correct model selection using the MJ statistic as a model selection criterion (given that the null hypothesis is not rejected at α) with three competing models.

CASE			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n		50	100	250	50	100	250	50	100	250	50	100	250	50	100	250	50	100	250
			true model: beta						**true model: simplex**						**true model: unit gamma**					
VI	Sce. 1	MJ_S	98.34	99.98	100.00	98.62	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	99.95	100.00	100.00
		MJ_G	98.48	99.99	100.00	98.71	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.96	100.00	100.00	99.95	100.00	100.00
		MJ_{LR}	98.59	99.99	100.00	98.85	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.96	100.00	100.00	99.95	100.00	100.00
	Sce. 2	MJ_S	99.86	100.00	100.00	99.93	100.00	100.00	93.99	99.67	100.00	94.88	99.76	100.00	97.36	99.98	100.00	98.01	100.00	100.00
		MJ_G	99.98	100.00	100.00	99.99	100.00	100.00	96.28	99.81	100.00	96.93	99.89	100.00	99.00	100.00	100.00	99.23	100.00	100.00
		MJ_{LR}	99.99	100.00	100.00	99.99	100.00	100.00	96.50	99.84	100.00	97.14	99.89	100.00	99.09	100.00	100.00	99.29	100.00	100.00
	Sce. 3	MJ_S	92.53	99.78	100.00	93.36	99.83	100.00	100.00	100.00	100.00	100.00	100.00	100.00	97.50	99.97	100.00	97.79	99.98	100.00
		MJ_G	98.68	99.99	100.00	98.88	99.99	100.00	99.61	100.00	100.00	99.61	100.00	100.00	98.75	100.00	100.00	98.91	100.00	100.00
		MJ_{LR}	98.95	99.99	100.00	99.08	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	98.81	100.00	100.00	98.99	100.00	100.00
	Sce. 4	MJ_S	71.19	82.82	95.80	70.92	83.02	95.94	99.78	100.00	100.00	99.85	100.00	100.00	69.93	82.40	94.40	69.94	82.21	94.52
		MJ_G	71.07	82.32	95.41	70.72	82.47	95.60	99.89	100.00	100.00	99.94	100.00	100.00	70.81	82.79	94.53	70.70	82.61	94.62
		MJ_{LR}	70.89	82.47	95.55	70.72	82.65	95.69	99.95	100.00	100.00	99.98	100.00	100.00	70.23	82.50	94.40	70.09	82.32	94.52
	Sce. 6	MJ_S	83.55	93.21	99.35	83.44	93.15	99.40	97.89	99.86	99.98	98.23	99.91	99.98	66.75	84.41	98.01	67.01	84.70	98.33
		MJ_G	76.69	90.50	99.12	77.09	90.68	99.21	84.32	98.14	100.00	84.19	98.19	100.00	70.06	86.23	98.21	70.29	86.45	98.50
		MJ_{LR}	79.43	90.71	99.08	79.48	90.78	99.18	98.97	99.98	100.00	99.05	99.99	100.00	68.27	85.80	98.22	68.88	86.10	98.41
			true model: beta						**true model: simplex**						**true model: Johnson S_B**					
VII	Sce. 1	MJ_G	98.48	99.96	100.00	98.73	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		MJ_{LR}	98.62	99.96	100.00	98.85	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 2	MJ_G	99.92	100.00	100.00	99.94	100.00	100.00	96.18	99.81	100.00	96.66	99.87	100.00	99.48	100.00	100.00	99.56	100.00	100.00
		MJ_{LR}	99.92	100.00	100.00	99.95	100.00	100.00	96.41	99.84	100.00	96.90	99.91	100.00	99.55	100.00	100.00	99.60	100.00	100.00
	Sce. 3	MJ_G	99.45	100.00	100.00	99.62	100.00	100.00	99.88	100.00	100.00	99.87	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		MJ_{LR}	99.68	100.00	100.00	99.71	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 4	MJ_G	99.45	100.00	100.00	99.62	100.00	100.00	99.88	100.00	100.00	99.87	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		MJ_{LR}	99.68	100.00	100.00	99.71	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Sce. 6	MJ_G	99.45	100.00	100.00	99.62	100.00	100.00	99.88	100.00	100.00	99.87	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		MJ_{LR}	99.68	100.00	100.00	99.71	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.19: Frequencies (%) of correct model selection using the MJ statistics as a model selection criterion (given that the null hypothesis is not rejected at nominal level α) with two competing models that only differ in the distribution.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	100	250	400	100	250	400	100	250	400	100	250	400
		true model: beta						**true model: simplex**					
I	MJ_S	79.15	94.45	98.13	78.75	94.52	98.38	80.65	94.01	97.38	80.88	94.17	97.67
	MJ_G	77.41	93.70	97.80	77.16	93.87	98.28	82.90	94.73	97.78	82.89	94.88	98.02
	MJ_{LR}	76.14	93.14	97.64	75.96	93.41	98.19	83.32	95.13	97.92	83.29	95.19	98.16
		true model: unit gamma						**true model: beta**					
II	MJ_S	67.14	72.57	75.70	67.42	72.42	75.49	54.33	66.08	71.88	53.73	65.61	71.42
	MJ_G	69.20	73.85	76.46	68.80	73.43	76.13	52.59	65.01	71.03	52.37	64.74	70.70
	MJ_{LR}	70.27	74.48	76.88	69.42	73.96	76.51	51.55	64.04	70.36	51.56	63.95	70.09
		true model: simplex						**true model: unit gamma**					
III	MJ_S	77.11	84.30	89.52	77.02	84.12	89.50	64.23	78.46	86.24	63.63	78.03	86.09
	MJ_G	78.47	85.07	90.32	77.92	84.78	90.29	62.69	77.58	85.55	62.32	77.26	85.37
	MJ_{LR}	79.41	85.68	90.70	78.69	85.35	90.65	61.47	76.90	85.06	61.34	76.68	84.93
		true model: beta						**true model: Johnson S_B**					
IV	MJ_G	61.26	76.55	84.14	60.78	76.25	84.04	73.85	80.64	85.42	73.28	80.26	85.26
	MJ_{LR}	60.24	75.91	83.67	60.05	75.67	83.56	74.55	81.05	85.90	73.82	80.60	85.66
		true model: Johnson S_B						**true Model: simplex**					
V	MJ_G	59.08	71.97	80.01	58.60	71.67	79.76	73.70	80.54	84.07	73.32	80.19	83.83
	MJ_{LR}	57.56	71.31	79.35	57.57	71.07	79.19	74.70	81.21	84.59	74.18	80.71	84.29

Table 4.20: Frequencies (%) of correct model selection using the MJ statistics as a model selection criterion (given that the null hypothesis is not rejected at nominal level α) with three competing models that only differ in the distribution.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			CASE		$\alpha = 5\%$			$\alpha = 10\%$		
	n	100	250	400	100	250	400		n	100	250	400	100	250	400
		true model: beta								**true model: beta**					
VI	MJ_S	63.71	70.43	75.95	62.92	69.75	75.38	VII							
	MJ_G	58.39	67.62	74.04	58.07	67.18	73.71		MJ_G	76.86	91.96	96.64	76.69	92.05	96.76
	MJ_{LR}	56.59	66.18	73.05	56.52	65.85	72.79		MJ_{LR}	75.65	91.53	96.49	75.51	91.66	96.61
		true model: simplex								**true model: simplex**					
	MJ_S	73.54	84.48	90.35	73.78	84.61	90.66								
	MJ_G	78.42	86.83	91.86	78.26	86.89	91.96		MJ_G	77.45	85.56	88.97	77.34	85.53	89.04
	MJ_{LR}	79.85	87.54	92.39	79.70	87.51	92.39		MJ_{LR}	79.17	86.60	89.61	79.11	86.40	89.57
		true model: unit gamma								**true model: Johnson S_B**					
	MJ_S	29.60	52.93	64.99	30.08	53.28	65.32								
	MJ_G	32.47	54.75	66.26	32.76	55.04	66.40		MJ_G	19.76	41.71	58.40	19.95	41.94	58.39
	MJ_{LR}	32.83	55.08	66.74	32.85	55.36	66.75		MJ_{LR}	19.55	41.61	58.16	19.77	41.85	58.25

4.6 Nonnull rejection rates for J and MJ tests

Finally, we report simulation results on the tests nonnull behavior (power). Tables 4.21, 4.22, 4.23, 4.24, 4.25, 4.26 and 4.27 contain the J test nonnull rejection rates under scenarios 1 through 7, respectively, based on all three criteria (score, gradient and likelihood ratio) for the 5% and 10% nominal leves and when there are two competing models. For scenarios 1, 2 and 3, where the competing models differ in covariates that they include, the J test proved to quite powerful even in small samples, since for $n = 50$ the nonnull rejection rates exceed 85% for all statistics and for all cases in consideration. In scenario 4 and case I when the true model is beta it is necessary a large sample size to achieve power greater than 70%. When competing models only differ in the link function on the precision/dispersion submodels (scenario 5), the J test has low power when the number observations is small, especially for cases II, IV and V, therefore, test becoming more powerful as the sample size increases. In scenario 6, when the models use different link functions in both submodels, the nonnull rejection rates are large for all samples sizes. In the last scenario (7), where the competing models only differ in the response distribution, the power of the J test increases slowly as the sample size grows.

The nonnull rejection rates for three competing models are presented in Tables 4.28 for case VI, 4.29 for case VII under scenarios 1 through 6, and in Table 4.30 for both cases but under scenario 7, when the models only differ in the response distribution. Overall, the J test is powerful even in small samples. Whenever the power is low, the test becomes powerful as the sample size increases. For instance, for case VII and scenario 6, under the beta (unit gamma) law, the $J_{UG}(J_B)$-gradient nonnull rejection rates are 28.31% (21.64%) for a sample of 50 observations, but quickly increases to 85.51% (80.71%) for samples of 250 observations. Again, as with two competing models, the power of the test under scenario 7 increases slowly as the samples size grows.

Tables 4.31, 4.32, 4.33 and 4.34 contain the MJ test nonnull rejection rates (powers) for $M = 2$ for scenarios 1 and 2, 3 and 4, 5 and 6 and 7, respectively; the results for $M = 3$ are presented in Tables 4.35 for scenarios 1 through 6 and in Table 4.36 for scenario 7. Recall that the test null hypothesis is that one of the models under consideration is the correct model. Thus, we performed Monte Carlo simulations using as true data processing process a model which was different from all considered models. For scenarios 1 through 3, where models differ in the explanatory variables used in the location or precision submodels, the true data generating process differs from the models under testing by a single covariate and the response values were generated from a distribution that differs from those of the models under scrutiny. Under scenarios 4 through 6, the true data generating mechanism uses link functions and response distribution that are different from those employed by the models under investigation. For the last scenario (7), the true data generating process has exactly the same regression structure as the competing models, but the response distribution is different.

The following setups were used: Case I: response is Johnson S_B-distributed and complementary log-log and identity link functions for the location and precision submodels, respectively; Case II: response distributed as Johnson S_B and logit and identity link functions for the location and precision submodels, respectively; Cases III and V: beta-distributed response and logit and identity link functions for the location and precision submodels, respectively; Case IV: response distributed as unit gamma and logit and identity link functions for the location and precision submodels, respectively: Cases VI and VII: Cauchy and square root link functions for location and precision submodels, respectively, and response distributed as Johnson S_B (VI) and unit gamma (VII).

Overall, the MJ test powers were similar for the different testing criteria (gradient, likelihood ratio and score). In some cases, the power of the test was considerably small when the sample size was small ($n = 50$), but increased as the sample size grew. Scenarios 5 ($M = 2$) and 7 ($M = 2$ and $M = 3$) proved to be particularly challenging, especially scenario 7. The nonnull rejection rates increase slowly as the sample size increases, more so with three competing models.[1] In some situations, however, the test displayed very high power (i.e., very small type II error probability) even in small samples. In several instances, for instance, the nonnull rejection rate exceeded 95% with only 50 observations in the sample.

Table 4.21: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 1 and two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	J_S	99.98	100.00	100.00	99.98	100.00	100.00	99.69	100.00	100.00	99.92	100.00	100.00
	J_G	99.98	100.00	100.00	99.98	100.00	100.00	99.74	100.00	100.00	99.92	100.00	100.00
	J_{LR}	99.98	100.00	100.00	99.99	100.00	100.00	99.77	100.00	100.00	99.94	100.00	100.00
		true model: unit gamma						**true model: beta**					
II	J_S	100.00	100.00	100.00	100.00	100.00	100.00	99.58	100.00	100.00	99.84	100.00	100.00
	J_G	100.00	100.00	100.00	100.00	100.00	100.00	99.67	100.00	100.00	99.86	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	99.72	100.00	100.00	99.88	100.00	100.00
		true model: simplex						**true model: unit gamma**					
III	J_S	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	J_G	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	J_G	99.97	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	J_{LR}	99.99	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		true model: Johnson S_B						**true model: simplex**					
V	J_G	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

[1] The results for $n = 1{,}000$ and $n = 3{,}000$ in Table 4.34 were obtained using 1,000 Monte Carlo replications.

Table 4.22: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 2 and two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
	J_S	99.76	100.00	100.00	99.94	100.00	100.00	88.98	99.94	100.00	96.38	99.97	100.00
I	J_G	99.95	100.00	100.00	99.99	100.00	100.00	97.40	99.98	100.00	98.73	100.00	100.00
	J_{LR}	99.97	100.00	100.00	100.00	100.00	100.00	97.82	99.98	100.00	98.81	100.00	100.00
		true model: unit gamma						**true model: beta**					
	J_S	95.35	100.00	100.00	98.91	100.00	100.00	90.85	99.99	100.00	97.26	100.00	100.00
II	J_G	99.64	100.00	100.00	99.85	100.00	100.00	98.77	100.00	100.00	99.52	100.00	100.00
	J_{LR}	99.71	100.00	100.00	99.87	100.00	100.00	98.96	100.00	100.00	99.58	100.00	100.00
		true model: simplex						**true model: unit gamma**					
	J_S	98.97	100.00	100.00	99.72	100.00	100.00	89.88	99.98	100.00	96.12	100.00	100.00
III	J_G	99.89	100.00	100.00	99.97	100.00	100.00	97.39	100.00	100.00	98.86	100.00	100.00
	J_{LR}	99.90	100.00	100.00	99.96	100.00	100.00	97.82	100.00	100.00	98.92	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	J_G	99.90	100.00	100.00	99.96	100.00	100.00	99.23	100.00	100.00	99.63	100.00	100.00
	J_{LR}	99.93	100.00	100.00	99.96	100.00	100.00	99.31	100.00	100.00	99.65	100.00	100.00
		true model: Johnson S_B						**true model: simplex**					
V	J_G	99.83	100.00	100.00	99.92	100.00	100.00	96.80	99.99	100.00	98.61	100.00	100.00
	J_{LR}	99.86	100.00	100.00	99.93	100.00	100.00	97.19	99.99	100.00	98.76	100.00	100.00

Table 4.23: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 3 and two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
	J_S	89.49	99.95	100.00	95.04	99.98	100.00	100.00	100.00	100.00	100.00	100.00	100.00
I	J_G	99.45	100.00	100.00	99.74	100.00	100.00	99.86	100.00	100.00	99.87	100.00	100.00
	J_{LR}	99.70	100.00	100.00	99.85	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		true model: unit gamma						**true model: beta**					
	J_S	96.86	99.98	100.00	98.66	100.00	100.00	89.09	99.93	100.00	94.52	99.98	100.00
II	J_G	99.75	100.00	100.00	99.88	100.00	100.00	99.22	100.00	100.00	99.76	100.00	100.00
	J_{LR}	99.79	100.00	100.00	99.92	100.00	100.00	99.68	100.00	100.00	99.87	100.00	100.00
		true model: simplex						**true model: unit gamma**					
	J_S	100.00	100.00	100.00	100.00	100.00	100.00	99.36	100.00	100.00	99.75	100.00	100.00
III	J_G	100.00	100.00	100.00	100.00	100.00	100.00	99.81	100.00	100.00	99.92	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	99.82	100.00	100.00	99.93	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	J_G	99.44	100.00	100.00	99.73	100.00	100.00	99.99	100.00	100.00	100.00	100.00	100.00
	J_{LR}	99.73	100.00	100.00	99.86	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		true model: Johnson S_B						**true model: simplex**					
V	J_G	99.98	100.00	100.00	99.99	100.00	100.00	99.57	100.00	100.00	99.64	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.24: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 4 and two competing models.

CASE		$\alpha=5\%$			$\alpha=10\%$			$\alpha=5\%$			$\alpha=10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	J_S	26.00	40.15	71.88	35.49	51.16	80.35	83.59	98.53	100.00	89.82	99.32	100.00
	J_G	23.03	37.09	69.66	33.19	48.84	78.78	97.86	99.96	100.00	98.59	99.99	100.00
	J_{LR}	24.46	37.94	70.16	34.23	49.52	79.08	99.97	100.00	100.00	99.97	100.00	100.00
		true model: unit gamma						**true model: beta**					
II	J_S	99.92	100.00	100.00	99.99	100.00	100.00	87.06	99.39	100.00	92.52	99.80	100.00
	J_G	99.98	100.00	100.00	100.00	100.00	100.00	88.47	99.46	100.00	93.33	99.85	100.00
	J_{LR}	99.98	100.00	100.00	100.00	100.00	100.00	89.09	99.49	100.00	93.56	99.85	100.00
		true model: simplex						**true model: unit gamma**					
III	J_S	100.00	100.00	100.00	100.00	100.00	100.00	99.69	100.00	100.00	99.89	100.00	100.00
	J_G	100.00	100.00	100.00	100.00	100.00	100.00	99.88	100.00	100.00	99.95	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	99.88	100.00	100.00	99.94	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	J_G	83.37	98.04	100.00	89.59	99.09	100.00	99.97	100.00	100.00	99.98	100.00	100.00
	J_{LR}	83.40	98.00	100.00	89.46	99.05	100.00	99.97	100.00	100.00	99.98	100.00	100.00
		true model: Johnson S_B						**true model: simplex**					
V	J_G	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.25: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 5 and two competing models.

CASE		$\alpha=5\%$			$\alpha=10\%$			$\alpha=5\%$			$\alpha=10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	J_S	81.05	98.75	100.00	85.82	99.27	100.00	19.70	50.38	92.93	38.24	67.80	96.93
	J_G	83.83	98.96	100.00	87.90	99.43	100.00	48.17	67.81	95.63	60.54	78.29	97.97
	J_{LR}	83.53	98.96	100.00	87.73	99.39	100.00	45.41	66.27	95.46	58.63	77.28	97.89
		true model: unit gamma						**true model: beta**					
II	J_S	25.11	52.41	91.62	33.27	62.49	95.20	25.43	68.40	99.02	48.93	83.55	99.65
	J_G	26.80	52.34	91.57	35.89	63.21	95.21	62.30	84.00	99.51	72.74	90.48	99.79
	J_{LR}	26.39	51.49	91.23	35.45	62.59	95.08	59.02	82.96	99.48	71.04	90.02	99.78
		true model: simplex						**true model: unit gamma**					
III	J_S	44.33	90.87	99.98	69.70	96.56	100.00	30.42	62.89	96.64	39.65	72.61	98.20
	J_G	81.86	97.21	99.98	88.96	98.77	100.00	33.47	64.39	96.82	43.81	73.98	98.33
	J_{LR}	79.75	96.81	99.98	87.94	98.67	100.00	32.98	63.58	96.66	43.31	73.56	98.27
		true model: beta						**true model: Johnson S_B**					
IV	J_G	23.54	46.74	85.97	33.32	57.26	91.27	33.25	50.12	84.04	44.89	62.32	90.66
	J_{LR}	23.09	46.08	85.58	33.04	56.75	91.05	31.16	48.50	83.62	43.21	61.39	90.40
		true model: Johnson S_B						**true model: simplex**					
V	J_G	29.64	60.91	96.01	39.54	70.84	97.71	87.48	92.51	98.76	89.28	93.92	99.08
	J_{LR}	29.09	60.12	95.85	39.04	70.21	97.69	28.77	39.01	71.66	40.60	52.40	81.90

Table 4.26: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 6 and two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	50	100	250	50	100	250	50	100	250	50	100	250
		true model: beta						**true model: simplex**					
I	J_S	68.12	94.78	100.00	76.09	96.91	100.00	22.16	46.92	89.88	34.32	60.72	95.27
	J_G	69.93	95.08	100.00	77.69	97.15	100.00	30.75	53.21	91.19	43.02	66.34	95.89
	J_{LR}	69.80	94.72	100.00	77.66	97.02	100.00	34.76	53.20	91.14	46.86	66.32	95.85
		true model: unit gamma						**true model: beta**					
II	J_S	100.00	100.00	100.00	100.00	100.00	100.00	85.62	99.82	100.00	93.87	99.92	100.00
	J_G	100.00	100.00	100.00	100.00	100.00	100.00	93.82	99.93	100.00	97.31	99.96	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	95.12	99.94	100.00	97.66	99.98	100.00
		true model: simplex						**true model: unit gamma**					
III	J_S	100.00	100.00	100.00	100.00	100.00	100.00	99.90	100.00	100.00	99.98	100.00	100.00
	J_G	100.00	100.00	100.00	100.00	100.00	100.00	99.94	100.00	100.00	99.99	100.00	100.00
	J_{LR}	100.00	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	99.99	100.00	100.00
		true model: beta						**true model: Johnson S_B**					
IV	J_G	92.33	99.79	100.00	95.49	99.94	100.00	71.24	98.68	100.00	85.31	99.66	100.00
	J_{LR}	91.76	99.76	100.00	95.27	99.92	100.00	79.53	98.96	100.00	89.28	99.72	100.00
		true model: Johnson S_B						**true model: simplex**					
V	J_G	98.47	100.00	100.00	99.28	100.00	100.00	99.97	100.00	100.00	99.98	100.00	100.00
	J_{LR}	98.31	100.00	100.00	99.22	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.27: Nonnull rejection rates for the J test at $\alpha = 5\%$ and 10% for Scenario 7 and two competing models.

CASE		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
	n	100	250	400	100	250	400	100	250	400	100	250	400
		true model: beta						**true model: simplex**					
I	J_S	45.62	85.89	96.90	54.66	90.41	98.29	20.29	62.72	87.27	34.35	76.70	93.37
	J_G	44.09	85.51	96.77	54.15	90.39	98.30	32.21	71.34	90.67	45.99	82.11	95.15
	J_{LR}	42.57	84.72	96.51	52.72	89.79	98.20	36.04	73.26	91.34	49.41	83.15	95.60
		true model: unit gamma						**true model: beta**					
II	J_S	4.59	9.63	15.44	11.43	19.22	26.73	8.83	15.37	21.38	14.89	23.88	31.21
	J_G	10.39	14.24	19.80	18.81	24.53	31.43	8.21	13.80	19.64	14.86	22.59	29.58
	J_{LR}	14.01	16.76	21.88	22.58	26.63	33.30	8.78	12.94	18.72	15.29	21.89	28.74
		true model: simplex						**true model: unit gamma**					
III	J_S	11.42	32.86	52.32	22.74	47.57	67.12	18.30	39.19	57.78	26.15	49.68	67.89
	J_G	21.07	39.71	57.75	32.49	53.01	70.89	16.74	36.79	55.95	25.06	48.10	66.65
	J_{LR}	25.43	42.29	59.82	36.57	55.31	72.33	16.09	35.32	54.40	24.70	46.76	65.34
		true model: beta						**true model: Johnson S_B**					
IV	J_G	15.12	33.57	49.82	23.08	44.60	61.67	15.25	26.23	39.86	25.25	38.36	54.01
	J_{LR}	14.72	31.80	48.29	22.73	43.41	60.36	18.78	28.36	42.15	28.24	40.37	55.75
		true model: Johnson S_B						**true model: simplex**					
V	J_G	12.03	25.36	39.44	20.14	35.77	51.17	14.09	25.73	37.23	24.25	38.08	51.09
	J_{LR}	12.08	24.34	38.40	20.34	35.09	50.23	17.77	28.15	39.53	28.19	40.18	52.75

Table 4.28: Nonnull rejection rates of the J test at $\alpha = 5\%$ and 10% for Case VI with three competing regression models for scenarios 1 through 4 and 6.

		$\alpha=5\%$			$\alpha=10\%$				$\alpha=5\%$			$\alpha=10\%$				$\alpha=5\%$			$\alpha=10\%$		
SCENARIO	n	50	100	250	50	100	250		50	100	250	50	100	250		50	100	250	50	100	250
		true model: beta							**true model: simplex**							**true model: unit gamma**					
1	J_{S_S}	92.72	99.92	100.00	96.64	99.99	100.00	J_{S_B}	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_B}	99.86	100.00	100.00	99.95	100.00	100.00
	$J_{S_{GU}}$	89.55	99.76	100.00	95.15	99.92	100.00	$J_{S_{UG}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_S}	99.94	100.00	100.00	99.96	100.00	100.00
	J_{G_S}	93.63	99.93	100.00	96.99	99.98	100.00	J_{G_S}	100.00	100.00	100.00	100.00	100.00	100.00	J_{G_B}	99.91	100.00	100.00	99.98	100.00	100.00
	$J_{G_{GU}}$	91.25	99.82	100.00	95.73	99.92	100.00	$J_{G_{UG}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{G_S}	99.97	100.00	100.00	99.98	100.00	100.00
	J_{LR_S}	95.21	99.94	100.00	97.62	99.99	100.00	J_{LR_B}	100.00	100.00	100.00	100.00	100.00	100.00	J_{LR_B}	99.96	100.00	100.00	99.98	100.00	100.00
	$J_{LR_{GU}}$	93.29	99.86	100.00	96.43	99.92	100.00	$J_{LR_{UG}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{LR_S}	99.97	100.00	100.00	99.98	100.00	100.00
2	J_{S_S}	99.31	100.00	100.00	99.75	100.00	100.00	J_{S_B}	65.98	99.64	100.00	85.35	99.90	100.00	J_{S_B}	72.73	99.93	100.00	92.25	100.00	100.00
	$J_{S_{GU}}$	98.28	100.00	100.00	99.69	100.00	100.00	$J_{S_{GU}}$	49.51	97.14	100.00	72.13	99.17	100.00	J_{S_S}	73.20	99.84	100.00	89.30	99.98	100.00
	J_{G_S}	99.92	100.00	100.00	99.99	100.00	100.00	J_{G_B}	92.53	99.94	100.00	96.47	99.97	100.00	J_{G_B}	97.35	100.00	100.00	98.95	100.00	100.00
	$J_{G_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	$J_{G_{GU}}$	85.51	99.44	100.00	92.25	99.76	100.00	J_{G_S}	95.98	99.98	100.00	98.17	100.00	100.00
	J_{LR_S}	99.97	100.00	100.00	99.99	100.00	100.00	J_{LR_B}	94.14	99.95	100.00	97.00	99.97	100.00	J_{LR_B}	98.08	100.00	100.00	99.19	100.00	100.00
	$J_{LR_{GU}}$	99.99	100.00	100.00	100.00	100.00	100.00	$J_{LR_{GU}}$	88.12	99.58	100.00	93.38	99.79	100.00	J_{LR_S}	97.00	99.98	100.00	98.56	100.00	100.00
3	J_{S_S}	59.37	97.59	100.00	74.23	99.06	100.00	J_{S_B}	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_B}	77.38	99.53	100.00	87.93	99.86	100.00
	$J_{S_{GU}}$	66.02	99.22	100.00	80.95	99.75	100.00	$J_{S_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_S}	88.92	99.91	100.00	94.48	99.97	100.00
	J_{G_S}	90.98	99.85	100.00	95.58	99.93	100.00	J_{G_B}	98.86	100.00	100.00	99.30	100.00	100.00	J_{G_B}	92.63	99.88	100.00	96.30	99.98	100.00
	$J_{G_{GU}}$	95.77	99.97	100.00	98.12	100.00	100.00	$J_{G_{GU}}$	99.67	100.00	100.00	99.84	100.00	100.00	J_{G_S}	96.80	99.99	100.00	98.44	100.00	100.00
	J_{LR_S}	94.76	99.89	100.00	97.24	99.96	100.00	J_{LR_B}	100.00	100.00	100.00	100.00	100.00	100.00	J_{LR_B}	94.70	99.93	100.00	97.20	99.99	100.00
	$J_{LR_{GU}}$	97.66	99.99	100.00	98.85	100.00	100.00	$J_{LR_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{LR_S}	97.74	99.99	100.00	98.79	100.00	100.00
4	J_{S_S}	63.70	93.42	99.99	75.84	96.40	100.00	J_{S_B}	98.04	100.00	100.00	99.29	100.00	100.00	J_{S_B}	18.50	36.70	79.94	30.24	51.13	88.13
	$J_{s_{GU}}$	29.29	49.89	85.75	40.01	60.57	91.28	$J_{S_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_S}	87.12	99.62	100.00	93.23	99.83	100.00
	J_{G_S}	68.72	93.86	100.00	79.07	96.45	100.00	J_{G_B}	99.33	100.00	100.00	99.77	100.00	100.00	J_{G_B}	21.64	39.27	80.71	33.16	52.99	88.56
	$J_{G_{GU}}$	28.31	48.74	85.51	39.47	59.81	91.14	$J_{G_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{G_S}	89.59	99.66	100.00	94.45	99.85	100.00
	J_{LR_S}	71.81	94.35	100.00	80.73	96.71	100.00	J_{LR_B}	99.60	100.00	100.00	99.86	100.00	100.00	J_{LR_B}	23.11	39.23	80.33	34.08	52.84	88.27
	$J_{LR_{GU}}$	28.75	48.25	84.99	39.86	59.30	90.87	$J_{LR_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	$J_{LR_{GU}}$	91.48	99.70	100.00	95.20	99.85	100.00
6	J_{S_S}	63.70	93.42	99.99	75.84	96.40	100.00	J_{S_B}	98.04	100.00	100.00	99.29	100.00	100.00	J_{S_B}	18.50	36.70	79.94	30.24	51.13	88.13
	$J_{s_{GU}}$	29.29	49.89	85.75	40.01	60.57	91.28	$J_{S_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{S_S}	87.12	99.62	100.00	93.23	99.83	100.00
	J_{G_S}	68.72	93.86	100.00	79.07	96.45	100.00	J_{G_B}	99.33	100.00	100.00	99.77	100.00	100.00	J_{G_B}	21.64	39.27	80.71	33.16	52.99	88.56
	$J_{G_{GU}}$	28.31	48.74	85.51	39.47	59.81	91.14	$J_{G_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{G_S}	89.59	99.66	100.00	94.45	99.85	100.00
	J_{LR_S}	71.81	94.35	100.00	80.73	96.71	100.00	J_{LR_B}	99.60	100.00	100.00	99.86	100.00	100.00	J_{LR_B}	23.11	39.23	80.33	34.08	52.84	88.27
	$J_{LR_{GU}}$	28.75	48.25	84.99	39.86	59.30	90.87	$J_{LR_{GU}}$	100.00	100.00	100.00	100.00	100.00	100.00	J_{LR_S}	91.48	99.70	100.00	95.20	99.85	100.00

Table 4.29: Nonnull rejection rates of the J test at $\alpha = 5\%$ and 10% for Case VII with three competing regression models for scenarios 1 through 4 and 6.

SCENARIO	n	$\alpha=5\%$ 50	100	250	$\alpha=10\%$ 50	100	250		$\alpha=5\%$ 50	100	250	$\alpha=10\%$ 50	100	250		$\alpha=5\%$ 50	100	250	$\alpha=10\%$ 50	100	250
		true model: beta							**true model: simplex**							**true model: Johnson S_B**					
1	J_{G_S}	76.42	98.54	100.00	92.42	99.75	100.00	J_{G_B}	96.16	99.88	100.00	100.00	100.00	100.00	J_{G_B}	100.00	100.00	100.00	100.00	100.00	100.00
	$J_{G_{J_S}}$	74.36	98.78	100.00	92.47	99.73	100.00	$J_{G_{J_S}}$	96.13	99.94	100.00	100.00	100.00	100.00	J_{G_S}	100.00	100.00	100.00	100.00	100.00	100.00
	J_{LR_S}	84.71	98.97	100.00	94.20	99.77	100.00	J_{LR_B}	96.78	99.91	100.00	100.00	100.00	100.00	J_{LR_B}	100.00	100.00	100.00	100.00	100.00	100.00
	$J_{LR_{J_S}}$	83.02	99.12	100.00	94.37	99.79	100.00	$J_{LR_{J_S}}$	96.97	99.95	100.00	100.00	100.00	100.00	J_{LR_S}	100.00	100.00	100.00	100.00	100.00	100.00
2	J_{G_S}	99.41	100.00	100.00	99.93	100.00	100.00	J_{G_B}	99.98	100.00	100.00	77.57	99.49	100.00	J_{G_B}	93.43	99.95	100.00	96.78	99.98	100.00
	$J_{G_{J_S}}$	98.80	100.00	100.00	99.85	100.00	100.00	$J_{G_{J_S}}$	99.95	100.00	100.00	55.42	95.53	100.00	J_{G_S}	82.49	99.32	100.00	90.46	99.77	100.00
	J_{LR_S}	99.66	100.00	100.00	99.96	100.00	100.00	J_{LR_B}	99.98	100.00	100.00	84.85	99.65	100.00	J_{LR_B}	94.90	99.96	100.00	97.40	99.98	100.00
	$J_{LR_{J_S}}$	99.37	100.00	100.00	99.91	100.00	100.00	$J_{LR_{J_S}}$	99.96	100.00	100.00	66.11	96.98	100.00	J_{LR_S}	86.06	99.51	100.00	92.11	99.82	100.00
3	J_{G_S}	87.75	99.88	100.00	96.91	99.98	100.00	J_{G_B}	98.71	100.00	100.00	98.78	100.00	100.00	J_{G_B}	99.79	100.00	100.00	99.86	100.00	100.00
	$J_{G_{J_S}}$	91.58	99.99	100.00	98.45	100.00	100.00	$J_{G_{J_S}}$	99.45	100.00	100.00	99.58	100.00	100.00	J_{G_S}	99.90	100.00	100.00	99.94	100.00	100.00
	J_{LR_S}	95.03	99.95	100.00	98.43	100.00	100.00	J_{LR_B}	99.20	100.00	100.00	99.99	100.00	100.00	J_{LR_B}	100.00	100.00	100.00	100.00	100.00	100.00
	$J_{LR_{J_S}}$	97.62	99.99	100.00	99.46	100.00	100.00	$J_{LR_{J_S}}$	99.74	100.00	100.00	100.00	100.00	100.00	J_{LR_S}	100.00	100.00	100.00	100.00	100.00	100.00
4	J_{G_S}	44.44	74.69	96.71	67.45	86.83	98.60	J_{G_B}	76.26	91.26	99.16	100.00	100.00	100.00	J_{G_B}	100.00	100.00	100.00	100.00	100.00	100.00
	$J_{G_{J_S}}$	20.64	47.73	92.54	43.29	71.04	97.77	$J_{G_{J_S}}$	56.71	80.30	98.62	55.41	94.00	100.00	J_{G_S}	80.40	98.63	100.00	88.73	99.49	100.00
	J_{LR_S}	49.94	76.00	96.71	69.29	87.10	98.60	J_{LR_B}	77.18	91.39	99.16	100.00	100.00	100.00	J_{LR_B}	100.00	100.00	100.00	100.00	100.00	100.00
	$J_{LR_{J_S}}$	26.04	50.51	92.73	46.80	72.05	97.75	$J_{LR_{J_S}}$	58.76	80.65	98.65	60.77	94.52	100.00	J_{LR_S}	82.24	98.68	100.00	89.38	99.52	100.00
6	J_{G_S}	76.16	97.94	100.00	86.67	99.12	100.00	J_{G_B}	90.98	99.45	100.00	52.77	99.40	100.00	J_{G_B}	84.03	99.95	100.00	92.19	99.98	100.00
	$J_{G_{J_S}}$	36.53	76.53	99.68	57.99	88.66	99.96	$J_{G_{J_S}}$	69.22	93.19	99.98	96.68	100.00	100.00	J_{G_S}	98.60	100.00	100.00	98.94	100.00	100.00
	J_{LR_S}	77.64	97.92	100.00	87.44	99.05	100.00	J_{LR_B}	91.22	99.43	100.00	89.22	99.90	100.00	J_{LR_B}	97.22	100.00	100.00	98.88	100.00	100.00
	$J_{LR_{J_S}}$	41.38	76.25	99.62	61.21	88.43	99.96	$J_{LR_{J_S}}$	71.50	92.97	99.98	100.00	100.00	100.00	J_{LR_S}	100.00	100.00	100.00	100.00	100.00	100.00

Table 4.30: Nonnull rejection rates of the J test at $\alpha = 5\%$ and 10% for Case VI and VII with three competing regression models for Scenario 7.

CASE	n	$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250		$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250		$\alpha = 5\%$ 50	100	250	$\alpha = 10\%$ 50	100	250
		true model: beta							**true model: simplex**							**true model: unit gamma**					
VI	J_{S_S}	73.70	97.69	99.82	79.83	98.44	99.88	J_{S_B}	20.29	61.85	87.01	34.19	75.38	93.36	J_{S_B}	6.18	11.00	14.80	13.44	19.96	25.26
	$J_{S_{GU}}$	9.92	15.53	20.46	17.14	23.16	29.97	$J_{S_{GU}}$	12.16	32.32	51.51	23.84	46.74	66.72	J_{S_S}	27.38	52.48	70.75	38.09	63.32	79.34
	J_{G_S}	72.60	97.43	99.82	79.48	98.41	99.90	J_{G_B}	33.81	70.76	90.67	48.37	81.77	95.43	J_{G_B}	11.04	14.85	19.11	20.53	25.09	30.55
	$J_{G_{GU}}$	10.16	13.83	18.24	17.77	21.59	27.80	$J_{G_{GU}}$	20.67	38.85	57.56	33.12	53.27	70.70	J_{G_S}	25.79	49.92	68.57	36.87	60.97	77.98
	J_{LR_S}	72.70	97.38	99.80	79.61	98.35	99.90	J_{LR_B}	40.89	73.89	91.74	54.40	83.74	95.97	J_{LR_B}	15.26	17.38	21.38	24.54	27.57	32.64
	$J_{LR_{GU}}$	11.95	13.62	17.73	19.54	21.56	27.07	$J_{LR_{GU}}$	26.24	42.35	60.12	39.08	56.06	72.51	J_{LR_S}	26.90	49.33	67.80	38.09	60.41	77.60
		true model: beta							**true model: simplex**							**true model: Johnson S_B**					
VII	J_{G_S}	70.90	97.59	99.87	77.88	98.51	99.92	J_{G_B}	32.76	67.91	89.15	47.06	79.42	94.49	J_{G_B}	12.94	21.48	33.31	22.48	33.35	46.87
	$J_{G_{J_S}}$	34.84	70.40	89.21	46.88	79.52	93.29	$J_{G_{J_S}}$	15.57	25.71	37.62	26.75	38.09	51.17	J_{G_S}	12.22	20.73	33.71	20.36	30.48	45.98
	J_{LR_S}	70.73	97.45	99.87	77.64	98.40	99.91	J_{LR_B}	39.74	71.54	90.41	53.05	81.44	95.20	J_{LR_B}	17.78	24.40	36.06	27.56	36.17	49.19
	$J_{LR_{J_S}}$	36.41	69.91	88.91	48.64	79.28	93.09	$J_{LR_{J_S}}$	21.14	28.60	40.27	32.19	41.02	53.49	J_{LR_S}	13.26	20.05	32.62	21.50	29.93	44.86

Table 4.31: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ and 10% for scenarios 1 and 2 and two competing regression models.

		$\alpha=5\%$			$\alpha=10\%$			$\alpha=5\%$			$\alpha=10\%$		
CASE	n	50	100	250	50	100	250	50	100	250	50	100	250
		SCENARIO 1						SCENARIO 2					
I	MJ_S	3.42	17.83	57.95	8.20	29.17	68.79	17.49	37.13	71.78	25.87	48.12	77.88
	MJ_G	3.36	18.26	58.72	8.11	29.61	69.51	12.93	31.17	65.32	21.19	42.02	73.53
	MJ_{LR}	3.21	18.19	58.95	7.98	29.64	69.56	12.51	30.98	65.37	20.82	42.08	73.49
II	MJ_S	2.82	13.96	42.12	7.33	24.00	56.37	2.53	15.55	36.11	8.88	25.27	46.44
	MJ_G	3.96	15.55	43.37	10.20	26.75	57.45	13.03	22.39	40.05	21.76	32.28	50.97
	MJ_{LR}	2.75	14.41	42.80	7.52	24.67	56.98	8.98	21.54	39.65	17.17	30.69	49.40
III	MJ_S	3.00	13.54	57.59	6.31	23.34	71.31	41.12	77.88	99.21	52.56	84.78	99.60
	MJ_G	2.71	13.35	57.55	6.17	23.10	71.32	39.58	80.07	99.54	52.05	86.49	99.76
	MJ_{LR}	2.62	13.06	57.43	6.06	22.85	71.30	39.19	79.80	99.54	51.73	86.30	99.76
IV	MJ_G	2.10	12.45	49.11	5.17	21.56	59.66	7.61	19.11	44.03	14.12	28.13	53.61
	MJ_{LR}	1.90	12.36	49.24	5.09	21.50	59.72	7.76	19.12	44.10	14.19	28.29	53.70
V	MJ_G	2.81	12.25	54.63	6.09	20.50	67.32	17.18	39.58	70.03	27.15	50.23	76.22
	MJ_{LR}	2.63	12.06	54.34	5.90	20.24	67.30	17.11	39.18	69.85	26.87	50.06	76.18

Table 4.32: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ and 10% for scenarios 3 and 4 and two competing regression models.

		$\alpha=5\%$			$\alpha=10\%$			$\alpha=5\%$			$\alpha=10\%$		
CASE	n	50	100	250	50	100	250	50	100	250	50	100	250
		SCENARIO 3						SCENARIO 4					
I	MJ_S	1.20	6.40	43.37	3.15	14.07	59.66	76.65	95.49	99.97	85.38	97.77	100.00
	MJ_G	0.28	4.38	39.25	1.37	11.00	56.46	78.68	95.91	99.98	86.43	97.88	100.00
	MJ_{LR}	0.19	3.95	38.72	1.05	10.46	56.25	78.03	95.78	99.98	86.10	97.84	100.00
II	MJ_S	0.00	13.95	99.89	0.24	62.27	99.93	96.85	99.99	100.00	98.93	100.00	100.00
	MJ_G	5.63	92.40	99.90	30.83	96.74	99.94	98.12	100.00	100.00	99.28	100.00	100.00
	MJ_{LR}	0.80	84.07	99.90	8.62	96.12	99.94	97.22	100.00	100.00	98.96	100.00	100.00
III	MJ_S	4.11	7.09	18.75	7.74	12.05	27.18	20.30	55.53	97.33	33.04	68.79	98.73
	MJ_G	3.51	5.94	15.94	6.95	10.67	24.25	18.72	55.47	97.41	31.79	68.76	98.77
	MJ_{LR}	2.78	5.65	15.58	6.15	10.13	24.03	17.23	54.10	97.25	30.27	68.03	98.74
IV	MJ_G	9.79	24.83	67.11	15.83	34.85	76.76	22.76	61.95	98.21	37.22	75.38	99.35
	MJ_{LR}	8.58	23.73	66.59	14.75	33.89	76.43	21.99	61.46	98.19	36.84	74.84	99.33
V	MJ_G	3.99	9.16	23.37	7.92	15.09	33.15	13.25	27.96	42.29	22.31	37.28	50.99
	MJ_{LR}	3.48	8.80	23.07	7.60	14.82	32.92	11.88	26.63	41.47	21.08	36.17	50.52

Table 4.33: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ and 10% for scenarios 5 and 6 and two competing regression models.

		$\alpha = 5\%$			$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 10\%$		
CASE	n	50	100	250	50	100	250	50	100	250	50	100	250
		SCENARIO 5						SCENARIO 6					
I	MJ_S	1.61	0.70	11.63	3.00	3.01	23.62	68.57	94.23	99.99	78.96	96.63	99.99
	MJ_G	0.61	0.22	11.05	1.38	1.21	24.61	65.94	93.65	99.99	76.23	96.26	99.99
	MJ_{LR}	0.89	0.27	11.03	1.77	1.32	24.22	65.76	93.40	99.97	75.63	96.10	99.99
II	MJ_S	2.17	9.40	45.13	6.95	21.10	60.23	40.72	94.11	100.00	61.72	97.69	100.00
	MJ_G	1.26	2.81	41.87	3.22	9.72	59.69	65.02	96.76	100.00	79.52	98.69	100.00
	MJ_{LR}	4.63	14.79	50.91	11.85	25.88	63.76	57.09	96.18	100.00	73.27	98.43	100.00
III	MJ_S	0.52	7.52	50.67	2.59	19.30	67.79	41.81	85.48	99.92	51.96	90.28	99.97
	MJ_G	0.34	9.54	59.07	2.24	22.49	72.25	33.04	83.79	99.95	44.94	89.76	99.98
	MJ_{LR}	0.23	7.44	57.30	1.57	20.08	71.44	32.10	82.81	99.95	44.10	89.15	99.98
IV	MJ_{LR}	15.63	25.15	45.14	25.12	34.71	53.56	27.62	77.21	99.88	42.06	86.55	99.97
	MJ_G	33.79	60.09	87.69	43.79	66.39	90.91	26.93	72.71	99.54	37.50	81.16	99.78
V	MJ_G	33.79	60.09	87.69	43.79	66.39	90.91	26.93	72.71	99.54	37.50	81.16	99.78
	MJ_{LR}	33.64	60.22	87.50	43.72	66.42	90.86	22.91	69.27	99.52	33.82	79.46	99.77

Table 4.34: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ and 10% for scenarios 7.

CASE		$\alpha = 5\%$			$\alpha = 10\%$		
	n	100	250	400	100	250	400
I	MJ_S	3.21	3.21	4.87	7.63	7.42	11.02
	MJ_G	4.45	3.65	5.21	8.66	7.97	11.60
	MJ_{LR}	4.87	3.87	5.42	9.64	8.12	11.42
II	MJ_S	10.48	35.72	58.97	21.56	51.42	71.62
	MJ_G	19.56	38.35	58.21	31.51	52.70	70.54
	MJ_{LR}	18.71	43.01	63.86	31.03	57.00	74.39
III	MJ_S	13.27	19.38	28.06	19.17	28.68	37.85
	MJ_G	8.24	15.46	23.84	13.76	24.49	34.50
	MJ_{LR}	6.16	13.21	21.53	11.27	22.23	32.20
IV	MJ_G	5.61	3.92	3.82	11.08	8.75	8.20
	MJ_{LR}	6.58	4.44	4.20	12.01	9.32	8.66
V	MJ_G	17.04	36.78	52.93	25.79	49.15	64.94
	MJ_{LR}	14.52	34.85	51.93	23.75	47.96	64.12

Table 4.35: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ for Scenarios 1 through 4 and 6 and three competing regression models.

SCENARIO	n	$\alpha = 5\%$			$\alpha = 10\%$				$\alpha = 5\%$			$\alpha = 10\%$		
		50	100	250	50	100	250		50	100	250	50	100	250
		CASE VI							CASE VII					
	MJ_S	2.93	45.72	99.57	10.93	65.42	99.92							
1	MJ_G	2.62	54.47	99.91	10.99	73.66	99.98	MJ_G	4.51	28.05	89.14	10.84	44.08	93.01
	MJ_{LR}	1.74	53.84	99.93	9.65	73.68	99.98	MJ_{LR}	4.56	28.41	89.25	10.96	44.65	93.01
	MJ_S	7.04	11.54	22.99	13.13	19.02	32.38							
2	MJ_G	6.65	11.09	21.05	11.76	17.79	30.84	MJ_G	6.59	11.45	19.06	12.43	19.91	29.18
	MJ_{LR}	6.61	11.02	21.16	11.84	17.67	30.79	MJ_{LR}	6.44	11.48	19.17	12.27	19.56	29.00
	MJ_S	0.99	7.14	74.09	2.22	15.32	84.41							
3	MJ_G	0.90	7.99	75.32	2.21	16.01	85.64	MJ_G	11.44	29.65	71.15	18.48	39.93	78.89
	MJ_{LR}	0.78	7.35	75.06	1.98	15.42	85.40	MJ_{LR}	10.11	28.70	70.45	16.71	38.99	78.44
	MJ_S	97.56	100.00	100.00	99.20	100.00	100.00							
4	MJ_G	98.62	100.00	100.00	99.55	100.00	100.00	MJ_G	19.82	69.54	99.18	37.82	82.29	99.70
	MJ_{LR}	98.52	100.00	100.00	99.58	100.00	100.00	MJ_{LR}	18.57	68.77	99.19	37.23	82.11	99.70
	MJ_S	97.56	100.00	100.00	99.20	100.00	100.00							
6	MJ_G	98.62	100.00	100.00	99.55	100.00	100.00	MJ_G	40.14	84.94	99.93	56.97	91.53	99.98
	MJ_{LR}	98.52	100.00	100.00	99.58	100.00	100.00	MJ_{LR}	49.84	87.62	99.93	64.10	92.66	99.98

Table 4.36: Nonnull rejection rates of the MJ test at $\alpha = 5\%$ for Cases VI and VII for scenario 7.

SCENARIO		CASE VI						CASE VII				
	n	100	250	400	1,000	3,000		100	250	400	1,000	3,000
	MJ_S	3.36	4.54	6.21	14.20	49.10						
7	MJ_G	3.79	5.37	6.70	15.20	48.70	MJ_G	2.46	3.38	4.87	10.50	67.00
	MJ_{LR}	3.30	5.71	7.10	14.70	49.10	MJ_{LR}	2.49	3.47	4.00	10.80	67.20

Chapter 5

Empirical applications

Resumo

Neste capítulo aplicamos os testes anteriormente avaliados em dois conjuntos de dados reais. O primeiro conjunto de dados foi apresentado por Smithson e Verkuilen (2006), e os dados tendo sido obtidos de Pammer e Kevan (2004). Trata-se de um estudo de habilidades de leitura em um grupo de 44 crianças disléxicas e não disléxicas que frequentaram a escola primária na Austrália. O segundo conjunto de dados provem de um estudo de transplante autólogo de células tronco do sangue periférico desenvolvido com 239 pacientes no Cross Cancer Instituto em Alabama, Canadá que receberam doses de quimioterapia com diferentes frequências. Esses dados foram analisados por Allan et al. (2002) e por Yang et al. (2005).

5.1 Introduction

In this chapter we shall present two empirical applications of the nonnested hypothesis testing in the classes of models for rates and proportions discussed in the previous sections. Maximum likelihood point estimation was carried out using the BFGS quasi-newton method with analytic first derivatives. All computations were performed using the `Ox` matrix programming language Doornik (2009). The plots were produced using the `R` statistical computing environment (Core and Team, 2013).

5.2 Reading skills data in children

We shall now analyze data from a study realized by School of Psychology of the Australian National University on reading ability of 44 Australian children that attended primary school. The data were collected by Pammer and Kevan (2004) and analyzed by Smithson and Verkuilen (2006) who used a beta regression model. Grün, Kosmidis and Zeileis (2012), Cribari-Neto and Queiroz (2014) and Cribari-Neto and Lucena (2015) also modeled such data using beta regressions.

The variable of interest $\mathbf{y}$ are scores on a test of reading accuracy and the explanations variables are: nonverbal intelligent quotient (IQ) scores converted to z scores ($\mathbf{x}_1$) and a dummy variable that equals 1 for dyslexics and -1 for non-dyslexics ($\mathbf{x}_2$). Grün, Kosmidis and Zeileis (2012) and Cribari-Neto and Queiroz (2014) modeled such data using a variable dispersion beta regression model. Cribari-Neto and Queiroz (2014) included IQ squared as an explanatory variable in the precision submodel whereas Grün, Kosmidis and Zeileis (2012) included the interaction between the two covariates in the precision submodel. Since such models are nonnested, Cribari-Neto and Lucena (2015) compared them using J and MJ tests. Their results favor the model used by Cribari-Neto and Queiroz (2014), which is given by

$$\log\left(\frac{\mu_t}{1-\mu_t}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t1} \times x_{t2}$$
$$\log(\phi_t) = \gamma_0 + \gamma_1 x_{t1} + \gamma_2 x_{t2} + \gamma_3 (x_{t1})^2.$$

Abdel-Fattah (forthcoming) modeled the same data using the following unit gamma regression model:

$$\log\left[-\log(1-\mu_t)\right] = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t1} \times x_{t2}$$
$$\log(\tau_t) = \gamma_0 + \gamma_1 x_{t1} + \gamma_2 x_{t2}.$$

In what follows we shall perform nonnested testing inference using the J and MJ test. We shall consider three candidate models, namely: the two models listed above (beta and unit gamma) and a simplex regression model. The latter was selected within the class of simplex models using the Akaike information criteria (AIC) and the Bayesian information criteria (BIC). The following simplex regression model was selected:

$$\log\left(\frac{\mu_t}{1-\mu_t}\right) = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t1} \times x_{t2}$$
$$\sqrt{\lambda_t} = \gamma_0 + \gamma_1 x_{t1} + \gamma_2 x_{t2} + \gamma_3 x_{t1} \times x_{t2}.$$

Notice that each candidate model belongs to a different class of regression models.

Figure 5.1 contains the simulated envelope plots for the residuals from the three models. Notice that none shows evidence of model misspecification.

Since the sample size is small, testing inference shall be based on the score test statistic. The J and MJ test p-values are presented in Table 5.1. No model is rejected by the J test. The null hypothesis that the one of the three models is the correct model is not rejected by the MJ test at the usual significance levels. Furthermore, the smallest J statistic corresponds to the unit gamma regression model. We thus conclude that unit gamma regression models used by Abdel-Fattah (forthcoming) yields the best fit.

5.3 Autologous peripheral blood stem cell transplants

The data contain information obtained from 239 patients at the Edmonton Hematopoietic Stem Cell Lab in Cross Cancer Institute (Alberta Health Services) and were analyzed

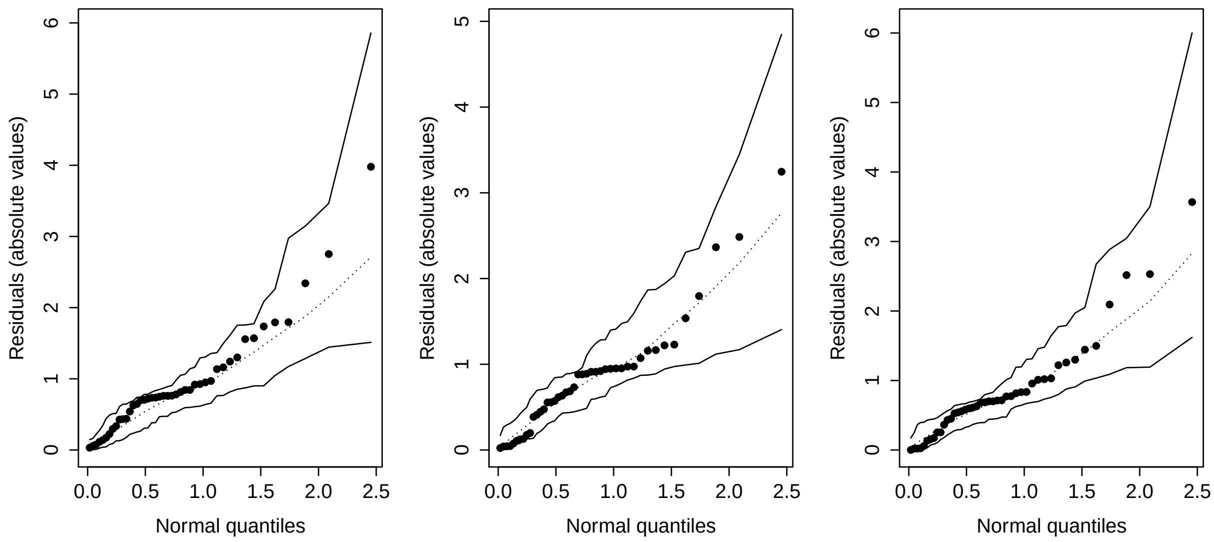

Figure 5.1: Simulated envelope plots for the beta (left panel), unit gamma (center panel) and simplex (right panel) regression models.

Table 5.1: J and MJ (score) statistics and corresponding p-values for reading accuracy data.

	J test			
	beta	unit gamma	simplex	MJ test
Statistic	5.9512	5.0567	6.0567	5.0567
p-value	0.2028	0.2815	0.1949	0.2815

by Allan et al. (2002) and Yang et al. (2005). The patients consented to autologous peripheral blood stem cell (PBSC) transplant after myeloablative doses of chemotherapy. The period analyzed ranges from 2003 through 2008.

We consider the recovery rates for viable CD34+ cells as the variable of interest ($\mathbf{y}$). One of the explanatory variables contains information on the patients adjusted age ($\mathbf{x}_1$), which equals the patient age minus 40 truncated at zero. The other covariate we consider is the chemotherapy protocol ($\mathbf{x}_2$), which is a dummy variable that equals 0 if a patient receives a chemotherapy on a one-day protocol and 1 if a patient receives it on a three-day protocol. Silva (2016) modeled such data using the following simplex regression model:

$$\begin{aligned} \mathrm{M}_s : \log\left(\frac{\mu_t}{1-\mu_t}\right) &= \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \\ \log(\lambda_t) &= \gamma_0 + \gamma_1 x_{t1}. \end{aligned} \tag{5.1}$$

In what follows we shall consider four candidate models, namely: the simplex (s) regression model in (5.1), and also beta (b), a unit gamma (ug) and a Johnson S_B (J) regression models. Notice that such models belong to separate classes of regression models.

We shall distinguish between them on the basis of J and MJ tests.

Using the AIC and the BIC, we arrived at follows competing regression models, namely:

$$\begin{aligned}\mathrm{M}_b : \tan\left[\pi(\mu_t - 0.5)\right] &= \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}\\ \log\left(\phi_t\right) &= \gamma_0 + \gamma_1 x_{t1},\end{aligned}$$

$$\begin{aligned}\mathrm{M}_{ug} : -\log\left[-\log(\xi_t)\right] &= \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}\\ \sqrt{\tau_t} &= \gamma_0 + \gamma_1 x_{t2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{M}_J : \tan\left[\pi(\mu_t - 0.5)\right] &= \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2}\\ \sqrt{\delta_t} &= \gamma_0 + \gamma_1 x_{t2}.\end{aligned}$$

Notice that these models use the same regressors in both submodels location and dispersion, but differ in the link functions. The simplex model (M_s) uses logit and log link, the beta model (M_b) uses Cauchy and log link, the unit gamma model (M_{ug}) uses log-log and square root link and the Johnson S_B model (M_J) uses Cauchy and square root links. Figure 5.2 contains residual simulated envelope plots for the three competing models. They indicate that such models are correctly specified and can be used to model the behavior of the recovery rates for viable CD34+ cells in patients that accepted autologous PBSC transplant.

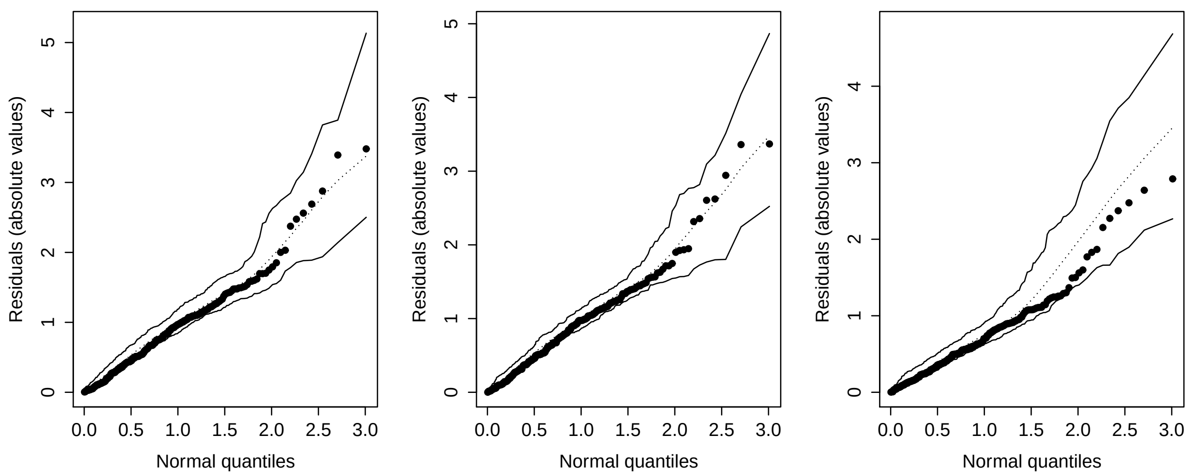

Figure 5.2: Simulated envelope plots for the M_b (left panel), M_{ug} (center panel) and M_J (right panel) regression models.

The competing models are nonnested and we shall distinguish between then using J and MJ testing inference. The test statistics (gradient and likelihood ratio) and the corresponding p-values are presented in Table 5.2. No model is rejected at the 10% nominal level. The MJ test indicates that one of the four models is the correct model at the usual nominal levels and the MJ model selection procedure selects the beta regression model with Cauchy and log links as the best fitting model.

Table 5.2: J and MJ statistics and corresponding p-values for autologous peripheral blood stem cell transplants data.

	J **test**				MJ test
	J_{M_s}	J_{M_b}	$J_{\mathrm{M}_{ug}}$	J_{M_J}	
Gradient	2.8025	7.8003	3.3395	3.2522	2.8025
	(0.8332)	(0.2531)	(0.7652)	(0.7766)	(0.8332)
Likelihood ratio	2.8906	8.0033	3.2969	3.3036	2.8906
	(0.8224)	(0.2379)	(0.7708)	(0.7699)	(0.8224)

Chapter 6

Conclusions

There are various alternative classes of regression models for rates and proportions. These models can differ in the regressors used, in the link functions and even in the response distribution. In this dissertation, we develop tests that can be used when different nonnested models under consideration belong to different classes of regression models. We extend the J and MJ tests, which were originally proposed for the linear regression model. We also implemented these tests based not only in the likelihood ratio, but also in score and gradient statistics. We use Monte Carlo simulations to evaluate the performance of the null and nonnull rejection rates on finite sample tests. We presented how such testing inference can be carried out. The J and MJ tests based on score statistic are shown to outperform the competition in small samples. Besides, the tests based on the statistics considered become more accurate as the sample size increases. The results revealed that the MJ test can be quite useful for selecting the best regression model when the response assumes values in the standard unit interval. It works well in all cases, including for small sample sizes. We present and discuss two empirical applications to illustrate these results.

References

ABDEL-FATTAH, M. Heteroscedastic generalized gamma regression for bounded continuous data. *Communications in Statistics, Theory and Methods*, forthcoming.

ALLAN, D. et al. Number of viable cd34+ cells reinfused predicts engraftment in autologous hematopoietic stem cell transplantation. *Bone Marrow Transplantation*, v. 29, n. 12, p. 967–972, 2002.

ATKINSON, A. C. A method for discriminating between models. *Journal of the Royal Statistical Society B*, v. 32, n. 3, p. 323–353, 1970.

ATKINSON, A. C. *Plots, Transformations, and Regression: An Introduction to Graphical Methods of Diagnostic Regression Analysis*. New York: Oxford University Press, 1985.

BARNDORFF-NIELSEN, O. E.; JØRGENSEN, B. Some parametric models on the simplex. *Journal of Multivariate Analysis*, v. 39, n. 1, p. 106–116, 1991.

CORE, R.; TEAM others. R: A language and environment for statistical computing. Vienna, Austria, 2013.

COX, D. R. Tests of separate families of hypotheses. In: *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. Berkeley: University of California Press, 1961. v. 1, p. 105–123.

COX, D. R. Further results on tests of separate families of hypotheses. *Journal of the Royal Statistical Society B*, v. 24, n. 2, p. 406–424, 1962.

CRIBARI-NETO, F.; LUCENA, S. E. F. Nonnested hypothesis testing in the class of varying dispersion beta regressions. *Journal of Applied Statistics*, v. 42, n. 5, p. 967–985, 2015.

CRIBARI-NETO, F.; LUCENA, S. E. F. Non-nested hypothesis testing inference for GAMLSS models. *Journal of Statistical Computation and Simulation*, v. 87, n. 6, p. 1189–1205, 2017.

CRIBARI-NETO, F.; QUEIROZ, M. P. F. On testing inference in beta regressions. *Journal of Statistical Computation and Simulation*, v. 84, n. 1, p. 186–203, 2014.

DASTOOR, N. K. Some aspects of testing non-nested hypotheses. *Journal of Econometrics*, v. 21, n. 2, p. 213–228, 1983.

DAVIDSON, R.; MACKINNON, J. G. Several tests for model specification in the presence of alternative hypotheses. *Econometrica*, v. 49, n. 3, p. 781–793, 1981.

DEATON, A. S. Model selection procedures, or, does the consumption function exist? In: CHOW, G.; CORSI, P. (Ed.). *Evaluating the reliability of macroeconomic models*. New York: Wiley, 1982. cap. 5, p. 43–65.

DOORNIK, J. A. Ox: An object-oriented matrix programming language. Timberlake Consultants Ltd, 2009.

FANG, K.-T.; ANDERSON, T. W. *Statistical Inference in Elliptically Contoured and Related Distributions*. Lincoln: Allerton Press, 1990.

FERRARI, S.; CRIBARI-NETO, F. Beta regression for modelling rates and proportions. *Journal of Applied Statistics*, v. 31, n. 7, p. 799–815, 2004.

GOURIEROUX, C.; MONFORT, A.; TROGNON, A. Testing nested or non-nested hypotheses. *Journal of Econometrics*, v. 21, n. 1, p. 83–115, 1983.

GRASSIA, A. On a family of distributions with argument between 0 and 1 obtained by transformation of the gamma and derived compound distributions. *Australian Journal of Statistics*, v. 19, n. 2, p. 108–114, 1977.

GRÜN, B.; KOSMIDIS, I.; ZEILEIS, A. Extended beta regression in R: Shaken, stirred, mixed and partitioned. *Journal of Statistical Software*, v. 48, n. 11, p. 1–25, 2012.

HAGEMANN, A. A simple test for regression specification with non-nested alternatives. *Journal of Econometrics*, v. 166, n. 2, p. 247–254, 2012.

JOHNSON, N. L. Systems of frequency curves generated by methods of translation. *Biometrika*, v. 36, n. 1/2, p. 149–176, 1949.

JØRGENSEN, B. *The Theory of Dispersion Models*. London: Chapman and Hall, 1997.

LAMPORT, L. *LaTeX: A Document Preparation System*. Reading, MA: Addison–Wesley Pub. Co., 1986.

LEMONTE, A. J.; BAZÁN, J. L. New class of Johnson SB distributions and its associated regression model for rates and proportions. *Biometrical Journal*, v. 58, n. 4, p. 727–746, 2016.

MIZON, G. E.; RICHARD, J.-F. The encompassing principle and its application to testing non-nested hypotheses. *Econometrica*, v. 54, n. 3, p. 657–678, 1986.

MORGAN, B.; PALMER, K.; RIDOUT, M. Negative score test statistic. *The American Statistician*, v. 61, n. 4, p. 285–288, 2007.

MOUSA, A.; EL-SHEIKH, A.; ABDEL-FATTAH, M. A gamma regression for bounded continuous variables. *Advances and Applications in Statistics*, v. 49, n. 4, p. 305–326, 2016.

NELDER, J. A.; WEDDERBURN, R. W. M. Generalized linear models. *Journal of the Royal Statistical Society A*, v. 135, n. 3, p. 370–384, 1972.

PAMMER, K.; KEVAN, A. The contribution of visual sensitivily, phonological processing and nonverbal IQ to children's reading. *Scientific Studies of Reading*, v. 11, n. 1, p. 33–53, 2004.

PESARAN, M. H.; WEEKS, M. Non-nested hypothesis testing: an overview. In: BALTAGI, B. H. (Ed.). *A Companion to Theoretical Econometrics*. Malden: Blackwell, 2001. cap. 13, p. 279–309.

SILVA, F. C. *Teste de Diagnóstico Baseado em Influência Local Aplicado ao Modelo de Regressão Simplex*. Dissertação (Mestrado) — Universidade Federal de Pernambuco, 2016.

SIMAS, A. B.; BARRETO-SOUZA, W.; ROCHA, A. V. Improved estimators for a general class of beta regression models. *Computational Statistics & Data Analysis*, v. 54, n. 2, p. 348–366, 2010.

SMITHSON, M.; VERKUILEN, J. A better lemon squeezer? Maximum-likelihood regression with beta-distributed dependent variables. *Psychological Methods*, v. 11, n. 1, p. 54, 2006.

SONG, P. X.-K.; QIU, Z.; TAN, M. Modelling heterogeneous dispersion in marginal models for longitudinal proportional data. *Biometrical Journal*, v. 46, n. 5, p. 540–553, 2004.

SONG, P. X.-K.; TAN, M. Marginal models for longitudinal continuous proportional data. *Biometrics*, v. 56, n. 2, p. 496–502, 2000.

YANG, H. et al. Association of post-thaw viable cd34+ cells and cfu-gm with time to hematopoietic engraftment. *Bone Marrow Transplantation*, v. 35, n. 9, p. 881–887, 2005.

Appendix A

Ox programs

In this Appendix we present two of the simulations programs in Ox code used to evaluate the finite sample performances of J and MJ test presented in Chapter 4.

A.1 Models differ in the location submodels regressors.

```
/*******************************************************************

PROGRAM: Simulação5.1_RV_Esc_Grad_PDT_VCEst.ox

USE: Calculates the null and nonnull rejection rates for the J and MJ
tests, based on the score, gradient and likelihood ratio statistics
for two competing models for rates and proportions, which are:

M_1: Unit gamma model
logit(m_t) = beta_0 + beta_1*x_1 + beta_2*x_2
log(tau_t) = gamma_0 + gamma_1*x_1

M_2: Beta model
logit(m_t) = beta_0 + beta_3*x_1 + beta_4*x_3
log(tau_t) = gamma_0 + gamma_1*x_1

AUTOR: Olivia Lizeth Leal Alturo.

OBS:  This program was developed from the program presented in the
master dissertation of Lucena (2013).

LAST MODIFIED: July 17, 2016.

*******************************************************************/

/* header files */
#include <oxstd.h>/* Standard library OX */
#include <oxprob.h>  /* Required for random number generation */
#include <oxfloat.h>  /* Required for PI number in density simplex */
#include <quadpack.h>/* Required to compute variance of simplex and Johnson S_B
distribution */
#import <maximize>  /* Required for nonlinear optimization */

/* global variables */
static decl s_vY; /* Random sample */
static decl s_mX1;  /* Matrix X1 */
static decl s_mX2;  /* Matrix X2 */
static decl s_mX3;  /* Matrix X3 - PDT */
static decl s_mZ;  /* Matrix Z */
static decl s_mCovMU;  /* Matrix of covariates of submodel for mu  */
static decl s_mCovDIS;  /* Matrix of covariates of submodel for precision/dispersion  */
static decl mu_var, lambda_var, n; /* Calculate of variance of simplex distribution */
static decl gama_var, delta_var;  /* Calculate of variance of Johnson S_B dist  */

/* definition of constants */
```

```
const decl NOBS = 250;  /* Sample size */
const decl NREP = 10000;   /* Number of Monte Carlo replications */
const decl rep = 50;   /* Step for increasing the sample size */
const decl k = 3;  /* Number of parameters in the submodel for mu */
const decl m = 2;  /* Number of parameters in the submodel for precision/dispersion */
const decl l = 2; /* Number of competing models*/
const decl beta1 = <1.1, -2.9, 0.7>; /* Parameter values of submodel for mu M_1 */
const decl beta2 = <1.1, -2.8, 0.7>;  /* Parameter values of submodel for mu M_2 */
const decl beta3 = <1.1, -2.7, 0.5>;  /* Parameter values of submodel for mu TP
(test power) */
const decl gama1 = <2.3, 5.2>;  /* Parameter values of submodel  for
precision/dispersion M_1 */
const decl gama2 = <2.3, 4.0>;  /* Parameter values of submodel for
precision/dispersion M_2 */
const decl gama3 = <1.3, 2.0>;  /* Parameter values of submodel for
precision/dispersion TP */

/* Unit gamma random number generator*/
rUnitGamma(const mu, const phi)
{
  decl alpha, rg, y;
  alpha = (mu^(1.0 / phi)) / (1.0 - (mu^(1.0/phi)));
  rg = rangamma(1,1,phi, alpha);
  y = exp(-rg);
  return(y);
}

/* Johnson S_B Random number generator (TP) */
rGJSnor(const gamma, const delta)
{
  decl u,z,y;
  u=ranu(1,1);
  z=(1/delta)*(quann(u)-gamma);
  y=exp(z)/(1+exp(z));
  return(y);
}

/* Moment of Johson S_B distribution */
MomGJSN(const u)
{
   return(1.0/sqrt(2*M_PI))*exp(-(u^2/2.0))*( exp( (u - gama_var)/delta_var )/
( 1.0 + exp( (u - gama_var)/delta_var ) ) )^n;
}

/* Variance of Johnson S_B distribution */
VarGJSNormal(const xi, const delta)
{
decl varG, i, txi, resM1, resM2,  resultIntM1, resultIntM2, erroabsM1, erroabsM2;
varG = zeros(NOBS, 1);
txi = log( xi ./ (1.0 - xi) );
for (i=0; i < NOBS; i++)
{
gama_var = -(delta[i]*txi[i]);
delta_var = delta[i];
n=1;
resM1=QAGI(MomGJSN, 1, 2, &resultIntM1, &erroabsM1);
n=2;
resM2=QAGI(MomGJSN, 1, 2, &resultIntM2, &erroabsM2);
if(resM1==0 && resM2==0)
{
  varG[i]= resultIntM2 - (resultIntM1)^2;
}
else{i--; continue}
}
return varG;
}

/* log-likelihood function of unit gamma models with logit(mu) and log(tau) links*/
floglik_logit_logGU(const vtheta, const adFunc, const avScore, const amHess)
{
decl y = s_vY;
decl k = sizec(s_mCovMU);  /* number of columns in the matrix of covariates of mu */
decl m = sizec(s_mCovDIS);  /* number of columns in the matrix of covariates of tau */
```

```
decl eta1 = s_mCovMU * vtheta[0:(k-1)];  /* used to obtain mu */
decl mu = exp(eta1) ./ (1.0 + exp(eta1));  /* inverse of logit link */
decl eta2 = s_mCovDIS * vtheta[k:(k+m-1)]; /* used to obtain tau */
decl tau = exp(eta2); /* inverse of log link */
decl d = ( mu.^(1.0 ./ tau)) ./ ( 1.0 - (mu.^ (1.0 ./ tau)) );  /* Decl of di*/
adFunc[0] = double(sumc( (tau .* log(d)) - loggamma(tau)
+ ((d - 1.0).* log(y))
  + ((tau - 1.0) .* log( -log(y))) ));  /* log-likelihood */

/* Parameters used for calculation of the firts derivatives */
decl tau1 = (1.0 ./ tau);
decl tau2 = ((1.0 ./ tau) + 1.0 );
decl a1 = ( d ./ (mu .^ tau2) );
decl a2 = (1.0 + ( tau1 .* d .* log(y) ) );
decl a = a1 .* a2;
decl T = diag( mu .* (1.0-mu));  /* 1/g'(mu) */
decl b1 = ( tau1 .* d .* log(mu) ) .* ( 1.0 + ( (d .* log(y) ) ./
( tau .* (mu .^ tau1) ) ) );
decl b2 = ( log( (mu .^ tau1) ./ d) ) + polygamma(tau,0);
decl b = log( -log(y)) - b1 - b2;
decl H = diag(tau); /* 1/h'(tau)*/

if(avScore)
{
(avScore[0])[0:(k-1)] =s_mCovMU'*T*a;
(avScore[0])[k:(k+m-1)] = s_mCovDIS'*H*b;
}

if( isnan(adFunc[0]) || isdotinf(adFunc[0]) )
return 0;
else
return 1; /* 1 indicates success */
}

/* EML function of unit gamma model with logit(mu) and log(tau) links */
EstMV_logit_logGU(const ystar, const mX, const mZ)
{
decl betaols, gamaols, vtheta, dfunc, converge;
decl l = sizec(mZ);
/* initial guess for beta */
if(k > 1)
ols2c(ystar, mX, &betaols);  /* OLS estimates */
else if(k == 1)
betaols = meanc(ystar);
/* Quantities for initial guess of gamma */
gamaols = zeros(l,1); /* initial guess for gamma*/
vtheta = betaols | gamaols;
s_mCovMU = mX;  /* cavariates of mu submodel */
s_mCovDIS = mZ; /* cavariates of tau submodel */

/* maximum likelihood estimation */
converge = MaxBFGS(floglik_logit_logGU, &vtheta, &dfunc, 0, 0);
return vtheta | dfunc | converge;
}

/* log-likelihood function of beta model with logit(mu) and log(phi) links*/
floglik_logit_logB(const vtheta, const adFunc, const avScore, const amHess)
{
decl y = s_vY;
decl k = sizec(s_mCovMU);  /* number of columns in the matrix of covariates of mu */
decl m = sizec(s_mCovDIS);  /* number of columns in the matrix of covariates of phi */
decl eta = s_mCovMU * vtheta[0:(k-1)];  /* used to obtain mu */
decl mu = exp(eta) ./ (1.0 + exp(eta));  /* inverse of logit link */
decl delta = s_mCovDIS * vtheta[k:(k+m-1)]; /* used to obtain phi*/
decl phi = exp(delta); /* inverse of log link */
decl i, funlog = zeros(NOBS, 1);
for(i = 0; i < NOBS; i++)  /* logarithm of the density */
funlog[i] = log(densbeta(y[i], mu[i] * phi[i], (1 - mu[i]) * phi[i]));
adFunc[0] = double( sumc( funlog ) );  /* log-likelihood */

/* Parameters used for calculation of the firts derivatives */
decl ystar = log(y ./ (1.0 - y));
decl mustar = polygamma(mu.*phi,0) - polygamma((1.0-mu).*phi,0);
```

```
decl ydag = log(1.0 - y);
decl mudag = polygamma((1.0 - mu).*phi,0) - polygamma(phi,0);
decl T = diag( mu .* (1.0-mu));  /* 1/g'(mu) */
decl F = diag(phi);
decl H = diag(phi); /* 1/h'(phi)*/
decl M = diag(mu);
if(avScore)
{
(avScore[0])[0:(k-1)] =s_mCovMU'*T*F*(ystar-mustar);
(avScore[0])[k:(k+m-1)] = s_mCovDIS'*H*(M*(ystar-mustar)+(ydag-mudag));
}

if( isnan(adFunc[0]) || isdotinf(adFunc[0]) )
return 0;
else
return 1; /* 1 indicates success */
}

/* EML function of beta model with logit(mu) and log(phi) links */
EstMV_logit_logB(const ystar, const mX, const mZ)
{
decl betaols, etaols, muols, varols, gamaols, phiols, phiolstar, vtheta,
dfunc, converge;
decl l = sizec(mZ);
/* initial guess for beta */
if(k > 1)
ols2c(ystar, mX, &betaols);  /* OLS estimates */
else if(k == 1)
betaols = meanc(ystar);
/* Quantities for initial guess of gamma */
etaols = mX * betaols;
muols = exp(etaols) ./ (1.0 + exp(etaols));  /* inverse of logit link */
varols = ( ((ystar-etaols)' * (ystar - etaols)) .* ((muols.*(1.0-muols)).^2) )
./(NOBS - k);
phiols = ( ( muols .* (1.0 - muols) ) ./ varols );
phiolstar = log(phiols);  /* evaluation of phiols at log link */
if(l > 1)
ols2c(phiolstar, mZ, &gamaols); /* Return in gamaolsr OLS estimates */
else if(l==1)
gamaols = meanc(phiols);
vtheta = betaols | gamaols;
s_mCovMU = mX;  /* cavariates of mu submodel */
s_mCovDIS = mZ; /* cavariates of phi submodel */

/* maximum likelihood estimation */
converge = MaxBFGS(floglik_logit_logB, &vtheta, &dfunc, 0, 0);
return vtheta | dfunc | converge;
}

/*Function to compute the score statistics for unit gamma model with logit and
log links, t=0 only parameter in the mean, t=1 only parameters in the precision
and t=2 parameters in both mean and precision */
Score_logit_logGU(const t, const Est)
{
decl cx = sizec(s_mCovMU);
decl cz = sizec(s_mCovDIS);
decl eta1 = s_mCovMU* Est[0:(cx-1)];
decl muest = (exp(eta1) ./ (1.0 + exp(eta1)));
decl eta2 = s_mCovDIS * Est[cx:(cx+cz-1)];
decl tauest = exp(eta2);
decl tauest1 = (1.0 ./ tauest);
decl tauest2 = ((1.0 ./ tauest) + 1.0);
decl d = ( muest .^ tauest1) ./ ( 1.0 - (muest.^ tauest1) );
/* Quantities for score vector */
decl a1 = ( d ./ (muest .^ tauest2) );
decl a2 = (1.0 + ( tauest1 .* d .* log(s_vY) ) );
decl a = a1 .* a2;
decl T = diag( muest .* (1.0-muest));  /* 1/g'(mu) */
decl b1 = ( tauest1 .* d .* log(muest) ) .* ( 1.0 + ( (d .* log(s_vY) ) ./
( tauest .* (muest .^ tauest1) ) ) );
decl b2 = ( log( (muest .^ tauest1) ./ d) ) + polygamma(tauest,0);
decl b = log( -log(s_vY)) - b1 - b2;
decl H = diag(tauest); /* 1/ h'(tau)*/
```

```
/* Quantities for Fisher's information matrix*/
decl t1 = (muest .* (1.0-muest)); /* 1/g'(mu) */
decl h = tauest;  /* 1/h'(tau) */
decl a_tilde = ( d ./ (muest .^ tauest2) );
decl b_tilde = -a_tilde;
decl c_tilde = (  (d .* log(muest) ) ./ (muest .^ tauest1 ) );
decl W_BB = diag( ( (a_tilde.^2.0)./ tauest).*(t1 .^ 2.0) );
decl W_BG = diag( (b_tilde ./ tauest) .* ( (c_tilde ./ tauest) + 1.0 ) .* t1 .* h );
decl W_GB = W_BG';
decl W_GG = diag( ( ( (2*c_tilde) ./ (tauest .^ 2.0) ) + ( (c_tilde.^2.0) ./
(tauest.^3.0) ) + polygamma(tauest,1) ) .* (h .^ 2.0) );
decl K_BB = s_mCovMU'*W_BB*s_mCovMU;
decl K_BG = s_mCovMU'*W_BG*s_mCovDIS;
decl K_GB = K_BG';
decl K_GG = s_mCovDIS'*W_GG*s_mCovDIS;
decl K = (K_BB ~ K_BG) | (K_GB ~ K_GG);
decl invK = invert(K);
decl U_B = s_mCovMU'T*a;
decl U_G = s_mCovDIS'*H*b;
decl Score;
/* score statistic */
if(t==0) { Score = (U_B[k][])'*(invK[k][k])*(U_B[k][]); }
if(t==1) { Score = (U_G[m][])'*(invK[k+m][k+m])*(U_G[m][]); }
if(t==2) { Score = ( U_B[k][] | U_G[m][] )'*( (invK[k][k]~invK[k][k+m+1]) |
(invK[k][k+m+1]~invK[k+m+1][k+m+1]) )*( U_B[k][] | U_G[m][] ); }
return Score;
}

/*Function to compute the score statistics for beta model with
logit and log links,
t=0 only parameter in the mean, t=1 only parameters in the precision and
t=2 parameters in both mean and precision */
Score_logit_logB(const t, const Est)
{
decl cx = sizec(s_mCovMU);
decl cz = sizec(s_mCovDIS);
decl eta1 = s_mCovMU* Est[0:(cx-1)];
decl muest = (exp(eta1) ./ (1.0 + exp(eta1)));
decl eta2 = s_mCovDIS * Est[cx:(cx+cz-1)];
decl phiest = exp(eta2);
/* Quantities for score vector and Fisher's information matrix */
decl ystar=log(s_vY ./ (1.0 - s_vY));
decl mueststar= polygamma(muest .* phiest,0) - polygamma((1.0 - muest) .* phiest,0);
decl ycruz = log(1.0 - s_vY);
decl muestcruz = polygamma((1.0 - muest) .* phiest,0) - polygamma(phiest,0);
decl PHI=diag(phiest);
decl M=diag(muest);
decl T=diag(muest .* (1.0 - muest)); /*1/g'(muest) */
decl H=diag(phiest); /*1/h'(phiest) */
decl VSTAR= diag(polygamma(muest .* phiest, 1) + polygamma((1.0 - muest) .* phiest,1));
decl C = diag(- polygamma((1.0 - muest) .* phiest,1));
decl VCRUZ = diag(polygamma((1.0 - muest) .* phiest,1) - polygamma(phiest,1));
decl K_BB = s_mCovMU'*PHI*T*VSTAR*T*PHI*s_mCovMU;
decl K_BG = s_mCovMU'*PHI*T*(M*VSTAR + C)*H*s_mCovDIS;
decl K_GB = K_BG';
decl K_GG = s_mCovDIS'*H*((M*VSTAR*M) + 2.0*M*C + VCRUZ)*H*s_mCovDIS;
decl K = (K_BB ~ K_BG) | (K_GB ~ K_GG);
decl invK = invert(K);
decl U_B = s_mCovMU'*PHI*T*(ystar - mueststar);
decl U_G = s_mCovDIS'*H*((M*(ystar - mueststar))+(ycruz - muestcruz));
decl Score;
/* score statistic */
if(t==0) { Score = (U_B[k][])'*(invK[k][k])*(U_B[k][]); }
if(t==1) { Score = (U_G[m][])'*(invK[k+m][k+m])*(U_G[m][]); }
if(t==2) { Score = ( U_B[k][] | U_G[m][] )'*( (invK[k][k]~invK[k][k+m+1]) |
(invK[k][k+m+1]~invK[k+m+1][k+m+1]) )*( U_B[k][] | U_G[m][] ); }
return Score;
}

/*Function to compute the gradient statistic for unit gamma model with
logit and log links */
Grad_logit_logGU(const Unres, const Res)
{
decl cx = sizec(s_mCovMU);
```

```
decl cz = sizec(s_mCovDIS);
decl eta1 = s_mCovMU* Res[0:(cx-1)];
decl muest = (exp(eta1) ./ (1.0 + exp(eta1)));
decl eta2 = s_mCovDIS * Res[cx:(cx+cz-1)];
decl tauest = exp(eta2);
decl tauest1 = (1.0 ./ tauest);
decl tauest2 = ((1.0 ./ tauest) + 1.0);
decl d = ( muest .^ tauest1) ./ ( 1.0 - (muest.^ tauest1) );
/* Quantities for score vector*/
decl a1 = ( d ./ (muest .^ tauest2) );
decl a2 = (1.0 + ( tauest1 .* d .* log(s_vY) ) );
decl a = a1 .* a2;
decl T = diag( muest .* (1.0-muest));  /* 1/g'(mu) */
decl b1 = ( tauest1 .* d .* log(muest) ) .* ( 1.0 + ( (d .* log(s_vY) ) ./
( tauest .* (muest .^ tauest1) ) ) );
decl b2 = ( log( (muest .^ tauest1) ./ d) ) + polygamma(tauest,0);
decl b = log( -log(s_vY)) - b1 - b2;
decl H = diag(tauest); /* 1/ h'(tau)*/
decl U_B = s_mCovMU'T*a;
decl U_G = s_mCovDIS'*H*b;
decl U = U_B | U_G;
decl Grad = U'*(Unres - Res);
return Grad;
}

/*Function to compute the gradient statistic for beta model with logit and log links */
Grad_logit_logB(const Unres, const Res)
{
decl cx = sizec(s_mCovMU);
decl cz = sizec(s_mCovDIS);
decl eta1 = s_mCovMU* Res[0:(cx-1)];
decl muest = (exp(eta1) ./ (1.0 + exp(eta1)));
decl eta2 = s_mCovDIS * Res[cx:(cx+cz-1)];
decl phiest = exp(eta2);
/* Quantities for score vector*/
decl ystar=log(s_vY ./ (1.0 - s_vY));
decl mueststar= polygamma(muest .* phiest,0) - polygamma((1.0 - muest) .* phiest,0);
decl ycruz = log(1.0 - s_vY);
decl muestcruz = polygamma((1.0 - muest) .* phiest,0) - polygamma(phiest,0);
decl PHI=diag(phiest);
decl M=diag(muest);
decl T=diag(muest .* (1.0 - muest)); /*1/g'(muest) */
decl H=diag(phiest); /*1/h'(phiest) */
decl U_B = s_mCovMU'*PHI*T*(ystar - mueststar);
decl U_G = s_mCovDIS'*H*((M*(ystar - mueststar))+(ycruz - muestcruz));
decl U = U_B | U_G;
decl Grad = U'*(Unres - Res);
return Grad;
}

main()
{
/* Declaration of variables used in the program */
decl dExecTime,i,j, poder,
falhaC1MCB, falhaC1MCGU,falhaC1AmpMCB, falhaC1AmpMCGU,
falhaC2MCB, falhaC2MCGU,falhaC2AmpMCB, falhaC2AmpMCGU,
acertoJC1BV, acertoJC1GUV, acertoMJC1V, acertoJC2BV, acertoJC2GUV, acertoMJC2V,
acertomC1V, acertomC2V,
acertoJC1BE, acertoJC1GUE, acertoMJC1E, acertoJC2BE, acertoJC2GUE, acertoMJC2E,
acertomC1E, acertomC2E,
acertoJC1BG, acertoJC1GUG, acertoMJC1G, acertoJC2BG, acertoJC2GUG, acertoMJC2G,
acertomC1G, acertomC2G,
acertoMJC1_QV, acertoMJC1_QE, acertoMJC1_QG, acertoMJC2_QV, acertoMJC2_QE, acertoMJC2_QG;
/* Models parameters and MLE */
decl eta1B, muB, eta2B, phi, eta1GU, muGU, eta2GU, tau, irC1B, irC1GU,irC1AmpB,irC1AmpGU,
irC2B, irC2GU,irC2AmpB,irC2AmpGU;
/* Tests statistics */
decl estaJC1_BV, estaJC1_GUV, estaMJC1V, estaJC2_BV, estaJC2_GUV, estaMJC2V,
estaJC1_BE, estaJC1_GUE, estaMJC1E, estaJC2_BE, estaJC2_GUE, estaMJC2E,
estaJC1_BG, estaJC1_GUG, estaMJC1G, estaJC2_BG, estaJC2_GUG, estaMJC2G,
EJBC1V, EJGUC1V, EMJC1V, EJBC2V, EJGUC2V, EMJC2V,
EJBC1E, EJGUC1E, EMJC1E, EJBC2E, EJGUC2E, EMJC2E,
EJBC1G, EJGUC1G, EMJC1G, EJBC2G, EJGUC2G, EMJC2G;
/* Tests critical values*/
decl VC_MJC1V, VC_MJC2V, VC_MJC1E, VC_MJC2E, VC_MJC1G, VC_MJC2G;
```

```
/* Auxiliar parameters*/
decl chi2, quantis, ystar, var_Beta, var_GU, v1, var_GJSN, file;

dExecTime = timer();  /* start clock */
ranseed("MWC_52");  /* Random number generator */
ranseed({6548,8984});   /* seed */
/* Initializations */
/* Sample vector */
s_vY = zeros(NOBS, 1);
/* MLE */
irC1B = irC1GU= irC2B = irC2GU =zeros((k+m+2), NREP);
irC1AmpB = irC1AmpGU = irC2AmpB = irC2AmpGU = zeros((k+m+(l-1)+2), NREP);
/* fail counter */
falhaC1MCB = falhaC1MCGU = falhaC1AmpMCB = falhaC1AmpMCGU = 0;
falhaC2MCB = falhaC2MCGU = falhaC2AmpMCB = falhaC2AmpMCGU = 0;
/* Tests statistics */
estaJC1_BV = estaJC1_GUV = estaMJC1V = estaJC2_BV = estaJC2_GUV =
estaMJC2V = zeros(NREP,1);
estaJC1_BE = estaJC1_GUE = estaMJC1E = estaJC2_BE = estaJC2_GUE =
estaMJC2E = zeros(NREP,1);
estaJC1_BG = estaJC1_GUG = estaMJC1G = estaJC2_BG = estaJC2_GUG =
estaMJC2G = zeros(NREP,1);
EJBC1V = EJGUC1V = EMJC1V = EJBC2V = EJGUC2V = EMJC2V = zeros(2,1);
EJBC1E = EJGUC1E = EMJC1E = EJBC2E = EJGUC2E = EMJC2E = zeros(2,1);
EJBC1G = EJGUC1G = EMJC1G = EJBC2G = EJGUC2G = EMJC2G = zeros(2,1);
/* Tests successes counter */
acertoJC1BV = acertoJC1GUV = acertoMJC1V = acertoJC2BV =
acertoJC2GUV = acertoMJC2V = acertomC1V = acertomC2V = zeros(3,1);
acertoJC1BE = acertoJC1GUE = acertoMJC1E = acertoJC2BE =
acertoJC2GUE = acertoMJC2E = acertomC1E = acertomC2E = zeros(3,1);
acertoJC1BG = acertoJC1GUG = acertoMJC1G = acertoJC2BG =
acertoJC2GUG = acertoMJC2G = acertomC1G = acertomC2G = zeros(3,1);
acertoMJC1_QV = acertoMJC1_QE = acertoMJC1_QG = acertoMJC2_QV =
acertoMJC2_QE = acertoMJC2_QG = zeros(3,1);
 /* Tests critical values */
quantis = <0.99, 0.95, 0.90>';
poder = 0;

/* Obtaining matrices of covariates */
s_mX1 = 1 ~ (ranu(rep,k-1));  /* Matrix of mean submodel unit gamma */
s_mX2 = s_mX1[][0:1] ~ ranu(rep,1);  /* Matrix of mean submodel beta  */
s_mX3 = s_mX1[][0:1] ~ ranu(rep,1);  /* Matrix of mean submodel - TP  */

for(i = rep; i < NOBS; i += rep)
{
s_mX1 = s_mX1 | s_mX1[0:(rep-1)][];
s_mX2 = s_mX2 | s_mX2[0:(rep-1)][];
s_mX3 = s_mX3 | s_mX3[0:(rep-1)][];
}

s_mZ = s_mX1[][0:1]; /* Matrix of precision submodel */

    :regresa if(poder==0){ println("\n Inicio Tamanho"); }
 if(poder==1){ println("\n Inicio Poder"); }

/* True parameter values in the size simulation - CASE I: unit gamma model
(whenever poder=0)*/
if(poder==0)
{
eta1GU = s_mX1 * beta1';
muGU = (exp(eta1GU) ./ (1.0 + exp(eta1GU))); /* inverse logit link */
eta2GU = s_mZ * gama1';
tau = exp(eta2GU); /* inverse log link */
v1 = (2.0 - muGU.^(1.0 ./ tau)).^(tau);
var_GU = muGU.*((1.0 ./ v1) - muGU); /* variance unit gamma */
chi2 = VC_MJC1V = VC_MJC2V = VC_MJC1E = VC_MJC2E = VC_MJC1G =
VC_MJC2G = quanchi( (quantis) , 1) ;  /* Critical values of X-squared*/
}
/* True parameter values in the power simulation - Johnson S_B model
(whenever poder=1)*/
if(poder==1)
{
eta1GU = s_mX3 * beta3';
muGU = (exp(eta1GU) ./ (1.0 + exp(eta1GU)));  /* inverse logit link */
eta2GU = s_mZ * gama3';
```

```
tau = exp(eta2GU); /* inverse log link */
v1 = (2.0 - muGU.^(1.0 ./ tau)).^(tau);
var_GU = muGU.*((1.0 ./ v1) - muGU); /* variance unit gamma */
var_GJSN= VarGJSNormal(muGU, tau); /* variance Johnson S_B - TP */

/* Critical values estimate in the size simulation */
/* LR test*/
VC_MJC1V = quantilec( estaMJC1V , quantis');
/* score test */
VC_MJC1E = quantilec( estaMJC1E , quantis');
/* gradient test */
VC_MJC1G = quantilec( estaMJC1G , quantis');

/* set counters equal to zero */
falhaC1MCB = falhaC1MCGU = falhaC1AmpMCB = falhaC1AmpMCGU = 0;
acertoJC1BV = acertoJC1GUV = acertoMJC1V = acertomC1V = zeros(3,1);
acertoJC1BE = acertoJC1GUE = acertoMJC1E = acertomC1E = zeros(3,1);
acertoJC1BG = acertoJC1GUG = acertoMJC1G = acertomC1G = zeros(3,1);
}

/* CASE I: DATA GENERATION */
println( "CasoI:  ");
for(i = 0; i < NREP; i++)
{
if(poder==0) /*UnitGamma(mu, tau)*/
{
for(j = 0; j < NOBS; j++)
{ s_vY[j] = rUnitGamma(muGU[j], tau[j]); }
}
if(poder==1) /*JohnsonS_B(mu, tau)*/
{
for(j = 0; j < NOBS; j++)
{
decl gama=tau[j]*log((1.0 - muGU[j])/muGU[j]);
s_vY[j] = rGJSnor(gama, tau[j]);
}
}

/* Obtaining y* */
ystar = log( s_vY ./ (1.0-s_vY) ); /* logit transformation */

/* Unit gamma model MLE */
irC1GU[][i] = EstMV_logit_logGU(ystar, s_mX1, s_mZ);
if(irC1GU[k+m+1][i] != MAX_CONV && irC1GU[k+m+1][i] != MAX_WEAK_CONV){
falhaC1MCGU++;
i--;
continue;
}

/* Beta model MLE  */
irC1B[][i] = EstMV_logit_logB(ystar, s_mX2, s_mZ);
if(irC1B[k+m+1][i] != MAX_CONV && irC1B[k+m+1][i] != MAX_WEAK_CONV){
falhaC1MCB++;
i--;
continue;
}

/* fitted unit gamma mu to included in beta estimation  */
decl eta1C1GU =s_mX1*irC1GU[0:(k-1)][i];
decl muC1GU =exp(eta1C1GU) ./ (1.0 + exp(eta1C1GU));

/* fitted beta mu to included in unit gamma estimation */
decl eta1C1B =s_mX2*irC1B[0:(k-1)][i];
decl muC1B =exp(eta1C1B) ./ (1.0 + exp(eta1C1B));

/* augmented model MLE (by beta) in unit gamma model */
irC1AmpGU[][i] = EstMV_logit_logGU(ystar, s_mX1~muC1B, s_mZ);
if(irC1AmpGU[k+m+(l-1)+1][i] != MAX_CONV && irC1AmpGU[k+m+(l-1)+1][i]
!= MAX_WEAK_CONV){
falhaC1AmpMCGU++;
i--;
continue;
}
```

```
/* augmented model MLE (by unit gamma) in beta model */
irC1AmpB[][i] = EstMV_logit_logB(ystar, s_mX2~muC1GU, s_mZ);
if(irC1AmpB[k+m+(l-1)+1][i] != MAX_CONV && irC1AmpB[k+m+(l-1)+1][i]
!= MAX_WEAK_CONV){
falhaC1AmpMCB++;
i--;
continue;
}

/* J and MJ statistics based on score and gradient statistics  */

/* Restricted theta */
decl ResC1GU = irC1GU[0:(k-1)][i] | 0 | irC1GU[k:(k+m-1)][i];
decl ResC1B = irC1B[0:(k-1)][i] | 0 | irC1B[k:(k+m-1)][i];

/* Unrestricted theta */
decl IrresC1GU = irC1AmpGU[0:(k+m)][i];
decl IrresC1B = irC1AmpB[0:(k+m)][i];

/* Calculation of the two statistics */
s_mCovDIS = s_mZ;
/* J1 test   */
s_mCovMU = s_mX1~muC1B;
estaJC1_GUE[i] = Score_logit_logGU(0, ResC1GU);
estaJC1_GUG[i] = Grad_logit_logGU(IrresC1GU, ResC1GU);
    /* J2 test  */
s_mCovMU = s_mX2~muC1GU;
estaJC1_BE[i] = Score_logit_logB(0, ResC1B);
estaJC1_BG[i] = Grad_logit_logB(IrresC1B, ResC1B);
/* MJ test  */
estaMJC1E[i] = min(estaJC1_GUE[i], estaJC1_BE[i]);
estaMJC1G[i] = min(estaJC1_GUG[i], estaJC1_BG[i]);

/*NUMBER OF NON-REJECTIONS OF THE GRADIENT TEST  */

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC1_GUG[i] < chi2[0] ) /* 1% sig. */
acertoJC1GUG[0]++;
if(estaJC1_GUG[i] < chi2[1]) /*  5% sig. */
acertoJC1GUG[1]++;
if(estaJC1_GUG[i] < chi2[2]) /* 10% sig. */
acertoJC1GUG[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC1_BG[i] < chi2[0] ) /* 1% sig. */
acertoJC1BG[0]++;
if(estaJC1_BG[i] < chi2[1]) /*  5% sig. */
acertoJC1BG[1]++;
if(estaJC1_BG[i] < chi2[2]) /* 10% sig. */
acertoJC1BG[2]++;

/* Number of non-rejections of the MJ test */
if(estaMJC1G[i] < VC_MJC1G[0]) /*  1% sig. */
acertoMJC1G[0]++;
if(estaMJC1G[i] < VC_MJC1G[1]) /*  5% sig. */
acertoMJC1G[1]++;
if(estaMJC1G[i] < VC_MJC1G[2]) /* 10% sig. */
acertoMJC1G[2]++;

if(poder==1)
{
/* Number of non-rejections of the MJ test - based on ChiQu*/
if(estaMJC1G[i] < chi2[0]) /*  1% sig. */
acertoMJC1_QG[0]++;
if(estaMJC1G[i] < chi2[1]) /*  5% sig. */
acertoMJC1_QG[1]++;
if(estaMJC1G[i] < chi2[2]) /* 10% sig. */
acertoMJC1_QG[2]++;
}

/* Counting of model=unitgamma when H0 is not rejected in MJ test */
if(estaMJC1G[i] < VC_MJC1G[0] && estaMJC1G[i] == estaJC1_GUG[i])
acertomC1G[0]++;
if(estaMJC1G[i] < VC_MJC1G[1] && estaMJC1G[i] == estaJC1_GUG[i])
```

```
acertomC1G[1]++;
if(estaMJC1G[i] < VC_MJC1G[2] && estaMJC1G[i] == estaJC1_GUG[i])
acertomC1G[2]++;

/* NUMBER OF THE NON-REJECTIONS OF THE SCORE TEST */

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC1_GUE[i] < chi2[0] ) /* 1% sig. */
acertoJC1GUE[0]++;
if(estaJC1_GUE[i] < chi2[1]) /*  5% sig. */
acertoJC1GUE[1]++;
if(estaJC1_GUE[i] < chi2[2]) /* 10% sig. */
acertoJC1GUE[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC1_BE[i] < chi2[0] ) /* 1% sig. */
acertoJC1BE[0]++;
if(estaJC1_BE[i] < chi2[1]) /*  5% sig. */
acertoJC1BE[1]++;
if(estaJC1_BE[i] < chi2[2]) /* 10% sig. */
acertoJC1BE[2]++;

/* Number of non-rejections of the MJ test */
if(estaMJC1E[i] < VC_MJC1E[0]) /*  1% sig. */
acertoMJC1E[0]++;
if(estaMJC1E[i] < VC_MJC1E[1]) /*  5% sig. */
acertoMJC1E[1]++;
if(estaMJC1E[i] < VC_MJC1E[2]) /* 10% sig. */
acertoMJC1E[2]++;

if(poder==1)
{
/* Number of non-rejections of the MJ test - based on ChiQu*/
if(estaMJC1E[i] < chi2[0]) /*  1% sig. */
acertoMJC1_QE[0]++;
if(estaMJC1E[i] < chi2[1]) /*  5% sig. */
acertoMJC1_QE[1]++;
if(estaMJC1E[i] < chi2[2]) /* 10% sig. */
acertoMJC1_QE[2]++;
}

/* Counting of model=unitgamma when H0 is not rejected in MJ test */
if(estaMJC1E[i] < VC_MJC1E[0] && estaMJC1E[i] == estaJC1_GUE[i])
acertomC1E[0]++;
if(estaMJC1E[i] < VC_MJC1E[1] && estaMJC1E[i] == estaJC1_GUE[i])
acertomC1E[1]++;
if(estaMJC1E[i] < VC_MJC1E[2] && estaMJC1E[i] == estaJC1_GUE[i])
acertomC1E[2]++;

/* J and MJ statistics based on likelihood ratio statistic  */

/* J1 test  */
estaJC1_GUV[i] = 2*(irC1AmpGU[k+m+(l-1)][i]-irC1GU[k+m][i]); /* M_1 */
/* J2 test  */
estaJC1_BV[i] = 2*(irC1AmpB[k+m+(l-1)][i]-irC1B[k+m][i]); /* M_2 */
/* MJ test  */
estaMJC1V[i] = min(estaJC1_BV[i], estaJC1_GUV[i]);

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC1_GUV[i] < chi2[0] ) /* 1% sig. */
acertoJC1GUV[0]++;
if(estaJC1_GUV[i] < chi2[1]) /*  5% sig. */
acertoJC1GUV[1]++;
if(estaJC1_GUV[i] < chi2[2]) /* 10% sig. */
acertoJC1GUV[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC1_BV[i] < chi2[0] ) /* 1% sig. */
acertoJC1BV[0]++;
if(estaJC1_BV[i] < chi2[1]) /*  5% sig. */
acertoJC1BV[1]++;
if(estaJC1_BV[i] < chi2[2]) /* 10% sig. */
acertoJC1BV[2]++;
```

```
/* Number of non-rejections of the MJ test */
if(estaMJC1V[i] < VC_MJC1V[0]) /*  1% sig. */
acertoMJC1V[0]++;
if(estaMJC1V[i] < VC_MJC1V[1]) /*  5% sig. */
acertoMJC1V[1]++;
if(estaMJC1V[i] < VC_MJC1V[2]) /* 10% sig. */
acertoMJC1V[2]++;

if(poder==1)
{
/* Number of non-rejections of the MJ test - based on ChiQu*/
if(estaMJC1V[i] < chi2[0]) /*  1% sig. */
acertoMJC1_QV[0]++;
if(estaMJC1V[i] < chi2[1]) /*  5% sig. */
acertoMJC1_QV[1]++;
if(estaMJC1V[i] < chi2[2]) /* 10% sig. */
acertoMJC1_QV[2]++;
}

/* Counting of model=unitgamma when H0 is not rejected in MJ test */
if(estaMJC1V[i] < VC_MJC1V[0] && estaMJC1V[i] == estaJC1_GUV[i])
acertomC1V[0]++;
if(estaMJC1V[i] < VC_MJC1V[1] && estaMJC1V[i] == estaJC1_GUV[i])
acertomC1V[1]++;
if(estaMJC1V[i] < VC_MJC1V[2] && estaMJC1V[i] == estaJC1_GUV[i])
acertomC1V[2]++;
}

/* descriptive statistics MJ - GRADIENT */
EJBC1G[0]=meanc(estaJC1_BG);
EJBC1G[1]=varc(estaJC1_BG);
EJGUC1G[0]=meanc(estaJC1_GUG);
EJGUC1G[1]=varc(estaJC1_GUG);
EMJC1G[0]=meanc(estaMJC1G);
EMJC1G[1]=varc(estaMJC1G);

/* descriptive statistics MJ - SCORE */
EJBC1E[0]=meanc(estaJC1_BE);
EJBC1E[1]=varc(estaJC1_BE);
EJGUC1E[0]=meanc(estaJC1_GUE);
EJGUC1E[1]=varc(estaJC1_GUE);
EMJC1E[0]=meanc(estaMJC1E);
EMJC1E[1]=varc(estaMJC1E);

/*descriptive statistics MJ - LR */
EJBC1V[0]=meanc(estaJC1_BV);
EJBC1V[1]=varc(estaJC1_BV);
EJGUC1V[0]=meanc(estaJC1_GUV);
EJGUC1V[1]=varc(estaJC1_GUV);
EMJC1V[0]=meanc(estaMJC1V);
EMJC1V[1]=varc(estaMJC1V);

/* True parameter values in the size simulation - CASE II: beta model
(whenever poder=0)*/
if(poder==0)
{
eta1B = s_mX2 * beta2';
muB = (exp(eta1B) ./ (1.0 + exp(eta1B)));  /* inverse logit link */
eta2B = s_mZ * gama2';
phi = exp(eta2B); /* inverse log link */
var_Beta = (muB.*(1.0 - muB)) ./ (1.0 + phi); /* variance beta*/
}

/* Go to - print result simulation of test power */
if(poder==1)
{
goto imprimir;
}

/* CASE II: DATA GENERATION */
println( "CasoII:  ");
for(i = 0; i < NREP; i++)
{
if(poder==0) /* beta(mu,phi) */
```

```
{
for(j = 0; j < NOBS; j++)
{
decl p,q;
p=muB[j]*phi[j];
q=(1.0-muB[j])*phi[j];
s_vY[j] = ranbeta(1,1,p,q);
}
}
if(poder==1) /*JohnsonS_B(mu, phi)*/
{
for(j = 0; j < NOBS; j++)
{
decl gama=phi[j]*log((1.0 - muB[j])/muB[j]);
s_vY[j] = rGJSnor(gama, phi[j]);
}
}

/* Obtaining y* */
ystar = log( s_vY ./ (1.0-s_vY) ); /* logit transformation */

/* Unit gamma model MLE */
irC2GU[][i] = EstMV_logit_logGU(ystar, s_mX1, s_mZ);
if(irC2GU[k+m+1][i] != MAX_CONV && irC2GU[k+m+1][i] != MAX_WEAK_CONV){
falhaC2MCGU++;
i--;
continue;
}

/* Beta model MLE */
irC2B[][i] = EstMV_logit_logB(ystar, s_mX2, s_mZ);
if(irC2B[k+m+1][i] != MAX_CONV && irC2B[k+m+1][i] != MAX_WEAK_CONV){
falhaC2MCB++;
i--;
continue;
}

/* fitted unit gamma mu to included in beta estimation  */
decl eta1C2GU =s_mX1*irC2GU[0:(k-1)][i];
decl muC2GU =exp(eta1C2GU) ./ (1.0 + exp(eta1C2GU));

/* fitted beta mu to included in unit gamma estimation  */
decl eta1C2B =s_mX2*irC2B[0:(k-1)][i];
decl muC2B =exp(eta1C2B) ./ (1.0 + exp(eta1C2B));

/* augmented model MLE (by beta) in unit gamma model */
irC2AmpGU[][i] = EstMV_logit_logGU(ystar, s_mX1~muC2B, s_mZ);
if(irC2AmpGU[k+m+(l-1)+1][i] != MAX_CONV && irC2AmpGU[k+m+(l-1)+1][i]
!= MAX_WEAK_CONV){
falhaC2AmpMCGU++;
i--;
continue;
}

/* augmented model MLE (by unit gamma) in beta model */
irC2AmpB[][i] = EstMV_logit_logB(ystar, s_mX2~muC2GU, s_mZ);
if(irC2AmpB[k+m+(l-1)+1][i] != MAX_CONV && irC2AmpB[k+m+(l-1)+1][i]
!= MAX_WEAK_CONV){
falhaC2AmpMCB++;
i--;
continue;
}

/* J and MJ statistics based on score and gradient statistics  */

/* Restricted theta */
decl ResC2GU = irC2GU[0:(k-1)][i] | 0 | irC2GU[k:(k+m-1)][i];
decl ResC2B = irC2B[0:(k-1)][i] | 0 | irC2B[k:(k+m-1)][i];

/* Unrestricted theta */
decl IrresC2GU = irC2AmpGU[0:(k+m)][i];
decl IrresC2B = irC2AmpB[0:(k+m)][i];

/* Calculation of the two statistics */
```

```
s_mCovDIS = s_mZ;
/* J1 test  */
s_mCovMU = s_mX1~muC2B;
estaJC2_GUE[i] = Score_logit_logGU(0, ResC2GU);
estaJC2_GUG[i] = Grad_logit_logGU(IrresC2GU, ResC2GU);
    /* J2 test  */
s_mCovMU = s_mX2~muC2GU;
estaJC2_BE[i] = Score_logit_logB(0, ResC2B);
estaJC2_BG[i] = Grad_logit_logB(IrresC2B, ResC2B);
/* MJ test  */
estaMJC2E[i] = min(estaJC2_GUE[i], estaJC2_BE[i]);
estaMJC2G[i] = min(estaJC2_GUG[i], estaJC2_BG[i]);

/* NUMBER OF NON-REJECTIONS OF THE GRADIENT TEST */

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC2_GUG[i] < chi2[0] ) /* 1% sig. */
acertoJC2GUG[0]++;
if(estaJC2_GUG[i] < chi2[1]) /*  5% sig. */
acertoJC2GUG[1]++;
if(estaJC2_GUG[i] < chi2[2]) /* 10% sig. */
acertoJC2GUG[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC2_BG[i] < chi2[0] ) /* 1% sig. */
acertoJC2BG[0]++;
if(estaJC2_BG[i] < chi2[1]) /*  5% sig. */
acertoJC2BG[1]++;
if(estaJC2_BG[i] < chi2[2]) /* 10% sig. */
acertoJC2BG[2]++;

/* Number of non-rejections of the MJ test */
if(estaMJC2G[i] < VC_MJC2G[0]) /*  1% sig. */
acertoMJC2G[0]++;
if(estaMJC2G[i] < VC_MJC2G[1]) /*  5% sig. */
acertoMJC2G[1]++;
if(estaMJC2G[i] < VC_MJC2G[2]) /* 10% sig. */
acertoMJC2G[2]++;

/* Counting of model=beta when H0 is not rejected in MJ test */
if(estaMJC2G[i] < VC_MJC2G[0] && estaMJC2G[i] == estaJC2_BG[i])
acertomC2G[0]++;
if(estaMJC2G[i] < VC_MJC2G[1] && estaMJC2G[i] == estaJC2_BG[i])
acertomC2G[1]++;
if(estaMJC2G[i] < VC_MJC2G[2] && estaMJC2G[i] == estaJC2_BG[i])
acertomC2G[2]++;

/* NUMBER OF NON-REJECTIONS OF THE SCORE TEST */

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC2_GUE[i] < chi2[0]) /*  1% sig. */
acertoJC2GUE[0]++;
if(estaJC2_GUE[i] < chi2[1]) /*  5% sig. */
acertoJC2GUE[1]++;
if(estaJC2_GUE[i] < chi2[2]) /* 10% sig. */
acertoJC2GUE[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC2_BE[i] < chi2[0]) /*  1% sig. */
acertoJC2BE[0]++;
if(estaJC2_BE[i] < chi2[1]) /*  5% sig. */
acertoJC2BE[1]++;
if(estaJC2_BE[i] < chi2[2]) /* 10% sig. */
acertoJC2BE[2]++;

/* Number of non-rejections of the MJ test */
if(estaMJC2E[i] < VC_MJC2E[0]) /*  1% sig. */
acertoMJC2E[0]++;
if(estaMJC2E[i] < VC_MJC2E[1]) /*  5% sig. */
acertoMJC2E[1]++;
if(estaMJC2E[i] < VC_MJC2E[2]) /* 10% sig. */
acertoMJC2E[2]++;

/* Counting of model=beta when H0 is not rejected in MJ test */
```

```
if(estaMJC2E[i] < VC_MJC2E[0] && estaMJC2E[i] == estaJC2_BE[i])
acertomC2E[0]++;
if(estaMJC2E[i] < VC_MJC2E[1] && estaMJC2E[i] == estaJC2_BE[i])
acertomC2E[1]++;
if(estaMJC2E[i] < VC_MJC2E[2] && estaMJC2E[i] == estaJC2_BE[i])
acertomC2E[2]++;

/* J and MJ statistics based on likelihood ratio statistic  */

/* J1 test */
estaJC2_GUV[i] = 2*(irC2AmpGU[k+m+(l-1)][i]-irC2GU[k+m][i]); /* M_1 */
/* J2 test */
estaJC2_BV[i] = 2*(irC2AmpB[k+m+(l-1)][i]-irC2B[k+m][i]); /* M_2 */
/* MJ test */
estaMJC2V[i] = min(estaJC2_BV[i], estaJC2_GUV[i]);

/* Number of non-rejections of the J test (tested model: unit gamma) */
if(estaJC2_GUV[i] < chi2[0]) /*  1% sig. */
acertoJC2GUV[0]++;
if(estaJC2_GUV[i] < chi2[1]) /*  5% sig. */
acertoJC2GUV[1]++;
if(estaJC2_GUV[i] < chi2[2]) /* 10% sig. */
acertoJC2GUV[2]++;

/* Number of non-rejections of the J test (tested model: beta) */
if(estaJC2_BV[i] < chi2[0]) /*  1% sig. */
acertoJC2BV[0]++;
if(estaJC2_BV[i] < chi2[1]) /*  5% sig. */
acertoJC2BV[1]++;
if(estaJC2_BV[i] < chi2[2]) /* 10% sig. */
acertoJC2BV[2]++;

/* Number of non-rejections of the MJ test */
if(estaMJC2V[i] < VC_MJC2V[0]) /*  1% sig. */
acertoMJC2V[0]++;
if(estaMJC2V[i] < VC_MJC2V[1]) /*  5% sig. */
acertoMJC2V[1]++;
if(estaMJC2V[i] < VC_MJC2V[2]) /* 10% sig. */
acertoMJC2V[2]++;

/* Counting of model= beta when H0 is not rejected in MJ test */
if(estaMJC2V[i] < VC_MJC2V[0] && estaMJC2V[i] == estaJC2_BV[i])
acertomC2V[0]++;
if(estaMJC2V[i] < VC_MJC2V[1] && estaMJC2V[i] == estaJC2_BV[i])
acertomC2V[1]++;
if(estaMJC2V[i] < VC_MJC2V[2] && estaMJC2V[i] == estaJC2_BV[i])
acertomC2V[2]++;
}

/* descriptive statistics MJ - GRADIENT */
EJBC2G[0]=meanc(estaJC2_BG);
EJBC2G[1]=varc(estaJC2_BG);
EJGUC2G[0]=meanc(estaJC2_GUG);
EJGUC2G[1]=varc(estaJC2_GUG);
EMJC2G[0]=meanc(estaMJC2G);
EMJC2G[1]=varc(estaMJC2G);

/* descriptive statistics MJ - SCORE */
EJBC2E[0]=meanc(estaJC2_BE);
EJBC2E[1]=varc(estaJC2_BE);
EJGUC2E[0]=meanc(estaJC2_GUE);
EJGUC2E[1]=varc(estaJC2_GUE);
EMJC2E[0]=meanc(estaMJC2E);
EMJC2E[1]=varc(estaMJC2E);

/* descriptive statistics MJ - LR */

EJBC2V[0]=meanc(estaJC2_BV);
EJBC2V[1]=varc(estaJC2_BV);
EJGUC2V[0]=meanc(estaJC2_GUV);
EJGUC2V[1]=varc(estaJC2_GUV);
EMJC2V[0]=meanc(estaMJC2V);
EMJC2V[1]=varc(estaMJC2V);

/* PRINTING RESULTS */
```

```
if(poder==0)
{
if(NOBS==50){ file = fopen("Res_Sim5.1_n50.txt", "w"); }
if(NOBS==100){ file = fopen("Res_Sim5.1_n100.txt", "w"); }
if(NOBS==250){ file = fopen("Res_Sim5.1_n250.txt", "w"); }
}

/* Print results of test power*/
:imprimir

fprintln( file, " \n PROGRAMA OX:  ", oxfilename(0) );

if(poder==0) {fprintln( file,  " \n Simulacao 5.1: Modelos com dispersão variável -
Diferenca nos regressores do parâmetro de locação." ); }
if(poder==1) {fprintln( file,  " \n Simulacao 5.1 (PDT): Modelos com dispersão variável -
Diferenca nos regressores do parâmetro de locação." );}

fprintln( file,  " \n Teste MJ baseado na est. Gradiente, Escore e
Razão de verossimilhança." );
fprintln( file,  " \n MODELOS: BETA E GAMA UNITARIO " );
if(poder==1) {fprintln( file, " \n MODELO GERADOR DOS DADOS - JOHNSON S_B." );}
fprintln( file,  "\n \n TAMANHO DA AMOSTRA: ", NOBS );
fprintln( file,  "\n NUMERO DE REPLICAS MC:", NREP );

fprintln( file, "\n \n \t \t \t CASO I  - MODELO VERDADEIRO: GAMA UNITARIO");
fprintln( file, "\n PARÂMETROS:" );
fprintln( file, "%c",{" beta0:", " beta1:", " beta2:", " gama0:", " gama1:" }, "%r",
{"Verdaderos: ", "Estimativas: ", "Var Est Estimativas: "},
"%cf",{"%12.4f","%12.4f","%12.4f","%12.4f","%12.4f"}, (beta1 ~ gama1) |
(meanr(irC1GU[0:k+m-1][]))' | (varr(irC1GU[0:k+m-1][]))'   );
fprintln( file, "%c",{"Mínimo \t","Máximo \t"}, "%r", {"Média(mu)", "Dispersão (tau)"},
"%cf",{"%15.6f","%15.6f"}, (min(muGU) ~ max(muGU)) | (min(tau) ~ max(tau)) );

if(poder==0){fprintln( file, "\n lambda1_GamaUnit: ", max(tau)/min(tau) );
 fprintln( file, "\n lambda2_GamaUnit: ", max(var_GU)/min(var_GU) ); }

if(poder==1){ fprintln( file, "\n lambda1_PDT: ", max(tau)/min(tau) );
  fprintln( file, "\n lambda2_GamaUnit ", max(var_GU)/min(var_GU) );
  fprintln( file, "\n lambda2_GJSN: ", max(var_GJSN)/min(var_GJSN) ); }

fprintln( file, "\n \t NUM. REPLICAS MC SEM CONVERGENCIA: " );
fprintln( file, "%c",{"Normal \t","Ampliado \t"}, "%r", {"GamaUnit", "Beta"},
"%cf", {"%8.1f","%8.1f"}, (falhaC1MCGU|falhaC1MCB )~(falhaC1AmpMCGU| falhaC1AmpMCB));

fprintln( file, "\n \n RESULTADOS ESTATÍSTICA GRADIENTE:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Mínimo", "Máximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ((EJGUC1G|min(estaJC1_GUG)|max(estaJC1_GUG) ) ~
(EJBC1G|min(estaJC1_BG)|max(estaJC1_BG))~(EMJC1G|min(estaMJC1G)|max(estaMJC1G))));
fprintln( file, "\n  ");

fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUG", "TesteJBG","TesteMJG", "AcertoMJG" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
 ( 100*(1 - (acertoJC1GUG/NREP)) )' | (100*(1 - (acertoJC1BG/NREP)))'
| (100*(1 - (acertoMJC1G/NREP)) )' | (100*(acertomC1G./acertoMJC1G))' ); }

if(poder==1){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"PoderMJGUG", "PoderMJGUG_Q" },"%cf",{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
 (100*(1 - (acertoMJC1G/NREP)) )' | (100*(1 - (acertoMJC1_QG/NREP)) )'  ); }

fprintln( file, "\n Valor Critico MJ_G:   ", VC_MJC1G');

fprintln( file, "\n \n RESULTADOS ESTATÍSTICA ESCORE:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Mínimo", "Máximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ( (EJGUC1E|min(estaJC1_GUE) | max(estaJC1_GUE) ) ~
(EJBC1E|min(estaJC1_BE)|max(estaJC1_BE))~(EMJC1E|min(estaMJC1E)|max(estaMJC1E))));
fprintln( file, "\n  ");
```

```
fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUE", "TesteJBE","TesteMJE", "AcertoMJE" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
 ( 100*(1 - (acertoJC1GUE/NREP)) )' | (100*(1 - (acertoJC1BE/NREP)))' |
(100*(1 - (acertoMJC1E/NREP)) )' | (100*(acertomC1E./acertoMJC1E))' ); }

if(poder==1){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"PoderMJGUE", "PoderMJGUE_Q" },"%cf",{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
 (100*(1 - (acertoMJC1E/NREP)) )' | (100*(1 - (acertoMJC1_QE/NREP)) )'  ); }

fprintln( file, "\n Valor Critico MJ_E:   ", VC_MJC1E');


fprintln( file, "\n \n RESULTADOS ESTATÍSTICA RAZÃO DE VEROSSIMILHANÇA:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Minimo", "Maximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ( (EJGUC1V |min(estaJC1_GUV) | max(estaJC1_GUV) ) ~
(EJBC1V|min(estaJC1_BV)|max(estaJC1_BV))~(EMJC1V|min(estaMJC1V)|max(estaMJC1V) )));
fprintln( file, "\n  ");

fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUV", "TesteJBV", "TesteMJV", "AcertoMJV" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
( 100*(1 - (acertoJC1GUV/NREP)) )' | (100*(1 - (acertoJC1BV/NREP)))' |
(100*(1 - (acertoMJC1V/NREP)) )' | (100*(acertomC1V./acertoMJC1V))' ); }

if(poder==1){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"PoderMJGUV", "PoderMJGUV_Q" },"%cf",{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
 (100*(1 - (acertoMJC1V/NREP)) )' | (100*(1 - (acertoMJC1_QV/NREP)) )'  ); }

fprintln( file, "\n Valor Critico MJ_V:   ", VC_MJC1V');

/*Go to - finish of program after print results of test power */
if(poder==1)
{
goto termina;
}

fprintln( file, "\n \n CASO II - MODELO VERDADEIRO: BETA");
fprintln( file, "\n PARÂMETROS:" );
fprintln( file, "%c",{" beta0:", " beta1:", " beta2:", " gama0:", " gama1:" }, "%r",
{"Verdaderos: ", "Estimativas: ", "Var Est Estimativas: "},
"%cf",{"%12.4f","%12.4f","%12.4f","%12.4f","%12.4f"}, (beta2 ~ gama2) |
(meanr(irC2B[0:k+m-1][]))' | (varr(irC2B[0:k+m-1][]))'   );
fprintln( file, "%c",{"Mínimo \t","Máximo \t"}, "%r", {"Média(mu)", "Dispersão (phi)"},
"%cf",{"%15.6f","%15.6f"}, (min(muB) ~ max(muB)) | (min(phi) ~ max(phi)) );

fprintln( file, "\n lambda1_Beta: ", max(phi)/min(phi) );
fprintln( file, "\n lambda2_Beta: ", max(var_Beta)/min(var_Beta) );

fprintln( file,"\n \t NUM. REPLICAS MC SEM CONVERGENCIA: " );
fprintln( file, "%c",{"Normal \t","Ampliado \t"}, "%r", {"GamaUnit", "Beta"},
"%cf",{"%8.1f","%8.1f"},(falhaC2MCGU|falhaC2MCB)~(falhaC2AmpMCGU|falhaC2AmpMCB) );

fprintln( file, "\n \n RESULTADOS ESTATÍSTICA GRADIENTE:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Mínimo", "Máximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ((EJGUC2G|min(estaJC2_GUG) | max(estaJC2_GUG)) ~
(EJBC2G|min(estaJC2_BG) | max(estaJC2_BG)) ~ (EMJC2G |min(estaMJC2G) | max(estaMJC2G))));
fprintln( file, "\n  ");

fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUG", "TesteJBG","TesteMJG", "AcertoMJG" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
( 100*(1 - (acertoJC2GUG/NREP)) )' | ( 100*(1 - (acertoJC2BG/NREP)) )' |
```

```
( 100*(1 - (acertoMJC2G/NREP)) )' | ( 100*(acertomC2G./acertoMJC2G) )' ); }

fprintln( file, "\n \n RESULTADOS ESTATÍSTICA ESCORE:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Mínimo", "Máximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ((EJGUC2E|min(estaJC2_GUE) | max(estaJC2_GUE)) ~
(EJBC2E|min(estaJC2_BE) | max(estaJC2_BE)) ~ (EMJC2E |min(estaMJC2E) | max(estaMJC2E))));
fprintln( file, "\n  ");

fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUE", "TesteJBE","TesteMJE", "AcertoMJE" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
( 100*(1 - (acertoJC2GUE/NREP)) )' | ( 100*(1 - (acertoJC2BE/NREP)) )' |
( 100*(1 - (acertoMJC2E/NREP)) )' | ( 100*(acertomC2E./acertoMJC2E) )' ); }

fprintln( file, "\n \n RESULTADOS ESTATÍSTICA RAZÃO DE VEROSSIMILHANÇA:   ");
fprintln( file, "%c",{"Est J_GamaUnit \t", "Est J_Beta \t", "Est MJ \t"}, "%r",
{"Média","Variância", "Minimo", "Maximo"},
"%cf",{"%15.6f","%15.6f", "%15.6f"}, ( (EJGUC2V |min(estaJC2_GUV) | max(estaJC2_GUV) ) ~
(EJBC2V|min(estaJC2_BV)|max(estaJC2_BV))~(EMJC2V|min(estaMJC2V)|max(estaMJC2V) )));
fprintln( file, "\n  ");

fprintln( file, "TAXA DE REJEIÇÃO (%) DOS TESTES.  ");
if(poder==0){
fprintln( file, "%c",{" 1% \t"," 5% \t", " 10% \t"}, "%r",
{"TesteJGUV", "TesteJBV", "TesteMJV", "AcertoMJV" },"%cf",
{"%15.2f","%15.2f", "%15.2f", "%15.2f"},
( 100*(1 - (acertoJC2GUV/NREP)) )' | (100*(1 - (acertoJC2BV/NREP)))' |
(100*(1 - (acertoMJC2V/NREP)) )' | ( 100*(acertomC2V./acertoMJC2V) )' ); }

/*Go to - simulation of test power */
if(poder==0)
{
poder = 1;
goto regresa;
}

/*Finish program*/
:termina

fprintln( file, "\n DATA: ", date() );
fprintln( file, "\n HORA: ", time() );
fprintln( file, "\n TEMPO DE EXECUCAO: ", timespan(dExecTime));
    fclose(file);
}
```