UNIVERSIDADE FEDERAL DE PERNAMBUCO

DEPARTAMENTO DE ENGENHARIA DE PRODUÇÃO

PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA DE PRODUÇÃO

THAÍS LIMA CORRÊA

Q-WEIBULL GENERALIZED RENEWAL PROCESS WITH RELIABILITY APPLICATIONS

RECIFE/2017

THAÍS LIMA CORRÊA

Q-WEIBULL GENERALIZED RENEWAL PROCESS WITH RELIABILITY APPLICATIONS

Master thesis presented to UFPE for the master's degree attainment as part of the requirements of the Programa de Pós-Graduação em Engenharia de Produção (Concentration Area: Operations Research).

Advisor: Isis Didier Lins, DSc.

RECIFE/2017

Catalogação na fonte
Bibliotecária Valdicea Alves, CRB-4 / 1260

C824q Corrêa, Thaís Lima.
Q-weibull generalized renewal process with reliability applications /This LimaCorrêa - 2017.
92 folhas, Il. e Tabs.

Orientadora: Prof.ª Drª Isis Didier Lins.

Dissertação (Mestrado) – Universidade Federal de Pernambuco. CTG. Programa de Pós-Graduação em Engenharia de Produção, 2017.
Inclui: Referências e Apêndices.

1. Engenharia de Produção. 2.Generalized renewal process.
3. Kijima type I. 4. Kijima type II. 5. q-Weibull.6.Bathtub curve.
I.Lins,Isis Didier.(Orientadora). II. Título.

UFPE

658.5 CDD (22. ed.) BCTG/2017-144

UNIVERSIDADE FEDERAL DE PERNAMBUCO
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA DE PRODUÇÃO

PARECER DA COMISSÃO EXAMINADORA
DE DEFESA DE DISSERTAÇÃO DE
MESTRADO ACADÊMICO DE

THAÍS LIMA CORRÊA

"Q-WEIBULL GENERALIZED RENEWAL PROCESS WITH RELIABILITY APPLICATIONS"

ÁREA DE CONCENTRAÇÃO: PESQUISA OPERACIONAL

A comissão examinadora composta pelos professores abaixo, sob a presidência do(a) primeiro(a), considera o candidato **THAÍS LIMA CORRÊA,APROVADA.**

Recife, 21 de fevereiro de 2017.

Profª. ISIS DIDIER LINS,Doutora (UFPE)

Prof. CRISTIANO ALEXANDRE VIRGÍNIO CAVALCANTE, Doutor (UFPE)

Prof.ABRAÃO DAVID COSTA DO NASCIMENTO, Doutor (UFPE)

To my father, Max Corrêa.

ACKNOWLEDGMENTS

At first I would like to thank God, who encourages me every day.

Also, I would like to acknowledge my father, Max, who is always there for me. My mother, Yara, for the education provided. My brother, Renan, my source of rationality and serenity.

My friends, especially the ones from PPGEP, Caio, Everton, Gisele and Patrícia, for all the support.

My advisor, Isis Didier Lins, for the endless help during the development of this work and also for the opportunity of being part of CEERMA.

Professor Márcio, for the valuable contributions and discussions about this dissertation.

PPGEP and CAPES, for offering educational and financial support.

ABSTRACT

Generalized Renewal Process (GRP) is a probabilistic model for repairable systems that can represent any of the five possible post-repair states of an equipment: as new condition, as old condition, as an intermediate state between new and old conditions, a better condition and a worse condition. GRP is often coupled with the Weibull distribution to model the equipment failure process and the Weibull-based GRP is able to accommodate three types of hazard rate functions: monotonically increasing, monotonically decreasing and constant. This work proposes a novel approach of GRP based on the q-Weibull distribution, which has the Weibull model as a particular case. The q-Weibull distribution has the capability of modeling two additional hazard rate behaviors, namely bathtub-shaped and unimodal curves. Such flexibility is related to a pair of parameters that govern the shape of the distribution, instead of a single parameter as in the Weibull model. In this way, the developed q-Weibull-based GRP is a more general framework that can model a variety of practical situations in the context of reliability and maintenance. The maximum likelihood problems associated with the q-Weibull-based GRP using Kijima's virtual age type I and II for the failure and time terminated cases are developed. The probabilistic and derivative-free heuristic Particle Swarm Optimization (PSO) is used to obtain the q-Weibull-based GRP paramaters' estimates. The proposed methodology is applied to examples involving equipment failure data from literature and the obtained results indicate that the q-Weibull-based GRP may be a promising tool to model repairable systems.

Keywords: Generalized renewal process. Kijima type I. Kijima type II.q-Weibull.Bathtub curve.

RESUMO

O Processo de Renovação Generalizado (PRG) pode ser definido como um modelo probabilístico de sistemas reparáveis capaz de representar os cinco possíveis estados do sistema após o reparo: condição de um equipamento novo, condição de um equipamento antigo, um estado intermediário entre novo e antigo, melhor do que novo e pior do que antigo. O PRG costuma ser comumente empregado junto com a distribuição Weibull para a modelagem do processo de falhas dos equipamentos, no entanto, o modelo de GRP baseado na distribuição Weibull é capaz de considerar três comportamentos de taxa de falha: monotonicamente crescente, monotonicamente decrescente e constante. Este trabalho propõe uma nova abordagem para o PRG baseado na distribuição q-Weibull, que apresenta como um de seus casos particulares a distribuição Weibull. A distribuição q-Weibull apresenta a capacidade de modelar dois comportamentos de falha adicionais, denominadas curva da banheira e curva unimodal. Esta flexibilidade está relacionada a dois parâmetros que definem o formato da distribuição, ao invés de um único parâmetro como no caso da Weibull. Dessa forma, o modelo de PRG baseado na q-Weibull pode ser considerado uma estrutura mais geral de modelagem de uma variedade de situações práticas no contexto da confiabilidade e manutenção. São desenvolvidos estimadores de máxima verossimilhança para os casos de PRG baseada na distribuição q-Weibull sendo utilizadas as idades virtuais Kijima tipo I e II para os casos de dados censurados e não censurados. A heurística probabilística e livre de derivadas denominada Otimização via Nuvem de Partículas (*Particle Swarm Optimization* - PSO) é utilizada para obter os estimadores de máxima verossimilhança do modelo. O modelo proposto é aplicado a exemplos envolvendo falhas de equipamentos retirados da literatura e os resultados obtidos indicam que o PRG baseado na q-Weibull é uma ferramenta promissora na modelagem de sistemas reparáveis.

Palavras-chave: Processo de renovação generalizado. Kijima tipo I. Kijima tipo II. q-Weibull. Curva da banheira.

FIGURE LIST

TABLE LIST

CONTENT

1 INTRODUCTION

1.1 Opening Remarks

A repairable system, characterized as being restorable with no need of complete replacement, can reach different states: as new condition, as old condition, as an intermediate state between new and old conditions, a better condition and a worse condition. These states can be addressed to the repair action performed: perfect repair, whosesystem's failure intensity behaves as an intensity of a brand new system; minimal repair, which the system's failure intensity remain unaltered since the last failure; the imperfect repair, which is a general repair that represents a classification between the perfect and minimal repair and may include them; worse repair, characterized by a maintenance action that makes the system failure intensity increases but the system does not break down (Pham & Wang, 1996); better repair, related to a system failure intensity decrease.

There are different point processes that can be used to model repairable systems, which are related to important notions of reliability analysis such as repairs, spare stocks, maintenance, preventive maintenance, optimal preventive maintenance and availability (KAMINSKIY, 2013). In all of them, repair times are supposed negligible when compared to operational times.

Traditional probabilistic models in literature of repairable system analysis account for the states as good as new and as bad as old, which are modeled by renewal process (RP) and non-homogeneous Poisson process (NHPP), respectively. Nevertheless, these states are often exceptions rather than rule, from the standpoint of practical reliability engineering (WANG & YANG, 2012).

In this context, Kijima & Sumita (1986) have proposed a probabilistic model, named generalized renewal process (GRP), which is able to attend all the post-repair states due to the inclusion of the parameter of repair effectiveness. This parameter, denoted in this work as r, represents the post-repair states through the notion of virtual age (JACOPINO *et al.*, 2004). Kijima (1989) proposed two types of virtual age, one that compensates only the damage accumulated during the last time between failures – type I; and the other which compensate the damages since the beginning of equipment's operation – type II (WANG & YANG, 2012). GRP has been applied using times to failure assumed to be Weibull random variables

(YAÑEZ *et al.*, 2002; WANG & YANG, 2012; KAMINSKIY & KRIVTSOV, 2000; MOURA *et al*., 2007).

Although the Weibull distribution has been widely used along with GRP, the q-Weibull probability distribution appears as an interesting alternative to be used in the GRP. The q-Weibull is proposed as a distribution that smoothly interpolates the q-Exponential and the Weibull distributions in order to generate a unified framework to accommodate different cases of data adjustment (PICOLI *et al*., 2003). The Weibull distribution can handle monotonically decreasing, constant and monotonically increasing hazard rate functions, whereas, besides these three behaviors, the q-Weibull distribution can model two additional ones with a single set of parameters: unimodal and U-shaped (bathtub curve) (ASSIS *et al.*, 2013).

Using the q-Weibull distribution for reliability analysis is an important step towards an efficient approach to handle equipment failure process dismissing previous limitations in terms of modelling the whole failure intensity behavior, specifically when unimodal or bathtub-shaped ones are presented. This ability of the q-Weibull distribution along with the GRP enables the modelling and analysis of repairable systems according to more realistic conditions, so the q-Weibull GRP is expected to be a tool that allows decisions about reliability, maintenance planning and evaluation to be performed in a more accurate way.

In this context, this research proposes a q-Weibull-based GRP model considering both Kijima type I and type IImodels.The time to the occurrence of the first failure is distributed according to a q-Weibull model, while the subsequent times to failurefollow a conditional q-Weibull distribution, meaning that the arrival of a subsequent failure is conditional on the virtual age.

Model parameters are estimated by the maximum likelihood (ML) method, due to the good statistical properties ofthe resulting estimators. The obtained estimators through ML are approximately unbiased, its limit variance is nearly as small as the variance resulting from other estimators(MONTGOMERY, 2003).ML estimates are obtained for both failure terminated and timeterminated cases.

The ML framework, when applied to the q-Weibull-based GRP, results in an intricate system of first derivatives and the estimators are very difficult to be analytically obtained. Due to the complicated first derivatives, along with constraints over parameters' values to assure the model probabilistic validity, derivative-based optimization methods may fail.

Alternatively, derivative-free nature-based heuristics can be used in the quest for proper parameters' estimates. In this work, the chosen estimation procedure is to maximize the log-likelihood function by means of a Particle Swarm Optimization (PSO) algorithm.

PSO is a derivative-free probabilistic heuristic based on the social behavior of biological organisms which has as a basic element a particle that can fly throughout the search space of the problem toward an optimum using its own information and the information provided by other particles within its neighborhood (BRATTON & KENNEDY, 2007).PSO procedure can be used to solve problems where correlations between model parameters are high, sensitivity of the objective function to model parameters is low and the objective function is discontinuous (SCHWAAB *et al*., 2008). PSO has been used for parameter estimation (Schwaab *et al*., 2008; Prata *et al*., 2009; Santos *et al*., 2015; Carneiro *et al*., 2016); in reliability context along with Support Vector Machine to reliability prediction (Lins *et al*., 2012); to solve allocation problem in distribution systems (Ramadan *et al*., 2017), among others.

1.2 Justification

Reliability modeling brings more knowledge of the design and evaluation of a system since it allows the understanding and prediction the success-failure behavior of systems (JAISINGH *et al.,* 1987).The application of GRP permits the estimate of reliability, maintainability of repairable systems, eolic models, atmospheric phenomena and various probabilistic processes (JIMÉNEZ & VILLALÓN, 2006).

The q-Weibull distribution has its advantage on offering more flexibility by modelling different hazard rate's behaviors, especially the bathtub curve, which is widely used in reliability engineering and that can be modelled directly instead of using three Weibull models to represent the lifecycle of equipment. This q-Weibull's flexibility is related to the q parameter, which controls the shape of the distribution along with the β parameter, while the Weibull distribution has just β affecting its shape.

GRP is able to model different post-repair states, reducing the model uncertainty from the repair assumptions required by NHPP and RP. Thus, attaching q-Weibull distribution to GRP may be an interesting approach to suit the realistic cases that are observed in practical situations. Thus, the proposed model in this master thesis is an evolution in terms of reliability and maintenance analysis, since it combines the ability of GRP in handling different repair possibilities with the flexibility of the q-Weibull distribution in modeling monotone and non-monotone failure intensities.The q-Weibull GRP is able to analyze a more diverse range of equipment failure-repair behaviors if compared to NHPP, RP and Weibull-based GRP.

1.3 Objectives

1.3.1 General Objective

This dissertation aims to develop probabilistic models for GRP based on the q-Weibull distribution and to apply them to reliability-related data.

1.3.2 Specific Objectives

In order to achieve the general objective, some specific targets are defined:

- Exploration of works involving probabilistic models for repairable systems, mainly GRP and researches about q-Weibull distribution;
- Development of the maximum likelihood problem related to the q-Weibull GRP for failure and time terminated cases as well as for the Kijima types I and II.
- Implementation of PSO algorithm to obtain the maximum likelihood estimates for the q-Weibull GRP parameters for each of the previous cases (failure and time terminated, Kijima types I and II);
- Development of asymptotic confidence intervals for the model parameters;
- Application of the proposed models to real data associated with reliability of equipment;
- Comparison of the q-Weibull GRP models' outputs with real data and with Weibull-based GRP results by means of Monte Carlo simulation.

1.4. Dissertation layout

Besides this introductory chapter, this dissertation presents four additional chapters briefly described in this section.

Chapter 2 presents the theoretical background. At first, the q-Weibull distribution is explained. Then, the GRP is detailed and the PSO is commented. Finally, a literature review about is exposed.

In Chapter 3the proposed q-Weibull-based GRP is presented for the virtual ages type I and type II, failure terminated and time terminated cases. The PSO methodology used in order to achieve the maximum likelihood estimators, solving the maximum likelihood optimization problem is described.

In Chapter 4 the model application to failure data obtained from literature is presented and analyzed.

Finally, in Chapter 5a summary of the main results obtained in this master thesis is provided. Also, limitations and suggestions for future works are presented.

2 THEORETICAL BACKGROUND AND LITERATURE REVIEW

This chapter presents definitions and explanation about the main topics of this dissertation: q-Weibull probabilistic distribution, Generalized Renewal Process and Particle Swarm Optimization. Also, a literature review is presented.

2.1 q-Weibull Distribution

Generalizations of the Weibull distribution often share the base on the exponential framework (Marshall & Olkin, 1997, Mudholkar & Srivastava, 1993). Exponentials are commonly used in non-interacting or weakly interacting systems, while power-laws dominates statistical distributions of complex systems, for instance systems that exhibit long-range (spatial) interactions, long-term (temporal) memory, among others (ASSIS *et al.*, 2013).

System's failure usually has multiple and interacting causes, therefore a complex behavior can possibly appear. Power-law like expressions are expected to substitute exponentials in the statistical description for these cases. The power law behavior may appear in the upper tail of the distributions (PRIETO & SARABIA, 2017).

The q-distributions are appropriate to describe diverse systems due to its ability of exhibit heavy tails and model power law phenomena, characteristics of complex systems (PICOLI *et al.*, 2009). The q-distributions emerge from nonextensive formalism proposed by Tsallis, a generalization of Boltzmann-Gibbs-Shannon (BGS) entropy, introducing the possibility to extend statistical mechanics to complex systems in a coherent and natural way (ASSIS *et al.*, 2013; PICOLI *et al.*, 2009).

In this context, the q-Weibull distribution can be seen as natural step forward to the Weibull distribution in the light of nonextensive statistics, as it is derived from the substitution of the Exponential function by a q-Exponential in the classic Weibull model, represented in Equation (2.1) (ASSIS *et al.*, 2013):

$$f_q(t) = (2-q)\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]. \tag{2.1}$$

in which parameters β and q, control the shape of the distribution, whereas α is the scale parameter and $q < 2, \alpha, \beta > 0$.

The q-Exponential function can be defined as:

$$exp_q(x) = \begin{cases} (1+(1-q)x)^{\frac{1}{(1-q)}}, & \text{if } (1+(1-q)x) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Differently from the classic Weibull the q-Weibull has two parameters that affect its shape, thus leadingto more flexibilityin modeling additional hazard rate behaviors.

The support of Equation (2.) changes depending on the value of q:

$$t \in \begin{cases} [0,\infty), & \text{for } 1 < q < 2, \\ [0, t_{max}], & \text{for } q < 1, \end{cases} \tag{2.3}$$

where $t_{max} = \alpha/(1-q)^{1/\beta}$ is the maximum allowed time so as to preserve the probabilistic properties of Equation (2.2) when $q < 1$. For these values the integration of $f_q(t)$ diverges for $t > t_{max}$(ASSIS *et al.*, 2013).

The q-Weibull cumulative distribution and reliability function are given by Equations (2.42.) and (2.5), respectively:

$$F_q(t) = 1 - \left[exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]^{2-q} = 1 - \left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{\frac{2-q}{1-q}}. \tag{2.4}$$

$$R_q(t) = \left[exp_q\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]^{2-q} = \left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{\frac{2-q}{1-q}}. \tag{2.5}$$

Assis *et al.* (2013) list the combination of β and q values representing the various types of hazard rate function behaviors that can be reproduced by the q-Weibull distribution (Table 2.1): monotonically decreasing, constant, monotonically increasing, unimodal and U-shaped (bathtub curve), which are illustrated in Figure 2.1. The hazard rate function is given by Equation (2.6)

$$h_q = \frac{f_q}{R_t} = (2-q)\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t}{\alpha}\right)^{\beta}\right]^{-1}. \tag{2.6}$$

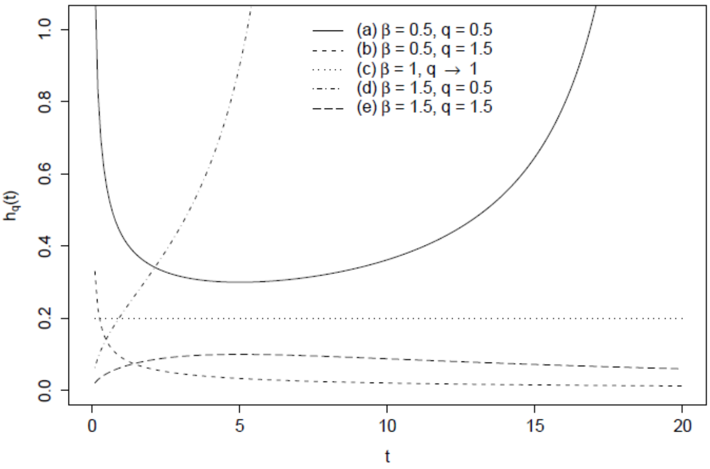
The q-Weibull distribution has other probability distributions as special cases: when $\beta = 1$, a q-Exponential distribution; for $q \to 1$, a Weibull distribution; for both $\beta = 1$ and $q \to 1$, an Exponential distribution. Figure 2.2 illustrates the coverage of q-Weibull distribution considering its special cases distributions and possible hazard rate behaviors.

Table 2.1 - Behaviors of hazard rate for the q-Weibull distribution

	$\boldsymbol{0 < \beta < 1}$	$\boldsymbol{\beta = 1}$	$\boldsymbol{\beta > 1}$
$\boldsymbol{q < 1}$	Bathtub curve	Monotonically increasing	Monotonically increasing
$\boldsymbol{q = 1}$	Monotonically decreasing	Constant	Monotonically increasing
$\boldsymbol{1 < q < 2}$	Monotonically decreasing	Monotonically decreasing	Unimodal

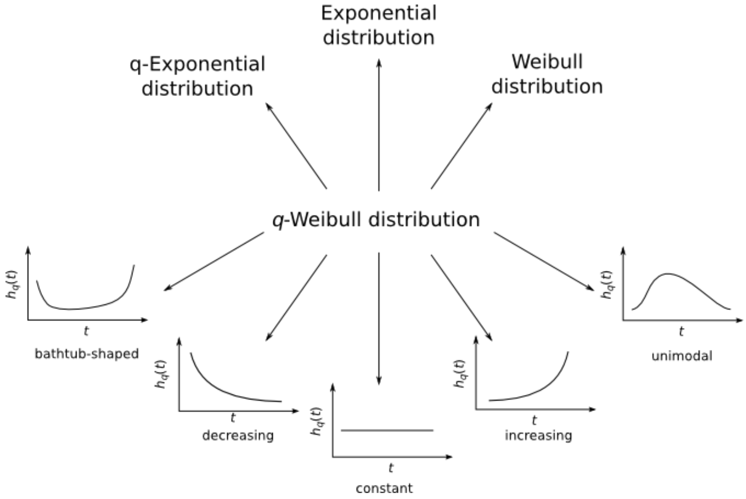
Source: Assis et al. (2013, p. 732)

Figure 2.1- Failure Intensity Behaviors Modelled By q-Weibull



Source: This research

Figure 2.2 - q-Weibull's Distribution Coverage



Source: This research

2.2 Generalized Renewal Process

Different point processes can be applied to model repairable systems and in this context often the time of repair is considered negligible when compared with the times between failures. This assumption enables applying different point processes as appropriate models for real life failure processes.

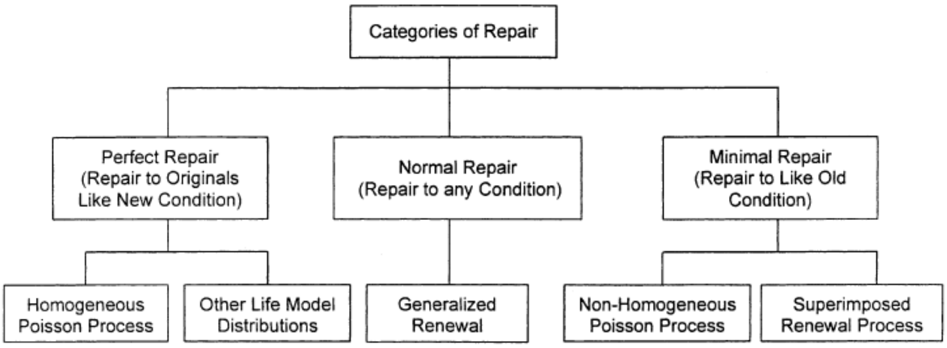
A point process can be defined as a mathematical model for events distributed randomly in time, and this notion plays the same role in the reliability of repairable systems as the notion of time to failure in the context of reliability of non-repairable systems. The major random variable of interest tothese processes is the number of failures observed in some time interval (0, t] (KAMINSKIY, 2013).

The GRP, probabilistic model proposed by Kijima & Sumita (1986), is able to incorporate the five post-repair states that a repairable system may assume. The models that have been mostly used in the reliability analysis of repairable systems are the RP and the NHPP, which can be considered particular cases of GRP (YAÑEZ *et al.*, 2002). However, RP and NHPP assume simplifying hypotheses which restrict its application to realistic cases.

The RP assumes that the failures are independent and identically distributed.Therefore, the system returns to an as new condition that may only occur when the system is completely replaced after the failure, resembling to non-repairable systems. In the NHPP, in turn, the time between failures follows a conditional Exponential probability distribution, meaning that the arrival of the ith failure is conditional on the cumulative operation time up to the failure $i - 1$. It is assumed that the system condition after repair is the same as the one immediately before occurrence of ith failure.

In contrast to the NHPP, the GRP considers a conditional probability function based on the system's virtual age, therefore the time to the next failure is related to the virtual age, instead of the real age used in NHPP. By covering major repair assumptions encountered in practice, GRP provides more flexibility in modeling real life failure occurrence processes (KAMINSKIY & KRIVTSOV, 2000). Figure 2.3 presents a categorization of stochastic point processes for modelling repairable systems.

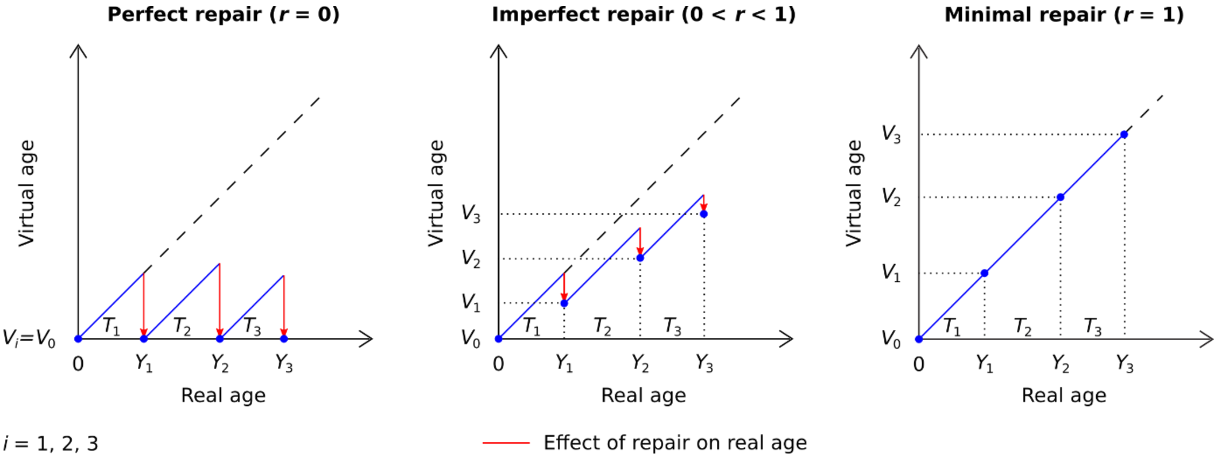
Figure 2.3 - Categories of Stochastic Point Processes for Repairable systems



Source: Yañez et al. (2002, p. 168)

GRP presents the concept of virtual age, which is illustrated in Figure 2.4. If Y_i and V_i, in Figure 2.4, are the equipment's calculated age before and after repair, respectively, and T_i is the chronological time, it is possible to verify the relation between virtual and real ages of the system, according to parameter r, defined as the repair effectiveness (MOURA *et al.*, 2007).

Figure 2.4 - Virtual Age and the Repair Effectiveness Parameter



Source: Adapted from Moura et al. (2007)

Pham & Wang (1996) classify the maintenance according to the degree to which the operating conditions of an item is restored. The perfect repair/maintenance is related to a system that after a repair has the same lifetime distribution and failure intensity as a brand new, generally there is a replacement of the failed item. The minimal repair/maintenance restores the system to the failure intensity it had just before it failed. The imperfect repair/maintenance is a general repair that can include the two extreme cases of minimal and perfect repair.

The values of the parameter r can be seen as an index for representing effectiveness and quality of repair. Assuming $r = 0$ leads to an RP (like new condition), while $r = 1$ leads to an NHPP (like old condition). The intermediate values $0 < r < 1$ lead to an intermediate condition between like old and like new ones and is related to imperfect repair (PHAM & WANG, 1996). When $r > 1$, the equipment's condition is worse than old; and when $r < 0$, the equipment's associated condition is better than new (MOURA, 1997; YAÑEZ *et al*, 2002).

According to Kijima (1989), two models can be constructed depending on how the repair activities affect the virtual age process. In the first model, it is assumed that the ith repair cannot remove the damages incurred before the $(i-1)$th repair. The virtual age type I is given by Equation (2.7):

$$V_i = V_{i-1} + rT_i. \tag{2.7}$$

In the second model, virtual age type II, at the ith failure the virtual age has been accumulated to $V_{i-1} + T_i$, which means that the repair can compensate the accumulated damage, as defined in Equation (2.8):

$$V_i = r(V_{i-1} + T_i). \tag{2.8}$$

Kijima type I assumes that the ith repair can only compensate for the damage accumulated during the period of time between the ith and $(i-1)$th failure, while Kijima type II assumes that the repair can compensate the system damage since the beginning of its operation (WANG & YANG, 2012). Jacopino *et al.* (2004) recommends that Kijima type II GRP should be used for complex systems, while individual components should be modelled using Kijima type I GRP model. Complex systems can be considered as a number of sub-systems and components which are purely mechanical, electrical/electronic or a hybrid of both elements such as, respectively, a landing gear, the flight computer and control actuators of an aircraft which combine electronic controllers and mechanical actuators. Therefore, the selection of the model is related to the physical failure modes that the component or system reaches over the number of renewals performed.

2.3 Maximum Likelihood Method

According to Montgomery (2003), the maximum likelihood method, proposed by Fisher, is one of the best methods of obtaining a point estimator of a parameter and can be used in situations where there are several unknown parameters to estimate. The optimal estimator that maximizes the likelihood functionis obtained by equating the first order partial derivatives to zero and solving the resulting system of equations.

Consider X as a random variable with probability density function$f(x|\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of parameters. Let $x_1, x_2, \dots, x_n$be the observed values in a random sample of size n. The likelihood function of the sample is given by:

$$L(\boldsymbol{\theta}|x) = f(x_1|\boldsymbol{\theta}) \cdot f(x_2|\boldsymbol{\theta}) \cdot \dots \cdot f(x_n|\boldsymbol{\theta}). \tag{2.9}$$

Under general conditions, when the sample size n is large and $\widehat{\boldsymbol{\theta}}$ is the maximum likelihood estimator of $\boldsymbol{\theta}$, the properties of $\widehat{\boldsymbol{\theta}}$ are that $\widehat{\boldsymbol{\theta}}$ is an approximately unbiased estimator for $\boldsymbol{\theta}$; the variance of $\widehat{\boldsymbol{\theta}}$ is nearly as small as the variance that could be obtained from other estimator and $\widehat{\boldsymbol{\theta}}$ has an approximate normal distribution.

Although the maximum likelihood estimator is one of the most applied techniques, complications may occur in its use. Sometimes it is not easy to maximize the likelihood function because the partial derivatives obtained can be difficult to solve. In addition, it may not always be possible to determine the maximum of the likelihood function directly using calculus methods.

2.3.1 Failure Terminated and Time Terminated GRP Likelihood Functions

Failure terminated cases are the occasions when failure data are available up to the time of the last failure occurrence. Considering that the first failure does not attend to the conditional probability function, then, the likelihood function is given by:

$$L_{FT} = f(t_1) \prod_{i=2}^{n} f(t_i|v_{i-1}). \tag{2.10}$$

Time terminated cases are related to an estimation at a time $t$after the occurrence of the last failure and before the next failure happens. In this case,a term related to the conditional reliability of the system at a time $t$is included. The maximum likelihood function is given by:

$$L_{TT} = f(t_1)\ \prod_{i=2}^{n} f(t_i|v_{i-1})R(t|v_n). \tag{2.11}$$

2.3.2 Variability Assessment of the Maximum Likelihood Estimators

Asymptotic confidence intervals are developed to evaluate the precision of the obtained maximum likelihood parameters' estimates of the proposed GRP based on the q-Weibull distribution. Asymptotic intervals rely on the central limit theorem, which means that they are suited for large samples and for small ones they may present inconsistent results.

Considering θas a parameter and $\hat{\theta}$as its estimator, if $\hat{\theta}$ has an approximate normal distribution, is approximately unbiased for θ and has standard error $\sigma_{\hat{\theta}}$ that can be estimated from the sample data, the quantity $(\hat{\theta} - \theta)/\sigma_{\hat{\theta}}$ has an approximate standard normal distribution. Thus, an approximate confidence interval (CI) for θ with $(1-\gamma)\cdot 100\%$of confidence is given by Equation (2.12) (MONTGOMERY, 2003).

$$CI[\theta, (1-\gamma)\cdot 100\%] = \left[\hat{\theta} + z_{\frac{\gamma}{2}}\sigma_{\hat{\theta}}; \hat{\theta} + z_{1-\frac{\gamma}{2}}\sigma_{\hat{\theta}}\right] \tag{2.12}$$

Maximum likelihood estimators usually present the necessary characteristics to use Equation (2.12) and the process of obtaining the confidence intervals are detailed thereafter.Asymptotic confidence intervals can be constructed for the parameters using the asymptotic normality property of maximum likelihood estimators. For the q-Weibull GRP, the asymptotic confidence intervals with $(1-\gamma)\cdot 100\%$ of confidence for α, β, q, r are given by, respectively:

$$CI[\alpha, (1-\gamma)\cdot 100\%] = \left[\hat{\alpha} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{11}}; \hat{\alpha} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{11}}\right], \tag{2.13}$$

$$CI[\beta, (1-\gamma)\cdot 100\%] = \left[\hat{\beta} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{22}}; \hat{\beta} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{22}}\right], \tag{2.14}$$

$$CI[q, (1-\gamma)\cdot 100\%] = \left[\hat{q} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{33}}; \hat{q} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{33}}\right], \tag{2.15}$$

$$CI[r, (1-\gamma)\cdot 100\%] = \left[\hat{r} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{44}}; \hat{r} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{44}}\right], \tag{2.16}$$

where $z_{\frac{\gamma}{2}}$, $z_{1-\frac{\gamma}{2}}$ are the $\frac{\gamma}{2}$ and $1-\frac{\gamma}{2}$ quantiles of the standard normal distribution and $\sqrt{\widehat{var}_{11}}$, $\sqrt{\widehat{var}_{22}}$, $\sqrt{\widehat{var}_{33}}$ and $\sqrt{\widehat{var}_{44}}$ are the diagonal elements of the covariance matrix associated with the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{q}, \hat{r}$presented in Equation 2.17.

$$\widehat{var}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t) = I^{-1}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t) \tag{2.17}$$

$$= -\begin{bmatrix} \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\alpha^2} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\alpha\partial\beta} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\alpha\partial q} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\alpha\partial r} \\ \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\beta\partial\alpha} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\beta^2} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\beta\partial q} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial\beta\partial r} \\ \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial q\partial\alpha} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial q\partial\beta} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial q^2} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial q\partial r} \\ \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial r\partial\alpha} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial r\partial\beta} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial r\partial q} & \frac{\partial^2\mathcal{L}(\hat{\alpha},\hat{\beta},\hat{q},\hat{r}|t)}{\partial r^2} \end{bmatrix}^{-1}.$$

The covariance matrix is approximated by the negative of the inverted Hessian (Millar, 2011), which is related to the second derivatives of the log-likelihood function with respect to each of the q-Weibull GRP parameters. The second-order derivatives will be detailed for each case of the q-Weibull GRP – failure terminated and time terminated using virtual ages Kijima type I and type II – in AppendicesA toD.

2.4 Particle Swarm Optimization

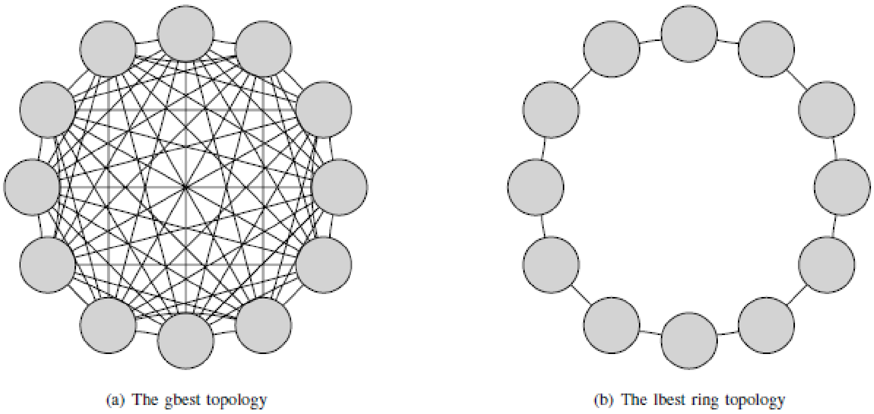
PSO is a probabilistic optimization heuristic inspired by the social behavior of biological organisms, specifically the ability of animal groups to work as a whole in order to find some desirable position. This seeking behavior is artificially modeled by PSO, which has been mainly used in the quest for solutionsfor non-linear optimization problems in a real-valued search space (BRATTON & KENNEDY, 2007).

For a problem with n-variables, each possible solution can be considered as a particle with a position vector of dimension n and the population of particles is defined as swarm (SAMANTA & NATARAJ, 2009). Each particle, represented by j, $j = 1, \dots, n_{part}$ is composed by the following features: current position in the search space (s_j), best position it has visited so far (p_j), velocity (v_j) and fitness, which is the value of the considered objective function (BRATTON & KENNEDY, 2007)

In PSO, particles move through the search space accordingly to the combination of the best solution they individually found and the best solution that any particle in the neighborhood found. A neighborhood can be defined as the subset of particles with which a given particle is able to communicate.

There are two approaches for the PSO algorithm, *gbest*and*lbest*, presented in Figure 2.5, according to the ability of particle’s communication with its neighborhood. In the case where every particle can communicate with every other particle, the *gbest* approach is adoptedand the number of particle’s neighbors is equal to the total number of particles.The *lbest* approach is adoptedwhen a particle can obtain information only from a subset of particles, thus the number of particle’s neighbors is less than the total number of particles in the swarm.

Figure 2.5 - PSO topologies



(a) The gbest topology (b) The lbest ring topology

Source: Adapted from Bratton & Kennedy (2007)

During the iterative process, the swarm evolution occurs as every particle has the velocity and position update equations applied to each dimension k, with $k = 1, \dots, n$(SHI & EBERHART, 1998; BRATTON & KENNEDY, 2007):

$$v_{jk}(m+1) = \chi\{v_{jk}(m) + c_1u_1\left[p_{jk}(m) - s_{jk}(m)\right] + c_2u_2\left[p_{gk}(m) - s_{jk}\right]\}, \tag{2.18}$$

$$s_{jk}(m+1) = s_{jk}(m) + v_{jk}(m+1), \tag{2.19}$$

where m is the iteration number, χ is the constriction factor that avoids velocity explosion during iterations, c_1 and c_2 are positive constants, u_1 and u_2 are independent uniform random numbers between 0 and 1, p_{gk} is the kth entry of vector p_g related to the best position that has been found by any neighbor of particle j. The updates of velocities and positions happen until a stop criterion is met.

During PSO iterations, particles may go outside the search space, then become infeasible. To confront that, the "let particles fly" procedure is adopted: their velocities and positions remain unaltered and the fitness evaluation step is skipped to avoid infeasible particles assuming best but non-acceptable positions. Thereby, infeasible particles may be drawn back to the search space by the influence of their own best or their neighborhood's and the algorithm performance is not affected (LINS *et al.*, 2012; BRATTON & KENNEDY, 2007). Details of the PSO algorithm used in this master thesis are given in Section 3.3.

2.5 Literature Review

GRP has been used for repairable system analysis, providing quantitative metrics for system evaluation andmaintenance planning.Various researches about GRP has utilized the Weibull distribution as the probability distribution for the times between failures, although modifications have been performed to the Weibull distribution in order to allow for non-monotone hazard rate functions.

Yañez *et al*. (2002), developed the maximum likelihood estimators of the parameters of the Weibull based GRP using virtual age Kijima type I and also proposed a Bayesian approach to estimate the parameters in cases where limited equipment failuredata is available.

Wang & Yang (2012) explore GRP model using Kijima type I and type II virtual ages through a nonlinear constrained programming numerical method and analyzes complex repairable systems with conditional Weibull distribution.

Ferreira *et al.* (2015) affirms that the traditional GRP modelling does not allow the weight of both models of Kijima type I and II in the analysis and historical data might involve an impact between short and long memory-impact interventions, referred to Kijima type I and II respectively.

To confront this, the authors developed a GRP model based on the Weibull distribution through a perspective of a mixed GRP, where the virtual age is constructed by coupling Kijima type I and type II virtual ages. This idea is justified by the different classes of intervention that the systems frequently suffer in a real context and the model proposed allows comparing the quality of the existing intervention types.

The coupling of the virtual ages is performed by appending the class label of each intervention and the respective coefficient in a linear combination between Kijima type I and II models using a variable that belongs to the interval [0,1]. If this variable assumes the value 1, it may characterize a corrective intervention and Kijima type I is applied, if it assumes 0, it may characterize a preventive intervention and Kijima type II is used and finally if it assumes a value between 0 and 1 the influence of the intervention impacts on the history of the system.

The adherence of the GRP model to the available performance data set plays the role for adequate support decision (OLIVEIRA et al, 2016). Although the Weibull distribution is widely used along with GRP, the authors of these researches have inadvertently assumed the Weibull distribution. Therefore, the authors perform a goodness of fit test, GOFT, of the Weibull-based GRP by transforming the times between interventions and the respective virtual ages via a power law function. The transformation generalizes the relationship between Weibull variables with scale and shape, α and β, and exponential ones in order to apply a GOFT for an exponential distribution.

Jiménez & Villalón (2006) developed a MAPLE coded algorithm to solve the maximum likelihood parameter estimation of the Weibull based GRP and applied the model to an electrical and eolic system.

Moura *et al*. (2014) developed a combined approach between GRP and the intensity proportional repair alert in order to turn the traditional GRP able to distinguish how different types of maintenance influence each other in addition to the already existent ability of capture the quality of the maintenance actions performed through the evaluation of the rejuvenation parameter.The competing risks is used to identify how preventive maintenance can modify the distribution of the time between corrective maintenances, how corrective maintenances can change the frequency of failures and the GRP is used to incorporate the possibility of an imperfect repair condition.

The above-mentioned works present different GRP models, but all of them have the characteristic of being based on times between failures governed by the Weibull probability distribution.

Modelling equipment’s through GRP enables decisions about maintenance planning and system evaluation:Pham & Wang (1996) relates directly the maintenance to the equipment condition in order to establish maintenance planning strategies;Stadje & Zuckerman (1992) elaborate optimal maintenance strategies that accounts to the possibilities of actions related to general maintenance/repair; Makis & Jardine (1993) formulate an optimal replacement policy through a replacement model with general repair; Kobbacy & Jeon (2002) uses the renewal process in order to model a preventive maintenance scheduling model.

Probabilistic approaches to handle non-monotone hazard rate functions have been proposed adapting the already stablished models of NHPP and RP, modifying the hazard rate function, adopting different virtual ages during the equipment's lifetime

Dijoux (2009) proposed a reliability model for complex systems that concerns a bathtub shaped ageing and imperfect maintenance. The models that consider perfect and minimal repair are suitable for reliability models with increasing initial failure intensity, but in practice the failure behavior can be decreasing or even present the bathtub shaped intensity. The author presents a model based on virtual age when the bathtub curve appears on the initial intensity.

The model makes an adaptation of the usual assumption on virtual ages, since the reduction of the Kijima virtual age leads to efficient maintenance only for degrading systems. For each period of the bathtub curve, a different virtual age is used. For the burn-in period, the minimal repair is considered; during the useful life and the wear-out period, Kijima virtual age I is used; and during the interfailure time overlapping the burn-in period and the useful lifethe reduction of age is assumed to be proportional to the time elapsed from the time of the end of the burn-in period.ML estimation is performed in order to obtain the parameters' estimates of the model. Due to the large number of parameters to be estimated, it is not possible to achieve consistent results.

Jiang *et al.* (2003) proposed a way to determine the aging property of a given unimodal failure rate model that can be extended to study other non-monotonic failure rate models. By doing this, the effectiveness of a burn-in or preventive maintenance procedure for a product whose lifetime can be represented by a unimodal failure rate can be judged.

The failure pattern of many products or systems can be represented by the bathtub curve, that includes an early failure phase represented by a decreasing failure rate; a normal use phase which is related to a constant failure rate and the wear-out phase that presents an increasing failure rate (Jiang *et al.*, 2003). Otherwise, when failures are caused by factors such as fatigue and corrosion, the failure times often follow models with the unimodal or reverse bathtub-shaped hazard rate behavior.

Considering a unimodal hazard rate behavior and $t_c$its mode, denoted as critical time. The failure rate function consists of two parts: an increasing part over the interval (0, t_c) and a decreasing part over the interval (t_c, ∞).The unimodal hazard rate can be seen as approximately decreasing if t_c is small, then the system can be defined as anti-aging; approximately increasing if t_c is large, the then the system can be classified as aging; and approximately constant if $t_c$is a midterm between the previous mentioned and when the curve is relatively flat. The quantitative analysis is made through the concept of aging intensity, which is defined as the ratio of instantaneous failure rate and a baseline future rate. Along with the notion of peakedness, which the smaller it is, the closer to a constant failure rate the unimodal failure rate is, depending on the model parameters, a unimodal can be related to the behaviors above-mentioned.

Spinato *et al.* (2009) presents a model for reliability analysis of wind turbines based on the concept of the bathtub shaped curve for a repairable system and the Power Law Process (PLP), its mathematical formulation.

In the proposed model a reliability growth analysis is performed based on the PLP, a specific case of NHPP, to analyze reliability of three subassemblies: generator, gearbox and converter. The PLP is used because of its flexibility to represent the three phases of the bathtub curve. The choice of the equipment was based on its crucial role in the operation of wind turbines. Through the results of the improving reliability of generators and converters, the operations and maintenance activities can be seen as effective and some other observations can be taken to improve some aspects of the machines. Although the reliability lead to conclusions about the activities performed by operations and maintenance, the NHPP model does not consider the imperfect repair, which could represent a quantitative evidence of the repair's effectiveness.

Wang *et al.* (2002) developed a model that is concerned with the behavior of hazard rate based on the failure mechanisms along the whole range of bathtub curve. The hazard rate function is composed by a summation of the terms based on unpredictable failure occurrence due to the intrinsic weakness or/and sudden changes in the environmental conditions, cumulative damage, machine interference and adaptation.

Jaisingh *et al.* (1987) proposed a model to predict the reliability of nonrepairable systems with uncensored data that present its hazard rate behavior as a bathtub curve through ML estimators and the least-squares methods.

Guida & Pulcini (2009) proposed the Bounded Bathtub Intensity Process (BIPP), a four-parameter NHPP model to analyze systems that experiences early failures, degradation phenomena and operating time so long that the intensity function approaches a finite asymptote.

The q-distributions, such as q-Exponential, q-Gaussian and q-Weibull, have been applied to various problems in the interdisciplinary field of complex systems (PICOLI *et al.*, 2009). These distributions have been used to model variables in different contexts: basketball baskets, cyclone victims, time to breakdown, brand name drugs by retail sales, highway length (NADARAJAH & KOTZ, 2006).

Complex systems have been successfully described by q-distributions, such as cyclones (Reynolds & Veneziani, 2004), gravitational systems (Abreu *et al*., 2014), stock market (Ivanov *et al*,2014; Gu *et al*., 2014), journal citations (Anastasiadis *et al*. 2010),cosmic rays (Beck, 2004), earthquakes (Vallianatos *et al*., 2014), financial markets (Namaki *et al*, 2013), internet (Li *et* al., 2006), mechanical stress (Vallianatos &Triantis, 2013), reliability (Sales Filho, 2016).

The q-Weibull distribution can be adopted to reliability issues such as stress-strengh analysis, optimal preventive maintenance problems, optimal system design and competitive risks (Xu *et al*., 2017).It has been used in reliability context to describe time-to-breakdown data of electronic devices (Costa *et al.*, 2006), failure rate of a compression unit in a natural gas recovery plant (Sartori *et al*., 2009), failure data of components of oil wells: oil pumps, pumping rods and production tubings (Assis *et al.*, 2013).

In order to achieve MLestimates for the q-Weibull parameters, Xu *et al.* (2017) proposed an adaptative hybrid artificial bee colony (AHABC) algorithm and Lins *et al.*(2015) used PSO.

Considering the limitations of the Weibull distribution in modelling non-monotone failure intensities and the different possibilities of approaches proposed in literature to model the unimodal and bathtub curve behavior, which needs many variables and considerations to be incorporated to the model, bringing complexity, the q-Weibull represents an efficient and simple alternative due to its three parameters and its suitability to model complex systems, such as equipment's failure occurrence. Additionally, the q-Weibull distribution has already been successfully used to model equipment lifetime data, representing a natural choice to be attached to GRP.

3 PROPOSED Q-WEIBULL GENERALIZED RENEWAL PROCESS

In this section, the basis to model the q-Weibull GRP is presented. Both Kijima types I and II approaches are considered. In order to estimate the model parameters and to obtain the related asymptotic confidence intervals, the maximum likelihood problem is formulated for the q-Weibull GRP and adapted for each Kijima model and for failure terminated and time terminated data.Additionally, a Monte Carlo procedure is devised to obtain the simulated expected number of failures within a given time interval, based on the parameters' estimates, so as to be compared with experimental data.

3.1 Conditional Probability Functions

The proposed q-Weibull GRP is based on the definition of conditional probability:

$$P(T \leq t|T > t_1) = \frac{F(t) - F\ (t_1)}{R(t_1)} = \frac{1 - R(t) - 1 + R(t_1)}{R(t_1)} = 1 - \frac{R(t)}{R(t_1)}. \tag{3.1}$$

Where $F(\cdot)$ and $R(\cdot)$ are, respectively, the cumulative probability functionfor failure times and the reliability function. Assuming a q-Weibull distribution, Equation (3.1) turns into:

$$F(t_i|v_{i-1}) = 1 - \left\{\frac{\left[1-(1-q)\left(\frac{t_i+v_{i-1}}{\alpha}\right)^{\beta}\right]}{\left[1-(1-q)\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]}\right\}^{\frac{2-q}{1-q}}. \tag{3.2}$$

The conditional q-Weibull probability density function is:

$$f(t_i|v_{i-1}) = (2-q)\frac{\beta}{\alpha^{\beta}}(t_i)^{\beta-1}\left[1-(1-q)\left(\frac{t_i+v_{i-1}}{\alpha}\right)^{\beta}\right]^{\frac{1}{1-q}} \left[1-(1-q)\left(\frac{v_{i-1}}{\alpha}\right)^{\beta}\right]^{\frac{q-2}{1-q}}. \tag{3.3}$$

Equations (3.1)-(3.3) are valid for the subsequent $i - 1$ observations after the first failure occurrence.

When Kijima virtual age type I (Equation 2.7) is introduced to Equation (3.3), it becomes:

$$f(t_i|v_{i-1}) = (2-q)\frac{\beta}{\alpha^\beta}\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^\beta\right]$$
$$\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^\beta\right]^{q-2}. \tag{3.4}$$

The same can be performed to Kijima virtual age type II (Equation 2.8). Using the expression, the Equation (3.3) turns:

$$f(t_i|v_{i-1}) = (2-q)\frac{\beta}{\alpha^\beta}\left(t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}\right)^{\beta-1}$$
$$\left[1-(1-q)\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right]\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right]^{q-2}. \tag{3.5}$$

The failure intensity considering the conditional probabilities is obtained for both models Kijima type I and II are given by Equations (3.6) and (3.7), respectively:

$$h_q = \frac{f_q}{R_t} = (2-q)\frac{\beta}{\alpha^\beta}\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^\beta\right]^{-1}, \tag{3.6}$$

$$h_q = \frac{f_q}{R_t} = (2-q)\frac{\beta}{\alpha^\beta}\left(t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}\right)^{\beta-1}$$
$$\left[1-(1-q)\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right]^{-1}. \tag{3.7}$$

3.2 q-Weibull GRP Maximum Likelihood Problem

The q-Weibull GRP maximum likelihood problemsare developed for failure terminated and time terminated data using the conditional q-Weibull density functions obtained for both virtual ages. They are presented inTables 3.1 and 3.2, respectively.The constraints utilized for the developed optimization problems are related to the validity of the q-Weibull probabilistic model: (i) non-negativity of the q-Weibull parameters (otherwise, negative probability densities would be obtained contradicting their definition); (ii) non-negativity of the argument of q-Exponential function.

The procedure adopted to find the maximum likelihood estimators would be to differentiate the log-likelihood function obtained from the maximum likelihood function with respect to each of the parameters, make the derivatives equal to zero and solve the resulting system of equations. However, the resulting system (AppendicesA toD) involves intricate nonlinear equations and analytical expressions for the estimators are very difficult to be obtained. Therefore, in this work, a constrained optimization method based on PSO heuristic is adopted.

Table 3.1 - Failure Terminated Optimization Problems

Failure Terminated	
Constraints	
$(2-q) > 0,$ (3.16) $\alpha, \beta > 0,$ (3.17) $\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right] \geq 0, \forall i,$ (3.18) $t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j} \geq 0, \forall i,$(3.19) $\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right], \forall i.$ (3.20)	$(2-q) > 0,$(3.10) $\alpha, \beta > 0,$(3.11) $\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \geq 0, \forall i,$(3.12) $t_i + r\sum_{j=1}^{i-1} t_j \geq 0, \forall i,$ (3.13) $\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right], \forall i.$(3.14)

Virtual age model	Objective function
Kijima type I	$\max_{\alpha,\beta,q,r} (-n\beta)\ln\alpha + n\,(\ln\beta + \ln(2-q)) + \sum_{i=1}^{n}\left\{\frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + (\beta-1)\ln\left(t_i + r\sum_{j=1}^{i-1} t_j\right)\right\} + \sum_{i=2}^{n}\frac{(q-2)}{(1-q)}\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right], \quad (3.9)$
Kijima type II	$\max_{\alpha,\beta,q,r} (-n\beta)\ln\alpha + n\,(\ln\beta + \ln(2-q)) + \sum_{i=1}^{n}\left\{\frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i + \sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right] + (\beta-1)\ln\left(t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}\right)\right\} + \sum_{i=2}^{n}\frac{(q-2)}{(1-q)}\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right] \quad (3.15)$

Source: This research

Table 3.2 - Time Terminated Optimization Problems

Time Terminated
Constraints
$(2-q) > 0$,(3.22) $\alpha, \beta > 0$,(3.23) $\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \geq 0, \forall i$,(3.24) $t_i + r\sum_{j=1}^{i-1} t_j \geq 0, \forall i$,(3.25) $\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right], \forall i$,(3.26) $\left[1-(1-q)\left(\frac{T - r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right] \geq 0$,(3.27) $\left[1-(1-q)\left(\frac{r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right] \geq 0$.(3.28)
$(2-q) > 0$,(3.30) $\alpha, \beta > 0$,(3.31) $\left[1-(1-q)\left(\frac{t_i + \sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right] \geq 0, \forall i$,(3.32) $t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j} \geq 0, \forall i$,(3.33) $\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right], \forall i$,(3.34) $\left[1-(1-q)\left(\frac{T - \sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right] \geq 0$,(3.35)

Virtual age model	Objective function
Kijima type I	$\max_{\alpha,\beta,q,r} (-n\beta)\ln\alpha + n\,(\ln\beta + \ln(2-q)) +$ $\sum_{i=1}^{n}\left\{\frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right] +\right.$ $\left.(\beta-1)\ln\left(t_i + r\sum_{j=1}^{i-1}t_j\right)\right\} + \sum_{i=2}^{n}\frac{(q-2)}{(1-q)}\ln\left[1-(1-\right.$ $\left.q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right] + \frac{(2-q)}{(1-q)}\left\{\ln\left[1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right] -\right.$ $\left.ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right]\right\}$(3.21)
Kijima type II	$\max_{\alpha,\beta,q,r} (-n\beta)\ln\alpha + n\,(\ln\beta + \ln(2-q)) +$ $\sum_{i=1}^{n}\left\{\frac{1}{(1-q)}\ln\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right] +\right.$ $\left.(\beta-1)\ln\left(t_i + \sum_{j=1}^{i-1}t_j\, r^{i-j}\right)\right\} + \sum_{i=2}^{n}\frac{(q-2)}{(1-q)}\ln\left[1-(1-\right.$ $\left.q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(2-q)}{(1-q)}\left\{-\ln\left[1-(1-\right.\right.$ $\left.q)\left(\frac{T-\sum_{j=1}^{n}t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right] + ln\left[1-(1-\right.$ $\left.\left.q)\left(\frac{\sum_{j=1}^{n}t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right]\right\}$(3.29)

Source: This research

3.3 PSO Algorithm to Solve the q-Weibull GRP ML Problem

In this work, the objective function used in PSO algorithm is the q-Weibull GRP log-likelihood function associated with the specific data type and virtual age model (e.g., failure terminated and Kijima type I). It has to be maximized by choosing optimal parameters' estimates $\hat{\alpha}$, $\widehat{\beta}$, $\hat{q}$, $\hat{r}$.The considered problem involves a four-dimensional search space where each dimension is related to the decision variables α, β, q and r. Therefore, each particle $j = 1,\dots,n_{part}$has s_j,p_j,v_j as four-dimensional vectors that have their entries associated with α, β, q and r. The *lbest* PSO and "let particles fly strategy" are used. Finally, two stop criteria are adopted: (i) maximum number of iterations and (ii) the global best particle is the same for 10% of the maximum number of iterations.

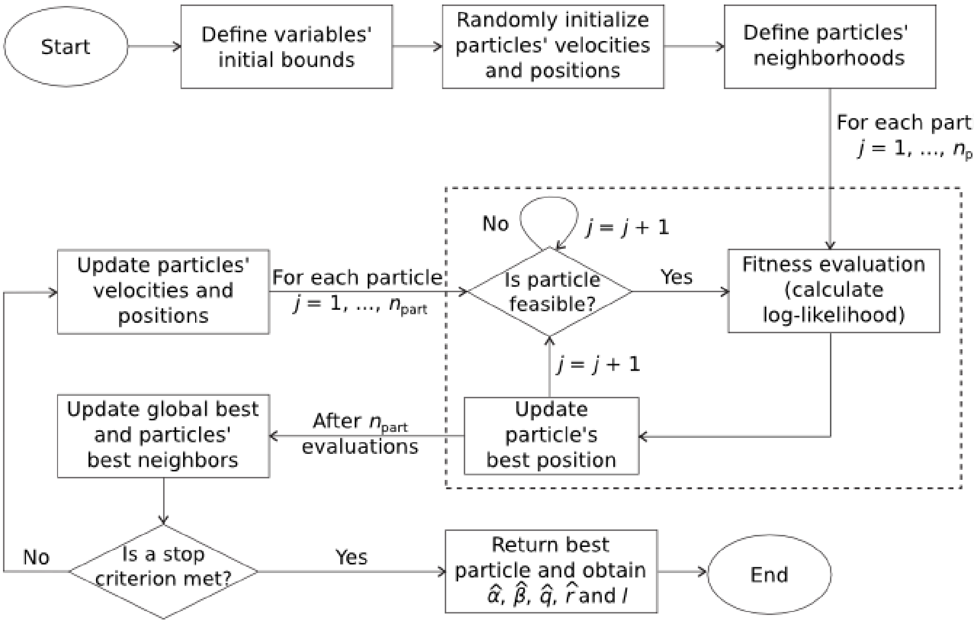
The steps of the PSO algorithm applied in this research are summarized in the flow chart presented in Figure 3.1. The particles' initialization procedure involves the random generation of particles' positions within ranges defined in the previous step ("Define variables' initial bounds", Figure 3.1). However, some combinations of values for α, β, q and r may result in infeasible particles with respect to the constraints related to the maximum log-likelihood problem considered (e.g. failure terminated and Kijima type I). In the initialization step, whenever an infeasible position is generated it is immediately discarded and a new position is then created. This stage ends when all particles' positions are feasible in the initial swarm.

The maximum velocities are defined as a fraction of 0.1 of the position limits of the parameters and the velocities are obtained from uniform random numbers in the maximum velocities interval defined. The positions and velocities are obtained from uniform distributions over the intervals of definition of the decisions variables and by setting a maximum velocity value $\mathrm{v}_{\max}$, respectively.

The particles' neighbors are defined considering the particles' generation order and not taking into account any sort of distance metrics. The particle i has $i-1$ and $i+1$ as neighbors. If $i = 1$, then the "left" neighbor is the last particle and, conversely, if the last particle is considered, its "right" neighbor is the first particle.

After the definition of particles' neighbors, the fitness evaluation step takes place. It consists in evaluating the log-likelihood function for each particle's position. Then, each particle has its own best position updated if such if the current position returns a better log-likelihood value. After the entire swarm is evaluated, the best neighbors and global best are updated. If any of the stop criteria is met, new velocities and positions are calculated through equations (2.9) and (2.10) and the fitness evaluation step is repeated only for feasible particles ("let particles fly" strategy). This procedure is repeated until a stop criterion is met. The algorithm's output of the best particle are the ML estimates.

Figure 3.1 - PSO flowchart



Source: This research

3.4 Monte Carlo Simulation Procedure for Model Validation

The parameters' estimates obtainedvia PSO are used in a Monte Carlo procedureto simulate the expected number of failures. The simulation results can be used to compare real data and the q-Weibull GRP response as a mode to evaluate its performance.

Consider a time interval (t_l, t_u), which is of interest to estimate the expected number of failures. The Monte Carlo simulation starts by the generation of uniform random numbers between 0 and 1 for the cumulative distribution function$F(t_i|v_{i-1})$.

The cumulative distribution function of the q-Weibull distribution solved for $t_i$for each virtual age model is shown in Table 3.3.

Table 3.3 - Cumulative Distribution Functions

Virtual age model	Cumulative Distribution Function (CDF)
Kijima type I	$t_i = \alpha \left\{ \frac{1-\left(1-F(t_i\|t_{i-1})\right)^{\frac{1-q}{2-q}}\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}{(1-q)} \right\}^{\frac{1}{\beta}} - r\sum_{j=1}^{i-1} t_j (3.36)$

Kijima type II	$t_i = \alpha \left\{ \frac{1-\left(1-F(t_i \vert t_{i-1})\right)^{\frac{1-q}{2-q}}\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} r^{i-j} t_j}{\alpha}\right)^{\beta}\right]}{(1-q)} \right\}^{\frac{1}{\beta}} - \sum_{j=1}^{i-1} r^{i-j} t_j$(3.37)

Source: This research

The uniform distribution is used for generating random values of$F(t_i|t_{i-1})$, considering the intervals composed by the accumulated failure time, and then random numbers of t_i are obtained through the Equations presented in Table 3.3. This value is added to the previous sum of generated times and compared to the period of interest (t_l, t_u) accordingto the rules of the pseudocode in Figure 3.2.

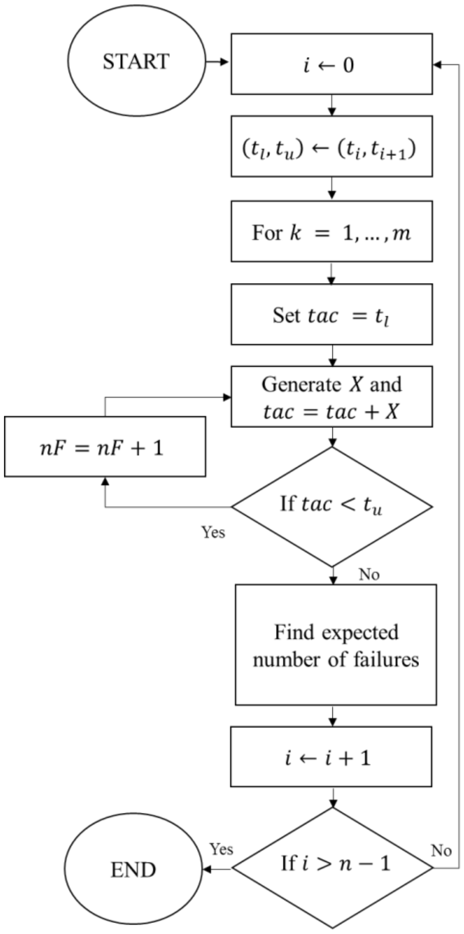
Figure 3.2 - Monte Carlo Simulation Pseudocode

```
Procedure MONTECARLOSIMULATION ($\hat{\alpha}, \hat{\beta}, \hat{q}, \hat{r}, m, t, nFail$)
► Monte Carlo simulation initialization
for $i = 0, \dots, nFail - 1$do
    for $k = 1, \dots, m$do
        set time interval of interest $(t_l, t_u) \leftarrow (t_i, t_{i+1})$
                    $tac = t_l$
        while $tac < t_u$
            calculate virtual age
            generate cdf from a uniform distribution
            calculate time of failure $X$ based on $\hat{\alpha}, \hat{\beta}, \hat{q}, \hat{r}$
            $tac = tac + X$
                if $tac < t_u$
                    $nF = nF + 1$                              ►Failure number update
                else
                    break
                end if
        end while
    end for
    $NF = nF/m$                                                ►Expected number of failures
end for
end procedure
```

Source: This research

The flowchart representing the algorithm of the Monte Carlo Simulation for each of the m repetitions is given in Figure 3.3.

Figure 3.3 - Monte Carlo Simulation Flowchart



Source: This research

4 RELIABILITY APPLICATIONS OF THE Q-WEIBULL GRP

This section presents the applications of the proposed q-Weibull based GRP models to reliability data of engineered systems taken from literature. More specifically there are threeapplication examples, both involving failure terminated data but one considers Kijima type I and the other is related to Kijima type II as virtual age model and the last one concerns a time terminated case of Kijima type I model.

The PSO implementation, which was performed using the parameters presented in Table 4.1 (χ, c_1 and c_2 values are from Bratton & Kennedy (2007)) and Monte Carlo simulations, repeated 10000 times, were conducted using MATLAB version 7.7. All experiments were run in a personal computer with 2.4 GHz, 8Gb of RAM and Windows 10 operating system.

Table 4.1 - PSO Parameters

Parameter	**Value**
Number of particles	30
Number of neighbors	2
Number of iterations	10000
Number of algorithm's replication	30
$c_1 = c_2$	2,050
χ	$7{,}298 \cdot 10^{-1}$

Source: This research

4.1 Example 1: Failure Times of the Powertrain System of Bus #514

In this first example, the data regard the failure times of the powertrain system of a bus that was employed in urban routes of the city of Naples extracted from Guida & Pulcini (2009). The data are referred to a failure terminated case andare presented in Table 4.2. The virtual age model applied is Kijima type I.

Table 4.2 - Powertrain of Bus #514 Failure Data

Failure number	TBF (km)	Failure number	TBF (km)	Failure number	TBF (km)
1	2.412	12	16.096	23	5.094
2	0.794	13	27.218	24	16.006
3	1.184	14	0.907	25	0.816
4	8.194	15	10.496	26	32.941
5	17.325	16	2.11	27	2.293
6	20.803	17	0.343	28	6.662
7	47.838	18	2.181	29	1.938
8	2.522	19	2.814	30	4.174
9	43.421	20	6.646	31	20.765
10	14.723	21	2.056	32	2.279
11	3.201	22	7.544	33	3.756

Source: Guida & Pulcini (2009, p.438)

The descriptive statistics of the results and the response of PSO's best particle are presented in Tables 4.3 and 4.4. The obtained failure intensity behavior of the model is a bathtub curve presenting $\hat{\beta} = 0.5846$ and $\hat{q} = 0.9314$, the repair effectiveness parameter $\hat{r} = 0.999$ is and approximately denotes minimal repair.

From the descriptive statistics (Table 4.3) small standard deviations are observed, which indicates the robustness of the PSO algorithm in providing approximately the same response in different algorithm replications.

Table 4.3 - Powertrain of Bus #514 PSO Descriptive Statistics

Powertrain Bus #514		Minimum	Maximum	Mean	Std. Dev
Parameters	$\hat{\alpha}$	4.3548	4.3548	4.3548	1.74E-06
	$\hat{\beta}$	0.5846	0.5846	0.5846	7.42E-08
	$\hat{q}$	0.9314	0.9314	0.9314	9.67E-09
	$\hat{r}$	0.9999	0.9999	0.9999	1.91E-13
Maximum log-likelihood		-110.1458	-110.1458	-110.1458	1.91E-13

Source: This research

Table 4.4 - Powertrain of Bus #514 PSO Best Particle

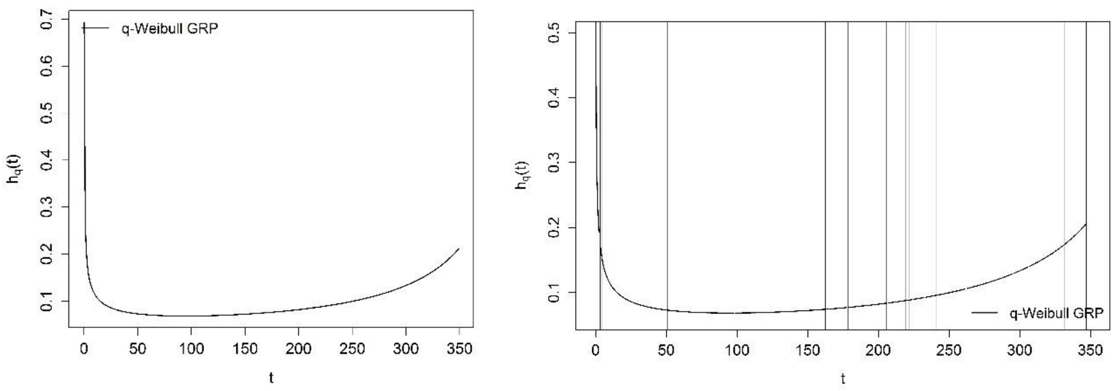
Powertrain Bus #514		q-Weibull GRP	ACI	Weibull GRP
Parameters	$\hat{\alpha}$	4.3548	[-1.1375, 9.8471]	8.4639
	$\hat{\beta}$	0.5846	[-184.2154, 185.3846]	0.8045
	$\hat{q}$	0.9314	[-490.0621, 491.9249]	-
	$\hat{r}$	0.9999	[-15.2056, 17.2054]	0.0041
Hazard Rate		Bathtub Curve	-	Monotonically decreasing

Source: This research

In the case of the bathtub curve, the relation between the repair effectiveness parameter to the equipment’s condition depends on each phase of the function. The parameters’ estimates are the obtained by the PSO’s best particle and its asymptotic confidence intervals with 90% of confidence are presented in Table 4.4. The intervals related to α and β contain invalid parameters’ values, since them cannot be negative, the $r$interval includes all post repair states. The large range of the intervals may be justified by the asymptotic confidence intervals suitability for large samples and thus for small ones, such the presented case, they may present inconsistent results.

The unconditional failure intensity function (associated with the time to first failure) and the conditional behavior resulting from the virtual age considerationare illustratedin Figure (4.1).

Figure 4.1 - Failure intensity for the case of Powertrain of Bus #514. (a) First failure; (b) Conditional failures.

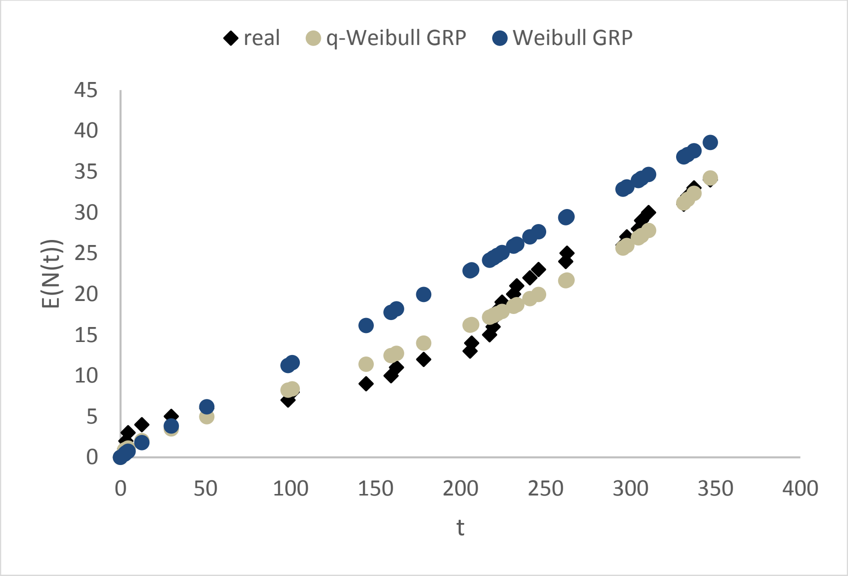


(a) (b)

Source: This research

Monte Carlo simulations were performed to stablish a comparison between the model proposed and the real failure occurrence and is presented in Figure 4.2. From the simulation's result, a similarity between the real and simulated failure occurrence can be noted, which indicates a good adjustment of the q-Weibull GRP model for this case, especially compared to the results obtained with Weibull GRP model, which presents a totally different effectiveness repair value and classification.

Figure 4.2 - Expected Number of Failures of Powertrain of Bus #514



Source: This research

4.2 Example 2: Failure Times of an NC Machine Tool

The second example comprehends the application of the Kijima type II model using the failure times encountered in Wang & Yang (2012), a failureterminated case.

The estimated parameters for the NC machine tool case from the best scenario of Wang & Yang (2012), which uses a GRP Kijima type I based on the Weibull distribution, were $\alpha = 30.58, \beta = 0.766$ and $r = 0.109$ that suggests a post-repair condition of better than new but worse than old since it has a monotonically decreasing failure intensity. The results obtained with q-Weibull, which are presented in Tables 4.6 and 4.7, of $\beta = 0.4502$ and $q =$

-2.1489 leads to a bathtub curve hazard rate (Figure 4.3) and $r = 0.2155$to the same repair effectiveness category as the work in comparison.

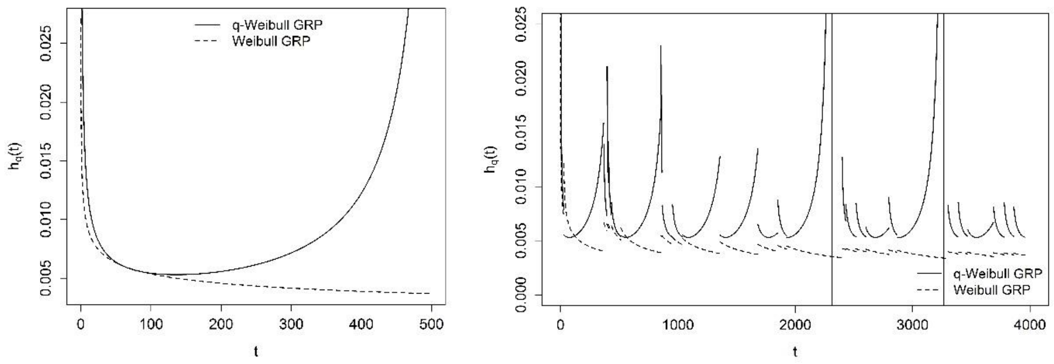
Table 4.5 - NC Machine Tool Failure Data

Failure number	TBF (h)	Failure number	TBF (h)	Failure number	TBF (h)	Failure number	TBF (h)
1	27.51	8	341.4	15	76.43	22	432.42
2	340.01	9	9.28	16	471.23	23	87.75
3	27	10	88.17	17	32.4	24	81.01
4	1.12	11	86.34	18	86.43	25	220.05
5	11.11	12	318.44	19	83.18	26	91.7
6	25.74	13	323.12	20	196.27	27	82.17
7	81.68	14	169.63	21	70.91	28	92.98

Source: Wang & Yang (2012, p.1131)

Figure 4.3 - Failure intensity for the case NC Machine Tool for q-Weibull GRP Kijima type II and Weibull GRP. (a) First failure; (b) Conditional failures.

(a) (b)



Source: This research

The descriptive statistics for the parameters and the maximum log-likelihood estimates for the 30 runs are presented in Table 4.6. The best particle's results were utilized for the estimates. The descriptive statistics of the NC machine tool presents relatively high standard deviations, even though they are still small when compared to the respective mean values. This may be explained by the estimates of q and β parameters, which satisfy $q<1$ and 0 <β< 1. According to experiments, values in these ranges for the shape parameters may incorporate instability to the log-likelihood function and make the search for the optimum more difficult. Current researches are focused on investigating these variations.

Table 4.6 - NC Machine Tool PSO Descriptive Statistics

NC Machine Tool		Minimum	Maximum	Mean	Std. Dev
Parameters	$\hat{\alpha}$	6206.5712	6587.8891	6358.8070	87.8355
	$\hat{\beta}$	0.4502	0.4526	0.4517	5.61E-04
	$\hat{q}$	-2.1489	-2.0829	-2.1102	1.58E-02
	$\hat{r}$	0.2148	0.2168	0.2158	1.53E-03
Maximum log-likelihood		-165.6476	-165.6475	-165.6476	4.23E-04

Source: This research

Table 4.7 - NC Machine Tool Comparison Between q-Weibull GRP and Weibull GRP

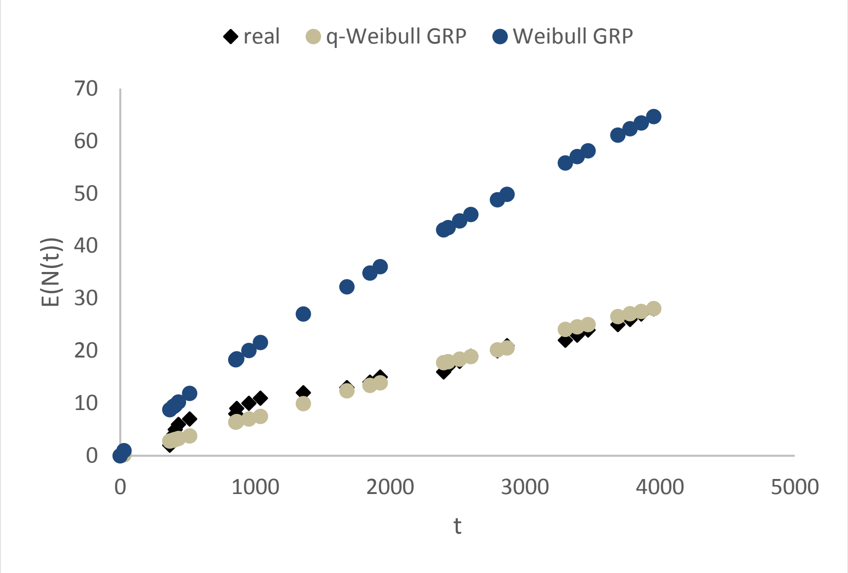
NC Machine Tool		q-Weibull GRP	ACI	Weibull GRP
Parameters	$\hat{\alpha}$	6587.8891	[6587.88, 6587.89]	0.0199
	$\hat{\beta}$	0.4502	[-177.53, 178.43]	0.870
	$\hat{q}$	-2.1489	[-23.59, 19.29]	-
	$\hat{r}$	0.2155	[-4.67, 5.10]	1
Hazard Rate		Bathtub Curve	-	Monotonically decreasing

Source: This research

The best particle's results, presented in Table 4.7 along with its asymptotic confidence intervals with 90% of confidence, were utilized for the estimates. As previously commented in Example 1, the intervals present a large range, providing invalid parameter's values for β and comprehending different post repair states.

The influence of the virtual age in the intensity failure rate can be observed in Figure 4.3 and the expected number of failures in Figure 4.4. The adjustment of the q-Weibull model has proven to be not so effective when compared to the real failure occurrence, but comparing to the results obtained with the Weibull based model the proposed model can be considered efficient.

Figure 4.4 - Expected Number of Failures of NC Machine Tool



Source: This research

4.3 Example 3: Failure Times of a Propulsion Motor

This example of a time terminated case using Kijima type I modeluses failure data encountered in Yañez *et al.* (2002) of the U.S.S. Halfbeak No. 3 main propulsion motor. The data is given in Table 4.8.

Table 4.8 - Propulsion Motor Failure Data

Failure number	TBF (h)	Failure number	TBF (h)	Failure number	TBF (h)	Failure number	TBF (h)
1	860	7	1278	13	367	19	490
2	1608	8	605	14	2758	20	945
3	1134	9	344	15	355	21	105
4	2703	10	1054	16	1084	22	127
5	645	11	680	17	855	23	61
6	95	12	405	18	280	24	326

Source: Yañez et al. (2012, p.177)

Tables 4.9 and 4.10 present the descriptive statistics of the results and the response of PSO's best particle along with the 90% asymptotic confidence intervals. The obtained failure intensity behavior of the model is a bathtub curve presenting $\hat{\beta} = 0.5424$ and $\hat{q} = 0.8783$, the repair effectiveness parameter $\hat{r} = 0.0058$ approximately denotes perfect repair. The inconsistencies encountered around the ACI in the previous examples remains in this case.

As observed in Example 2, the descriptive statistics of the propulsion motor presents relatively high standard deviations, even though they are still small when compared to the respective mean values. The estimated values of q and β parameters, which satisfy $q<1$ and 0 <β< 1, are in the ranges where the log-likelihood may present instabilities that difficult the quest for optimal estimates.

Table 4.9 - Propulsion Motor PSO Descriptive Statistics

Propulsion Motor		Minimum	Maximum	Mean	Std. Dev
Parameters	$\hat{\alpha}$	790.5135	2551.7426	1782.9440	5.09E+02
	$\hat{\beta}$	0.3088	0.6282	0.4518	9.53E-02
	$\hat{q}$	0.5699	0.8928	0.7889	9.79E-02
	$\hat{r}$	0.0004	0.0119	0.0050	3.00E-03
Maximum log-likelihood		-165.6476	-198.2526	-196.6727	5.18E-01

Source: This research

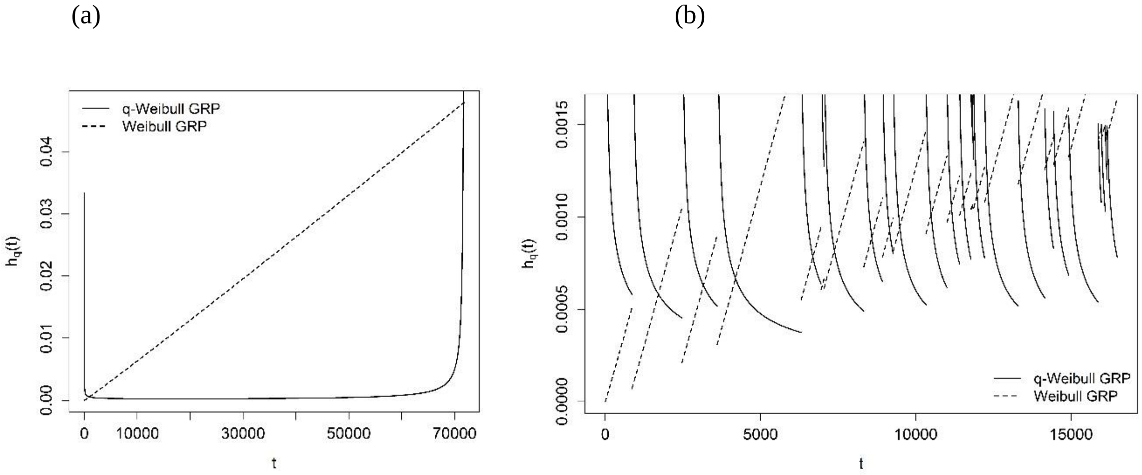
Table 4.10 – Propulsion Motor Comparison Between q-Weibull GRP and Weibull GRP

Propulsion Motor		q-Weibull GRP	ACI	Weibull GRP
Parameters	$\hat{\alpha}$	1479.2134	[1479.21, 1479.22]	1828
	$\hat{\beta}$	0.5424	[-20.42, 21.51]	2.026
	$\hat{q}$	0.8783	[-26.78, 28.53]	-
	$\hat{r}$	0.0058	[-70.68, 70.69]	0.146
Hazard Rate		Bathtub Curve	-	Monotonically increasing

Source: This research

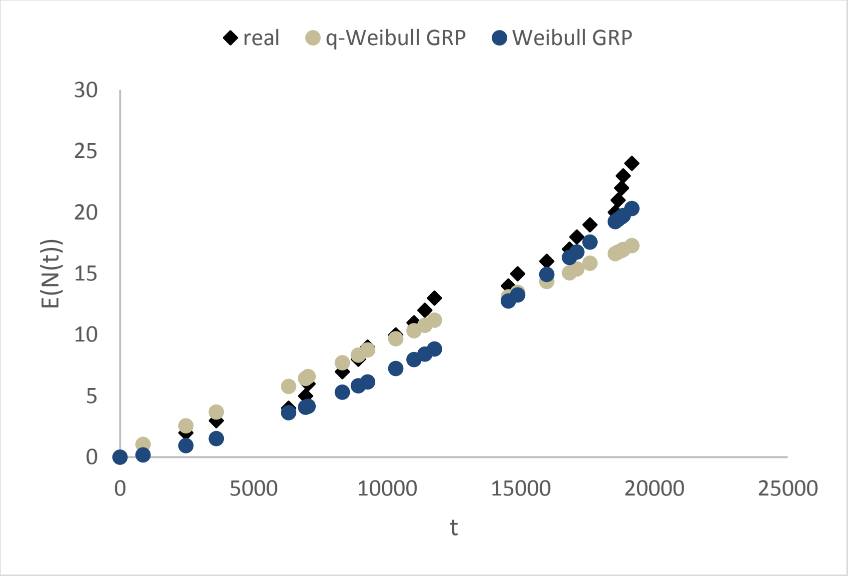
The influence of the repair effectiveness parameter close to 0, which indicates an as new condition to the equipment after repair and the expected number of failures can be observed in Figures 4.5 and 4.6, respectively. The difference between the expected number of failures and the real failure occurrence can be related to the difficulties around the optimization of the maximum log-likelihood function, mentioned previously. Although the total predicted number of failures in q-Weibull GRP model is smaller compared to the Weibull GRP model, the proposed model presents a good adjustment in almost a half of the first failures, while the Weibull model presents a better adjustment in the last failures.Due to the $\hat{r} = 0.0058$ and the constant behavior phase of the bathtub curve until 60000h, the linear behavior encountered in the Monte Carlo simulation of q-Weibull GRP is understandable.When compared to real data, the proposed model super estimates the beginning of equipment's deterioration stage. The q-Weibull GRP would become more adequate if a larger number of failure occurrence data were provided.

Figure 4.5 - Failure intensity for the case Propulsion Motor for q-Weibull GRP Kijima type I and Weibull GRP. (a) First failure; (b) Conditional failures.



Source: This research

Figure 4.6 - Expected Number of Failures of Propulsion Motor



Source: This research

4.4 Results Discussion

Through the different experiments performed in order to obtain the parameters' estimates of the model, a limitation regarding the ability of the model arise. For various failure data tested, the model's response of the effectiveness repair parameter, r , was near the values 0 or 1. This results leads to the cases of the RP and NHPP and may occur due to a limitation of the proposed model to evaluate the whole behavior of the failure intensity.

Dijoux (2009) confronted this problem with the Weibull based GRP making assumptions of repairs and using different virtual ages to each phase of the bathtub curve resulting in a complex system with an elevated number of parameters to be estimated, which is not viable.

The value of the repair effectiveness parameter r was constrained to the interval $[0,1]$, but it is not necessarily in this interval, although it is rather common in practical situations and literature, leading to the situations where the system condition is restored to an intermediate state, depending on the failure intensity considered. Values of $r < 0$ may result in complex numbers in the Monte Carlo simulation and it would be an issue to be solved in future applications.

There are different interpretations to the repair, which is assumed as a unique value and may not be the same for all periods of equipment's lifecycle. The different interpretations are described in Table 4.11.

Table 4.11 - Different Repair Effectiveness Interpretations

Failure Intensity	$r$value	Interpretation
Bathtub curve	$r = 0$ and $r < 0$	Burn-in phase: non effective repair; Wear out phase: effective repair;
	$r = 1$ and $r > 1$	Burn-in phase: effective repair; Wear out phase: non effective repair;
Unimodal	$r = 0$ and $r < 0$	Decreasing phase: non effective repair; Increasing phase: effective repair;
	$r = 1$ and $r > 1$	Decreasing phase: effective repair; Increasing phase: non effective repair
Monotonically Increasing	$r = 0$ and $r < 0$	Effective repair
	$r = 1$ and $r > 1$	Non effective repair
Monotonically Decreasing	$r = 0$ and $r < 0$	Non effective repair
	$r = 1$ and $r > 1$	Effective repair

Source: This research

Regarding the asymptotic confidence intervals, the obtained results of the application examples may be considered inaccurate due to the reduced sample size of failure times data that indicates the requirement of developing an alternative confidence interval such as bootstrap-based ones, which can be combined to PSO to provide interval estimates.

5 CONCLUDING REMARKS

5.1 Conclusions

In this master thesis, a GRP elaborated using a q-Weibull distribution for the timesto failure is proposed. In fact, a total of four models were developed, since failure and time terminated data were considered along with Kijima type I and type II virtual ages. In order to obtain the q-Weibull GRP parameters' estimates, the maximum log-likelihood problem associated with each data type and virtual age model was derived.

Although the Weibull distribution has been widely used with GRP, the q-Weibull brings more flexibility to the model because of the q parameter affecting the shape as well as β parameter. Therefore, the GRP becomes able to incorporate additional failure intensity behaviors that could not be modelled with the Weibull based GRP.

Due to the complexity of the maximum log-likelihood optimization problem, a probabilistic heuristic is used instead of numerically solving the corresponding system of derivatives. The chosen method, PSO, proved to be a great tool in this context of application: it provided coherent estimates for the parameters of the q-Weibull GRP and several replications for a given example indicated small variations for the estimates.

Some results found in the applications performed were compared with Weibull GRP applications and differences between the hazard rate's behaviors were observed.The numerical experiments indicate the PSO ability in providing very similar solutions for the q-Weibull maximum log-likelihood problem related to a specific failure data set in different runs.

Moreover, Monte Carlo simulations were developed to compare the outputs of the estimated q-WeibullGRP model against the real failure occurrences. For the application examples considered in this work, the proposed model presented relatively good data adjustment compared to the results obtained in previous works based on Weibull GRP model.

5.2 Limitations and Future Works

A step forward to the q-Weibull based GRP proposed in this master thesis is to evaluate manners to consider the different failure intensity behaviors in order to obtain the most realistic parameters to represent equipment’s conditions during its life cycle.

The PSO is also a subject of possible improvement, such as the initialization to reduce time and computational effort; and the evaluation of the optimization when complex log-likelihood functions occurs.

A step forward to this research would be to numerically validate the model to evaluate its accuracy and precision and develop confidence intervals using methods such as Bootsrap, Jackknife, Nelder Mead and Broyden-Fletcher-Goldfarb-Shannon (BFGS) algorithm, since the obtained asymptotic confidence intervals rely on the central limit theorem, suited for large samples, and for small ones may present inconsistent results.

The obtained results from q-Weibull GRP can be used in practical enterprise situations to maintenance planning and evaluation,representing an interesting tool to support operational decisions.

REFERENCES

ABREU E. M. C., NETO J. A., GODINHO C. F. L., Nonextensive statistics, entropic gravity and gravitational force in a non-integer dimensional space, *Phys. A Stat. Mech. its Appl*., vol. 411, p. 118–127, 2014.

ANASTASIADIS, A. D., DE ALBUQUERQUE, M. P. DE ALBUQUERQUE, M. P., MUSSI, D. B., Tsallis q-exponential describes the distribution of scientific citations-a new characterization of the impact, *Scientometrics*, vol. 83, no. 1, p. 205–218, 2010

ASSIS, E.M., BORGES, E.P., MELO, S.A.B.V., Generalized q-Weibull model and the bathtub curve, *International Journal of Quality & Reliability Management*, vol. 30, issue 7, p. 720-736, 2013.

BRATTON, D., KENNEDY, J., Defining a Standard for Particle Swarm Optimization, *Proceedings of the 2007 IEEE Swarm Optimization Symposium*, 2007.

BECK, C., Generalized statistical mechanics of cosmic rays, *Phys. A Stat. Mech. its Appl*., v. 331, no. 1–2, p. 173–181, 2004.

CARNEIRO, T.C., MELO, S.P., CARVALHO, P.C.M., BRAGA, A.P.S. Particle Swarm Optimization method for estimation of Weibull parameters: A case study for the Brazilian northeast region, *Renewable Energy*, vol.86, p.751-759, 2016.

COSTA, U.M.S., FREIRE, V.N., MALACARNE, L.C., MENDES, R.S., PICOLI JR, S., VASCONCELOS, E.A., SILVA JR., E.F. An improved description of the dieletric breakdown in oxides based on a generalized Weibull distribution, *Physica A*, vol.361, p.209-215, 2006.

DIJOUX, Y. A Virtual Age Model Based on a Bathtub Shaped Initial Intensity, vol. 94, p. 982-989, 2009.

EBERHART, R.C., SHI, Y., Particle Swarm Optimization: Developments, Applications and Resources, *Proceedings of IEEE Congress on Evolutionary Computation 2001 IEEE service center*, Piscataway, NJ., Seoul, Korea., 2001.

FERREIRA, R.J., FIRMINO, P.R.A., CRISTINO, C.T. A Mixed Kijima Model Using the Weibull-Based Generalized Renewal Process, *PLoS ONE*, 2015.

GU, G., XIONG, X., ZHANG, W., ZHANG, Y., ZHOU W., Empirical properties of intercancellation durations in the Chinese stock market, vol. 2, p. 1–12, 2014.

GUIDA, M., PULCINI, G. Reliability Analysis of Mechanical Systems With Bounded and Bathtub Shaped Intensity Function, *IEEE Transactions on Reliability*, v.58, p.423-443, 2009.

IVANOV P. C., YUEN, A., PERAKAKIS, P., Impact of stock market structure on intertrade time and price dynamics, *PLoS One*, vol. 9, no. 4, 2014.

JACOPINO, A., GROEN, F., MOSLEH, A. Behavioural Study of the General Renewal Process, *Annual Reliability and Maintenability Symposium*, p.237-242, 2004.

JAISINGH, L.R., KOLARIK, W.J., DEY, D.K. A Flexible Bathtub Hazard Model for Non-Repairable Systems with Uncensored Data, *Microelectronical Reliability*, v.27, p.87-103, 1987.

JIANG, R., JI, P., XIAO, X. Aging property of unimodal failure rate models, *Reliability Engineering and System Safety*, vol. 79, p. 113-116, 2003.

JIMÉNEZ, P., VILLALON, R. Generalized Renewal Process as an Adaptive Probabilistic Model, *Transmission & Distribution Conference and Exposition*: Latin America, 2006.

KAMINSKIY, M.P. *Reliability Models for Engineers and Scientists*, CRC Press: Boca Raton, 2013.

KAMINSKIY, M.P., KRIVTSOV, V.V., G-Renewal Process as a Model for Statistical Warranty Claim Predictions, *Annual Reliability and Maintainability Symposium*, p. 276-280, 2000.

KIJIMA, M., SUMITA, N. A useful generalization of renewal theory: counting process governed by non-negative Markovian increments, *Journal of Applied Probability*, vol. 23, p.71-88, 1986.

KIJIMA, M., Some results for Repairable Systems with General Repair, *Journal of Applied Probability*, vol. 26, p. 89-102, 1989.

KOBBACY, K.A.H., JEON, J. Generalized stationary models for scheduling single and multiple preventive maintenance routines, *IMA Journal of Management Mathematics*, v.13, p.81-94, 2002.

LI, W., WANG, Q. A., NIVANEN, L., LE MÉHAUTÉ, A., How to fit the degree distribution of the air network?, *Phys. A Stat. Mech. its Appl.*, vol. 368, no. 1, pp. 262–272, 2006.

LINS, I.D., MOURA, M.C.M., ZIO, E., DROGUETT, E.L., A Particle Swarm-Optimized Support Vector Machine for Reliability Prediction, *Quality and Reliability Engineering International*, vol. 28, p.141-158, 2012.

LINS, I.D., DROGUETT, E.A.L., SALES FILHO, R.L.M., MOURA, M.C. Estimation for the parameters of the q-Weibull distribution by maximum likelihood and particle swarm optimization. In: *European Safety & Reliability Association Conference – ESREL*, Zurich, Switzerland, p.1315-1322, 2015.

MAKIS, M., JARDINE, A.K.S. A note on optimal replacement policy under general repair, *European Journal of Operational Research*, v.69, p.75-82, 1993.

MARSHALL, A.W., OLKIN, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika, vol.84, p.641-652, 1997.

MILLAR, R.B. *Maximum Likelihood Estimation and Inference With Examples in R, SAS and ADMB*, United Kingdom: John Wiley & Sons, Ltd., 2011.

MONTGOMERY, D.C.; RUNGER, G.C. *Applied Statistics and Probability for Engineers*. 3 ed. New York: John Wiley & Sons, Inc., 2003.

MOURA, M.C., DROGUETT, E.L., FIRMINO, P.R.A., FERREIRA, R.J. A competing risk model for dependent and imperfect condition-based preventive and corrective maintenances, *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol.228, p.590-605, 2014.

MOURA, M.C., ROCHA, S.P.V., DROGUETT, E.L., JACINTO, C.M., Avaliação bayesiana da eficácia da manutenção via processo de renovação generalizado, *Pesquisa Operacional*, vol. 27, issue 3, p. 569-589, 2007.

MUDHOLKAR, G.S., SRIVASTAVA, D.K. Exponentiated Weibull Family for Analyzing Bathtub Failure-Rate Data, IEEE Transactions On Reliability, vol.42, p.299-302, 1993.

NADARAJAH, S., KOTZ, S., On the q-type distributions, *Physica A*, vol. 377, p.465-468, 2007.

NAMAKI, A., KOOHI LAI, Z., JAFARI, G. R., RAEI, R., TEHRANI, R., Comparing emerging and mature markets during times of crises: A non-extensive statistical approach, *Phys. A Stat. Mech. its Appl.*, vol. 392, no. 14, pp. 3039–3044, 2013.

OLIVEIRA, C.C.F., CRISTINO, C.T., FIRMINO, P.R.A. In the kernel of modelling repairable systems: a goodness of fit test for Weibull-based generalized renewal process, *Journal of Cleaner Production*, vol.133, p.358-367, 2016.

PHAM, H., WANG, H. Imperfect Maintenance, *European Journal of Operational Research*, vol. 94, p.425-438, 1996.

PICOLI, S., MENDES, R.S., MALACARNE, L.C., q-Exponential, Weibull and q-Weibull distributions: an empirical analysis, *Physica A Statistical Mechanics and its Applications*, vol. 324, p.678-688, 2003.

PICOLI, S., MENDES, R.S., MALACARNE, L.C., SANTOS, R.P.B. q-distributions in complex systems: a brief review, *Brazilian Journal of Physics*, vol.39, p.468-474, 2009.

PRATA, D.M., SCHWAAB, M., LIMA, E.L., PINTO, J.C. Nonlinear dynamic data reconciliation and parameter estimation through particle swarm optimization: Application for an industrial polypropylene reactor, *Chemical Engineering Science*, vol.64, p.3953-3967, 2009.

PRIETO, F., SARABIA, J.M. A generalization of the power law distribution with nonlinear exponent, *Commum Nonlinear Sci Numer Simulat*, vol.42, p.215-228, 2017.

RAMADAN, H.S., BENDARY, A.F., NAGY, S. Particle swarm optimization algorithm for capacitor allocation problem in distribution systems with wind turbine generators, *Electrical Power and Energy Systems*, vol.84, p.143-152, 2017.

REYNOLDS, A. M., VENEZIANI, M., Rotational dynamics of turbulence and Tsallis statistics, *Phys. Lett. A*, vol. 327, no. 1, p. 9–14, 2004.

SALES FILHO, R.L.M., A novel q-exponential based stress-strength reliability model and applications to fatigue life with extreme values, 24/02/2016, 127. Tese (Doutorado) – Universidade Federal de Pernambuco. CTG. Programa de Pós-graduação em Engenharia de Produção, 2016.

SAMANTA, B., NATARAJ, C., Use of particle swarm optimization for machinery fault detection, *Engineering Applications of Artificial Intelligence*, vol. 22, p. 308-316, 2009.

SANTOS, L.C., TAVARES, F.W., AHÓN, V.R.R. Modeling MEA with the CPA equation of state: A parameter estimation study adding local search to PSO algorithm, *Fluid Phase Equilibria*, vol.400, p.76-86, 2015.

SARTORI, I., ASSIS, E.M., SILVA, A.L., MELO, R.L.F.V., BORGES, E.P., MELO, S.A.B.V. Reliability Modeling of a Natural Gas Recovery Plant Using q-Weibull Distribution, *Computer Aided Chemical Engineering*, vol.27, p.1797-1802, 2009.

SCHWAAB, M., BISCAIA, E.C., MONTEIRO JR., J.L., PINTO, J.C. Nonlinear parameter estimation through particle swarm optimization, Chemical Engineering Science, vol.63, p.1542-1552, 2008.

SPINATO, F., TAVNER, P.J., VAN BUSSEL, G.J.W., KOUTOULAKOS, E. Reliability of Wind Turbines Subassemblies, *IET Renewable Power Generation*, vol. 3, iss. 4, p. 387-401.

STADJE, W., ZUCKERMAN, D. Optimal repair policies with general degree of repair in two maintenance models, *Operations Research Letters*, v.11, p.77-80, 1992.

VALLIANATOS, F., TRIANTIS, D. A non-extensive view of the Pressure Stimulated Current relaxation during repeated abrupt uniaxial load-unload in rock samples, *EPL (Europhysics Lett.*, vol. 104, no. 6, p. 68002, 2013.

VALLIANATOS, F., MICHAS, G., PAPADAKIS, G., Non-extensive and natural time analysis of seismicity before the Mw6.4, October 12, 2013 earthquake in the South West segment of the Hellenic Arc, *Phys. A Stat. Mech. its Appl.*, vol. 414, pp. 163–173, 2014.

WANG, K.S., HSU, F.S., LIU, P.P. Modeling the bathtub shape hazard rate function in terms of reliability, *Reliability Engineering and System Safety*, vol. 75, p. 397-406, 2002.

WANG, Z., YANG, J., Numerical method for Weibull generalized renewal process and its applications in reliability analysis of NC machine tools, *Computers & Industrial Engineering*, vol.63, p. 1128-1134, 2012.

XU, M., DROGUETT, E.A.L., LINS, I.D., MOURA, M.C.M., On The q-Weibull Distribution for Reliability Applications: An Adaptive Hybrid Artificial Bee Colony Algorithm for Parameter Estimation, Reliability Engineering & System Safety, vol. 158, p.93-105, 2017.

YAÑEZ, M., JOGLAR, F., MODARRES, M., Generalized renewal process for analysis of repairable systems with limited failure experience, *Reliability Engineering and System Safety*, vol. 77, p. 167-180, 2002.

APPENDIX A – Kijima type I – Failure terminated case

First partial derivatives failure terminated Kijima I

$$\frac{\partial ln\mathcal{L}}{\partial\alpha}= \frac{-n\beta}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}+(q-2)\frac{\beta}{\alpha}\sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}$$

$$\frac{\partial ln\mathcal{L}}{\partial\beta}= \frac{n}{\beta}-n\ln(\alpha)+\sum_{i=1}^{n}\ln\left(t_i+r\sum_{j=1}^{i-1}t_j\right)-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}-(q-2)\sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}$$

$$\frac{\partial ln\mathcal{L}}{\partial q}=\frac{-n}{(2-q)}+\frac{1}{(1-q)^2}\sum_{i=1}^{n}ln\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]+\frac{1}{(1-q)}\sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}$$

$$+\frac{(q-2)}{(1-q)^2}\sum_{i=2}^{n}ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]$$

$$+\frac{1}{(1-q)}\sum_{i=2}^{n}\left\{ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]+\frac{(q-2)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right\}$$

$$\frac{\partial ln\mathcal{L}}{}=(\beta-1)\sum^{n}\frac{\sum_{j=1}^{i-1}t_j}{}-\frac{\beta}{}\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{}\right)^{\beta-1}\frac{\sum_{j=1}^{i-1}t_j}{}-(q$$

Second partial derivatives failure terminated Kijima type I

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha \partial r} = \frac{\partial \ln L}{\partial r \partial \alpha} = -\beta \sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \Bigg\{ \Bigg[-\frac{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} - \frac{(\beta-1)\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} \Bigg] \Bigg[1 \\
- (1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg] + \frac{\beta(q-1)\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{2\beta-1}}{\alpha^{2\beta+1}} \Bigg\} \\
- (q \\
- 2) \sum_{i=2}^{n} \frac{\sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \Bigg\{ \Bigg[\frac{-\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} - \frac{(\beta-1)\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} \Bigg] \Bigg[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg] \\
+ \frac{\beta(q-1)\left(r\sum_{j=1}^{i-1} t_j\right)^{2\beta-1}}{\alpha^{2\beta+1}} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \frac{\left[ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)\right]^2 \left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \Bigg\{ \Bigg[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg] - (q-1)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg\} \\
- (q-2) \sum_{i=2}^{n} \frac{\left[ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)\right]^2 \left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \Bigg\{ \Bigg[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg] - (q-1)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \Bigg\}
\end{aligned}
$$

Second partial derivatives failure terminated Kijima type I

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta \partial r} = \frac{\partial \ln L}{\partial r \partial \beta} = \sum_{i=1}^{n} & \frac{\sum_{j=1}^{i-1} t_j}{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)} \\
& - \frac{1}{\alpha^{\beta}} \sum_{i=1}^{n} \frac{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1} \sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[\beta \ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right) + 1\right] \left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \right. \\
& \left. - (q-1)\frac{\beta}{\alpha^{\beta}} \left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta} \ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right) \right\} \\
& - \frac{(q-2)}{\alpha^{\beta}} \sum_{i=2}^{n} \frac{\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1} \sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[\beta \ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right) + 1\right] \left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \right. \\
& \left. - (q-1)\frac{\beta}{\alpha^{\beta}} \left(r\sum_{j=1}^{i-1} t_j\right)^{\beta} \ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right) \right\}
\end{aligned}
$$

$$
\frac{\partial lnL}{\partial r^2} = \sum_{i=1}^{n} \left(- \frac{(\beta-1)\left(\sum_{j=1}^{i-1} t_j\right)^2}{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^2} - \frac{\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \left(\sum_{j=1}^{i-1} t_j\right)^2 \beta^2}{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^2 \left(1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right)} \right.
$$

Second partial derivatives failure terminated Kijima type I

$$\frac{\partial \ln L}{\partial q^2} = \frac{-n}{(2-q)^2} + \frac{(1-q)}{(1-q)^4}\sum_{i=1}^{n}\left\{\ln\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{(1-q)\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2}\left\{\frac{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}{(1-q)} - \left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right\}$$

$$+\frac{1}{(1-q)^4}\sum_{i=2}^{n}\left\{\left[\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}\right]\right.$$

$$\left. -\,2(q-1)(q-2)\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]\right]\Bigg\}$$

$$+\frac{1}{(1-q)^2}\sum_{i=2}^{n}\left\{\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{(q-1)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{}\left\{-\frac{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}{} - \frac{(q-2)}{}\left(\frac{r\sum_{j=1}^{i-1} t_j}{}\right)^{\beta}\right\}$$

Second partial derivatives failure terminated Kijima type I
$\frac{\partial lnL}{\partial\alpha^2}=\frac{n\beta}{\alpha^2}+\sum_{i=1}^{n}\left(-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)$ $+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)$ $\frac{\partial lnL}{\partial\alpha\partial\beta}=\frac{\partial lnL}{\partial\beta\partial\alpha}=-\frac{n}{\alpha}$ $+\sum_{i=1}^{n}\left(\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}+\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}\right.$ $\left.+\frac{\left(\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2\beta(1-q)\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)$ $\sum^{n}\left((q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)\beta \qquad (q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right.$

APPENDIX B – Kijima type II – Failure terminated case

First partial derivatives failure terminated Kijima II

$$\frac{\partial ln\mathcal{L}}{\partial\alpha} = \frac{-n\beta}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}$$

$$+ (q-2)\frac{\beta}{\alpha}\sum_{i=2}^{n}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}$$

$$\frac{\partial ln\mathcal{L}}{\partial\beta} = \frac{n}{\beta} - n\ln(\alpha) + \sum_{i=1}^{n}\ln\left(t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}\right) - \left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\ln\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)$$

$$-(q-2)\sum_{i=2}^{n}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\ln\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right),$$

$$\frac{\partial ln\mathcal{L}}{\partial q} = \frac{-n}{(2-q)} + \frac{1}{(1-q)^2}\sum_{i=1}^{n} ln\left[1-(1-q)\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right]$$

$$+\frac{1}{(1-q)}\sum_{i=1}^{n}\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}$$

$$+\frac{(q-2)}{(1-q)^2}\sum_{i=2}^{n} ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right]$$

Second partial derivatives failure terminated Kijima type II

$$\frac{\partial lnL}{\partial r\partial \alpha}=\sum_{i=1}^{n}\left(\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$+\sum_{i=2}^{n}\left(\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}\right.$$

$$\left.+\frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$\frac{\partial \ln L}{\partial \beta^2}=-\frac{n}{\beta^2}-\sum_{i=1}^{n}\frac{\left[ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)\right]^2\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]^2}\left\{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j\,r^{i-j}}{\alpha}\right)^{\beta}\right]-(q-1)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j\,r^{i-j}}{\alpha}\right)^{\beta}\right\}$$

$$-(q-2)\sum_{i=2}^{n}\frac{\left[ln\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)\right]^2\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right]^2}\left\{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j\,r^{i-j}}{\alpha}\right)^{\beta}\right]-(q-1)\left(\frac{\sum_{j=1}^{i-1}t_j\,r^{i-j}}{\alpha}\right)^{\beta}\right\}$$

$$n\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)^{2\beta}\ln\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)$$

Second partial derivatives failure terminated Kijima type II
$\frac{\partial lnL}{\partial r\partial\beta}=\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)}-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)\ln\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)$

Second partial derivatives failure terminated Kijima type II

$$\frac{\partial \ln L}{\partial q^2} = \frac{-n}{(2-q)^2} + \frac{(1-q)}{(1-q)^4} \sum_{i=1}^{n} \left\{ \ln\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]} \right\}$$

$$+ \sum_{i=1}^{n} \frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{(1-q)\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]^2} \left\{ \frac{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}{(1-q)} - \left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \right\}$$

$$+ \frac{1}{(1-q)^4} \sum_{i=2}^{n} \left\{ \left[\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]} \right] \right.$$

$$\left. - 2(q-1)(q-2)\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] \right\}$$

$$+ \frac{1}{(1-q)^2} \sum_{i=2}^{n} \left\{ \ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(q-1)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]} \right\}$$

$$+ \sum^{n} \frac{\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{} \left\{ -\frac{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}{} - \frac{(q-2)}{} \left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{}\right)^{\beta} \right\}$$

Second partial derivatives failure terminated Kijima type II

$$\frac{\partial lnL}{\partial \alpha^2}=\frac{n\beta}{\alpha^2}+\sum_{i=1}^{n}\left(-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}\right.$$

$$\left.-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}\right.$$

$$\left.-\frac{\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$\frac{\partial lnL}{\partial \alpha\partial \beta}=-\frac{n}{\alpha}+\sum_{i=1}^{n}\left(\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}\right.$$

Second partial derivatives failure terminated Kijima type II

$$\frac{\partial lnL}{\partial r^2}=\sum_{i=1}^{n}\left(\frac{(\beta-1)\left(\sum_{j=1}^{i-1}\left(\frac{t_jr^{i-j}(i-j)^2}{r^2}-\frac{t_jr^{i-j}(i-j)}{r^2}\right)\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)}-\frac{(\beta-1)\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2}\right.$$

$$-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2\beta^2}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\left(\frac{t_jr^{i-j}(i-j)^2}{r^2}-\frac{t_jr^{i-j}(i-j)}{r^2}\right)\right)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}$$

$$\left.+\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2(1-q)}{\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}\right.$$

$$-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\left(\frac{t_jr^{i-j}(i-j)^2}{r^2}-\frac{t_jr^{i-j}(i-j)}{r^2}\right)\right)}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)^2}{\left(\sum_{j=1}^{i-1}t_jr^{i-j}\right)^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}$$

$$(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{}\right)^2(1-q)\Bigg)$$

APPENDIX C – Kijima type I – Time terminated case

First partial derivatives time terminated Kijima I

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{-n\beta}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n} \frac{\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]} + (q-2)\frac{\beta}{\alpha}\sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}$$

$$+\frac{(2-q)\left(-\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}\right)}{1-q}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n}{\beta} - n\ln(\alpha) + \sum_{i=1}^{n}\ln\left(t_i + r\sum_{j=1}^{i-1} t_j\right) - \frac{\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]} - (q -$$

$$2)\sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}+\frac{(2-q)\left(\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\ln\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}-\frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}\right)}{1-q}$$

First partial derivatives time terminated Kijima I
$\frac{\partial \mathcal{L}}{\partial q} = \frac{-n}{(2-q)} + \frac{1}{(1-q)^2}\sum_{i=1}^{n} ln\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{1}{(1-q)}\sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]} + \frac{(q-2)}{(1-q)^2}\sum_{i=2}^{n} ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{1}{(1-q)}\sum_{i=2}^{n}\left\{ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] + \frac{(q-2)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right\} - \frac{-\ln\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right) + \ln\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}{1-q} + \frac{(2-q)\left(-\ln\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)\right) + \ln\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}{(1-q)^2} + \frac{(2-q)\left(-\frac{\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}} + \frac{\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}\right)}{1-q}$

Second partial derivatives time terminated Kijima type I
$\frac{\partial \ln L}{\partial \alpha \partial r} = \frac{\partial \ln L}{\partial r \partial \alpha} = -\beta \sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[-\frac{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} - \frac{(\beta-1)\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} \right] \left[1 \right.\right.$ $\left.\left. - (1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \right] + \frac{\beta(q-1)\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{2\beta-1}}{\alpha^{2\beta+1}} \right\}$ $-(q$ $-2)\sum_{i=2}^{n} \frac{\sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[\frac{-\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} - \frac{(\beta-1)\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1}}{\alpha^{\beta+1}} \right] \left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta} \right] \right.$ $\left. + \frac{\beta(q-1)\left(r\sum_{j=1}^{i-1} t_j\right)^{2\beta-1}}{\alpha^{2\beta+1}} \right\}$ $+ \frac{(2-q)}{(1-q)} \left(\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \beta^2\left(\sum_{j=1}^{n} t_j\right)}{\left(T-r\sum_{j=1}^{n} t_j\right)\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)} \right.$ $\left. + \frac{(1-q)^2\left(\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)^2 \beta^2\left(\sum_{j=1}^{n} t_j\right)}{} \right)$

Second partial derivatives time terminated Kijima type I

$$\frac{\partial \ln L}{\partial\beta\partial r} = \frac{\partial \ln L}{\partial r\partial\beta} = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i-1} t_j}{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)}$$

$$- \frac{1}{\alpha^{\beta}} \sum_{i=1}^{n} \frac{\left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta-1} \sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[\beta \ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right) + 1\right] \left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \right.$$

$$\left. - (q-1)\frac{\beta}{\alpha^{\beta}} \left(t_i + r\sum_{j=1}^{i-1} t_j\right)^{\beta} \ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right) \right\}$$

$$- \frac{(q-2)}{\alpha^{\beta}} \sum_{i=2}^{n} \frac{\left(r\sum_{j=1}^{i-1} t_j\right)^{\beta-1} \sum_{j=1}^{i-1} t_j}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{ \left[\beta \ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right) + 1\right] \left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] \right.$$

$$\left. - (q-1)\frac{\beta}{\alpha^{\beta}} \left(r\sum_{j=1}^{i-1} t_j\right)^{\beta} \ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right) \right\}$$

$$+ \frac{1}{(1-q)} \left((2-q) \left(- \frac{(1-q)\left(\frac{T - r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \beta\left(\sum_{j=1}^{n} t_j\right) \ln\left(\frac{T - r\sum_{j=1}^{n} t_j}{\alpha}\right)}{\left(T - r\sum_{j=1}^{n} t_j\right)\left(1-(1-q)\left(\frac{T - r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)} \right.\right.$$

Second partial derivatives time terminated Kijima type I

$$\frac{\partial \ln L}{\partial q^2}=\frac{-n}{(2-q)^2}+\frac{(1-q)}{(1-q)^4}\sum_{i=1}^{n}\left\{\ln\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]+\frac{(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{(1-q)\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]^2}\left\{\frac{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}{(1-q)}-\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right\}$$

$$+\frac{1}{(1-q)^4}\sum_{i=2}^{n}\left\{\left[\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]+\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}\right]-2(q-1)(q-2)\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]\right\}$$

$$+\frac{1}{(1-q)^2}\sum_{i=2}^{n}\left\{\ln\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]+\frac{(q-1)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]^2}\left\{-\frac{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]}{(1-q)^2}-\frac{(q-2)}{(1-q)}\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right\}$$

$$-\frac{2}{(1-q)}\left(-\frac{\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right)$$

$$-\frac{2\left(-\ln\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)+\ln\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)\right)}{}$$

Second partial derivatives time terminated Kijima type I
$\frac{\partial lnL}{\partial\alpha^2}=\frac{n\beta}{\alpha^2}+\sum_{i=1}^{n}\left(-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)+\frac{1}{(1-q)}\left((2-q)\left(\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)^2\left(\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2}-\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)^2\left(\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2}\right)\right)$

Second partial derivatives time terminated Kijima type I
$\frac{\partial \ln L}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \frac{\left[ln\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)\right]^2 \left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{\left[1-(1-q)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] - (q-1)\left(\frac{t_i + r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right\} - (q-2)\sum_{i=2}^{n} \frac{\left[ln\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)\right]^2 \left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]^2} \left\{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right] - (q-1)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right\} + \frac{1}{(1-q)}\left((2-q)\left(\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \ln\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)} + \frac{(1-q)^2\left(\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)^2 \ln\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)^2} - \frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)} - \frac{(1-q)^2\left(\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)^2 \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)^2}\right)\right)$

Second partial derivatives time terminated Kijima type I

$$
\begin{aligned}
\frac{\partial lnL}{\partial\alpha\partial q} = {} & \sum_{i=1}^{n}\left(-\frac{\left(\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}}\right)+\sum_{i=2}^{n}\left(\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}}\right) \\
& -\frac{1}{(1-q)}\left(-\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right) \\
& +\frac{(2-q)}{(1-q)}\left(\frac{\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}}-\frac{\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right. \\
& \left.-\frac{(1-q)\left(\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}}\right)+\frac{(2-q)}{(1-q)^{2}}\left(-\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right)
\end{aligned}
$$

Second partial derivatives time terminated Kijima type I

$$\frac{\partial lnL}{\partial\alpha\partial\beta}=-\frac{n}{\alpha}+\sum_{i=1}^{n}\Bigg(\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}+\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}$$

$$+\frac{\left(\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta(1-q)\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\alpha\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}}\Bigg)$$

$$+\sum_{i=2}^{n}\Bigg(\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}$$

$$+\frac{(q-2)\left(\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta(1-q)\ln\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\alpha\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}}\Bigg)$$

$$+\frac{1}{(1-q)}\Bigg((2-q)\Bigg(-\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}$$

Second partial derivatives time terminated Kijima type I

$$
\begin{aligned}
\frac{\partial lnL}{\partial q \partial r} = \sum_{i=1}^{n} & \left(\frac{\left(\left(\frac{t_i + r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \right)^2 \beta \sum_{j=1}^{i-1} t_j}{\left(1 - (1-q) \left(\frac{t_i + r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \right)^2 \left(t_i + r \sum_{j=1}^{i-1} t_j \right)} \right) \\
& + \sum_{i=2}^{n} \left(- \frac{\left(\frac{r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \beta}{r \left(1 - (1-q) \left(\frac{r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \right)} + \frac{(q-2) \left(\left(\frac{r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \right)^2 \beta}{r \left(1 - (1-q) \left(\frac{r \sum_{j=1}^{i-1} t_j}{\alpha} \right)^{\beta} \right)^2} \right) \\
& + \frac{\left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \beta \left(\sum_{j=1}^{n} t_j \right)}{\left(T - r \sum_{j=1}^{n} t_j \right) \left(1 - (1-q) \left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \right)} - \frac{(2-q) \left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \beta \left(\sum_{j=1}^{n} t_j \right)}{(1-q) \left(T - r \sum_{j=1}^{n} t_j \right) \left(1 - (1-q) \left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \right)} \\
& + \frac{(2-q)}{(1-q)} \left(\frac{\left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \beta \left(\sum_{j=1}^{n} t_j \right)}{\left(T - r \sum_{j=1}^{n} t_j \right) \left(1 - (1-q) \left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \right)} + \frac{\left(\left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \right)^2 (1-q) \beta \left(\sum_{j=1}^{n} t_j \right)}{\left(T - r \sum_{j=1}^{n} t_j \right) \left(1 - (1-q) \left(\frac{T - r \sum_{j=1}^{n} t_j}{\alpha} \right)^{\beta} \right)^2} \right)
\end{aligned}
$$

Second partial derivatives time terminated Kijima type I

$$\frac{\partial lnL}{\partial r^2}=\sum_{i=1}^{n}\left(-\frac{(\beta-1)\left(\sum_{j=1}^{i-1}t_j\right)^2}{\left(t_i+r\sum_{j=1}^{i-1}t_j\right)^2}-\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}t_j\right)^2\beta^2}{\left(t_i+r\sum_{j=1}^{i-1}t_j\right)^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}\right.$$

$$\left.+\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}t_j\right)^2}{\left(t_i+r\sum_{j=1}^{i-1}t_j\right)^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{\left(\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}t_j\right)^2(1-q)}{\left(t_i+r\sum_{j=1}^{i-1}t_j\right)^2\left(1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$+\sum_{i=2}^{n}\left(-\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta^2}{r^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\beta}{r^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2\beta^2(1-q)}{r^2\left(1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^2}\right)$$

$$+\frac{(2-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{n}t_j\right)^2}{\left(T-r\sum_{j=1}^{n}t_j\right)^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(2-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{n}t_j\right)^2}{\left(T-r\sum_{j=1}^{n}t_j\right)^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}$$

$$+\frac{(2-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{n}t_j\right)^2(1-q)}{\left(T-r\sum_{j=1}^{n}t_j\right)^2\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2}$$

Second partial derivatives time terminated Kijima type I

$$
\begin{aligned}
\frac{\partial \ln L}{\partial\beta\partial q} = \frac{\partial \ln L}{\partial q\partial\beta} &= \sum_{i=1}^{n}\frac{\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{2\beta}\ln\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]^2}\\
&- \sum_{i=2}^{n}\frac{\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]^2}\left\{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right]-(q-2)\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right\}\\
&- \frac{1}{(1-q)}\left(\frac{(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right)\\
&+ \frac{1}{(1-q)}\left((2-q)\left(-\frac{\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)\left(\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2\ln\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-r\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2}\right.\right.\\
&\left.\left.+\frac{\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2\ln\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^2}\right)\right)
\end{aligned}
$$

APPENDIX D – Kijima type II – Time Terminated case

First partial derivatives time terminated Kijima II

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{-n\beta}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}$$

$$+ (q-2)\frac{\beta}{\alpha}\sum_{i=2}^{n}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}$$

$$+ \frac{(2-q)\left(-\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)} + \frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}\right)}{1-q}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n}{\beta} - n\ln(\alpha) + \sum_{i=1}^{n}\ln\left(t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}\right) - \left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\ln\left(\frac{t_i + \sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)$$

$$-(q-2)\sum_{i=2}^{n}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta} \frac{1}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\ln\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)$$

$$(2-q)\left((1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta} \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta} - (1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \right)$$

First partial derivatives time terminated Kijima II

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q} = \frac{-n}{(2-q)} &+ \frac{1}{(1-q)^2}\sum_{i=1}^{n} ln\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] \\
&+ \frac{1}{(1-q)}\sum_{i=1}^{n}\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\frac{1}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]} \\
&+ \frac{(q-2)}{(1-q)^2}\sum_{i=2}^{n} ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] \\
&+ \frac{1}{(1-q)}\sum_{i=2}^{n}\left\{ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(q-2)}{\left[1-(1-q)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right]}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right\} \\
&- \frac{-\ln\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)+\ln\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}{1-q} \\
&+ \frac{(2-q)\left(-\ln\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)\right)+\ln\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}{(1-q)^2} \\
&\quad (2-q)\left(-\frac{\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}}+\frac{\left(\frac{r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{1-(1-q)\left(\frac{r\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}\right)
\end{aligned}
$$

Second partial derivatives time terminated Kijima type II
$\frac{\partial lnL}{\partial r\partial \alpha}=\frac{\partial lnL}{\partial \alpha\partial r}=\sum_{i=1}^{n}\left(\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)+\sum_{i=2}^{n}\left(\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_jr^{i-j}(i-j)}{r}\right)(1-q)}{\alpha\left(t_i+\sum_{j=1}^{i-1}t_jr^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\right)-\frac{(2-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\beta^2\left(\frac{\sum_{j=1}^{n}t_jr^{n-j+1}(n-j+1)}{r}\right)}{\alpha\left(T-\sum_{j=1}^{n}t_jr^{n-j+1}\right)\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)}$ $(2-q)\left(\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\frac{\sum_{j=1}^{n}t_jr^{n-j+1}(n-j+1)}{r}\right)(1-q)$

Second partial derivatives time terminated Kijima type II

$$
\begin{aligned}
\frac{\partial lnL}{\partial r\partial\beta} = \sum_{i=1}^{n}\Bigg(& \frac{\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_j r^{i-j}\right)} - \frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)(1-q)\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)}{\left(t_i+\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\Bigg) \\
+\sum_{i=2}^{n}\Bigg(& -\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} - \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)(1-q)\ln\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)}{\left(\sum_{j=1}^{i-1}t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\Bigg)
\end{aligned}
$$

$$(2-q)\left(T-\sum_{j=1}^{n}t_j r^{n-j+1}\right)^{\beta}\ln\left(T-\sum_{j=1}^{n}t_j r^{n-j+1}\right)\beta\left(\sum_{j=1}^{n}t_j r^{n-j+1}(n-j+1)\right)$$

Second partial derivatives time terminated Kijima type II

$$\frac{\partial \ln L}{\partial q^2} = \frac{-n}{(2-q)^2} + \frac{(1-q)}{(1-q)^4}\sum_{i=1}^{n}\left\{\ln\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum_{i=1}^{n}\frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{(1-q)\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]^2}\left\{\frac{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}{(1-q)} - \left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right\}$$

$$+\frac{1}{(1-q)^4}\sum_{i=2}^{n}\left\{\left[\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\right]\right.$$

$$\left. - 2(q-1)(q-2)\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right]\right\}$$

$$+\frac{1}{(1-q)^2}\sum_{i=2}^{n}\left\{\ln\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] + \frac{(q-1)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}\right\}$$

$$+\sum_{i=2}^{n}\frac{\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]^2}\left\{-\frac{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]}{(1-q)^2} - \frac{(q-2)}{(1-q)}\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right\}$$

$$-\frac{2}{\quad}\left(-\frac{\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}}{\quad} + \frac{\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}}{\quad}\right)$$

Second partial derivatives time terminated Kijima type II

$$\frac{\partial lnL}{\partial\alpha^2}=\frac{n\beta}{\alpha^2}+\sum_{i=1}^{n}\Bigg(-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}$$

$$-\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\Bigg)$$

$$+\sum_{i=2}^{n}\Bigg(-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}$$

$$-\frac{\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{2\beta}\beta^2(1-q)}{\alpha^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^2}\Bigg)$$

$$+\frac{1}{(1-q)}\Bigg((2-q)\Bigg(\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\beta^2}{\alpha^2\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\beta}{\alpha^2\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)}$$

Second partial derivatives time terminated Kijima type II

$$
\begin{aligned}
\frac{\partial lnL}{\partial r^2} = \sum_{i=1}^{n} \Bigg(& \frac{(\beta-1)\left(\sum_{j=1}^{i-1}\left(\frac{t_j r^{i-j}(i-j)^2}{r^2} - \frac{t_j r^{i-j}(i-j)}{r^2}\right)\right)}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)} - \frac{(\beta-1)\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)^2} - \frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2 \beta^2}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\left(\frac{t_j r^{i-j}(i-j)^2}{r^2} - \frac{t_j r^{i-j}(i-j)}{r^2}\right)\right)}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} + \frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2(1-q)}{\left(t_i + \sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^2}\Bigg) \\
& + \sum_{i=2}^{n}\Bigg(- \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta^2\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} - \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\left(\frac{t_j r^{i-j}(i-j)^2}{r^2} - \frac{t_j r^{i-j}(i-j)}{r^2}\right)\right)}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} \\
& + \frac{(q-2)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} - \frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^2\beta^2\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)^2(1-q)}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)^2\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^2}\Bigg) \\
& - \frac{(2-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\beta^2\left(\frac{\sum_{j=1}^{n} t_j r^{n-j+1}(n-j+1)}{r}\right)^2}{\left(T-\sum_{j=1}^{n} t_j r^{n-j+1}\right)^2\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)} + \frac{(2-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{n}\left(\frac{t_j r^{n-j+1}(n-j+1)^2}{r^2} - \frac{t_j r^{n-j+1}(n-j+1)}{r^2}\right)\right)}{\left(T-\sum_{j=1}^{n} t_j r^{n-j+1}\right)\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)} \\
& (2-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{}\right)^{\beta}\beta\left(\frac{\sum_{j=1}^{n} t_j r^{n-j+1}(n-j+1)}{}\right)^2 \qquad (2-q)\left(\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{}\right)^{\beta}\right)^2\beta^2\left(\frac{\sum_{j=1}^{n} t_j r^{n-j+1}(n-j+1)}{}\right)^2(1-q)
\end{aligned}
$$

Second partial derivatives time terminated Kijima type II
$\frac{\partial lnL}{\partial\alpha\partial\beta}=\frac{\partial lnL}{\partial\alpha\partial\beta}=-\frac{n}{\alpha}+\sum_{i=1}^{n}\left(\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta(1-q)\ln\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)+\sum_{i=2}^{n}\left(\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\ln\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)}+\frac{(q-2)\left(\left(\frac{r\sum_{j=1}^{i-1}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta(1-q)\ln\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)+\frac{1}{(1-q)}\left((2-q)\left(-\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n}t_j r^{n-j+1}}{\alpha}\right)^{\beta}\ln\left(\frac{T-\sum_{j=1}^{n}t_j r^{n-j+1}}{\alpha}\right)^{\beta}\beta}{\left(\quad\left(T-\sum_{j=1}^{n}t_j r^{n-j+1}\right)^{\beta}\right)}-\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n}t_j r^{n-j+1}}{\alpha}\right)^{\beta}}{\left(\quad\left(T-\sum_{j=1}^{n}t_j r^{n-j+1}\right)^{\beta}\right)}\right.\right.$

Second partial derivatives time terminated Kijima type II

$$
\begin{aligned}
\frac{\partial lnL}{\partial\alpha\partial q}=\frac{\partial lnL}{\partial\alpha\partial q}=&\sum_{i=1}^{n}\left(-\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)\\
&+\sum_{i=2}^{n}\left(\frac{\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)}-\frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1}t_jr^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right)\\
&-\frac{1}{(1-q)}\left(-\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}\right)\\
&+\frac{(2-q)}{(1-q)}\left(\frac{\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)}+\frac{(1-q)\left(\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n}t_jr^{n-j+1}}{\alpha}\right)^{\beta}\right)^{2}}\right.\\
&\left.-\frac{\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)}-\frac{(1-q)\left(\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}\beta}{\alpha\left(1-(1-q)\left(\frac{\sum_{j=1}^{n}t_j}{\alpha}\right)^{\beta}\right)^{2}}\right)
\end{aligned}
$$

Second partial derivatives time terminated Kijima type II

$$
\begin{aligned}
\frac{\partial lnL}{\partial r\partial q} = \frac{\partial lnL}{\partial q\partial r} = {} & \sum_{i=1}^{n}\left(\frac{\left(\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(t_i+\sum_{j=1}^{i-1} t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right) \\
& + \sum_{i=2}^{n}\left(-\frac{\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)} + \frac{(q-2)\left(\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\sum_{j=1}^{i-1}\frac{t_j r^{i-j}(i-j)}{r}\right)}{\left(\sum_{j=1}^{i-1} t_j r^{i-j}\right)\left(1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right)^{2}}\right) \\
& - \frac{\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\beta\left(\frac{\sum_{j=1}^{n} t_j r^{n-j+1}(n-j+1)}{r}\right)}{\left(T-\sum_{j=1}^{n} t_j r^{n-j+1}\right)\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)} \\
& - \frac{(2-q)\left(\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)^{2}\beta\left(\frac{\sum_{j=1}^{n} t_j r^{n-j+1}(n-j+1)}{r}\right)}{\left(T-\sum_{j=1}^{n} t_j r^{n-j+1}\right)\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)^{2}}
\end{aligned}
$$

Second partial derivatives time terminated Kijima type II
$\frac{\partial \ln L}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \frac{\left[ln\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)\right]^2 \left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^\beta}{\left[1-(1-q)\left(\frac{t_i+r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^\beta\right]^2} \left\{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right] - (q-1)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right\}$ $- (q-2)\sum_{i=2}^{n} \frac{\left[ln\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)\right]^2 \left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^\beta}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^\beta\right]^2} \left\{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right]\right.$ $\left. - (q-1)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^\beta\right\} \frac{1}{(1-q)} \left((2-q)\left(\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^\beta \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^\beta\right)}\right.\right.$ $+ \frac{(1-q)^2\left(\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^\beta\right)^2 \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^\beta\right)^2} - \frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^\beta \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^\beta\right)}$ $\left.\left. - \frac{(1-q)^2\left(\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^\beta\right)^2 \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^2}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^\beta\right)^2}\right)\right)$

Second partial derivatives time terminated Kijima type II

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta \partial q} = \frac{\partial \ln L}{\partial q \partial \beta} = & \sum_{i=1}^{n} \frac{\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{2\beta} \ln\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)}{\left[1-(1-q)\left(\frac{t_i+\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]^2} \\
& - \sum_{i=2}^{n} \frac{\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta} \ln\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)}{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j r^{i-j}}{\alpha}\right)^{\beta}\right]^2} \left\{\left[1-(1-q)\left(\frac{\sum_{j=1}^{i-1} t_j\, r^{i-j}}{\alpha}\right)^{\beta}\right] - (q-2)\left(\frac{r\sum_{j=1}^{i-1} t_j}{\alpha}\right)^{\beta}\right\} \\
& - \frac{1}{(1-q)}\left(\frac{(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta} \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)} - \frac{(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}\right) \\
& + \frac{1}{(1-q)}\left((2-q)\left(-\frac{\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta} \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)}\right.\right. \\
& - \frac{(1-q)\left(\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)^2 \ln\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)}{\left(1-(1-q)\left(\frac{T-\sum_{j=1}^{n} t_j r^{n-j+1}}{\alpha}\right)^{\beta}\right)^2} + \frac{\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta} \ln\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)}{\left(1-(1-q)\left(\frac{\sum_{j=1}^{n} t_j}{\alpha}\right)^{\beta}\right)}
\end{aligned}
$$