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ADVANCES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION

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ADVANCES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION

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GIANNINI ITALINO ALVES VIEIRA

ADVANCES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION

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Abstract

In this thesis we present some advances obtained in the graph model for conflict resolution (GMCR). The first one is a new stability concept, called *symmetric sequential stability* (*SSEQ*), which was proposed for conflicts involving n decision makers (DMs) and the relationships between this new concept and the existing concepts in GMCR is analyzed. In addition, an extension of this concept to other preference structures is proposed. The second advance was to propose matrix representations to facilitate the obtaining of stable states according to the stability definitions proposed in the GMCR with probabilistic preferences and also according to the SSEQ notion proposed for such model. The third advance was to modify the GMCR allowing the DMs to have iterated levels of unawareness about the options available to them in a conflict, i.e., we consider that DMs may be unaware of some of their options, or some options of their opponents and, therefore, may have only partial knowledge of the state space of the conflict. Finally, the fourth and final advance of this thesis is to present an alternative definition of the stability concept *generalized metarationality* for conflicts with n-DMs. Our motivation to propose such an alternative definition lies on the fact that, unlike the definition of *generalized metarationality* for n-DMs in the literature, our definition coincides with the *generalized metarationality* for conflicts involving only two DMs. In addition, we have pointed out some problems in results that relate this definition to other solution concepts in the GMCR and analyze which properties are satisfied by the alternative definition that we propose.

Keywords: Conflicts. Graph Model. Stability Concepts. Unawareness.

Resumo

Nesta tese apresentamos alguns avanços obtidos no modelo de grafos para resolução de conflitos (GMCR). O primeiro deles é um novo conceito de estabilidade, chamado *symmetric sequential stability* ($SSEQ$), o qual foi proposto para conflitos envolvendo n decisores (DMs) e analisamos as relações entre esse novo conceito e os conceitos existentes no GMCR, além de estendermos tal conceito para outros GMCR com diferentes estruturas de preferências. O segundo avanço foi propor representações matriciais para facilitar a obtenção de estados estáveis de acordo com as definições de estabilidades propostas no GMCR com preferências probabilísticas e também de acordo com a noção de $SSEQ$ proposta para tal modelo. O terceiro avanço foi modificar o GMCR permitindo que os DMs possam ter níveis iterados de falta de consciência sobre as opções disponíveis para estes em um conflito, isto é, consideramos que os DMs podem estar inconscientes sobre algumas de suas opções, ou sobre as opções de seus oponentes e, portanto, podem ter apenas conhecimento parcial a respeito do espaço de estados do conflito. Finalmente, o quarto e último avanço dessa tese consiste em apresentar uma definição alternativa do conceito de estabilidade *generalized metarationality*, para conflitos com n-DMs. Nossa motivação para propor tal definição alternativa reside no fato de que, ao contrário da definição de *generalized metarationality* para n-DMs na literatura, nossa definição coincide com a definição *generalized metarational* no caso em que o conflito tem apenas dois DMs. Além disso, apontamos alguns problemas em resultados que relacionam tal definição com outros conceitos de solução no GMCR e analisamos quais propriedades são satisfeitas pela definição alternativa que propomos.

Palavras-chave: Conflitos. Modelo de Grafos. Conceitos de Estabilidade. Falta de Consciência.

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List of Abbreviations and Acronyms

GMCR	Graph Model for Conflict Resolution
DMs	Decision makers
GMCRP	Graph Model for Conflict Resolution with Probabilistic Preferences
Nash	Nash stability
GMR	General metarationality stability
SMR	Symmetric metarationality stability
SEQ	Sequential stability
$SSEQ$	Symmetric sequential stability
L_h	Limited-move stability with horizon h
$CGMR$	Coalitional general metarationality stability
$CSMR$	Coalitional symmetric metarationality stability
$CSEQ$	Coalitional sequential stability
$CSSEQ$	Coalitional symmetric sequential stability
α-Nash	α-Nash stability
(α, β)-GMR	(α, β)-metarationality stability
(α, β)-SMR	(α, β)-symmetric metarationality stability
(α, β, γ)-SEQ	(α, β, γ)-Sequential stability
(α, β, γ)-$SSEQ$	(α, β, γ)-Symmetric sequential stability
$GGMR$	Generalized general metarationality stability
$GSMR$	Generalized symmetric metarationality stability
$GSEQ$	Generalized sequential stability

$GSSEQ$	Generalized symmetric sequential stability
MR_h	Metarational stability with horizon h
CMR_h	Credible metarational stability with horizon h
MR_r	i-Metarational stability with r rounds
CMR_r	i-Credible metarational stability with r rounds
$\overline{MR}_r$	$\bar{i}$-Metarational stability with r rounds
$\overline{CMR}_r$	$\bar{i}$-Credible metarational stability with r rounds

List of Symbols

N	Set of Decision Makers
H	Subset of DMs, i.e, a coalition
S	Set of States
s	State s
D_i	Directed graph for DM i
A_i	Set of arcs for DM i
$\succ_i$	Strict preference relation for DM i
$\succeq_i$	Non-strict preference relation for DM i
$\sim_i$	Indifference relation for DM i
$R_i(s)$	Set of all reachable states from s for DM i
$R_i^+(s)$	Set of all unilateral improvements states from s for DM i
$R_H(s)$	Set of all reachable states from s for coalition H
$R_H^+(s)$	Set of all unilateral improvements states from s for coalition H
$\Omega_H(s, s_1)$	Subset of H whose members are DMs who make the last move to reach s_1 in a legal sequence of moves from s
$\Omega_H^+(s, s_1)$	Subset of H whose members are DMs who make the last move to reach s_1 in a legal sequence of unilateral improvements from s
$K_i(s)$	Cardinality of the set of states that are worse than s for DM i
$G_h(i, s)$	Anticipation vector of DM i from s
$\varphi(N)$	Class of all coalitions of DMs in N
C	Class of coalitions

$R_H^{++}(s)$	Coalition improvement list from s by coalition H
$R_C(s)$	Set of reachable states by class C from s
$\Omega_C(s, s_1)$	Subset of C whose members are the sets of DMs that make the final legal sequence of movements to achieve state s_1 from s
$R_C^{++}(s)$	Class coalitional improvement list from state s by class C
$\Omega_C^{++}(s, s_1)$	Subset of C whose members are the subsets of DMs that make the last improvement movement to achieve s_1 in a legal sequence of movements from s
U_i	Uncertain preferences of DM i
$R_i^U(s)$	DM i's reachable list from state s by a unilateral uncertain move
$R_i^{+,U}(s)$	DM i's reachable list from state s by a unilateral improvement or a unilateral uncertain move
$R_H^{+,U}(s)$	Set of unilateral improvements or unilateral uncertain moves by coalition H
$\Omega_H^{+,U}(s, s_1)$	Set of all last DMs in unilateral improvements or uncertain moves from s to s_1
A	Matrix that represents the Fuzzy preferences
$a^k(s_i, s_j)$	The preference degree of state s_i over s_j for DM k
$\hat{R}_{k,\gamma_k}^{+}(s)$	Fuzzy unilateral improvement list for DM k
$\hat{\Omega}_{H,\gamma_H}^{+}(s, s_1)$	Set of all last DMs who make the last fuzzy improvement move in a legal sequence from s to s_1.
α	Parameter lying in the interval $[0, 1]$
β	Parameter lying in the interval $[0, 1]$
γ	Parameter lying in the interval $[0, 1]$
$P_i(s, q)$	Chance with which DM i prefers state s over q
$R_i^{+\gamma}(s)$	Set of all γ-improvements for DM i when the current state is s
$R_H^{+\gamma}(s)$	Set all γ-unilateral improvement by coalition H from state s
$\Omega_H^{+\gamma}(s, s_1)$	Set of all last DMs in a legal sequence of unilateral γ-improvement from s to s_1
$\Phi_i^{+\gamma}(s)$	set of all states that DM i strictly prefers to state s with probability greater that γ
J_i	Accessibility Matrix for DM i
$J_i^{+\gamma}$	Matrix γ unilateral improvements for DM i
Y	Matrix with all elements equal to 1
e_k	The $\|s\|$-dimensional column vector with k^{th} element equal to 1 and all other elements equal to 0

$"\circ"$	Hadamard product
$\text{sign}[K]$	Matrix signal of matrix K
$Q_i^{+\gamma}$	Matrix with element (s,q) equal to 1 if state q is strictly preferred by DM i over state s with probability greater than γ, and 0 otherwise
$Q_i^{=\gamma}$	Matrix with element (s,q) equal to 1 if state q is strictly preferred by DM i over state s with probability equal than γ, and 0 otherwise
$Q_i^{-\gamma}$	Matrix with element (s,q) equal to 1 if state q is strictly preferred by DM i over state s with probability smaller than γ, and 0 otherwise
$Q_i^{-,=,\gamma}$	Matrix with element (s,q) equal to $1 - Q_i^{+\gamma}(s,q)$
M_H	Matrix of reachable states for coalition H
M_H^{γ}	Matrix of unilateral γ-improvement by DMs in coalition H
$\mathcal{A}$	Set of all options available to all DMs in the conflict
$\mathcal{A}^*$	Some non-empty subset of the power set of $\mathcal{A}$
$\sum$	Union of spaces
$S_{\alpha'}$	State space associated to the subset of the options $\alpha^{'}$
$r_S^{S^{'}}$	Surjection that associates each state in a more refined state space (S) with some state in a less refined state space $(S^{'})$
$(S_\alpha, A_i^{S_\alpha})$	Directed graphs defined in space S_α
$\succ_i^{S_\alpha}$	Preference relation defined in space S_α
$R_i^{S_\alpha}(s)$	Set of reachable states from s by DM i in space S_α
$R_i^{+,S_\alpha}(s)$	Set of unilateral improvements from s by DM i in space S_α
$\prod_i$	awareness function of DM i
Φ	Standard GMCR defined in space S
Φ'	Canonical representation of Φ as a GMCR with interactive unawareness
$U_i^{S_{\alpha'}}(s_1)$	Subset consisting of all states in $S_{\alpha'}$ reachable for DM i from state s_1 in one step considering that at s_1, DM i may not be aware of all options in α'
$U_j^{+,S_{\alpha'}}(s_1)$	Subset consisting of all states in $S_{\alpha'}$ that are unilateral improvement moves from s_1 by DM j, considering that at s_1, DM j may not be aware of all options in α'

$U_H^{S_{\alpha'}}(s)$	Set of all states in space $S_{\alpha'}$ that can be reached for te coalition H, considering that the DMs in H may not be aware of all options in α' while moving in the sequence
$\Omega_H^{S_{\alpha'}}(s, s_1)$	Subset of H whose members are DMs that make the last move to reach s_1 in a legal sequence of moves from s, considering that DMs may be unaware of some options in α' while moving
$U_H^{+,S_{\alpha'}}(s)$	Set of all states in space $S_{\alpha'}$ that can be reached from a legal sequence of unilateral improvements by coalition H, considering that the DMs in H may not be aware of all options in α' while moving in the sequence
$\Omega_H^{+,S_{\alpha'}}(s, s_1)$	Subset of H whose members are DMs that make the last move to reach s_1 in a legal sequence of unilateral improvement from s, considering that DMs may be unaware of some options in α' while moving
P_i	*Policy* of DM $i \in N$
$P_i^c(s)$	Credible *Policy* of DM $i \in N$
h	Horizon or length
r	Rounds
$\mathcal{A}_i^r(s)$	Metarational tree based on P_j, $j \in N - \{i\}$, for DM i from state s with r rounds

Contents

CHAPTER 1

Introduction

1.1 Game Theory and Conflict analysis

Game theory is an important mathematical theory whose main objective is to analyze situations involving strategic interactions, such as war problems or financial and economic speculations. At the beginning of the 20th century, Emile Borel [1] and Von Neumann [2] began to analyze situations like these from the skill of the agents involved, not just from the lucky factor. In the middle of 20th century, after the publication of book *Theory of Games and Economic Behavior* (Von Neumann and Morgenstern [3]) and the works of John Nash *Equilibrium Points in n-Person Games* [4], *Non-cooperative Games* [5], *The Bargaining Problem* [6] and *Two Person Cooperative Games* [7], this theory gained considerable prominence in economics and applied mathematics due to the mathematical techniques employed, which enabled mathematical formalism in the analysis of strategic situations among multiple agents.

A branch of game theory that is devoted to conflict analysis began to be broadly developed from Howard [8] pioneering work on metagame analysis. Since then, several contributions have emerged in the area of conflict analysis such as [9] and [10]. Fraser and Hipel [11] proposed a model that was based on concepts of game theory and conflict analysis, such a model is called the graph model for conflict resolution (GMCR). The GMCR is a model that basically describes a set of possible states (outcomes) that can arise in a conflict according to actions that can be taken by individuals involved in a conflict, called decision makers (DMs). DMs may change the conflict state by changing some of their actions taking into account their preferences over the set of possible states in the conflict and the countermoves of the other DMs.

Since DMs can behave in different ways, there are several stability definitions (solution concepts) which determine whether or not a DM has incentive to move away from a given state. In the GMCR, there are a number of stability concepts used in conflict resolution. Some of these concepts are: Nash stability [5], general metarationality (GMR) [8], symmetric metarationality (SMR) [12], sequential stability (SEQ) [12], limited-move stability of horizon h (L_h) [13] and metarational stable states of r rounds (MR_r) [14]. Such stability concepts are defined for a given DM, called focal DM, which is considering whether or not to move away from a given state. Theses concepts differ in what are the sanctions allowed for the opponents of the focal DM and in how far ahead the focal DM foresees the conflict. If a state is stable for all DMs in the conflict according to a particular solution concept, then it is called an equilibrium according to such concept.

In view of the many works that have been developed on the GMCR in order to extend the model to capture the most diverse situations of conflict that can arise in the real world, we present in this thesis advances that we have made on the GMCR. The contributions range from the proposal of new solution concepts, the demonstration of mathematical results that facilitate the calculation of stable states in the GMCR with probabilistic preferences [15] and the formulation of a GMCR that allows for lack of awareness among the DMs involved in the conflict.

1.2 Objectives

This thesis aims to present advances that we made in the GMCR. We present the objectives below.

(1) Since several solution concepts have been proposed in order to represent the most varied human behaviors, the first objective of this thesis was to propose a concept of stability that

is a type of SEQ stability, but that allows a counter-reaction of the focal DM. This solution concept, called symmetric sequential stability ($SSEQ$) and presented in Chapter 3, was proposed for bilateral and multilateral conflicts. We obtained results relating $SSEQ$ with the previously mentioned solution concepts that are usually used in the GMCR. We also proposed the $SSEQ$ stability definition for a coalition and obtained its relationship with the classical stability definitions in coalitional analysis. Additionally, we extended this new solution concept for n-DM GMCR with uncertain [16], probabilistic [15] and fuzzy preferences [17].

(2) Stability analysis is a fundamental procedure in conflict analysis. Many of the usual solution concepts in the GMCR are complicated or require a lot of calculation to be used in conflicts involving many DMs and states. In the works of Xu et al. [18] and [26], matrix representations were proposed to facilitate the achievement of stable states according to the usual definitions of GMCR stability (Nash, GMR, SMR and SEQ). [15] provided an extension of the GMCR in which probabilistic preferences are adopted in the model. For this model, new notions of stability were proposed, similar to the usual definitions in the GMCR. The second objective of this thesis was to propose matrix results similar to those obtained in [18] to facilitate the calculation of stable states in the GMCR with probabilistic preferences. Additionally, we propose a matrix representation for the $SSEQ$ concept defined in the GMCR with probabilistic preferences.

(3) An assumption commonly adopted in most part of the GMCR literature is the common knowledge of the DMs involved about who are the DMs in the conflict, what are the states of the conflict, what are the reachable states and preferences of the DMs over the set of states. The third objective of this thesis was to propose a GMCR in which this assumption was removed, i.e., we modify the standard GMCR to allow for the possibility that DMs may be unaware of some of the options available in the conflict. Our motivation for proposing this model is that in some conflicts having an available option that your opponents is unaware of can be crucial to determine what kinds of conflict resolutions can be achieved.

For example, in a war setting developing a new weapon technology which the adversary is unaware of can be crucial in defining the war resolution.

(4) As we study the concept of generalized metarationality for n-DMs [14], we observed that it is not an extension of the generalized metarationality concept, proposed in [19], for conflicts with 2 DMs. Moreover, we also observed that some results presented in [14] are false. For example, there is a problem with the result that SMR is equivalent to a particular case of generalized metarationality for n-DM conflicts. In view of these problems, the fourth and final objective of this thesis is to present an alternative definition for generalized metarationality that coincides with the definition previously proposed by [19] in the case $n = 2$ and verify which of the results stated in [14] are still valid with the alternative definition proposed.

1.3 Methods and procedures

The method used to elaborate this thesis consisted in making a study of the works that have been proposed in the literature on GMCR. In particular, we investigated what were the most recent developments in the theory, that is, what was at the frontier of science with respect to such model. From our studies, we saw the need to propose a new definition ($SSEQ$), which would be a refinement of SMR and SEQ, reducing the number of stable states of the conflict. When analyzing the relations between this new concept ($SSEQ$) and the concepts of stabilities existing in the GMCR, we found some problems in the literature, more specifically in the generalized metarationality concept of Zeng et al. [14]. These problems motivated us to try to correct them by proposing alternative generalized metarationality definitions to overcome some of them.

By studying the GMCR with probabilistic preferences, proposed by Rêgo and Santos [15], we felt the need to propose a more efficient way to calculate the parameter regions of stability for the conflict states. The works of Xu et al. [18] and [26] provided us ideas that were adapted to our desired situation. Finally, we saw that there were some GMCR that allowed for the possibility of misperception in the GMCR and that motivated us to use our familiarity with the theory of unawareness in the game theory literature, specially with the work of Heifetz et al. [54], and use it to investigate the impact of unawareness in the stability analysis of the GMCR.

1.4 Thesis Organization

This thesis is divided into 6 chapters, including this introductory chapter. In Chapter 2, we recall the GMCR and the mostly used solution concepts. In this chapter, we also present an overview of the main theoretical and applied works that have been done in the GMCR literature. In Chapter 3, we present the $SSEQ$ stability and several results establishing relationships between $SSEQ$ and other solution concepts used in the GMCR. Moreover, we provide a definition of $SSEQ$ stability for coalitions. Finally, we finish this chapter extending the $SSEQ$ definition to other GMCR models with different preference structures, such as GMCR with uncertain preferences [16], GMCR with probabilistic preferences [15] and GMCR with fuzzy preferences [17].

In Chapter 4, we have developed matrix results, similar to those obtained in [18], to find stable states in the GMCR with probabilistic preferences with n decision makers. The matrix methods are used to determine more easily the stable states according to four stability definitions proposed for this model, namely: α-Nash stability, (α, β)-metarationality, (α, β)-symmetric metarationality and (α, β, γ)-sequential stability. Additionally, we have also proposed a matrix result to determine (α, β, γ)-$SSEQ$ stable states more efficiently in the GMCR with probabilistic preferences.

In Chapter 5 we generalize the GMCR to allow for interactive unawareness of the DMs in bilateral and multilateral conflicts. More specifically, we consider a GMCR, where a DM, in some given state, can be unconscious about some of his options, or about the options of his opponents, and therefore, may have only a partial knowledge of the state space. Additionally, we generalize standard solution concepts for this model.

In Chapter 6 we show that the concept of generalized metarationality for n-DMs proposed in [14] is not an extension of the generalized metarationality concept proposed in [19], for the particular case where $n = 2$. Such observation led us to seek an alternative definition for generalized metarationality stability for n-DM conflicts that coincides with the definition proposed earlier by [19] in the case $n = 2$. Moreover, we show that some of the results stated in [14] for n-DM conflicts relating generalized metarationality and other solution concepts are not valid. In particular, there is a problem with the result that SMR is equivalent to a particular case of generalized metarationality for n-DM conflicts, as stated in [14]. In this chapter we proposed an alternative generalized metarationality definition for n-DM conflicts that, unlike to the original definition, captures the concept of SMR as a special case and coincides with the definition proposed in [19] for conflicts involving only two DMs.

1.5 Computer Support

To write this thesis we use the typographic system LaTeX[1], which is a tool for the production of mathematical and scientific texts due to its high typographic quality. The MikTeX software was adopted: an implementation of LaTeX for use in the Windows environment. In addition, to do the matrix algorithms, we use the statistical software R[2] which is a tool used in statistical data analysis.

[1]For more information and details on the typography system LaTeX see De Castro (2003) or visit http://www.tex.ac.uk/CTAN/latex.

[2]The software R can be found at https://cran.r-project.org/bin/windows/base/

CHAPTER 2

Theoretical background

2.1 Introduction

The graph model for conflict resolution (GMCR) was originally proposed in [20] and is a mathematical tool used in conflict analysis. In the GMCR, there is a set of decision makers (DMs) that may take some actions and a set of states (possible conflict resolutions) that may arise according to the actions taken by DMs. DMs can change the state of the conflict by changing some of their actions. DMs have preferences over the set of states and may change states taking into account such preferences and the countermoves reachable to other DMs that participate in the conflict.

Various notions of stability (solution concepts) have been proposed in the GMCR literature aiming to model the various types of behavior that can arise in a strategic conflict. When a state of a conflict satisfies a particular stability definition for all DMs involved in the conflict, this state is considered an equilibrium according to that particular definition. In the GMCR, there are several solution concepts, some of these are: Nash stability [5], general metarationality (GMR) [8], symmetric metarationality (SMR) [12], sequential stability (SEQ) [12], and limited-move stability of horizon h (L_h) [13].

Next, we present an overview about the GMCR literature, recalling the basic idea of the GMCR and its main solution concepts for conflicts with 2-DMs, n-DMs and for coalitions, which are important for the good comprehension of the results that will be presented in this thesis.

2.2 GMCR and Solution Concepts

In this section, we recall the basic idea of the GMCR and the following stability definitions: Nash stability, GMR stability, SMR stability, SEQ stability, L_h stability and CMR_r stability. Additionally, we also recall some standard stability definitions of coalitional analysis. The main objective to review these solution concepts is to establish relationships between these stability notions and the new concepts that are presented in this thesis.

2.2.1 *GMCR*

The GMCR was introduced by Kilgour *et al.* [20] and consists of a set of DMs N, with cardinality equal to n, a set of possible states or conflict scenarios, $S = \{s_1, \ldots, s_m\}$, and, for each DM $i \in N$, a preference relation over S and a directed graph $D_i = \{S, A_i\}$, where $A_i \subseteq S \times S$ determines for each state s to what states DM i can lead the conflict, called reachable states from s in one step.

The GMCR provides a framework in which to analyze strategic interactions among DMs, based on the information of the options available to DMs and on what their preferences about the conflict states are. As in most game theoretic models, in GMCR it is assumed that the preferences of a DM i can be expressed by a binary relation on S, denoted by $\succ_i$, where $s \succ_i s_1$ indicates that DM i strictly prefers state s to state s_1. Additionally, one can also derive the weak preference relation $\succeq_i$, where $s \succeq_i s_1$ means that DM i does not strictly prefer state s_1 to state s, and the indifference relation $\sim_i$, where $s \sim_i s_1$ means that DM i does not strictly prefer state s to state s_1 and does not strictly prefer state s_1 to state s.

2.2.2 *Solution concepts in the GMCR*

The study of possible moves and countermoves made by DMs in strategic conflicts is called stability analysis. Several different behaviors can arise in conflict situations, so many concepts of stability have been proposed and are still being obtained. In this section, we review six stability concepts used in the GMCR, namely: Nash stability, GMR stability, SMR stability, SEQ stability and L_h stability. In order to present such definitions, we need to describe some basic components that are useful in their formalization.

Let $i \in N$ and denote by $R_i(s)$ the set of all states in S that are reachable in one step for DM i when the current state is s, i.e., $R_i(s) = \{s_1 \in s : (s, s_1) \in A_i\}$, and denote by $R_i^+(s)$ the set of all states that are attainable for DM i when the current state is s and that are preferable, for DM i, to state s, i.e. $R_i^+(s) = \{s_1 \in R_i(s) : s_1 \succ_i s\}$. As usual in the GMCR literature, we assume that $s \notin R_i(s)$, $\forall s \in S$ and $\forall i \in N$.

Solution concepts in the GMCR with two DMs

In some chapters of this thesis we first present GMCR extensions for conflicts with two DMs and then generalize to conflicts with n-DMs. Therefore, it is necessary to recall the stability concepts for conflicts with two or more DMs. Next, we recall the usual stability concepts in the GMCR involving only two DMs that can be found in more details in [21].

Let $N = \{1, 2\}$ and let $i, j \in N$, such that $i \neq j$, then the solutions concepts Nash, GMR, SMR, SEQ and L_h stability are defined as follows.

Definition 2.2.1. *A state $s \in S$ is Nash stable for DM $i \in N$ iff $R_i^+(s) = \emptyset$.*

Intuitively, if a DM i is in a Nash stable state, then he or she has no incentive to move away from it in a single step. Note that the Nash solution concept does not depend on the behavior of the opponents of the focal DM. Thus, this concept is the same for conflicts with n-DMs.

Definition 2.2.2. *A state $s \in S$ is GMR stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$.*

Definition 2.2.3. *A state $s \in S$ is SMR stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

Definition 2.2.4. *A state $s \in S$ is SEQ stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$.*

Intuitively, if a DM i is in a GMR stable state, he has no incentive to move away from it, because he foresees a reaction of his opponent leading the conflict to a no better situation. In an SMR stable state, DM i cannot escape from this latter no better situation. Finally, in an SEQ stable state, the move in the reaction of the opponent of DM i is beneficial to him or her, but no requirement to whether DM i may counter-react is made.

In what follows, we present the limited-move stability definition for conflicts with two DMs. For this definition, we assume that DMs are not indifferent between any pair of states. In order to define the limited-move stability notion we need to introduce some concepts (that can be found in more details in [21]). Let $K_i(s)$ be the cardinality of the set of states that are worse than s for DM i, i.e., $K_i(s) = \#\{s_1 \in S : s \succ_i s_1\}$. Let h be a positive integer number. A DM who foresees a sequence of length at most h is said to be a DM with horizon h. Let $G_h(i,s) \in S$, $i \in N$, be the state that DM i believes that will be the final state of the conflict when he or she foresees a horizon h, the conflict starts at state s and DM i moves first. Then $(G_h(i,s), s \in S)$ is the anticipation vector of DM i and, for convenience, $G_0(i,s) = s$. The anticipation vector of DM i is constructed, inductively, in the following way: If $R_i(s) = \emptyset$, then state s is stable for DM i, because DM i is unable to move from this state. If $R_i(s) \neq \emptyset$ and $h \geq 1$, then $G_h(i,s)$ is constructed as follows:

$$G_h(i,s) = \begin{cases} s \;\; if \;\; R_i(s) = \emptyset \;\; or \;\; if \;\; K_i(s) \geq A_h(i,s), \\ G_{h-1}(j, M_h(i,s)) \;\; if \;\; R_i(s) \neq \emptyset \;\; and \;\; K_i(s) < A_h(i,s), \end{cases}$$

where $M_h(i,s)$ is the unique state $s_1^* \in R_i(s)$ which satisfies $K_i(G_{h-1}(j,s_1^*)) = \max\{K_i(G_{h-1}(j,s_1)) : s_1 \in R_i(s)\}$, $j \neq i$, and $A_h(i,s) = K_i(G_{h-1}(j, M_h(i,s)))$.

Having defined $G_h(i,s)$, the definition of L_h stability in this case is given as follows:

Definition 2.2.5. *A state $s \in S$ is limited-move stable with horizon h for DM $i \in N$ iff $G_h(i,s) = s$.*

Solution concepts in GMCR with n-DM

As previously mentioned, the Nash concept for conflicts with n-DMs is defined exactly as in Definition 2.2.1. In order to present GMR, SMR and SEQ stability definitions, we need to recall the concept of a legal sequence of movements and of unilateral improvement for a group of DMs $H \subseteq N$.

Let $H \subseteq N$ be a subset of DMs, called a coalition, and let $R_H(s) \subseteq S$ denote the set of states that can be reached by any *legal sequence* of movements, where a sequence of movements is legal if any DM may move more than once, but not twice consecutively. Let $\Omega_H(s, s_1)$ be the subset of H whose members are DMs who make the last move to reach s_1 in a legal sequence of moves from s. $R_H(s)$ and $\Omega_H(s, \cdot)$ are the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$ and $s_1 \in R_i(s)$, then $s_1 \in R_H(s)$ and $i \in \Omega_H(s, s_1)$, and (2) if $s_1 \in R_H(s)$, $i \in H$, $\Omega_H(s, s_1) \neq \{i\}$ and $s_2 \in R_i(s_1)$, then $s_2 \in R_H(s)$ and $i \in \Omega_H(s, s_2)$. Let $R_H^+(s) \subseteq S$ be the set of all states that result from a *legal sequence of unilateral improvements*, starting at state s, where a sequence unilateral improvements is legal if any DM may make unilateral improvements more than once, but not twice consecutively. Similarly, if $s_1 \in R_H^+(s)$, then $\Omega_H^+(s, s_1)$ is the set of all last DMs in a legal sequence of unilateral improvements from s to s_1. We have that $R_H^+(s)$ and $\Omega_H^+(s, \cdot)$ are defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$ and $s_1 \in R_i^+(s)$, then $s_1 \in R_H^+(s)$ and $i \in \Omega_H^+(s, s_1)$, and (2) if $s_1 \in R_H^+(s)$, $i \in H$, $\Omega_H^+(s, s_1) \neq \{i\}$ and $s_2 \in R_i^+(s_1)$, then $s_2 \in R_H^+(s)$ and $i \in \Omega_H^+(s, s_2)$.

We can now state the definitions of GMR, SMR and SEQ stability, respectively, as follows:

Definition 2.2.6. *A state $s \in S$ is GMR stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$.*

Definition 2.2.7. *A state $s \in S$ is SMR stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

Definition 2.2.8. *A state $s \in S$ is SEQ stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}^+(s_1)$ such that $s \succeq_i s_2$.*

Analogously to the respective definitions presented in the previous subsection, we have that, intuitively, if a DM i is in a GMR stable state, he or she has no incentive to move away from it, because he foresees a reaction of his opponents leading the conflict to a no better situation.

In an SMR stable state, DM i cannot escape from this latter no better situation. Finally, in a SEQ stable state, all the moves in the reaction of the opponents of DM i are beneficial to them, but no requirement to whether DM i may counter-react is made.

2.2.3 *Coalition Stability Analysis*

In conflict situations, DMs can act together in order to achieve mutual benefits [56]. A coalition is a set of DMs acting together to achieve results which are desirable for all DMs in the set. Below, we recall the solution concepts in GMCR which take into account the possibility of coalitions formation. The coalition stability concepts recalled in this subsection are due to [56] and [23].

Let $\emptyset \neq H \subseteq N$ be a coalition of DMs in N and $\varphi(N)$ be the class of all coalitions of DMs in N. In the coalitional stability analysis, the coalition improvement list from s by coalition H is defined by $R_H^{++}(s) = \{s_1 \in S : s_1 \in R_H(s) \text{ and } s_1 \succ_i s \text{ for all } i \in H\}$. In this setting, we have two notions of stability: for a coalition and for a DM. Coalitional Nash stability can be defined as follows.

Definition 2.2.9. *(Coalitional Nash Stability for a Coalition) Let $H \in \varphi(N)$. A state $s \in S$ is coalitional Nash stable for coalition H if and only if $R_H^{++}(s) = \emptyset$.*

Definition 2.2.10. *(Coalitional Nash Stability for a DM) Let $i \in N$. A state $s \in S$ is coalitional Nash stable for DM i if and only if s is coalitional Nash stable for all coalitions $H \in \varphi(N)$ such that $i \in H$.*

In order to define the coalitional versions of GMR, SMR and SEQ, it is necessary to review the concepts of reachable states and of coalitional improvement by a class of coalitions of DMs.

Let C be a class of coalitions and let $R_C(s)$ be the set of reachable states by class C from s by a legal sequence of movements. Let $\Omega_C(s, s_1)$ be the subset of C whose members are the sets of DMs that make the final legal sequence of movements to achieve state s_1 from s. Formally, $R_C(s)$ and $\Omega_C(s, \cdot)$ are the smallest sets (in the sense of inclusion), satisfying: (i) if $H \in C$ and $s_1 \in R_H(s)$, then $s_1 \in R_C(s)$ and $H \in \Omega_C(s, s_1)$, (ii) if $s_1 \in R_C(s)$, $H \in C$, $\Omega_C(s, s_1) \neq \{H\}$ and

$s_2 \in R_H(s_1)$, then $s_2 \in R_C(s)$ and $H \in \Omega_C(s, s_2)$.

The class coalitional improvement list from state s by class C, denoted by $R_C^{++}(s)$, and the subset of C whose members are the subsets of DMs that make the last improvement movement to achieve s_1 in a legal sequence of movements from s, denoted by $\Omega_C^{++}(s, s_1)$, are defined as the smallest sets (in the sense of inclusion), satisfying: (i) if $H \in C$ and $s_1 \in R_H^{++}(s)$, then $s_1 \in R_C^{++}(s)$ and $H \in \Omega_C^{++}(s, s_1)$, (ii) if $s_1 \in R_C^{++}(s)$, $H \in C$, $\Omega_C^{++}(s, s_1) \neq \{H\}$ and $s_2 \in R_H^{++}(s_1)$, then $s_2 \in R_C^{++}(s)$ and $H \in \Omega_C^{++}(s, s_2)$.

Definition 2.2.11. *(Coalitional GMR Stability for a Coalition) Let $H \in \varphi(N)$. A state $s \in S$ is coalitional GMR (CGMR) stable for coalition H if and only if for every $s_1 \in R_H^{++}(s)$, there exists $s_2 \in R_{\varphi(N-H)}(s_1)$ such that $s \succeq_i s_2$ for some $i \in H$.*

Definition 2.2.12. *(Coalitional GMR Stability for a DM) Let $i \in N$. A state $s \in S$ is CGMR stable for DM i if and only if s is CGMR stable for all coalitions $H \in \varphi(N)$ such that $i \in H$.*

Definition 2.2.13. *(Coalitional SMR Stability for a Coalition) Let $H \in \varphi(N)$. A state $s \in S$ is coalitional SMR (CSMR) stable for coalition H if and only if for every $s_1 \in R_H^{++}(s)$, there exists $s_2 \in R_{\varphi(N-H)}(s_1)$ such that $s \succeq_i s_2$ for some $i \in H$ and for every $s_3 \in R_H(s_2)$, $s \succeq_j s_3$ for some $j \in H$.*

Definition 2.2.14. *(Coalitional SMR Stability for a DM) Let $i \in N$. A state $s \in S$ is CSMR stable for DM i if and only if s is CSMR stable for all coalitions $H \in \varphi(N)$ such that $i \in H$.*

Definition 2.2.15. *(Coalitional SEQ Stability for a Coalition) Let $H \in \varphi(N)$. A state $s \in S$ is coalitional SEQ (CSEQ) stable for coalition H if and only if for every $s_1 \in R_H^{++}(s)$, there exists $s_2 \in R_{\varphi(N-H)}^{++}(s_1)$ such that $s \succeq_i s_2$ for some $i \in H$.*

Definition 2.2.16. *(Coalitional SEQ Stability for a DM) Let $i \in N$. A state $s \in S$ is CSEQ stable for DM i if and only if s is CSEQ stable for all coalitions $H \in \varphi(N)$ such that $i \in H$.*

2.3 Overview about the GMCR literature

The method used to elaborate this thesis consisted in making a study of the works that have been proposed in the literature on GMCR. In particular, we investigated what were the most recent developments in the theory, that is, what was at the frontier of science with respect to such model. From our studies, we saw the need to propose a new definition ($SSEQ$), which would be a refinement of SMR and SEQ, reducing the number of stable states of the conflict. When analyzing the relations between this new concept ($SSEQ$) and the concepts of stabilities existing in the GMCR, we found some problems in the literature, more specifically in the generalized metarationality concept of Zeng et al. [14]. These problems motivated us to try to correct them by proposing alternative generalized metarationality definitions to overcome some of them.

By studying the GMCR with probabilistic preferences, proposed by Rêgo and Santos [15], we felt the need to propose a more efficient way to calculate the parameter regions of stability for the conflict states. The works of Xu et al. [18] and [26] provided us ideas that were adapted to our desired situation. Finally, we saw that there were some GMCR that allowed for the possibility of misperception in the GMCR and that motivated us to use our familiarity with the theory of unawareness in the game theory literature, specially with the work of Heifetz et al. [54], and use it to investigate the impact of unawareness in the stability analysis of the GMCR.

The GMCR has been the object of study by several researchers and has gained a lot of attention due to the flexibility of the model and the different situations in which it can be applied. Several extensions of the GMCR have been proposed aiming to better capture the particularities of real situations, for example [20], [21], [23] analyze the advantages of agents relating to each other taking into account different forms of behavior, i.e., according to different stability notions.

As DMs involved in the conflict can behave in a variety of ways, several solution concepts have been proposed in the GMCR, as previously mentioned. In conflicts with many DMs and states, it is computationally challenging to obtain stable states according to some of the solution concepts in the GMCR. Some papers in the literature on GMCR propose alternative simpler methods to find stable states according to some stability concepts, as [18], [24], [25], [26] and [27].

In the GMCR, the DMs involved have preferences over the set of states of the conflict. Several works on GMCR extend the usual preferences of this model to other types of preference structures. For example, [28], [29] extends the usual preference relation for uncertain preference, to handle situations in which a DM may have strict preference for one state over the other, be indifferent between two states or be unable to compare two states. [30] proposes definitions based on grey numbers to capture uncertainty in preferences, [31], [32] use fuzzy preferences in the GMCR and in [15] and [33] the usual preference structure in the GMCR is replaced by precise and imprecise probabilistic preference structures, respectively.

In most works on GMCR, the concepts of stabilities are proposed for conflicts with two or n-DMs. However, given that, in conflict situations, a group of DMs can form coalitions to respond to a particular DM or another coalition, some works, such as [22] and [34], extend the usual stability concepts (Nash, GMR, SMR and SEQ) for coalition analysis.

The GMCR is a very flexible model and has been applied in aquaculture to analyze a problem about a moratorium imported by the British Columbia government on salmon farming expansion [35], in the effective investigation of the strategic interactions that have occurred between an owner and a general contractor on the financing of a construction project [36] and in water resources management [37], where the GMCR is employed to analyze a contamination conflict of Groundwater. In the GMCR literature, it is also possible to find applications in problems related to sustainable development [38], water exports [39] and Military Support and Peace [40].

For the advances proposed in this thesis, the following papers were used as starting points. Adapted ideas from the works of Xu et al. [18] and [26] were used to provide a more efficient way of determining parameter regions of stability for the GMCR with probabilistic preferences, as proposed by [15]. Heifetz et al. [54] introduced a generalized state space model that allos the modeling of non-trivial unawareness among several individuals and ideas from this model were adapted to analyze the impact of unawareness in the stability analysis of the GMCR. Finally, in [19] and [14], a new solution concept, called generalized metarationality, is proposed in the GMCR for two and n-DMs, respectively. Motivated by some problems found in the work of Zeng et al. [14], we used the ideas developed in [19] to present an alternative definition to the concept proposed in [14], for conflicts with n decision makers, that overcame some of the problems found in [14].

In the following chapter, we start by presenting a new solution concept for the GMCR that uses ideas from the SMR and SEQ stability concepts, being a refinement of such concepts.

CHAPTER 3

Symmetric Sequential Stability in the GMCR

Abstract

In this chapter, a new solution concept, called symmetric sequential stability ($SSEQ$), for the GMCR is proposed for conflicts with two and n DMs. For conflicts with two DMs, we present the relationship of this new concept with four stability definitions commonly used in the GMCR, namely: Nash, GMR, SMR and SEQ. Next, we generalize the $SSEQ$ stability definition for n-DM conflicts and we obtain new results relating this new concept to the definitions previously mentioned. We also present the $SSEQ$ stability definition for a coalition and its relations with the classical stability definitions of coalitional analysis. Finally, $SSEQ$ stability is extended for GMCR with uncertain, probabilistic and fuzzy preferences.

3.1 Introduction

In the stability analysis of the GMCR, there may be several states satisfying a certain number of stability definitions. Thus, the proposal of new solution concepts which may reduce the number of stable states in some conflicts and accommodates the way in which DMs behave in actual conflicts is an active area of research in the GMCR literature. In this chapter we present a new solution concept in GMCR, called symmetric sequential stability ($SSEQ$). In this definition, as in the case of SMR stability, the conflict is analyzed up to three steps ahead from the current state and it is required that the countermove(s) is(are) also beneficial to the opponent(s), as in the case of SEQ stability notion. Moreover, it is assumed that for stability the focal DM cannot escape to a preferred state once the countermove is taken, as in the SMR stability notion.

We obtain results relating the $SSEQ$ stability concept with four stability definitions in the GMCR, namely: Nash stability, general metarational stability, symmetric metarational stability, sequential stability, limited-move stability of horizon 3 and credible metarational stable states of 2 rounds. We also present the $SSEQ$ stability definition for a coalition and its relations with the classical stability definitions of coalitional analysis. Finally, we extended this new solution concept for n-DM conflicts in the GMCR with uncertain [16], probabilistic [15] and fuzzy preferences [17], and present two applications to illustrate the usefulness of this new concept.

The notion of $SSEQ$ stability for conflicts with two DMs was published in the *Proceedings of the 2015 Conference on Group Decision and Negotiation*, see reference [41], and the notion $SSEQ$ for conflicts with n-DMs is published online in the *Journal Group Decision and Negotiation* [42]. The contents of this chapter were extracted from these papers.

This chapter is organized as follows. In Section 3.2, we present the $SSEQ$ stability concept for conflicts involving two DMs and its relations with other solution concepts in GMCR. In Section 3.3, the $SSEQ$ concept is generalized for conflicts with n-DMs and we introduce this definition for a coalition. In Section 3.4, we extend the $SSEQ$ stability definition for GMCR with other preference structures. Finally, in Section 3.5, we present two applications to illustrate the usefulness of this new concept.

3.2 Symmetric Sequential Stability in the GMCR with two DMs

In this section, we present the $SSEQ$ stability concept for conflicts involving two DMs. This definition, as the name implies, is a type of sequential stability in which a player, while planning to move, consider not only the reaction of his or her opponent, but also his own counter-reaction.

Definition 3.2.1. *A state $s \in S$ is symmetric sequentially (SSEQ) stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

It is important to emphasize that the counter-reaction does not need to be a unilateral improvement for the DM; it is required that the resulting state cannot be better than the current state for every possible reachable counter-reaction. This is specially important in cases where preferences are not negatively transitive, where a preference relation $\succ$ is negatively transitive if $s \not\succ t$ and $t \not\succ q$ implies that $s \not\succ q$ [43]. We now show that if DM i's preferences is negatively transitive, then $R_i(s_2)$ could be replaced by $R_i^+(s_2)$ in the $SSEQ$ definition, by showing that in this case for every $s_3 \in R_i(s_2) - R_i^+(s_2)$ $s \succeq_i s_3$. If $s_3 \in R_i(s_2) - R_i^+(s_2)$, then $s_3 \not\succ_i s_2$. Thus, as $s_2 \not\succ_i s$, if $\succ_i$ were negatively transitive, it would follow that $s_3 \not\succ_i s$, as desired.

3.2.1 *Relationships with other solution concepts*

In the GMCR, there are well known relationships between the four standard stability concepts. Next, we establish some relationships of the $SSEQ$ stability with some of the existing solution concepts.

Theorem 3.2.1. *The following statements are true in the GMCR:*

(a) If state s is Nash stable for DM i, then s is SSEQ stable for DM i.

(b) If state s is SSEQ stable for DM i, then s is SEQ stable for DM i.

(c) If state s is SSEQ stable for DM i, then s is SMR stable for DM i.

Proof:

For (a), if s is Nash stable for DM i, then $R_i^+(s) = \emptyset$ which implies that s is $SSEQ$ stable for DM i.

For (b), suppose that s is $SSEQ$ stable for DM i. Thus, for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$. Therefore, it is true that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$, which implies that s is SEQ stable for DM i.

For (c) suppose that s is $SSEQ$ stable for DM i. Thus, for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$. Since $R_j^+(s_1) \subseteq R_j(s_1)$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$, which implies that s is SMR stable for DM i. □

3.3 Symmetric sequential stability in GMCR with n-DM

In this section we generalize the $SSEQ$ stability definition proposed in Section 3.2, to conflicts with n-DMs and introduce this definition for a coalition. Additionally, we present results which relate $SSEQ$ and coalitional $SSEQ$ to other stability definitions commonly used in GMCR and we extended this new solution concept for n-DM GMCR with uncertain [16], probabilistic [15] and fuzzy preferences [17].

3.3.1 *Symmetric sequential stability*

Definition 3.3.1. *A state $s \in S$ is symmetric sequentially (SSEQ) stable for DM $i \in N$ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}^+(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

In other words, a state s is classified as stable $SSEQ$ to DM i if for any unilateral improvement from the state s so that DM can do, there is an reaction of his or her opponents which leads to a state worse than the state s and from that state the counter-reaction of DM i is always to go to a state that is also not preferable to s.

3.3.2 *Relations with other solution concepts*

In what follows, we present generalizations of the results established in Section 3.2 and new ones relating $SSEQ$ stability with L_3 and CMR_2.

Theorem 3.3.1 states the relationship between $SSEQ$, Nash, SMR and SEQ stability.

Theorem 3.3.1. *The following statements are true in the GMCR:*

(a) If state s is Nash stable for DM i, then s is SSEQ stable for DM i.

(b) If state s is SSEQ stable for DM i, then s is SEQ stable for DM i.

(c) If state s is SSEQ stable for DM i, then s is SMR stable for DM i.

Proof:

For (a), if s is Nash stable for DM i, then $R_i^+(s) = \emptyset$ which implies that s is $SSEQ$ stable for DM i.

For (b), suppose that s is $SSEQ$ stable for DM i. Thus, for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R^+_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$. Therefore, it is true that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R^+_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$, which implies that s is SEQ stable for DM i.

For (c) suppose that s is $SSEQ$ stable for DM i. Thus, for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R^+_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$. Since $R^+_{N-\{i\}}(s_1) \subseteq R_{N-\{i\}}(s_1)$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$, which implies that s is SMR stable for DM i. □

The following hypothetical example illustrates that, in general, SMR and SEQ stability together do not imply $SSEQ$ stability.

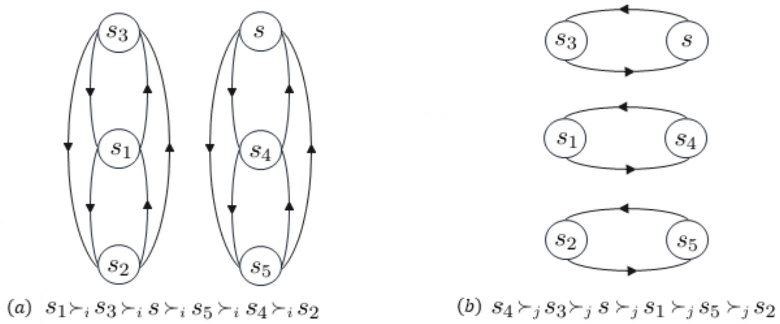


Figure 3.1: Conflict in the graph form: (a) DM i; (b) DM j.

Example 3.3.1. *Consider the hypothetical conflict shown in Figure 3.1. In this example, state s is SMR and SEQ, but it is not $SSEQ$ for DM j. Indeed, it is SMR since state $s_3 \in R_j^+(s)$ is the unique unilateral improvement from s for DM j and state s_2 is accessible for DM i from s_3 such that $s \succeq_j s_2$ and $s \succeq_j s_5$, where s_5 is unique accessible state for DM j from s_2. Also, we have that s is SEQ for DM j. Indeed, since from state $s_3 \in R_j^+(s)$, there exists a unique state $s_1 \in R_i^+(s_3)$ and such state satisfies $s \succeq_j s_1$. But state s is not $SSEQ$ for DM j because $s_4 \succeq_j s$, where s_4 is accessible for DM j from s_1.*

The next theorem describes a particular case where $SSEQ$ is equivalent to SMR and SEQ together.

Theorem 3.3.2. *Suppose that a strategic conflict is composed of* 2 *DMs. If for every* $s \in S$ *and* $i \in N$*, the cardinality of* $R_i(s)$ *is at most equal to one, then a state is SSEQ if, and only if, it is* SMR *and* SEQ*.*

Proof:

Indeed, by Theorem 3.3.1, if state s is $SSEQ$ then it is SMR and SEQ. Suppose that s is SMR and SEQ for DM i. Thus, either $R_i^+(s) = \emptyset$, in which case s is also $SSEQ$, or $R_i^+(s) = \{s_1\}$. By hypothesis and since no state is accessible to itself, the fact that s is SEQ implies that there exists a unique state $s_2 \in S$, such that $R_j(s_1) = R_j^+(s_1) = \{s_2\}$ and $s \succeq_i s_2$. Thus, since s is SMR it follows that $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. Therefore, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$, and $s \succeq_i s_3$ for all $s_3 \in R_i(s_2)$, which implies that s is $SSEQ$ for DM i. □

The following example illustrates that Theorem 3.3.2 is not true if the strategic conflict has more than two DMs.

Example 3.3.2. *Consider a hypothetical conflict situation in which there are* 3 *DMs,* i*,* j *and* k*. Suppose that in this conflict there are five states, states* s*,* s_1*,* s_2*,* s_3 *and* s_4*. Assume that the preferences of DM* i*, DM* j *and DM* k *are, respectively, given by* $s_1 \succ_i s_4 \succ_i s \succ_i s_2 \succ_i s_3$*,* $s_1 \succ_j s_2 \succ_j s \succ_j s_3 \succ_j s_4$ *and* $s_3 \succ_k s_1 \succ_k s \succ_k s_4 \succ_k s_2$*. Consider also that* $R_i(s) = \{s_1\}$*,* $R_j(s_1) = \{s_2\}$*,* $R_i(s_2) = R_k(s_1) = \{s_3\}$*,* $R_i(s_3) = \{s_4\}$ *and that* $R_i(s_1) = R_i(s_4) = R_j(s) = R_j(s_2) = R_j(s_3) = R_j(s_4) = R_k(s) = R_k(s_2) = R_k(s_3) = R_k(s_4) = \emptyset$*, as illustrated in Figure 3.2.*

We now show that state s *is* SMR *and* SEQ *for DM* i*, but it is not SSEQ. First, it is* SMR *since from the unique unilateral improvement for DM* i *from* s*, state* s_1*, DM* j *can lead the conflict to state* s_2 *and from* s_2 *DM* i *can only move to state* s_3*, but states* s_2 *and* s_3 *are worse than* s *for DM* i*. It is SEQ for DM* i*, since DM* k *has a unilateral improvement leading the conflict to state* s_3*, which is worse than* s *for DM* i*.*

On the other hand, state s *is not SSEQ for DM* i*, since the unique unilateral improvement from state* s_1 *for coalition* $\{j,k\}$ *is state* s_3*, but from* s_3 *DM* i *can move to state* s_4*, which is preferred to state* s *by DM* i*.*

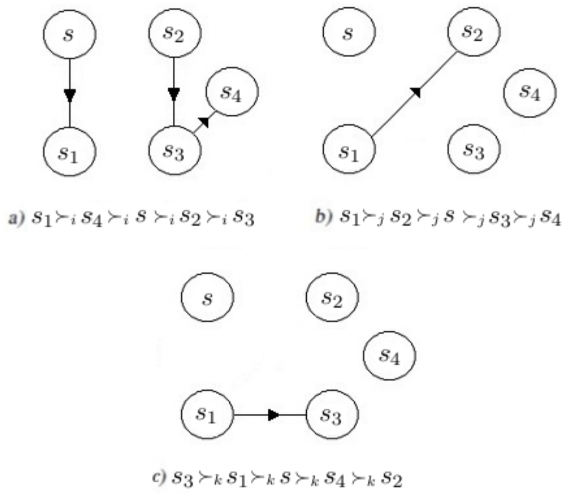


Figure 3.2: Conflict in the graph form: (a) DM i; (b) DM j and (c) DM k.

Examples 3.3.3 and 3.3.4 below show that there is no relation between the stability concepts $SSEQ$ and L_3. More specifically, Example 3.3.3 shows that if a state s is $SSEQ$ stable, then this state may not be L_3 stable. Conversely, Example 3.3.4 illustrates the case of a state that is L_3 stable but is not $SSEQ$.

Example 3.3.3. *Consider the conflict illustrated in Figure 3.3. Its state space is given by* $S = \{s, s_1, s_2, s_3\}$ *and it is composed of two DMs,* i *and* j*. Suppose that* $R_i(s) = \{s_1\}$*,* $R_i(s_1) = \{s\}$ *and* $R_i(s_2) = R_i(s_3) = \emptyset$*. For DM* j*, suppose that* $R_j(s) = R_j(s_2) = R_j(s_3) = \emptyset$ *and* $R_j(s_1) = \{s_2, s_3\}$*. Consider also that the preference relation of DM* i *is given by* $s_3 \succ_i s_1 \succ_i s \succ_i s_2$ *and the preference relation of DM* j *is given by* $s_3 \succ_j s_2 \succ_j s \succ_j s_1$*. We now argue that state* s *is* $SSEQ$ *for DM* i*, because the unique improvement for DM* i *from* s *is* s_1*, but, from state* s_1*, DM* j *can sanction DM* i *going to state* s_2 *such that* s_2 *is better than* s_1 *to DM* j *and* s_2 *is worse than* s *to DM* i*. As DM* i *can not get out of* s_2*, we have that* s *is* $SSEQ$ *for DM* i*. On the other hand, state* s *is not* L_3 *stable for DM* i *because from* s_1*, the anticipated state for DM* j *is* s_3 *and not* s_2*. Since* s_3 *is better than* s *to DM* i*, it follows that he intends to move away from* s*.*

Example 3.3.4. *The following example illustrates that* L_3 *stability does not imply* $SSEQ$ *stability. An example that illustrates this fact is the chicken game described in [44]. In this game, two*

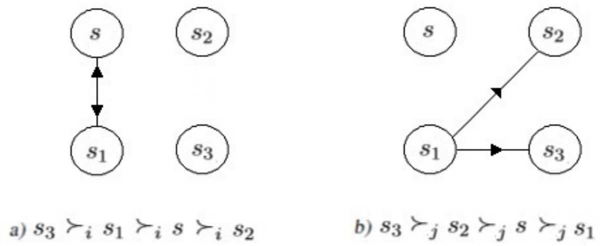


Figure 3.3: Conflict in the graph form: (a) DM i; (b) DM j.

DMs, called DM i and DM j, have the choice of either swerving, denoted by S, thereby avoiding a collision, or continuing to drive straight ahead and hence selecting the strategy of not swerving, D. The graph form of the chicken game is shown in Figure 3.4. The preference relation of DM i is given by $DS \succ_i SS \succ_i SD \succ_i DD$, *and the preference relation of DM j is given by* $SD \succ_j SS \succ_j DS \succ_j DD$. *Using backward induction, working from the bottom to the top of the diagram in Figure 3.5, we have that state SS is* L_3 *stable for DM i. On the other hand, this state is not SSEQ stable for DM i, because from state SS, the unique improvement of DM i is state DS. But from DS, there is no reachable improvement for DM j. Therefore, state SS is not SSEQ stable for DM i.*

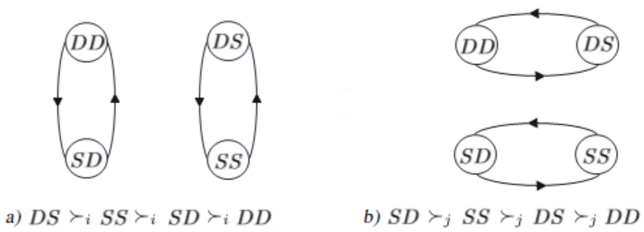


Figure 3.4: Conflict in the graph form: (a) DM i; (b) DM j.

We have the following relationship, established in Theorem 3.3.3, between the concepts of $SSEQ$ and CMR_2.

Theorem 3.3.3. *If a state s is SSEQ for DM i, then s is* CMR_2 *stable for DM i.*

Proof:

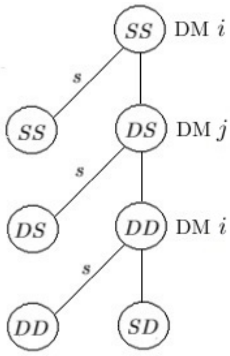


Figure 3.5: L_3 stability analysis of state SS for DM i. Source: [21].

Suppose that s is $SSEQ$ stable for DM $i \in N$. Let us consider two cases: (a) $R_i^+(s) = \emptyset$ or (b) $R_i^+(s) \neq \emptyset$. If (a) occurs, then the unique credible metarational tree which has DM i moving at the root s has one round and ends once DM i stays at s. Thus, s is CMR_2 stable for DM i in that case. If (b) occurs and $s_1 \in R_i^+(s)$, then there exists a state $s_2 \in R_{N-\{i\}}^+(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$, for all $s_3 \in R_i(s_2)$. Thus, there is a sequence of unilateral improvement moves by $N - \{i\}$ from s_1 leading the conflict to s_2, which is not preferred to s by DM i. If there are more than of such sequences, choose one of them whose length is minimum, call it x. In x, every DM moves at most once in every state in that sequence. Let us use x to define credible policies for DM $j \in N - \{i\}$, as follows

$$P_j^c(s_t) = \begin{cases} s_u & \text{if there is a move by DM } j \text{ from } s_t \text{ to } s_u \text{ in } x, \\ s_t & \text{otherwise.} \end{cases}$$

Given that set of credible policies, since from s_2 DM i cannot move to a state that is preferred to s, it follows that there exists an i-sequence that starts with DM i moving from s to s_1 of 2 rounds and results in a state that is not preferred to s by DM i. Thus, also in this case, s is CMR_2 stable for DM i. □

The following example illustrates that the reciprocal of Theorem 3.3.3 is not true.

Example 3.3.5. *Consider a hypothetical conflict with 2 DMs, DM i and DM j, five states, namely, s, s_1, s_2, s_3 and s_4, and suppose that accessibility between the states are $R_i(s) = \{s_1\}$, $R_i(s_2) = \{s_3, s_4\}$ and $R_j(s_1) = \{s_2\}$, as illustrated in Figure 3.6.*

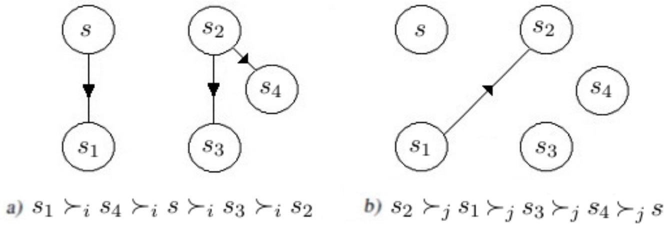


Figure 3.6: A conflict where s is CMR_2 stable but not $SSEQ$ stable for DM i.

Assume that preference relations are given by $s_1 \succ_i s_4 \succ_i s \succ_i s_3 \succ_i s_2$, and $s_2 \succ_j s_1 \succ_j s_3 \succ_j s_4 \succ_j s$. Suppose that DM i is in state s. State s is not SSEQ for DM i, since from s, DM i can move to a better state s_1, and, from s_1, the unique reaction of DM j is to lead the conflict to state s_2 which is not preferred to s by DM i but it is preferred to s_1 by DM j. However, from s_2, DM i can move to states s_3 and s_4, and state s_4 is better than s for DM i. On the other hand, s is CMR_2 stable for DM i, since there is an credible policy of DM j satisfying $P_j(s_1) = s_2$, such that the sequence $(s, i, s_1, j, s_2, i, s_3)$ is an i-sequence of round 2 such that DM i does not prefer the result of this sequence to state s.

The relationships obtained between the $SSEQ$ stability concept and existing solution concepts in the GMCR, namely: Nash stability, GMCR stability, SMR stability, SEQ stability, L_3 stability and CMR_2 stability are summarized in Figure 3.7. As one can see, the $SSEQ$ stability concept reduces the number of stable states in comparison to SEQ, SMR and CMR_2. This is specially useful in conflicts having multiple stable states.

3.3.3 *Coalitional SSEQ*

The coalitional stability analysis in the GMCR has been studied in recent works [56] extending the stability analysis to situations in which DMs can act together forming a coalition. Thus, in this context, it is possible for DMs to achieve improvements that are not possible to achieve if they were acting individually.

Definition 3.3.2. *(Coalitional SSEQ Stability for a Coalition) Let $H \in \varphi(N)$, a state $s \in S$ is coalitional SSEQ (CSSEQ) stable for coalition H if and only if for every $s_1 \in R_H^{++}(s)$, there*

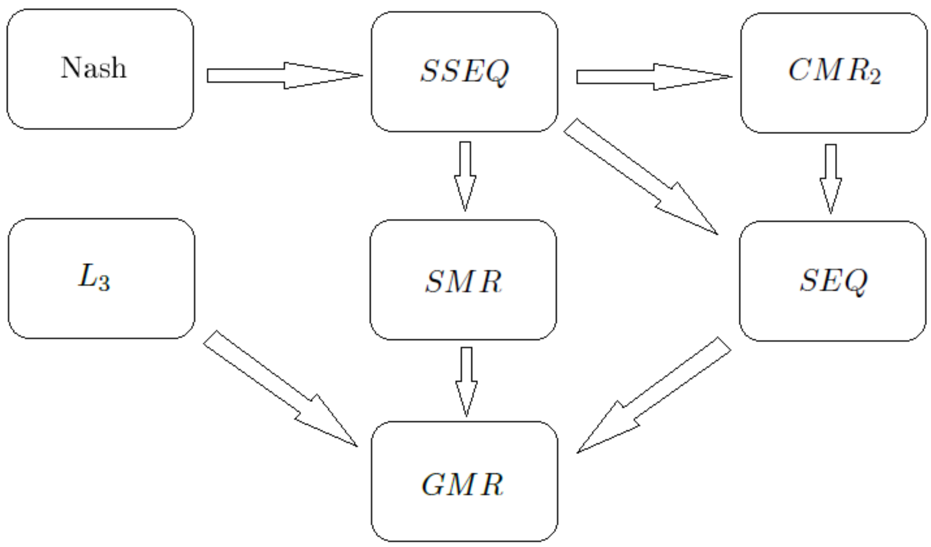


Figure 3.7: Implications among SSEQ and other stability definitions

exists $s_2 \in R^{++}_{\varphi(N-H)}(s_1)$ *such that* $s \succeq_i s_2$ *for some* $i \in H$ *and for every* $s_3 \in R_H(s_2)$, $s \succeq_j s_3$ *for some* $j \in H$.

Definition 3.3.3. *(Coalitional SSEQ Stability for a DM) For* $i \in N$, *a state* $s \in S$ *is CSSEQ stable for DM* i *if and only if* s *is CSSEQ for all coalitions* $H \in \varphi(N)$ *such that* $i \in H$.

Similar results as those of Theorem 3.3.1 remain valid for a coalition $H \subseteq N$.

Theorem 3.3.4. *The following statements are true in the GMCR:*

(a) If state s *is coalitional Nash stable for coalition* H, *then* s *is CSSEQ stable for this coalition.*

(b) If state s *is CSSEQ stable for coalition* H, *then* s *is CSEQ stable for this coalition.*

(c) If state s *is CSSEQ stable for coalition* H, *then* s *is CSMR stable for this coalition.*

Proof:

The proof of this theorem is similar to proof of Theorem 3.3.1. The only necessary changes are to replace R_i^+ by R_H^{++}, $R_{N-\{i\}}$ by $R_{\varphi(N-H)}$ and $R^+_{N-\{i\}}$ by $R^{++}_{\varphi(N-H)}$ in that proof. $\square$

3.4 *SSEQ* in GMCR with other preference structures

In this section, we extend the *SSEQ* stability definition for the GMCR with uncertain [16], probabilistic [15], and fuzzy preference [17] structures. In what follows, we review, briefly, these models and present the corresponding adapted version of *SSEQ* to each one of these three preference structures.

3.4.1 *The SSEQ stability in the GMCR with uncertain preferences*

Li et al.[16] proposed to use a new preference structure in the GMCR in which DM's preferences are expressed by a triple of relations $\{\succ_i, \sim_i, U_i\}$, were $s \succ_i s_1$ and $s \sim_i s_1$ are the strict preference and indifference relations, and sU_is_1 means that DM i is uncertain as to whether he or she prefers state s to state s_1, prefers s_1 to s, or is indifferent between s and s_1.

Let $R_i^U(s) = \{s_1 \in R_i(s) : s_1U_is\}$ be the DM i's reachable list from state s by a unilateral uncertain move. Let $R_i^{+,U}(s) = R_i^+(s) \cup R_i^U(s) = \{s_1 \in R_i(s) : s_1 \succ_i s \text{ or } s_1U_is\}$ be the DM i's reachable list from state s by a unilateral improvement or a unilateral uncertain move. Let $R_H^{+,U}(s)$ denote the set of unilateral improvements or unilateral uncertain moves by coalition $H \subseteq N$. If $s_1 \in R_H^{+,U}(s)$, then $\Omega_H^{+,U}(s, s_1)$ is the set of all last DMs in unilateral improvements or uncertain moves from s to s_1. These sets can be formally defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$ and $s_1 \in R_i^{+,U}(s)$, then $s_1 \in R_H^{+,U}(s)$ and $i \in \Omega_H^{+,U}(s, s_1)$, and (2) if $s_1 \in R_H^{+,U}(s)$, $i \in H$, $\Omega_H^{+,U}(s, s_1) \neq \{i\}$ and $s_2 \in R_i^{+,U}(s_1)$, then $s_2 \in R_H^{+,U}(s)$ and $i \in \Omega_H^{+,U}(s, s_2)$.

Then, based on this extended preference structure we have the following *SSEQ* definitions. First, if DM i has an incentive to move to states with uncertain preferences relative to the status quo, but, when assessing possible sanctions, he will not consider states with uncertain preferences, then we have the following definition.

Definition 3.4.1. *A state $s \in S$ is SSEQ stable for DM $i \in N$, denoted by $SSEQ_a$, in a GMCR with uncertain preferences iff for every $s_1 \in R_i^{+,U}(s)$, there exists $s_2 \in R_{N-\{i\}}^{+,U}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

Second, if DM i would only move from the status quo to preferred states and would be sanctioned only by less preferred or equally preferred states relative to the status quo, then we have the following definition:

Definition 3.4.2. *A state $s \in S$ is SSEQ stable for DM $i \in N$, denoted by $SSEQ_b$, in a GMCR with uncertain preferences iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}^{+,U}(s_1)$ such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$.*

Third, if preference uncertainty is allowed when DM i considers both incentives to leave a state and sanctions to deter him or her from doing so, then we have the following definition:

Definition 3.4.3. *A state $s \in S$ is SSEQ stable for DM $i \in N$, denoted by $SSEQ_c$, in a GMCR with uncertain preferences iff for every $s_1 \in R_i^{+,U}(s)$, there exists $s_2 \in R_{N-\{i\}}^{+,U}(s_1)$ such that $s \succeq_i s_2$ or sU_is_2 and $s \succeq_i s_3$ or sU_is_3 for every $s_3 \in R_i(s_2)$.*

Finally, if DM i is not willing to move to a state with uncertain preference relative to the status quo, but is deterred by sanctions to states that have uncertain preference relative to the status quo, then we have the following definition:

Definition 3.4.4. *A state $s \in S$ is SSEQ stable for DM $i \in N$, denoted by $SSEQ_d$, in a GMCR with uncertain preferences iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}^{+,U}(s_1)$ such that $s \succeq_i s_2$ or sU_is_2 and $s \succeq_i s_3$ or sU_is_3 for every $s_3 \in R_i(s_2)$.*

3.4.2 *The SSEQ stability in the GMCR with probabilistic preferences*

In Rêgo and Santos [15], the authors replace the usual preference notion used in GMCR by adopting probabilistic preferences [45]. According to this model, whenever a DM must state preferences between two particular objects, he or she may do so with a certain probability. Thus, in the GMCR with probabilistic preferences, for any two states s and s_1, $P_i(s, s_1)$ expresses the chance with which DM i strictly prefers state s over s_1. This probability is defined on $S \times S$ and must satisfy:

(1) $P_i(s, s) = 0, \forall s \in S$;

(2) $P_i(s, s_1) \geq 0, \forall s, s_1 \in S$;

(3) $P_i(s, s_1) + P_i(s_1, s) \leq 1, \forall s, s_1 \in S$.

Consider parameters α, β, γ lying in the interval $[0, 1]$. Let $R_i^{+\gamma}(s)$ be the set of γ-unilateral improvements from state s for DM i, where a state s_2 is a γ-unilateral improvement from state s_3 for DM i, if $s_2 \in R_i(s_3)$ and $P_i(s_2, s_3) > \gamma$. In order to define the notion of *SSEQ* stability for the GMCR with probabilistic preferences, we need to present the definition of γ-unilateral improvement by a coalition. Let $\Omega_H^{+\gamma}(s, s_1)$ be the subset of H whose members are the DMs who make the last γ improvement move to reach s_1 in a legal sequence of γ improvement moves from state s. The sets $R_H^{+\gamma}(s)$ and $\Omega_H^{+\gamma}(s, \cdot)$ are defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$, $s_1 \in R_i(s)$ and $P_i(s_1, s) > \gamma$, then $s_1 \in R_H^{+\gamma}(s)$ and $i \in \Omega_H^{+\gamma}(s, s_1)$, and (2) if $s_1 \in R_H^{+\gamma}(s)$, $i \in H$, $s_2 \in R_i(s_1)$, $\Omega_H^{+\gamma}(s, s_1) \neq \{i\}$ and $P_i(s_2, s_1) > \gamma$, then $s_2 \in R_H^{+\gamma}(s)$ and $i \in \Omega_H^{+\gamma}(s, s_2)$. Additionally, also consider $\Phi_i^{+\gamma}(s) = \{s_1 \in S : P_i(s_1, s) > \gamma\}$ as defined in [15]. In this model, we have the following *SSEQ* definition:

Definition 3.4.5. *A state $s \in S$ is (α, β, γ)-SSEQ stable for DM $i \in N$ iff for every $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists $s_2 \in R_{N-\{i\}}^{+\gamma}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$ such that $R_i(s_2) \cap \Phi_i^{+(1-\alpha)}(s) = \emptyset$.*

3.4.3 *The SSEQ stability in the GMCR with fuzzy preferences*

In Hipel et al. [17] is proposed the use of fuzzy preferences in the GMCR to indicate the degree of uncertainty that a DM can have when comparing two states. Fuzzy preferences over the set of states, S, is a fuzzy relation in S represented by the matrix $A = (a_{ij})_{m \times m}$, with membership function $\mu_A : S \times S \to [0, 1]$, where $\mu_A(s_i, s_j) = a_{ij}$, the degree of preference for s_i over s_j, satisfies $a_{ij} + a_{ji} = 1$, and $a_{ii} = 0.5$, for all $i, j = 1, 2, \ldots, m$.

The authors define DM k's fuzzy relative certainty of preference for state s_i over s_j as $\alpha^k(s_i, s_j) = a^k(s_i, s_j) - a^k(s_j, s_i)$, where $a^k(s_i, s_j)$ denotes the preference degree of state s_i over s_j for DM k. In this model a state $s_i \in R_k(s)$, where $k \in N$, is called a fuzzy unilateral improvement from s by DM k if and only if $\alpha^k(s_i, s) \geq \gamma_k$, where γ_k is the fuzzy satisficing threshold for DM k. Let $\hat{R}_{k,\gamma_k}^{+}(s) = \{s_i \in R_k(s) : \alpha^k(s_i, s) \geq \gamma_k\}$ be the fuzzy unilateral improvement list for DM k. In order to define the notion of $SSEQ$ stability for the GMCR with fuzzy preferences, we need to present the definition of the fuzzy unilateral improvement list by a coalition. Let $\hat{\Omega}^{+}_{H,\gamma_H}(s, s_1)$, where $\gamma_H = \times_{i \in H} \gamma_i$, be the set of all last DMs who make the last fuzzy improvement move in a legal sequence from s to s_1.

The sets $\hat{R}^{+}_{k,\gamma_k}(s)$ and $\hat{\Omega}^{+}_{H,\gamma_H}(s, \cdot)$ are defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$ and $s_1 \in \hat{R}^{+}_{i,\gamma_i}(s)$, then $s_1 \in \hat{R}^{+}_{H,\gamma_H}(s)$ and $i \in \hat{\Omega}^{+}_{H,\gamma_H}(s, s_1)$, and (2) if $s_1 \in \hat{R}^{+}_{H,\gamma_H}(s)$, $i \in H$, $s_2 \in \hat{R}^{+}_{i,\gamma_i}(s_1)$ and $\hat{\Omega}^{+}_{H,\gamma_H}(s, s_1) \neq \{i\}$, then $s_2 \in \hat{R}^{+}_{H,\gamma_H}(s)$ and $i \in \hat{\Omega}^{+}_{H,\gamma_H}(s, s_2)$. Then, we have the following $SSEQ$ definition:

Definition 3.4.6. *A state $s \in S$ is SSEQ fuzzy stable for DM $i \in N$ iff for every $s_1 \in \hat{R}^{+}_{i,\gamma_i}(s)$, there exists $s_2 \in \hat{R}^{+}_{N-\{i\},\gamma_{N-\{i\}}}(s_1)$ such that $\alpha^i(s_2, s) < \gamma_i$, and $\alpha^i(s_3, s) < \gamma_i$ for all $s_3 \in R_i(s_2)$.*

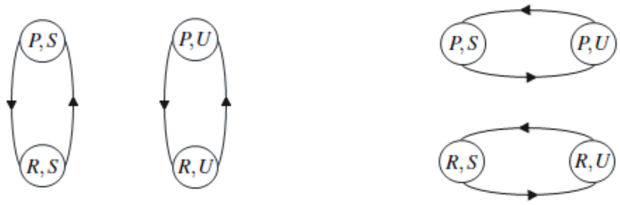
3.5 Applications

In this section, the definition of $SSEQ$ stability is applied in two examples to illustrate its usefulness. The first one is a hypothetical environmental conflict involving 2 DMs and the second one is the Rafferty-Alameda dams conflict, which is a real-life case involving 4 DMs.

3.5.1 *Hypothetical Environmental Conflict*

We now present a modified version of a hypothetical conflict proposed by [23] to illustrate an application of the $SSEQ$ stability. In this conflict, there are two DMs: environmentalist (E) and developers (D). Environmentalists may choose to be proactive (P) in promoting environmental responsibility or not, in this case they are called reactive (R). Developers may choose to be sustainable (S) or not, which is represented by (U). The set of possible states of the conflict is: $s =(P, S)$, $s_1 =(P, U)$, $s_2 =(R, S)$ and $s_3 =(R, U)$. Figure 3.8 represents the graph model for this strategic conflict.

Table 3.1 shows the stable states, for each DM, according to the usual stability definitions and also according to $SSEQ$ stability. Each cell in the array specifies for which DMs, if any, the column state is stable according to the stability definition of the corresponding line. As it can



a) $(P,S) \succ_E (R,S) \succ_E (P,U) \succ_E (R,U)$ b) $(R,U) \succ_D (P,U) \succ_D (R,S) \succ_D (P,S)$

Figure 3.8: Conflict in the graph form: a) DM E; b) DM D.

be seen, although $SSEQ$ concept has more requirements than GMR, SMR and SEQ concepts, in this conflict such concepts coincide, what strengthens the stability properties of the states.

Table 3.1: Stable states according to five stability definitions

	s	s_1	s_2	s_3
Nash	E	E,D		D
GMR	E	E,D	E	D
SMR	E	E,D	E	D
SEQ	E	E,D	E	D
$SSEQ$	E	E,D	E	D

3.5.2 *The Rafferty-Alameda Dams Conflict*

We will make now the $SSEQ$ stability analysis for a conflict with four DMs. This conflict, known as Rafferty-Alameda dams conflict, is a problem of dams construction in Canada that occurred in early 1986, and can be found in more details in [46], [26] and [47].

The history of that conflict begins when Canadian Province of Saskatchewan, seeking to provide improvements such as the reduction of flooding and water supply to cool a plant that produced energy through coal, decided to build Rafferty and Alameda dams. After the license granted by the Minister of Environment of the Federal Government of Canada, various Environmental Groups were opposed to the construction project and appealed to the Federal Court. The Federal Court suspended the license granted and the Federal Environmental Review Panel was responsible for evaluating the project and making an environmental assessment and review. After starting the project assessments, the panel noted that the project was still being developed and decided to contact the Federal Government to complain, but got no answer and the panel had decided to suspend its review.

In order to model Rafferty-Alameda dams conflict using the GMCR, four DMs are considered as acting in this conflict, namely: Federal Government of Canada (**Federal**), called DM F, **Saskatchewan**, called DM S, Environment Groups (**Groups**), called DM G, and Federal Environmental Review Panel (**Panel**), called DM P. The graph form of this conflict is illustrated in Figure 3.9. The options of DM F are: (1) seek a court order to halt the project (Court order) or (2) to lift the license (Lift). The option of DM S is only to go ahead at full speed (Full speed). The option of DM G is only to threaten court action to halt the project (Court action). The options of DM P is only to resign (Resign).

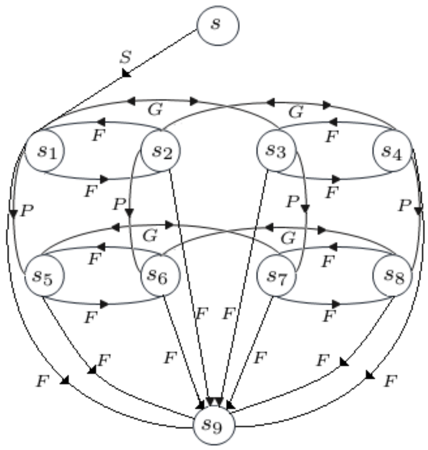


Figure 3.9: Graph form of Rafferty-Alameda dams conflict

In this conflict, the number of all states possible is 2^5. However, the set of feasible states contains only 10 states (s, s_1, s_2, ..., s_9) which are determined by means of the options shown in Table 3.2. The notation Y indicates that the DM that controls the corresponding option takes it, while the notation N indicates that the DM that controls the corresponding option does not take it.

We also have that the sets of reachable states and the usual DMs' preferences in this conflict are summarized in Table 3.3.

Table 3.2: States in the Rafferty-Alameda dams conflict

Federal										
1. Court order	-	N	Y	N	Y	N	Y	N	Y	N
2. Lift	-	N	N	N	N	N	N	N	N	Y
Saskatchewan										
3. Full speed	N	Y	Y	Y	Y	Y	Y	Y	Y	-
Groups										
4. Court action	-	N	N	Y	Y	N	N	Y	Y	-
Panel										
5. Resign	-	N	N	N	N	Y	Y	Y	Y	-
State	s	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9

Table 3.3: Set of reachable states and payoff

State Number	Federal		Saskatchewan		Groups		Panel	
	R_F	p_F	R_S	p_S	R_G	p_G	R_P	p_D
s	$\emptyset$	10	s_1	1	$\emptyset$	9	$\emptyset$	10
s_1	s_2, s_9	7	$\emptyset$	10	s_3	1	s_5	1
s_2	s_1, s_9	9	$\emptyset$	6	s_4	3	s_6	3
s_3	s_4, s_9	6	$\emptyset$	9	s_1	5	s_7	2
s_4	s_3, s_9	8	$\emptyset$	5	s_2	7	s_8	4
s_5	s_6, s_9	3	$\emptyset$	8	s_7	2	$\emptyset$	6
s_6	s_5, s_9	5	$\emptyset$	4	s_8	4	$\emptyset$	8
s_7	s_8, s_9	2	$\emptyset$	7	s_5	6	$\emptyset$	7
s_8	s_7, s_9	4	$\emptyset$	3	s_6	8	$\emptyset$	9
s_9	$\emptyset$	1	$\emptyset$	2	$\emptyset$	10	$\emptyset$	5

The notations p_F, p_S, p_G and p_P indicate the preference order of DMs F, S, G and P, respectively, where a higher number indicates a more desired state.

Table 3.4 represents the stable states in Rafferty-Alameda dams conflict, for each DM, according to the notions of Nash, GMR, SMR, SEQ and $SSEQ$ stability. Each cell in the array specifies for which DMs, if any, the line state is stable according to the stability definition of the corresponding column.

Like the results of the previous conflict, GMR, SMR, SEQ and $SSEQ$ coincides in this conflict. Thus, even though opponents moves according to $SSEQ$ stability are restricted in comparison to those according to GMR or SMR, or the focal DM gets the opportunity to counter-react, as opposed to what is allowed in SEQ, stability of the states remain the same.

Table 3.4: Stable states according to five stability definitions

	Nash	GMR	SMR	SEQ	$SSEQ$
s	F,G,P	F,G,P	F,G,P	F,G,P	F,G,P
s_1	S	F,S	F,S	F,S	F,S
s_2	F,S	F,S	F,S	F,S	F,S
s_3	S,G	F,S,G	F,S,G	F,S,G	F,S,G
s_4	F,S,G	F,S,G	F,S,G	F,S,G	F,S,G
s_5	S,P	S,P	S,P	S,P	S,P
s_6	F,S,P	F,S,P	F,S,P	F,S,P	F,S,P
s_7	S,G,P	S,G,P	S,G,P	S,G,P	S,G,P
s_8	F,S,G,P	F,S,G,P	F,S,G,P	F,S,G,P	F,S,G,P
s_9	F,S,G,P	F,S,G,P	F,S,G,P	F,S,G,P	F,S,G,P

3.6 Conclusion

This chapter presents the notion of $SSEQ$ stability and extends this concept for n-DM conflicts in the GMCR. The $SSEQ$ stability is a kind of sequential stability in which the DM who moves first considers not only the reaction of his or her opponents, but also his own counter-reaction. We also present the relationships of $SSEQ$ with six existing solution concepts in the literature.

Additionally, we introduced the $SSEQ$ concept for coalitional analysis and extended $SSEQ$ stability for GMCR with uncertain, probabilistic and fuzzy preferences in n-DM conflicts.

$SSEQ$ stability can be applied to other preferences structures that have been recently proposed to be used in the GMCR, such as Gray Preference [48] and Upper and Lower Probabilistic Preferences [33]. The idea to extend this definition to models with other preference structures is that for a state s to be $SSEQ$ stable for DM i, it must be such that for every improvement s_1 reachable from s to DM i, there is a series of improvements for the other DMs that leads the conflict from state s_1 to a state s_2 such that s_2 is not preferred to s by DM i and from s_2 DM i cannot reach a state s_3 which is preferred to s by DM i, where the notion of improvement depends on the preference structure adopted.

The $SSEQ$ concept enriches the SEQ and the SMR concepts providing for both DMs and analysts more information regarding stability of states. It enhances the SEQ concept by allowing DMs to analyze the conflict one further step and the SMR concept by restricting focal DM opponents to use only unilateral improvement moves. Such enhancements may help DMs make better decisions since, in general, they can reduce the number of stable states, which is useful in conflicts having multiple stable states.

In future research, we plan to investigate how to extend the $SSEQ$ stability notion allowing for more rounds of conflict analysis. We also leave for future work, the question of existence of $SSEQ$ equilibrium in finite conflicts.

CHAPTER 4

Matrix representations of solutions concepts in GMCR with probabilistic preferences

Abstract

In this chapter, matrix methods are developed to determine stable states in the graph model for conflict resolution with probabilistic preferences with n decision makers. The matrix methods are used to determine more easily the stable states according to four stability definitions proposed for this model, namely: α-Nash stability, (α, β)-metarationality, (α, β)-symmetric metarationality and (α, β, γ)-sequential stability. Additionally, we propose a matrix representation for the $SSEQ$ concept defined in the GMCR with probabilistic preferences.

4.1 Introduction

In the GMCR with probabilistic preferences (GMCRP) [15], DMs do not simply prefer one state over another one, but they do it with a certain probability. The authors proposed four stability definitions for this model, namely: α-Nash stability, (α, β)-metarationality, (α, β)-symmetric metarationality and (α, β, γ)-sequential stability. In this chapter, we follow the same line of reasoning of that used by [26], where matrix representations were used to facilitate the identification of stable states in the GMCR, and we propose matrix methods to determine more easily the stable states according to four stability definitions proposed in the GMCRP for n-DM conflicts and to the definition of SSEQ stability proposed for this model. The matrix methods to determine more easily the stable states according to definitions proposed in the GMCRP for conflicts with two DMs were published in the *Proceedings of the 2015 Conference on Group Decision and Negotiation*, see reference [49]. This chapter generalizes these methods to conflicts involving n-DMs.

This chapter is organized as follows. In Section 4.2, the GMCRP and corresponding stability definitions are recalled. In Section 4.3, we present matrix representations that provide a means to determine stable states in the GMCRP for n-DM conflicts. In Section 4.4, we present an application to illustrate the utility of the matrix representation proposed here. Finally, in Section 4.5, we finish the chapter with the main conclusions found and directions for future work.

4.2 GMCR with probabilistic preferences and solution concepts

Recently, [15] replaced the usual preference notion used in the GMCR by adopting probabilistic preferences [45]. According to a probabilistic preference model, whenever a DM must state preferences between two particular objects, it may do so with a certain probability. Thus, in the GMCRP, for any two states s and q, $P_i(s,q)$ expresses the chance with which DM i prefers state s over q. This probability is defined on $S \times S$ and must satisfy:

(1) $P_i(s,q) \geq 0, \ \forall s \in S$,

(2) $P_i(s,s) = 0, \ \forall s,q \in S$,

(3) $P_i(s,q) + P_i(q,s) \leq 1, \ \forall s,q \in S$.

The expression in (1) says that for any two states in S, we have necessarily that DM i prefer one state to another with probability greater or equal to zero, (2) states that no DM i can strictly prefer one state over itself with positive probability and the expression (3) says that the sum of the probabilities that some DM i strictly prefers some state s to some other state q and strictly prefers q over s is at most equal to 1. The difference $1 - P_i(s,q) - P_i(q,s)$ represents the probability with which DM i is indifferent between s and q.

4.2.1 *Stability Definitions in the GMCRP*

In this subsection, we recall the solution concepts in the GMCRP proposed in [15]. Consider parameters α, β, γ lying in the interval $[0,1]$. Let $R_i^{+\gamma}(s) = \{q \in R_i(s) : P(q,s) > \gamma\}$ be the set of all γ-improvements for DM i when the current state is s, i.e., a state q is a γ-improvement for DM i from state s if q is reachable for DM i from s and DM i prefers state q over state s with probability greater than γ. The solution concept called α-Nash stability is defined as follows.

Definition 4.2.1. *A state $s \in S$ is α-Nash stable for DM $i \in N$ iff $R_i^{+(1-\alpha)}(s) = \emptyset$.*

Intuitively, if DM i is in a α-Nash stable state, then he has no incentive to move away from it in a single step with a sufficiently high probability.

In order to present (α,β)-GMR, (α,β)-SMR, (α,β,γ)-SEQ and (α,β,γ)-$SSEQ$ stability definitions, we need define the set of all unilateral γ-improvement by coalition $H \subseteq N$ from state s.

Let $H \subseteq N$, and let $R_H(s) \subseteq S$ denote the set of states that can be reached by any legal sequence of movements, as defined in Section 2.2.2. Let $R_H^{+\gamma}(s) \subseteq S$ be the set all γ-unilateral improvement by coalition H from state s. If $s_1 \in R_H^{+\gamma}(s)$, then $\Omega_H^{+\gamma}(s,s_1)$ is the set of all last DMs in a legal sequence of unilateral γ-improvement from s to s_1. We have that $R_H^{+\gamma}(s)$ and $\Omega_H^{+\gamma}(s,\cdot)$ are defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $i \in H$ and $s_1 \in R_i^{+\gamma}(s)$, then $s_1 \in R_H^{+\gamma}(s)$ and $i \in \Omega_H^{+\gamma}(s,s_1)$, and (2) if $s_1 \in R_H^{+\gamma}(s)$, $i \in H$, $\Omega_H^{+\gamma}(s,s_1) \neq \{i\}$ and $s_2 \in R_i^{+\gamma}(s_1)$, then $s_2 \in R_H^{+\gamma}(s)$ and $i \in \Omega_H^{+\gamma}(s,s_2)$. Let also $\Phi_i^{+\gamma}(s) = \{q \in S : P_i(q,s) > \gamma\}$ be the set of all states that DM i strictly prefers to state s with probability greater that γ. Then, we can now present the definitions of (α,β)-GMR, (α,β)-SMR, (α,β,γ)-SEQ, proposed in the GMCRP [15], and the definition (α,β,γ)-$SSEQ$ stability that we presented for GMCRP in the previous chapter.

Definition 4.2.2. *A state $s \in S$ is (α,β)-GMR stable for DM $i \in N$ iff for every $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists s_2 such that $s_2 \in R_{N-\{i\}}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$.*

Definition 4.2.3. *A state $s \in S$ is (α,β)-SMR stable for DM $i \in N$ iff for every $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists s_2 such that $s_2 \in R_{N-\{i\}}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$ and $R_i(s_2) \cap \Phi_i^{+(1-\alpha)}(s) = \emptyset$.*

Definition 4.2.4. *A state $s \in S$ is (α, β, γ)-SEQ stable for DM $i \in N$ iff for every $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists s_2 such that $s_2 \in R_{N-\{i\}}^{+\gamma}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$.*

Definition 4.2.5. *A state $s \in S$ is (α, β, γ)-SSEQ stable for DM $i \in N$ iff for every $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists s_2 such that $s_2 \in R_{N-\{i\}}^{+\gamma}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$ and $R_i(s_2) \cap \Phi_i^{+(1-\alpha)}(s) = \emptyset$*

Intuitively, if a state s is (α, β)-GMR stable for DM i, he has no incentive to move away from it, because for all state s_1 that i strictly prefers over s with probability greater $1 - \alpha$, there exists a reachable state s_2 for the opponents of i such that i does not strictly prefer s_2 over s with probability greater than $1 - \beta$. Besides that, in an (α, β)-SMR stable state, DM i cannot scape from this latter situation for a state that he strictly prefers over s with probability greater $1 - \alpha$. In an (α, β, γ)-SEQ stable state, all the moves in the reaction of the opponents of DM i are γ-unilateral improvements, but no requirement to whether DM i may counter-react is made. Finally, an (α, β, γ)-$SSEQ$ stable state, both all the moves in the reaction of the opponents of DM i are γ-unilateral improvements and DM i cannot scape from the state to which his opponents lead the conflict to a state preferred over s with probability greater than $1 - \alpha$.

4.3 Matrix Representations of Solution Concepts of GMCRP

In what follows, we make appropriate adjustments in the matrices proposed by [26] that are used to find results similar to those obtained by those authors, i.e., we propose a way to determine stable states, through matrix operations, according to the five stability notions for GMCRP presented in the previous subsection.

Consider the $|S| \times |S|$, 0-1 matrices J_i and $J_i^{+\gamma}$ defined, respectively, as follows.

$$J_i(s, q) = \begin{cases} 1, & \text{if } q \in R_i(s), \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that the element (s, q) of matrix J_i, called accessibility matrix, receives value 1 if state q is reachable by DM i from state s, and receives value 0 otherwise.

Matrix $J_i^{+\gamma}$ is defined as

$$J_i^{+\gamma}(s,q) = \begin{cases} 1, & \text{if } q \in R_i(s) \text{ and } P_i(q,s) > \gamma, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Similarly to matrix J_i, the element (s,q) of matrix $J_i^{+\gamma}$ receives value 1 if state q is reachable from state s, and if DM i strictly prefers state q over s with probability greater than γ. Otherwise, the element (s,q) receives value 0.

Matrix J_i is defined exactly as in [26]. On the other hand, matrix $J_i^{+\gamma}$ is different since its corresponding matrix defined in [26], denoted by J_i^{+}, is the matrix whose element (s,q) receives value 1 if state q is reachable from state s, and if DM i strictly prefers state q over s. Otherwise, the element (s,q) receives value 0.

We now recall some matrices as defined in [26]. Consider Y an $|S| \times |S|$ matrix with all elements equal to 1, and let e_k denote the $|s|$-dimensional column vector with k^{th} element equal to 1 and all other elements equal to 0. Let M and N be $|S| \times |S|$ matrices and define $W = M \circ N$ as the $|S| \times |S|$ matrix with (s,q) entry $W(s,q) = M(s,q) \cdot N(s,q)$. Let the matrix $H = M \vee N$, an $|S| \times |S|$ matrix with entry (s,q) defined as 1 if $M(s,q) + N(s,q) \neq 0$, and 0 otherwise. If K is an arbitrary $|S| \times |S|$ matrix, then the matrix signal of K, denoted by $\text{sign}(K)$, is an $|S| \times |S|$ matrix with (s,q) entry defined as follows

$$\text{sign}[K(s,q)] = \begin{cases} -1, & \text{if } K(s,q) < 0, \\ 0, & \text{if } K(s,q) = 0, \\ 1, & \text{if } K(s,q) > 0. \end{cases}$$

In [26], the authors define preference matrices, which are useful in determining what are the stable states according to various stability definitions. These matrices, denoted by P_i^{+}, P_i^{-} and $P_i^{=}$, have element (s,q) equal to 1 if $q \succ_i s$, $s \succ_i q$ and $q \sim_i s$, respectively, and have element (s,q) equal to zero otherwise. In addition, these authors also propose a less than or equal preference matrix, denoted by $P_i^{-,=}$, which has element (s,q) equal to $1 - P_i^{+}(s,q)$ if $s \neq q$ and zero otherwise. Note that all elements in the main diagonal of $P_i^{-,=}$ are equal to zero. As we show in Section 4.3.1, such definition causes some problems in some results presented in [26].

Here, we propose similar matrices, but considering probabilistic preferences. The corresponding matrices are defined as follows:

$$Q_i^{+\gamma}(s,q) = \begin{cases} 1, & \text{if } P_i(q,s) > \gamma, \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

$$Q_i^{-\gamma}(s,q) = \begin{cases} 1, & \text{if } P_i(q,s) < \gamma, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

$$Q_i^{=\gamma}(s,q) = \begin{cases} 1, & \text{if } P_i(q,s) = \gamma, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

Matrix $Q_i^{+\gamma}$ has element (s,q) equal to 1 if state q is strictly preferred by DM i over state s with probability greater than γ, and has element (s,q) equal to 0 otherwise. The matrix $Q_i^{-\gamma}$ has element (s,q) equal 1 if state q is strictly preferred by DM i over state s with probability smaller than γ. Finally, in matrix $Q_i^{=\gamma}$, the element (s,q) is equal to 1 if DM i prefers state q over s with probability exactly equal to γ. Note that in matrix $Q_i^{+\gamma}$ state q does not need to be achievable from state s. Finally, matrix $Q_i^{-,=,\gamma}(s,q)$ can be obtained in terms of matrix $Q_i^{+\gamma}$ as follows:

$$Q_i^{-,=,\gamma}(s,q) = 1 - Q_i^{+\gamma}(s,q). \tag{4.6}$$

Note that as opposed to the definition of $P_i^{-,=}$, all elements in the main diagonal of $Q_i^{-,=,\gamma}(s,q)$ are equal to one.

For a coalition $H \subseteq N$, let also the matrices $M_H(s,q)$ and $M_H^{+\gamma}(s,q)$ be defined, respectively, by

$$M_H(s,q) = \begin{cases} 1, & \text{if } q \in R_H(s), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M_H^{+\gamma}(s,q) = \begin{cases} 1, & \text{if } q \in R_H^{+\gamma}(s), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the element (s,q) of matrix M_H receives value 1 if state q is reachable by means of a legal sequence of movements, made by DMs in H, from state s, and this element is equal to 1 in matrix $M_H^{+\gamma}$ if q is a unilateral γ-improvement by DMs in H from state s.

Matrix M_H is defined exactly as in [26]. On the other hand, matrix $M_H^{+\gamma}$ is different since its corresponding matrix defined in [26], denoted by M_H^{+}, is the matrix whose element (s,q) receives value 1 if q is a unilateral improvement by DMs in H from state s and value 0, otherwise.

Xu *et al.* [26] shows how to obtain matrix M_H in terms of the accessibility matrices J_i, $i \in H$. If δ is the number of legal moves required for obtaining all states in the list $R_H(s)$ (δ is upper bounded by the total number of one step moves for all DMs in the conflict) and M_i^t is a matrix with entry (s,q) equal to 1 if q is reachable from state s in exactly t legal moves with first move made by DM i and 0 otherwise, then it follows that

$$M_i^t = \text{sign}\left(J_i \cdot \left(\bigvee_{j \in N-i} M_j^{(t-1)}\right)\right)$$

and

$$M_H = \bigvee_{t=1}^{\delta} \bigvee_{i \in H} M_i^{(t)},$$

where for all $i \in N$, $M_i^1 = J_i$.

Using a similar idea of the one used by Xu et al. [26], one can obtain the $M_H^{+\gamma}$ matrix in terms of $J_i^{+\gamma}$, for $i \in N$. Let δ_γ be the number of legal unilateral improvements required to find all states in the list $R_H^{+\gamma}$ (δ_γ is upper bounded by the total number of unilateral improvement moves for all DMs in the conflict) and $M_i^{(t,+\gamma)}$ be a matrix with entry (s,q) equal to 1 if q is reachable from state s in exactly t legal γ-unilateral improvement moves with first move made by DM i and 0 otherwise. This result is provided by the following theorem.

Theorem 4.3.1. *The matrix $M_H^{+\gamma}$ can be found inductively in the following way*

$$M_H^{+\gamma} = \bigvee_{t=1}^{\delta_\gamma} \bigvee_{i \in H} M_i^{(t,+\gamma)},$$

where

$$M_i^{t,+\gamma} = \text{sign}\left(J_i^{+\gamma} \cdot \left(\bigvee_{j \in N-i} M_j^{(t-1,+\gamma)}\right)\right)$$

and $M_i^{1,+\gamma} = J_i^{+\gamma}$.

The proof of the result is analogous to the proof of the correspondent results proposed in [26], just replacing the matrices J_i^+ and $M_j^{(t-1)}$ by $J_i^{+\gamma}$ and $M_j^{(t-1,+\gamma)}$, respectively.

Using the above matrices, results analogous to those obtained by [26] remain valid for the GMCRP. These results are given by the following four theorems:

Theorem 4.3.2. *Let $i \in N$. A state s is α-Nash stable for DM i iff $e_s^\top \cdot J_i^{+(1-\alpha)} = \vec{0}^\top$.*

Theorem 4.3.3. *Let $i \in N$. A state $s \in S$ is (α,β)-metarational stable for DM i iff $M_i^{(\alpha,\beta)-GMR}(s,s) = 0$, where $M_i^{(\alpha,\beta)-GMR} = J_i^{+(1-\alpha)}\left[Y - \text{sign}\left(M_{N-i} \cdot (Q_i^{-,=,(1-\beta)})^\top\right)\right]$.*

Theorem 4.3.4. *Let $i \in N$. A state $s \in S$ is (α,β)-symmetric metarational stable for DM i iff $M_i^{(\alpha,\beta)-SMR}(s,s) = 0$, where $M_i^{(\alpha,\beta)-SMR} = J_i^{+(1-\alpha)}[Y - \text{sign}(M_{N-i} \cdot W)]$, and $W = (Q_i^{-,=,(1-\beta)})^\top \circ \left[Y - sign\left(J_i \cdot (Q_i^{+(1-\alpha)})^\top\right)\right]$.*

Theorem 4.3.5. *Let $i \in N$. A state $s \in S$ is (α,β,γ)-sequential stable for DM i iff $M_i^{(\alpha,\beta,\gamma)-SEQ}(s,s) = 0$, where $M_i^{(\alpha,\beta,\gamma)-SEQ} = J_i^{+(1-\alpha)}\left[Y - \text{sign}\left(M_{N-i}^{+\gamma} \cdot (Q_i^{-,=,(1-\beta)})^\top\right)\right]$.*

The proof of the four above results are analogous to the proof of the results proposed in [26], just replacing the matrices J_i^+, P_i^+, $P_i^{-,=}$ and M_H^+ proposed by these authors by, respectively, the adjusted matrices $J_i^{+\gamma}$, $Q_i^{+\gamma}$, $Q_i^{-,=,\gamma}$ and $M_H^{+\gamma}$ presented in this work.

Here, we add a new result providing a matrix representation for the (α,β,γ)-symmetric sequential stability concept.

Theorem 4.3.6. *Let $i \in N$. A state $s \in S$ is (α,β,γ)-symmetric sequentially stable for DM i iff $M_i^{(\alpha,\beta,\gamma)-SSEQ}(s,s) = 0$, where $M_i^{(\alpha,\beta,\gamma)-SSEQ} = J_i^{+(1-\alpha)}[Y - \text{sign}(M_{N-i}^{+\gamma} \cdot W)]$, and $W = (Q_i^{-,=,(1-\beta)})^\top \circ \left[Y - \text{sign}\left(J_i \cdot (Q_i^{+(1-\alpha)})^\top\right)\right]$.*

Proof: Suppose without loss of generality that $|S| = t$. Then, we have that the diagonal element (s,s) of matrix $M_i^{(\alpha,\beta,\gamma)-SSEQ}$ can be written as

$$\begin{aligned} M_i^{(\alpha,\beta,\gamma)-SSEQ}(s,s) &= \left\langle (J_i^{+(1-\alpha)})^\top e_s, (Y - \text{sign}(M_{N-i}^{+\gamma} \cdot W))e_s \right\rangle \\ &= \sum_{s_1=1}^{t} J_i^{+(1-\alpha)}(s,s_1)\left[1 - \text{sign}\left(\left\langle (M_{N-i}^{+\gamma})^\top e_{s_1}, We_s \right\rangle\right)\right], \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors. Thus $M_i^{(\alpha,\beta,\gamma)-SSEQ}(s,s) = 0$ iff

$$\sum_{s_1=1}^{t} J_i^{+(1-\alpha)}(s,s_1)\left[1 - \text{sign}\left(\left\langle (M_{N-i}^{+\gamma})^\top e_{s_1}, We_s \right\rangle\right)\right] = 0.$$

Since all terms are non-negative, the above condition is equivalent to

$$J_i^{+(1-\alpha)}(s,s_1)\left[1 - \text{sign}\left(\left\langle (M_{N-i}^{+\gamma})^\top e_{s_1}, We_s \right\rangle\right)\right] = 0, \text{ for all } s_1 \in S. \tag{4.7}$$

Note that (4.7) is true iff,

$$(e_{s_1}^\top M_{N-i}^{+\gamma}) \cdot (We_s) > 0, \text{ for all } s_1 \in R_i^{+(1-\alpha)}(s). \tag{4.8}$$

Let $W(s_2, s)$ denote the (s_2, s) entry of matrix W. Thus it follows that

$$(e_{s_1}^\top M_{N-i}^{+\gamma}) \cdot (We_s) = \sum_{s_2=1}^{t} M_{N-i}^{+\gamma}(s_1,s_2) \cdot W(s_2,s)$$

Therefore, (4.8) holds iff, for all $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists $s_2 \in R_{N-i}^{+\gamma}(s_1)$ such that $W(s_2, s) \neq 0$.

Note that the element (s_2, s) of matrix W can be written

$$W(s_2,s) = Q_i^{-,=,(1-\beta)}(s,s_2)\left[1 - \text{sign}\left(\sum_{s_3=1}^{t} J_i(s_2,s_3) Q_i^{+(1-\alpha)}(s,s_3)\right)\right].$$

Thus $W(s_2, s) \neq 0$ is equivalent to

$$Q_i^{-,=,(1-\beta)}(s,s_2) \neq 0 \tag{4.9}$$

and

$$\sum_{s_3=1}^{t} J_i(s_2,s_3) Q_i^{+(1-\alpha)}(s,s_3) = 0. \tag{4.10}$$

Thus $M_i^{(\alpha,\beta,\gamma)-SSEQ}(s,s) = 0$ iff for all $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists $s_2 \in R_{N-i}^{+\gamma}(s_1)$ such that $Q_i^{-,=,(1-\beta)}(s,s_2) \neq 0$ and $Q_i^{+(1-\alpha)}(s,s_3) = 0$ for every $s_3 \in R_i(s_2)$. Therefore, we have that $M_i^{(\alpha,\beta,\gamma)-SSEQ}(s,s) = 0$ iff, for all $s_1 \in R_i^{+(1-\alpha)}(s)$, there exists s_2 such that $s_2 \in R_{N-i}^{+\gamma}(s_1) \cap (\Phi_i^{+(1-\beta)}(s))^c$ and $R_i(s_2) \cap \Phi_i^{+(1-\alpha)}(s) = \emptyset$. $\square$

4.3.1 *A problem in the paper of Xu et al. [26]*

The results obtained in the previous section are analogous to the ones obtained by Xu et al. [26]. However, in this section, we show that there are problems in the results related to the GMR and SEQ solution concepts obtained by such authors. We show that the GMR and SEQ matrix results, obtained in [26], are false by means of a counter-example.

The definition of matrix $Q_i^{-=\gamma}(s,q)$ in the case of GMCRP, is similar to the definition of matrix $P_i^{-,=}(s,q)$ proposed in Xu et al. [26]. The main difference between such matrices is that while the elements of the main diagonal of $P_i^{-,=}$ are zero, those of $Q_i^{-=\gamma}$ are equal to one. Since a state cannot be strictly preferred to itself, we find that our definition is more appropriate. Moreover, using the definition of $P_i^{-,=}$ makes the GMR and SEQ results presented in Xu et al. [26] false.

In order to show that, we recall the results presented in [26] for the case of a 2-DM conflict next.

Theorem 4.3.7. *Let $i \in N$. A state $s \in S$ is GMR stable for DM i iff $M_i^{GMR}(s,s) = 0$, where $M_i^{GMR}(s,s) = J_i^+ \left[Y - sign\left(J_j \cdot (P_i^{-,=})^\top\right)\right]$.*

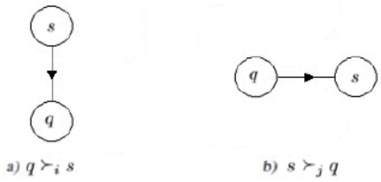
Theorem 4.3.8. *Let $i \in N$. A state $s \in S$ is SEQ stable for DM i iff $M_i^{SEQ}(s,s) = 0$, where $M_i^{SEQ}(s,s) = J_i^+ \left[Y - sign\left(J_j^+ \cdot (P_i^{-,=})^\top\right)\right]$.*

The counter-example presented below illustrates a state s which by definition is GMR and SEQ stable, event though $M_i^{GMR}(s,s) \neq 0$ and $M_i^{SEQ}(s,s) \neq 0$.

Example 4.3.1. *(Counter-Example) Consider the following conflict with 2 DMs, i and j. Suppose that $R_i(s) = \{q\}$, $R_j(q) = \{s\}$, $R_i(q) = R_j(s) = \emptyset$ and that the ordinal preferences are $q \succ_i s$ and $s \succ_j q$, as shown in Figure 4.3.1.*

It is easy to see that, by definition, state s is GMR and SEQ stable for DM i. However, we argue that $M_i^{GMR}(s,s) \neq 0$ and $M_i^{SEQ}(s,s) \neq 0$. Indeed, we have that matrices J_i, J_j, J_i^+, J_j^+ and $P_i^{-=}$ are given, respectively, by

Figure 4.1: Counter-Example



$$J_i = J_i^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$J_j = J_j^+ = P_i^{-=} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Using the definitions according to Theorems 4.3.7 and 4.3.8, respectively, we have that $M_i^{GMR} = M_i^{SEQ} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Therefore, $M_i^{GMR}(s,s) = M_i^{SEQ}(s,s) = 1$, *which according to Theorem 4.3.7 (resp., 4.3.8) implies that state* s *is not* GMR *and (resp., not SEQ) stable for DM* i*, which is a contradiction.*

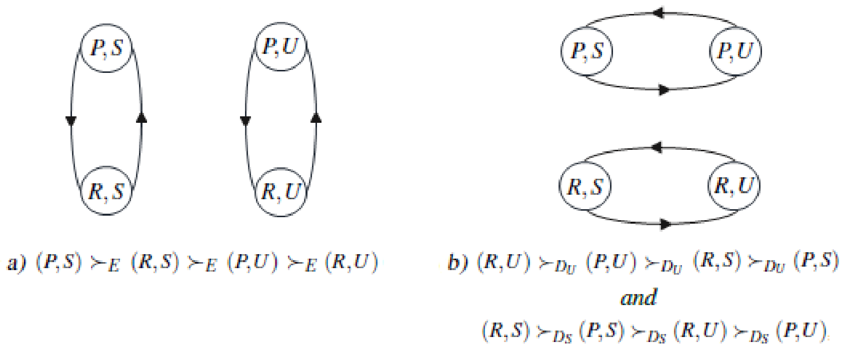
It is easy to see that Theorems 4.3.7 and 4.3.8 would become true if either the main diagonal of $P_i^{-=}$ have all elements equal to one instead of zero or if we add the restriction that $J_i \circ J_{N-i}^T = \hat{0}$, where $\hat{0}$ is the null matrix of same dimension of J_i and J_{N-i}. This latter restriction implies the opponents of DM i cannot return the conflict to its initial state after DM i's first move, which for us is a too demanding requirement in general. We emphasize that this assumption is not mentioned in Xu et al. [26].

4.4 Application

We now present a modified version, presented in [50], of a hypothetical conflict proposed by [51] to illustrate an application of how to obtain the stable states using the matrix representations described in the previous section. In this conflict, there are two DMs: environmentalist

(E) and developers (D). Environmentalists may choose to be proactive (P) in promoting environmental responsibility or not, in this case they are called reactive (R). Developers may choose to be sustainable (S), or not, which is represented by U. The set of possible states of the conflict is: (P, S), (P, U), (R, U), and (R, S). Figure 4.4 represents the graph model for this strategic conflict.

Figure 4.2: Conflict in the graph form: a) DM E; b) DM D.



Consider that DM D has two possible types, denoted by D_S and D_U, and consider the probability distribution which describes the chance that the developers are of one of these types is given by $P(D = D_S) = 0.3$ and $P(D = D_U) = 0.7$. We have that the matrices J_E, and J_D are given, respectively, by

$$J_E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$J_D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

where, (P, S), (R, S), (P, U) and (R, U) are represented in lines (columns) 1, 2, 3 and 4, respectively.

The probabilistic preferences of the DMs E and D, are given in the Tables 4.1 and 4.2, respectively.

Table 4.1: Probabilistic preferences of DM E

DM E	(P,S)	(R,S)	(P,U)	(R,U)
(P,S)	0.0	1.0	1.0	1.0
(R,S)	0.0	0.0	1.0	1.0
(P,U)	0.0	0.0	0.0	1.0
(R,U)	0.0	0.0	0.0	0.0

Table 4.2: Probabilistic preferences of DM D

DM E	(P,S)	(R,S)	(P,U)	(R,U)
(P,S)	0.0	0.0	0.3	0.3
(R,S)	1.0	0.0	0.3	0.3
(P,U)	0.7	0.7	0.0	0.0
(R,U)	0.7	0.7	1.0	0.0

In Tables 4.1 and 4.2, each cell expresses the probability that the respective DM prefers the line state over the column state.

Considering, for example, the parameter values $\alpha = 0.3$, $\beta = 0.8$ and $\gamma = 0.5$, we have that the matrices $J_E^{+(1-\alpha)}$, $J_D^{+\gamma}$, $Q_E^{+(1-\beta)}$ and $Q_E^{+(1-\alpha)}$ are given, respectively, by

$$J_E^{+(0.7)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$J_D^{+(0.5)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Q_E^{+(0.2)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q_E^{+(0.7)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

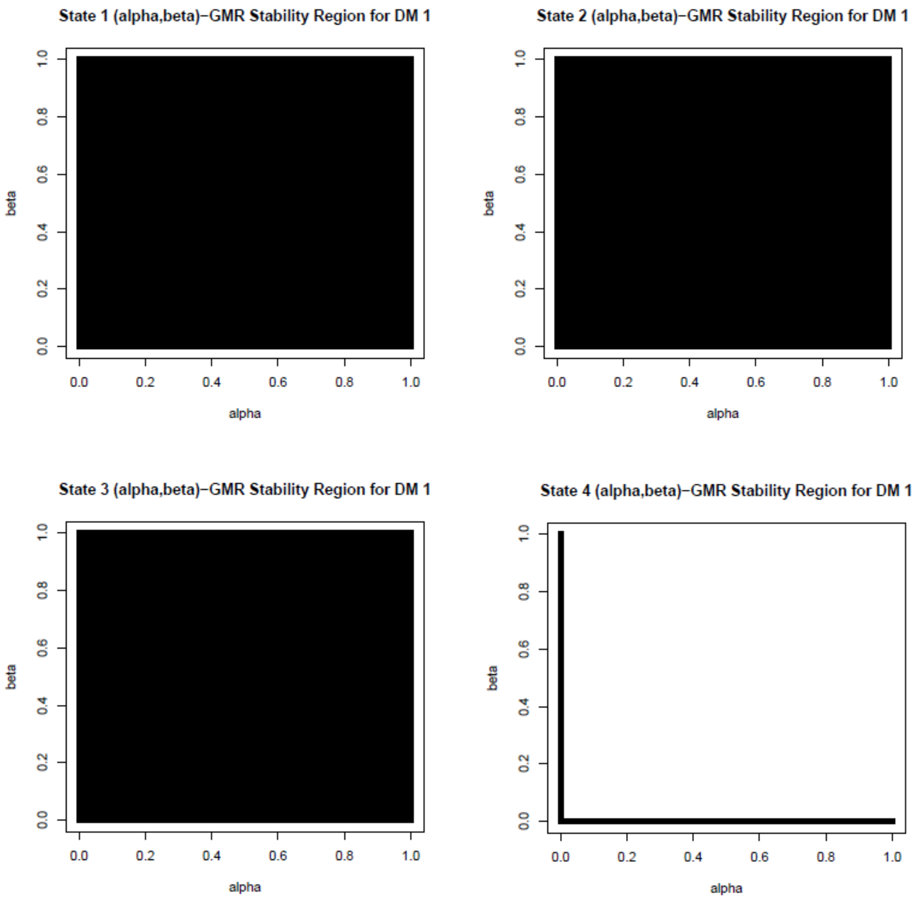
Using Theorem 4.3.2, we conclude that states (P, S) and (P, U) are 0.3-Nash stable for DM E, because the rows of the matrix $J_E^{+(0.7)}$, corresponding to these states, are all null. And using Theorems 4.3.3, 4.3.4, 4.3.5 and 4.3.6, we conclude that $\text{diag}(M) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. Thus, states (P, S), (R, S) and (P, U) are $(0.3, 0.8)$-metarational, $(0.3, 0.8)$-symmetric metarational, $(0.3, 0.8, 0.5)$-sequentially stable and $(0.3, 0.8, 0.5)$-symmetric sequentially stable for DM D, but state (R, U) is not.

More generally, in Figures 4.3, 4.4, 4.5, 4.6, 4.7, 4.8 4.9 and 4.10 we have established parameter regions in which the above conflict states are stable according to the definitions (α, β)-metarationality, (α, β)-symmetric metarationality, (α, β, γ)-sequential stability and (α, β, γ)-sy mmetric sequential stability, respectively, for DM E. The dark regions in the graphs refer to α, β and γ values, for which the states (P, S), (R, S), (P, U) and (R, U), denoted in the graphs by 1, 2, 3 and 4, respectively, are stable according to these four stability definitions for DM E and DM D, denoted by DM 1 and DM 2, respectively.

4.5 Conclusion

Following a similar idea as that used by Xu et al. [26], we propose matrix representations to determine stable states in 2-DM and n-DM conflicts in the GMCR with probabilistic preferences, according to the definitions proposed in [15] and [42], namely: α-Nash stability, (α, β)-metarationality, (α, β)-symmetric metarationality, (α, β, γ)-sequential and (α, β, γ)-symmetric sequential stability. The methodology presented in this chapter can help to find conflict resolutions using the GMCR. It combines the advantages of probabilistic preference models, which are more flexible to accommodate preference features of DMs in real conflicts, and of matrix representations of solution concepts in GMCR, which are more effective in determining stabilities and in predicting equilibria, especially in complex conflict models with many feasible states. Using the approached proposed here, one can more easily determine for which set of parameters' values a given state is stable and, as suggested by [15], such information can be relevant to compare the equilibrium robustness of the states. We are currently investigating an extension of matrix representations to other solution concepts, such as limited-move stability [21], nonmyopic stability

Figure 4.3: (α, β)-GMR stability region for DM E



[52] and Stackelberg equilibrium [53].

Figure 4.4: (α, β)-GMR stability region for DM D

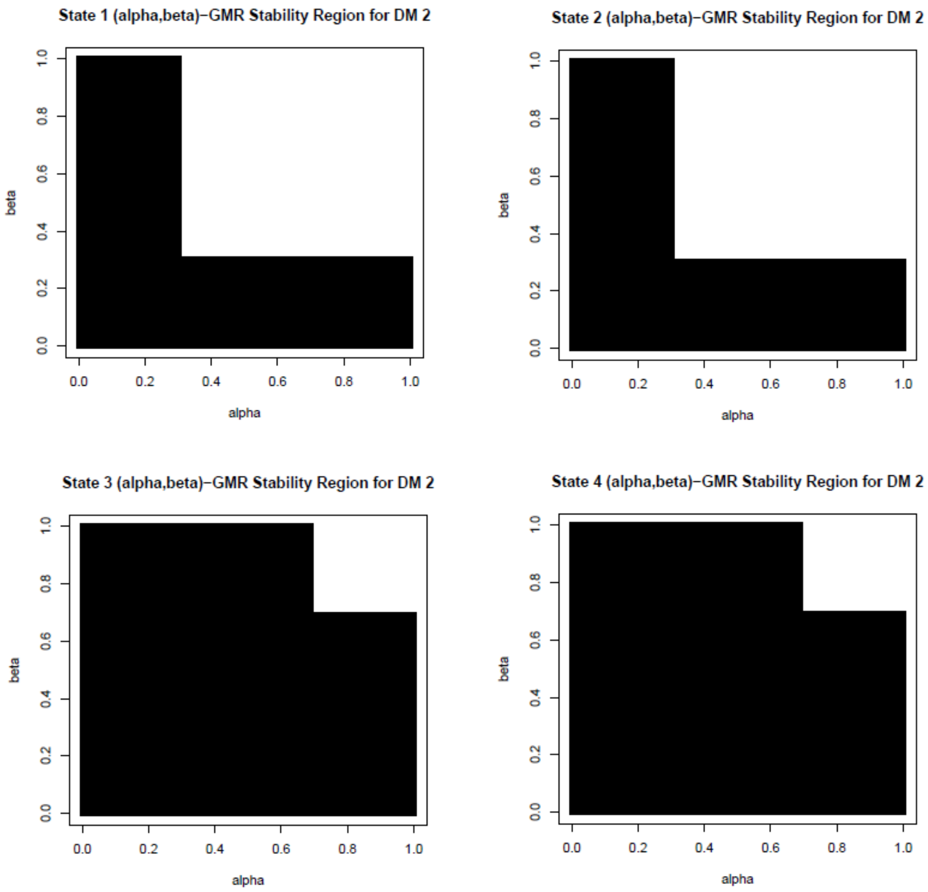


Figure 4.5: (α, β)-SMR stability region for DM E

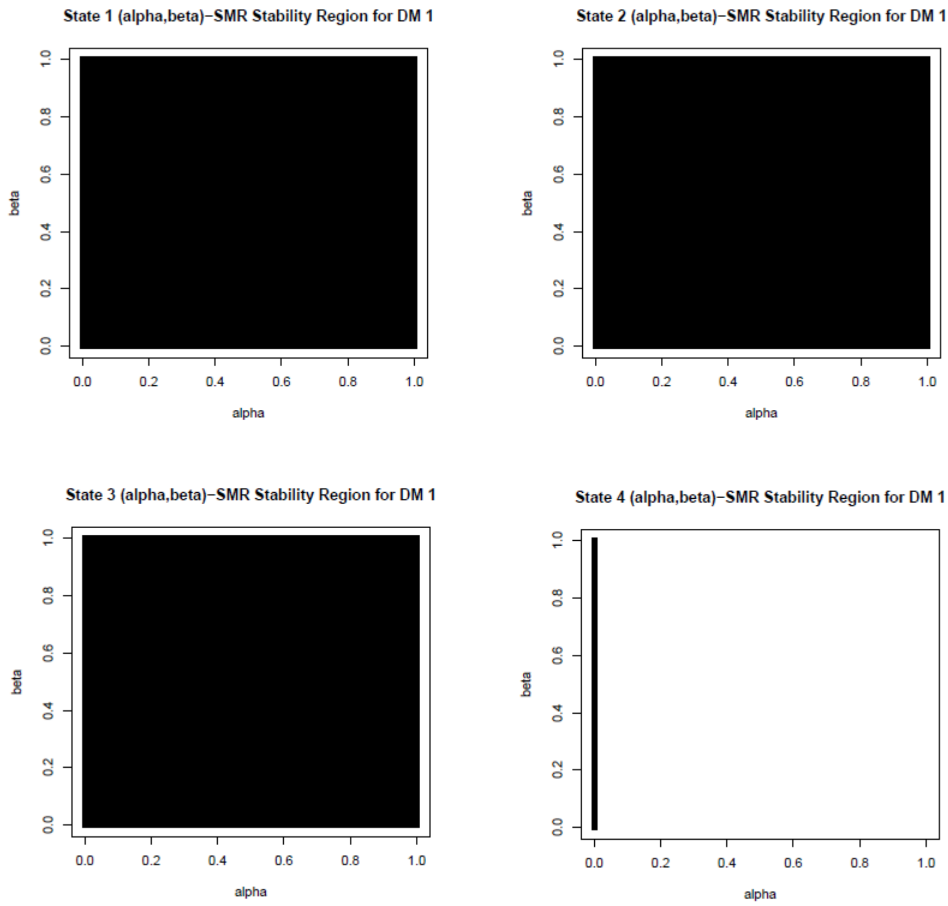


Figure 4.6: (α, β)-SMR stability region for DM D

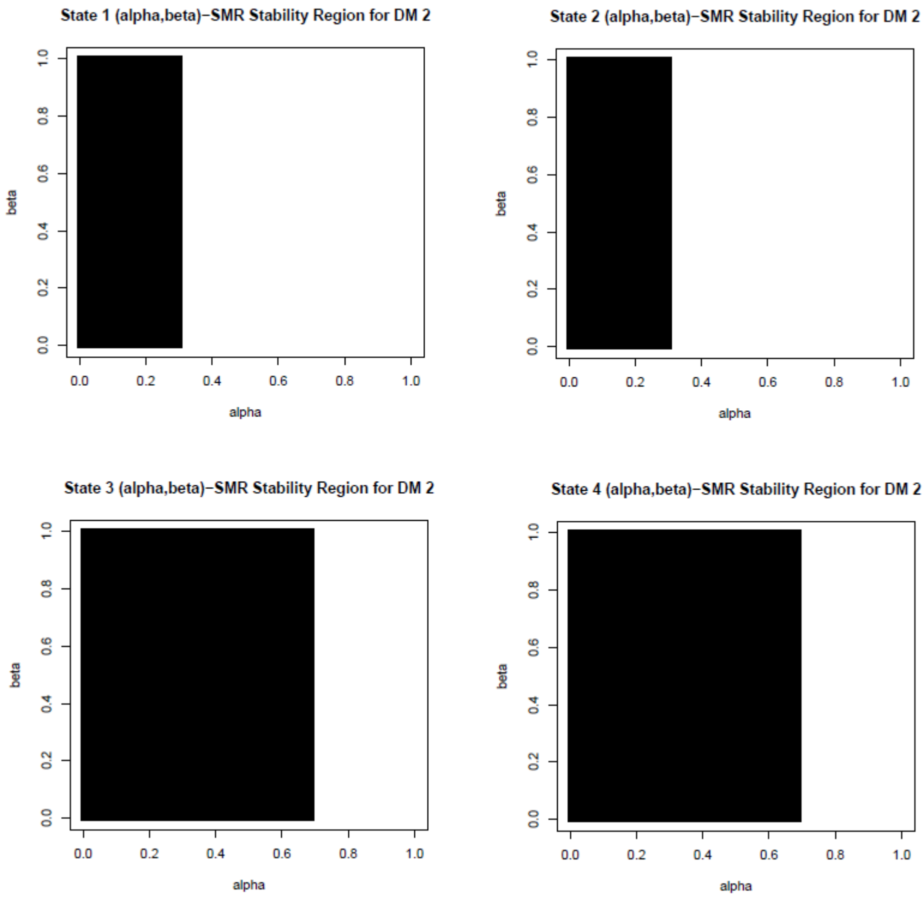


Figure 4.7: (α, β, γ)-SEQ stability region for DM E

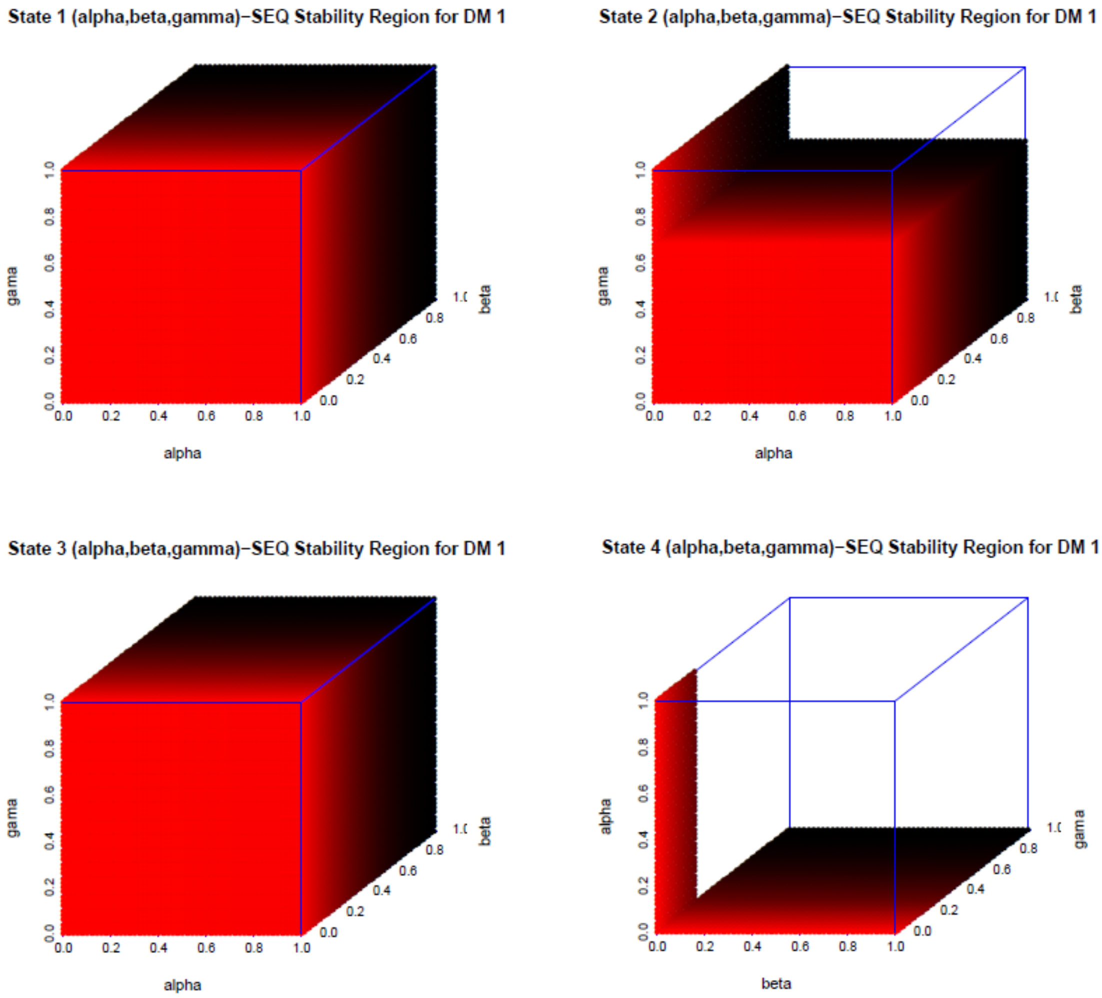


Figure 4.8: (α, β, γ)-SEQ stability region for DM D

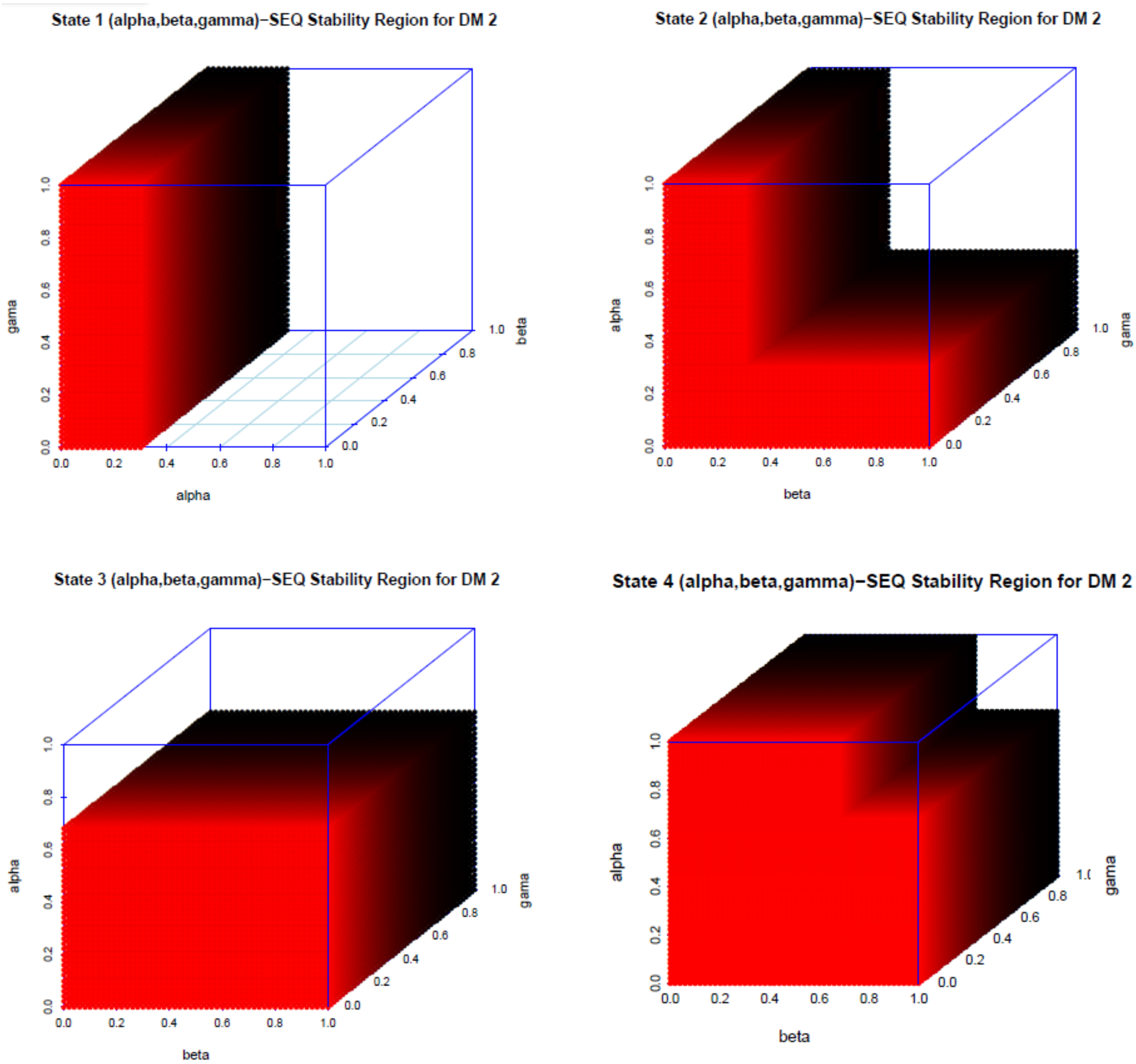


Figure 4.9: (α, β, γ)-$SSEQ$ stability region for DM E

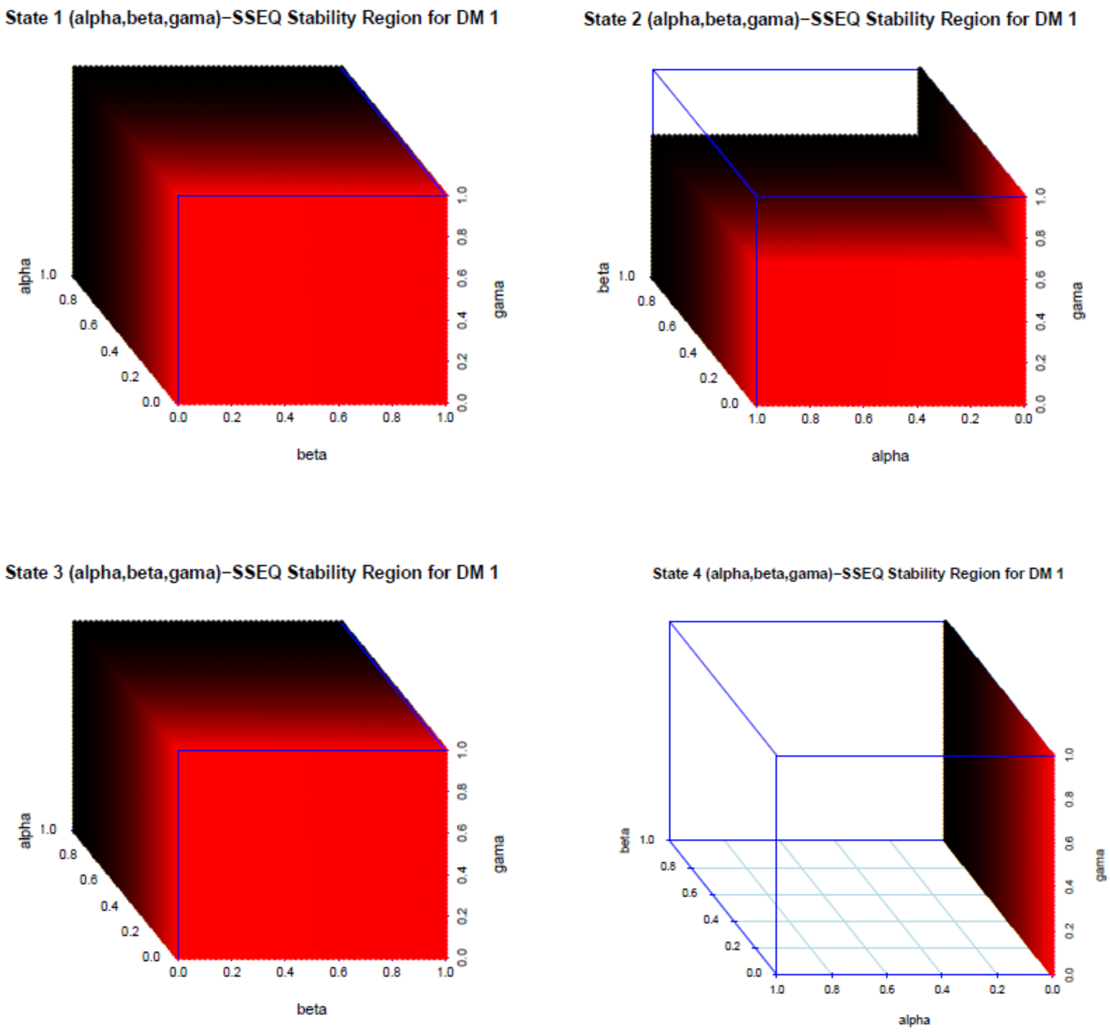
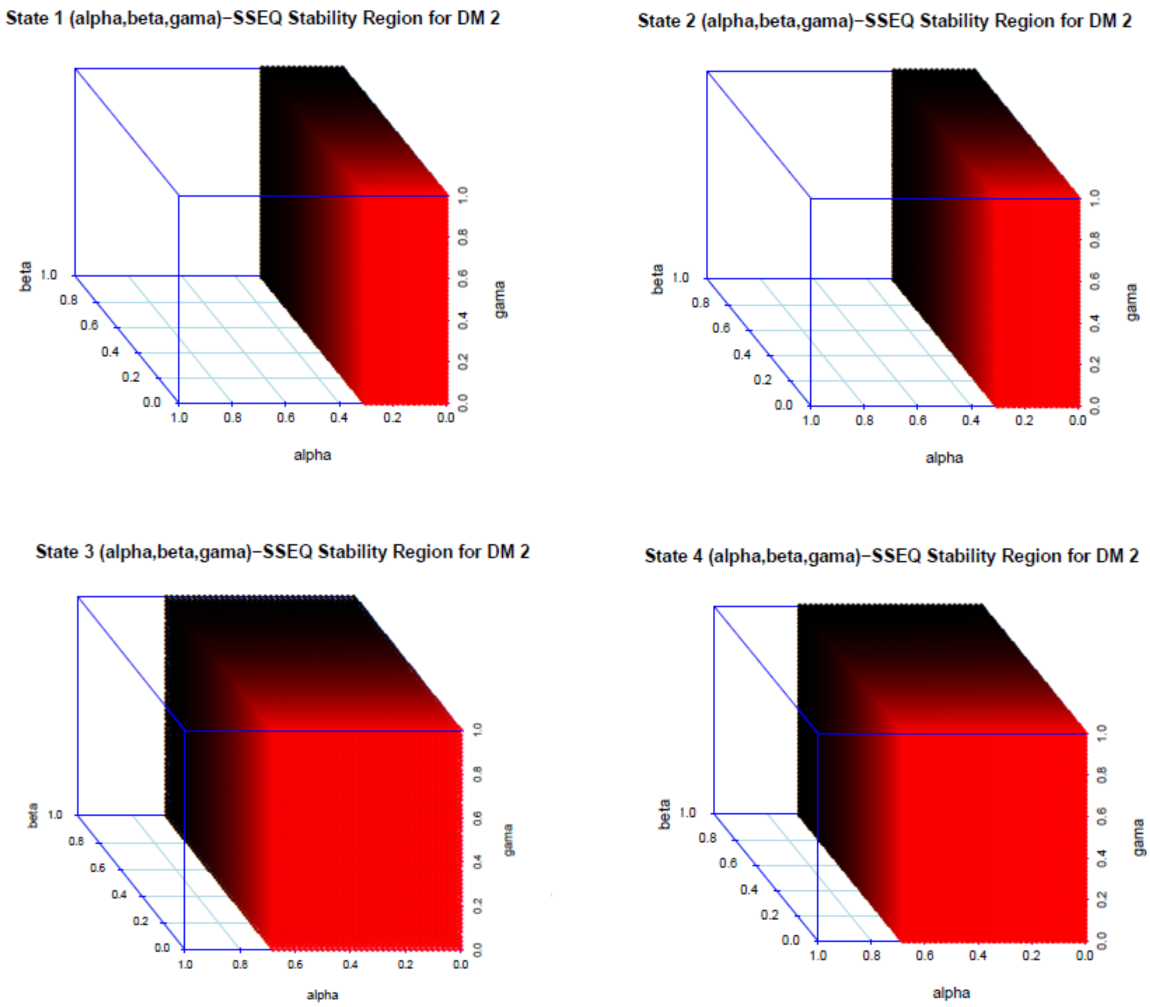


Figure 4.10: (α, β, γ)-$SSEQ$ stability region for DM D



CHAPTER 5

Interactive Unawareness in the Graph Model for Conflict Resolution

Abstract

In this Chapter, we present a generalization of the Graph Model for Conflict Resolution (GMCR) to model interactive unawareness of decision makers (DMs) about the options available to them in the conflict. More specifically, we consider a GMCR with two and n DMs in which a DM, in some given state, can be unconscious about some of his options, or about the options available to his opponent(s), and therefore, may have only a partial knowledge of the state space. We present generalizations of the usual stability concepts in the GMCR and we have obtained some results relating such new concepts.

5.1 Introduction

We modify the standard GMCR model to allow for the possibility that DMs may be unaware of some of the options available in the conflict. Our motivation for proposing this model is that in some conflicts having an available option that your opponents is unaware of can be crucial to determine what kinds of conflict resolutions can be achieved. For example, in a war setting developing a new weapon technology which the adversary is unaware of can be crucial in defining the war resolution.

Our approach is to adapt a model of interactive unawareness proposed by [54] to the GMCR setting. In this proposed setting, instead of a single state space common to all DMs involved in the conflict, there are several state spaces and associated to each one of them, there is a set of options available to the DMs in the conflict, according to the viewpoint of a DM who believes the conflict is described by such state space. Thus, if a DM is in a given state in some state space, he may believe that he is in another state space, because he might not be aware of all options available. Moreover, even if he is aware of all options available he might believe that his opponent is not.

In this model, we allow for an arbitrary number of levels of iterated unawareness, in the sense that DM i may be unaware that DM j is unaware that DM i is unaware that DM j has a certain option available, and so on. We discuss the case with two and n-DMs and generalize the usual notions of stability for the GMCR with interactive unawareness. The GMCR with interactive unawareness for conflicts with two DMs was published in the *Proceedings of the 16th Meeting on Group Decision and Negotiation*, see reference [55]. In this chapter, we extend this model further to deal with n-DM conflicts.

This chapter is organized as follows. In Section 5.2, we present the GMCR with interactive unawareness. In Section 5.3, we present the solution concepts for conflicts with two DMs and establish some relationships between the proposed solution concepts. In Section 5.4, we presents the solution concepts for conflicts with n-DMs and establish relationships between the proposed solution concepts. In Section 5.5, we present an application of the proposed model to highlight its usefulness. Finally, in Section 5.6, we finish the paper with the main conclusions found.

5.2 Interactive Unawareness in the GMCR

The study of misperceptions in game theory has a long history starting with the work of Bennett [61] who defined the notion of hypergames to model how DMs perceive a conflict. Takahashi et al. [62] developed a methodology to analyze hypergames using an adaption of the sequential stability notion. Wang et al. [63, 64] present definitions and properties of solution concepts used in hypergames analysis.

In the last decade, the game theory literature has devoted increased attention to the modeling of a particular kind of misperception, called unawareness. In such literature, DMs might not conceive all relevant aspects (contingencies, actions, DMs involved) of a strategic situation, but once they are aware of something, they cannot hold arbitrary false beliefs about what they are aware of. The idea is to understand what are the implications that unawareness can have on strategic behavior. For example, Feinberg [65] showed that even with a small uncertainty about unawareness of actions, rational DMs can cooperate in the finitely repeated prisoner's dilemma. Halpern and Rêgo [59] proposed a model of extensive form games, where players may be unaware of some actions available in the game. Chen and Zhao [60] proposed a framework to analyze the behavior of unaware agents in the classical principal-agent model. Heifetz et al. [54] introduced a generalized state-space model that allows for non-trivial unawareness among several individuals.

Aligned with what has been done in the game theory literature, in this chapter, we want to propose a model that is able to analyze what are the implications of unawareness as the only source of misperception in the GMCR stability analysys. In our model, DMs may be unaware about some options available in the conflict but not false beliefs about what they are aware of. As opposed to the perceptual graph model, in this setting, we allow for higher order unawareness levels, so that we can model a situation where DM i may be unaware that DM j is aware that DM i is aware of option a. We only discuss the case of conflicts with two DMs and extend the usual notions of stability for the GMCR with interactive unawareness.

Although in a practical scenario, it is likely that besides being unaware of some options, DMs may have wrong perceptions about the true conflict, the aim of this chapter is to understand the impact of unawareness in the stability analysis of conflicts modeled by the GMCR. As we said before, other kinds of misperceptions have already being modeled in the GMCR by other works.

5.2.1 *Modeling Interactive Unawareness in GMCR*

Our approach is to adapt a model of interactive unawareness proposed by Heifetz et al. [54] to the GMCR setting. We suppose instead of a single state space common to all DMs involved in the conflict, there are several state spaces and associated to each one of them, there is a set of options available to the DMs in the conflict. Thus, if a DM believes that he is in some state space, then he is only aware of the options available in such space. Therefore, we suppose that if a DM is in a given state in some state space, he may believe that he is in another state space, because he might not be aware of all options available. We present formally the model as follows.

Let $\mathcal{A}$ be the set of all options available to all DMs in the conflict. Let $\mathcal{A}^*$ be some non-empty subset of the power set of $\mathcal{A}$. In order to allow for interactive unawareness, we need several states spaces, $\mathbb{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}^*}$, where with each state space is associated a unique subset of the options available in the conflict. Denote by $\sum = \cup_{\alpha \in \mathcal{A}^*} S_\alpha$ the union of these spaces. If $\alpha^{'} \subseteq \alpha$, then S_α is considered as more refined than $S_{\alpha'}$, i.e., it describes better the conflict. $S_{\alpha'}$ is said to be less expressible than S_α and this is denoted by $S_\alpha \geq S_{\alpha'}$. As in [54], we define a surjection $r_S^{S'} : S^{'} \to S$ that associates each state in a more refined state space with some state in a less refined state space, which is the restriction of the more refined state to the options available in the less refined state space.

In each state space S_α, we define a usual GMCR with a set of directed graphs with common state space S_α, $(S_\alpha, A_i^{S_\alpha})$, and a preference relation on S_α, denoted by $\succ_i^{S_\alpha}$, for each DM $i \in N$, which represent their possible moves and preferences among the states in S_α if they were aware of all the options available in α. As opposed to the usual GMCR, states now describe not only the options taken by DMs but also the options, expressible in the state, that they are aware of. For each $s \in S_\alpha$, let $R_i^{S_\alpha}(s)$ and $R_i^{+,S_\alpha}(s)$ be the set of reachable states from s by DM i and of unilateral improvements from s by DM i, respectively.

The awareness level of DM i, $i \in N$, in a given state s is modeled by an awareness function, $\prod_i : \sum \to \sum$, which specifies what state DM i believes to be in, while at state s. Such awareness function must satisfy some conditions in order to explicit capture unawareness as the only source of misperception of the DMs. The conditions that we require on the awareness function are

(a) Confinedness: If $s \in S_\alpha$, then $\prod_i(s) \in S_{\alpha'}$, for some $S_\alpha \geq S_{\alpha'}$;

(b) If $s^{'} \in R_i^{S_{\alpha''}}(s)$, $\prod_i(s^{'}) \in S_{\alpha'}$ and $\prod_i(s) \in S_\alpha$, then $S_\alpha = S_{\alpha'}$;

(c) If $s^{'} \in R_j^{S_{\alpha''}}(s)$, $\prod_i(s^{'}) \in S_{\alpha'}$ and $\prod_i(s) \in S_\alpha$, then $S_{\alpha'} \geq S_\alpha$;

(d) Stationarity: $\prod_i(s) = \prod_i(\prod_i(s))$;

(e) Coherent Accessibility: If $S_{\alpha'} \geq S_\alpha$, $s \in S_{\alpha'}$, $\prod_i(s) \in S_\alpha$ and $t \in R_i^{S_\alpha}(\prod_i(s))$, then there is a unique state $s' \in S_{\alpha'}$ such that $s' \in R_i^{S_{\alpha'}}(s)$ and $r_{S_\alpha}^{S_{\alpha'}}(s') = t$.

(f) Generalized Reflexivity: If $s \in S_\alpha$ and $\prod_i(s) \in S_{\alpha'}$, then $\prod_i(s) = r_{S'_\alpha}^{S_\alpha}(s)$;

(g) Projections Preserve Ignorance: If $s \in S_{\alpha'}$, and $S_{\alpha'} \geq S_\alpha$, then $\prod_i^\uparrow(s) \subseteq \prod_i^\uparrow(r_{S_\alpha}^{S'_\alpha}(s))$, where for any $s^* \in S_\alpha$, $(s^*)^\uparrow = \{s \in \sum : r_{S_\alpha}^{S_{\alpha'}}(s) = s^*$, for some $S_{\alpha'}\}$.

(h) Projections Preserve Knowledge: If $S_{\alpha''} \geq S_{\alpha'} \geq S_\alpha$, $s \in S_{\alpha''}$, and $\prod_i(s) \in S_{\alpha'}$, then $\prod_i(r_{S_\alpha}^{S_{\alpha''}}(s)) = r_{S_\alpha}^{S_{\alpha'}}(\prod_i(s)) = r_{S_\alpha}^{S''_\alpha}(s)$;

For the results of this chapter, the only necessary conditions are (a), (c), (d) and (e). Such conditions are not so strong. Confinedness (a) requires that for any S_α, at a given state $s \in S_\alpha$, DMs cannot be aware of any non-existing option in α. As shown by Heifeltz et al. [54] and Halpern and Rêgo [66], Stationarity (d) means that knowledge satisfies positive introspection, i.e., if a DM knows some option is (or is not) taken at a given state, then he knows that he knows it. Coherent accessibility (e) implies that DMs with higher awareness level can understand the moves that other DMs with lower awareness levels believe that they can make (as we see, in Section 5.3, Condition (e) is necessary for extending the notions of GMR, SMR, SEQ and $SSEQ$ to this model).

The other conditions, although not necessary for the results, capture our intuition regarding unawareness. Condition (b) states that no DM can believe that he can reach a state where he is aware of options that he is currently unaware of (otherwise, the DM would already be aware of those options). Conditions (f), (g) and (h) are taken from Heiftez et al. [54]. Condition (f) implies that DMs cannot have false beliefs about what they are aware of. As opposed to Heiftez et al. [54] that modeled both uncertain and unawareness, here we only allow for unawareness, since Π_i is a function and not a correspondence. Thus, *Generalized Reflexivity* implies that projections to lower state spaces determine the awareness function of agents.

Heiftez et al. [54] interpreted conditions (g) and (h) in terms of projections, but they do not made explicit what would these conditions imply in terms of DMs awareness and beliefs. We do that in the following paragraphs.

Thus, consider condition (g). Assume that $s \in S_{\alpha'}$ and $\Pi_j(s) \in S_\alpha$, by Confinedness, it follows that $S_{\alpha'} \geq S_\alpha$ and by *Generalized Reflexivity*, we know that $\Pi_j(s) = r^{S_{\alpha'}}_{S_\alpha}(s)$. Therefore, condition (g) implies that $\Pi_i(s)$ be projected in a state space at least as rich as the one in which $\Pi_i(\Pi_j(s))$ is projected. Since $\Pi_i(\Pi_j(s))$ represents what DM j believes that DM i is aware of, it follows that DM j cannot believe that DM i is aware of options that DM i is unaware of.

Regarding condition (h), assume that $S_{\alpha''} \geq S_{\alpha'} \geq S_\alpha$, $s \in S_{\alpha''}$, $\Pi_i(s) \in S_{\alpha'}$ and $\Pi_j(s) \in S_\alpha$, thus DM i is aware of more options than DM j. By *Generalized Reflexivity*, $\Pi_i(s) = r^{S_{\alpha''}}_{S_{\alpha'}}(s)$ and $\Pi_j(s) = r^{S_{\alpha''}}_{S_\alpha}(s)$. Thus, condition (h), can be rewritten as $\Pi_i(\Pi_j(s)) = \Pi_j(s)$, which implies that DM j believes that DM i is aware of the same options that he (DM j) is aware of.

Heiftez et al. [54] also mention another property, called *projections preserve awareness*, but, as they observe, it follows from the assumption that projections preserve knowledge, so we do not consider it in this thesis.

Here it is worth pointing out that, in general, only the analyst may know the set of all DMs' options in the conflict. Indeed, a DM does not have the same model as the analyst, but from his perspective the conflict can be described by another GMCR with interactive unawareness, where the more refined state space describes only the options that such DM is aware of.

It is easy to verify that a standard GMCR $(S, (A_i)_{i\in N}, (\succ_i)_{i\in N})$ can be represented by a GMCR with interactive unawareness, where $\mathcal{A}^* = \{\mathcal{A}\}$, $S_\mathcal{A} = S$, $\succ_i^{S_\mathcal{A}}$ is the same preference relation as $\succ_i$, $\forall i \in N$, and $\prod_i(s) = s$, $\forall s \in S_\mathcal{A}$ and $i \in N$. We call such model the canonical GMCR with interactive unawareness.

5.3 Stability in the GMCR with Int. Unawareness with two DMs

In terms of such awareness function, we generalize five stability notions for the GMCR with interactive unawareness with two DMs, namely: Nash, GMR, SMR, SEQ and $SSEQ$ stability.

Definition 5.3.1. *(GNash) A state* $s \in S_\alpha$ *is generalized Nash stable for DM i iff* $R_i^{+,S_{\alpha'}}(\prod_i(s)) = \emptyset$*, where* $\prod_i(s) \in S_{\alpha'}$.

Definition 5.3.2. *(GGMR) A state* $s \in S_\alpha$ *is generalized GMR stable for DM i iff for every* $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$*, where* $\prod_i(s) \in S_{\alpha'}$*, there exists* $u \in R_j^{S_{\alpha''}}(\prod_j(q))$*, where* $\prod_j(q) \in S_{\alpha''}$*, such that* $\prod_i(s) \succeq_i^{S_{\alpha'}} v$*, where v is the unique state such that* $v \in R_j^{S_{\alpha'}}(q)$ *and* $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$.

Definition 5.3.3. *(GSMR) A state* $s \in S_\alpha$ *is generalized SMR stable for DM i iff for every* $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$*, where* $\prod_i(s) \in S_{\alpha'}$*, there exists* $u \in R_j^{S_{\alpha''}}(\prod_j(q))$*, where* $\prod_j(q) \in S_{\alpha''}$*, such that* $\prod_i(s) \succeq_i^{S_{\alpha'}} v$*, where v is the unique state such that* $v \in R_j^{S_{\alpha'}}(q)$ *and* $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$*, and* $\prod_i(s) \succeq_i^{S_{\alpha'}} w$*, for all* $w \in R_i^{S_{\alpha'}}(v)$.

Definition 5.3.4. *(GSEQ) A state* $s \in S_\alpha$ *is generalized SEQ stable for DM i iff for every* $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$*, where* $\prod_i(s) \in S_{\alpha'}$*, there exists* $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$*, where* $\prod_j(q) \in S_{\alpha''}$*, such that* $\prod_i(s) \succeq_i^{S_{\alpha'}} v$*, where v is the unique state such that* $v \in R_j^{S_{\alpha'}}(q)$ *and* $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$.

Definition 5.3.5. *(GSSEQ) A state* $s \in S_\alpha$ *is generalized SSEQ stable for DM i iff for every* $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$*, where* $\prod_i(s) \in S_{\alpha'}$*, there exists* $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$*, where* $\prod_j(q) \in S_{\alpha''}$*, such that* $\prod_i(s) \succeq_i^{S_{\alpha'}} v$*, where v is the unique state such that* $v \in R_j^{S_{\alpha'}}(q)$ *and* $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$*, and* $\prod_i(s) \succeq_i^{S_{\alpha'}} w$*, for all* $w \in R_i^{S_{\alpha'}}(v)$.

Figure 5.1 illustrates that states the appear in the definitions of the generalized stability concepts proposed here. Intuitively, if a DM i is in a GNash stable state, s, then he has no incentive to move away in a single step from the state that he believes to be in, $\prod_i(s)$, which is determined by his awareness function. Moreover, if a DM i is in a *GGMR* stable state s, he has no incentive to move away from the state that he believes to be in, $\prod_i(s)$, because for every possible unilateral improvement move that he believes to have, q, his opponent believes to have a reachable state from $\prod_j(q)$ leading the conflict to a state u, which corresponds to a unique state v according to DM i's description of the conflict (the existence and uniqueness of such state v is guaranteed by the Coherent Accessibility property of the awareness function) and v is no better than $\prod_i(s)$ for DM i. Here is worth pointing out that property (b) of the awareness function guarantees that $\prod_i(q) = q$ and that properties (c) and (d) ensures that $\prod_i(v) = v$, making it reasonable to compare state $\prod_i(s)$ and v, since they are in the same state space from the point of view of DM i. In a *GSMR* stable state, s, DM i cannot escape from this latter no better situation v to a better state w. Since $\prod_i(v) = v$, property (b) guarantees that $\prod_i(w) = w$, making it reasonable to compare state $\prod_i(s)$ and w, since they are in the same state space from the point of view of DM i. In a *GSEQ* stable state, the reaction of DM i's opponent which leads the conflict to u is also beneficial to the opponent, but no requirement is made as to whether DM i may counter-react. Finally, in a *GSSEQ* stable state, s, it is required both that the reaction of DM i's opponent must be beneficial to the opponent and that DM i has no counter-reaction that leads the conflict from v to a situation better than what he believes to be the initial state, $\prod_i(s)$, for him.

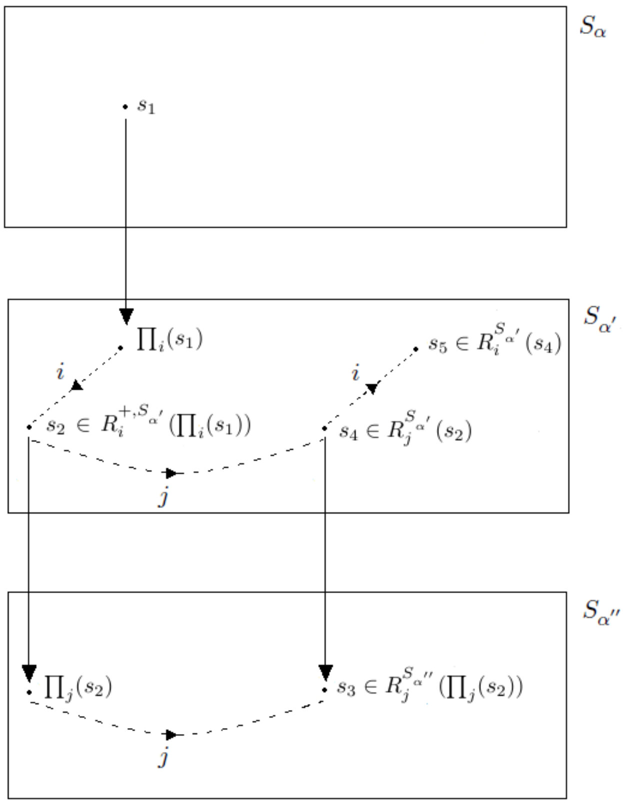


Figure 5.1: Illustration of states in the definitions of generalized stability concepts.

5.3.1 *Results*

In the GMCR, there are well known relationships between the five standard stability concepts mentioned above. Analogous results for the generalized stability definitions in the GMCR with interactive unawareness remain valid. Theorem 5.3.1 summarizes the results.

Theorem 5.3.1. *In the GMCR with interactive unawareness, there exist the following relationships between the stability concepts:*

(a) If state s is GNash stable for DM i, then s is GGMR, GSMR, GSEQ and GSSEQ stable for DM i.

(b) If state s is GSMR stable for DM i, then s is GGMR stable for DM i.

(c) If state s is GSEQ stable for DM i, then s is GGMR stable for DM i.

(d) If state s is GSSEQ stable for DM i, then s is GSEQ stable for DM i.

(e) If state s is GSSEQ stable for DM i, then s is GSMR stable for DM i.

Proof: For (a), if s is GNash stable for DM i, then $R_i^{+,S_{\alpha'}}(\Pi_i(s)) = \emptyset$, where $\Pi_i(s) \in S_{\alpha'}$, which implies that s is *GGMR*, *GSMR*, *GSEQ* and *GSSEQ* stable for DM i.

For (b), if $s \in S_\alpha$ is *GSMR* stable for DM i iff for every $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$. Therefore, it follows that for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, which implies that s is *GGMR* for DM i.

For (c), suppose that s is *GSEQ* stable for DM i. Thus, for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$. Since $R_j^{+,S_{\alpha'}}(\prod_j(q)) \subseteq R_j^{S_{\alpha'}}(\prod_j(q))$, it follows that for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in$

$R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, which implies that s is $GGMR$ stable for DM i.

For (d), suppose that s is $GSSEQ$ stable for DM i. Thus, for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$. Therefore, it follows that for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, which implies that s is $GSEQ$ stable for DM i.

For (e) suppose that s is $GSSEQ$ stable for DM i. Thus, for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$. Since $R_j^{+,S_{\alpha'}}(\prod_j(q)) \subseteq R_j^{S_{\alpha'}}(\prod_j(q))$, it follows that for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$, which implies that s is $GSMR$ stable for DM i. □

Figure 5.3.1 summarizes the relationships between the stability concepts provided by Theorem 5.3.1. The arrows represent the implications of the solution concepts.

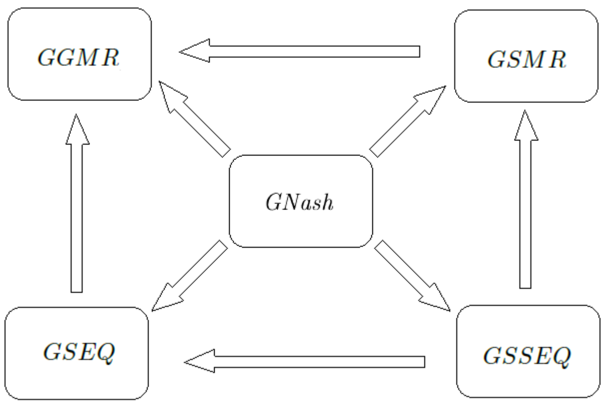


Figure 5.2: Implications among the generalized stability definitions

We also obtained results relating stability of a given state with the stability of the state that a DM believes to be true.

Theorem 5.3.2. *State $s \in S_\alpha$ is stable for DM i according to some stability notion iff $\prod_i(s)$ is stable for DM i according to the same stability notion.*

Proof: The demonstration of this result follows from the Stationarity property of the Awareness function, i.e., $\prod_i(s) = \prod_i(\prod_i(s))$. Next, we prove the theorem for each one of the solution concepts presented in Subsection 5.3.

(1) (GNash) State $s \in S_\alpha$ is GNash stable for DM i iff $R_i^{+,S_{\alpha'}}(\prod_i(s)) = \emptyset$, where $\prod_i(s) \in S_{\alpha'}$. By Stationarity, we have that $\prod_i(s) = \prod_i(\prod_i(s))$, i.e., $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$ $= \emptyset$. Therefore s is GNash stable for DM i iff $\prod_i(s)$ is GNash stable for DM i.

(2) (*GGMR*) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is *GGMR* stable for DM i iff for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being *GGMR* stable for DM i.

(3) (*GSMR*) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is *GSMR* stable for DM i iff for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\Pi_i(\Pi_i(s)) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being *GSMR* stable for DM i.

(4) (*GSEQ*) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is *GSEQ* stable for DM i iff for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$.

Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being *GSEQ* stable for DM i.

(5) (*GSSEQ*) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is *GSSEQ* stable for DM i iff for all $q \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $u \in R_j^{S_{+,\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$, and $\Pi_i(\Pi_i(s)) \succeq_i^{S_{\alpha'}} w$, for all $w \in R_i^{S_{\alpha'}}(v)$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being *GSSEQ* stable for DM i.

□

Theorem 5.3.3 shows that if both DMs have the same awareness level at some state, then generalized stability of such state is equivalent to the corresponding standard stability of the state they believe to be in with respect to the standard GMCR, whose state space is the one that they believe to be in.

Theorem 5.3.3. *If $\prod_i(s) = \prod_j(s) \in S_{\alpha'}$, then $s \in S_\alpha$ is equilibrium according to some generalized stability notion iff $\prod_i(s)$ is an equilibrium according to the corresponding standard stability notion in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$.*

Proof: Next, we prove the result for each one of the solution concepts presented in Subsection 5.3.

(1) (GNash) State $s \in S_\alpha$ is equilibrium according to the GNash concept iff $R_i^{+,S_{\alpha'}}(\prod_i(s)) = R_j^{+,S_{\alpha'}}(\prod_j(s)) = \emptyset$. If $\prod_i(s) = \prod_j(s)$, then s is equilibrium according to this concept iff $\prod_i(s)$ is equilibrium in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$.

(2) (*GGMR*) State $s \in S_\alpha$ is equilibrium according to the *GGMR* concept iff:

(i) For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r_{S_{\alpha''}}^{S_{\alpha'}}(v)$.

(ii) For all $q \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$, there exists $u \in R_i^{S_{\alpha''}}(\prod_i(q))$, where $\prod_i(q) \in S_{\alpha''}$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_i^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$.

Since $\prod_i(s) = \prod_j(s) \in S_{\alpha'}$, we now prove that $S_{\alpha'} = S_{\alpha''}$ in the previous stability definitions. First, note that by Stationarity, $\prod_j(\prod_i(s)) = \prod_j(\prod_j(s)) = \prod_j(s) \in S_{\alpha'}$. Since $S_{\alpha''}$ is the state space containing $\prod_j(q) \in S_{\alpha''}$, where $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, property (c) of the awareness function implies that $S_{\alpha''} \geq S_{\alpha'}$. On the other hand, by Confinedness, it follows that $S_{\alpha'} \geq S_{\alpha''}$. Therefore, $S_{\alpha'} = S_{\alpha''}$, which implies that $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v) = v$.

Thus, we have that $GGMR$ stability for DM i can be rewritten as:

– For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, there exists $u \in R_j^{S_{\alpha'}}(q)$ such that $\prod_i(s) \succeq_i^{S_{\alpha'}} u$.

Therefore, $\prod_i(s)$ is GMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is GMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$ for DM j.

(3) ($GSMR$) State $s \in S_\alpha$ is equilibrium according to the $GSMR$ concept iff:

(i) For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$ for all $w \in R_i^{S_{\alpha'}}(v)$.

(ii) For all $q \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$, there exists $u \in R_i^{S_{\alpha''}}(\prod_i(q))$, where $\prod_i(q) \in S_{\alpha''}$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_i^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$, and $\prod_j(s) \succeq_j^{S_{\alpha'}} w$ for all $w \in R_j^{S_{\alpha'}}(v)$.

Using a similar argument to that of part (2) of this theorem, we have that $GSMR$ stability for DM i can be rewritten as

– For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, there exists $u \in R_j^{S_{\alpha'}}(q)$ such that $\prod_i(s) \succeq_i^{S_{\alpha'}} u$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$ for all $w \in R_i^{S_{\alpha'}}(u)$.

Therefore, $\prod_i(s)$ is SMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is SMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM j.

(4) ($GSEQ$) State $s \in S_\alpha$ is equilibrium according to the $GSEQ$ concept iff:

(i) For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$.

(ii) For all $q \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$, there exists $u \in R_i^{+,S_{\alpha''}}(\prod_i(q))$, where $\prod_i(q) \in S_{\alpha''}$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_i^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$.

Using a similar argument to that of part (2) of this theorem, we have that $GSEQ$ stability for DM i can be rewritten as

– For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, there exists $u \in R_j^{+,S_{\alpha'}}(q)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} u$.

Therefore, $\prod_i(s)$ is SEQ in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.
Similarly, we conclude that $\prod_i(s)$ is SEQ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM j.

(5) ($GSSEQ$) State $s \in S_\alpha$ is equilibrium according to the $GSSEQ$ concept iff:

(i) For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $u \in R_j^{+,S_{\alpha''}}(\prod_j(q))$, where $\prod_j(q) \in S_{\alpha''}$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_j^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$, and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$ for all $w \in R_i^{S_{\alpha'}}(v)$.

(ii) For all $q \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$, there exists $u \in R_i^{+,S_{\alpha''}}(\prod_i(q))$, where $\prod_i(q) \in S_{\alpha''}$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} v$, where v is the unique state such that $v \in R_i^{S_{\alpha'}}(q)$ and $u = r^{S_{\alpha'}}_{S_{\alpha''}}(v)$, and $\prod_j(s) \succeq_j^{S_{\alpha'}} w$ for all $w \in R_j^{S_{\alpha'}}(v)$.

Using a similar argument to that of part (2) of this theorem, we have that $GSSEQ$ stability for DM i can be rewritten as

- For all $q \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, there is $u \in R_j^{+,S_{\alpha'}}(q)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} u$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} w$ for all $w \in R_i^{S_{\alpha'}}(u)$.

Therefore, $\prod_i(s)$ is $SSEQ$ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is $SSEQ$ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM j.

□

Theorem 5.3.4 shows that the standard solution concepts for the standard GMCR are special cases of the generalized solution concepts proposed here for the GMCR with interactive unawareness.

Theorem 5.3.4. *State s satisfies some stability notion in the standard GMCR $\Phi = (S, (A_i)_{i\in N}, (\succeq_i)_{i\in N})$ for DM i iff it satisfies the corresponding generalized stability notion in the canonical representation of Φ as a GMCR with interactive unawareness, denoted by Φ'.*

Proof:

In order to prove this theorem we consider individually each of the five usual stability concepts in the GMCR presented in Section 2.2.2 and the corresponding generalized concepts in the GMCR with interactive unawareness presented in Section 5.3.

(1) We prove that state s is Nash stable for DM $i \in N$ in Φ iff it is GNash for DM $i \in N$ in Φ'. Indeed, state s is Nash stable for DM $i \in N$ in Φ iff $R_i^+(s) = \emptyset$. As we have that $\prod_i(s) = s$ for all $s \in S_{\mathcal{A}}$, then $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s) = \emptyset$, i. e., s is Nash stable in Φ iff it is GNash stable in Φ'.

(2) We prove that state s is GMR stable for DM $i \in N$ in Φ iff it is $GGMR$ for DM $i \in N$ Φ'. Indeed, state s is GMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists

$s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$. But as, in Φ', we have that $\prod_i(s) = s$ and $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $R_j^{S_{\mathcal{A}}}(\prod_j(s)) = R_j(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is GMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in R_j^{S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i^{S_{\mathcal{A}}} s_2$, where s_2 is the unique state such that $s_2 \in R_j^{S_{\mathcal{A}}}(s_1)$ and $s_2 = r_{S_{\mathcal{A}}}^{S_{\mathcal{A}}}(s_2)$, which is equivalent to s being $GGMR$ stable for DM i in Φ'.

(3) We prove that state s is SMR stable for DM $i \in N$ in Φ iff it is $GSMR$ for DM $i \in N$ Φ'. Indeed, state s is SMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j(s_1)$ such that $s \succeq_i s_2$ and $s_2 \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. But as, in Φ', we have that $\prod_i(s) = s$, $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{S_{\mathcal{A}}}(\prod_i(s)) = R_i(s)$, $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $R_j^{S_{\mathcal{A}}}(\prod_j(s)) = R_j(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is SMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in R_j^{S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i s_2$, where s_2 is the unique state such that $s_2 \in R_j^{S_{\mathcal{A}}}(s_1)$ and $s_2 = r_{S_{\mathcal{A}}}^{S_{\mathcal{A}}}(s_2)$ and $\prod_i(s) \succeq_i s_3$ for all $s_3 \in R_i^{S_{\mathcal{A}}(s_2)}$, which is equivalent to s being $GSMR$ stable for DM i in Φ'.

(4) We prove that state s is SEQ stable for DM $i \in N$ in Φ iff it is $GSEQ$ for DM $i \in N$ Φ'. Indeed, state s is SEQ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$. But as, in Φ', we have that $\prod_i(s) = s$ and $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $R_j^{+,S_{\mathcal{A}}}(\prod_j(s)) = R_j^+(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is SEQ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in R_j^{+,S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i^{S_{\mathcal{A}}} s_2$, where s_2 is the unique state such that $s_2 \in R_j^{+,S_{\mathcal{A}}}(s_1)$ and $s_2 = r_{S_{\mathcal{A}}}^{S_{\mathcal{A}}}(s_2)$, which is equivalent to s being $GSEQ$ stable for DM i in Φ'.

(5) We prove that state s is $SSEQ$ stable for DM $i \in N$ in Φ iff it is $GSSEQ$ for DM $i \in N$ Φ'. Indeed, state s is $SSEQ$ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succeq_i s_2$ and $s_2 \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. But as, in Φ', we have that $\prod_i(s) = s$, $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{S_{\mathcal{A}}}(\prod_i(s)) = R_i(s)$, $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) =$

$R_i^+(s)$ and $R_j^{+,S_\mathcal{A}}(\prod_j(s)) = R_j^+(s)$ for all $s \in S_\mathcal{A}$. Thus, state s is $SSEQ$ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_\mathcal{A}}(\prod_i(s))$ there exists $s_2 \in R_j^{+,S_\mathcal{A}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i s_2$, where s_2 is the unique state such that $s_2 \in R_j^{+,S_\mathcal{A}}(s_1)$ and $s_2 = r_{S_\mathcal{A}}^{S_\mathcal{A}}(s_2)$ and $\prod_i(s) \succeq_i s_3$ for all $s_3 \in R_i^{S_\mathcal{A}(s_2)}$, which is equivalent to s being $GSSEQ$ stable for DM i in Φ'.

□

5.4 Stability in the GMCR with Int. Unawareness with n-DMs

In order to generalize the stability definitions and results presented in Section 5.3 for conflicts with n-DMs, it is necessary to define some sets, such as the set of accessible states and improvements for a particular DM and for a group of DMs. Next, we formally define these sets and present the solution concepts and some results obtained for the GMCR with interactive unawareness with multiple DMs.

Let $U_j^{S_{\alpha'}}(s_1) = \{s_2 \in R_j^{S_{\alpha'}}(s_1) : \exists s_3 \in R_j^{S_{\alpha''}}(\prod_j(s_1))$ such that $\prod_j(s_1) \in S_{\alpha''}$, and $s_3 = r_{S_{\alpha''}}^{S_{\alpha'}}(s_2)\}$ be the subset of $R_j^{S_{\alpha'}}(s_1)$ consisting of all states in $S_{\alpha'}$ reachable for DM j from state s_1 in one step considering that at s_1, DM j may not be aware of all options in α'. Let also $U_j^{+,S_{\alpha'}}(s_1) = \{s_2 \in R_j^{S_{\alpha'}}(s_1) : \exists s_3 \in R_j^{+,S_{\alpha''}}(\prod_j(s_1))$ such that $\prod_j(s_1) \in S_{\alpha''}$, and $s_3 = r_{S_{\alpha''}}^{S_{\alpha'}}(s_2)\}$ be the subset of $R_j^{S_{\alpha'}}(s_1)$, consisting of all states in $S_{\alpha'}$ that are unilateral improvement moves from s_1 by DM j, considering that at s_1, DM j may not be aware of all options in α'.

We are now able to define a legal sequence of movements in a GMCR with interactive unawareness. Let H be some coalition and let $U_H^{S_{\alpha'}}(s)$ denote the set of all states in space $S_{\alpha'}$ that can be reached by any *legal sequence* of movements, considering that the DMs in H may not be aware of all options in α' while moving in the sequence. Let also $\Omega_H^{S_{\alpha'}}(s, s_1)$ be the subset of H whose members are DMs that make the last move to reach s_1 in a legal sequence of moves from s, considering that DMs may be unaware of some options in α' while moving. Formally, $U_H^{S_{\alpha'}}(s)$ and $\Omega_H^{S_{\alpha'}}(s, \cdot)$ are the smallest sets (in the sense of inclusion) satisfying: (1) if $j \in H$ and $s_1 \in U_j^{S_{\alpha'}}(s)$, then $s_1 \in U_H^{S_{\alpha'}}(s)$ and $j \in \Omega_H^{S_{\alpha'}}(s, s_1)$, and (2) if $s_1 \in U_H^{S_{\alpha'}}(s)$, $j \in H$,

$\Omega_H^{S_{\alpha'}}(s, s_1) \neq \{j\}$ and $s_2 \in U_j^{S_{\alpha'}}(s_1)$, then $s_2 \in U_H^{S_{\alpha'}}(s)$ and $j \in \Omega_H^{S_{\alpha'}}(s, s_2)$.

Similarly, let $U_H^{+,S_{\alpha'}}(s) \subseteq S$ be the set of all states that result from a *legal sequence of unilateral improvements*, starting at state s, taking into account that DMs in H may not be aware of all options in α'. Finally, if $s_1 \in U_H^{+,S_{\alpha'}}(s)$, then $\Omega_H^{+,S_{\alpha'}}(s, s_1)$ is the set of all last DMs in a legal sequence of unilateral improvements from s to s_1, considering that DMs in H may be unaware of all options in α'. We have that $U_H^{+,S_{\alpha'}}(s)$ and $\Omega_H^{+,S_{\alpha'}}(s, \cdot)$ are defined as the smallest sets (in the sense of inclusion) satisfying: (1) if $j \in H$ and $s_1 \in U_j^{+,S_{\alpha'}}(s)$, then $s_1 \in U_H^{+,S}(s)$ and $j \in \Omega_H^{+,S_{\alpha'}}(s, s_1)$, and (2) if $s_1 \in U_H^{+,S_{\alpha'}}(s)$, $j \in H$, $\Omega_H^{+,S_{\alpha'}}(s, s_1) \neq \{j\}$ and $s_2 \in U_j^{+,S_{\alpha'}}(s_1)$, then $s_2 \in U_H^{+,S_{\alpha'}}(s)$ and $j \in \Omega_H^{+,S_{\alpha'}}(s, s_2)$.

We are now able to provide stability definitions for the GMCR with interactive unawareness and multiple DMs.

Definition 5.4.1. *(GNash) A state $s \in S_\alpha$ is generalized Nash stable for DM i iff $R_i^{+,S_{\alpha'}}(\prod_i(s)) = \emptyset$, where $\prod_i(s) \in S_{\alpha'}$.*

Definition 5.4.2. *(GGMR) A state $s \in S_\alpha$ is generalized GMR stable for DM i iff for every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.*

Definition 5.4.3. *(GSMR) A state $s \in S_\alpha$ is generalized GMR stable for DM i iff for every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.*

Definition 5.4.4. *(GSEQ) A state $s \in S_\alpha$ is sequential SEQ stable for DM i iff for every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.*

Definition 5.4.5. *(GSSEQ) A state $s \in S_\alpha$ is symmetric sequential SSEQ stable for DM i iff for every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.*

In other words, if a DM i is in a GNash stable state, s, then he has no incentive to move away in a single step from the state that he believes to be in, $\prod_i(s)$, which is determined by his awareness function. Moreover, if a DM i is in a *GGMR* stable state s, he has no incentive to move away from the state that he believes to be in, $\prod_i(s)$, because for every possible unilateral improvement move that he believes to have, s_1, his opponents may react reaching a state s_2 that is no better than $\prod_i(s)$ for DM i. In a $GSMR$ stable state, s, DM i cannot escape from this latter no better situation s_2. In a $GSEQ$ stable state, the reactions of DM i's opponents which lead the conflict to s_2 are also beneficial to the opponents, but no requirement is made as to whether DM i may counter-react. Finally, in a $GSSEQ$ stable state, s, it is required both that the reactions of DM i's opponents must be beneficial to them and that DM i has no counter-reaction that leads the conflict from s_2 to a situation better for him than what he believes to be the initial state, $\prod_i(s)$.

5.4.1 *Results*

Analogous results to the obtained in Section 5.2 for the generalized stability definitions in the GMCR with interactive unawareness with multiple DMs remain valid. Theorem 5.4.1 summarizes the results.

Theorem 5.4.1. *In the GMCR with interactive unawareness with multiple DMs, there exist the following relationships between the stability concepts:*

(a) If state s is GNash stable for DM i, then s is $GGMR$, $GSMR$, $GSEQ$ and $GSSEQ$ stable for DM i.

(b) If state s is $GSMR$ stable for DM i, then s is $GGMR$ stable for DM i.

(c) If state s is $GSEQ$ stable for DM i, then s is $GGMR$ stable for DM i.

(d) If state s is $GSSEQ$ stable for DM i, then s is $GSEQ$ stable for DM i.

(e) If state s is $GSSEQ$ stable for DM i, then s is $GSMR$ stable for DM i.

Proof: The proof of this theorem are similar to the respective theorem presented in previous section.

We also obtained results relating stability of a given state with the stability of the state that a DM believes to be true. This results generalized the respective result obtained in Rêgo and Vieira for conflicts with n-DMs.

Theorem 5.4.2. *State $s \in S_\alpha$ is stable for DM i according to some stability notion iff $\prod_i(s)$ is stable for DM i according to the same stability notion.*

Proof: The demonstration of this result follows from the Stationarity property of the Awareness function, i.e., $\prod_i(s) = \prod_i(\prod_i(s))$. Next, we prove the theorem for each one of the solution concepts presented in Subsection 2.2.2.

(1) State $s \in S_\alpha$ is GNash stable for DM i iff $R_i^{+,S_{\alpha'}}(\prod_i(s)) = \emptyset$, where $\prod_i(s) \in S_{\alpha'}$. By Stationarity, we have that $\prod_i(s) = \prod_i(\prod_i(s))$, i.e., $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s)) = \emptyset$. Therefore s is GNash stable for DM i iff $\prod_i(s)$ is GNash stable for DM i.

(2) ($GGMR$) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is $GGMR$ stable for DM i iff for all $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} s_2$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being $GGMR$ stable for DM i.

(3) ($GSMR$) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is $GSMR$ stable for DM i iff for all $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} s_2$, and $\Pi_i(\Pi_i(s)) \succeq_i^{S_{\alpha'}} s_3$, for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being $GSMR$ stable for DM i.

(4) ($GSEQ$) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is $GSEQ$ stable for DM i iff for all $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$, where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} s_2$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being $GSEQ$ stable for DM i.

(5) ($GSSEQ$) By Stationarity, $\prod_i(s) = \prod_i(\prod_i(s))$, which implies that $R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s))) = R_i^{+,S_{\alpha'}}(\prod_i(s))$. Thus, $s \in S_\alpha$ is $GSSEQ$ stable for DM i iff for all $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(\prod_i(s)))$,

where $\prod_i(\prod_i(s)) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(\prod_i(s)) \succeq_i^{S_{\alpha'}} s_2$, and $\Pi_i(\Pi_i(s)) \succeq_i^{S_{\alpha'}} s_3$, for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$. Finally, this last statement is equivalent to the definition of $\Pi_i(s)$ being $GSSEQ$ stable for DM i.

□

Theorem 5.4.3 shows that if all DMs in an coalition H have the same awareness level at some state, then generalized stability of such state is equivalent to the corresponding standard stability of the state they believe to be in with respect to the standard GMCR, whose state space is the one that they believe to be in.

Theorem 5.4.3. *If* $\prod_i(s) = \prod_j(s) \in S_{\alpha'}$ *for all* $j \in N - \{i\}$, *then* $s \in S_\alpha$ *is equilibrium according to some generalized stability notion iff* $\prod_i(s)$ *is an equilibrium according to the corresponding standard stability notion in the standard GMCR* $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$.

Proof: Next, we prove the result for each one of the solution concepts presented in Subsection 5.3.

(1) (GNash) We have that $s \in S_\alpha$ is equilibrium according to the GNash concept iff $R_i^{+,S_{\alpha'}}(\prod_i(s)) = R_j^{+,S_{\alpha'}}(\prod_j(s)) = \emptyset$, for all $j \in N - \{i\}$. If $\prod_i(s) = \prod_j(s)$ for all $j \in N - \{i\}$, then s is equilibrium according to this concept iff $\prod_i(s)$ is equilibrium in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$.

(2) ($GGMR$) State $s \in S_\alpha$ is equilibrium according to the $GGMR$ concept iff:

(i) For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.

(ii) For every $s_1 \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$ and $j \in N - \{i\}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} s_2$.

Since $\prod_i(s) = \prod_j(s) \in S_{\alpha'}$, we now prove that $U_H^{S_{\alpha'}}(s_1) = R_H^{S_{\alpha'}}(s_1)$ for all $i \in N$. First, note that by Stationarity, $\prod_j(\prod_i(s)) = \prod_j(\prod_j(s)) = \prod_j(s) \in S_{\alpha'}$ $\forall j \in N - \{i\}$. Since $S_{\alpha''}$ is the state space containing $\prod_j(s_1)$, for a arbitrary DM $j \in N - \{i\}$ where

$s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, property (c) of the awareness function implies that $S_{\alpha''} \geq S_{\alpha'}$. On the other hand, by Confinedness, it follows that $S_{\alpha'} \geq S_{\alpha''}$. As $S_{\alpha'} = S_{\alpha''}$ then $\prod_j(s_1) = r_{S_{\alpha'}}^{S_{\alpha'}}(s_1) = s_1$. Thus, $\prod_j(s_1) = s_1$ for all $j \in N - \{i\}$. Let now $s_2 \in R_j^{S_{\alpha'}}(q)$, if $k \in N - \{j\}$ is an DM moving from s_2, then similarly we have that $\prod_k(s_2) = s_2 \in S_{\alpha'}$ for all $k \in N - \{j\}$, and so on.

Thus, with similar reasoning to employee above, we can show that in every state, the awareness levels of DMs moving in such state is the same and the states that they believe to be the true state of conflict is always in the state space $S_{\alpha'}$, which ensures that $U_H^{S_{\alpha'}}(s_1) = R_H^{S_{\alpha'}}(s_1)$ for all $i \in N$.

Thus, we have that $GGMR$ stability for DM i can be rewritten as:

- For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in R_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.

Therefore, $\prod_i(s)$ is GMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is GMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i \in N}, (\succeq_i^{S_{\alpha'}})_{i \in N})$ for every DM $j \in N - \{i\}$.

(3) ($GSMR$) State $s \in S_\alpha$ is equilibrium according to the $GSMR$ concept iff:

(i) For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.

(ii) For every $s_1 \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$ and $j \in N - \{i\}$, there exists $s_2 \in U_H^{S_{\alpha'}}(s_1)$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} s_2$ and $\prod_j(s) \succeq_j^{S_{\alpha'}} s_3$ for all $s_3 \in R_j^{S_{\alpha'}}(s_2)$.

Using a similar argument to that of part (2) of this theorem, we have that $GSMR$ stability for DM i can be rewritten as

– For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in R_H^{S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.

Therefore, $\prod_i(s)$ is SMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is SMR stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for every DM $j \in N - \{i\}$.

(4) ($GSEQ$) State $s \in S_\alpha$ is equilibrium according to the $GSEQ$ concept iff:

(i) For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.

(ii) For every $s_1 \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$ and $j \in N - \{i\}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} s_2$.

Using a similar argument to that of part (2) of this theorem, we have that $GSEQ$ stability for DM i can be rewritten as

– For all $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, there exists $s_2 \in R_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$.

Therefore, $\prod_i(s)$ is SEQ in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is SEQ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM j.

(5) ($GSSEQ$) State $s \in S_\alpha$ is equilibrium according to the $GSSEQ$ concept iff:

(i) For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.

(ii) For every $s_1 \in R_j^{+,S_{\alpha'}}(\prod_j(s))$, where $\prod_j(s) \in S_{\alpha'}$ and $j \in N - \{i\}$, there exists $s_2 \in U_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_j(s) \succeq_j^{S_{\alpha'}} s_2$ and $\prod_j(s) \succeq_j^{S_{\alpha'}} s_3$ for all $s_3 \in R_j^{S_{\alpha'}}(s_2)$.

Using a similar argument to that of part (2) of this theorem, we have that $GSSEQ$ stability for DM i can be rewritten as

– For every $s_1 \in R_i^{+,S_{\alpha'}}(\prod_i(s))$, where $\prod_i(s) \in S_{\alpha'}$, there exists $s_2 \in R_H^{+,S_{\alpha'}}(s_1)$, such that $\prod_i(s) \succeq_i^{S_{\alpha'}} s_2$ and $\prod_i(s) \succeq_i^{S_{\alpha'}} s_3$ for all $s_3 \in R_i^{S_{\alpha'}}(s_2)$.

Therefore, $\prod_i(s)$ is $SSEQ$ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for DM i.

Similarly, we conclude that $\prod_i(s)$ is $SSEQ$ stable in the standard GMCR $(S_{\alpha'}, (A_i^{S_{\alpha'}})_{i\in N}, (\succeq_i^{S_{\alpha'}})_{i\in N})$ for every DM $j \in N - \{i\}$.

□

Theorem 5.4.4 shows that the standard solution concepts for the standard GMCR are special cases of the generalized solution concepts proposed here for the GMCR with interactive unawareness with n-DMs.

Theorem 5.4.4. *State s satisfies some stability notion in the GMCR $\Phi = (S, (A_i)_{i\in N}, (\succeq_i)_{i\in N})$ for DM i iff it satisfies the corresponding generalized stability notion in the canonical representation of Φ as a GMCR with interactive unawareness, denoted by Φ'.*

Proof: In order to prove this theorem we consider individually each of the five usual stability concepts in the GMCR presented in Section 2.2.2 and the corresponding generalized concepts in the GMCR with interactive unawareness with n-DMs presented in Section 5.4.

(1) We prove that state s is Nash stable for DM $i \in N$ in Φ iff it is GNash for DM $i \in N$ in Φ'. Indeed, state s is Nash stable for DM $i \in N$ in Φ iff $R_i^+(s) = \emptyset$. As we have that $\prod_i(s) = s$ for all $s \in S_{\mathcal{A}}$, then $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s) = \emptyset$, i. e., s is Nash stable in Φ iff it is GNash stable in Φ'.

(2) We prove that state s is GMR stable for DM $i \in N$ in Φ iff it is $GGMR$ for DM $i \in N$ Φ'. Indeed, state s is GMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$. But as, in Φ', we have that $\prod_i(s) = s$ for all $i \in N$ and

$s \in S_{\mathcal{A}}$, implying that $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $U_H^{S_{\mathcal{A}}}(\prod_j(s)) = R_H(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is GMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in U_H^{S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i^{S_{\mathcal{A}}} s_2$, which is equivalent to s being $GGMR$ stable for DM i in Φ'.

(3) We prove that state s is SMR stable for DM $i \in N$ in Φ iff it is $GSMR$ for DM $i \in N$ Φ'. Indeed, state s is SMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s_2 \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. But as, in Φ', we have that $\prod_i(s) = s$, $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{S_{\mathcal{A}}}(\prod_i(s)) = R_i(s)$, $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $U_H^{S_{\mathcal{A}}}(\prod_j(s)) = R_H(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is SMR stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in U_H^{S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i s_2$, and $\prod_i(s) \succeq_i s_3$ for all $s_3 \in R_i^{S_{\mathcal{A}}(s_2)}$, which is equivalent to s being $GSMR$ stable for DM i in Φ'.

(4) We prove that state s is SEQ stable for DM $i \in N$ in Φ iff it is $GSEQ$ for DM $i \in N$ Φ'. Indeed, state s is SEQ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$. But as, in Φ', we have that $\prod_i(s) = s$ and $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $U_H^{+,S_{\mathcal{A}}}(\prod_j(s)) = R_H^+(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is SEQ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in U_H^{+,S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i^{S_{\mathcal{A}}} s_2$, which is equivalent to s being $GSEQ$ stable for DM i in Φ'.

(5) We prove that state s is $SSEQ$ stable for DM $i \in N$ in Φ iff it is $GSSEQ$ for DM $i \in N$ Φ'. Indeed, state s is $SSEQ$ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$ such that $s \succeq_i s_2$ and $s_2 \succeq_i s_3$ for all $s_3 \in R_i(s_2)$. But as, in Φ', we have that $\prod_i(s) = s$, $\prod_j(s) = s$ for all $s \in S_{\mathcal{A}}$, implying that $R_i^{S_{\mathcal{A}}}(\prod_i(s)) = R_i(s)$, $R_i^{+,S_{\mathcal{A}}}(\prod_i(s)) = R_i^+(s)$ and $U_H^{+,S_{\mathcal{A}}}(\prod_j(s)) = R_H^+(s)$ for all $s \in S_{\mathcal{A}}$. Thus, state s is $SSEQ$ stable for DM $i \in N$ in Φ iff for every $s_1 \in R_i^{+,S_{\mathcal{A}}}(\prod_i(s))$ there exists $s_2 \in U_H^{+,S_{\mathcal{A}}}(\prod_j(s_1))$, such that $\prod_i(s) \succeq_i s_2$ and $\prod_i(s) \succeq_i s_3$ for all $s_3 \in R_i^{S_{\mathcal{A}}(s_2)}$, which is equivalent to s being $GSSEQ$ stable for DM i in Φ'.

□

5.5 Application

In what follows, we provide an application that illustrates the usefulness of the model proposed in this work.

Hypothetical Conflict

Consider a hypothetical conflict with two decision makers, Country 1 (C_1) and Country 2 (C_2). Suppose that C_2 intends to invade C_1. Admit that C_1 to defend its territory can either use a conventional weapon (DC) or a secret weapon (DS). On the other hand, C_2 has only a conventional weapon (AC) to attack. Suppose that C_1 is aware of all options available in the conflict, while C_2 is unaware of the secret weapon. Moreover, suppose that C_2 can learn about the secret weapon either if he attacks C_1 and C_1 uses it or if C_1 decides to reveal that he has such secret weapon. Finally, suppose that C_2 has a successful attack iff C_1 does not use DS.

Thus, the set of options available in this conflict is $\mathcal{A} = \{DC, DS, AC\}$. We need two state spaces to represent such conflict. The richer state space, where all options are available, and the less expressible state space, where C_2 is unaware of DS. The richer state space is described in Table 5.1. Note that in the richer state space, the sates describe which options are taken by the DMs and also what options they are aware of at those states. At state s_3, C_2 becomes aware of DS because it attacks C_1 and C_1 uses the secret weapon. On the other hand, the states s_7 to $s_1 1$ represent the situations where C_1 informs C_2 that it has the option to defend itself with the secret weapon even if it does not plan to use it, as it is the case in all these states except for state s_9.

From the point view of C_2, if he is unaware of DS, then the state space that such country considers possible is described in Table 5.2. Note that at all states in the less expressible state space, DMs are unaware of DS.

Table 5.3 provides the reachable states and the preference ranking in the richer space of the two countries involved in the conflict, where higher numbers indicate more preferable states.

Table 5.1: The Richer State Space

C_1											
1.DC	Y	Y	N	N	N	N	Y	Y	N	N	N
2.DS	N	N	Y	Y	N	N	N	N	Y	N	N
$\mathcal{A}_1$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$
C_2											
1.AC	Y	N	Y	N	Y	N	Y	N	N	Y	N
$\mathcal{A}_2$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$
State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}

Table 5.2: Less Expressible State Space

C_1				
1. DC	Y	Y	N	N
$\mathcal{A}_1$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}-DS$
C_2				
1. AC	Y	N	Y	N
$\mathcal{A}_2$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}-DS$	$\mathcal{A}-DS$
State	$s_1^{'}$	$s_2^{'}$	$s_3^{'}$	$s_4^{'}$

Table 5.3: Reachable states and preference ranking - richer space

State Number	Country 1		Country 2	
	R_1	p_1	R_2	p_2
s_1	s_3, s_5, s_7, s_{10}	4	s_2	8
s_2	s_4, s_6, s_8, s_9, s_{11}	9	s_1	3
s_3	s_7, s_{10}	5	s_9	1
s_4	s_2, s_6, s_8, s_9, s_{11}	8	s_3	4
s_5	s_1, s_3, s_7, s_{10}	2	s_6	10
s_6	s_2, s_4, s_8, s_9, s_{11}	11	s_5	2
s_7	s_3, s_{10}	3	s_8	9
s_8	s_9, s_{11}	7	s_7	6
s_9	s_8, s_{11}	6	s_3	7
s_{10}	s_3, s_7	1	s_{11}	11
s_{11}	s_8, s_9	10	s_{10}	5

Table 5.4 provides the reachable states and preference ranking in the less expressible space of the two countries involved in the conflict, where higher numbers indicate more preferable states. Figure 5.3 illustrates the awareness function of Country C_2, where self-loops are omitted.

Table 5.4: Reachable states and preference ranking - less expressible space

State Number	Country 1		Country 2	
	R_1	p_1	R_2	p_2
s_1'	s_3'	2	s_2'	3
s_2'	s_4'	3	s_1'	2
s_3'	s_1'	1	s_4'	4
s_4'	s_2'	4	s_3'	1

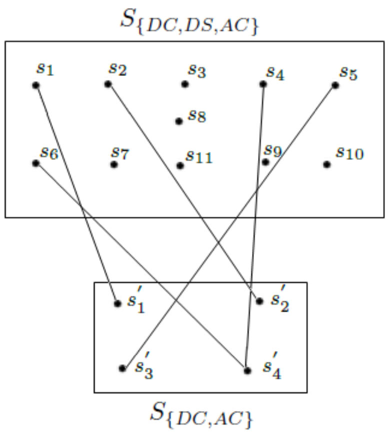


Figure 5.3: States spaces and the Awareness function of C_2 (Π_2)(self loops are omitted).

Table 5.5: Stability Analysis - richer space

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}
GNash	C_2		C_1		C_2	C_1	C_2		C_2	C_2	C_1
$GGMR$	C_2	C_1	C_1	C_1	C_2	C_1	C_2	C_1, C_2	C_1, C_2	C_2	C_1, C_2
$GSMR$	C_2	C_1	C_1	C_1	C_2	C_1	C_2	C_1	C_1, C_2	C_2	C_1
$GSEQ$	C_2	C_1	C_1	C_1	C_2	C_1	C_2	C_1, C_2	C_1, C_2	C_2	C_1, C_2
$GSSEQ$	C_2	C_1	C_1	C_1	C_2	C_1	C_2	C_1	C_1, C_2	C_2	C_1

Since Country C_1 is always aware of all options available, it follows that $\prod_1(s) = s$, $\forall s \in \Sigma$.

Table 5.5 summarizes for which country a particular state in the richer state space satisfies each of the generalized stability definitions proposed for the GMCR with interactive unawareness, while Table 5.6 does the same for the less expressible state space.

Note that s_9 is the state in the richer state space that is equilibrium according to a greater number of generalized stability notions and represents the situation where C_1 tells C_2 about DS, plan to use DS, if attacked, and C_2 does not attack.

On the other hand, in the less expressible state space, which represents the conflict from

Table 5.6: Stability Analysis - less expressible space

	s'_1	s'_2	s'_3	s'_4
GNash	C_1, C_2		C_2	C_1
$GGMR$	C_1, C_2	C_1	C_2	C_1
$GSMR$	C_1, C_2	C_1	C_2	C_1
$GSEQ$	C_1, C_2	C_1	C_2	C_1
$GSSEQ$	C_1, C_2	C_1	C_2	C_1

the viewpoint of an unaware C_2, he falsely believes that s'_1 is an equilibrium according to all generalized notions of stability. Such state represents a situation where C_2 attacks and C_1 defends himself using DC.

5.6 Conclusion

In this chapter, we propose a modification in the GMCR, for conflicts with two and n-DMs, in order to allow the representation of conflicts where DMs may be unaware of some options available for them or for their opponents in the conflict. We define the model adapting ideas from Heifetz et al. [54] to the GMCR setting. This model is more flexible, in the extent that it does not require that all DMs have the same awareness level about the options available in the conflict. We propose five notions of stability in the GMCR with interactive unawareness, providing results that relate such notions and also showed that standard solution concepts for the GMCR are special cases of the notions proposed here where no lack of awareness is presented.

CHAPTER 6

Generalized Metarationalities for n-DM Conflicts Revisited

Abstract

In this chapter, we present an alternative definition for the generalized metarational stability concept for conflicts with n-decision makers (DMs) in the context of the graph model for conflict resolution. Our motivation to present this proposal lies on the fact that unlike the original definition of generalized metarationality for n-DMs, our definition coincides with the definition of generalized metarationality in the particular case where the conflict has only two DMs. Moreover, this work points out problems in some results that relate the concept of generalized metarationality for n-DMs with other solution concepts in the GMCR and analyzes which properties are satisfied by the proposed alternative definition.

6.1 Introduction

In [19] and [14], a new solution concept, called generalized metarationality, is proposed in the GMCR for two and n-DMs, respectively. The importance of this concept lies on the fact that, for conflicts with 2 DMs, it generalizes some common concepts, such as Nash stability, GMR, SMR and SEQ.

This concept takes into account variable horizons for the focal DM and is a flexible tool in the stability analysis insofar as it takes into account various movements of reaction and counter-reaction of DMs involved in the conflict. Thus, the understanding of the possible extensions of such solution concept to conflicts with n-DMs is important.

In this chapter, we show that the concept of generalized metarationality for n-DMs proposed in [14] is not a generalization of the concept proposed in [19], for the particular case where $n = 2$, which led us to seek an alternative definition for generalized metarationality that coincides with the definition proposed earlier by [19] in the case $n = 2$. Moreover, we show that some of the results obtained in [14] for conflicts with n-DMs relating generalized metarationality and other solution concepts are not valid. In particular, there is a problem with the result that SMR is equivalent to a particular case of generalized metarationality for n-DMs, proposed in [14]. Unlike to the original definition, our alternative definition captures the concept of SMR as a special case.

This chapter is organized as follows. In Section 6.2, we recall the definitions of the generalized metarationality concept and three problems involving this concept for conflicts with n-DMs, proposed in [14], are pointed out. In Section 6.3, an alternative definition of generalized metarationality concept for n-DM conflicts is presented. In Section 6.4, a study of the properties of this new solution concept and of the relationship with other existing solution concepts is made, and finally, in Section 6.5, we finish the article with the main conclusions found.

6.2 Generalized Metarationality

Similarly to the concept of strategy in an extensive form game, Zeng *et al.* [19] define the notion of policy in GMCR as a function that determines what each DM does in every possible state of the conflict. Formally, a *policy* of DM $i \in N$, denoted by P_i, is a function $P_i : S \to S$ such that $P_i(s) \in R_i(s) \cup \{s\}$. Thus, for example, if DM i is the first to move from the initial state s_0, then according to his or her policy, P_i, the conflict is taken to state $s_1 = P_i(s_0)$. After such initial move made by DM i, if DM $j \in N - \{i\}$ is the second one to move in the conflict, then the conflict is taken to state $s_2 = P_j(P_i(s_0))$, and so on. A policy of DM $i \in N$ is said to be *credible*, denoted by P_i^c, if it only allows DM i to move to states more preferable than the current state, i.e., $P_i^c(s) \in R_i^+(s) \cup \{s\}$.

Given an initial state s_0, a set of policies, P_i, $i \in N$, and a sequence of DMs $I = (i_1, i_2, i_3, \ldots)$ such that $i_k \neq i_{k+1}$, $k = 1, 2, \ldots$, we have that the sequence of states that the conflict must go through if it starts in state s_0, the DMs move according to the order in I using policies P_i's, is the sequence $(s_0, P_{i_1}(s_0), P_{i_2}(P_{i_1}(s_0)), \ldots)$. Whenever some DM stays in some state, the conflict terminates at that state. In order to comprehend the notion of generalized metarationality, an analysis of the possible alternate sequences of states and DMs that can arise in a conflict is necessary. Such alternating sequence of states and DMs is called a *sequence of moves* and is formally defined next.

Definition 6.2.1. *Given a set of policies, P_i, $i \in N$, $(P_i)_{i \in N}$-based sequence of moves is an alternate list of states and DMs such that:*

(1) Every sequence of moves starts in some state.

(2) Every finite sequence of moves ends in some state.

(3) If the triplet (s, i, s_1) appears in some part of the sequence of moves, then $s_1 = P_i(s)$.

(4) There is no triplet of the form (i, s_1, i) in any part of any sequence of moves.

(5) A triplet of the form (s, i, s) can only appear at the end of a sequence of moves, which, in this case, is called a terminated sequence.

(6) Every infinite sequence of moves is also called terminated.

In other words, Definition 6.2.1 establishes that, in a sequence of moves, the DMs always move from one state to another state in accordance with pre-established policies. Moreover, it is not permitted to any DM to move twice consecutively in any sequence of moves and if some DM stays in a given state the sequence ends at that state.

A particular DM i, while examining the possibility of moving away from the current state s, can consider all possible sequences of states that can arise given a particular set of policies, P_j, $j \in N - \{i\}$, for his or her opponents, and that DM i always has the option of choosing any reachable state while moving, without the restriction to use any policy, P_i. Thus, we need the following definition:

Definition 6.2.2. *Given a set of policies, P_j, $j \in N - \{i\}$, $(P_j)_{j \in N-\{i\}}$-based sequence of moves for DM i is an alternate list of states and DMs such that:*

(1) Every sequence of moves for DM i starts in some state and has i as its second element.

(2) Every finite sequence of moves for DM i ends in some state.

(3) If the triplet (s, i, s_1) appears in some part of the sequence of moves for DM i, then $s_1 \in R_i(s)$.

(4) If the triplet (s, j, s_1), $j \neq i$, appears in some part of the sequence of moves for DM i, then $s_1 = P_j(s)$.

(5) There is no triplet of the form (j, s_1, j), $j \in N$, in any part of any sequence of moves for DM i.

(6) A triplet of the form (s, j, s), $j \in N$, can only appear at the end of a sequence of moves for DM i, which, in this case, is called a terminated sequence.

(7) Every infinite sequence of moves for DM i is also called terminated.

Similarly, given a set of credible policies, P_j^c, $j \neq i$, a credible sequence moves for DM i based on P_j^c, $j \neq i$, can be defined by replacing P_j and R_i by P_j^c and R_i^+, respectively, in Definition 6.2.2.

The result of a sequence of moves (for DM i) is given by the final state if the sequence is finite, or by the first state s^* that repeats for the first time followed by the same DM i^*, i.e., s^* is the first state from which the conflict repeats itself in infinite cycles (the fact that N and S are finite guarantee the existence of such cycles in every infinite sequence of moves).

The length or horizon of a sequence of moves is given by the number of times that states appear in the sequence less 1. A sequence of moves for DM i is said to be of r rounds if DM i

appears r times in the sequence. A sequence of moves of r rounds for DM i is called an i-sequence of r rounds if it ends with the last movement made by DM i, i.e., it ends with the triplet (s_1, i, s_2) for some $s_1, s_2 \in S$, and is called an $\bar{i}$-sequence of r rounds, otherwise.

Note that, even with the same fixed policies, different sequences of moves that differ from one another according to the order in which the DMs move in the sequence can arise. In order to fix a certain order of DMs' moves, the notion of a metarational tree is given in Definition 6.2.3. Such notion is used in the alternative generalized metarational stability definition proposed in this paper.

Definition 6.2.3. *Given a set of policies P_j, $j \in N - \{i\}$, a metarational tree, $\mathcal{A}_i^r(s)$, based on P_j, $j \in N - \{i\}$, for DM i from state s with r rounds is a set of all possible sequences of moves based on P_j, $j \neq i$, for DM i starting in s such that:*

(1) If $(s, i, ..., s_n)$ is not a terminated sequence in the tree, then there is a unique DM j such that $(s, i, ..., s_n, j, s_{n+1})$ is the initial part of some sequence of the tree.

(2) If $(s, i, ..., s_n)$ is a terminated sequence in the tree, then there is no other sequence in the tree which contain $(s, i, ..., s_n)$ as its first part.

(3) No sequence of the tree has the DM i appearing more than r times.

In other words, policies, P_j, $j \in N - \{i\}$, determine how other DMs move in the states in the metarational tree for DM i and the tree branches out every time DM i moves considering that he or she can move to any state in $R_i(s_1) \cup \{s_1\}$, while moving at an arbitrary state s_1 in the tree. Condition (1) establishes that in a metarational tree there is no doubt about who moves in every state of some sequence of movements for DM i, i.e., each tree determines who moves in each state every moment. Condition (2) states that once any DM stays at a given state, the conflict ends at that state. Finally, Condition (3) establishes that no sequence of moves for DM i can have more than r rounds in a metarational tree for DM i with r rounds.

A metarational tree for DM i is said to be credible if all of its sequences of moves for DM i are credible.

Note that there are several metarational trees based on the same policies P_j, $j \neq i$, for DM i from s which differ according to the order in which the DMs move in the states. In the case $n = 2$, this tree is unique because DMs alternate moves and there is no doubt about the order in which DMs move.

In case of conflicts with more than two DMs, it is also necessary to define a particular type of metarational tree for DM i which guarantees that all sequences end in states to which DM i moved, i.e., guarantees that DM i always has the last word.

Definition 6.2.4. *Given a set of policies P_j, $j \in N - \{i\}$, a metarational tree, based on P_j, $j \neq i$, for DM i from state s with r rounds is said to be* ***regular*** *if*

(1) There is no sequence in the tree that contains a part of the form (j, s_n, k, s_n), in that j and k are different from i.

(2) There is no infinite sequence in the tree.

Note that with the required conditions for regularity, a sequence may only be terminated by DM i or in the first movement of a DM $j \neq i$, after DM i's move, which means that DM j stays in the state for which DM i moved to. Although this condition does not appear explicitly in [14], it is necessary for the equivalence of some solution concepts, as discussed below. Note that, in the case $n = 2$, every metarational tree with r rounds is regular.

We are now able to review the generalized metarational stability definitions for conflicts with 2-DMs and n-DMs, proposed in Zeng *et al.* [19] and [14], respectively. In the case of conflicts with 2 DMs, there are two notions of stability, one that makes use of a metarational tree and another that makes use of a credible metarational tree.

Definition 6.2.5 ([19])**.** *A state s is (resp., credibly) metarationally stable with horizon h (MR_h) (resp., CMR_h) for DM i, denoted by $s \in S_i^{MR_h}$ (resp., $s \in S_i^{CMR_h}$), if there is a (resp., credible) policy P_j (resp., P_j^c) of DM j with, $j \neq i$ and $P_j(s) = s$ (resp., $P_j^c(s) = s$), such that the result of every sequence of length h and every terminated sequence of length smaller than h in the (resp., credible[1]) metarational tree for DM i is not preferable to s by DM i.*

[1] We believe that this is the correct definition of CMR_h stable states intended by Zeng et al. [19] according to

In conflicts with n-DMs, one not only considers whether sequences are credible or not, but also analyzes if the last move is from DM i or from his or her opponents. Therefore, we have the following definitions:

Definition 6.2.6 ([14]). *A state s is i-metarationally (resp., i-credibly metarationally) stable with r rounds for DM i, denoted by $s \in S_i^{MR_r}$ (resp., $s \in S_i^{CMR_r}$), if for every $s_1 \in R_i(s)$ (resp., $s_1 \in R_i^+(s)$), there is a set of (resp., credible) policies P_j (resp., P_j^c), for every DM j, $j \in N - \{i\}$, and an (resp., a credible) i-sequence, based on P_j (resp., P_j^c), $j \neq i$, for DM i starting with (s, i, s_1) of r rounds or less such that DM i does not prefer the result of this sequence to state s.*

Definition 6.2.7 ([14]). *A state s is $\bar{i}$-metarationally (resp., $\bar{i}$-credibly metarationally) stable with r rounds for DM i, denoted by $s \in S_i^{\overline{MR_r}}$ (resp., $s \in S_i^{\overline{CMR_r}}$), if for every $s_1 \in R_i(s)$ (resp., $s_1 \in R_i^+(s)$), there is a set of (resp., credible) policies P_j (resp., P_j^c), for every DM j, $j \in N - \{i\}$, and an (resp., a credible) $\bar{i}$-sequence, based on P_j (resp., P_j^c), $j \neq i$, for DM i starting with (s, i, s_1) of r rounds or less such that DM i does not prefer the result of this sequence to state s.*

In [14], two other equilibrium concepts for conflicts with n-DMs called Policy Equilibrium and Credible Policy Equilibrium were proposed. These concepts determine conflict equilibria in terms of a set of (credible) policies of the DMs involved in the conflict. Such definitions are formalized as follows:

Definition 6.2.8 (Policy Equilibrium). *The (resp., credible) policies $P_1, P_2, \ldots, P_n$ (resp., P_1^c, P_2^c, $\ldots, P_n^c$) form an equilibrium in (resp., credible) policies with respect to the current state s if the following occurs:*

(i) $P_i(s) = s$ (resp., $P_i^c(s) = s$), for all $i = 1, 2, \ldots, n$;

(ii) For all $i \in N$ and (resp., credible) policies P_i^* (resp., $P_i^{c,*}$) such that $P_i^*(s) \neq s$ (resp., $P_i^{c,*}(s) \neq s$), there is a (resp., credible) sequence of moves based on the policies $P_1, P_2, \ldots, P_{i-1}$,

what is illustrated in Figure 7 of [19], even though this is not what is written in the formal definition CMR_h in [19].

$P_i^*, P_{i+1}, \ldots, P_n$ (resp., $P_1^c, P_2^c, \ldots, P_{i-1}^c, P_i^{c,*}, P_{i+1}^c, \ldots, P_n^c$) starting with (s,i)[2]

such that the result of this sequence is not preferable to state s by DM i.

The set of all possible states for which there are (resp., credible) policies of DMs that form an equilibrium in (resp., credible) policies with respect to them is denoted by S^{PSS} (resp., S^{PSS^c}).

6.2.1 *Clarifying some results*

Three problems that were observed in some results in [14] are pointed out here. The first one refers to the fact that one of the motivations highlighted in [14] was to generalize the definition of MR_h (CMR_h) stability proposed in [19], to conflicts with two or more DMs. However, we show by means of a counter-example that the definition proposed in [14] in case where $n = 2$ is not equivalent to the definition in [19]. The second problem refers to a result that establishes an equivalence between MR_2 and SMR stable states. We show, by means of a counter-example, that such result is false if we consider the definition proposed in [14]. The third problem refers to the fact that, in [14], there is a theorem that establishes that $S^{MR_r} \subseteq S^{PSS}$. However, also by means of a counter-example, we illustrate this claim is not valid.

First Problem

[14] intended to define a generalization of the concept of MR_h stable states for conflicts with n-DMs. They claim that Definitions 6.2.6 and 6.2.7 are generalizations of Definition 6.2.5 to conflicts with n-DMs. In order to note that this is not true, we must first observe that in case $n = 2$, an i-sequence (resp., $\bar{i}$-sequence) with r rounds has length $h = 2r - 1$ (resp., $h = 2r$). Therefore, if Definitions 6.2.6 and 6.2.7 were generalizations of Definition 6.2.5 to conflicts with n-DMs, then the following equalities would have to be true: $S^{MR_r} = S^{MR_{h=2r-1}}$ and $S^{\overline{MR}_r} = S^{MR_{h=2r}}$.

Example 6.2.1 shows that $S^{MR_{r=2}} \neq S^{MR_{h=3}}$ and $S^{\overline{MR}_{r=2}} \neq S^{MR_{h=4}}$.

[2] Although in the original definition, it is not explicit that DM i must be the one who moves at s, we believe that is what the authors intended to define, otherwise every state would trivially satisfy the definition, since if another DM moves at s, he or she will stay at s and the sequence will end at s.

Example 6.2.1. *Consider a hypothetical conflict with two DMs, i and j, and seven states, s, s_1, s_2, s_3, s_4, s_5 and s_6. We show that s is $\overline{MR}_{r=2}$ but not $MR_{h=4}$ stable for DM i. Admit that DMs i and j's reachability and preferences relations are, respectively, given by $R_i(s) = s_1$, $R_i(s_2) = \{s_3, s_4\}$, $s_5 \succ_i s_3 \succ_i s_1 \succ_i s \succ_i s_4 \succ_i s_6 \succ_i s_2$ and $R_j(s_1) = \{s_2\}$, $R_j(s_3) = \{s_5\}$, $R_j(s_4) = \{s_6\}$ and $s_2 \succ_j s \succ_j s_1 \succ_j s_6, \succ_j s_4 \succ_j s_5 \succ_j s_3$.*

This conflict, in the graph form, is illustrated in Figure 6.1.

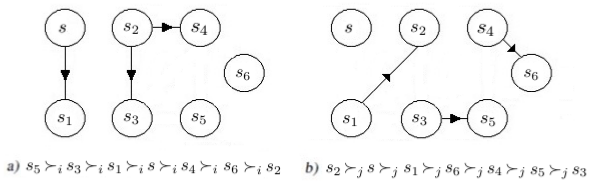


Figure 6.1: Conflict in the graph form: a) DM i and b) DM j.

Figure 6.2 illustrates the metarational tree for DM i based on P_j, where P_j is the policy where DM j always moves away from the current state. There are other metarational trees for DM i, where DM j could stay in some of the states s_1, s_3 or s_4.

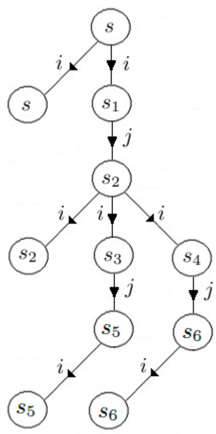


Figure 6.2: Metarational tree for DM i based on P_j, where $P_j(s_1) = s_2$, $P_j(s_3) = s_5$ and $P_j(s_4) = s_6$.

The sequence $(s, i, s_1, j, s_2, i, s_4)$ based on the policy $P_j(s_1) = s_2$ is enough to ensure that state s is $MR_{r=2}$ stable for DM i. However, s is not $MR_{h=3}$ stable for DM i, because there is no *policy P_j such that $P_j(s) = s$ and the result of every sequence of length 3 and of every terminated sequence with length smaller than 3 is not preferable to s by DM i. In order to see this, note that if $P_j(s_1) = s_1$, then the terminated sequence (s, i, s_1, j, s_1) has result s_1 which is preferable to s by DM i, and if $P_j(s_1) = s_2$, then the sequence of length 3 $(s, i, s_1, j, s_2, i, s_3)$ results in s_3 which is also preferable to s by DM i.*

Moreover, the sequence $(s, i, s_1, j, s_2, i, s_4, j, s_6)$ based on the policy $P_j(s_1) = s_2$, $P_j(s_4) = s_6$ is enough to ensure that state s is $\overline{MR}_{r=2}$ stable for DM i. However, s is not $MR_{h=4}$ stable for DM i, because there is no policy P_j such that $P_j(s) = s$ and the result of every sequence of length 4 and every terminated sequence of length smaller than 4 is not preferable to s by DM i. Note that analyzing all possibilities for P_j, there exists a sequence that ends in s_1, s_3 or s_5 and all of these states are preferable to state s by DM i. Thus s is not $MR_{h=4}$ stable for DM i.

Since the sequences in Example 6.2.1 are all credible, these same examples illustrate that $S^{CMR_{r=2}} \neq S^{CMR_{h=3}}$ and $S^{\overline{CMR}_{r=2}} \neq S^{CMR_{h=4}}$, respectively.

It is worth pointing out that the main reason that makes these equivalences invalid is that while in the definition of MR_h (resp., CMR_h) stability the result of *every* sequence of length h and *every* terminated sequence of length smaller than h is required not to be preferable to s by DM i, in the definitions of MR_r and $\overline{MR}_r$ (resp., CMR_r and $\overline{CMR}_r$) stabilities, it is only required the *existence* of an i-sequence and of an $\bar{i}$-sequence, respectively, whose result is not preferable to s by DM i.

Second Problem

The third part of Theorem 1 in [14] states that there is an equivalence between the notion of $MR_{r=2}$ and SMR stability. However, this result is not true if we consider Definition 6.2.6. For example, state s in Example 6.2.1 is $MR_{r=2}$ but not SMR stable for DM i.

It is worth pointing out that the main reason that makes the equivalence between SMR and $MR_{r=2}$ invalid is that while in the definition of SMR stability for each unilateral improvement from s by DM i, there exists a legal sequence of unilateral moves from the opponents of DM i leading the conflict to a state s_2 that is not preferable to s by DM i and from such state s_2, the result of *every* unilateral move from DM i is also not preferable to s by DM i, in the definition of $MR_{r=2}$ stability for each unilateral improvement from s by DM i, there exists a legal sequence of unilateral moves from the opponents of DM i leading the conflict to a state s_2 that is not preferable to s by DM i and from such state s_2, *there exists* a unilateral move from DM i whose result is also not preferable to s by DM i.

Third Problem

Theorem 3 in [14] states that $S^{MR_r} \subseteq S^{PSS}$, for all $r \geq 1$. Example 6.2.2 illustrates that this claim is not valid.

Example 6.2.2. *Consider a hypothetical conflict with three DMs, i, j and k, and state space given by $S = \{s, s_1, s_2, s_3\}$. Suppose that $R_i(s) = R_k(s) = \{s_1\}$, $R_j(s) = R_j(s_2) = R_j(s_3) = \emptyset$,*

$R_i(t) = R_k(t) = \emptyset$, for all $t \in \{s_1, s_2, s_3\}$ and $R_j(s_1) = \{s_2, s_3\}$. Consider also that DMs i, j and k's preference relations are given by $s_3 \succ_i s_1 \succ_i s \succ_i s_2$, $s_3 \succ_j s_2 \succ_j s_1 \succ_j s$, and $s_2 \succ_k s_1 \succ_k s \succ_k s_3$. Figure 6.3 illustrates that conflict.

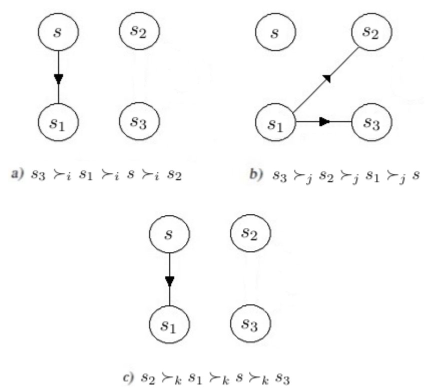


Figure 6.3: Conflict in the graph form: a) DM i; b) DM j and c) DM k.

Let us prove that s is not a policy equilibrium, but it is an equilibrium according to MR_2. In fact, note that in this example DM j has three possible policies that differ in what he or she does in state s_1. Such policies are (a) $P_j^1(s_1) = s_1$, (b) $P_j^2(s_1) = s_2$ or (c) $P_j^3(s_1) = s_3$. Note also that DM i (resp., k) has only two possible policies that differ in what he or she does in state s: (a) $P_i^1(s) = s$ (resp., $P_k^1(s) = s$) and (b) $P_i^2(s) = s_1$ (resp., $P_k^2(s) = s_1$).

Part (a) of Figure 6.4 illustrates a metarational tree for DM i based on P_j^2 *and an arbitrary* P_k*. On part (b) of Figure 6.4, we have a metarational tree for DM k based on* P_j^3 *and an arbitrary* P_i*. There are other metarational trees for DMs i and k that differ in the other in which DMs move and also in the policies used by the DMs.*

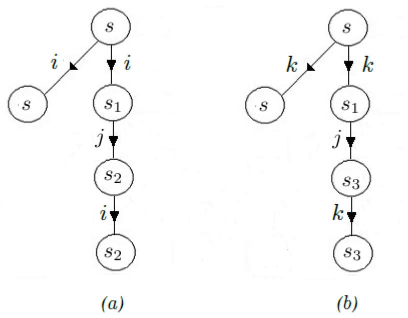


Figure 6.4: (a) Metarational tree for DM i based on P_j^2 and an arbitrary P_k and (b) Metarational tree for DM k based on P_j^3 and an arbitrary P_i.

Then, by definition of policy equilibrium, we have to verify if some DM has an incentive to deviate from using a policy that stays in state s. Let us first consider the case where DM j uses policy P_j^1*. In this case, both DMs i and k have an incentive to use their policies* P_i^2 *and* P_k^2*, respectively, because it always results in sequences whose final state is* s_1 *that is preferable to s for both DMs i and k. If DM j uses policy* P_j^2 *(resp.,* P_j^3*), then DM k (resp., DM i) has an incentive to use policy* P_k^2 *(resp.,* P_i^2*), which results either in state* s_1 *or in state* s_2 *(resp.,* s_3*) which are preferable to state s by DM k (resp., i). Therefore,* $s \notin S^{PSS}$*.*

On the other hand, s is MR_1 *and consequently* MR_2 *stable for DM j and* MR_2 *stable for DMs i and k. In order to verify that s is* MR_2 *stable for DM i (resp., k), consider the policy* P_j^2 *(resp.,* P_j^3*) of DM j and the metarational tree illustrated in Part (b) (resp. (b)) of Figure 6.4, which results in state* s_2 *(resp.,* s_3*), which is not preferable to s by DM i (resp., k).*

Note that as every possible policy in Example 6.2.2 is credible, this same example illustrates that S^{CMR_2} is not a subset of S^{PSS^c}.

It is worth pointing out that the main reason that makes the inclusion $S^{MR_r} \subseteq S^{PSS}$ invalid is that while analyzing MR_r stability for different DMs, there may be no relation about the policies being used by their opponents. On the other hand, in order to be a policy equilibrium, we need to fix a whole set of policies for all DMs such that none of them has incentive for deviating from such fixed policy.

6.3 A New Generalized Metarationality

Motivated by the problems found in the definition of MR_r and $\overline{MR}_r$ stable states proposed in [14], we present the following alternative definitions of generalized metarational stable states for conflicts with n DMs, which solves the first and the second problems pointed out in Section 6.2.1.

Definition 6.3.1 (Alternatives to MR_r and CMR_r). *A state $s \in S$ is i-metarationally (resp., i-credibly metarationally) stable with r rounds for DM i, denoted by $s \in S_i^{MR_r^{new}}$ (resp., $s \in S_i^{CMR_r^{new}}$), if there is a set of (resp., credible) policies P_j (resp., P_j^c), for all $j \in N - \{i\}$, and a (resp., credible)* **regular metarational tree***, based on P_j (resp., P_j^c), $j \neq i$, of r rounds such that the result of* **every** *(resp., credible) i-sequence of r rounds and of every (resp., credible) terminated sequence with less than r rounds is not preferable to s by DM i.*

Intuitively, in a MR_r (resp., CMR_r) stable state for DM i, there exists a set of (resp. credible) policies for the opponents of DM i such that there is no way that he can move (resp. using unilateral improvements) at most r times and finish in a more preferred state, where opponents can respond to at most $r-1$ moves of DM i and can stay only at states to which DM i moved to.

Definition 6.3.2 (Alternatives to $\overline{MR}_r$ and $\overline{CMR}_r$). *A state $s \in S$ is $\bar{i}$-metarationally (resp., i-credibly metarationally) stable with r rounds for DM i, denoted by $s \in S_i^{\overline{MR}_r^{new}}$ (resp., $s \in S_i^{\overline{CMR}_r^{new}}$), if there is a set of (resp., credible) policies P_j (resp., P_j^c), for all $j \in N - \{i\}$, and a (resp., credible)* **metarational tree***, based on P_j (resp., P_j^c), $j \neq i$, of r rounds such that the result of* **every** *(resp., credible) $\bar{i}$-sequence of r rounds,* **which is not an initial part of another (resp., credible) $\bar{i}$-sequence of r rounds***, of every (resp., credible) terminated i-sequence of r rounds and every (resp., credible) terminated sequence with less than r rounds is not preferable to s by DM i.*

Intuitively, in a $\overline{MR_r}$ (resp., $\overline{CMR_r}$) stable state for DM i, there exists a set of (resp. credible) policies for the opponents of DM i such that there is no way that he can move (resp. using unilateral improvements) at most r times and finish in a more preferred state, where opponents can respond to at most r moves of DM i.

6.4 Properties of these new Solution Concepts

In this section, in order to investigate which results stated in [14] remain valid for our alternative definitions, we analyze the relationship between the alternative definitions proposed in Section 6.3 and the solution concepts of Nash stability, GMR, SMR, SEQ, $SSEQ$ and PSS. We also obtain some results involving the various alternative solution concepts proposed in Section 6.3.

Theorem 6.4.1 states that Definitions 6.3.1 and 6.3.2 are generalizations of Definition 6.2.5 for conflicts with n-DMs.

Theorem 6.4.1. *If $n = 2$, then: (a) $S_i^{MR_{h=2r-1}} = S_i^{MR_r^{new}}$ and (b) $S_i^{MR_{h=2r}} = S_i^{\overline{MR}_r^{new}}$.*

Proof: Let us consider part (a) first. If $n = 2$, then note that the set of all sequences of length $h = 2r - 1$ is equal to the set of all i-sequences with r rounds. Furthermore, the set of terminated sequences with length smaller than $2r - 1$ is equal to the set of all terminated sequences with less than r rounds. Thus, if s is $MR_{h=2r-1}$ stable for DM i in a conflict with two DMs, i and j, then there exists a policy P_j of DM j with $P_j(s) = s$ such that the result of every sequence of length equal to $2r - 1$ and every terminated sequence with length smaller than $2r - 1$, i.e., of every i-sequence with r rounds and of every terminated sequence with less than r rounds is not preferable to state s by DM i. Thus, s is also MR_r^{new} stable for DM i.

In order to prove the other direction of part (a), consider that state s is MR_r^{new} stable for DM i. Thus, there exists a policy P_j of DM j and a regular metarational tree of r rounds such that the result of every i-sequence with r rounds or every terminated sequence with less than r rounds is not preferable to s by DM i. Define the policy $P_j^{\#}$ such that $P_j^{\#}(t) = t$ if $s \succeq_i t$ and $P_j^{\#}(t) = P_j(t)$, otherwise. Thus, it follows that $P_j^{\#}(s) = s$. By definition of $P_j^{\#}$, if some sequence of moves is in the metarational tree based on $P_j^{\#}$ of r rounds for DM i that starts in state s but not in the metarational tree based on P_j of r rounds for DM i that starts in state s, then the result of this sequence is not preferable to s by DM i. Moreover, since the result of every sequence of length $2r-1$ or terminated sequence of length smaller than $2r-1$ that are in the metarational tree based on P_j is not preferable to s by DM i, we have that s is $MR_{h=2r-1}$ stable for DM i.

The proof of part (b) is similar and is omitted. □

Note that with an argument similar to the one used in the proof of Theorem 6.4.1, only changing the metarational tree by a credible metarational tree for DM i and DM j policy by a credible policy, we can get that if $n=2$, then $S_i^{CMR_{h=2r-1}} = S_i^{CMR_r^{new}}$ and $S_i^{CMR_{h=2r}} = S_i^{\overline{CMR}_r^{new}}$, i.e., CMR_r^{new} and $\overline{CMR}_r^{new}$ are generalizations for conflicts with n-DMs of the notion of CMR_h stability for h odd and even, respectively.

Theorem 6.4.2 establishes an equivalence between the set of MR_1^{new} (resp., CMR_1^{new}) stable states with the set of Nash stable states.

Theorem 6.4.2. *A state s is MR_1^{new} (resp., CMR_1^{new}) stable for DM i iff it is Nash stable for DM i.*

Proof: The proof of this result follows a similar idea of the proof of the corresponding result obtained in [14]. Thus, we omit it here. □

Theorem 6.4.3 establishes an equivalence between the set of $\overline{MR}_1^{new}$ stable states with the set of GMR stable states.

Theorem 6.4.3. *A state s is $\overline{MR}_1^{new}$ stable for DM i iff it is GMR stable for DM i.*

Proof: If state s is $\overline{MR}_1^{new}$ stable for DM i, then there exists a set of policies P_j, $j \in N-\{i\}$, and a metarational tree of 1 round, based on P_j, $j \in N-\{i\}$, such that the result of every $\bar{i}$-sequence with 1 round, which is not an initial part of another $\bar{i}$-sequence in the metarational tree, is not preferable to s by DM i. As $R_i^+(s) \subseteq R_i(s)$, then for each state $s_1 \in R_i^+(s)$, there exists a state $s_2 \in R_{N-\{i\}}(s_1)$, determined by the policies of DMs j, $j \in N-\{i\}$, and the metarational tree, such that $s \succeq_i s_2$. Therefore, s is GMR stable for DM i.

Suppose now that state s is GMR stable for DM i. Let $R_i^+(s) = \{s_1, s_2, \ldots, s_W\}$. Thus, for each s_w, $w = 1, 2, \ldots, W$, there exists $s'_w \in R_{N-i}(s_w)$ such that s'_w is not preferable to s by DM i. Therefore, there is a legal sequence of moves of DMs j, $j \in N - \{i\}$, which takes the conflict from state s_w to state s'_w, for $w = 1, 2, \ldots, W$. Consider the shortest sequence of legal moves of DMs j, $j \in N - \{i\}$, denoted by s_x^w, that takes the conflict from s_w to some state s'_w such that s'_w is not preferable to s by DM i. In s_x^w, there is no cycles and, furthermore, all states appearing in s_x^w before s'_w must be preferable to s by DM i, otherwise s_x^w would not be a sequence with the shortest length that takes the conflict from s_w to some state that is not preferable to s by DM i. Define DMs j, $j \in N - \{i\}$, policies as follows:

(i) For all $u \in S$ and DM j, $j \in N - \{i\}$, if the pair (u, j) does not appear in any of the sequences s_x^w, for $w = 1, 2, \ldots, W$, then $P_j(u) = u$;

(ii) Let w^* be the smallest w value such that the pair (u, j) appear in the sequence s_x^w. Since $s_x^{w^*}$ does not contain cycles, then there is only one state $t \in S$ such that (u, j, t) is a triplet in $s_x^{w^*}$, then define $P_j(u) = t$.

From the above policy definition, we have that $P_j(u) = u$ for every $u \in S$ and DM $j \neq i$ such that $s \succeq_i u$, since the pair (u, j) does not appear in any of the sequences s_x^w. Thus, consider the metarational tree based on P_j, $j \neq i$, consisting of sequences of the form (and their initial parts) (s, i, s_x^w), for $w = 1, 2, \ldots, W$, together with sequences of the form (and their initial parts) (s, i, u, j, u), for every $u \in R_i(s) \cap (R_i^+(s))^c$ and some DM j, $j \in N - \{i\}$. Then, there is a metarational tree of 1 round based on $P_j, j \neq i$, for DM i starting at state s such that the result of every $\bar{i}$-sequence of 1 round, which is not an initial part of another $\bar{i}$-sequence of 1 round in the metarational tree, and every terminated i-sequence of 1 round is not preferable s by DM i. Therefore, s is $\overline{MR}_1^{new}$ stable for DM i. □

Theorem 6.4.4 establishes an equivalence between the set of MR_2^{new} stable states and the set of SMR stable states.

Theorem 6.4.4. *A state s is MR_2^{new} stable for DM i iff it is SMR stable for DM i.*

Proof: The proof of this result follows a similar idea as that of the proof of the Theorem 6.4.3 and is left to the Appendix. □

Theorem 6.4.5 establishes an equivalence between the set of $\overline{CMR}_1^{new}$ stable states and the set of SEQ stable states.

Theorem 6.4.5. *A state s is $\overline{CMR}_1^{new}$ stable for DM i iff it is SEQ stable for DM i.*

Proof: It is similar to the proof of Theorem 6.4.3, only changing the metarational tree by a credible metarational tree for DM i and DMs j, $j \neq i$, policies by credible policies. □

Next, we recall the definition of a negative transitive binary relation [43], which will be useful for the comprehension of the result of the following theorem.

Definition 6.4.1. *Let X be a set of outcomes and let B an binary relation on X. B is a negative transitive relation if $\neg xBy$ and $\neg yBz$ implies $\neg xBz$, where the notation $\neg xBy$ means $(x, y) \notin B$.*

Theorem 6.4.6 establishes a relationship between the concepts of $SSEQ$ and CMR_2^{new} stability.

Theorem 6.4.6. *If a state s is $SSEQ$ stable for DM i, then s is CMR_2^{new} stable for DM i. The reciprocal is true if DM i's preference is negatively transitive.*

Proof: The proof that $SSEQ$ implies CMR_2^{new} stability is similar to the proof that SMR implies MR_2^{new} stability and is omitted.

For the reciprocal, suppose that $\succ_i$ is negatively transitive. Thus, we have that $\succeq_i$ is transitive. Suppose s is CMR_2^{new} stable for DM i, then there is a set of credible policies P_j^c, $j \neq i$, and a credible regular metarational tree with two rounds based on P_j^c such that the result of every credible i-sequence with two rounds and of every credible terminated sequence with one round is not preferable to s by DM i. Since the credible metarational tree is regular, there is a unique credible terminated sequence of 1 round, which is (s, i, s). Moreover, for each state $s_1 \in R_i^+(s)$, there exists a state $s_2 \in R_{N-\{i\}}^+(s_1)$, determined by the credible policies of DMs j, $j \neq i$, and the credible regular metarational tree, such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for all $s_3 \in R_i^+(s_2)$, since s is CMR_2^{new} stable for DM i. It remains to consider the case where $s_4 \in R_i(s_2) - R_i^+(s_2)$. In this case, we have $s_2 \succeq_i s_4$ and by the transitivity of $\succeq_i$, we have that $s \succeq_i s_4$. Thus, s is $SSEQ$ stable for DM i, if $\succ_i$ is negatively transitive. □

The following example illustrates that if DM i's preference relation is not negatively transitive, then the converse of Theorem 6.4.6 is not true. This often occurs in real conflicts, where DM i, for some reason, does not know how to compare some states.

Example 6.4.1. *Consider a hypothetical conflict with* 2 *DMs, i and j, and four sates, s, s_1, s_2 and s_3. Suppose that the relations are given by $R_i(s) = s_1$, $R_i(s_2) = s_3$ and $R_j(s_1) = s_2$. The conflict is illustrated in Figure 6.5.*

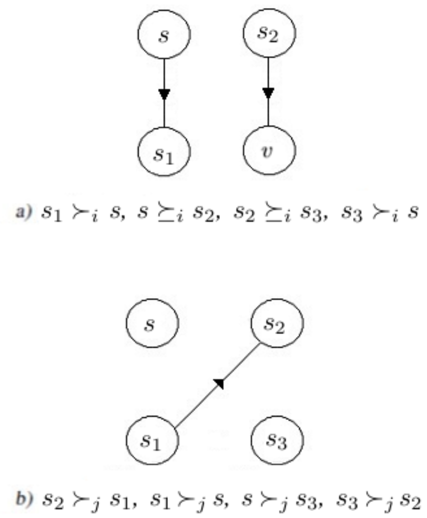


Figure 6.5: Conflict in the graph form: $a)$ DM i and $b)$ DM j.

Assume that in this conflict DM i does not have a very great knowledge about the state s_2 and does not know how to compare it to any other state of the conflict. Then, assume that DMs i and j's preference of relations are given by $s_1 \succ_i s$, $s \succeq_i s_2$, $s_2 \succeq_i s_3$, $s_3 \succ_i s$ and $s_2 \succ_j s_1$, $s_1 \succ_j s$, $s \succ_j s_3$, $s_3 \succ_j s_2$, respectively. Suppose DM i is in state s. Note that state s is not $SSEQ$ stable for DM i, because from s DM i can move to a better state s_1 and from s_1 the unique unilateral improvement reaction of DM j is to go to state s_2 that is not preferable to s by DM i, and from s_2 DM i can move to s_3, which is preferable to s by DM i. Moreover, s is CMR_2^{new} stable for DM i, since if $P_j^c(s_1) = s_2$, the conflict ends at s_2, since there is no unilateral improvement from

s_2 for DM i and s_2 is not preferable to s by DM i. Figure 6.6 illustrates the credible metarational tree for DM i based on $P_j^c(s_1) = s_2$.

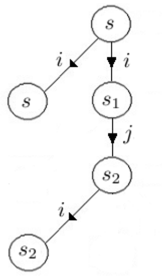


Figure 6.6: Credible metarational tree for DM i based on $P_j^c(s_1) = s_2$.

Here it is worth pointing out that if in the definition of CMR_r^{new} stability, we replace the existence of a credible regular metarational tree by the existence of a regular metarational tree, then we would have equivalence between CMR_2^{new} and $SSEQ$ stability. However with such modification, if $n = 2$, then CMR_r^{new} would only be equivalent to $CMR_{h=2r-1}$ if preferences were negatively transitive. Since our main objective was to propose a new definition of generalized metarationality that coincides with Definition 6.2.5 for conflicts with 2 DMs, we preferred to use Definition 6.3.1.

Theorem 6.4.7 establishes that MR_r^{new} stability implies $\overline{MR}_r^{new}$ stability, for all $r \geq 1$.

Theorem 6.4.7. *If a state s is MR_r^{new} stable for DM i, then it is $\overline{MR}_r^{new}$ stable for DM i, for all $r \geq 1$.*

Proof: Suppose that s is MR_r^{new} stable for DM i, then there are policies P_j, $j \in N - \{i\}$, and a regular metarational tree with r rounds, denoted by $\mathcal{A}_r$, based on P_j, $j \in N - \{i\}$, for DM i starting at state s such that the result of every i-sequence of r rounds and every terminated sequence of less than r rounds is not preferable to s by DM i. Then consider the modified policies for DMs j, $j \in N - \{i\}$, such that $P_j^{\#}(t) = t$, if $s \succeq_i t$ and $P_j^{\#}(t) = P_j(t)$, otherwise. Let $\mathcal{A}_r^{\#}$ be a metarational tree of r rounds, based on $P_j^{\#}$, $j \in N - \{i\}$, for DM i starting at state s such that:

(*i*) If $s_x \in \mathcal{A}_r$ is a sequence such that for all states t, in which some DM j, $j \in N - \{i\}$, moves in s_x, t is preferable to s by DM i, then $s_x \in \mathcal{A}_r^{\#}$;

(*ii*) If $s_x = (s, i, s_1, j_1, s_2, j_2, \ldots, s_m) \in \mathcal{A}_r$ and there is some s_w, for $w = 1, 2, \ldots, m-1$ such that $j_w \neq i$ and $s \succeq_i s_w$, then let w^* be the smallest w value for which s_w satisfy these conditions. Thus, we have that the sequence $(s, i, s_1, j_1, \ldots, j_{w^*-1}, s_{w^*}, j_{w^*}, s_{w^*}) \in \mathcal{A}_r^{\#}$, for some $j_{w^*} \neq j_{w^*-1}$.

Note that all sequences in $\mathcal{A}_r^{\#}$ that are not in $\mathcal{A}_r$ end in some state s_{w^*} that is not preferable to s by DM i. Note also that the result of every terminated sequence of less than r rounds in $\mathcal{A}_r^{\#} \cap \mathcal{A}_r$ is not preferable to s by DM i, because s is MR_r^{new} stable for DM i. Finally, suppose a terminated sequence of r rounds in $\mathcal{A}_r^{\#} \cap \mathcal{A}_r$. If this sequence is an i-sequence, then its result is not preferable to s by DM i, since s is MR_r^{new} stable for DM i. If this sequence is an $\bar{i}$-sequence, it must finish in the final state of the i-sequence of r rounds that is equal to its initial part, since $\mathcal{A}_r$ is regular. Thus, since all the final states of the i-sequences of r rounds in $\mathcal{A}_r$ are not preferable to s by DM i, we have that the final results of the $\bar{i}$-sequences of r rounds in $\mathcal{A}_r^{\#}$ are not preferable to s by DM i. Therefore, s is $\overline{MR}_r^{new}$ stable for DM i. □

Similarly, we can obtain a theorem that states that if a state is CMR_r^{new} stable for DM i, then it is $\overline{CMR}_r^{new}$ stable for DM i, for all $r \geq 1$. The proof of this fact is similar to the proof of Theorem 6.4.7, just changing the regular metarational tree for DM i by a credible regular metarational tree for DM i and DMs j's, $j \in N - \{i\}$, policies P_j by credible policies P_j^c.

Theorem 6.4.8 relates $\bar{i}$-metarational stability with r rounds with $\bar{i}$-metarational stability with a smaller number of rounds.

Theorem 6.4.8. *If a state s is $\overline{MR}_r^{new}$ stable for DM i, then it is $\overline{MR}_l^{new}$ stable for DM i, for all $1 \leq l \leq r-1$.*

Proof: The proof os this result follows a similar idea of proof of the corresponding theorem obtained in [14] □

Theorem 6.4.8 remains valid if we replace $\overline{MR}_r^{new}$ by $\overline{CMR}_r^{new}$ stability. The proof of this fact is similar to the proof of Theorem 6.4.8, just changing the metarational tree for DM i by a credible metarational tree for DM i and DMs j's, $j \in N - \{i\}$, policies P_j by credible policies P_j^c.

According to the alternative definition of states MR_r^{new} stability proposed in this paper, it is not true that if a state is MR_r^{new} stable for DM i, then it is MR_{r+1}^{new} stable for DM i, for all positive integer r in conflicts with n-DM. This relationship fails because of the regularity condition required in the definition of MR_r^{new} stability.

We point out that if this regularity condition were removed from the MR_r^{new} stability definition, then it would follow that MR_r^{new} implies MR_{r+1}^{new} stability, but on the other hand the equivalence results between SMR and MR_2^{new} and $SSEQ$ and CMR_2^{new} would no longer be true.

Example 6.4.2 illustrates this fact, by showing that MR_3^{new} stability does not imply MR_4^{new} stability.

Example 6.4.2. *Consider a hypothetical conflict composed by three DMs, i, j and k, and seven states $\{s, s_1, s_2, s_3, s_4, s_5, s_6\}$. Suppose that DMs reachability relations are given by: $R_i(s) = \{s_1, s_6\}$, $R_i(s_3) = \{s_4\}$, $R_i(s_5) = \{s_2\}$, $R_j(s_1) = R_k(s_6) = \{s_2\}$, $R_j(s_2) = \{s_3\}$, $R_k(s_2) = \{s_3\}$, $R_k(s_4) = \{s_5\}$ and that the reachability relations of all DMs in all states that are not specified above are equal to the empty set. Suppose also that DMs' preferences are given by: $s_6 \succ_i s_4 \succ_i s_1 \succ_i s \succ_i s_2 \succ_i s_3 \succ_i s_5$, $s_4 \succ_j s_6 \succ_j s_3 \succ_j s_2 \succ_j s \succ_j s_1 \succ_j s_5$ and $s_3 \succ_k s_2 \succ_k s_5 \succ_k s_1 \succ_k s_4 \succ_k s_6 \succ_k s$. Consider the policies of DMs j and k that always move out from the state where they are. Thus, there is a regular metarational tree of 3 rounds for DM i starting at s as illustrated in Figure 6.7*

Note that the result of every i-sequence of 3 rounds and of every terminated sequence of less than 3 rounds is not preferable to s by DM i, i.e., s is MR_3^{new} stable for DM i. On the other hand, for all policies of DMs j and k and all regular metarational tree for DM i based on these policies of 4 rounds starting at s, there exists a sequence of moves that either ends in s_1, s_6 or s_4 if one of the DMs j or k stays in those states or there exists an i-sequence of 4 rounds with final state s_4, i.e., for any policies of DMs j and k and any regular metarational tree based on

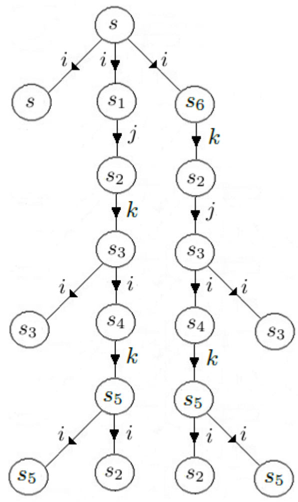


Figure 6.7: Metarational tree for DM i based on P_j and P_k, where $P_j(s_1) = s_2$, $P_j(s_2) = P_k(s_2) = s_3$, $P_k(s_4) = s_5$ and $P_k(s_6) = s_2$.

these policies, there is an i-sequence of 4 *rounds or a terminated sequence of less than* 4 *rounds whose result is preferable to s by DM i. Thus, s is not* MR_4^{new} *stable for DM i.*

Since in Example 6.4.2 all sequences are based on unilateral improvements of DM i and credible policies of DMs j, $j \in N - \{i\}$, the same example illustrates that CMR_3^{new} stability does not imply CMR_4^{new} stability.

Theorem 6.4.9 establishes an implication between the set of CMR_r^{new} stable states and the set of MR_r^{new} stable states.

Theorem 6.4.9. *If a state s is* CMR_r^{new} *stable for DM i, then it is* MR_r^{new} *stable for DM i.*

Proof: This proof is similar to the proof of Theorem 6.4.7 and is left to the Appendix. □

Theorem 6.4.10 establishes an implication between the set of $\overline{CMR}_r^{new}$ stable states and the set of $\overline{MR}_r^{new}$ stable states.

Theorem 6.4.10. *If a state s is* $\overline{CMR}_r^{new}$ *stable for DM i, then it is* $\overline{MR}_r^{new}$ *stable for DM i.*

Proof: The proof of this result follows a similar idea of proof of Theorem 6.4.7 and is left to the Appendix. □

Theorem 6.4.11 establishes a relationship between $\overline{MR}_r^{new}$ stability and policy equilibrium.

Theorem 6.4.11. *$S^{PSS} \subseteq S^{\overline{MR}_r^{new}}$, for all $r \geq 1$.*

Proof: Let $s \in S^{PSS}$. Thus, there is a set of policies P_i^*, $i \in N$, satisfying $P_i^*(s) = s$, such that for every policy $P_i^{\#}$ satisfying $P_i^{\#}(s) \neq s$, there is a sequence that starts with (s, i) based on policies $P_i^{\#}(s)$ and $P_j^*(s)$, for all $j \in N - \{i\}$, whose result is not preferable to state s by DM i. Thus, for every $s_w \in R_i(s)$, there is a legal sequence of moves, s_x^w beginning with (s, i, s_w) and ending in s'_w such that s'_w is not preferable to s by DM i. Let $\mathcal{A}$ be the metarational tree for DM i starting at s and based on policies $P_j^*(s)$, for all $j \in N - \{i\}$, that contains all sequences of the type s_x^w, along with the sequence (s, i, s). Denote by $\mathcal{A}_r$ the metarational tree that results from removing all sequences in $\mathcal{A}$ with more than r rounds. Then, every terminated sequence in $\mathcal{A}_r$ is also terminated in $\mathcal{A}$ and must result in a state non-preferable to s by DM i.

Furthermore, for each $\bar{i}$-sequence of r rounds in $\mathcal{A}_r$, which is not the initial part of another $\bar{i}$-sequence of r rounds in $\mathcal{A}_r$, denoted by s_x, one of the following situations must occur: (a) DM i move at the final state of s_x in $\mathcal{A}$ or (b) s_x is a finite or infinite terminated sequence in $\mathcal{A}$. In the case (b), it follows that the result of the sequence is not preferable to s by DM i, since $s \in S^{PSS}$. Finally, in case (a), the final state of s_x is not preferable to s by DM i, otherwise, since DM i always has the opportunity to stay in the final state, this would result in a terminated sequence in $\mathcal{A}$ whose result is preferable to s by DM i, which would be a contradiction since $s \in s^{PSS}$. Therefore, for every $i \in N$, s must also be $\overline{MR}_r^{new}$ stable for DM i. □

With a proof similar to that of Theorem 6.4.11, one can show that $S^{PSS^c} \subseteq S^{\overline{CMR}_r^{new}}$, for all $r \geq 1$.

On the other hand, Example 6.2.2 also illustrates a conflict where $S^{MR_2^{new}} \not\subseteq S^{PSS}$.

6.5 Conclusion

This chapter presents some problems found in the definition of generalized metarational stability proposed in [14] for conflicts with n-DMs. In particular, it shows that such definition is not a generalization of generalized metarational stability for conflicts with 2 DMs, as proposed in [19]. Motivated by that fact, we introduce an alternative definition for generalized metarationality for conflicts with n-DMs that overcomes this problem. We proved many results that relate our proposed solution concept to other solution concepts common in the GMCR literature. A summary of those results can be found in Tables 6.1 and 6.2.

Table 6.1: Equivalences between solution concepts in the GMCR

Equivalences
$S_i^{MR_{h=2r-1}} = S_i^{MR_r^{new}}$, if $n = 2$.
$S_i^{MR_{h=2r}} = S_i^{\overline{MR}_r^{new}}$, if $n = 2$.
$S_i^{CMR_{h=2r-1}} = S_i^{CMR_r^{new}}$, if $n = 2$.
$S_i^{CMR_{h=2r}} = S_i^{\overline{CMR}_r^{new}}$, if $n = 2$.
$S_i^{Nash} = S_i^{MR_1^{new}} = S_i^{CMR_1^{new}}$.
$S_i^{GMR} = S_i^{\overline{MR}_1^{new}}$.
$S_i^{SMR} = S_i^{MR_2^{new}}$.
$S_i^{SEQ} = S_i^{\overline{CMR}_1^{new}}$.

Table 6.2: Implications between solution concepts in the GMCR

Implications
$S_i^{SSEQ} \subseteq S_i^{CMR_2^{new}}$.
$S_i^{CMR_2^{new}} \subseteq S_i^{SSEQ}$, if DM i's preference is negatively transitive.
$S_i^{MR_r^{new}} \subseteq S_i^{\overline{MR}_r^{new}}$, for all $r \geq 1$.
$S_i^{CMR_r^{new}} \subseteq S_i^{\overline{CMR}_r^{new}}$, for all $r \geq 1$.
$S_i^{\overline{MR}_r^{new}} \subseteq S_i^{\overline{MR}_l^{new}}$, for all $1 \leq l \leq r-1$.
$S_i^{\overline{CMR}_r^{new}} \subseteq S_i^{\overline{CMR}_l^{new}}$, for all $1 \leq l \leq r-1$.
$S_i^{CMR_r^{new}} \subseteq S_i^{MR_r^{new}}$, for all $r \geq 1$.
$S_i^{\overline{CMR}_r^{new}} \subseteq S_i^{\overline{MR}_r^{new}}$, for all $r \geq 1$.
$S^{PSS} \subseteq S^{\overline{MR}_r^{new}}$, for all $r \geq 1$.

Having a better understanding of such solution concept is of key importance to the stability analysis of conflicts with n-DMs, since such concept generalizes the most usual solution concepts of the literature to allow for DMs to analyze the conflict with variable horizons.

In future research, we plan to investigate whether there exists a computationally efficient way of finding such generalized metarationally stable states.

CHAPTER 7

Conclusions and Directions for future work

7.1 Conclusions

In this thesis, we present various advances in the graph model for conflict resolution. Such advances range from the proposal of new concepts of stability, efficient methods to obtain stability in the GMCR with probabilistic preferences and a generalization of the GMCR to handle possibly unaware players. More specifically, we propose the notion of $SSEQ$ stability and extended this concept for n-DM conflicts in the GMCR. We also presented the relationships among $SSEQ$ with six solution concepts commonly used in the GMCR. Additionally, we introduced the $SSEQ$ concept for coalitional analysis and extended $SSEQ$ stability for the GMCR with uncertain, probabilistic and fuzzy preferences in n-DM conflicts.

We adapted matrix methods proposed by Xu et al. [18] and [24] to determine stable states in 2-DM and n-DM conflicts in the GMCRP according to five stability definitions that have been proposed for such model. We also proposed a generalization of the GMCR, for conflicts with two and n-DMs, in order to allow the representation of conflicts where DMs may be unaware of some options available for them or for their opponents in the conflict. We propose five notions of stability in the GMCR with interactive unawareness, providing results that relate such notions and also showed that standard solution concepts for the GMCR are special cases of the notions proposed.

We also present some problems found in the definition of generalized metarational stability proposed in Zeng et al. [14] for conflicts with n-DMs and motivated by that fact, we introduced an alternative definition for generalized metarationality for conflicts with n-DMs that overcomes some of the problems in Zeng et al. [14]. We also study some properties of our proposed definition.

7.2 Directions for future work

In future research we intend:

(1) Propose the $SSEQ$ stability definition for an arbitrary horizon, i.e., considering several moves of reaction and counter-reaction according to this concept.

(2) Extend the GMCR with iterative unawareness by adopting other preference structures, such as the probability structure of Rêgo and Santos [15], uncertain preference of Li et al. [16] and Fuzzy preference of Hipel et al. [17].

(3) Propose matrix representations to facilitate the obtaining of stable states according to the stability concepts proposed in the GMCR with interactive unawareness.

References

[1] BOREL, E. La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes Rendus de l'Academie des Sciences*, v. 173, p. 58, 1921.

[2] NEUMANN, J.V. Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, v. 100, n. 1, p. 295-320, 1928.

[3] NEUMANN, J. V.; MORGENSTERN, O. Theory of Games and Economic Behaviour. *Princeton: University Press*, 776 p., 1944.

[4] NASH, J. F. The bargaining problem. *Econometrica: Journal of the Econometric Society*, p. 155-162, 1950.

[5] NASH, J. F. Equilibrium points in n-person games. *Proceedings of the national academy of sciences*, v. 36, p. 48-49, 1950.

[6] NASH, J. F. Non-cooperative games. *Annals of Mathematics*, p. 286-295, 1951.

[7] NASH, J. F. Two-person cooperative games. Econometrica: Journal of the Econometric Society, p. 128-140, 1953.

[8] HOWARD, N. *Paradoxes of rationality: theory of metagames and political behavior*. New York: MIT press, 1971.

[9] FRASER, N. M.; HIPEL, K. W. *Conflict analysis: models and resolutions.* North-Holland, 1984.

[10] HIPEL, K. W.; FRASER, N. M. Systems Management: Conflict Analysis, updated invited paper in Concise Encyclopaedia of Information Processing in Systems and Organizations, edited by A.P. Sage, Pergamon Press, Oxford, p. 490-496, 1990.

[11] HIPEL, K. W.; FRASER, N. M. *Conflict analysis models and resolutions.* North-Holland, 1984.

[12] FRASER, N. M.; HIPEL, K. W. Solving complex conflicts. *IEEE Transactions on Systems, Man, and Cybernetics*, v. 9, n. 12, p. 805-816, 1979.

[13] KILGOUR, D. M. Anticipation and stability in two-person noncooperative games. *Dynamic models of international conflict*, p. 26-51, 1985.

[14] ZENG, D. Z.; FANG, L.; HIPEL, K. W.; KILGOUR, D. M. Policy equilibrium and generalized metarationalities for multiple decision-maker conflicts. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, v. 37, n. 4, p. 456-463, 2007.

[15] RÊGO, L. C.; DOS SANTOS, A. M. Probabilistic preferences in the graph model for conflict resolution. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, v. 45, n. 4, p. 595-608, 2015.

[16] LI, K. W.; HIPEL, K. W.; KILGOUR, D. M.; FANG, L. Preference uncertainty in the graph model for conflict resolution. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, v. 34, n. 4, p. 507-520, 2004.

[17] HIPEL, K. W.; KILGOUR, D. M.; BASHAR, M. A. Fuzzy preferences in multiple participant decision making. *Scientia Iranica*, v. 18, no. 3, p. 627-638, 2011.

[18] XU, H.; HIPEL, K. W.; KILGOUR, D. M. Matrix representation of conflicts with two decision-makers. *Systems, Man and Cybernetics,* 2007. *ISIC. IEEE International Conference on*, IEEE, 2007. p. 1764-1769.

[19] ZENG, D. Z.; FANG, L.; HIPEL, K. W.; KILGOUR, D. M. Generalized metarationalities in the graph model for conflict resolution. *Discrete applied mathematics*, v. 154, n. 16, p. 2430-2443, 2006.

[20] D. M. KILGOUR, K. W. HIPEL, and L. FANG, The graph model for conflicts. *Automatica*, v. 23, n. 1, p. 41-55, 1987.

[21] FANG, L.; HIPEL, K. W.; KILGOUR, D. M. *Interactive decision making: the graph model for conflict resolution*. John Wiley & Sons, 1993.

[22] INOHARA, T.; HIPEL, K. W. Coalition analysis in the graph model for conflict resolution. *Systems Engineering*, v. 11, n. 4, p. 343-359, 2008.

[23] KILGOUR, D. M.; HIPEL, K. W.; FANG, L.; PENG, X. Coalition analysis in group decision support. *Group Decision and Negotiation*, v. 10, n. 2, p. 159-1754, 2001.

[24] XU, H.; HIPEL, K. W.; KILGOUR, D. M. Matrix Representation of Solution Concepts in Graph Models for Two Decision-Makers with Preference Uncertainty. *Dynamics of Continuous, Discrete and Impulsive Systems*, v. 14, p. 703-707, 2007.

[25] XU, H.; HIPEL, K. W.; KILGOUR, D. M. Matrix Representation of Conflict Resolution in Multiple-Decision-Maker Graph Models with Preference Uncertainty. *Group Decision and Negotiation*, v. 20, p. 755-779, 2011.

[26] XU, H.; HIPEL, K. W.; KILGOUR, D. M. Matrix Representation of Solution Concepts in Multiple Decision Maker Graph Models. *IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans*, v. 39, n. 1, p. 96-108, 2009.

[27] XU, H.; HIPEL, K. W.; KILGOUR, D. M. Matrix representation and extension of coalition analysis in group decision support. *Computers & Mathematics with Applications*, v. 60, p. 1164-1176, 2010.

[28] LI, K.W.; HIPEL, K.W.; KILGOUR, D.M.; NOAKES, D.J. Integrating Uncertain Preferences into Status Quo Analysis with Application to an Environmental Conflict. *Group Decision and Negotiation*, v. 14, n. 6, p. 461-479, 2005.

[29] LI, K.W.; HIPEL, K.W.; KILGOUR, D.M.; FANG, L. Preference Uncertainty in the Graph Model for Conflict Resolution. *IEEE Transactions on Systems, Man, and Cybernetics, Part A*, v. 34, n. 4, p. 507-520, 2004.

[30] KUANG, H.; BASHAR, M.A.; KILGOUR, D.M.; HIPEL, K.W. Strategic Analysis of a Brownfield Revitalization Conflict Using the Grey-based Graph Model for Conflict Resolution. *EURO Journal on Decision Processes*, v. 3, n. 3-4, p. 219-248, 2015.

[31] BASHAR, M. A.; KILGOUR, D. M.; HIPEL, K. W. Fuzzy preferences in the graph model for conflict resolution. *IEEE Transactions on Fuzzy Systems*, v. 20, n. 4, p. 760-770, 2012.

[32] BASHAR, M. A.; KILGOUR, D. M.; HIPEL, K. W.; OBEIDI, A. Coalition Fuzzy Stability Analysis in the Graph Model for Conflict Resolution. *Journal of Intelligent and Fuzzy Systems*, v. 29, n. 2, p. 593-607, 2015

[33] DOS SANTOS, A. M.; RÊGO, L. C. Graph model for conflict resolution with upper and lower probabilistic preferences. *The proccedings of the 14th international conference on group decision and negotiation*, v. 1, p. 208-215, 2014.

[34] INOHARA, T.; HIPEL, K. W. Interrelationships among Noncooperative and Coalition Stability Concepts. *Journal of Systems Science and Systems Engineering*, v. 17, n. 1, p. 1-29, 2008.

[35] KASSAB, M.; HIPEL, K.; HEGAZY, T. Conflict resolution in construction disputes using the graph model. *Journal of construction engineering and management*, v. 132, n. 10, p. 1043-1052, 2006.

[36] NOAKES, D. J.; FANG, L.; HIPEL, K. W.; KILGOUR, D. M. An examination of the salmon aquaculture conflict in British Columbia using the graph model for conflict resolution. *Fisheries Management and Ecology*, v. 10, n. 3, p. 123-137, 2003.

[37] HIPEL, K. W.; KILGOUR, D. M.; FANG, L.; PENG, X. The decision support system GMCR II in negotiations over groundwater contamination. *Systems, Man, and Cybernetics, 1999. IEEE SMC'99 Conference Proceedings. 1999 IEEE International Conference on.* IEEE, 1999. p. 942-948.

[38] HIPEL, K.W.; OBEIDI, A. Trade versus the Environment: Strategic Settlement from a Systems Engineering Perspective. *Systems Engineering*, v. 8, p. 211-233, 2005.

[39] OBEIDI, A.; HIPEL, K.W.; KILGOUR, D.M. Canadian Bulk Water Exports: Analyzing the Sun Belt Conflict using the Graph Model for Conflict Resolution, *Knowledge, Technology, and Policy*, v. 14, p. 145-163, 2002.

[40] HIPEL, K.W. A Systems Engineering Approach to Conflict Resolution in Command and Control. *The International C2 Journal*, v. 5, p. 1-56, 2011.

[41] RÊGO, L. C.; G. I. A. VIEIRA. Symmetric Sequential Stability in the Graph Model for Conflict Resolution. *The* 15*th International Conference on Group Decision and Negotiation Letters*, v. 1, 231-238, 2015.

[42] RÊGO, L. C.; G. I. A. VIEIRA. Symmetric Sequential Stability in the Graph Model for Conflict Resolution with Multiple Decision Makers. *Group Decision and Negotiation*, p. 1-18, 2016.

[43] KREPS, D. *Notes on the Theory of Choice.* Westview press, 1988.

[44] RAPOPORT, A.; GUYER, M. A taxonomy of 2×2 games, 1967.

[45] LUCE, R. D. A probabilistic theory of utility. *Econometrica: Journal of the Econometric Society*, p. 193-224, 1958.

[46] ROBERTS, D. Rafferty-Alameda: The tangled history of 2 dams projects. *The Globe and Mail*, p. A4, 1990.

[47] HIPEL, K. W.; LIPING, F.; KILGOUR, D. M. Game theoretic models in engineering decision making. *Doboku Gakkai Ronbunshu*, v. 1993, n. 470, p. 1-16, 1993.

[48] KUANG, H.; BASHAR, M.A.; KILGOUR, D.M.; HIPEL, K.W. Grey-Based preference in a graph model for conflict resolution with multiple decision makers. *Systems, Man, and Cybernetics: Systems, IEEE Transactions on*, v. 45, n. 9, 1254-1267, 2015.

[49] RÊGO, L. C.; G. I. A. VIEIRA. Matrix Representation of Solution Concepts in the Graph Model for Conflict Resolution with Probabilistic Preferences, *The 15th International Conference on Group Decision and Negotiation Letters*, v. 1, 239-244, 2015.

[50] SANTOS, A. M. Aplicações de Modelos de Grafos na Análise de Conflitos e de Redes Sociais. 2014.

[51] HIPEL, K.W. Conflict Resolution. *M.K. Tolba (eds.), Our fragile World: Challenges and Opportunities for Sustainable Development(Forerunner to the Encyclopedia of Life Support Systems)*, v. 1, Oxford: EOLSS Publishers, p. 935-952, 2001.

[52] BRAMS, S. J.; WITTMAN, D. Nonmyopic equilibria in 2×2 games. *Conflict Management and Peace Science*, v. 6, n. 1, p. 39-62, 1981.

[53] VON STACKELBERG, H. *Marktform und gleichgewicht.* J. springer, 1934.

[54] HEIFETZ, A.; MEIER, M.; SCHIPPER, B. C. Interactive unawareness. *Journal of economic theory*, v. 130, n. 1, p. 78-94, 2006.

[55] RÊGO, L. C.; G. I. A. VIEIRA. Interactive Unawareness in the Graph Model for Conflict Resolution. *16th Meeting on Group Decision and Negotiation*, 2016.

[56] ALJEFRI, Y. M.; FANG, L.; HIPEL, K. Misperception of Preferences in the Graph Model for Conflict Resolution. *Group Decision and Negotiation* 2014 *GDN* 2014*: Proceedings of*

the Joint International Conference of the INFORMS GDN Section and the EURO Working Group on DSS. EWG-DSS, 2014. p. 200.

[57] OBEIDI, A.; KILGOUR, D. M.; HIPEL, K. W. Perceptual graph model systems. *Group Decision and negotiation*, v. 18, n. 3, p. 261-277, 2009.

[58] FEINBERG, Y. Subjective reasoning-games with unawareness, 2004.

[59] HALPERN, J. Y.; RÊGO, L. C. Extensive games with possibly unaware players. *Mathematical Social Sciences*, v. 70, p. 42-58, 2014.

[60] CHEN, Y.; ZHAO, X. Solution concepts of principal-agent models with unawareness of actions. *Games*, v. 4, n. 3, p. 508-531, 2013.

[61] BENNETT, P. Toward a theory of hypergames. *OMEGA*, v. 5, p. 749-751, 1977.

[62] TAKAHASHI, M.; FRASER, N.; HIPEL, K. A procedure for analyzing hypergames. *European Journal of Operational Research*, v. 18, n.1, p. 111-122, 1984.

[63] WANG, M.; HIPEL, K.; FRASER, N. Modeling misperceptions in games. *Journal of Behavioral Science*, v. 33, n. 3, p. 207-223, 1988.

[64] WANG, M.; HIPEL, K.; FRASER, N. Solution concepts in hypergames. *Applied Mathematics and Computation*, v. 34, n. 3, p. 147-171, 1989.

[65] FEINBERG, Y. Subjective reasoning-games with unawareness. *mimeo Stanford University*, 2004. https://www.gsb.stanford.edu/gsb-cmis/gsb-cmisdownload-auth/318361.

[66] HALPERN, J.; Rêgo, L. Interactive unawareness revisted. *Games and Economic Behavior*, v. 62, n. 1, p. 232-262, 2008.

APPENDIX A

List of Publications

Publications in Periodic

1 - RÊGO, L. C.; G. I. A. VIEIRA. Symmetric Sequential Stability in the Graph Model for Conflict Resolution with Multiple Decision Makers. *Group Decision and Negotiation*, p. 1-18, 2016. DOI: 10.1007/s10726-016-9520-8.

Publications in Conferences

1 - RÊGO, L. C.; G. I. A. VIEIRA. Symmetric Sequential Stability in the Graph Model for Conflict Resolution. *The 15th International Conference on Group Decision and Negotiation Letters*, v. 1, 231-238, 2015.

2 - RÊGO, L. C.; G. I. A. VIEIRA. Matrix Representation of Solution Concepts in the Graph Model for Conflict Resolution with Probabilistic Preferences, *The 15th International Conference on Group Decision and Negotiation Letters*, v. 1, 239-244, 2015.

3 - RÊGO, L. C.; G. I. A. VIEIRA. Interactive Unawareness in the Graph Model for Conflict Resolution. *16th Meeting on Group Decision and Negotiation*, 2016.

Publication Submitted to periodic

1 - RÊGO, L. C.; G. I. A. VIEIRA. Interactive Unawareness in the Graph Model for Conflict Resolution. Submitted for publication in *IEEE, Transactions on Systems, Man and Cybernetics: Systems.*

2 - RÊGO, L. C.; G. I. A. VIEIRA. Generalized Metarationalities for Multiple Decision-Maker Conflicts Revisited. Submitted for publication in *IEEE, Transactions on Systems, Man and Cybernetics: Systems.*

3 - RÊGO, L. C.; G. I. A. VIEIRA. Matrix Representations of Solutions Concepts in GMCR with Probabilistic Preferences. To be submitted for publication in *IEEE, Transactions on Systems, Man and Cybernetics: Systems.*

APPENDIX B

Proofs

- Proof of Theorem 6.4.4.

Proof: If state s is MR_2^{new} stable for DM i, then there is a set of policies P_j, $j \neq i$, and a regular metarational tree with two rounds, based on P_j, such that the result of every i-sequence with two rounds and every terminated sequence with one round is not preferable to s by DM i. Since the metarational tree is regular, every terminated sequence of 1 round either has length one if the DM i stays in s, or two, if some DM j, $j \neq i$, stays in a state $s_1 \in R_i(s) - R_i^+(s)$. Moreover, for each state $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-\{i\}}(s_1)$, determined by policies of DMs j, $j \in N - \{i\}$, and the regular metarational tree, such that $s \succeq_i s_2$ and $s \succeq_i s_3$ for every $s_3 \in R_i(s_2)$, since s is MR_2^{new} stable for DM i. Thus, s is SMR stable for DM i.

Suppose now that s is SMR stable for DM i. Let $R_i^+(s) = \{s_1, s_2, \ldots, s_W\}$. Thus, for each state s_w, $w = 1, 2, \ldots, W$, there is $s'_w \in R_{N-\{i\}}(s_w)$ such that s'_w is not preferable to s by DM i and for all $s''_w \in R_i(s'_w)$, s''_w is not preferable to s by DM i. Therefore, there is a legal sequence of moves of DMs j, $j \in N - \{i\}$, taking the conflict from state s_w to s'_w, for $w = 1, 2, \ldots, W$. Consider the shortest sequence of legal moves of DMs j, $j \in N - \{i\}$, denoted by s_x^w, that takes the conflict from s_w to some state s'_w such that s'_w is not preferable to s by DM i and for all $s''_w \in R_i(s'_w)$, s''_w is not preferable to s by DM i.

In s_x^w, there is no cycles and, moreover, for every state in t in s_x^w that appears before to s'_w, either it is preferable to s by DM i, or there must be some state at $R_i(t)$ that is preferable to s by DM i, otherwise s_x^w would not be a sequence with the shortest length that takes the conflict from s_w to any state that is not preferable to s by DM i and such that from this state DM i can not go to a state preferable to s by DM i. Define DMs j, $j \in N - \{i\}$, policies as follows:

(i) For all state $u \in S$ and DM j, $j \in N - \{i\}$, if the pair (u, j) does not appear in any of the sequences s_x^w, for $w = 1, 2, \ldots, W$, then $P_j(u) = u$;

(ii) Let w^* be the smallest w value such that the pair (u, j) appears in the sequence s_x^w. Since $s_x^{w^*}$ does not contain cycles, then there is only one state $t \in S$ such that (u, j, t) is a triplet in $s_x^{w^*}$, then define $P_j(u) = t$.

Consider the regular metarational tree based on $P_j, j \neq i$, consisting of sequences of the form (and their initial parts) (s, i, s_x^w, i, s''_w), for $s''_w \in R_i(s'_w) \cup \{s'_w\}$ and $w = 1, 2, \ldots, W$, along with sequences of the form (and their initial parts) (s, i, u, j, u), for some DM j, $j \in N - \{i\}$, and all $u \in R_i(s) \cap (R_i^+(s))^c$ such that u does not appear in any of the sequences s_x^w, for $w = 1, 2, \ldots, W$, and sequences of the form (and their initial parts) $(s, i, u_x^{w^*}, i, s''_{w^*})$, where $s''_{w^*} \in R_i(s'_{w^*}) \cup \{s'_{w^*}\}$ and w^* is the smallest w value such that the pair (u, j) appears in the sequence s_x^w and $u_x^{w^*}$ is the subsequence of $s_x^{w^*}$ that starts in u and ends at the same state, s'_{w^*}, as $s_x^{w^*}$, for all $u \in R_i(s) \cap (R_i^+(s))^c$ such that u appears in at least one of the sequences s_x^k.

Therefore, there is a regular metarational tree of 2 rounds based on P_j, $j \neq i$, for DM i starting at state s such that the result of every i-sequence of 2 rounds and every terminated sequence of 1 round is not preferable to s by DM i. Therefore, s is MR_2^{new} stable for DM i. □

- Proof of Theorem 6.4.8.

Proof: Suppose by way of contradiction that state s is not $\overline{MR}_l^{new}$ DM i, for some

$l \in \{1, 2, \ldots, r-1\}$. Then, given any set of policies P_j, for all DMs j, $j \in N - \{i\}$, and all metarational trees of l rounds that starts at s based on P_j, there exists (a) an $\bar{i}$-sequence of l rounds, which is not the initial part of another $\bar{i}$-sequence of l rounds, or (b) a terminated i-sequence of l-rounds or (c) a terminated sequence of less than l rounds that ends in a state s_1 such that $s_1 \succ_i s$. In cases (b) and (c) it is evident that there is a terminated sequence of less than r rounds which ends in a preferable state to the s to DM i. In case (a), there are 2 possibilities: (a1) $n = 2$ and DM i must move in the final state of the $\bar{i}$-sequence of l rounds, (a2) the $\bar{i}$-sequence of l rounds is terminated. In case (a2), there is a terminated sequence of less than r rounds which terminates in a state preferable to s by DM i. Finally, in case (a1) as DM i always has the option to stay in the final state of the $\bar{i}$-sequence, then there is an i-sequence of r rounds or less whose result is preferable to s by DM i. Therefore, s is not $\overline{MR}_r^{new}$ stable for DM i. □

- Proof of Theorem 6.4.9.

Proof:

If a state s is CMR_r^{new} stable for DM i, then there is a set of credible policies P_j^c, for all $j \in N - \{i\}$, and a credible regular metarational tree, denoted by $\mathcal{A}_r$, based on P_j^c, $j \neq i$, of r rounds such that the result of every credible i-sequence of r rounds and of every credible terminated sequence with less than r rounds is not preferable to s by DM i. Consider the set of policies $P_j^{\#}$, for all $j \in N - \{i\}$, defined in the following way: $P_j^{\#}(t) = P_j^c(t)$ if $t \succ_i s$ and $P_j^{\#}(t) = t$, otherwise. Let $\mathcal{A}_r^{\#}$ be the regular metarational tree based on $P_j^{\#}$, $j \neq i$, for DM i starting at state s such that:

(*i*) If $s_x \in \mathcal{A}_r$ is a sequence such that for all states t, where some DM j, $j \in N - \{i\}$, moves in s_x, t is preferable to s by DM i, then $s_x \in \mathcal{A}_r^{\#}$;

(*ii*) If $s_x = (s, i, s_1, j_1, s_2, j_2, \ldots, s_m) \in \mathcal{A}_r$ and there is some s_w, for $k = 1, 2, \ldots, m-1$ such that $j_w \neq i$ and $s \succeq_i s_w$, then let w^* be the smallest w value for which s_w satisfy these conditions. Thus, we have that the sequence $(s, i, s_1, j_1, \ldots, j_{w^*-1}, s_{w^*}, j_{w^*}, s_{w^*}) \in \mathcal{A}_r^{\#}$, for some $j_{w^*} \neq j_{w^*-1}$.

Note that all sequences in $\mathcal{A}_r^{\#}$ that are not in $\mathcal{A}_r$ end in some state s_{w^*} that is not preferable to s by DM i. Note also that the result of every i-sequence of r rounds and of every terminated sequence of less than r rounds in $\mathcal{A}_r^{\#} \cap \mathcal{A}_r$ is not preferable to s by DM i, since s is CMR_r^{new} stable for DM i. Therefore, s is MR_r^{new} stable for DM i. $\square$

- Proof of Theorem 6.4.10.

Proof: If state s is $\overline{CMR}_r^{new}$ stable for DM i, then there is a set of credible policies P_j^c, for all $j \in N - \{i\}$, and a credible metarational tree, denoted by $\mathcal{A}_r$, based on P_j^c, $j \neq i$, of r rounds such that the result of every credible $\bar{i}$-sequence of r rounds, which is not an initial part of another credible $\bar{i}$-sequence of r rounds, of every credible terminated i-sequence with r rounds and of every credible terminated sequence with less than r rounds is not preferable to s by DM i. Consider the set of policies $P_j^{\#}$, for all $j \in N - \{i\}$, defined in the following way: $P_j^{\#}(t) = P_j^c(t)$ if $t \succ_i s$ and $P_j^{\#}(t) = t$, otherwise. Let $\mathcal{A}_r^{\#}$ be the metarational tree based on $P_j^{\#}$, $j \neq i$, for DM i starting at state s such that:

(i) If $s_x \in \mathcal{A}_r$ is a sequence such that for all states t, in which some DM j, $j \in N - \{i\}$, moves in s_x, t is preferable to s by DM i, then $s_x \in \mathcal{A}_r^{\#}$;

(ii) If $s_x = (s, i, s_1, j_1, s_2, j_2, \ldots, s_m) \in \mathcal{A}_r$ and there is some s_w, for $w = 1, 2, \ldots, m-1$ such that $j_w \neq i$ and $s \succeq_i s_w$, then let w^* be the smallest w value for which s_w satisfy these conditions. Thus, we have that the sequence $(s, i, s_1, j_1, \ldots, j_{w^*-1}, s_{w^*}, j_{w^*}, s_{w^*}) \in \mathcal{A}_r^{\#}$, for some $j_{w^*} \neq j_{w^*-1}$.

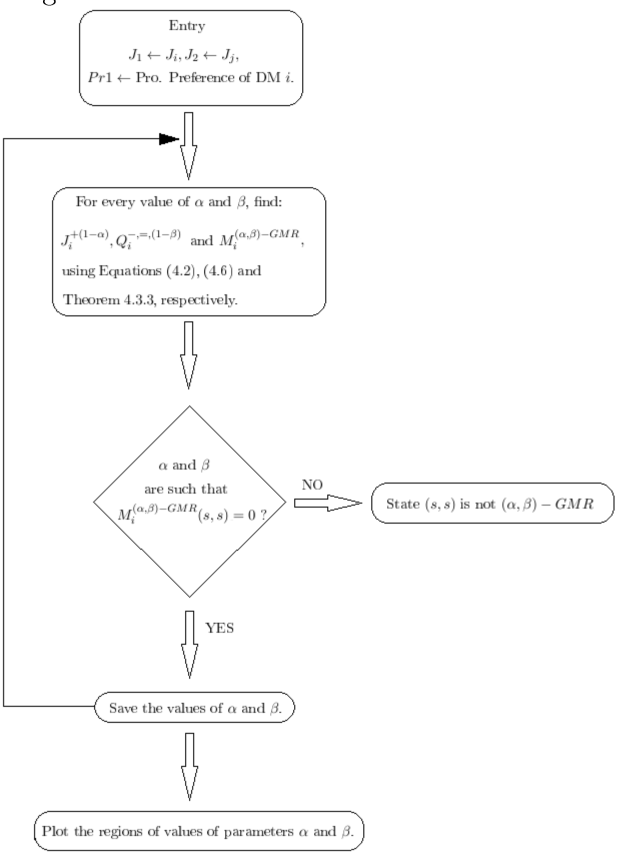
Note that all sequences in $\mathcal{A}_r^{\#}$ that are not in $\mathcal{A}_r$ end in some state s_{w^*} that is not preferable to s by DM i. Note also that the result of every $\bar{i}$-sequence of r rounds, which is not an initial part of another $\bar{i}$-sequence of r rounds, of every terminated i-sequence with r rounds and of every terminated sequence with less than r rounds in $\mathcal{A}_r^{\#} \cap \mathcal{A}_r$ is not preferable to s by DM i, since s is $\overline{CMR}_r^{new}$ stable for DM i. Therefore, s is $\overline{MR}_r^{new}$ stable for DM i. $\square$

APPENDIX C

Computational Codes

C.1 GMR Code

Figure C.1: Flowchart of GMR Code



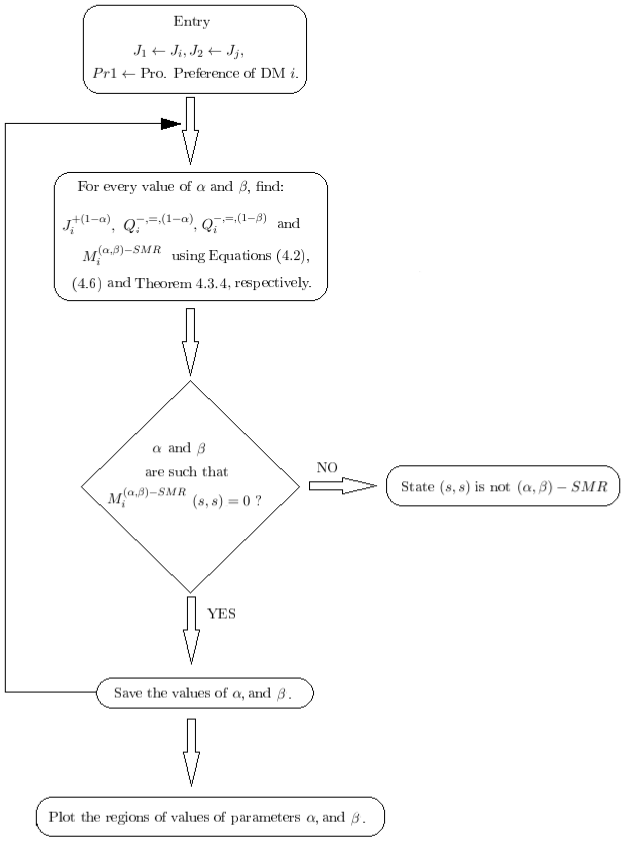
```
res <- function(J1, J2, P1, a, b)
  {
     N <- sqrt(length(J))
     J1 <- matrix(J, nrow=N, ncol=N, byrow=TRUE)
     J2 <- matrix(J2, nrow=N, ncol=N, byrow=TRUE)
     J_I=Q <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
     P1 <- matrix(P, nrow=N, ncol=N, byrow=TRUE)
     Y <- matrix(1, nrow=N, ncol=N, byrow = TRUE)



     for(i in 1:N)
      {
      for(j in 1:N)
        {
        if(J1[i,j]==1 && P1[j, i] > 1-a) {J_I[i,j] <- 1}
        else{J_I[i,j] <- 0}
        if(P1[j, i] > 1-b) {Q[i,j] <- 1}
        else{Q[i,j] <- 0}
        }
      }
     SINAL <- sign(J2%*%t(Y-Q))
     M <- diag(J_I %*%( Y- SINAL))
     return(M)
  }

gmr <- function(J1, J2, P1, P2, a, b, state, dm)
```

```
# Função que plota os pontos que o estado "state" é GMR estável para o
# DM "dm"
# J1: Matriz Acessibilidade do DM 1 (J_i)
# J2: Matriz Acessibilidade do DM 2 (J_j)
# P1: Matriz de Preferência do DM 1 (Pr_i)
# P2: Matriz de Preferência do DM 2 (Pr_j)
# a: valores de alpha para testar
# b: valores de beta para testar
# state: estado para testar
# dm: DM focal


{
   N <- sqrt(length(J1))
   Na <- length(a)
   Nb <- length(b)
   alpha <- matrix()
   beta <- matrix()

   if(state>=1 && state <=N && dm >= 1 && dm <= 2) {
     if(dm==1) {
     for(i in 1:Na){
      for(j in 1:Nb){
         R <- res(J1,J2,P1,a[i],b[j])
         if(R[state]==0) {
           alpha=rbind(alpha,a[i])
           beta=rbind(beta,b[j])
         }
```

```
    }
   }
   plot(alpha,beta,type= "p", main = paste("State", state, "(alpha,beta)-GMR Stability
Region for DM 1"), col = "black", cex = .8, pch = 15, lwd = 1)
  }
  else{
   for(i in 1:Na){
    for(j in 1:Nb){
      R <- res(J2,J1,P2,a[i],b[j])
      if(R[state]==0) {
       alpha=rbind(alpha,a[i])
       beta=rbind(beta,b[j])
      }
    }
   }
   plot(alpha,beta,type= "p", main = paste("State", state, "(alpha,beta)-GMR Stability
Region for DM 2"), col = "black", cex = .8, , pch = 15, lwd = 1)
  }
  }
  else{
   print("Erro na Entrada")}
  }
```

C.2 SMR Code

Figure C.2: Flowchart of SMR Code



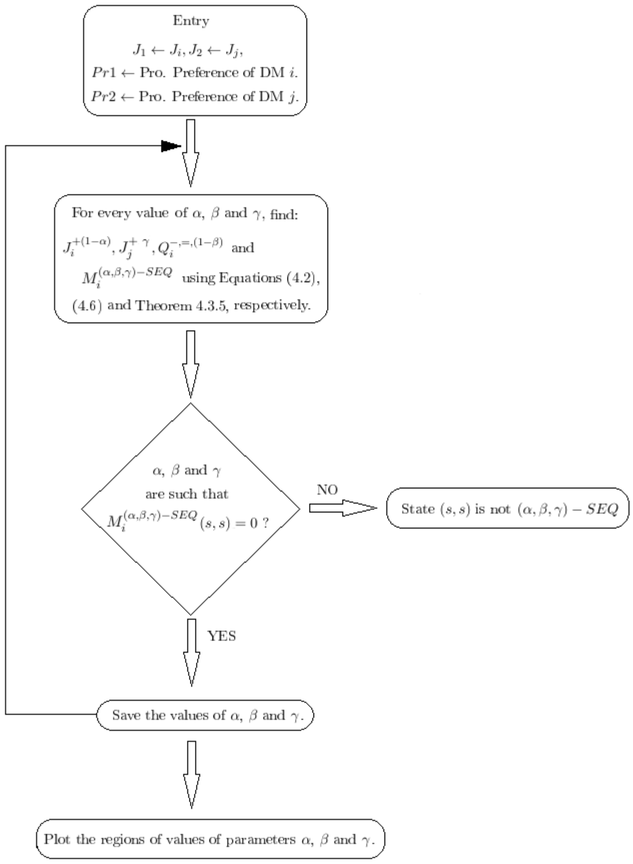
```
res <- function(J1, J2, P1, a, b)

{
N <- sqrt(length(J))
J1 <- matrix(J, nrow=N, ncol=N, byrow=TRUE) # matriz de acessibilidade do DM i
J2 <- matrix(J2, nrow=N, ncol=N, byrow=TRUE) # matriz de acessibilidade do DM j
J_I=Q_1 = Q_2 <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
P1 <- matrix(P, nrow=N, ncol=N, byrow=TRUE) # matriz de probabilidades
Y <- matrix(1, nrow=N, ncol=N, byrow = TRUE) # matriz de uns


for(i in 1:N)
  {
   for(j in 1:N)
    {
    if(J1[i,j]==1 && P1[j, i] > 1-a) {J_I[i,j] <- 1}
    else{J_I[i,j] <- 0}
    if(P1[j, i] > 1-a) {Q_1[i,j] <- 1}
    else{Q_1[i,j] <- 0}
    if(P1[j, i] > 1-b) {Q_2[i,j] <- 1}
    else{Q_2[i,j] <- 0}
    }
  }
SINALa <- sign(J1%*%t(Q_1)) # ultima matriz sinal do teorema
SINALb <- sign(J2%*%(t(Y-Q_2)*(Y-SINALa)))
M <- diag(J_I %*%( Y- SINALb))
 return(M)
}
```

```
smr <- function(J1, J2, P1, P2, a, b, state, dm)
    # Função plota os pontos que o estado "state" SMR estável para o
    # DM "dm"
    # J1: Matriz Acessibilidade do DM 1
    # J2: Matriz Acessibilidade do DM 2
    # P1: Matriz de Preferência do DM 1
    # P2: Matriz de Preferência do DM 2
    # a: valores de alpha para testar
    # b: valores de beta para testar
    # state: estado para testar
    # dm: DM focal
{

    {
        N <- sqrt(length(J1))
        Na <- length(a)
        Nb <- length(b)
        alpha <- matrix()
        beta <- matrix()

        if(state>=1 && state <=N && dm >= 1 && dm <= 2) {
          if(dm==1)
          {
            for(i in 1:Na) {
            for(j in 1:Nb) {
              R <- res(J1,J2,P1,a[i],b[j])
              if(R[state]==0) {
```

```
        alpha=rbind(alpha,a[i])
        beta=rbind(beta,b[j])


          }
      }
     }
     plot(alpha,beta,type= "p", xlim=c(0,1), ylim=c(0,1), main = paste("State", state,
"(alpha,beta)-SMR Stability
Region for DM 1"), col = "black", cex = .8, pch = 15, lwd = 1)
   }
   else{
     for(i in 1:Na){
       for(j in 1:Nb){
         R <- res(J2,J1,P2,a[i],b[j])
         if(R[state]==0) {
           alpha=rbind(alpha,a[i])
           beta=rbind(beta,b[j])
         }
      }
     }
    plot(alpha,beta,type= "p", xlim=c(0,1), ylim=c(0,1), main = paste("State", state, "(alpha,beta)-
SMR Stability
Region for DM 2"), col = "black", cex = .8, , pch = 15, lwd = 1)
   }
   }
   else{
    print("Erro na Entrada")}
   }
```

C.3 *SEQ* Code

Figure C.3: Flowchart of *SEQ* Code



```
require(scatterplot3d)

res <- function( J1, K, P1, S, a, b, d)

    {
     N <- sqrt(length(J))
     J1 <- matrix(J, nrow=N, ncol=N, byrow=TRUE)
     K <- matrix(K, nrow=N, ncol=N, byrow=TRUE)
     J_I <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
     Q <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
     J_J <- matrix(NA, nrow=N, ncol=N, byrow=TRUE) #matriz de melhoria do DM j
     S <- matrix(S, nrow=N, ncol=N, byrow=TRUE) #matriz de probabilidade do DM j
     P1 <- matrix(P, nrow=N, ncol=N, byrow=TRUE) #matriz de probabilidade do DM i
     Y <- matrix(1, nrow=N, ncol=N, byrow = TRUE)



    for(i in 1:N)
      {
      for(j in 1:N)
        {
         if(J1[i,j]==1 && P1[j, i] > 1-a) {J_I[i,j] <- 1}
         else{J_I[i,j] <- 0}
         if(K[i,j]==1 && S[j, i] > d) {J_J[i,j] <- 1}
         else{J_J[i,j] <- 0}
         if(P[j, i] > 1-b) {Q[i,j] <- 1}
         else{Q[i,j] <- 0}
        }
```

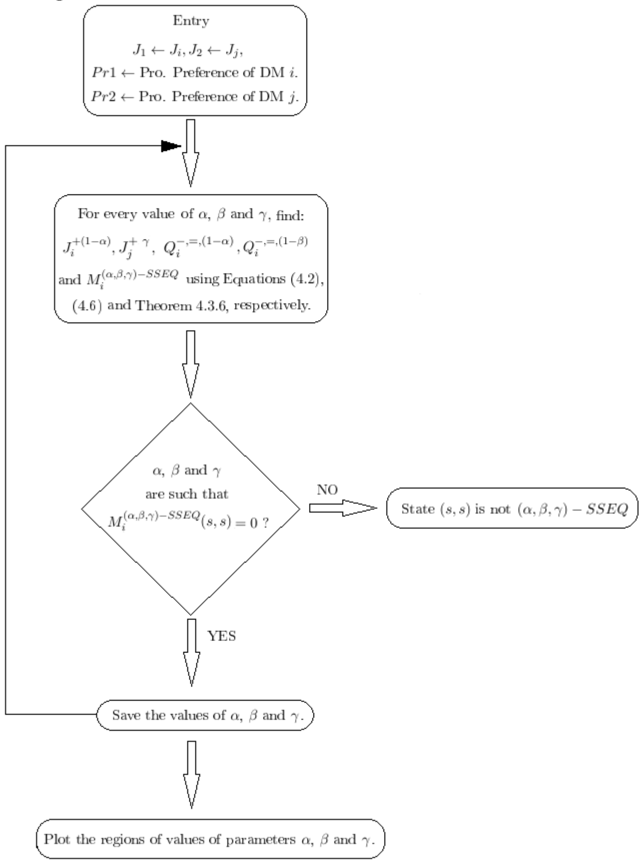
```
    }
  SINAL <- sign(J_J%*%t(Y-Q))
  M <- diag(J_I %*%( Y-SINAL))
  return(M)
}

SEQ <- function(J1, J2, P1, P2, a, b, d, state, dm)

  # Função plota os pontos que o estado "state" SEQ estável para o
  # DM "dm"
  # J1: Matriz Acessibilidade do DM 1
  # J2: Matriz Acessibilidade do DM 2
  # P1: Matriz de Preferência do DM 1
  # P2: Matriz de Preferência do DM 2
  # a: valores de alpha para testar
  # b: valores de beta para testar
  # d: valores de gama para testar
  # state: estado para testar
  # dm: DM focal


  {
    N <- sqrt(length(J1))
    Na <- length(a)
    Nb <- length(b)
    Nd <- length(d)
    alpha <- matrix()
    beta <- matrix()
```

```
gama <-matrix()

if(state>=1 && state <=N && dm >= 1 && dm <= 2) {
 if(dm==1) {
 for(i in 1:Na){
  for(j in 1:Nb) {
   for(k in 1:Nd) {
      R <- res(J1,J2,P1,P2,a[i],b[j],d[k])
      if(R[state]==0) {
      alpha=rbind(alpha,a[i])
      beta=rbind(beta,b[j])
      gama=rbind(gama,d[k])
     }
    }
   }
 }
  scatterplot3d(alpha, beta, gama, highlight.3d=TRUE, col.axis="blue", xlim=c(0,1),
ylim=c(0,1), zlim=c(0,1),
col.grid="lightblue", main = paste("State", state, "(alpha,beta,gama)-SEQ Stability Region for
DM 1"),
pch=20)
}
else{
 for(i in 1:Na){
  for(j in 1:Nb) {
   for(k in 1:Nd) {
      R <- res(J2,J1,P2,P1,a[i],b[j],d[k])
      if(R[state]==0) {
```

```
        alpha=rbind(alpha,a[i])
        beta=rbind(beta,b[j])
        gama=rbind(gama,d[k])
       }
      }
     }
    }

        scatterplot3d(alpha, beta, gama, highlight.3d=TRUE, col.axis="blue", xlim=c(0,1),
ylim=c(0,1), zlim=c(0,1),
col.grid="lightblue", main = paste("State", state, "(alpha,beta,gama)-SEQ Stability Region for
DM 2"),
pch=20)
    }
    }
    else{
     print("Erro na Entrada")}
    }
```

C.4 *SSEQ* Code

Figure C.4: Flowchart of *SSEQ* Code



```
require(scatterplot3d)


res <- function( J, K, P, S, a, b, d)
  {
     N <- sqrt(length(J))
     J <- matrix(J, nrow=N, ncol=N, byrow=TRUE)
     K <- matrix(K, nrow=N, ncol=N, byrow=TRUE)
     J_I=Q_1 = Q_2 <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
     Q <- matrix(NA, nrow=N, ncol=N, byrow=TRUE)
     J_J <- matrix(NA, nrow=N, ncol=N, byrow=TRUE) #matriz de melhoria do DM j
     S <- matrix(S, nrow=N, ncol=N, byrow=TRUE) #matriz de probabilidade do DM j
     P <- matrix(P, nrow=N, ncol=N, byrow=TRUE) #matriz de probabilidade do DM i
     Y <- matrix(1, nrow=N, ncol=N, byrow = TRUE)




    for(i in 1:N)
     {
     for(j in 1:N)
       {
        if(J[i,j]==1 && P[j, i] > 1-a) {J_I[i,j] <- 1}
        else{J_I[i,j] <- 0}
        if(K[i,j]==1 && S[j, i] > d) {J_J[i,j] <- 1}
        else{J_J[i,j] <- 0}
        if(P[j, i] > 1-a) {Q[i,j] <- 1}
        else{Q[i,j] <- 0}
        if(K[i,j]==1 && S[j, i] > d) {J_J[i,j] <- 1}
        else{J_J[i,j] <- 0 }
```

```
          if(P[j, i] > 1-b) {Q_2[i,j] <- 1}
          else{Q_2[i,j] <- 0}


            }
        }
      SINALa <- sign(J%*%t(Q_1)) # ultima matriz sinal do teorema
       SINALb <- sign(J_J%*%(t(Y-Q_2)*(Y-SINALa)))
      M <- diag(J_I %*%( Y-SINAL))
      return(M)
  }

sseq <- function(J1, J2, P1, P2, a, b, d, state, dm)

   # Função plota os pontos que o estado "state" SSEQ estável para o
   # DM "dm"
   # J1: Matriz Acessibilidade do DM 1
   # J2: Matriz Acessibilidade do DM 2
   # P1: Matriz de Preferência do DM 1
   # P2: Matriz de Preferência do DM 2
   # a: valores de alpha para testar
   # b: valores de beta para testar
   # d: valores de gama para testar
   # state: estado para testar
   # dm: DM focal



   {
      N <- sqrt(length(J1))
```

```
Na <- length(a)
Nb <- length(b)
Nd <- length(d)
alpha <- matrix()
beta <- matrix()
gama <-matrix()


if(state>=1 && state <=N && dm >= 1 && dm <= 2) {
 if(dm==1) {
 for(i in 1:Na){
  for(j in 1:Nb) {
   for(k in 1:Nd) {
      R <- res(J1,J2,P1,P2,a[i],b[j],d[k])
      if(R[state]==0) {
      alpha=rbind(alpha,a[i])
      beta=rbind(beta,b[j])
      gama=rbind(gama,d[k])
     }
    }
   }
 }
  scatterplot3d(alpha, beta, gama, angle=125, highlight.3d=TRUE, col.axis="blue", xlim=c(0,1),
ylim=c(0,1), zlim=c(0,1),
col.grid="lightblue", main = paste("State", state, "(alpha,beta,gama)-SSEQ Stability Region
for DM 1"),
pch=20)
}
```

```
else{
 for(i in 1:Na){
  for(j in 1:Nb) {
   for(k in 1:Nd) {
      R <- res(J2,J1,P2,P1,a[i],b[j],d[k])
      if(R[state]==0) {
      alpha=rbind(alpha,a[i])
      beta=rbind(beta,b[j])
      gama=rbind(gama,d[k])
     }
    }
   }
 }

      scatterplot3d(alpha, beta, gama, angle=125, highlight.3d=TRUE, col.axis="blue",
xlim=c(0,1), ylim=c(0,1), zlim=c(0,1),
col.grid="lightblue", main = paste("State", state, "(alpha,beta,gama)-SSEQ Stability Region
for DM 2"),
pch=20)
}
}
else{
 print("Erro na Entrada")}
}
```