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RENATA ROJAS GUERRA

SOME GENERALIZED BURR XII DISTRIBUTIONS WITH APPLICATIONS TO INCOME AND LIFETIME DATA

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RENATA ROJAS GUERRA

SOME GENERALIZED BURR XII DISTRIBUTIONS WITH APPLICATIONS TO INCOME AND LIFETIME DATA

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Abstract

The proposal of new continuous distributions by adding one or more shape parameter(s) to baseline models has attract researchers of many areas. Several generators have been studied in recent years that can be described as special cases of the transformed-transformer (T-X) method. The gamma generalized families ("*gamma-G*" for short), called *Zografos-Balakrishnan-G* (Zografos and Balakrishnan, 2009) and *Ristić-Balakrishnan-G* (Ristić and Balakrishnan, 2012), are important univariate distributions sub-families of the T-X generator. They are generated by gamma random variables. It was found that eighteen distributions have been studied as baselines in the gamma generalized families. Another known family of univariate distributions is generated by extending the Weibull model applied to the odds ratio $G(x)/[1 - G(x)]$, called the *Weibull-G* (Bourguignon *et al.*, 2014). It was found that seven distributions have been studied in the context of the *Weibull-G* family. The *logistic-X* (Tahir *et al.*, 2016a) is also a sub-family on the T-X generator that was recently introduced in the literature. Considering this approach, we discuss in this thesis the *gamma-G, logistic-X and Weibull-G* families by taking the Burr XII distribution as baseline. We present density expansions, quantile functions, ordinary and incomplete moments, generating functions, estimation of the model parameters by maximum likelihood and provide applications to income and lifetime real data sets for the proposed distributions. We show that the new distributions yield good adjustments for the considered data sets and that they can be used effectively to obtain better fits than other classical models and Burr XII generated families.

Keywords: Burr XII distribution. Gamma-G family. Logistic-X family. Weibull-G family.

Resumo

A ideia de obter novas distribuições contínuas adicionando um ou mais parâmetros a uma distribuição de base (*baseline*) tem atraído pesquisadores de diversas áreas. Muitos geradores de distribuições têm sido estudados nos últimos anos. Diversos deles podem ser descritos como casos especias da família de geradores transformed-transformer (T-X). Neste contexto, as famílias gama generalizadas (*gamma-G*), denominadas *Zografos-Balakrishnan-G* (Zografos and Balakrishnan, 2009) e *Ristić-Balakrishnan-G* (Ristić and Balakrishnan, 2012), são importantes sub-famílias de distribuições univariadas do gerador T-X, as quais são obtidas a partir de variáveis aleatórias com distribuição gama. Na literatura foi possível encontrar dezoito distribuições que foram estudadas como baselines nestas famílias. Outra conhecida sub-família do gerador T-X é gerada a partir da distribuição de Weibull aplicada à razão de chances $G(x)/[1 - G(x)]$, denominada família *Weibull-G*. Na literatura foi possível encontrar sete distribuições estudadas como baselines na família *Weibull-G* (Bourguignon *et al.*, 2014). A família *logistic-X* (Tahir *et al.*, 2016a) é também uma sub-família do gerador T-X recentemente introduzida na literatura. Nesta tese serão discutidas as famílias *gamma-G, logistic-X e Weibull-G*, considerando a distribuição Burr XII como *baseline*. Serão apresentadas expansões para a função de densidade, a função quantílica, momentos ordinários incompletos, funções geradoras de momentos e estimação por máxima verossimilhança. Também são realizadas aplicações das novas distribuições a conjuntos de dados reais de renda e de análise de sobrevivência. As distribuições propostas obtiveram ajustes adequados para as bases de dados consideradas, podendo ser utilizadas como alternativas efetivas a outros modelos clássicos e, também, a outras generalizações da distribuição Burr XII.

Palavras-chave: Distribuição Burr XII. Família gamma-G. Família logistic-X. Família Weibull-G.

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List of Abbreviations and Acronyms

AIC	Akaike information criteria
BIC	Bayesian information criteria
BXII	Burr XII distribution
BBXII	Beta Burr XII distribution
BEBXII	Beta exponentiated Burr XII distribution
CAIC	consistent Akaike information criteria
cdf	cumulative distribution function
EBXII	exponentiated Burr XII distribution
ExBXII	extended Burr XII distribution
exp-G	exponentiated-G family
EW	exponentiated Weibull distribution
GPW	generalized power Weibull distribution
HQIC	Hannan-Quinn information criteria
hrf	hazard rate function
KwBXII	Kumaraswamy Burr XII distribution
KS	Kolmogorov-Smirnov statistic
LBXII	logistic Burr XII distribution
LL	log-logistic distribution
mgf	moment generating function
McBXII	McDonald Burr XII distribution
MLE	maximum likelihood estimator
pdf	probability density function
qf	quantile function
RBXII	Ristíc-Balakrishnan Burr XII distribution
RMSEs	root mean squared errors
SD	standard deviation
TBXII	Transmuted Burr XII distribution
ZBXII	Zografos-Balakrishnan Burr XII distribution
W	Weibull distribution

WBXII	Weibull Burr XII distribution
ZBXII	Zografos-Balakrishnan Burr XII distribution

Contents

Chapter 1

Introduction

New distributions can often result by introducing one or more additional shape parameters to an existing lifetime distribution (say, a baseline model). They have been defined as the generalized (or generated) G-classes of distributions. According to Tahir and Nadarajah (2015), there are some reasons why the G-classes attract researchers of several areas. One reason might be the computational refinement of symbolic and numerical programming software. It becomes easier to derive some important mathematical and statistical properties. In addition, the structure of the new generators also allows exploring its tail properties. Another reason is that the extra parameters inducted from the G baseline models have presented evidence to improve the quality of fit. Pescim *et al.* (2010) also showed that the G-classes might provide better fits than the common distributions for skewed data.

Several generators have been studied in recent years. We refer the reader to Lee *et al.* (2013) for a review of some generated methods and Tahir and Nadarajah (2015) for a more updated and detailed survey on well-established and widely used generalized classes of continuous univariate distributions. Many of these classes can be defined as special cases of the transformed-transformer (T-X) method introduced by Alzaatreh *et al.* (2013). This technique allows deriving families of distributions by using any probability density function (pdf) as a generator.

Let $r(t)$ be the pdf of a random variable $T \in [a, b]$ for $-\infty < a < b < \infty$. Let $G(x)$ be the baseline cumulative distribution function (cdf) of a random variable X such that $W[G(x)]$ satisfies the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is differentiable and monotonically non-decreasing;
- $W[G(x)] \to a$ when $x \to -\infty$ and $W[G(x)] \to b$ when $x \to +\infty$.

The T-X family cdf is defined by

$$F(x) = \int_a^{W[G(x)]} r(t)dt. \tag{1.1}$$

The pdf corresponding to (1.1) is given by

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r\{W[G(x)]\}.$$

The *T*-*X* family of distributions can be classified into subfamilies. One subfamily has the same X distribution but different T distributions, other has the same T distribution but different X distributions. Different functions $W(\cdot)$, such as $W(x) = -\log(1-x)$, $x/(1-x)$, $\log(x/1-x)$, $\log[-\log(x)]$ for $x \in (0,1)$, will also define different subfamilies.

For example, consider $W(x) = G(x)$. If T is a beta random variable, we have the beta-generated family pionnered by Eugene *et al.* (2002). The Kumaraswamy generalized family (Cordeiro and de Castro, 2011) follows when T is a Kumaraswamy random variable. The *gamma-G* families (Zografos and Balakrishnan, 2009; Ristić and Balakrishnan, 2012), *log-gamma* generated families (Amini *et al.*, 2014), McDonald-generalized family (Alexander *et al.*, 2012), *Weibull-G* family (Bourguignon *et al.*, 2014) and exponentiated half-logistic-G (Cordeiro *et al.*, 2014a) family are also *T*-*X* special models.

The gamma generalized family, called the *gamma-G* is an important class of univariate distributions generated by gamma random variables. Zografos and Balakrishnan (2009) proposed the first type *gamma-G* family (ZB-G for short). They defined the ZB-G family with pdf $f(x)$ and cdf $F(x)$ by

$$f(x) = \frac{g(x)}{\Gamma(a)} \left\{-\log\left[1-G(x)\right]\right\}^{a-1}, \tag{1.2}$$

and

$$F(x) = \frac{\gamma\left(a, -\log\left[1-G(x)\right]\right)}{\Gamma(a)},$$

respectively, where $G(x)$ is any baseline cdf, $g(x) = \mathrm{d}G(x)/\mathrm{d}x$, $x \in I\!R$, $\gamma(a,z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function, $\Gamma(.)$ denotes the gamma function and $a > 0$ is a shape parameter. For $a = 1$, equation (1.2) reduces to the baseline pdf. The hazard rate function (hrf) corresponding to (1.2) becomes

$$h(x) = \frac{\{-\log[1 - G(x)]\}^{a-1}\, g(x)}{\Gamma(a, -\log[1 - G(x)])},$$

were $\Gamma(a,z) = \int_z^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ is the upper incomplete gamma function.

Zografos and Balakrishnan (2009) presented several motivations for the *gamma-G* family, which are also valid for the ZBXII distribution: if $X_{L(1)}, ..., X_{L(n)}$ are lower record values from a sequence of independent random variables with common pdf $g(\cdot)$, then the pdf of the nth lower record value has the form (1.2); if Z is a gamma random variable with unit scale parameter and shape parameter $a > 0$, then $X = F^{-1}(\exp(Z))$ has the pdf (1.2); and, if Z is a log-gamma random variable, then $X = F^{-1}(\exp\{-\exp(Z)\})$ has the pdf (1.2).

A different type *gamma-G* family was introduced by Ristić and Balakrishnan (2012), so-called the Ristić-Balakrishnan-G (RB-G) family, having pdf, cdf and hrf given by

$$f(x) = \frac{g(x)}{\Gamma(a)}\left\{-\log\left[G(x)\right]\right\}^{a-1}, \tag{1.3}$$

$$F(x) = 1 - \frac{\gamma\left(a, -\log\left[G(x)\right]\right)}{\Gamma(a)}, \tag{1.4}$$

and

$$h(x) = \frac{\{-\log[G(x)]\}^{a-1}\, g(x)}{\Gamma(a, -\log[G(x)])}$$

respectively. Note that for $a = 1$, equation (1.3) reduces to the baseline pdf. The ZB-G and RB-G families have the same parameters of the baseline cdf $G(x)$ plus an additional shape parameter $a > 0$. Letting $r(t)$ be the gamma distribution, we have that both ZB-G and RB-G families are *T-X* special models with, respectively, $W[G(x)] = -\log[1 - G(x)]$ and $W[G(x)] = -\log[G(x)]$.

Castellares and Lemonte (2016) pointed out the relationship between the ZB-G and RB-G families by discussing some similarities and differences between the gamma dual Weibull model and gamma exponentiated Weibull model. The authors proved that if the baseline $G(x)$ is absolutely continuous and symmetric about the origin, then X follows the ZB-G distribution and $-X$ has the RB-G distribution.

It was found that eighteen distributions have been studied as baselines in the *gamma-G* families. The list of authors contributions is presented in Table 1.1.

Table 1.1: Contributed work on the Gamma-G family of distributions.

Distribution	Author
Gamma exponentiated exponential	Ristić and Balakrishnan (2012)
Gamma Exponentiated Weibull	Pinho *et al.* (2012)
	Castellares and Lemonte (2015)
Gamma Pareto	Alzaatreh *et al.* (2012)
Gamma uniform	Torabi and Hedesh (2012)
Gamma extended Fréchet	Silva *et al.* (2013)
Gamma half normal	Alzaatreh and Knight (2013)
Gamma Dagum	Oluyede *et al.* (2013)
Gamma log-logistic	Ramos *et al.* (2013)
Gamma extended Weibull	Nascimento *et al.* (2014)
Gamma linear failure rate	Cordeiro *et al.* (2014b)
Gamma logistic	Alzaatreh *et al.* (2014b)
Gamma normal	Alzaatreh *et al.* (2014a)
	Lima *et al.* (2015)
Gamma Weibull Poisson	Percontini *et al.* (2014)
Gamma Birnbaum Saunders	Cordeiro *et al.* (2015b)
Gamma Lindley	Lima (2015)
Gamma Lomax	Cordeiro *et al.* (2015c)
Gamma Modified Weibull	Cordeiro *et al.* (2015a)
Gamma Nadarajah Haghighi	Bourguignon *et al.* (2015)
	Ortega *et al.* (2015)

Bourguignon *et al.* (2014) pioneered a family of univariate distributions generated by extending the Weibull model applied to the odds ratio $G(x)/[1-G(x)]$. For any baseline cdf $G(x)$, they defined the *Weibull-G* family for $x \in \mathcal{D} \subseteq \mathbb{R}$ with pdf and cdf given by

$$f(x) = \alpha\,\beta\,g(x)\frac{G(x)^{\beta-1}}{\overline{G}(x)^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x}{\overline{G}(x)}\right]^{\beta}\right\}. \tag{1.5}$$

and

$$F(x) = \int_0^{\frac{G(x)}{1-G(x)}} \alpha\,\beta\,t^{\beta-1}\mathrm{e}^{-\alpha\,t^{\beta}}\mathrm{d}t = 1-\exp\left\{-\alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right\}, \tag{1.6}$$

respectively. The hrf corresponding to (1.5) is given by

$$h(x) = \frac{\alpha\,\beta\,g(x)G(x)^{\beta-1}}{\overline{G}(x)^{\beta+1}} = \frac{\alpha\,\beta\,G(x)^{\beta-1}}{\overline{G}(x)^{\beta}}\tau(x),$$

where $\tau(x) = g(x)/\overline{G}(x)$. The multiplying quantity $\alpha\,\beta\,G(x)^{\beta-1}/\overline{G}(x)^{\beta}$ works as a corrected factor for the hrf of the baseline model.

The *Weibull-G* family has the same parameters of the G distribution plus two shape parameters $\alpha > 0$ and $\beta > 0$. According with Bourguignon *et al.* (2014), these additional parameters are sought as a manner to furnish more flexible distribution. If $\beta = 1$, it gives the exponential-generator (Gupta *et al.*, 1998). It was found that seven distributions have been studied in the context of the *Weibull-G* family. Other two types of *Weibull-G* families have also been considered. The list of authors contributions is presented in Table 1.2.

Table 1.2: Contributed work on the Weibull-G family of distributions.

Distribution	Author(s)
Weibull exponential	Oguntunde *et al.* (2015)
Weibull Lomax	Tahir *et al.* (2015)
Weibull Rayleigh	Merovci and Elbatal (2015)
Weibull Dagum	Tahir *et al.* (2016c)
Weibull Frechet	Afify *et al.* (2016)
Weibull Pareto	Tahir *et al.* (2016b)
Weibull Birnbaum-Saunders	Benkhelifa (2016)
Second Weibull-G family	Cordeiro *et al.* (2015d)
Third Weibull-G family	Tahir *et al.* (2016d)

Recently, Tahir *et al.* (2016a) proposed a class of univariate distributions generated by extending the logistic distribution, called the logistic-X class ("LX" for short). The LX family is a special model of the *T*-*X* family defined by $W(x) = \log\{-\log[1 - G(x)]\}$ in equation (1.1) by taking a logistic random variable for T. The cdf and pdf of T are given by (for $t \in I\!R$) $R(t) = (1 + \mathrm{e}^{-\lambda t})^{-1}$ and $r(t) = \lambda\,\mathrm{e}^{-\lambda t}(1 + \mathrm{e}^{-\lambda t})^{-2}$, respectively, where $\lambda > 0$. Thus, the LX family cdf is defined by

$$F(x) = \left[1 + \{-\log[1 - G(x)]\}^{-\lambda}\right]^{-1} \tag{1.7}$$

and its pdf is given by

$$f(x) = \frac{\lambda\, g(x)}{1-G(x)}\,\left[1+[-\log(1-G(x)]^{-\lambda}\right]^{-(\lambda+1)}\left\{1+[-\log(1-G(x)]^{-\lambda}\right\}^{-2}, \qquad (1.8)$$

where $G(x)$ is any baseline cdf and $g(x) = \mathrm{d}G(x)/\mathrm{d}x$. The LX family has the same parameter of the baseline distribution plus an additional shape parameter $\lambda > 0$. Note that the baseline distribution is not a special case of the LX family. However, it can be interpreted as a compounding model between the logistic and the baseline distributions. According to Tahir *et al.* (2016a) this family may allow: to construct distributions with symmetric, left-skewed, right-skewed and/or reversed-J shaped; to define models with more types for the hrf; and, to provide competitive models to other generated families under the same baseline distribution, among other characterizations.

This thesis is composed by independent chapters. We propose four new models defined in the *Zografos-Balakrishnan-G*, *Ristić-Balakrishnan-G*, *Weibull-G* and *logistic-X* families by taking the three-parameter Burr XII distribution as baseline. For each introduced distribution, we present density expansions, quantile function, moments, incomplete moments, generating functions, estimation of the model parameters by maximum likelihood and provide applications to income and/or lifetime real data sets. A background for the Burr XII distribution and some generalizations is presented in Chapter 2. In Chapter 3, we study the Zografos-Balakrishnan Burr XII distribution. In Chapter 4, we propose the Ristić-Balakrishnan Burr XII distribution. Chapter 5 introduces the Weibull Burr XII distribution. In Chapter 4, the logistic Burr XII distribution is investigated. Chapter 7 presents the final conclusion and outlines some future research lines.

Chapter 2

On the Burr XII distribution and generalizations: a survey

Resumo

Este capítulo apresenta uma revisão de literatura sobre a distribuição Burr XII e algumas de suas generalizações propostas recentemente. São apresentadas quinze generalizações da Burr XII, as quais foram introduzidas através de diferentes geradores de distribuições ou do método de composição. Aqui, relacionamos algumas propriedades matemáticas da distribuição Burr XII, tais como momentos, momentos incompletos e função geradora de momentos. Também é obtida uma expressão alternativa para os momentos incompletos desta distribuição. Estes resultados podem ser úteis para obter as propriedades matemáticas de algumas generalizações da distribuição Burr XII a partir de combinações lineares.

Palavras-chave: Distribuição Burr XII. Distribuição Singh-Maddala. Generalizações da Distribuição Burr XII. Momentos incompletos.

Abstract

This chapter presents a review on the Burr XII distribution and some of its generalizations introduced in the recent literature. We cite telegraphically fifteen distributions obtained by different gene-rated families or compounding approaches on the Burr XII distribution. We reviewed some Burr XII mathematical properties, including moments, incomplete moments and generat-

ing function. We also derive an alternative expression for the Burr XII incomplete moments. These results were used to obtain the properties of some Burr XII generalizations from linear representations.

Key-words: Burr XII distribution. Burr XII generalizations. Incomplete moments. Singh Maddala distribution.

2.1 Introduction

The Burr system of distributions was pioneered by Burr (1942). It is based on cdf's that satisfy the general differential equation

$$F'(x) = F(x)\,[1 - F(x)]\,g(x),$$

where $g(\cdot)$ is a nonnegative function.

This system is defined by taking twelve choices of $g(\cdot)$, which are usually referred to by number and yields a variety of density shapes. For example, the Burr I distribution is the well-known uniform model, obtained by taking $g(x) = F(x)\,[1 - F(x)]$. The Burr III, Burr X and Burr XII (BXII) are the most commonly used distributions on the Burr system.

In this chapter, we present a survey on the BXII distribution and some of its generalizations published in the last years. We discuss some mathematical properties of the BXII distribution that have been used to obtain the properties of its generalizations from linear representations.

A random variable X is said to have the BXII distribution if its cdf and pdf are given by

$$G(x; c, d, s) = 1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-d} \tag{2.1}$$

and

$$g(x; c, d, s) = c\,d\,s^{-c}\,x^{c-1}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-d-1}, \tag{2.2}$$

respectively, where $d > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter. If $c > 1$, the pdf in (2.2) is unimodal with mode at

$$x = s\left[\frac{c-1}{cd+1}\right]^{1/c}$$

and it is L-shaped if $c = 1$. The quantile function of the BXII distribution is

$$Q(u) = s\left[(1-u)^{-1/d} - 1\right]^{1/c},$$

for $u \in (0, 1)$. Hence, if U is a uniform random number from the interval $(0, 1)$, then $X = Q(U)$ follows the BXII distribution.

The hazard rate function of X is given by

$$h(x; c, d, s) = c\, d\, s^{-c}\, x^{c-1}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-1}.$$

It allows for monotonic and upside-down bathtub shaped hazard rates. However, it does not exhibit bathtub shape. According with Nadarajah *et al.* (2011) this is a weakness because most empirical life systems have bathtub shapes for their hrf. The particular case for $s = m^{-1}$ and $d = 1$ gives the log-logistic (LL) distribution. The particular case for $c = 1$ gives the Lomax distribution. Shao (2004) pointed out that the BXII model also has the Weibull and Pareto distributions as limiting cases.

Various authors employed an alternative parametrization of this distribution by taking the scale parameter $s = 1$. The called two-parameter BXII distribution may be more convenient in some scale-free applications. In addition a location parameter may also be required some situations, for example, see Shah and Gokhale (1993).

Rodriguez (1977) showed that the two-parameter BXII distribution has shape characteristics similar to the exponential, gamma, logistic, log-normal, normal and some Pearson type distributions in the skewness-kurtosis plane. Tadikamalla (1980) gives relationships among the BXII distribution and other well known models.

The BXII distribution has also appeared in the literature under other names. It was referred to as the Pareto type IV distribution by Arnold (1983) and as the beta-p distribution by Mielke and Johnson (1974). In economic context, it is known as Singh-Maddala distribution because Singh and Maddala (1975, 1976) derived it through a model for the hazard rates of continuous distributions and presented it as an alternative for modeling income data. It was also referred as the generalized log-logistic distribution by El-Saidi *et al.* (1990) and q-Weibull by Brouers and Sotolongo-Costa (2005).

Applications of the BXII distribution have been widespread. Kleiber and Kotz (2003) listed various empirical studies that applied this model over the second half of the twentieth century. We can also quote some recent researches that considered the BXII distribution in different fields. Most of these are concerned in model situations characterized by power laws behavior.

The work of Weron and Kotulski (1997) has given a physical interpretation of the BXII distribution in the theory of relaxation and reaction in complex systems. Brouers *et al.* (2004), Brouers and Sotolongo-Costa (2005) and Brouers (2014a) were other related works that used this distribution in physics context. It has also been employed by Brouers (2014b) for developing empirical isotherms, which are used in the literature to represent the sorption data of a great number of solid-gas and solid-liquid sorbate-sorbent couples. Brouers (2015) gave a novel interpretation of the concept of nonextensivity based on the BXII distribution.

Brzeziński (2014) considered the BXII distribution for modeling journal impact factors. Thupeng (2016) applied it in maximum levels of nitrogen dioxide. In meteorology literature, Papalexiou and Koutsoyiannis (2012) considered it to modeling daily rainfall records in different places across the world and Li *et al.* (2015) concluded that the BXII distribution is suitable for modeling the precipitation over two river basins in China.

Moore and Papadopoulos (2000), Khan and Pareek (2012) and Kumar *et al.* (2013) considered this distribution to model variables on reliability context. In addition, applications of the two-parameter BXII distribution under diferent censoring schemes have been provided by Lee *et al.* (2009), Tomer *et al.* (2015), Panahi and Sayyareh (2016), Asl *et al.* (2017) and Belaghi and Asl (2017).

Table 2.1 lists some studies that considered the BXII model in economic and actuarial applications. In this context, it has been used mostly to model individual and household income distributions (ID), but also for poverty measures, inpatient costs and size distributions, among others situations.

The rest of the chapter is organized as follows. In Section 2.2, we cite telegraphically fifteen distributions obtained by different generated families and compounding approaches on the Burr XII distribution. We review the BXII properties of moments (Section 2.3), incomplete moments

Table 2.1: Some studies conducted considering the BXII distribution in economic applications

Purpose of the paper	Author(s)
ID in United states	Majumder and Chakravarty (1990)
	Łukasiewicz *et al.* (2010)
	Tanak *et al.* (2015)
ID in United Kingdom	Jäntti and Jenkins (2010)
ID in some European countries	Brzeziński (2013)
ID in Pakistan	Shakeel *et al.* (2015)
Size distribution of Italian firms by age	Cirillo (2010)
Poverty measures	Chotikapanich *et al.* (2013)
Inpatient cost in English hospitals	Jones *et al.* (2014)
Pricing of critical illness insurance	Dodd *et al.* (2015)

(Section 2.4) and generating function (Section 2.5).Section 2.6 concludes this survey chapter.

2.2 Burr XII generalizations and related distributions

In this section, we provide a survey on some different BXII generalizations and related distributions introduced in at last seventeen years or so. Table 2.2 presents a summary on these distributions in chronological order, which are derived using various well-established generating methods. Wang (2000) introduced the additive Burr XII distribution given by the cdf

Table 2.2: Some Burr XII generalizations introduced in recent literature.

Distribution	Author(s)
Aditive Burr XII	Wang (2000)
Extended Burr XII	Shao (2004)
Exponentiated Burr XII	Al-Hussaini and Hussein (2011a)
Beta Burr XII	Paranaíba *et al.* (2011)
Kumaraswamy Burr XII	Paranaíba *et al.* (2013)
Marshal-Olkin extended Burr XII	Al-Saiari *et al.* (2014)
Beta exponentiated Burr XII	Mead (2014)
McDonald Burr XII	Gomes *et al.* (2015)
Transmuted Burr XII	Al-Khazaleh (2016)
Kumaraswamy exponentiated BXII	Mead and Afify (2017)
Burr XII negative binomial	Ramos *et al.* (2015)
Burr XII geometric	Lanjoni *et al.* (2015)
Burr XII power series	Silva and Cordeiro (2015)
Exponentiated Burr XII Poisson	Silva *et al.* (2015)
Complementary exponentiated Burr XII Poisson	Muhammad (2017)

$$F(x) = 1 - \left\{ \left[1 + \left(\frac{x}{s_1}\right)^{c_1}\right]^{-d_1} - \left[1 + \left(\frac{x}{s_2}\right)^{c_2}\right]^{-d_2} \right\}, \;\; x > 0,$$

where c_1, c_2, d_1 and d_2 are positive shape parameters and $s_1 > 0$ and $s_2 > 0$ are scale parameters. The additive Burr XII model combines two BXII distributions, one has a decreasing and another an increasing failure rates. This distribution allows for monotonic and bathtub shaped hazard rates, which are common situations for many mechanical and electronic components.

Another model closely related to the BXII model is the three-parameter extended Burr XII (ExBXII) distribution, given by the pdf

$$\begin{aligned} f(x) &= c\, s^{-c}\, x^{c-1} \left[1 + d\, \left(\frac{x}{s}\right)^{c}\right]^{\frac{1}{d}-1}, \;\; d \neq 0, \\ &= c\, s^{-c}\, x^{c-1} \mathrm{e}^{-(x/s)^c}, \;\; d = 0. \end{aligned}$$

For $d \leq 0$ we have $x \geq 0$ and for $d > 0$ we have that $0 \leq x \leq s d^{-1/c}$. Also, for $d = 0$, the ExBXII distribution yields the Weibull distribution. Shao *et al.* (2004) claim to have developed this distribution. However, Nadarajah and Kotz (2006) point out that in fact it was originally introduced and named as the generalized Weibull distribution by Mudholkar *et al.* (1996) and Mudholkar and Sarkar (1999).

Hao and Singh (2009) presented two methods based on the principle of maximum entropy and applied it for estimating the parameters of the ExBXII distribution. Usta (2013) evaluated the performance of six different estimation methods on the ExBXII distribution. Shao *et al.* (2004), Hao and Singh (2009) and Usta (2013) illustrated the usefulness of the ExBXII by means of applications in hydrological data sets.

The four-parameter exponentiated Burr XII (EBXII) distribution has pdf given by

$$f(x) = \alpha\, c\, d\, s^{-c}\, x^{c-1} \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-d-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}^{-\alpha-1}, \;\; x > 0, \qquad (2.3)$$

where $\alpha > 0$ is the additional shape parameter. Al-Hussaini and Hussein (2011a) obtained the Bayesian predictive probability density function for the three-parameter EBXII distribution, which is obtained by taking $s = 1$ in (2.3). Al-Hussaini and Hussein (2011b) investigated the maximum likelihood and Bayes estimators on the three-parameter EBXII distribution.

Paranaíba *et al.* (2011) introduced the five-parameter Beta Burr XII (BBXII) distribution given by the pdf

$$f(x) = \frac{c\,dx^{c-1}}{s^c B(a,b)}\left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{a-1}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-(db+1)}, \;\; x > 0, \tag{2.4}$$

where $a > 0$ and $b > 0$ are the aditional shape parameters and $B(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1}\mathrm{d}t$ is the beta function. This distribution allows for increasing, decreasing, upside down bathtub shaped and decreasing-increasing-decreasing shaped hazard rates. Domma and Condino (2017) studied the four parameter BBXII distribution under the name of Beta Singh-Maddala. It is obtained by taking $s = 1$ in (2.4).

Paranaíba *et al.* (2013) introduced the five-parameter Kumaraswamy Burr XII (KwBXII) distribution given by the pdf

$$f(x) = a\,b\,c\,d\,s^{-c}x^{c-1}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1}\left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{a-1}$$
$$\times\left[1 - \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^a\right]^{b-1}, \;\; x > 0, \tag{2.5}$$

where $a > 0$ and $b > 0$ are the aditional shape parameters. This distribution allows for increasing, unimodal shaped, decreasing and bathtub shaped hazard rates.

Al-Saiari *et al.* (2014) introduced the three-parameter Marshall-Olkin extended Burr XII distribution given by the pdf

$$f(x) = \frac{\alpha\,c\,d\,x^{c-1}\,(1+x^c)^{-d-1}}{[1-(1-\alpha)(1+x^c)^{-d}]^2}, \;\; x > 0,$$

where α, c and d are positive shape parameters. This distribution extends the two-parameter BXII distribution on the family pioneered by Marshall and Olkin (1997). It allows for decreasing and unimodal shaped hazard rates.

Mead (2014) introduced the five-parameter beta exponentiated Burr XII (BEBXII) distribution given by the pdf (for $x > 0$)

$$f(x) = \frac{\alpha\,c\,dx^{c-1}}{B(a,b)}\,(1+x^c)^{-d-1}\left[1-(1+x^c)^{-d}\right]^{a\alpha-1}\left\{1-\left[1-(1+x^c)^{-d}\right]\right\}^{b-1}, \tag{2.6}$$

where α, a, b, c and d are positive shape parameters. This distribution extends the three-parameter EBXII distribution on the family pioneered by Eugene *et al.* (2002).

Gomes *et al.* (2015) introduced the six-parameter McDonald Burr XII (McBXII) distribution given by the pdf

$$f(x) = \frac{\alpha\, c\, d\, x^{c-1}}{s^c B(a,b)} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{a\alpha-1}$$
$$\times \left[1 - \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{\alpha}\right]^{b-1}, \quad x > 0,$$

where a, b, c, d and α are positive shape parameters and $s > 0$ is a scale parameter. This distribution allows for monotonic, unimodal shaped and bathtub shaped hazard rates. Gomes *et al.* (2015) listed fourteen distributions as McBXII special models, including the BBXII and KwBXII distributions. We can note that the five-parameter BEBXII density in (2.6) is also a special model of the McBXII distribution, which is obtained by taking $s = 1$. Cordeiro *et al.* (2016) proposed an extended regression model based on the logarithm of a McBXII random variable.

Al-Khazaleh (2016) introduced the four-parameter transmuted Burr XII (TBXII) distribution given by the pdf

$$f(x) = \left\{c\, d\, s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1}\right\} \left\{1 - \lambda + 2\lambda \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}, \quad x > 0,$$

where c, d and λ are positive shape parameters and $s > 0$ is a scale parameter. The author presents an expansion for the TBXII distribution, the generating function, ordinary moments and order statistics. Under the name of Transmuted Singh-Maddala distribution, Shahzad *et al.* (2017) provided some other properties of the TBXII distribution and also applications to bladder cancer patients and to the Pakistani annual household expenditure.

Mead and Afify (2017) introduced the four parameter Kumaraswamy exponentiated Burr XII distribution given by the pdf

$$f(x) = \frac{a\, b\, c\, d\, \beta\, x^{c-1}}{(1 + x^c)^{d+1}} \left[1 - (1 + x^c)^{-d}\right]^{a\beta-1} \left\{1 - \left[1 - (1 + x^c)^{-d}\right]^{a\beta}\right\}^{b-1}, \quad x > 0,$$

where a, b, c, d and β are positive shape parameters. This distribution extends the three-parameter EBXII distribution on the family pioneered by Cordeiro and de Castro (2011).

Some BXII generalizations were also proposed using the discrete-continuous compounding approach, pioneered by Adamidis and Loukas (1998). Ramos *et al.* (2015) introduced the five-parameter Burr XII negative binomial distribution given by the pdf

$$f(x) = \frac{a\,\beta\,c\,d\,s^{-c}}{[(1-\beta)^{-a}-1]} x^{c-1} \left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d-1} \left\{1-\beta\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}^{-a-1}, \; x>0,$$

where $s > 0$ is a scale parameter and $a > 0, c > 0, d > 0$ and $\beta \in (0, 1)$ are shape parameters. Its hrf can be increasing, decreasing, constant and unimodal shaped. It includes as special models the LL, BXII, Burr XII geometric (Lanjoni *et al.*, 2015) and Burr XII Poisson distributions.

Silva and Cordeiro (2015) introduced the three-parameter Burr XII power series distribution given by the pdf

$$f(x) = \theta\,c\,d\,(1+x^{c})^{-d-1} \frac{C'[(1+x^{c})^{-d}]}{C[\theta(1+x^{c})^{-d}]}, \; x>0,$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, $\theta > 0$ is such that $C(\theta)$ is finite and the coefficients a_n's depend only on n.

Silva *et al.* (2015) introduced the exponentiated Burr XII Poisson distribution given by the pdf

$$f(x) = \frac{\alpha\,\lambda\,c\,d\,s^{-c}\,x^{c-1}}{1-\mathrm{e}^{-\lambda}} \left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d-1} \left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}^{\alpha-1}$$
$$\times \exp\left[-\lambda\left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}\right]^{b-1}, \; x>0,$$

where $c, d, s, \alpha, \lambda > 0$. This distribution has decreasing and unimodal shaped hazard rate. This model was motivated by applications in failures of a system. For example, suppose that a system has N serial sub-systems functioning independently at a give time, where N is a truncated Poisson random variable. Let Z denote the time of failure of the first out of the N functioning systems defined by the independent random variables $Y_1, \ldots, Y_N$ with pdf given by (2.3). Then, $Z = \min(Y_1, \ldots, Y_N)$ has the exponentiated Burr XII Poisson distribution.

Muhammad (2017) introduced the complementary exponentiated Burr XII Poisson distribution given by the pdf

$$f(x) = \frac{\alpha\,\lambda\,c\,d\,x^{c-1}}{\mathrm{e}^{\lambda}-1}\,(1+x^c)^{-d-1}\left[1-(1+x^c)^{-d}\right]^{\alpha-1}$$
$$\times\exp\left\{\lambda\left[1-(1+x^c)^{-d}\right]\right\}^{b-1},\ x>0,$$

where $c, d, s, \alpha, \lambda > 0$. Analogously to the exponentiated Burr XII Poisson distribution, this model is obtained by taking the maximum of N random variables that follows the EBXII distribution. However, on this proposal the authors considerered three-parameter EBXII distribution, while Silva *et al.* (2015) used the four-parameter EBXII distribution in their study.

2.3 Moments

Suppose X is a random variable having the pdf (2.2). It follows from Zimmer *et al.* (1998) the hth moment of X exists for $h < c\,d$ and is given by

$$\mu'_h = s^h\,d\,B(d-h\,c^{-1}, 1+h\,c^{-1}). \tag{2.7}$$

Rodriguez (1977) presented a similar expression for the moments of the two-parameter BXII distribution.

Paranaíba *et al.* (2011, 2013), Mead (2014), Gomes *et al.* (2015), Ramos *et al.* (2015), Lanjoni *et al.* (2015), Silva and Cordeiro (2015), Silva *et al.* (2015), Cordeiro *et al.* (2016), Domma and Condino (2017), Mead and Afify (2017) and Muhammad (2017) used the result in (2.7) to obtain the moments of their proposed distributions by linear combinations of the BXII model.

2.4 Incomplete moments

Let $T_h(y) = \int_0^y x^h\,f(x)\mathrm{d}x$ be the hth incomplete moment of X. It can be written as

$$T_h(y) = c\,d\int_0^y x^{h-1}\left(\frac{x}{s}\right)^c\left[1+\left(\frac{x}{s}\right)^c\right]^{-d-1}\mathrm{d}x. \tag{2.8}$$

Setting $t = \left[1+\left(\frac{x}{s}\right)^c\right]^{-1}$ in the last equation, we have

$$T_h(y) = d\,s^h\int_{s^c/(s^c+y^c)}^1 t^{d-\frac{h}{c}-1}\,(1-t)^{\frac{h}{c}}\mathrm{d}t.$$

Hence, the hth incomplete moment of X reduces to (for $h < c\,d$)

$$T_h(y) = d\,s^h\,B_{s^c/s^c+y^c}\left(d - h\,c^{-1}, 1 + h\,c^{-1}\right), \tag{2.9}$$

where $B_z(a,b) = \int_z^1 t^{a-1}\,(1-t)^{b-1}\mathrm{d}t$ is the upper incomplete beta function. By setting $h = 1$, we obtain the first incomplete moment of X.

Mead (2014), Gomes *et al.* (2015), Ramos *et al.* (2015), Silva and Cordeiro (2015), Silva *et al.* (2015) and Mead and Afify (2017) used the result in (2.9) to obtain the incomplete moments of their proposed distributions by linear combinations of the BXII model.

Alternatively, taking $u = (x/s)^c$ in equation (2.8), we can write

$$T_h(y) \quad = \quad d\,s^h \int_0^{\left(\frac{y}{s}\right)^c} u^{h/c}(1+u)^{-d-1}\mathrm{d}u.$$

The following integral (for $y > -1$ and $a > -1$) is calculated using `Mathematica`

$$\begin{aligned} J(y,a,b) &= \int_0^y z^a(z+1)^{-b}\mathrm{d}z \\ &= \frac{y^{a+1}\,{}_2F_1(a+1,b;a+2;-y)}{a+1}, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function defined by ${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}$, where $|x| < 1$, $c = 0, -1, -2, \ldots$ and $(z)_n$ is the Pochhammer polynomial. Thus, the hth incomplete moment of X can also be written as

$$T_h(y) = d\,s^h\,J\left(y, \frac{h}{c}, d+1\right). \tag{2.10}$$

The expression in (2.10) is new and previously unknown. Equations (2.9) and (2.10) are the main results of this section.

2.5 Generating function

Paranaíba *et al.* (2011, 2013) provided a representation for the moment generating function (mgf), $M(t)=\int_{-\infty}^{\infty}\mathrm{e}^{t\,x}f(x)dx$, of X. It may require the Meijer-G function defined by

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{array}{l}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j+t\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-t\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j+t\right)\prod_{j=m+1}^{p}\Gamma\left(1-b_j-t\right)}x^{-t}\mathrm{d}t,$$

where L denotes an integration path, see Section 9.3 in Gradshteyn and Ryzhik (2000). Many integrals with elementary and special functions are particular cases of the Meijer-G function (Prudnikov *et al.*, 1986).

Thus, by assuming that $c=m/d$, where m and d are positive integers, $\mu>-1$, $p>0$ and $t<0$, Paranaíba *et al.* (2011, 2013) proved that

$$M(t)=m\,I\left(-s\,t,\frac{m}{d}-1,\frac{m}{d},-d-1\right), \tag{2.11}$$

where

$$\begin{aligned}I\left(p,\mu,\frac{m}{d},\nu\right) &= \int_0^{\infty}\mathrm{e}^{-px}\,x^{\mu}\,(1+x^{\frac{m}{d}})^{\nu}\mathrm{d}x\\ &= \frac{d^{-\nu}m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{(m-1)}{2}}\Gamma(-\nu)p^{\mu+1}}\times\\ &\quad G_{d+m,d}^{d,d+m}\left(\frac{m^m}{p^m}\,\middle|\,\begin{array}{l}\Delta(m,-\mu),\Delta(d,\nu+1)\\ \Delta(d,0)\end{array}\right),\end{aligned}$$

$\mathrm{i}=\sqrt{-1}$ is the complex unit and $\Delta(d,a)=\frac{a}{d},\frac{a+1}{d},\cdots,\frac{a+d}{d}$. The condition over the parameter c is not restrictive since every positive real number can be approximated by a rational number.

Paranaíba *et al.* (2011, 2013), Gomes *et al.* (2015), Ramos *et al.* (2015),Lanjoni *et al.* (2015), Silva and Cordeiro (2015), Silva *et al.* (2015) and Cordeiro *et al.* (2016) used the result in (2.11) to obtain the mgf of their proposed distributions by linear combinations of the BXII model.

Furthermore, Paranaíba *et al.* (2011, 2013) demonstrated that for $c=1$ the mgf of X reduces to

$$M(t)=d(-st)^d\exp(-st)\Gamma(-d,-st),$$

where $\Gamma(v,x)=\int_x^{\infty}t^{v-1}\exp(-t)dt$ is the complementary incomplete gamma function. For $c=2$

and $t < 0$, the authors obtained that

$$M(t) = {}_1F_2\Big(1; \frac{1}{2}; 1-d; \frac{s^2t^2}{4}\Big) + \frac{st}{2} B\Big(2, d-\frac{1}{2}\Big)\, {}_1F_2\Big(1; \frac{3}{2}; d+\frac{7}{2}; \frac{-s^2t^2}{4}\Big) + \frac{\Gamma(-2d)}{(-st)^{-2d}},$$

where

$${}_1F_2(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!}$$

is a generalized hypergeometric function and $(a)_k = a(a+1)\dots(a+k-1)$ denotes the ascending factorial.

2.6 Concluding remarks

The Burr XII (BXII) distribution is one of the most commonly used distributions on the Burr system. It has also appeared in the literature under other names, such as Pareto type IV, beta-p, generalized log-logistic and Singh and Maddala distributions. Applications have been provided in several areas, especially in situations characterized by power law behavior. Various BXII generalizations and related distributions have been introduced in recent years. In this chapter, we have listed fifteen distributions obtained by different generated families and compounding approaches on the BXII distribution. We discuss the BXII properties of moments, incomplete moments and generating function and also derive an alternative expression for the Burr XII incomplete moments. We noted that these results were used to obtain the properties of some Burr XII generalizations from linear representations.

Chapter 3

The Zografos-Balakrishnan Burr XII distribution

Resumo

Neste capítulo propomos uma nova distribuição de quatro parâmetros, denominada Zografos-Balakrishnan Burr XII. Esta distribuição pode ser uma alternativa útil para modelar dados de renda e também pode ser aplicada em outras áreas, tais como ciências atuariais, finanças, telecomunicações e em análise de sobrevivência. A distribuição proposta tem como casos especiais alguns modelos conhecidos como, por exemplo, as distribuições log-logística, Weibull, Lomax e Burr XII. São investigadas algumas propriedades estruturais da distribuição proposta. O método de máxima verossimilhança é utilizado para estimar os parâmetros do modelo. Um estudo de simulação é conduzido. Além disso, são realizadas duas aplicações para ilustrar a flexibilidade da nova distribuição.

Palavras-chave: Distribuição Burr XII. Família gamma-G. Método de máxima verossimilhança. Família Zografos-Balakrishnan.

Abstract

We propose a four-parameter distribution, called the Zografos-Balakrishnan Burr XII distribution. The new distribution may be a useful alternative to describe income distributions and can also be applied in actuarial science, finance, telecommunications and modeling lifetime data, for example. It contains as special models some well-known distributions, such as the log-logistic, Weibull, Lomax and Burr XII distributions, among others. Some of its structural properties are investigated. The method of maximum likelihood is used for estimating the model parameters and a simulation study is conducted. We provide two applications to real data to illustrate the flexibility of the proposed distribution.

Keywords: Burr XII distribution. Gamma-G family. Maximum likelihood estimation. Zografos-Balakrishnan family.

3.1 Introduction

The three-parameter BXII distribution has cdf and pdf in (2.1) and (2.2) respectively. This statistical model was originally proposed by Burr (1942). However, the BXII distribution also appeared in the literature under different names, such as Pareto type IV (Arnold, 1983), beta-p (Mielke and Johnson, 1974) and Singh-Maddala (Singh and Maddala, 1975, 1976).

In reliability context, the BXII distribution may be useful as a failure model under various loss functions (Moore and Papadopoulos, 2000) and for modeling the strength of a manufactured (Khan and Pareek, 2012), among other applications. Schrnittlein (1983), Wingo (1983), Wingo (1993), Wang *et al.* (1996), Watkins (1999) and Shao (2004) investigated the behaviors of the BXII maximum likelihood estimators based on uncensored and censored data.

Motivated by the extensive usage of the BXII distribution, some generalizations were proposed such as the beta Burr XII (Paranaíba *et al.*, 2011) and Kumaraswamy Burr XII (Paranaíba *et al.*, 2013) distributions. We also refer the reader to the exponentiated Burr XII (Al-Hussaini and Hussein, 2011b), Marshal-Olkin extended Burr XII (Al-Saiari *et al.*, 2014; Kumar, 2016) and McDonald Burr XII (Gomes *et al.*, 2015) distributions, among others. These distributions are obtained through different generalized (or generated) G families of continuous univariate distributions.

Following the Zografos and Balakrishnan (2009) proposal, we introduce a new four-parameter distribution called the Zografos-Balakrishnan Burr XII (ZBXII) distribution. Once the proposed distribution is quite flexible regarding pdf and hrf, it may provide an interesting alternative to describe lifetime data and can also be applied in actuarial science, economy, finance, bioscience, telecommunications and physics, for example.

The rest of the chapter is outlined as follows. In Section 3.2, we define the ZBXII distribution and present some special models. In Section 3.3, we derive expansions for the pdf and cdf of the ZBXII distribution. A range of its mathematical properties are derived in Sections 3.4-3.6. The maximum likelihood estimation is used to estimate the model parameters in Section 3.7. A simulation study is carried out in Section 3.8. Section 3.9 provides two applications to real lifetime data sets. Finally, Section 3.10 presents the chapter concluding remarks.

3.2 Model definition

The ZBXII distribution is defined be taking $G(x)$ in (1.2) to be the cdf (2.1) of the BXII distribution. This distribution contain as special models several well-known distributions. The BXII distribution is a particular case for $a = 1$. For $s = m^{-1}$ and $d = 1$, it reduces to the gamma log-logistic distribution (Ramos *et al.*, 2013). For $a = 1$, $s = m^{-1}$ and $d = 1$, it becomes the log-logistic (LL) distribution. If $d \to \infty$, it is identical to the gamma Weibull distribution. If $d \to \infty$ in addition to $a = 1$, it becomes the Weibull distribution. For $c = 1$ and $a = c = 1$, it reduces to the gamma Lomax (Cordeiro *et al.*, 2015c) and Lomax distributions, respectively.

The ZBXII distribution has cdf and pdf given by

$$F(x) = \frac{\gamma\left(a,\, d\log\left[1+\left(\frac{x}{s}\right)^{c}\right]\right)}{\Gamma(a)} \tag{3.1}$$

and

$$f(x) = \frac{c\, d^{a}\, x^{c-1}}{s^{c}\,\Gamma(a)}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d-1}\left\{\log\left[1+\left(\frac{x}{s}\right)^{c}\right]\right\}^{a-1}, \tag{3.2}$$

respectively, where $a > 0$, $d > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter. If X is a random variable with density function (3.2), we write $X \sim \text{ZBXII}(a, s, d, c)$. The ZBXII hrf is given by

$$h(x) = \frac{c\, d^{a}\, x^{c-1}}{s^{c}\,\Gamma\left(a,\, d\log\left[1+\left(\frac{x}{s}\right)^{c}\right]\right)}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d-1}\left\{\log\left[1+\left(\frac{x}{s}\right)^{c}\right]\right\}^{a-1},$$

where $\Gamma(a, z) = \Gamma(a) - \gamma(a, z) = \int_z^\infty t^{a-1} \mathrm{e}^{-t} \mathrm{d}t$ is the upper incomplete gamma function.

Figure 3.1 reveals the versatility of the proposed density for some parameter values. The ZBXII pdf can take various forms, including power-law tails. This kind of tail is very common in economical systems and a important characteristic in income distributions. Figure 3.2 displays plots of the hrf for some parameter values. It reveals that the ZBXII distribution can have decreasing, decreasing-increasing-decreasing and upside-down bathtub hazard functions. This feature reveals that the ZBXII distribution is quite competitive with the BBXII model, which has the same forms for the hrf but one additional shape parameter.

3.3 Useful expansions

We derive some useful expansions for equations (3.1) and (3.2). For any real parameter m and $z \in (0, 1)$, the following formula holds

$$[-\log(1-z)]^m = z^m + \sum_{i=0}^{\infty} p_i(m)\, z^{i+m+1}, \tag{3.3}$$

where $p_0(m) = m/2$, $p_1(m) = m\,(3m+5)/24$, $p_2(m) = m\,(m^2+5m+6)/48$, $p_3(m) = m\,(15m^3 + 150m^2+485m+502)/5760$, etc, are Stirling polynomials. Castellares and Lemonte (2015) gave a recursive expression for these coefficients. The proof is given in details by Flajolet and Odlyzko (1990) (see Theorem 3A, page 227) and Flajolet and Sedgewick (2009) (see Theorem VI.2, page 385). By inserting (3.3) in equation (1.2), the ZBXII density can be expressed as

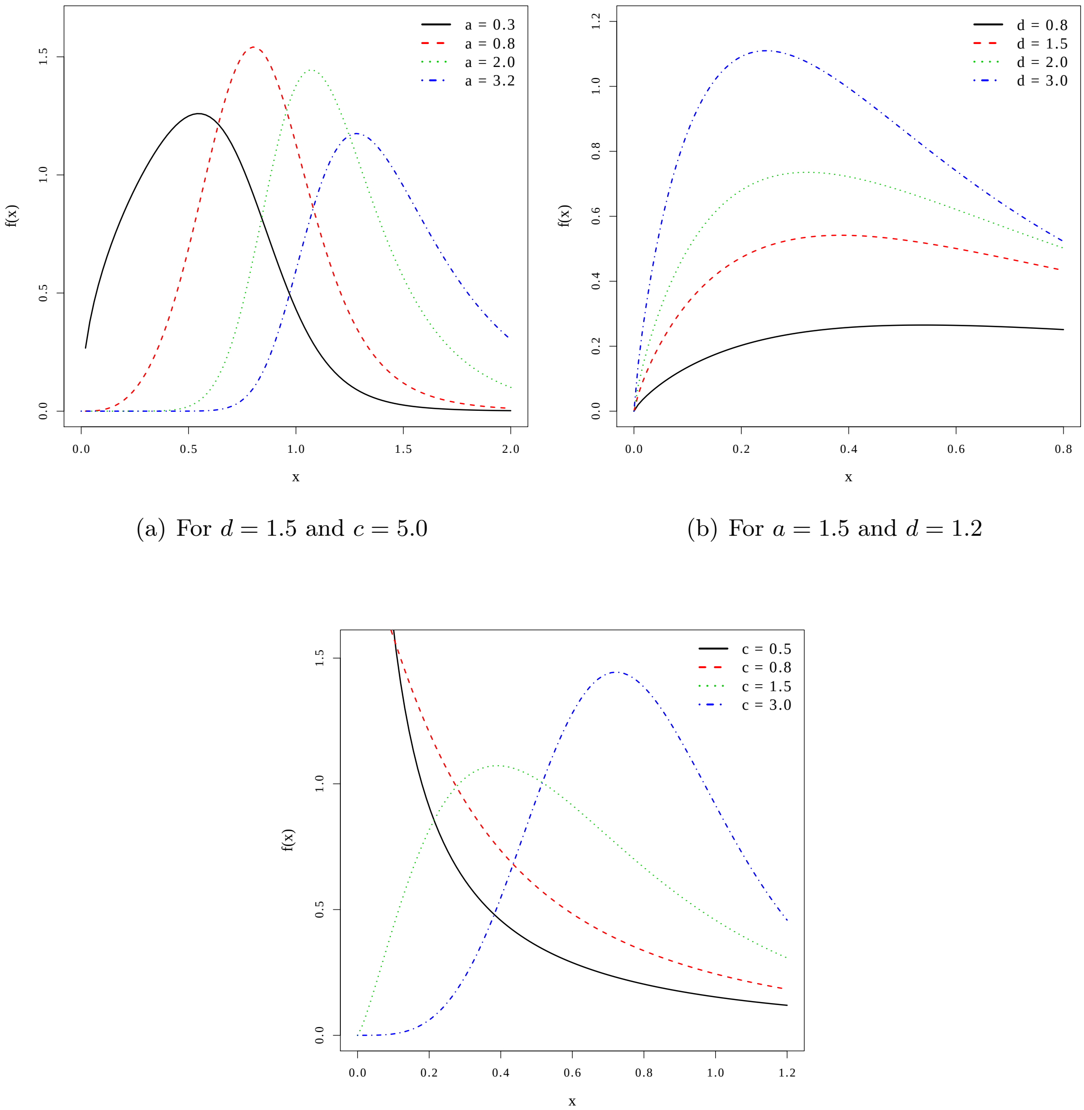
$$f(x) = c\,d\,s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1} \sum_{k=0}^{\infty} b_k \left\{1 - \left(1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{a+k-1}, \tag{3.4}$$

where $b_0 = 1/\Gamma(a)$, $b_1 = p_0(a-1)/\Gamma(a)$, $b_2 = p_1(a-1)/\Gamma(a)$, $b_3 = p_2(a-1)/\Gamma(a)$, etc. If $|z| < 1$ and $b > 0$ is real non-integer, the power series holds

$$(1-z)^{b-1} = \sum_{r=0}^{\infty} (-1)^r \binom{b-1}{r} z^r.$$

Using the above series for $\left\{1 - \left[1 + (x/s)^c\right]^{-d}\right\}^{a+k-1}$ in (3.4) and after some algebraic manipulation, we have

$$f(x) = \sum_{r=0}^{\infty} w_r\, g(x; s, (r+1)d, c), \tag{3.5}$$



(a) For $d = 1.5$ and $c = 5.0$

(b) For $a = 1.5$ and $d = 1.2$

(c) For $a = 1.5$ and $d = 3.0$

Figure 3.1: Pdf plots for the ZBXII model with $s = 1$.

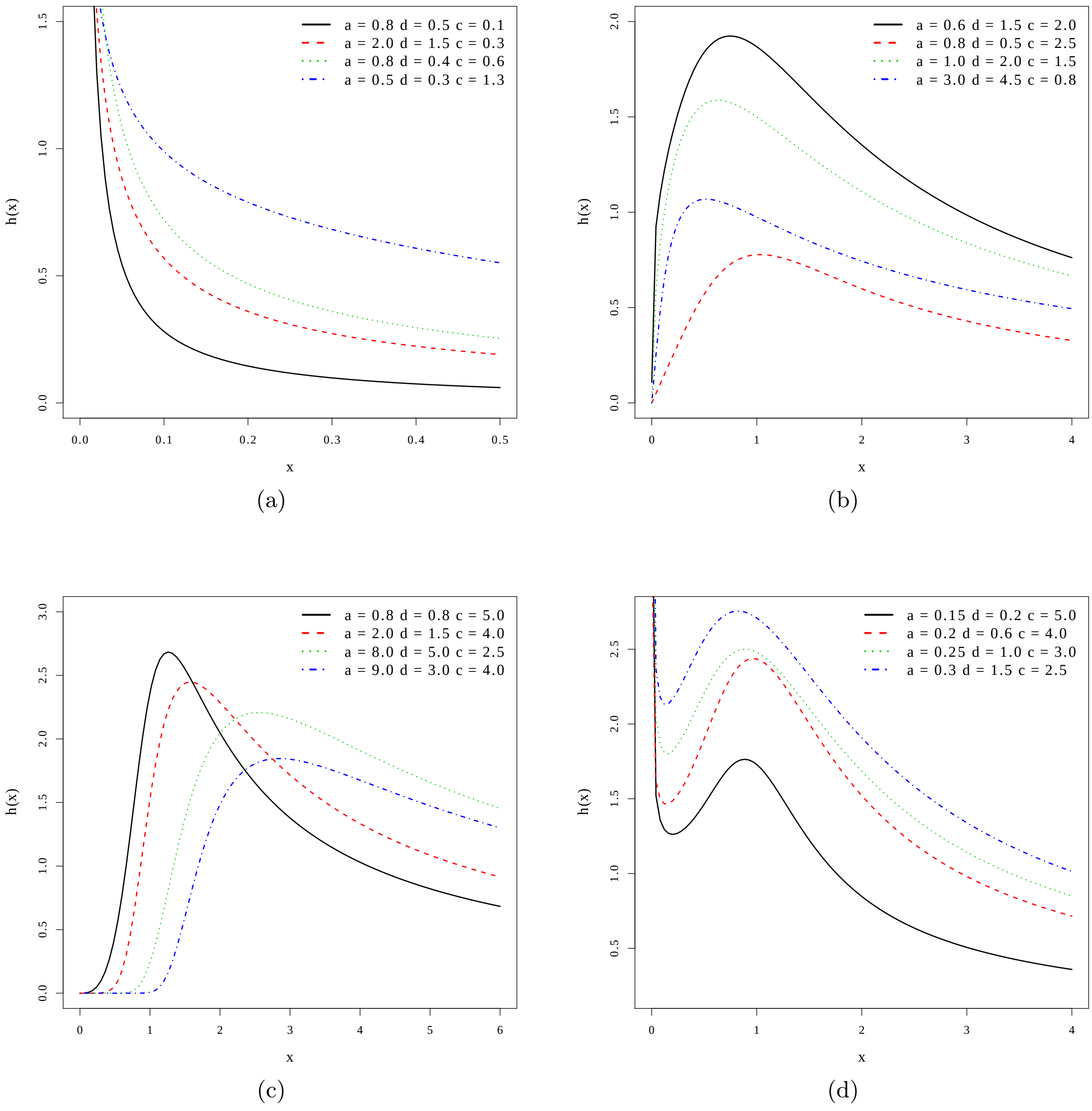


Figure 3.2: Hrf plots for the ZBXII model.

where

$$w_r = \sum_{k=0}^{\infty} \frac{(-1)^r\, b_k\, \Gamma(a+k)}{\Gamma(a+k-r)r!(r+1)}$$

and $g(x; s, (r+1)d, c)$ is the BXII density function with scale parameter s and shape parameters c and $(r+1)d$. Equation (3.5) reveals that the ZBXII density is an infinite linear combination of BXII densities. So, several structural properties of the ZBXII distribution can follow from those BXII properties. By integrating equation (3.5) gives

$$F(x) = \sum_{r=0}^{\infty} w_r\, G(x; s, (r+1)d, c). \tag{3.6}$$

Equations (3.5) and (3.6) are the main results of this section.

3.4 Quantile function

By inverting (3.1), we obtain an explicit expression for the quantile function (qf) of the ZBXII distribution, say $Q(u)$, as

$$Q(u) = s\left\{\exp\left[\frac{1}{d}Q^{-1}(a, 1-u)\right] - 1\right\}^{1/c}, \tag{3.7}$$

where $Q^{-1}(a, u)$ is the inverse function of $Q(a, x) = 1 - \gamma(a, x)/\Gamma(a)$. Quantities of interest can be obtained from (3.7) by substituting appropriate values for u.

Expressions for the skweness and kurtosis may be obtained from (3.7). Kenney and Keeping (1962) proposed the Bowley's skewness, which is based on quartiles and is defined as

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \tag{3.8}$$

whereas the Moors' kurtosis (Moors, 1988), based on octiles, is given by

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{3.9}$$

Plots of the skewness and kurtosis of X as functions of a and d for selected values of c and $s = 1$ are displayed in Figure 3.3. It suggests that Bowley's skewness and Moors' kurtosis increases and stabilizes when the parameters a and d increase and c is not very large. For $c = 0.2$, the Bowley's skewness seems to be little affected, while Moors' kurtosis takes more elevated values. For higher values of c, we note that Bowley's skewness gets negative values and the Moors' kurtosis decreases towards to zero.

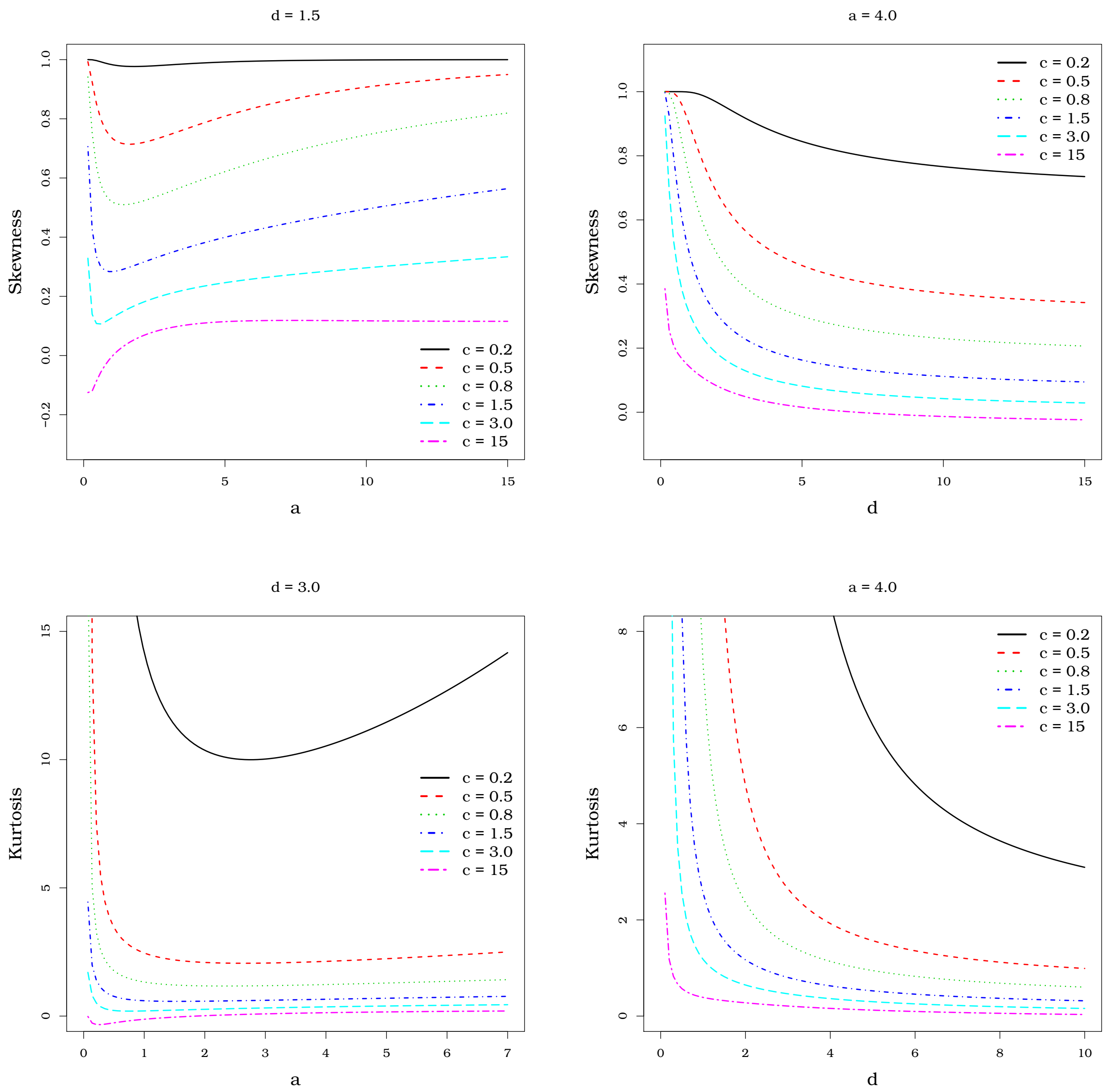


Figure 3.3: Skewness and kurtosis of the ZBXII model for some parameter values.

3.5 Moments and generating function

The hth moment of X follows directly from (3.5) using the result in equation (2.7). For $h < c\,d$, we have

$$\mu'_h = s^h\, d \sum_{r=0}^{\infty} (r+1) w_r\, B((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}), \tag{3.10}$$

where $B(\cdot,\cdot)$ is the beta function. The central moments (μ_s) and cumulants (κ_s) of X can be expressed from (3.10) as

$$\mu_s = \sum_{i=0}^{s} \binom{s}{i} (-1)^i\, \mu_1'^s\, \mu'_{s-i} \qquad \text{and} \qquad \kappa_s = \mu'_s - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \kappa_i\, \mu'_{s-i},$$

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be calculated from the third and fourth standardized cumulants.

The hth incomplete moment of X is defined by $T_h(y) = \int_0^y x^h\, f(x)dx$. By using (3.5), we obtain

$$T_h(y) = c\,d \sum_{r=0}^{\infty} (r+1)\, w_r \int_0^y x^{h-1} \left(\frac{x}{s}\right)^c \left[1 + \left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1} dx.$$

Setting $t = \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}$ in the last equation, we have

$$T_h(y) = d\, s^h \sum_{r=0}^{\infty} (r+1)\, w_r \int_{s^c/(s^c+y^c)}^{1} t^{(r+1)d - \frac{h}{c} - 1}\, (1-t)^{\frac{h}{c}}\, dt.$$

Hence, the hth incomplete moment of X reduces to (for $h < c\,d$)

$$T_h(y) = d\, s^h \sum_{r=0}^{\infty} (r+1)\, w_r\; B_{s^c/s^c+y^c}\left((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}\right), \tag{3.11}$$

where $B_z(a,b) = \int_z^1 t^{a-1}\,(1-t)^{b-1} dt$ is the upper incomplete beta function.

Let $M_d(t)$ be the mgf of the BXII(s,d,c) distribution. The mgf, $M(t)$, of X can be obtained from (3.5) as an infinite weighted sum

$$M(t) = \sum_{r=0}^{\infty} w_r\, M_{(r+1)d}(t), \tag{3.12}$$

where $M_{(r+1)d}(t)$ is the mgf of the BXII$(s,(r+1)d,c)$ distribution. For $t<0$, Paranaíba *et al.* (2011, 2013) provided the representation for $M_d(t)$ given in equation (2.11). Hence, for $t<0$, the generating function of X follows from (3.12) as

$$M(t)=m\sum_{r=0}^{\infty}w_r\,I\left(-s\,t,\frac{m}{(r+1)d}-1,\frac{m}{(r+1)d},-(r+1)d-1\right). \tag{3.13}$$

Equations (3.10)-(3.13) are the main results of this section.

3.6 Mean deviations

The deviations from the mean and the median are commonly used as measures of spread in a population. They can be determined as

$$\delta_1=2\mu_1'F(\mu_1')-2\,T_1(\mu_1') \qquad \text{and} \qquad \delta_2=\mu_1'-2\,T_1(M),$$

respectively, where $\mu_1'=I\!E(X)$, the median M of X follows from (3.7) by $M=Q(1/2)$, $F(\mu_1')$ is easily obtained from (3.1) and $T_1(y)=\int_0^y x\,f(x)\mathrm{d}x$ is the first incomplete moment. Hence, it follows from (3.11) that

$$T_1(y)=d\,s\sum_{r=0}^{\infty}(r+1)\,w_r\;B_{s^c/s^c+y^c}((r+1)d-\,c^{-1},1+\,c^{-1}).$$

An alternative expression for $T_1(y)$, using (3.5), takes the form

$$T_1(y)=cds^{-c}\sum_{r=0}^{\infty}(r+1)w_r\int_0^y x^c\left[1+\left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1}\mathrm{d}x.$$

Setting $u=(x/s)^c$, we obtain

$$\begin{aligned}T_1(y) &= d\,s\sum_{r=0}^{\infty}(r+1)w_r\int_0^{\left(\frac{y}{s}\right)^c}u^{1/c}(1+u)^{-(r+1)d-1}\mathrm{d}u.\\ &= \frac{c\,d\,s\,y^{c+1}}{1+c}\sum_{r=0}^{\infty}(r+1)\,w_r\,{}_2F_1\left[1+\frac{1}{c},\;(r+1)d+1;\;2+\frac{1}{c};\;-\left(\frac{y}{s}\right)^c\right],\end{aligned}$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $|x|<1$, $c=0,-1,-2,\ldots$ and $(z)_n$ is the Pochhammer polynomial.

The previous results are related to the Bonferroni and Lorenz curves. For a given probability π, they are defined as $B(\pi) = T_1(q)/(\pi\mu'_1)$ and $L(\pi) = T_1(q)/\mu'_1$, respectively, where $q = Q(\pi)$ is given by (3.7). If π is the proportion of units whose income is lower than or equal to q, the values of $L(\pi)$ yield fractions of the total income and $B(\pi)$ refers to the relative income levels. These curves are important in economics for studying income and poverty, but can be useful in demography, reliability, insurance, medicine and several other fields. The Lorenz curve also allows to obtain the Gini concentration (C_G), given by $C_G = 1 - 2\int_0^1 L(\pi)\mathrm{d}u$, and represents the area between the curve $L(\pi)$ and the straight line.

3.7 Maximum-likelihood estimation

This section addresses the estimation of the unknown parameters of the ZBXII distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample of size n from the ZBXII(a, c, d, s) distribution. Let $\boldsymbol{\theta} = (a, c, d, s)^T$ be the parameter vector of interest. The log-likelihood function for $\boldsymbol{\theta}$ can be expressed as

$$\begin{aligned}\ell(\boldsymbol{\theta}) =& n \log\left[\frac{c\, d^a}{s\,\Gamma(a)}\right] + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1) \\ &- (d+1)\sum_{i=1}^{n}\log(u_i) + (a-1)\sum_{i=1}^{n}\log\left[\log(u_i)\right],\end{aligned} \tag{3.14}$$

where $u_i = 1 + \left(\frac{x_i}{s}\right)^c$. The components of the score vector $\boldsymbol{U(\theta)}$ are given by

$$\begin{aligned}U_a(\boldsymbol{\theta}) &= n\left[\log(d) - \psi(a)\right] + \sum_{i=1}^{n}\log\left[\log(u_i)\right], \\ U_c(\boldsymbol{\theta}) &= \frac{n}{c} + c^{-1}\sum_{i=1}^{n}\log(u_i - 1) - \frac{d+1}{c}\sum_{i=1}^{n}\frac{(u_i - 1)\log(u_i - 1)}{u_i} \\ &\quad + \frac{a-1}{c}\sum_{i=1}^{n}\frac{(u_i - 1)\log(u_i - 1)}{u_i\log(u_i)}, \\ U_d(\boldsymbol{\theta}) &= \frac{an}{d} - \sum_{i=1}^{n}\log(u_i),\end{aligned}$$

and

$$U_s(\boldsymbol{\theta}) = \frac{-nc}{s} + \frac{c\,(d+1)}{s}\sum_{i=1}^{n}\frac{(u_i-1)}{u_i} - \frac{c\,(a-1)}{s}\sum_{i=1}^{n}\frac{(u_i-1)}{u_i\log(u_i)},$$

where $\psi(.)$ is the digamma function.

Setting these expressions to zero, say $\boldsymbol{U(\theta)} = \mathbf{0}$, and solving them simultaneously yields the maximum likelihood estimators (MLEs) of the unknown parameters. These equations can not be solved analytically but we can use iterative techniques.

For fixed c, d and s, the MLE of a is given by

$$\hat{a}(\hat{c}, \hat{d}, \hat{s}) = \psi^{-1}\left(\sum_{i=1}^{n}\frac{\log[\log(u_i)]}{n} - \log(d)\right), \tag{3.15}$$

where $\psi^{-1}(.)$ is the inverse digamma function. By replacing a by $\hat{a}$ in equation (3.14), we obtain the profile log-likelihood function for $\boldsymbol{\theta_{p_1}} = (c, d, s)$, expressed by

$$\begin{aligned}
\ell(\boldsymbol{\theta_{p_1}}) = {} & (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i-1) + n\left[\log\left(\frac{c}{s}\right) + \log(d)\,\psi^{-1}\left(\sum_{i=1}^{n}\frac{\log[\log(u_i)]}{n} - \log(d)\right)\right] \\
& - n\,\log\left\{\Gamma\left[\psi^{-1}\left(\sum_{i=1}^{n}\frac{\log[\log(u_i)]}{n} - \log(d)\right)\right]\right\} - (d+1)\sum_{i=1}^{n}\log(u_i) \\
& + \left[\psi^{-1}\left(\sum_{i=1}^{n}\frac{\log[\log(u_i)]}{n} - \log(d)\right) - 1\right]\sum_{i=1}^{n}\log\left[\log(u_i)\right].
\end{aligned} \tag{3.16}$$

This first profile log-likelihood may be helpful for obtaining the initial values for the parameters by fitting the BXII distribution and using the relationship in (3.15) for a. Alternatively, for fixed a, c and s, we obtain the MLE of d as

$$\hat{d}(\hat{a}, \hat{c}, \hat{s}) = a\,n\left[\sum_{i=1}^{n}\log(u_i)\right]^{-1} \tag{3.17}$$

It is easy to observe in (3.17) that, fixed on $x_1, \ldots, x_n$,

- $\hat{d} \to \infty$ when $\hat{a} \to \infty$,
- $\hat{d} \to \hat{0}^+$ when $\hat{a} \to \hat{0}^+$,
- $\hat{d} \to \hat{0}^+$ when $\hat{c} \to \hat{0}^+$, and $x_i > \hat{s}$ for some $i < n$,
- $\hat{d} \to \infty$ when $\hat{c} \to \infty$, and $x_i < \hat{s}$, $\forall i < n$.

Thus, we can think of the use of more refined procedures for estimation under small values of a and d.

By replacing d by (3.17) in equation (3.14), we have the profile log-likelihood function for $\boldsymbol{\theta_{p_2}} = (a, c, s)$. It can be expressed as

$$\ell(\boldsymbol{\theta_{p_2}}) = n\left\{\log\left[\frac{c}{s\,\Gamma(a)}\left(\frac{1}{a\,n}\sum_{i=1}^{n}\log(u_i)\right)^{-a}\right] - a\right\} + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1)$$
$$+ (a-1)\sum_{i=1}^{n}\log\left[\log(u_i)\right] - \sum_{i=1}^{n}\log(u_i). \tag{3.18}$$

We can note that (3.18) is simpler than (3.16) and might be a useful option for the parameter estimation of the ZBXII distribution. For interval estimation of the components of $\boldsymbol{\theta}$, we can adopt the observed information matrix $\boldsymbol{J(\theta)}$ given by

$$\boldsymbol{J(\theta)} = -\frac{\partial^2\,\ell(\theta)}{\partial\theta\,\partial\theta^T} = \begin{pmatrix} J_{aa} & J_{as} & J_{ad} & J_{ac} \\ . & J_{ss} & J_{sd} & J_{sc} \\ . & . & J_{dd} & J_{dc} \\ . & . & . & J_{cc} \end{pmatrix},$$

whose elements can be obtained from the authors upon request. Under standard regularity conditions, the multivariate normal $N_4(0, \boldsymbol{J(\widehat{\theta})}^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters.

We are able to compute the maximized unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing goodness-of-fit of the ZBXII model with its sub-models. For example, we may use LR statistics to check if the fitted ZBXII distribution for a given data set is statistically “superior" to the fitted BXII, LL and Weibull distributions. In any case, hypothesis tests of the type $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *versus* $H : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ can be performed using LR statistics. For example, the LR statistic for testing $H_0 : a = 1$ *versus* $H : a \neq 1$, which is equivalent to compare the ZBXII and BXII distributions, is given by

$$w = 2\{l(\widehat{a}, \widehat{s}, \widehat{d}, \widehat{c}) - l(1, \widetilde{s}, \widetilde{d}, \widetilde{c})\},$$

where $\widehat{a}$, $\widehat{s}$, $\widehat{d}$ and $\widehat{c}$ are the MLEs under H and $\widetilde{s}$, $\widetilde{d}$ and $\widetilde{c}$ are the estimates under H_0.

3.8 Simulation study

We perform a Monte Carlo simulation to evaluate some asymptotic properties of the MLEs for the parameters of the ZBXII distribution. We generate the ZBXII model for three different combinations of a, c, d and s with samples of sizes $n = 250, 500$ and repeat the simulation $N = 10,000$ times. We use the subroutine `optim` in `R` and the simulated annealing (SANN) algorithm for maximizing the log-likelihood in (3.14). Table 3.1 gives the mean estimates of the MLEs and their root mean squared errors (RMSEs). As expected, the MLEs tend to be closer to the true parameters and the RMSEs decrease as the sample size n increases.

Table 3.1: Monte Carlo results for the mean estimates and RMSEs of the ZBXII distribution.

		Mean				RMSE			
$\boldsymbol{\theta}$	n	$\hat{a}$	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{a}$	$\hat{c}$	$\hat{d}$	$\hat{s}$
$(0.3, 1.7, 0.2, 0.2)$	250	0.345	1.710	0.255	0.205	0.164	0.489	0.206	0.069
	500	0.319	1.703	0.222	0.202	0.085	0.338	0.089	0.043
$(0.5, 1.2, 0.1, 0.3)$	250	0.537	1.232	0.109	0.323	0.162	0.315	0.043	0.193
	500	0.517	1.213	0.104	0.310	0.097	0.160	0.027	0.127
$(0.9, 1.5, 0.3, 0.1)$	250	1.125	1.599	0.348	0.099	0.764	0.905	0.203	0.050
	500	0.984	1.519	0.323	0.099	0.337	0.271	0.114	0.035

3.9 Applications

In this section, we present two applications to real data sets for illustrating the potentiality of the new distribution for modeling positive data. First, we consider the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The real data set consists in 101 data points such that we obtain the exact failure times in hours. Andrews and Herzberg (1985), Cooray and Ananda (2008) and, more recently, Paranaíba *et al.* (2013) also analyzed these data.

The second data set represents the strengths of 1.5 cm glass fibres measured at the National Physical Laboratory, England. It has 51 observations and is available for download at `http://www.stat.ncsu.edu/research/sas/sicl/data/`. These data were previously analysed by Smith and Naylor (1987), Cordeiro and Lemonte (2011) and Paranaíba *et al.* (2013), among others.

We compare the ZBXII distribution with some other competitive models. One of these models is the KwBXII distribution, whose pdf is given by (2.5). The BBXII model has pdf given by (2.4). Introduced by Mudholkar and Srivastava (1993), the exponentiated Weibull (EW) distribution is a popular distribution in lifetime data, whose pdf is given by

$$g(t) = \alpha\,\beta\,\lambda\,t^{\alpha-1}\exp\left(-\lambda\,t^{\alpha}\right)\left[1-\exp\left(-\lambda\,t^{\alpha}\right)\right]^{\beta-1}, \quad t>0,$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter. The Weibull (W) model arises from the EW model when $\beta = 1$.

We estimate the model parameters of the ZBXII in (3.2), BXII and LL models and the above competitive models by the maximum likelihood method. Furthermore, we use the Akaike information criteria (AIC), consistent Akaike information criteria (CAIC), Bayesian information criteria (BIC), Hannan-Quinn information criteria (HQIC) and Kolmogorov-Smirnov (KS) statistic as goodness-of-fit statistics for these models. The lower are these, the better is the adjustment to the data. The MLEs and goodness-of-fit statistics are obtained using SANN algorithm for otimization and the `AdequacyModel` script in `R` software (Marinho *et al.*, 2016).

3.9.1 Stress data

Table 3.2 provides a descriptive summary of the stress data. It has positive skewness and large kurtosis. The mean and median are close but smaller than standard deviation (SD) and variance. Moreover, the amplitude is elevated if compared with the other descriptive statistics.

Table 3.2: Descriptive statistics for stress data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
1.03	0.80	1.12	1.25	3.05	14.51	0.01	7.89

Table 3.3 lists the MLEs (and the corresponding standard errors of the estimates) of the unknown parameters for the fitted models. We note that the estimated standard errors of $\hat{s}$ in the BXII distribution is large.

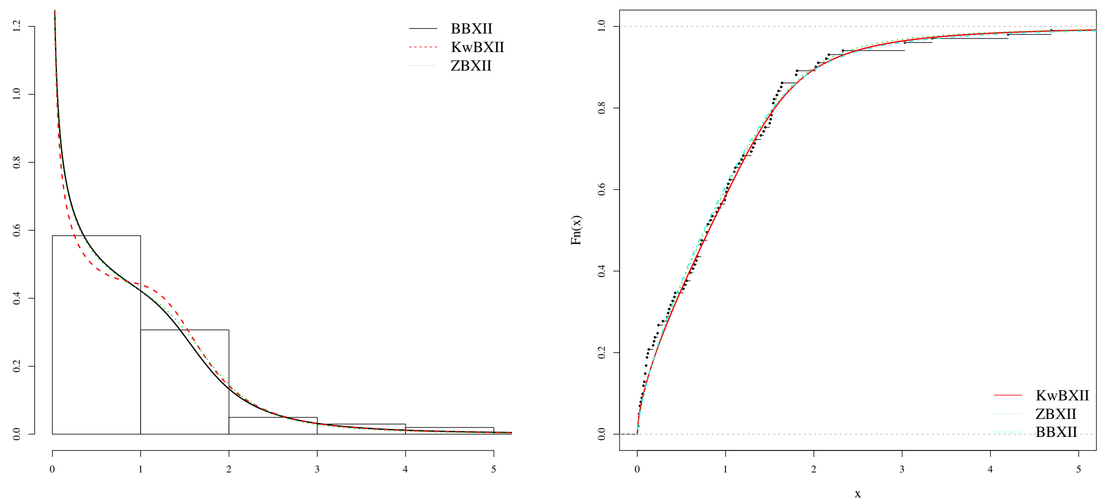
Table 3.3: The MLEs of the model parameters for stress data and corresponding standard errors in parentheses.

	c	d	s	a	b
BBXII	6.7846	0.5625	1.6275	0.1042	0.6052
	(0.0023)	(0.0032)	(0.0198)	(0.0111)	(0.1172)
KwBXII	6.81899	0.54959	1.53837	0.09036	0.68182
	(0.0155)	(0.0051)	(0.0155)	(0.0131)	(0.0835)
ZBXII	6.7147	0.2472	1.7822	0.1059	
	(0.0606)	(0.0557)	(0.0163)	(0.0109)	
BXII	0.9956	6.0925	5.2208		
	(0.0781)	(2.9914)	(2.9612)		
	c	m			
LL	1.270	0.624			
	(0.1069)	(0.0849)			
	λ	α	β		
EW	0.8212	1.0605	0.7931		
	(0.2658)	(0.2404)	(0.2877)		
W	0.9900	0.9258			
	(0.1118)	(0.0726)			

Table 3.4: Goodness-of-fit statistics for the fitted models for stress data.

	AIC	CAIC	BIC	HQIC	KS
BBXII	206.8734	207.5050	219.9490	212.1668	0.0637
KwBXII	206.9635	207.5951	220.0391	212.2569	**0.0619**
ZBXII	**204.0965**	**204.5132**	**214.5570**	**208.3312**	0.0667
BXII	211.9727	212.2201	219.8181	215.1487	0.0910
LL	229.3724	229.4948	234.6026	231.4898	0.1113
EW	211.5743	211.8218	219.4197	214.7504	0.0844
W	209.9536	210.0761	215.1839	212.0710	0.0906

According to the goodness-of-fit statistics (Table 3.4), the ZBXII distribution provides a good fit and is quite competitive with the BBXII and KwBXII models for these data. Considering the AIC, CAIC, BIC and HQIC, the ZBXII distribution yields a better fit than the other distributions. Based on the KS statistic, it is quite competitive to the KwBXII model. Figure 3.4.a displays some plots of the estimated densities and Figure 3.4.b the estimated and empirical cumulative functions for the most competitive models. They reveal a good adjustment for these data of the estimated densities and estimated and empirical cumulative functions of the BBXII, KwBXII and ZBXII distributions. The last one presents better results for the information criteria and has one less parameter. In conclusion, these results reveal that the ZBXII distribution can be used effectively to provide better fits than other lifetime models and it is a competitive alternative for the W, LL and BXII distributions, among others.



(a) (b)

Figure 3.4: (a) Estimated densities of the BBXII, KwBXII and ZBXII models for stress data; (b) estimated and empirical cumulative functions of these models for stress data

3.9.2 Fibres data

Table 3.5 gives a descriptive summary for the fibres data. The mean and median are close and we have small values for SD and variance. It has negative skewness, positive kurtosis and lower variability than the first data set. We can also note that the amplitude is 1.69 for these data.

Table 3.5: Descriptive statistics for fibres data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
1.44	1.52	0.33	0.11	-0.64	0.80	0.55	2.24

The MLEs of the model parameters for fibres data (with standard errors of the estimates) are listed in Table 3.6 for the fitted models. Table 3.7 presents the goodness-of-fit statistics and reveals that the ZBXII distribution yields a good adjustment for the fibres data. It has the lower values for all statistics but BIC and is quite competitive to the W distribution and other BXII generalizations. The plots of the estimated densities and estimated empirical cumulative functions for the most competitive models are displayed in Figure 3.5. We can note that the ZBXII distribution is more accurate in the central measurements, being superior to the W and KwBXII distributions for modeling this data set. Then, we can conclude that the ZBXII distribution provides a better adjustment than the other current distributions and then it is a good alternative for modeling these data.

Table 3.6: The MLEs of the model parameters for fibres data and corresponding standard errors in parentheses.

	c	d	s	a	b
BBXII	7.5794	17.7507	1.9162	0.4901	0.2338
	(0.6201)	(0.5416)	(0.0989)	(0.0739)	(0.0916)
KwBXII	8.7788	16.8998	1.7892	0.3769	0.1756
	(1.0605)	(1.8126)	(0.0026)	(0.1460)	(0.0298)
ZBXII	21.5835	0.5030	1.6843	0.2241	
	(1.8088)	(0.2335)	(0.0530)	(0.0447)	
BXII	5.916	5.886	2.084		
	(0.8826)	(3.8288)	(0.3186)		
	c	m			
LL	7.5390	1.4570			
	(0.9255)	(0.0456)			
	λ	α	β		
EW	0.6204	5.8931	0.8255		
	(0.0453)	(1.5640)	(0.3679)		
W	1.5639	5.2152			
	(0.0440)	(0.5615)			

Table 3.7: Goodness-of-fit statistics for the fitted models for fibres data.

	AIC	CAIC	BIC	HQIC	KS
BBXII	34.8875	36.2208	44.5466	38.5785	0.1993
KwBXII	31.3166	32.6500	40.9758	35.0077	0.1477
ZBXII	**26.9100**	**27.7796**	**34.6373**	**29.8628**	**0.1437**
BXII	33.3953	33.9059	39.1908	35.6099	0.1653
LL	40.8169	41.0669	44.6805	42.2933	0.1871
EW	33.4823	33.9930	39.2778	35.6969	0.1680
W	31.4875	31.7375	**35.3511**	32.9639	0.1747

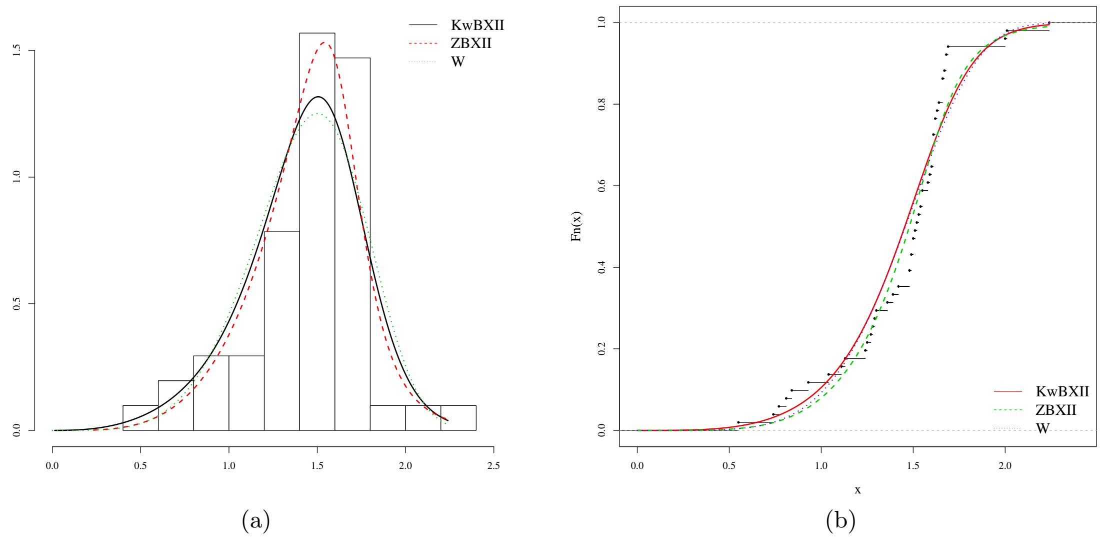


Figure 3.5: (a) Estimated densities of the BBXII, KwBXII and ZBXII models for fibres data; (b) estimated and empirical cumulative functions of these models for fibres data

3.10 Concluding remarks

In this chapter, we introduce a four-parameter distribution, called the Zografos-Balakrishnan Burr XII (ZBXII) distribution. Its hazard rate function allows decreasing, decreasing-increasing-decreasing and upside-down bathtub shapes and provides a Burr XII generalization that may be useful to still more complex situations. The new distribution may be an interesting alternative for modeling lifetime data, among other applications. We obtain some properties of the ZBXII distribution, provide the estimation of the parameters by maximum likelihood, a simulation study and present two applications to real data. We illustrate that the ZBXII distribution yields a good adjustment for both data sets and that it can be used effectively to obtain better fits than other classical lifetime models.

Chapter 4

The Ristić-Balakrishnan Burr XII distribution

Resumo

Uma nova distribuição contínua de quatro parâmetros, chamada Ristić-Balakrishnan Burr XII, é proposta e estudada. São fornecidas algumas propriedades matemáticas da nova distribuição, incluindo expressões explícitas para a função quantílica, momentos, momentos incompletos e função geradora de momentos. O método de máxima verossimilhança é empregado na estimação dos parâmetros do modelo. Um estudo de simulação é realizado. Uma aplicação é realizada para ilustrar que a distribuição proposta pode apresentar melhores ajustes que outras generalizações das distribuições Burr XII e Weibull. *Palavras-chave:* Distribuição Burr XII distribution.

Família gamma-G. Método de máxima verossimilhança. Família Ristć-Balakrishnan.

Abstract

A new four-parameter continuous distribution called the Ristić-Balakrishnan Burr XII distribution is defined and studied. We provide some of its mathematical properties, including explicit expressions for the quantile function, moments, incomplete moments and generating function. The maximum likelihood method is employed for estimating the model parameters. A simulation study is performed. An application is presented for illustrate that the proposed distribution may present consistenly better fits than other Burr XII and Weibull generalizations. *Keywords:* Burr

XII distribution. Gamma-G family. Maximum likelihood method. Ristiíc-Balakrishnan family.

4.1 Introduction

In a similar framework to the Pearson's system, Burr (1942) introduced a system of twelve diferent distributions based on solutions of a differential equation. The BXII distribution is a frequently used model of this system. The three-parameter BXII pdf and cdf are given by (2.2) and (2.1), respectively. It was investigated by Zimmer *et al.* (1998). Rodriguez (1977) and Tadikamalla (1980) summarized some properties of the two-parameter BXII distribution and studied its relation with other models. AL-Hussaini (1991) also gives some characterizations of this distribution.

The BXII distribution has been used as a lifetime model by several authors. Gupta *et al.* (1996) used the BXII model to analyze fibre failure strengths data. Tomer *et al.* (2015) obtain the MLEs for the BXII parameters and reliability function under type-I progressive hybrid censoring scheme. Panahi and Sayyareh (2016) developed the MLEs for the BXII distribution under unified hybrid censoring, which is a mixture of the generalized type I and type II hybrid censoring schemes. Asl *et al.* (2017) propose the use of expectation-maximization (EM) algorithm to compute the MLEs of the BXII distribution under progressive type-II hybrid censored data. Belaghi and Asl (2017) employ EM and stochastic EM algorithm for obtaining the MLEs based on progressively type-I hybrid censored.

In this chapter, we introduce the four-paramerer Ristić-Balakrishnan Burr XII (RBXII) distribution . It is obtained by inserting (2.1) and (2.2) in (1.4) and (1.3). Thus, the RBXII distribution has cdf and pdf given by

$$F(x) = 1 - \frac{\gamma\left(a, -\log\left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}\right)}{\Gamma(a)}, \tag{4.1}$$

and

$$f(x) = \frac{c\, d\, x^{c-1}}{s^c\, \Gamma(a)} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1} \left\{-\log\left(1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right)\right\}^{a-1}, \tag{4.2}$$

respectively. Figure 4.1 displays plots of the RBXII pdf for some parameter values. Note that its pdf allows power-law tails.

If X is a random variable with density function (4.2), we write $X \sim \text{RBXII}(a,s,d,c)$. The RBXII hrf is given by

$$h(x) = \frac{c\,d\,x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-d-1}}{s^c\,\gamma\left(a, -\log\left\{1-[1+(x/s)^c]^{-d}\right\}\right)}\left\{-\log\left(1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-d}\right)\right\}^{a-1}.$$

Figure 4.2 displays plots of the hrf for some parameter values. The RBXII hrf can have decreasing, decreasing-increasing-decreasing and upside-down bathtub hazard functions.

The chapter unfolds as follows. In Section 4.2, we obtain expansions for the RBXII pdf and cdf as linear combination of the BXII model. In Sections 4.3 and 4.4, we present explicit expressions for the quantile function, moments, incomplete moments and generating function of the RBXII model. Section 4.5 is devoted to the RBXII maximum likelihood estimators. In Section 4.6, we carry out a simulation experiment to study the performance of these estimates. In Section 4.7, we illustrate the potentiality of the new distribution by means of an application to real lifetime data. Finally, Section 4.8 concludes the chapter.

4.2 Useful expansion

Useful expansions can be derived for the cdf and pdf in (4.1) and (4.2), respectively. Let $z = \left[1+\left(\frac{x}{s}\right)^c\right]^{-d}$ and consider the power series in (3.3). Thus, we can rewrite the RBXII density as

$$f(x) = c\,d\,s^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-d-1}\sum_{k=0}^{\infty} b_k\left[1+\left(\frac{x}{s}\right)^c\right]^{-d(a+k-1)},$$

where $b_0 = 1/\Gamma(a)$, $b_1 = p_0(a-1)/\Gamma(a)$, $b_2 = p_1(a-1)/\Gamma(a)$, $b_3 = p_2(a-1)/\Gamma(a)$, etc.

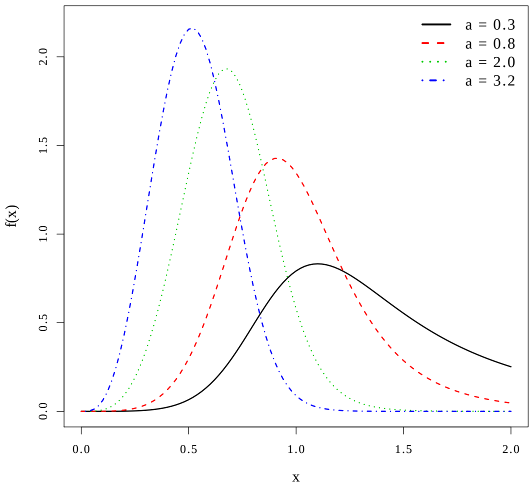
After some algebra, we obtain

$$f(x) = \sum_{k=0}^{\infty} v_k\, g(x; s, (a+k)d, c), \tag{4.3}$$

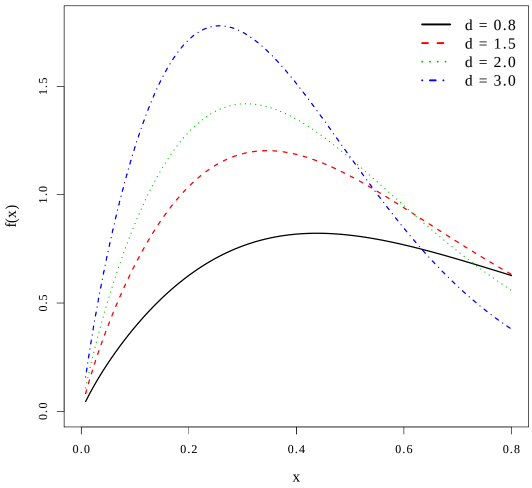
where $v_k = \frac{b_k}{a+k}$ and $g(x; s, (a+k)d, c)$ is the BXII density function with scale parameter s and shape parameters c and $d(a+k)$. Equation (4.3) reveals that the RBXII density is an infinite linear combination of BXII densities. By integrating equation (4.3) gives

$$F(x) = \sum_{r=0}^{\infty} v_r G(x; s, (a+k)d, c). \tag{4.4}$$

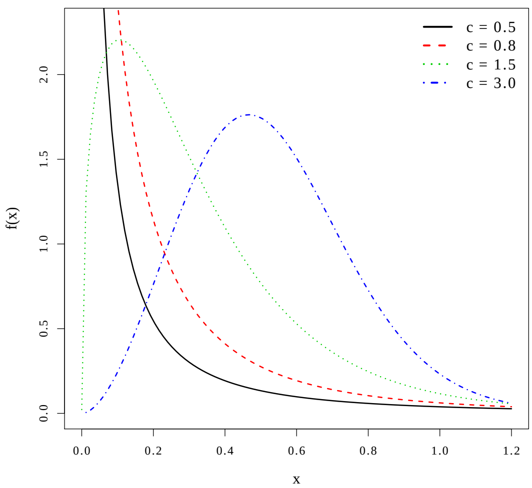
Equations (4.3) and (4.4) are de main result of this section.



(a) For $d = 1.5$ and $c = 5.0$



(b) For $a = 1.5$ and $c = 2.0$



(c) For $a = 1.5$ and $d = 3.0$

Figure 4.1: Pdf plots for the RBXII for $s = 1$.

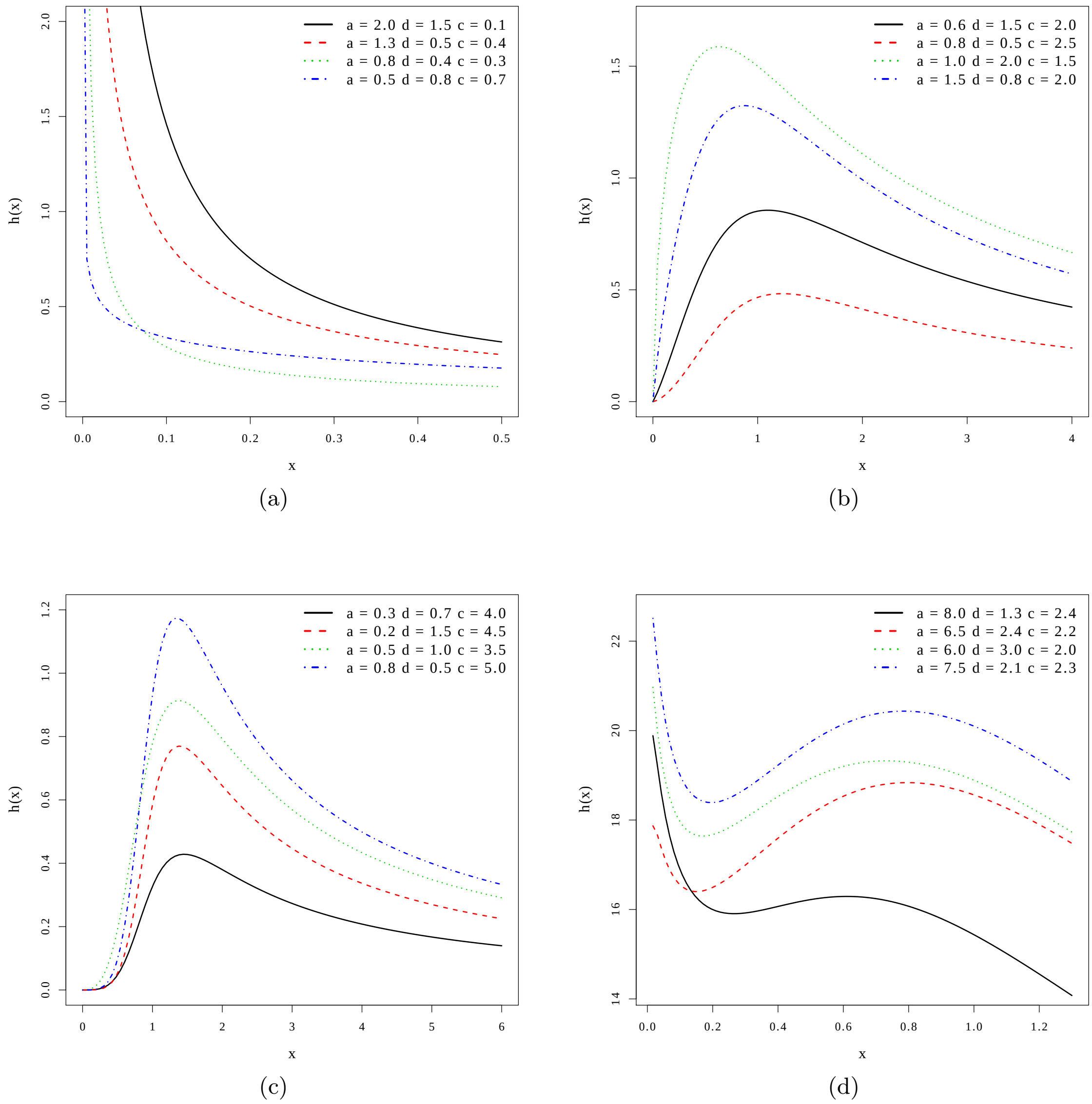


Figure 4.2: Hrf plots for the RBXII.

4.3 Quantile function

The explicit expression for the qf of the RBXII distribution, say $Q(u)$, is

$$Q(u) = s\left\{\left[1 - \frac{1}{\exp\left(Q^{-1}(a,u)\right)}\right]^{-\frac{1}{d}} - 1\right\}^{1/c}, \tag{4.5}$$

where $Q^{-1}(a,u)$ is the inverse function of $Q(a,x) = 1 - \gamma(a,x)/\Gamma(a)$. The median M and other quantiles of the RBXII distribution follow from (4.5) by replacing appropriate values for u. Therefore, it is easy to simulate the RBXII distribution. Let U be a continuous uniform variable on the unit interval. Using the inverse transformation method, the random variable $X = Q(U)$ has the RBXII.

The qf is also a useful tool to obtain alternative expressions for the skewness and kurtosis coefficients. The Bowley's skewness (Kenney and Keeping, 1962) and Moors' kurtosis (Moors, 1988) are given by (3.8) and (3.9), respectively.

4.4 Moments and generating function

In this section, we derive expressions for hth ordinary and incomplete moments and the generating function of the RBXII distribution. Let μ'_h be hth moment of X and μ'_{d_h} be the hth moment of the BXII(c,d,s) distribution. We can provide an expression for the hth moment of X from (4.3) as

$$\mu'_h = \sum_{k=0}^{\infty} v_k\, \mu'_{(a+k)d_h}.$$

Thus, using the result from Zimmer *et al.* (1998) given in equation (2.7), we have (for $h < c\,d$)

$$\mu'_h = s^h\, d \sum_{k=0}^{\infty} (a+k) v_r\, B((a+k)d - h\,c^{-1}, 1 + h\,c^{-1}), \tag{4.6}$$

where $B(\cdot,\cdot)$ is the beta function. By setting $h = 1$, we obtain the mean of X. The moments are also helpful to determine the central moments and cumulants. Thus, the skewness and kurtosis coefficients can be obtained from well-known relationships.

Let $T_h(y)$ and $T_{d_h}(y)$ be the hth incomplete moment of the RBXII(a,c,d,s) and BXII(c,d,s) distributions, respecively. A formula for the hth moment of X can be derived directly from (4.3) as

$$T_h(y) = \sum_{k=0}^{\infty} v_k \, T_{(a+k)d_h}(y). \tag{4.7}$$

By replacing (2.9) in equation (4.7), we obtain (for $h < c\,d$)

$$T_h(y) = s^h d \sum_{k=0}^{\infty} (a+k) v_k \, B_{s^c/s^c+y^c}\left((a+k)d - h\,c^{-1}, 1 + h\,c^{-1}\right). \tag{4.8}$$

Alternatively, the hth incomplete moment can be obtained by inserting (2.10) in equation (4.7). Thus, it can also be expressed as

$$T_h(y) = s^h d \sum_{k=0}^{\infty} (a+k) v_k \, J\left(y, \frac{h}{c}, (a+k)d + 1\right), \tag{4.9}$$

where

$$\begin{aligned} J(y,a,b) &= \int_0^y z^a (z+1)^{-b} \mathrm{d}z \\ &= \frac{y^{a+1} \, {}_2F_1(a+1, b; a+2; -y)}{a+1} \end{aligned}$$

and ${}_2F_1$ is the hypergeometric function.

The deviations from the mean and the median are important measures that apply the concept of incomplete moments. They can be determined as $\delta_1 = 2\mu_1' F(\mu_1') - 2\,T_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\,T_1(M)$, respectively. The quantity $\mu_1' = I\!E(X)$ is the mean of X, the median M follows from (3.7), $F(\mu_1')$ is easily obtained from (4.1) and $T_1(y) = \int_0^y x\,f(x)\mathrm{d}x$ is the first incomplete moment, which arises from $T_h(y)$ by taling $h = 1$. The Bonferroni and Lorenz curves are other related applications. They are defined by $B(\pi) = T_1(q)/(\pi\mu_1')$ and $L(\pi) = T_1(q)/\mu_1',$, where π is a given probability and $q = Q(\pi)$ is obtained from (3.7).

We can obtain the mgf of X can be obtained from (3.5) as

$$M(t) = \sum_{k=0}^{\infty} v_k \, M_{(a+k)d}(t), \tag{4.10}$$

where $M_{(r+1)d}(t)$ is the mgf of the BXII$(s, (r+1)d, c)$ distribution. For $t < 0$, Paranaíba *et al.* (2011, 2013) provided a representation for the BXII mgf, which is given in equation (2.11). By inserting this result in (4.10), we have that the mgf of X can be expressed as (for $t < 0$)

$$M(t) = m \sum_{k=0}^{\infty} v_k \, I\left(-s\,t, \frac{m}{(a+k)d} - 1, \frac{m}{(a+1)d}, -(a+k)d - 1\right). \tag{4.11}$$

Equations (4.6), (4.7), (4.9) and (4.11) are the main results of this section.

4.5 Maximum likelihood estimation

Here, we consider the estimation of the unknown parameters of the RBXII distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample of size n from the RBXII(a, c, d, s) distribution. The log-likelihood function for the parameter vector $\boldsymbol{\theta} = (a, c, d, s)^T$ is

$$\begin{aligned} l(\boldsymbol{\theta}) = & n \log \left[\frac{c\, d}{s\, \Gamma(a)} \right] + (c-1)c^{-1} \sum_{i=1}^{n} \log(u_i - 1) - (d+1) \sum_{i=1}^{n} \log(u_i) \\ & + (a-1) \sum_{i=1}^{n} \log \left[-\log(1 - u_i^{-d}) \right], \end{aligned} \tag{4.12}$$

where $u_i = 1 + \left(\frac{x_i}{s}\right)^c$. The components of the score vector $\boldsymbol{U(\theta)}$ are given by

$$\begin{aligned} U_a(\boldsymbol{\theta}) &= -n\psi(a) + \sum_{i=1}^{n} \log \left[-\log(1 - u_i^{-d}) \right], \\ U_c(\boldsymbol{\theta}) &= \frac{n}{c} + \sum_{i=1}^{n} \log(u_i - 1) - \frac{d+1}{c} \sum_{i=1}^{n} \frac{(u_i - 1) \log(u_i - 1)}{u_i} \\ &+ \frac{(a-1)\, d}{c} \sum_{i=1}^{n} \frac{(u_i - 1)\, u_i^{d-1} \log(u_i - 1)}{(1 - u_i^d) \log(1 - u_i^d)}, \\ U_d(\boldsymbol{\theta}) &= \frac{n}{d} - \sum_{i=1}^{n} \log(u_i) - (a-1) \sum_{i=1}^{n} \frac{u_i^d \log(u_i)}{(1 - u_i^d) \log(1 - u_i^d)}, \end{aligned}$$

and

$$U_s(\boldsymbol{\theta}) = \frac{-nc}{s} + \frac{c\,(d+1)}{s} \sum_{i=1}^{n} \frac{(u_i - 1)}{u_i} + \frac{c\, d\,(a-1)}{s} \sum_{i=1}^{n} \frac{(u_i - 1)\; u_i^{d-1}}{(1 - u_i^d) \log(1 - u_i^d)},$$

where $\psi(.)$ is the digamma function. Setting these expressions, $\boldsymbol{U(\theta)} = \mathbf{0}$, and solving them simultaneously yields the MLEs of the unknown parameters for the RBXII distribution.

Note that the we can express MLE of a can be expressed in a semi-closed form. For fixed c, d and s, we obtain

$$\hat{a}(\hat{c}, \hat{d}, \hat{s}) = \psi^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \log \left[-\log(1 - u_i^{-d}) \right] \right).$$

By replacing a by $\hat{a}$ in equation (4.12), we have the profile log-likelihood function for $\boldsymbol{\theta_p} = (c, d, s)$. It can be expressed as

$$\ell(\boldsymbol{\theta_p}) = n\log(c\,d) + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1) - (d+1)\sum_{i=1}^{n}\log(u_i)$$
$$+\left[\psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\log\left[-\log(1-u_i^{-d})\right]\right) - 1\right]\sum_{i=1}^{n}\log\left[-\log(1-u_i^{-d})\right]$$
$$-\,n\,\log\left\{s\,\Gamma\left[\psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\log\left[-\log(1-u_i^{-d})\right]\right)\right]\right\}.$$

where $\psi^{-1}(.)$ is the inverse digamma function.

The observed information matrix of X, denoted by $\boldsymbol{J(\theta)}$, can be expressed as

$$\boldsymbol{J(\theta)} = -\frac{\partial^2\,\ell(\theta)}{\partial\theta\,\partial\theta^T} = \begin{pmatrix} J_{aa} & J_{as} & J_{ad} & J_{ac} \\ . & J_{ss} & J_{sd} & J_{sc} \\ . & . & J_{dd} & J_{dc} \\ . & . & . & J_{cc} \end{pmatrix}.$$

The elements $\boldsymbol{J(\theta)}$ can be obtained from the authors upon request and are useful for interval estimation and hypothesis test.

4.6 Simulation study

In this section, a simulation study is carried out in order to evaluate the asymtotic performance of the MLEs for the parameters of the RBXII distribution. We consider five parameter combinations and the process is repeated 10,000 times. The subroutine `optim` and SANN algorithm in `R` software is used for maximizing the log-likelihood in (4.12) for the sample sizes $n = 100, 250$ and 500. The mean estimates and the RMSEs of the MLEs are listed in Table 4.1. We note that the maximum likelihood method performs well for estimating the model parameters. As expected, the biases and RMSEs of the MLEs decrease as the sample size increases.

4.7 Application

In this section, we use a data set corresponding to the failure times of 20 mechanical components. It is reported in Murthy *et al.* (2004). We provide some statistics for these data in

Table 4.1: Monte Carlo results for the mean estimates and RMSEs of the RBXII distribution.

		Mean				RMSE			
$\boldsymbol{\theta}$	n	$\hat{a}$	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{a}$	$\hat{c}$	$\hat{d}$	$\hat{s}$
$(8, 5, 3, 0.1)$	100	8.393	5.169	3.699	0.103	2.567	0.907	2.264	0.030
	250	8.214	5.081	3.382	0.101	1.833	0.645	1.732	0.021
	500	8.125	5.045	3.231	0.100	1.425	0.505	1.458	0.016
$(3.5, 0.2, 4.5, 9)$	100	4.129	0.219	4.477	9.110	1.787	0.051	2.260	2.843
	250	3.839	0.209	4.485	9.061	1.279	0.036	1.821	2.202
	500	3.737	0.206	4.442	9.068	1.002	0.027	1.517	1.868
$(0.5, 7, 3.5, 0.4)$	100	1.262	8.024	3.443	0.428	1.877	1.889	2.131	0.0915
	250	0.656	7.282	3.454	0.404	0.412	0.803	1.636	0.026
	500	0.642	7.230	3.338	0.402	0.391	0.635	1.405	0.017
$(0.9, 2.5, 0.1, 0.2)$	100	0.920	2.807	0.197	0.247	0.727	1.079	0.246	0.184
	250	0.808	2.665	0.177	0.208	0.533	0.697	0.165	0.078
	500	0.779	2.560	0.175	0.199	0.464	0.456	0.154	0.054
$(7.5, 0.2, 5, 3)$	100	7.848	0.207	5.101	3.168	1.887	0.036	2.175	2.115
	250	7.645	0.202	5.044	3.092	1.351	0.024	1.742	1.786
	500	7.581	0.201	5.020	3.055	1.045	0.018	1.453	1.525

Table 4.2. It has positive skewness and kurtosis. Its mean is higher than the median and we can note that the amplitude is 0.41 for this data set.

Table 4.2: Descriptive statistics for the mechanical components failure time data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
0.12	0.10	0.09	0.01	3.59	12.20	0.07	0.48

For modeling these data, we consider the RBXII distribution and other six competitive models described as follows (for $x > 0$):

- The BXII distribution, with pdf given in (2.2).
- The BBXII distribution, with pdf given in (2.4).
- The KwBXII distribution, with pdf given in (2.5).
- The generalized power Weibull (GPW) distribution, with pdf given by

$$g(t) = \alpha\lambda\gamma t^{\gamma-1}(1+\lambda t^{\alpha})^{\beta-1}\exp\{1-(1+\lambda t^{\alpha})^{\beta}\}, \tag{4.13}$$

where $\lambda > 0$ is a scale parameter and α and β are positive shape parameters.

- The EW distribution, with pdf given by

$$g(t) = \alpha\,\beta\,\lambda\,x^{\alpha-1}\exp\left(-\lambda\,x^{\alpha}\right)\left[1-\exp\left(-\lambda\,x^{\alpha}\right)\right]^{\beta-1},$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter.

- The W distribution, wich arises from the GPW and EW distribution by taking $\beta = 1$. It is also a special case of the RBXII distribution for $d \to \infty$ and $a = 1$.

Table 4.3 presents the MLEs of the model parameters. Note that the parameters of all fitted distributions are significant. The MLEs and goodness-of-fit statistics are determined using SANN algorithm in the `AdequacyModel` script in the `R` software Marinho *et al.* (2016). We evaluate AIC, CAIC, BIC, HQIC and KS statistics for the fitted models (Table 4.4). The lower are these statistics, the better is the fit to the data.

Table 4.3: The MLEs of the model parameters for the mechanical components failure time data and corresponding standard errors in parentheses.

	c	d	s	a	b
BBXII	2.3098	3.7239	0.0919	3.1998	0.3627
	(0.5843)	(1.8053)	(0.0232)	(1.5787)	(0.1745)
KwBXII	4.7744	3.1877	0.0861	2.1213	0.2324
	(0.3387)	(0.5130)	(0.0094)	(0.5582)	(0.0867)
RBXII	9.4081	1.5709	0.0743	0.1755	
	(0.1135)	(0.1696)	(0.0057)	(0.0482)	
BXII	5.54265	0.51927	0.0868		
	(1.4591)	(0.2106)	(0.0107)		
	λ	α	β		
GPW	0.0666	5.4595	0.1809		
	(0.0083)	(1.6523)	(0.0593)		
EW	1.9999	17.8650	0.0391		
	(0.0024)	(0.0137)	(0.0087)		
W	0.1374	1.6519			
	(0.0198)	(0.2316)			

According to all good-of-fit statistics, the RBXII distribution provides the smallest values for them. So, this distribution gives the best fit among the fitted models under these statistics. Thus, the RBXII is quite competitive with the Weibull and BXII generalizations. More information are provided by a visual comparison among the histogram and the three estimated densities

Table 4.4: Goodness-of-fit statistics for the fitted models for the mechanical components failure time data.

	AIC	CAIC	BIC	HQIC	KS
BBXII	-60.3299	-56.0442	-55.3513	-59.3580	0.1810
KwBXII	-65.7308	-61.4451	-60.7521	-64.7589	0.1129
RBXII	**-69.5934**	**-66.9268**	**-65.6105**	**-68.8159**	**0.0979**
BXII	-68.0884	-66.5884	-65.1012	-67.5053	0.1413
GPW	-61.7213	-60.2213	-58.7341	-61.1382	0.2335
W	-48.8432	-48.1373	-46.8517	-48.4544	0.2631
EW	-26.3297	-24.8297	-23.3425	-25.7465	0.4990

with best fits (Figure 4.3.a). The fitted cdfs of these models are also displayed in Figure 4.3.b. The plots confirm that the WBXII, Weibull and BXII models provide good fits to these data. Clearly, the RBXII presents a better adjustment for both plots. Therefore, we can conclude that the RBXII distribution is a competitive alternative for other BXII and Weibull generalizations in real applications. This distribution can be chosen as the best model for this data.

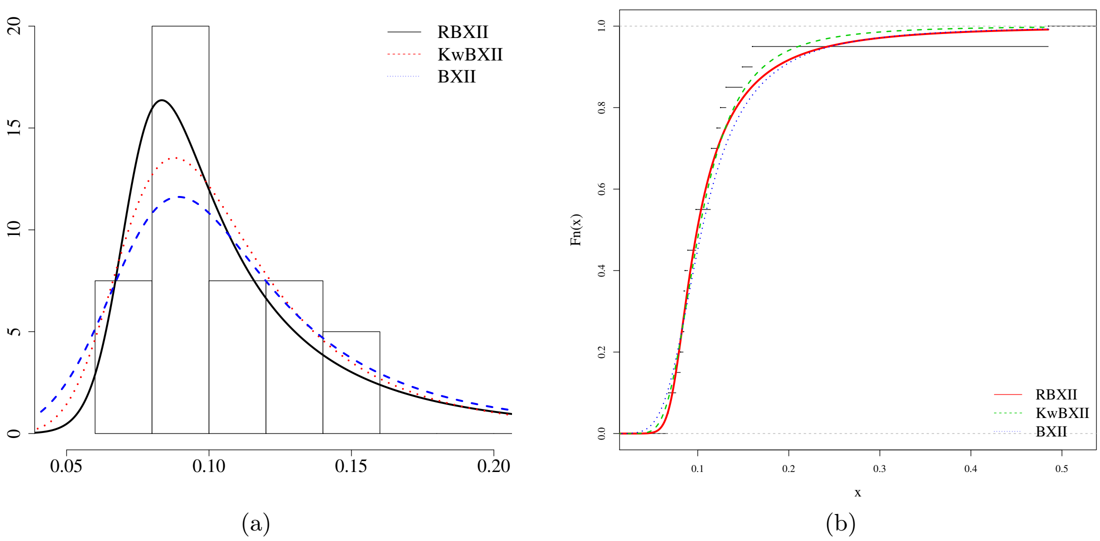


Figure 4.3: (a) Estimated densities of the BBXII, KwBXII and RBXII models for the mechanical components failure time data; (b) estimated and empirical cumulative functions of these models for the mechanical components failure time data

4.8 Concluding remarks

In this chapter, we propose a new four-parameter distribution, called the Ristić-Balakrishnan BXII (RBXII) distribution. We demonstrate that the RBXII density and cumulative distribution functions can be expressed as infinite linear combinations of the BXII baseline. We also study some of its mathematical properties, providing explicit expressions for the quantile function, moments, incomplete moments and generating function. The maximum likelihood method is employed for estimating the model parameters and a simulation study is performed. An application is presented for illustrative purposes. The proposed distribution presented consistenly better fits than other competing models. Therefore, the RBXII may be a competitive alternative for other BXII and Weibull generalizations.

Chapter 5

The Weibull Burr XII distribution

Resumo

Neste capítulo propomos uma nova distribuição de cinco parâmetros, denominada Weibull Burr XII. O novo modelo pode ser uma alternativa útil para descrever a distribuição de renda, podendo também ser aplicada em ciências atuariais, finanças, análise de sobrevivência e diversas outras áreas. A distribuição proposta contém como casos especiais as distribuições Weibull log-logística, Weibull Lomax e *generalized power Weibull.* São investigadas algumas propriedades do modelo proposto. O método de máxima verossimilhança é utilizado para a estimação dos parâmetros do modelo. Um estudo de simulação é realizado. Também são apresentadas duas aplicações em conjuntos de dados reais, as quais ilustram a utilidade da distribuição proposta para modelar dados de renda e também de análise de sobrevivência. Além disso, as aplicações mostram que o novo modelo é bastante competitivo com generalizações das distribuições Burr XII e Weibull.

Palavras-chave: Distribuição Burr XII. Distributição de renda. Método de máxima verossimilhança. Família Weibull-G.

Abstract

In this chapter, we introduce a five-parameter model called the Weibull Burr XII distribution. The new model may be a useful alternative to describe income distributions and can also be applied in actuarial science, finance and lifetime data and several other areas. It contains as special models the Weibull log-logistic, Weibull Lomax and generalized power Weibull distributions,

among others. Some of its properties are investigated. The method of maximum likelihood is used for estimating the model parameters. A simulation study is provided. We also present two applications to real data sets. They illustrate the usefulness of the proposed distribution for modeling income and lifetime data and also show that the new distribution is quite competitive with other Burr XII and Weibull generalizations.

Keywords: Burr XII distribution. Income distributions. Maximum likelihood method. Weibull-G family.

5.1 Introduction

The BXII model has a wide usage in the context of income distributions, see (Jäntti and Jenkins, 2010; Brzeziński, 2013; Shakeel *et al.*, 2015; Tanak *et al.*, 2015) for recent examples. Cirillo (2010) also applied this model for analysing the size distribution of Italian firms by age. Chotikapanich *et al.* (2013) considered it for calculating poverty measures in countries from South and Southeast Asia. Kumar *et al.* (2013) estimated the BXII distribution on reliability context.

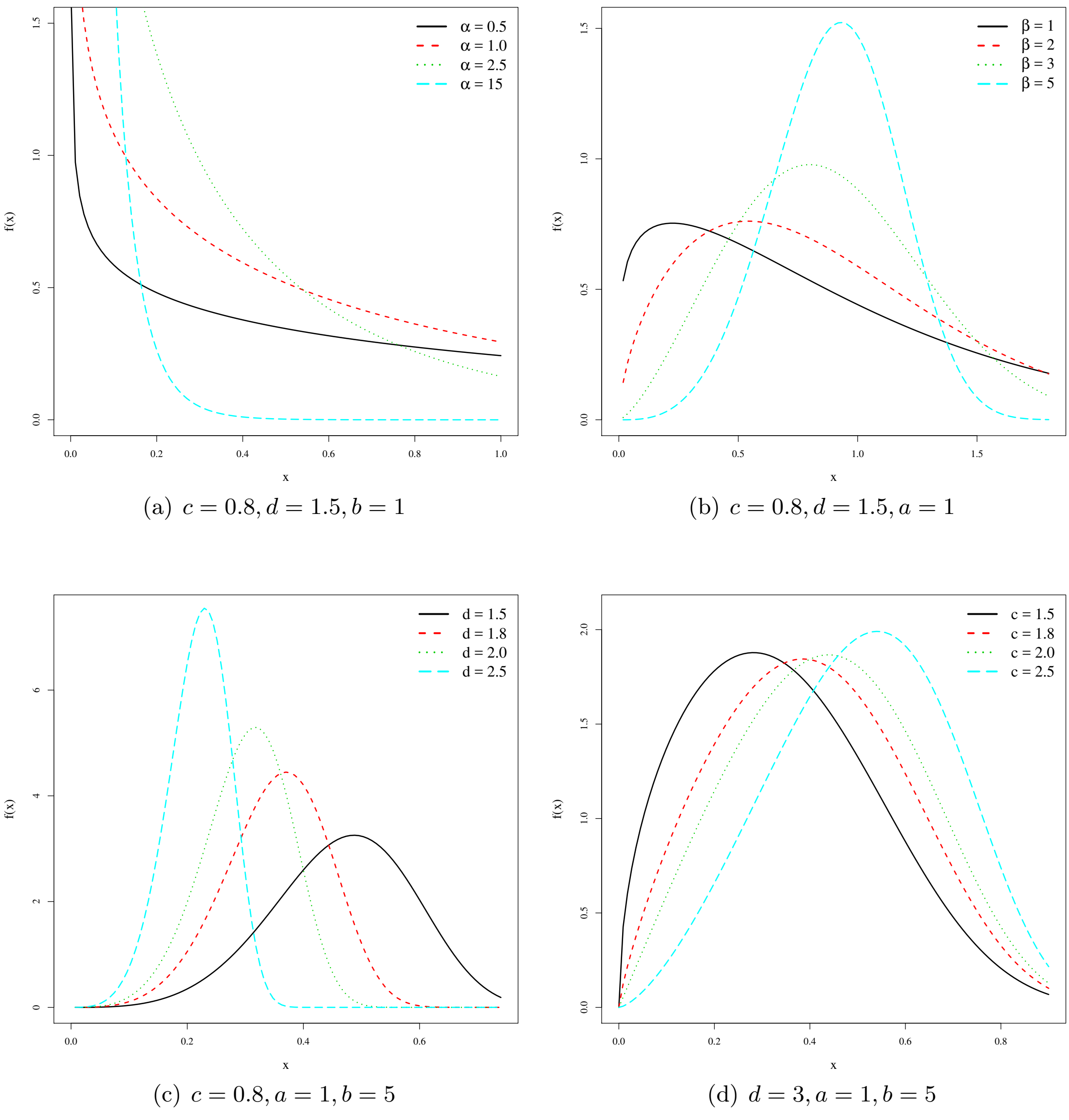
In this chapter, we define the *Weibull-Burr XII* (WBXII) distribution by inserting (2.1) and (2.2) in equations (1.5) and (1.6). Thus, the WBXII pdf reduces to (for $x > 0$)

$$f(x) = \frac{\alpha\,\beta\,c\,d\,s^{-c}x^{c-1}}{[1+(x/s)^c]^{1-d}} \exp\left\{-\alpha\left[(1+(x/s)^c)^d-1\right]^{\beta}\right\}\left[(1+(x/s)^c)^d-1\right]^{\beta-1}, \qquad (5.1)$$

where $\alpha > 0$, $\beta > 0$, $d > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter. The corresponding cdf is given by

$$F(x) = 1 - \exp\left\{-\alpha\left[(1+(x/s)^c)^d-1\right]^{\beta}\right\}. \qquad (5.2)$$

If X is a random variable with density function (5.1), we write $X \sim \text{WBXII}(c, d, s, a, b)$. Plots of the WBXII density function for selected parameter values are displayed in Figure 5.1. It can take various forms and has as special models the GPW (Nikulin and Haghighi, 2006) for $\alpha = \beta = 1$ and $\lambda = 1/s$, Weibull log-logistic for $d = 1$ and $s = m^{-1}$ and Weibull Lomax (Tahir *et al.*, 2015) for $c = 1$.



(a) $c = 0.8, d = 1.5, b = 1$

(b) $c = 0.8, d = 1.5, a = 1$

(c) $c = 0.8, a = 1, b = 5$

(d) $d = 3, a = 1, b = 5$

Figure 5.1: Pdf plots for the WBXII distribution with $s = 1$

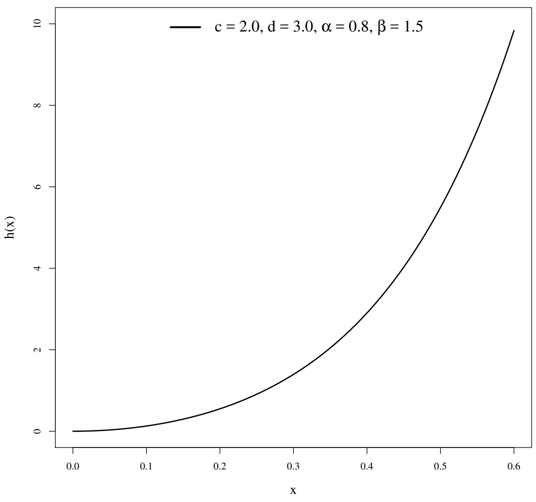
The hrf of X reduces to

$$h(x) = \alpha\,\beta\,c\,d\,s^{-c}x^{c-1}\,[1+(x/s)^c]^{d-1}\,[(1+(x/s)^c)^d-1]^{\beta-1}.$$

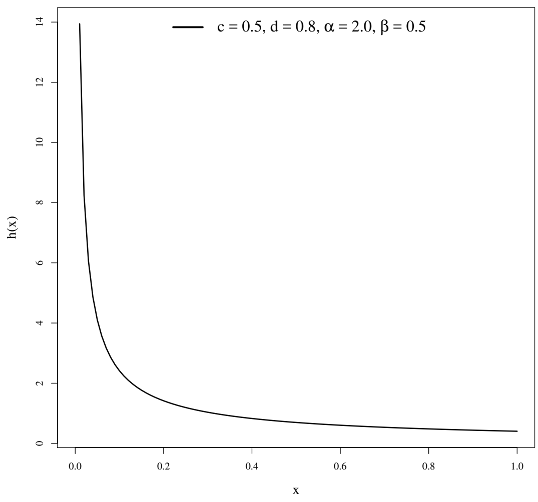
Figure 5.2 provides some plots of the hrf for selected parameter values. It reveals that the proposed distribution can have decreasing, increasing, upside-down bathtub and bathtub-shaped hazard functions. This characteristic makes this distribution very attractive for real applications. It can be applied in the context of engineering, income distributions, finance and lifetime data, for example.

Because the WBXII distribution has pdf and hrf quite flexible, it can also be a useful alternative to the BXII model and its generalizations. Considering different generalized (or generated) G families of continuous univariate distributions, we can cite the beta Burr XII (Paranaíba *et al.*, 2011), Kumaraswamy Burr XII (Paranaíba *et al.*, 2013), exponentiated Burr XII (Al-Hussaini and Hussein, 2011b), Marshal-Olkin extended Burr XII (Al-Saiari *et al.*, 2014) and McDonald Burr XII (Gomes *et al.*, 2015) distributions, among others. Therefore, the introduced distribution may provide an interesting alternative to describe still more complex situations. For example, the characterization and understanding of the income distribution, which is still an open problem in economic science (Moura Jr and Ribeiro, 2009).

The rest of the chapter is organized as follows. Useful expansions for the WBXII cdf and pdf are derived in Section 5.2. In Section 5.3, we investigate some of its mathematical properties such as the qf, ordinary and incomplete moments, mean deviations and generating function. In Section 5.5, we perform a simulation study. The maximum likelihood method is used to estimate the model parameters in Section 5.4. Two applications to lifetime and income data are addressed in Section 5.6. Section 5.7 offers some concluding remarks.



(a)



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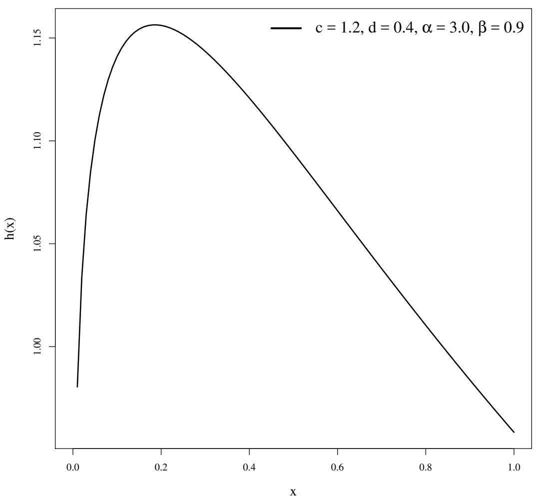
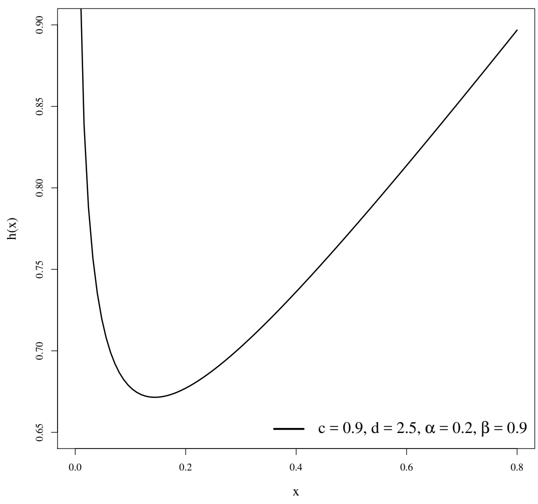


(c)



(d)

Figure 5.2: Hrf plots for the WBXII distribution with $s = 1$

5.2 Useful expansions

Some useful expansions for equations (5.1) and (5.2) can be derived by using power series. It follows from Bourguignon *et al.* (2014) that the *Weibull-G* density function can be expressed as

$$f(x) = \sum_{j,k=0}^{\infty} \upsilon_{j,k}\, g(x) G(x)^{[\beta(k+1)+j-1]-1},$$

where

$$\upsilon_{j,k} = \frac{(-1)^k\, \beta\, \alpha^{k+1}\, \Gamma(\beta(k+1)+j+1)}{k!\, j!\, \Gamma(\beta(k+1)+1)}.$$

By replacing $G(x)$ for (2.1) and $g(x)$ for (2.2), we have

$$f(x) = c\, d\, s^{-c} \sum_{j,k=0}^{\infty} \upsilon_{j,k}\, x^{c-1} u^{-d-1} \left(1-u^{-d}\right)^{\beta(k+1)+j-2}, \tag{5.3}$$

where $u = 1 + \left(\frac{x}{s}\right)^c$. Note that if $|z| < 1$ and $b > 0$ is a real non-integer, the power series holds

$$(1-z)^{b-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(b)}{r!\, \Gamma(b-r)}\, z^r.$$

Using the above expansion for $\left(1-u^{-d}\right)^{[\beta(k+1)+j-1]-1}$ in equation (5.3) and after some algebraic manipulation, we obtain

$$f(x) = \sum_{r=0}^{\infty} w_r\, g(x; c, (r+1)d, s), \tag{5.4}$$

where (for $r = 0, 1, \ldots$)

$$w_r = \sum_{k,j=0}^{\infty} \frac{(-1)^r\, \upsilon_{j,k}\, \Gamma(\beta(k+1)+j-1)}{\Gamma(\beta(k+1)+j-r-1) r!\, (r+1)}$$

and $g(x; c, (r+1)d, s)$ is the BXII density function with scale parameter s and shape parameters $(r+1)d$ and c. Equation (5.4) reveals that the WBXII density is an infinite linear combination of BXII densities. So, several structural properties of the WBXII distribution can follow from those BXII properties. By integrating equation (5.4) gives

$$F(x) = \sum_{r=0}^{\infty} w_r\, G(x; c, (r+1)d, s). \tag{5.5}$$

Equations (5.4) and (5.5) are the main results of this section.

5.3 Mathematical properties

In this section, we investigate some mathematical properties of the WBXII distribution including quantile and random number generation, ordinary and incomplete moments, mgf, mean deviations and Bonferroni and Lorenz curves. By determining analytical expressions for those mathematical quantities for the WBXII distribution can be more efficient than computing them directly by numerical integration of its density function.

5.3.1 Quantile function and random number generation

The qf of X follows by inverting (5.2) as

$$Q(u) = s\left\{\left[\left(\frac{-\log(1-u)}{\alpha}\right)^{1/b} + 1\right]^{1/d} - 1\right\}^{1/c}. \qquad (5.6)$$

By setting $u = 0.5$ in (5.6) gives the median M of X. Other quantiles of interest can also be obtained from (5.6) by replacing appropriate values for u. Simulating the WBXII random variable is straightforward by using the inverse transformation method. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ has pdf given by (5.1).

5.3.2 Moments

The nth ordinary moment of X can be determined from (5.4) as

$$\mu'_n = E(X^n) = \sum_{r=0}^{\infty} w_r \int_0^{\infty} x^n\, g(x; d, (r+1)d, s)\mathrm{d}x,$$

where $B(\cdot,\cdot)$ is the beta function.

Using a result in Zimmer *et al.* (1998) in (2.7), we have (for $n < c\,d$)

$$\mu'_n = s^n\, d \sum_{r=0}^{\infty} (r+1) w_r\, B((r+1)d - n\,c^{-1}, 1 + n\,c^{-1}), \qquad (5.7)$$

where $B(\cdot,\cdot)$ is the beta function. The central moments (μ_s) and cumulants (κ_s) of X follow recursively from (5.7) as

$$\mu_s = \sum_{i=0}^{p} \binom{s}{i} (-1)^i\, \mu_1'^{s}\, \mu'_{s-i} \qquad \text{and} \qquad \kappa_s = \mu'_s - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \kappa_i\, \mu'_{s-i},$$

respectively, where $\kappa_1 = \mu'_1$. The skewness and kurtosis measures can be evaluated from the ordinary moments using well-known relationships.

Alternative expressions for the skweness and kurtosis can be based on quantile measures, i.e., they can be obtained from (5.6). These measures are less sensitive to outliers and may exist even for distributions without moments. Figure 5.3 displays plots of Bowley's skewness (3.8) and Moors' kurtosis (3.9) for some parameter values. They indicate that the proposed distribution is quite flexible in terms of variation of the skewness and kurtosis. Note that the WBXII distribution allows negative and positive values for the skewness.

5.3.3 Incomplete moments

The hth incomplete moment of X is defined by $T_h(y) = \int_0^y x^h\, f(x)dx$. It can be expressed as

$$T_h(y) = c\,d \sum_{r=0}^{\infty} (r+1)\, w_r \int_0^y x^{h-1} \left(\frac{x}{s}\right)^c \left[1 + \left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1} dx.$$

By setting $t = \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}$ in the last equation, we obtain

$$T_h(y) = d\,s^h \sum_{r=0}^{\infty} (r+1)\, w_r \int_{s^c/(s^c+y^c)}^1 t^{(r+1)d-\frac{h}{c}-1}\,(1-t)^{\frac{h}{c}}\, dt.$$

Hence, the hth incomplete moment of X reduces to (for $h < c\,d$)

$$T_h(y) = d\,s^h \sum_{r=0}^{\infty} (r+1)\, w_r\; B_{s^c/s^c+y^c}((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}), \tag{5.8}$$

where $B_z(a,b) = \int_z^1 t^{a-1}\,(1-t)^{b-1}dt$ is the upper incomplete beta function.

5.3.4 Mean deviations

An important application of the first incomplete moment refers to the mean deviations about the mean and the median defined by

$$\delta_1 = 2\mu'_1 F(\mu'_1) - 2\,T_1(\mu'_1) \qquad \text{and} \qquad \delta_2 = \mu'_1 - 2\,T_1(M),$$

respectively, where $\mu'_1 = E(X)$, the median M of X can by determined from (5.6) by $M = Q(1/2)$, $F(\mu'_1)$ is easily obtained from (5.2) and (for $c\,d > 1$) $T_1(y)$ is the first incomplete

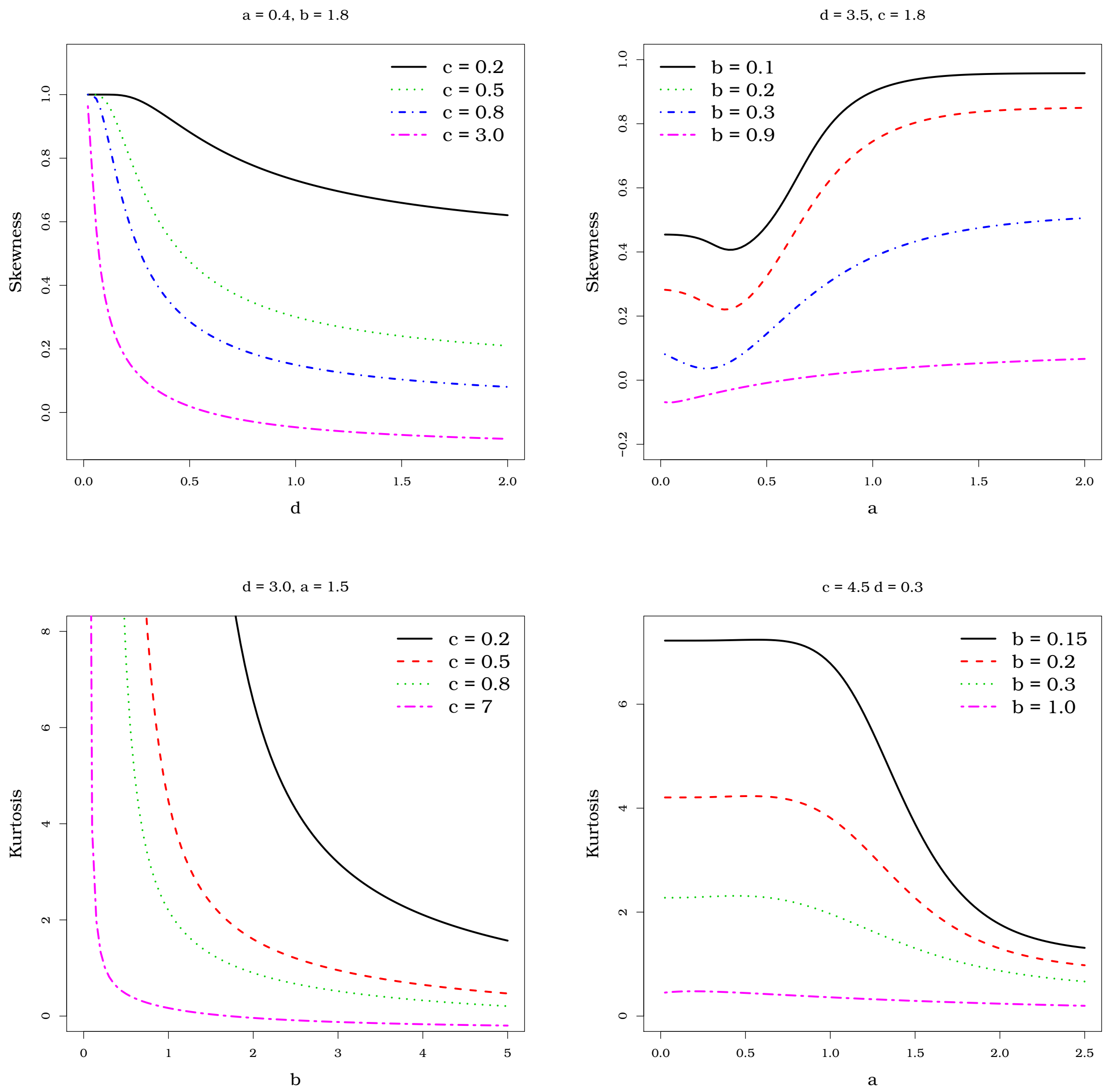


Figure 5.3: Skewness and kurtosis of WBXII for some parameter values.

moment given by (5.8) with $h = 1$. An alternative expression for the first incomplete moment comes from (5.4) as

$$T_1(y) = cds\sum_{r=0}^{\infty}(r+1)w_r\int_0^y x^c\left[1+\left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1}\mathrm{d}x.$$

Setting $z = (x/s)^c$, we obtain

$$\begin{aligned} T_1(y) &= d\,s\sum_{r=0}^{\infty}(r+1)w_r\int_0^{\left(\frac{y}{s}\right)^c} z^{1/c}(1+z)^{-(r+1)d-1}\mathrm{d}z. \\ &= \frac{c\,d\,s\,y^{c+1}}{1+c}\sum_{r=0}^{\infty}(r+1)\,w_r\,{}_2F_1\left[1+\frac{1}{c},\ (r+1)d+1;\ 2+\frac{1}{c};\ -\left(\frac{y}{s}\right)^c\right], \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $|x| < 1$, $c = 0, -1, -2, \ldots$ and $(z)_n$ is the Pochhammer polynomial.

The above results are related to the Bonferroni and Lorenz curves. These curves are important in economics for studying income and poverty, but can be useful in demography, reliability, insurance, medicine and several other fields. For a given probability π, they are defined by $B(\pi) = T_1(q)/(\pi\mu_1')$ and $L(\pi) = T_1(q)/\mu_1'$, respectively, where $q = Q(\pi)$ is given by (5.6). If π is the proportion of units whose income is lower than or equal to q, the values of $L(\pi)$ yield fractions of the total income and $B(\pi)$ refers to the relative income levels.

The Lorenz curve allows us to obtain the Gini concentration (C_G) defined by $C_G = 1 - 2\int_0^1 L(u)\mathrm{d}u$. It represents the area between the curve $L(u)$ and the straight line. Clearly, C_G can be evaluated by numerical integration.

5.3.5 Generating function

The mgf of X is defined by $M(t) = E(\mathrm{e}^{tX})$. We denote by $M_d(t)$ the mgf of the BXII(c, d, s) distribution. Thus, the mgf of the WBXII distribution can be obtained from (5.4) as

$$M(t) = \sum_{r=0}^{\infty} w_r\, M_{(r+1)d}(t), \tag{5.9}$$

where $M_{(r+1)d}(t)$ is the mgf of the BXII$(s, (r+1)d, c)$ distribution. For $t < 0$, we have

$$M_d(t) = c\,d\int_0^{\infty}\exp(y\,s\,t)\,y^{c-1}\,(1+y^c)^{-(d+1)}dy.$$

The above representation is provided by Paranaíba *et al.* (2011, 2013). The authors also demonstrated that $M_d(t)$ can be expressed in terms of the Meijer G-function. Hence, for $t < 0$, the mgf of X can be obtained by inserting (2.11) in (5.9) as

$$M(t) = m \sum_{r=0}^{\infty} w_r\, I\left(-s\,t, \frac{m}{(r+1)d} - 1, \frac{m}{(r+1)d}, -(r+1)d - 1\right). \tag{5.10}$$

Equations (5.9) and (5.10) are the main results of this section.

5.4 Maximum likelihood estimation

In this section, we estimate the five parameters of the WBXII distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample of size n from the WBXII(c, d, s, α, β) distribution. Let $\boldsymbol{\theta} = (c, d, s, \alpha, \beta)^T$ be the parameter vector. The log-likelihood function for $\boldsymbol{\theta}$ can be expressed as

$$\begin{aligned}\ell(\boldsymbol{\theta}) =& \, n \log(\alpha\,\beta\,c\,d\,s^{-1}) + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)\sum_{i=1}^{n}\log u_i \\ & - \alpha \sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta} + (\beta - 1)\sum_{i=1}^{n}\log\left(u_i^d - 1\right),\end{aligned} \tag{5.11}$$

where $u_i = 1 + \left(\frac{x_i}{s}\right)^c$. The estimates of the model parameters can be obtained by maximizing (5.11).

Alternatively, we can differentiating (5.11) and solving the resulting nonlinear likelihood equations. The components of the score vector $\boldsymbol{U(\theta)}$ are given by

$$\begin{aligned}U_c(\boldsymbol{\theta}) =& \, n\,c^{-1} + c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)\,c^{-1}\sum_{i=1}^{n}(u_i - 1)\log(u_i - 1)\,u_i^{-1} \\ & - \alpha\,\beta\,d\,(c\,s)^{-1}\sum_{i=1}^{n}(u_i - 1)\log(u_i - 1)\,u_i^{d-1}\left(u_i^d - 1\right)^{\beta - 1} \\ & + d\,(\beta - 1)\,(c\,s)^{-1}\sum_{i=1}^{n}(u_i - 1)\log(u_i - 1)\,u_i^{d-1}\left(u_i^d - 1\right)^{-1},\end{aligned}$$

$$
\begin{aligned}
U_d(\boldsymbol{\theta}) = & \, n\, d^{-1} + \sum_{i=1}^{n} \log u_i + (\beta-1) \sum_{i=1}^{n} u_i^d \log u_i \left(u_i^d - 1\right)^{-1} \\
& - \alpha\beta \sum_{i=1}^{n} u_i^d \log u_i \left(u_i^d - 1\right)^{\beta-1},
\end{aligned}
$$

$$
\begin{aligned}
U_s(\boldsymbol{\theta}) = & - c\, n\, s^{-1} + c(d-1)s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{-1} \\
& - c\, d\, (\beta-1)\, s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{d-1} \left(u_i^d - 1\right)^{-1} \\
& + \alpha\, \beta\, c\, d\, s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{d-1} \left(u_i^d - 1\right)^{\beta-1},
\end{aligned}
$$

$$
U_\alpha(\boldsymbol{\theta}) = n\, \alpha^{-1} - \sum_{i=1}^{n} \left(u_i^d - 1\right)^{\beta},
$$

and

$$
U_\beta(\boldsymbol{\theta}) = n\, \beta^{-1} + \sum_{i=1}^{n} \log\left(u_i^d - 1\right) - \alpha \sum_{i=1}^{n} \left(u_i^d - 1\right)^{\beta} \log\left(u_i^d - 1\right).
$$

Setting these expressions to zero, $\boldsymbol{U(\theta)} = \mathbf{0}$, and solving them simultaneously yields the MLEs of the five parameters. These equations cannot be solved analytically but statistical software can be used to solve them numerically using iterative methods such as the quasi-Newton BFGS and Newton-Raphson type algorithms, see Press *et al.* (2007).

Note that, for fixed c, d, s and β, the MLE of α is given by

$$
\hat{\alpha}(\hat{c}, \hat{d}, \hat{s}, \hat{\beta}) = \frac{n}{\sum_{i=1}^{n} (u_i^{-d} - 1)^{\beta}} \tag{5.12}
$$

It is easy to observe in (5.12) that, fixed on $x_1, \ldots, x_n$,

- $\hat{\alpha} \to 1$ when $\hat{\beta} \to \hat{0}^{+}$,
- $\hat{\alpha} \to \infty$ when $\hat{s} \to \infty$,
- $\hat{\alpha} \to \hat{0}^{+}$ when $\hat{s} \to \hat{0}^{+}$,

- $\hat{\alpha} \to \hat{0}^{+}$ when $\hat{d} \to \infty$,

- $\hat{\alpha} \to \infty$ when $\hat{d} \to \hat{0}^{+}$

Thus, we can think of the use of more refined procedures for estimation under small values of a and s. By replacing α by (5.12) in equation (5.11) and letting $\boldsymbol{\theta_p} = (c, d, s, \beta)$, the profile log-likelihood function for $\boldsymbol{\theta_p}$ can be expressed as

$$\ell(\boldsymbol{\theta_p}) = n\log(n\,\beta\,c\,d\,s^{-1}) + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)\sum_{i=1}^{n}\log u_i \qquad (5.13)$$
$$- n\log\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta} + (\beta-1)\sum_{i=1}^{n}\log\left(u_i^d - 1\right) - n,$$

The corresponding score vector of (5.13), $\boldsymbol{U(\theta_p)}$, has components

$$U_c(\boldsymbol{\theta_p}) = n\,c^{-1} + c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)c^{-1}\sum_{i=1}^{n}(u_i - 1)\log(u_i - 1)u_i^{-1}$$
$$- n\,\beta\,d\,c^{-1}\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{d-1}\left(u_i^d - 1\right)^{\beta-1}\log\left(u_i - 1\right)$$
$$+ d\,(\beta-1)\,c^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{d-1}\left(u_i^d - 1\right)^{-1}\log\left(u_i - 1\right),$$
$$U_d(\boldsymbol{\theta_p}) = n\,d^{-1} + \sum_{i=1}^{n}\log u_i + (\beta-1)\sum_{i=1}^{n}u_i^d\left(u_i^d - 1\right)^{-1}\log u_i$$
$$- n\beta\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}u_i^d\left(u_i^d - 1\right)^{\beta-1}\log u_i,$$

$$U_s(\boldsymbol{\theta_p}) = -n\,c\,s^{-1} + c(d-1)s^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{-1}$$
$$- c\,d\,(\beta-1)\,s^{-1}\sum_{i=1}^{n}(u_i - 1)\,u_i^{d-1}\left(u_i^d - 1\right)^{-1}$$
$$+ n\,\beta\,c\,d\,s^{-1}\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}(u_i - 1)\,u_i^{d-1}\left(u_i^d - 1\right)^{\beta-1},$$

and

$$U_{\beta}(\boldsymbol{\theta_p}) = n\,\beta^{-1} + \sum_{i=1}^{n} \log\left(u_i^d - 1\right) - n\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1} \sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta} \log\left(u_i^d - 1\right).$$

Solving the equations $\boldsymbol{U(\theta_p)} = \boldsymbol{0}$ simultaneously yields the MLEs of c, d, s and β. The MLE of α is just $\hat{\alpha}(\hat{c}, \hat{d}, \hat{s}, \hat{\beta})$. The maximization of the profile log-likelihood might be simpler since it involves only four parameters.

For interval estimation on the model parameters, we require the observed information matrix $\boldsymbol{J(\theta)}$ given by

$$\boldsymbol{J(\theta)} = -\frac{\partial^2\,\ell(\theta)}{\partial\theta\,\partial\theta^T} = \begin{pmatrix} J_{cc} & J_{cd} & J_{cs} & J_{c\alpha} & J_{c\beta} \\ . & J_{dd} & J_{ds} & J_{d\alpha} & J_{d\beta} \\ . & . & J_{ss} & J_{s\alpha} & J_{s\beta} \\ . & . & . & J_{\alpha\alpha} & J_{\alpha\beta} \\ . & . & . & . & J_{\beta\beta} \end{pmatrix},$$

whose elements can be obtained from the authors upon request. Under standard regularity conditions, the approximate confidence intervals for the individual model parameters can be constructed based the multivariate normal $N_5(0, \boldsymbol{J(\widehat{\theta})}^{-1})$ distribution.

5.5 Simulation study

In this section, our aim is to evaluate the MLEs of the parameters of the WBXII distribution. We conduct a Monte Carlo simulation experiment based on 10,000 replications. We considered five different parameter combinations and set the sample size at $n = 100, 250$ and 500. The simulation study is conducted using the subroutine `optim` and SANN algorithm in `R` software for maximizing the log-likelihood in (5.11). Table 5.1 reports the empirical mean estimates and corresponding RMSEs. We note that, for all parameter combinations the empirical biases and RMSEs decrease as the sample size increases. Considering the asymptotic properties of the MLEs, these results are expected.

5.6 Applications

In this section, we illustrate the usefulness of the WBXII distribution for modeling income and lifetime data. The first data set represents the time to failure (10^3 h) of 40 suits of turbochargers in one type of diesel engine (Xu *et al.*, 2003). These data was previously considered by Benkhelifa (2016). The second data set consists in annual salaries of 862 professional baseball players of the Major League Baseball for the season 2016. The data is mesuared in American dollars and is avaiable for download at `https://www.usatoday.com/sports/mlb/salaries/2016/player/all/`.

We use these two data sets to compare the fits of the WBXII distribuition with other six related models, i.e., the BBXII, KwBXII, BXII, LL, GPW, and W distributions. Their densities are given by equations (2.4), (2.5), (2.2), by taking $s = m^{-1}$ and $d = 1$ in (2.2), (4.13) and by taking $\beta = 1$ in (4.13), respectively.

In each case, the parameters are estimated by maximum likelihood using the `AdequacyModel` script in `R` software (Marinho *et al.*, 2016). We report the MLEs, their corresponding standard errors the statistics AIC, CAIC, BIC, HQIC and KS. The lower values of these statistics are associated with the better fits.

5.6.1 Turbochargers failure time

Table 5.2 describes some descreptive statistics of the turbochargers failure time data. Note that these data presents negative skewness and kurtosis coefficients and have an amplitude of 7.4. We also have close values for the mean and median. This descriptive summary indicates that the turbochargers data set follows a power law distribution with left-skewed tail.

Table 5.3 lists the MLEs for the fitted models and their corresponding standard errors. For all considered models, the parameter estimates are significant. Table 5.4 gives the goodness-of-fit statistics. The WBXII distribution has the lowest values for all statistics but BIC. Note that for the BIC, the WBXII is quite competitive with the W distribution. However, the W model may not be an effective alternative for modeling left-skewed data. Figure 5.4 displays the histogram and the estimated densities with lower values for goodness-of-fit statistics. We note that the WBXII yields a good adjustment for the current data. It is more accurate than the W distribution for modeling the left tail and is quite competitive with the BBXII distribution. Thus, we can conclude from Figure 5.4 and Table 5.4 that the WBXII model provides the better fit for the turbocharges failure time data.

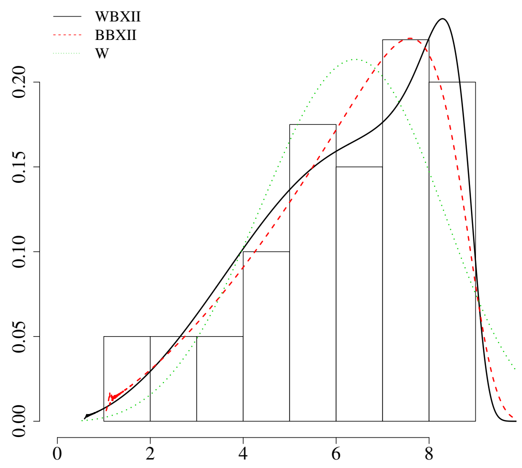


Figure 5.4: Histogram and estimated densities of the WBXII, BBXII and W models for the turbochargers failure time data.

5.6.2 Baseball players salaries

Some descriptive statistics for the baseball players data are provided in Table 5.5. These data presents positive values for the skewness and kurtosis coefficients, indicating right-skew data. We have high amplitude, variance and SD. We also note that the mean and median are not so close. This bevavior is quite common in income data sets.

Table 5.6 provides the MLEs and their standard errors for the seven models fitted for the baseball players data set. We have significant estimates for all the parameters of these models. Table 5.7 lists the goodness-of-fit measures considered. The WBXII distribution presents the lowest values for all statistics. These results indicate that the WBXII distribution performs better fits than the other competitive models considered for the baseball players data set. Figure 5.5 displays the fitted WBXII, BBXII and KwBXII densities and the histogram for the baseball players data. They confirm that the WBXII model yields a better fit. Finally, we can conclude that the WBXII is an effective alternative to modeling lifetime (see the first data set) and income (see the second data set) data, specially when they present power law tails. It is quite competitive to the classical Weibull lifetime distribution and to other BXII generalizations.

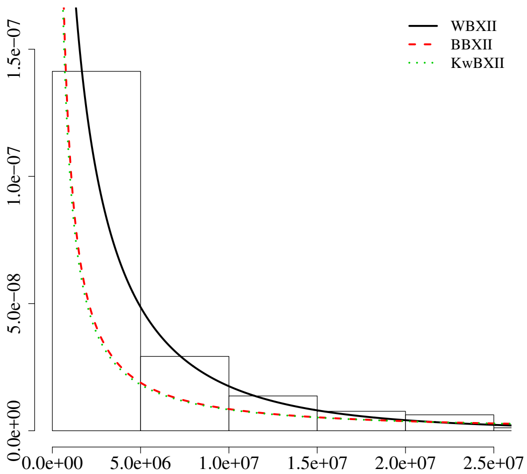


Figure 5.5: Histogram and estimated densities of the WBXII, BBXII and KwBXII models for the baseball players data.

5.7 Concluding remarks

The five-parameter *Weibull Burr XII* distribution is introduced and studied. Its hazard rate function can be increasing, decreasing, upside-down bathtub and bathtub-shaped. It is also very flexible in terms of the density function and skewness and kurtosis coefficients. Some mathematical properties of the proposed model are presented including the ordinary and incomplete moments, quantile and generating functions and mean deviations. We estimate the model parameters using maximum likelihood and present the components of the score vector. A simulation study is carried out. We also present two applications to real lifetime and income data sets. They illustrate the usefulness of the proposed distribution for modeling these kind of data and also show that the WBXII distribution is quite competitive with other Burr XII and Weibull generalizations.

Table 5.1: Monte Carlo results for the mean estimates and RMSEs of the WBXII distribution.

		Mean					RMSE				
$\boldsymbol{\theta}$	n	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$
$(0.1, 0.4, 2.5, 3, 1.5)$	100	0.216	0.586	2.763	2.379	1.376	0.285	0.480	1.635	1.631	0.830
	250	0.136	0.466	2.569	2.666	1.472	0.138	0.193	1.072	1.150	0.547
	500	0.109	0.430	2.526	2.832	1.511	0.058	0.110	0.761	0.824	0.366
$(1.5, 3, 0.2, 2, 5)$	100	1.409	4.700	0.907	2.557	0.835	0.928	2.892	1.452	1.880	0.799
	250	1.537	3.338	0.231	2.121	5.121	0.362	1.136	0.122	1.039	1.059
	500	1.525	3.209	0.217	2.067	5.073	0.292	0.909	0.084	0.862	0.848
$(1, 5, 5, 2, 2.3)$	100	2.117	5.968	3.756	3.468	1.823	2.011	3.394	2.905	3.225	1.297
	250	1.608	5.472	4.137	3.066	2.022	1.288	2.597	2.450	2.539	1.042
	500	1.362	5.275	4.380	2.755	2.122	0.900	2.031	2.072	2.030	0.860
$(1, 5, 3, 2, 0.5)$	100	1.430	5.925	3.203	2.184	0.590	1.029	2.822	2.336	1.327	0.583
	250	1.288	5.575	3.125	2.080	0.546	0.759	2.206	1.918	0.916	0.447
	500	1.183	5.318	3.089	2.037	0.517	0.561	1.770	1.622	0.714	0.314
$(0.4, 0.2, 1.8, 3, 4)$	100	0.674	0.199	2.198	2.833	4.097	0.581	0.125	1.333	1.488	0.803
	250	0.543	0.198	1.953	2.860	4.093	0.379	0.098	0.999	1.144	0.584
	500	0.475	0.197	1.887	2.913	4.062	0.247	0.067	0.835	0.926	0.428

Table 5.2: Descriptive statistics for turbochargers data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
6.25	6.50	1.96	3.82	-0.66	-0.36	1.60	9.00

Table 5.3: The MLEs of the model parameters for the turbochargers failure time data and corresponding standard errors in parenthese.

	c	d	s	α	β
WBXII	13.4956	7.5404	8.8931	1.1128	0.2216
	(2.7613)	(3.6805)	(0.7060)	(0.3671)	(0.0576)
	c	d	s	a	b
BBXII	15.4893	11.1316	11.2702	0.1666	4.5249
	(0.0395)	(0.1854)	(0.1994)	(0.0282)	(2.0589)
KwBXII	15.1758	6.2322	9.2966	0.1559	0.7550
	(0.1931)	(0.9355)	(0.3827)	(0.0398)	(0.2306)
BXII	3.8290	3.9620	9.6190		
	(0.5506)	(1.8934)	(1.5960)		
	c	m			
LL	4.8540	6.2340			
	(0.6551)	(0.3477)			
	α	λ	β		
GPW	3.5830	7.7010	1.3300		
	(0.5466)	(1.4125)	(0.6152)		
W	3.8720	6.9240			
	(0.5174)	(0.2951)			

Table 5.4: Goodness-of-fit statistics for the fits to the turbochargers failure time data.

	AIC	CAIC	BIC	HQIC	KS
WBXII	**165.8103**	**167.5750**	174.2547	**168.8635**	**0.0532**
BBXII	166.9631	168.7278	175.4075	170.0163	0.0744
KwBXII	167.0753	168.8400	175.5197	170.1286	0.0579
BXII	174.8080	175.4746	179.8746	176.6399	0.1029
LL	181.4142	181.7386	184.7920	182.6355	0.1440
GPW	169.6197	170.2864	174.6863	171.4516	0.1066
W	168.9513	169.2756	**172.3290**	170.1725	0.1069

Table 5.5: Descriptive statistics for baseball players data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
4,529,859.69	1.5×10^6	6,070,096	3.684606×10^{13}	1.98	3.74	507,500	34,416,666

Table 5.6: The MLEs of the model parameters for baseball players data and corresponding standard errors in parentheses.

	c	d	s	α	β
WBXII	0.5527	0.0796	1.8716	2.4141	7.4298
	(0.0686)	(0.0115)	(0.8292)	(1.0993)	(0.3079)
	c	d	s	a	b
BBXII	1.8134	0.0487	5.7723	12.3094	6.2716
	(0.1742)	(0.0046)	(0.7939)	(0.6308)	(0.5546)
KwBXII	3.92390	0.03251	2.59116	9.16545	4.0435
	(0.2031)	(0.0016)	(0.4065)	(0.5123)	(0.2463)
BXII	6.8459	0.0102	2.6500		
	(0.6858)	(0.0010)	(0.3534)		
	c	m			
LL	0.1289	14.2324			
	(0.0036)	(1.7615)			
	α	λ	β		
GPW	1.6123	10.1363	0.0400		
	(0.1926)	(1.2913)	(0.0047)		
W	0.0646	9.9377			
	(0.0015)	(1.2611)			

Table 5.7: Goodness-of-fit statistics for the fitted models for baseball players data.

	AIC	CAIC	BIC	HQIC	KS
WBXII	**28023.5004**	**28023.5705**	**28047.2967**	**28032.6096**	**0.2168**
BBXII	28977.3246	28977.3947	29001.1209	28986.4338	0.3812
KwBXII	29077.1495	29077.2196	29100.9458	29086.2586	0.3858
BXII	31195.1989	31195.2268	31209.4766	31200.6643	0.5745
LL	31825.4901	31825.5040	31835.0086	31829.1337	0.7943
GPW	30421.9845	30422.0125	30436.2623	30427.4500	0.6359
W	32146.5307	32146.5447	32156.0492	32150.1744	0.8667

Chapter 6

The logistic Burr XII distribution

Resumo

Neste capítulo propomos uma nova distribuição de quatro parâmetros, denominada logistic Burr XII. Esta distribuição é obtida inserindo a distribuição Burr XII com três parâmetros no gerador *logistic-X*. O modelo proposto é uma alternativa útil para modelar dados de renda, podendo também ser aplicada em outras áreas do conhecimento. Nós mostramos que a nova distribuição pode assumir as formas decrescente e banheira invertida para a função de risco e que sua densidade pode ser escrita como combinação linear da densidade da Burr XII. Algumas propriedades matemáticas da distribuição proposta são apresentadas, tais como a função quantílica, momentos ordinários e incompletos e função geradora de momentos. Nós também obtemos os estimadores de máxima verossimilhança para os parâmetros do modelo e realizamos um estudo de simulação de Monte Carlo. A potencialidade da nova distribuição é ilustrada através de duas aplicações em dados de renda.

Palavras-chave: Distribuição Burr XII. distribuição de renda. Estimação de máxima verossimilhança. Família *logistic-X*. Momentos.

Abstract

In this chapter, we introduce the four-parameter *logistic Burr XII* distribution. It is obtained by inserting the three-parameter Burr XII distribution as baseline in the logistic-X family and may be a useful alternative to model income distribution and applied to other areas. We prove that the new distribution can have decreasing and upside-down bathtub hazard functions and that its density function is an infinite linear combination of Burr XII densities. Some mathematical properties of the proposed model are determined such as the quantile function, ordinary and incomplete moments and generating function. We also obtain the maximum likelihood estimators of the model parameters and perform a Monte Carlo simulation study. The potentiality of the new distribution is illustrated by means of two applications to income data sets.

Keywords: Burr XII distribution. Income distribution. Logistic-X family. Maximum likelihood estimation. moments.

6.1 Introduction

The Burr XII (BXII) distribution first appears as part of the Burr system of distributions. This system was introduced in 1942 by Irving W. Burr and comprises twelve distributions which yield a variety of density shapes, see Burr (1942). In the economic context, the BXII distribution is known under the name of Singh-Maddala, see Singh and Maddala (1975, 1976). Since then, it has received special attention in the literature of income distributions.

Several studies have been conducted considering the BXII distribution for modeling personal or family incomes in different countries, such as Czech Republic (Brzeziński, 2013), Hungary (Brzeziński, 2013), Pakistan (Shakeel *et al.*, 2015), Poland (Brzeziński, 2013), Slovak Republic (Brzeziński, 2013), United Kingdom (Henniger and Schmitz, 1989; Jäntti and Jenkins, 2010) and United States (Majumder and Chakravarty, 1990; McDonald and Xu, 1995; Łukasiewicz *et al.*, 2010; Tanak *et al.*, 2015). Brzeziński (2014) also suggested that the BXII distribution is useful for empirical modelling of the distribution of journal impact factors. Jones *et al.* (2014) applied it to modelling inpatient cost in English hospitals.

The three-parameter BXII distribution has cdf and pdf given by (2.1) and (2.2) respectively. Shao (2004) and Shao *et al.* (2004) studied the maximum likelihood estimation and the models for extremes for the three-parameter BXII distribution, respectively. Wu *et al.* (2007) discussed the estimation problems using this distribution based on progressive type II censoring with random removals. Silva *et al.* (2008) proposed a location-scale regression model based on this distribution.

In this chapter, we introduce a new four-parameter distribution called de *logistic-Burr XII* (LBXII) distribution. It is defined by inserting the three-parameter Burr-XII (BXII) distribution as baseline in equations (1.7) and (1.8). Our purpose is to provide a BXII generalization that may be useful to model power law situations. For example, Pareto suggested that income distribution follows a power law for those with high income (Pareto, 1987). However, subsequent studies found that this conjecture applies only to a small percentage of the population (Guo and Gao, 2012). Therefore, the characterization and understanding of the income distribution for the remaining majority of the population is still an open problem (Moura Jr and Ribeiro, 2009).

The LBXII distribution has cdf given by (for $x > 0$)

$$F(x) = \left\{1 + \left[d \log\left\{1 + \left(\frac{x}{s}\right)^c\right\}\right]^{-\lambda}\right\}^{-1}, \tag{6.1}$$

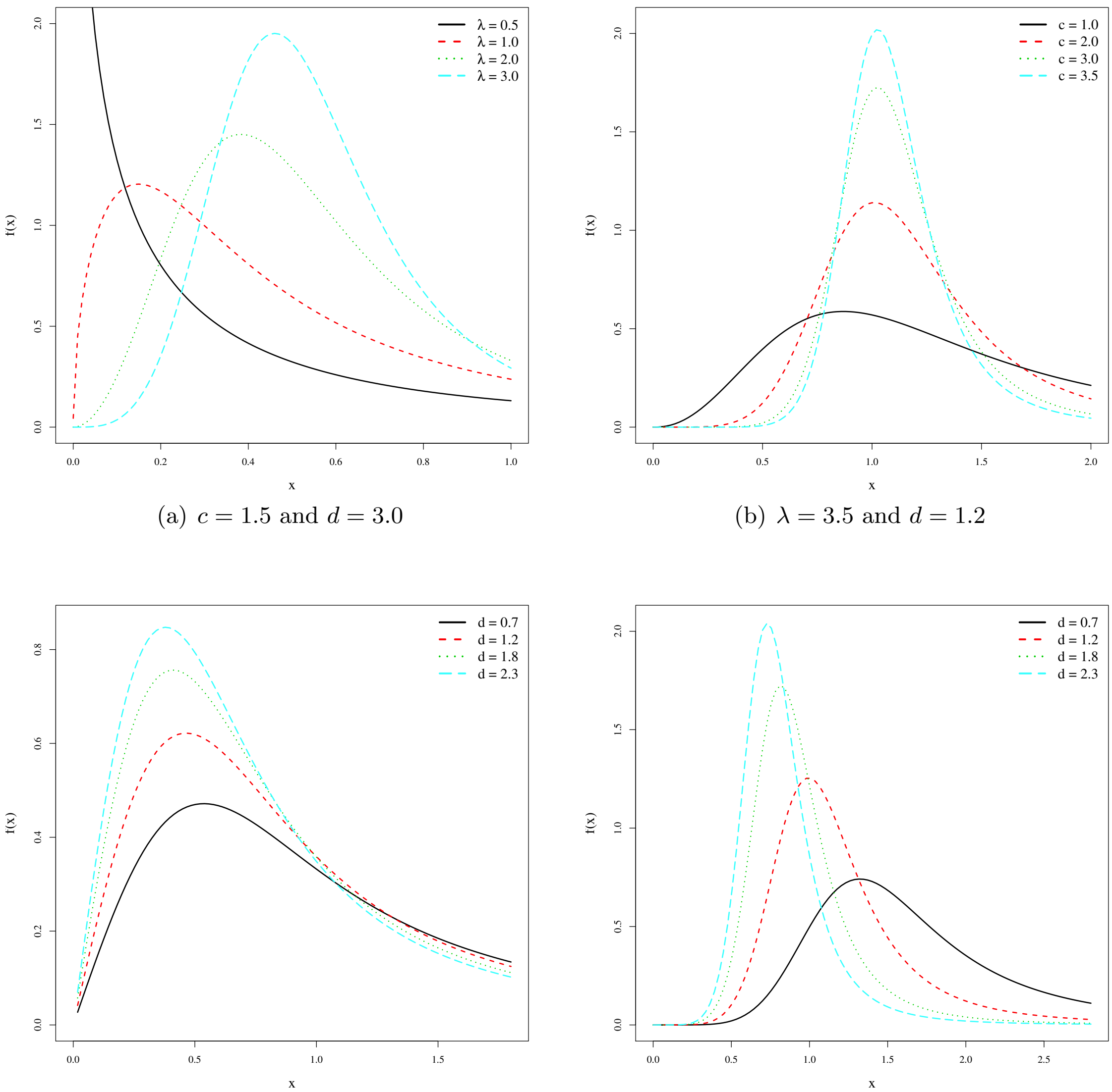
where $\lambda > 0$, $d > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter. The corresponding pdf is given by

$$f(x) = \frac{\lambda\, c\, d\, s^{-c} x^{c-1}}{1 + (x/s)^c} \left[d \log\left\{1 + \left(\frac{x}{s}\right)^c\right\}\right]^{-(\lambda+1)} \left\{1 + \left[d \log\left\{1 + \left(\frac{x}{s}\right)^c\right\}\right]^{-\lambda}\right\}^{-2}. \tag{6.2}$$

Henceforth, if X is a random variable with density function (6.2), we write $X \sim \text{LBXII}(c, d, s, \lambda)$. Figure 6.1 displays plots of the LBXII density function for selected parameter values. It can take various forms and has as special models some well-known distributions. For $d = 1$ and $s = m^{-1}$, we have the logistic-log-logistic distribution. The log-logistic (LL) distribution is obtained when $\lambda = 1$, $d = 1$ and $s = m^{-1}$. For $c = 1$ and $\lambda = c = 1$, it becomes the logistic-Lomax (LLo) and Lomax models, respectively. The hrf of X can be expressed as

$$h(x) = \frac{\lambda\, c\, s^{-c} x^{c-1}}{\log\left[1 + (x/s)^c\right]} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1} \left\{1 + \left[d\ \log\left\{1 + \left(\frac{x}{s}\right)^c\right\}\right]^{-\lambda}\right\}^{-1}.$$

Figure 6.2 provides plots of the hrf for some parameter values. It reveals that the LBXII distribution can have decreasing and upside-down bathtub hazard functions. The proposed distribution is quite flexible regarding the pdf and hrf and may be a useful alternative to the BXII



(a) $c = 1.5$ and $d = 3.0$

(b) $\lambda = 3.5$ and $d = 1.2$

(c) $\lambda = 0.8$ and $c = 2.5$

(d) $\lambda = 3.0$ and $c = 2.5$

Figure 6.1: Plots of the LBXII density for $s = 1$.

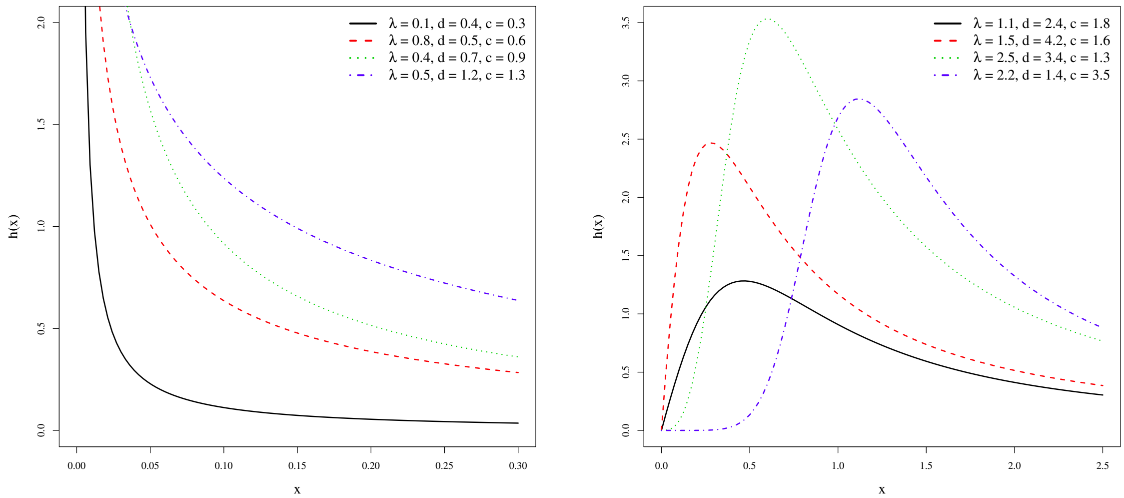


Figure 6.2: Plots of the LBXII hrf for $s = 1$.

model and its generalizations. Therefore, it can be considered for modeling income distribution and also in actuarial science, bioscience and lifetime data, among other areas.

The rest of the chapter is organized as follows. We derive useful expansions for the cdf and pdf of the new distribution in Section 6.2. In Section 6.3, some mathematical properties of the LBXII distribution are investigated. In Section 6.4, the maximum likelihood method is presented to estimate the model parameters. A simulation study is performed in Section 6.5. In Section 6.6, we illustrate the flexibility of the new model using two real income data sets. Some concluding remarks are offered in Section 6.7.

6.2 Useful expansions

Tahir *et al.* (2016a) demonstrated that the LX pdf can be expressed as an infinite linear combination of exponentiated-G (exp-G) densities. See Mudholkar and Srivastava (1993) for the definition of the exp-G distribution. In this section, we derive useful expansions for the LBXII pdf not from exponentiated models but based on our baseline model. Inserting (2.1) in equation (1.7), the LBXII cdf can be rewritten as

$$F(x) = \frac{1}{1 + \left[-\log\left(1 - \left\{1 - [1 + (x/s)^c]^{-d}\right\}\right)\right]^{-\lambda}}. \tag{6.3}$$

Using the `Mathematica` software, we obtain a power series for $w = 1 + [-\log(1-y)]^a$ as

$$\begin{aligned} w =& 1 + \left[1 + \frac{a}{2}\,y + \frac{1}{24}(3a^2 + 5a)\,y^2 + \frac{1}{48}(a^3 + 5a^2 + 6a)\,y^3 \right. \\ &+ \left. \frac{1}{5760}(15a^4 + 150a^3 + 485a^2 + 502a)\,y^4\right] y^a + O(y^{a+5}). \end{aligned}$$

Applying this power series for $y = 1 - [1 + (x/s)^c]^{-d}$ in (6.3) and after some algebraic manipulation, we have

$$F(x) = \frac{\left\{1 - [1 + (x/s)^c]^{-d}\right\}^{\lambda}}{\left\{1 - [1 + (x/s)^c]^{-d}\right\}^{\lambda} + \sum_{k=0}^{\infty} p_k \left\{1 - [1 + (x/s)^c]^{-d}\right\}^{k}}, \tag{6.4}$$

where the p_k's are $p_0 = 1, p_1 = \lambda/2,\ p_2 = \lambda\,(3\lambda + 5)/24,\ p_3 = \lambda\,(\lambda^2 + 5\lambda + 6)/48,\ p_4 = \lambda\,(15\lambda^3 + 150\lambda^2 + 485\lambda + 502)/5760$, etc. For any $\lambda > 0$ real non-integer, the following expansion holds since the left-side expression is a cdf

$$\{1 - [1 + (x/s)^c]\}^{\lambda} = \sum_{k=0}^{\infty} q_k \,\{1 - [1 + (x/s)^c]\}^{k},$$

where

$$q_k = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{\lambda}{j} \binom{j}{k}.$$

Thus, equation (6.4) can be rewritten as

$$F(x) = \frac{\sum_{k=0}^{\infty} q_k \,\{1 - [1 + (x/s)^c]\}^k}{\sum_{k=0}^{\infty} \upsilon_k \,\{1 - [1 + (x/s)^c]\}^k}, \tag{6.5}$$

where $\upsilon_k = q_k + p_k$. The coefficients of the quotient of the two power series in (6.5) can be determined from the recurrence equation (for $k \geq 0$)

$$\omega_k = \frac{1}{\upsilon_0}\left(q_k - \frac{1}{\upsilon_0}\sum_{l=0}^{k} \upsilon_r\,\omega_{k-l}\right)$$

and then equation (6.5) reduces to

$$F(x) = \sum_{k=0}^{\infty} \omega_k\, H_k(x), \tag{6.6}$$

where $H_k(x) = \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^k$. By differentiating (6.6), we obtain

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \omega_{k+1}\, h_{k+1}(x) \\ &= \omega_1\, g(x; s, d, c) \\ &+ \sum_{k=1}^{\infty} \omega_{k+1}\, (k+1)\, c\, d\, s^{-c}\, x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^k, \end{aligned} \tag{6.7}$$

where $h_{k+1}(x)$ is the exp-BXII pdf with power parameter $k+1$. Using the binomial theorem (for $k \geq 1$), we can write

$$\left\{1 - \left[1 + (x/s)^c\right]^{-d}\right\}^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \left[1 + (x/s)^c\right]^{-r\,d} \tag{6.8}$$

Inserting (6.8) in equation (6.7) and after some algebra, we obtain

$$f(x) = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^r\,(k+1)\,\omega_{k+1}}{r+1} \binom{k}{r}\, g(x; s, (r+1)d, c),$$

where $g(x; s, (r+1)d, c)$ is the BXII density function with scale parameter s and shape parameters c and $(r+1)d$. Since the sums in the above expressions vary in equal sets of indices, we can change $\sum_{k=0}^{\infty} \sum_{r=0}^{k}$ by $\sum_{r=0}^{\infty} \sum_{k=r}^{\infty}$. Therefore, the LBXII pdf can be reduced to

$$f(x) = \sum_{r=0}^{\infty} \rho_r\, g(x; s, (r+1)d, c), \tag{6.9}$$

where

$$\rho_r = \sum_{k=r}^{\infty} \frac{(-1)^r\,(k+1)\,\omega_{k+1}}{r+1} \binom{k}{r}.$$

Equation (6.9) is the main result of this section. So, the LBXII pdf is an infinite linear combination of BXII densities. Thus, some mathematical properties of X can be derived from those BXII properties.

6.3 Mathematical properties

In this section, we obtain some structural properties of the LBXII distribution by establishing algebraic expansions. It might be better than computing those directly by numerical integration of the density function of X. We obtain quantile function, ordinary and incomplete moments and generating function.

6.3.1 Quantile function

The qf of X is determined by inverting (6.1). We have

$$Q(u) = s\left[\exp\left\{\frac{1}{d}\left(\frac{1}{u}-1\right)^{-\frac{1}{\lambda}}\right\}-1\right]^{\frac{1}{c}}. \tag{6.10}$$

If U has the uniform distribution in $(0, 1)$, the random variable $X = Q(U)$ has the LBXII distribution. Thus, simulating the random variable X is straightforward by using the inverse transform method. We can also have any quantiles of interest by setting appropriate values of u. For example, $u = 1/2$ in (6.10) gives de median M of X.

Further, we have alternative expressions for the skewness and kurtosis coefficients based on quantile measures that can be obtained from (6.10). The Bowley's skewness (Kenney and Keeping, 1962) is given by (3.8). The Moors' kurtosis (Moors, 1988) is defined by (3.9). Some plots of B are displayed in Figure 6.3. They reveal the variation of the skewness for different shape parameters.

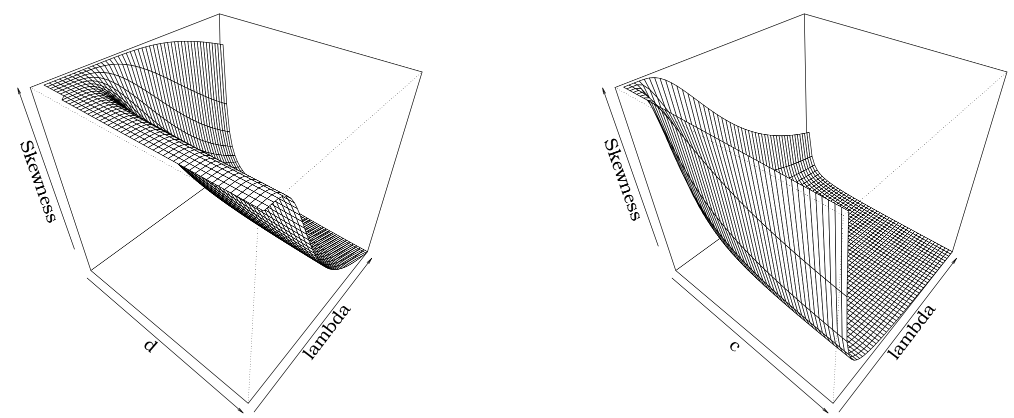


Figure 6.3: Skewness of the LBXII distribution for some parameter values.

6.3.2 Ordinary moments

A result from Zimmer *et al.* (1998) gives the hth ordinary moment of the BXII distribution as (2.7). Thus, the hth ordinary moment of X can be expressed directly from (6.9) as (for $h < c\,d$)

$$\mu'_h = s^h\, d \sum_{r=0}^{\infty} (r+1)\rho_r\, B((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}), \tag{6.11}$$

where $B(\cdot,\cdot)$ is the beta function.

By setting $h = 1$, we obtain the mean of X. The moments are most commonly taken about the mean. These so-called central moments (μ_s) follow recursively from (6.11) as

$$\mu_s = \sum_{i=0}^{s} \binom{s}{i} (-1)^i\, \mu_1'^{s}\, \mu'_{s-i}.$$

The central cumulants (κ_s) of X can also be determined recursively as

$$\kappa_s = \mu'_s - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \kappa_i\, \mu'_{s-i},$$

where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc.

6.3.3 Incomplete moments

Let $T_h(y)$ be the hth incomplete moment of X. It can be derived using the linear representation (6.9) as

$$T_h(y) = \sum_{r=0}^{\infty} \rho_r\, T_{(r+1)d_h}(y), \tag{6.12}$$

where $T_{(r+1)d_h}(y)$ is the hth incomplete moment of a BXII$(c, (r+1)d, s)$ random variable.

Hence, using the result in (2.9), the hth incomplete moment of X reduces to (for $h < c\,d$)

$$T_h(y) = d\, s^h \sum_{r=0}^{\infty} (r+1)\rho_r\; B_{s^c/s^c+y^c}\left((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}\right).$$

The first incomplete moment of X is obtained by setting $h = 1$. In equation (2.10) is presented an alternative expression for the hth incomplete moment of the BXII distribution. By inserting (2.10) in (6.12) we can also write

$$T_h(y) = d\, s^h \sum_{r=0}^{\infty} (r+1)\rho_r\, J\left(y, \frac{h}{c}, (r+1)d + 1\right).$$

One important application of the first incomplete moment refers to the mean deviations about the mean and the median of X. They are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\,T_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\,T_1(M)$, respectively. The quantity $F(\mu_1')$ is easily obtained from (6.1), $T_1(\mu_1')$ is the first incomplete moment of X at the mean μ_1' and $T_1(M)$ at the median M. Other useful applications are the Bonferroni and Lorenz curves. For a given probability π, they are defined by $B(\pi) = T_1(q)/(\pi\mu_1')$ and $L(\pi) = T_1(q)/\mu_1'$, respectively. The quantity $q = Q(\pi)$ is obtained from (6.10). These curves are useful in economics for studying income and poverty, but can be applied in several other fields.

6.3.4 Generating function

Let $M_d(t)$ be the mgf of the BXII(c, d, s) distribution. Here, we provide a formula for the mgf, $M(t) = \int_{-\infty}^{\infty} \mathrm{e}^{t\,x} f(x)dx$, of X. Clearly, it can be obtained from (6.9) as

$$M(t) = \sum_{r=0}^{\infty} (r+1)\rho_r\, M_{(r+1)d}(t), \tag{6.13}$$

where $M_{(r+1)d}(t)$ is the mgf of the BXII$(c, (r+1)d, s)$ distribution.

By assuming that $c = m/d$, where m and d are positive integers, $\mu > -1$, $p > 0$ and $t < 0$, the BXII mgf can be expressed as in (2.11) (Paranaíba *et al.*, 2011, 2013). Note that the condition over the parameter c is not restrictive since every positive real number can be approximated by a rational number. Inserting (2.11) in (6.13), we obtain the generating function of X (for $t < 0$) as

$$M(t) = m \sum_{r=0}^{\infty} (r+1)\rho_r\, I\left(-s\,t, \frac{m}{(r+1)d} - 1, \frac{m}{(r+1)d}, -(r+1)d - 1\right).$$

Furthermore, Paranaíba *et al.* (2011, 2013) also presented the special cases $c = 1$ and $c = 2$.

6.4 Maximum likelihood estimation

In this section, we determine the MLEs of the model paremeters for the proposed distribution. Let $\boldsymbol{\theta} = (\lambda, s, d, c)^T$ be the vector of the model parameters of the LBXII(λ, s, d, c) distribution and let $x_1, \ldots, x_n$ be a random sample of size n from this distribution. The log-likelihood function for $\boldsymbol{\theta}$ is given by

$$\ell(\boldsymbol{\theta}) = n\log(\lambda\, c\, d\, s^{-1}) - n\,(\lambda+1)\log d - \sum_{i=1}^{n}\log u_i + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i-1) \qquad (6.14)$$
$$-\ (\lambda+1)\sum_{i=1}^{n}\log\,\log u_i - 2\sum_{i=1}^{n}\log\left[1+(d\,\log u_i)^{-\lambda}\right],$$

where $u_i = 1 + \left(\frac{x_i}{s}\right)^c$. The MLE of $\boldsymbol{\theta}$ can be evaluated numerically by maximizing (6.14).

The components of $\boldsymbol{U(\theta)}$ are given by

$$U_\lambda(\boldsymbol{\theta}) = n\,\lambda^{-1} - n\,\log d - \sum_{i=1}^{n}\log\,\log u_i + 2\sum_{i=1}^{n}\frac{\log\left[d\,\log u_i\right]}{1+(d\,\log u_i)^{\lambda}},$$

$$U_c(\boldsymbol{\theta}) = n\,c^{-1} + \left[\frac{c^3+2c-1}{c^2} - 1\right]\sum_{i=1}^{n}\log(u_i-1) + c^{-1}\sum_{i=1}^{n}(u_i-1)\log\,(u_i-1)\,u_i^{-1}$$
$$-\ (\lambda+1)\,c^{-1}\sum_{i=1}^{n}\frac{(u_i-1)\log(u_i-1)}{u_i\log u_i} + 2\,\lambda\,c^{-1}\sum_{i=1}^{n}\frac{(u_i-1)\log(u_i-1)}{u_i+u_i\,(d\,\log u_i)^{\lambda}},$$

$$U_d(\boldsymbol{\theta}) = 2\,\lambda\,d^{-1}\sum_{i=1}^{n}\frac{1}{1+(d\log u_i)^{\lambda}} - \lambda\,n\,d^{-1}$$

and

$$U_s(\boldsymbol{\theta}) = c\,s^{-1}\sum_{i=1}^{n}(u_i-1)u_i^{-1}\left[1+(\lambda+1)\sum_{i=1}^{n}\frac{1}{\log u_i} - 2\,d\,\lambda\sum_{i=1}^{n}\frac{1}{1+(d\,\log u_i)^{\lambda}}\right]$$
$$-\ (n+c-1)s^{-1}.$$

Setting the score vector $\boldsymbol{U(\theta)}$ equal to zero and solving the equations simultaneously yields the MLEs of the four parameters. These equations cannot be solved analytically but there are routines for numerical maximization that may be used. For interval estimation and testing of hypotheses, we require the asymptotic normality of the MLEs. Under standard regularity conditions, the distribution of $\sqrt{n}(\hat{\lambda}-\lambda, \hat{s}-s, \hat{d}-d, \hat{c}-c)$ can be approximated by a multivariate normal $N_4(0, \boldsymbol{J}(\widehat{\boldsymbol{\theta}})^{-1})$ distribution. Here, $\boldsymbol{J}(\boldsymbol{\theta})$ is the observed information matrix given by

$$\boldsymbol{J}(\boldsymbol{\theta}) = -\frac{\partial^2\,\ell(\theta)}{\partial\theta\,\partial\theta^T} = \begin{pmatrix} J_{\lambda\lambda} & J_{\lambda c} & J_{\lambda d} & J_{\lambda s} \\ . & J_{cc} & J_{cd} & J_{cs} \\ . & . & J_{dd} & J_{ds} \\ . & . & . & J_{ss} \end{pmatrix},$$

whose elements can be obtained from the authors upon request.

6.5 Simulation study

In this section, we conduct a Monte Carlo experiment to investigate some asymptotic properties of the MLEs for the parameters of the LBXII distribution. Based on the LBXII qf, we use the inverse transform method to generate five different combinations of parameters λ, c, d and s for the LBXII model. Four samples sizes are considered ($n = 50, 100, 250, 500$) and the number of replications is $10,000$. For maximizing the log-likelihood (6.14), we use the subroutine `optim` and SANN algorithm in `R`. Table 6.1 presents the mean estimates of the MLEs and their RMSEs. As expected, the MLEs tend to be closer to the true parameters and the RMSEs decrease when the sample size n increases.

6.6 Applications

In this section, we present two examples to illustrate the potentiality of the LBXII distribution for modeling income data. The first data set consists in the annual salaries of professional hockey players of the American National Hockey League for the season 2012-2013. It has 714 observations in American dollars and is available for download at `https://www.usatoday.com/sports/nhl/salaries/`. The second example represents the individual payroll income of 5,024 Italian households with positive income. These data are obtained from a Survey of Household Income and Wealth (SHIW) of the Bank of Italy for 2014. The observations are measured in euros.

We fit the LBXII model for both data sets and compare with other six competitive models. They are defined bellow (for $x > 0$):

Table 6.1: Monte Carlo simulation results for the LBXII mean estimates and RMSEs.

		Mean				RMSE			
$\boldsymbol{\theta}$	n	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{s}$
$(3, 0.2, 2.5, 5)$	50	3.270	0.243	3.029	5.259	1.616	0.142	2.130	3.507
	100	3.278	0.224	2.791	5.172	1.443	0.104	1.428	3.040
	250	3.192	0.212	2.638	5.080	1.135	0.073	0.839	2.437
	500	3.155	0.206	2.566	5.048	0.918	0.056	0.561	1.984
$(6, 4, 5, 0.5)$	50	6.656	4.094	5.534	0.535	2.069	1.305	2.458	0.157
	100	6.423	4.080	5.280	0.518	1.670	1.088	1.984	0.109
	250	6.244	4.044	5.198	0.510	1.265	0.844	1.539	0.073
	500	6.137	4.044	5.119	0.505	1.013	0.693	1.227	0.055
$(9, 1.7, 5, 0.1)$	50	9.162	1.774	5.326	0.108	1.527	0.396	1.598	0.044
	100	9.054	1.752	5.210	0.104	1.224	0.300	1.270	0.028
	250	9.027	1.725	5.093	0.102	0.901	0.203	0.911	0.018
	500	8.993	1.716	5.067	0.100	0.668	0.148	0.699	0.013
$(10.5, 4.2, 6.5, 0.2)$	50	10.602	4.295	6.580	0.201	1.106	0.620	1.157	0.017
	100	10.539	4.263	6.540	0.200	0.881	0.474	0.955	0.012
	250	10.517	4.231	6.528	0.200	0.694	0.343	0.747	0.010
	500	10.511	4.217	6.517	0.200	0.557	0.266	0.597	0.007
$(14.1, 0.5, 0.1, 5.4)$	50	14.539	0.545	0.175	5.544	1.282	0.481	0.157	1.401
	100	14.438	0.540	0.164	5.487	0.995	0.445	0.138	1.194
	250	14.325	0.526	0.152	5.470	0.714	0.395	0.115	0.943
	500	14.255	0.522	0.143	5.466	0.551	0.352	0.100	0.798

- The KwBXII density is given by

$$\begin{aligned} f(x) &= a\,b\,c\,d\,s^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d-1}\left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}^{a-1}\times \\ & \left[1-\left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-d}\right\}^{a}\right]^{b-1}, \end{aligned}$$

where $a>0$ and $b>0$ are shape parameters.

- The BBXII density is given by

$$f(x)=\frac{c\,d\,x^{c-1}}{s^{c}B(a,b)}\left\{1-\left[1+(x/s)^{c}\right]^{-d}\right\}^{a-1}\left[1+(x/s)^{c}\right]^{-(d\,b+1)},$$

where $a>0$ and $b>0$ are shape parameters.

- The BXII density is given in (2.2).

- The exponentiated Weibull density (Mudholkar and Srivastava, 1993) is given by

$$g(t)=\alpha\,\beta\,\lambda\,x^{\alpha-1}\exp\left(-\lambda\,x^{\alpha}\right)\left[1-\exp\left(-\lambda\,x^{\alpha}\right)\right]^{\beta-1},$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter.

- The W density, which arises from the EW density when $\beta = 1$.
- The LL density obtained from the BXII density with $s = m^{-1}$ and $d = 1$.

We use the `AdequacyModel` script in `R` software (Marinho *et al.*, 2016) to obtain the MLEs and goodness-of-fit statistics. The statistics considered for these models are: AIC, CAIC, BIC, HQIC and KS. The lower are the goodness-of-fit statistics, the better is the distribution adjustment to the data.

6.6.1 Hockey players salaries

Table 6.2 provides a descriptive summary of the hockey players data. We have a higher value for the SD and an amplitude of 13,475,000. This indicates that the current data have great variability. The skewness is positive, and the kurtosis is large. Further, the mean and median are not so close. These statistics suggest that the hockey players salaries follow a power law distribution, which is very common in income data sets.

Table 6.2: Descriptive statistics for hockey players data.

Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
2,450,815.39	1.675×10^{-6}	2,112,878	1.61	3.35	5.25×10^{5}	1.4×10^{7}

The MLEs and their standard errors for all fitted distributions are listed in Table 6.3. We note that the parameter estimates are significant for all considered models. Table 6.4 presents the goodness-of-fit statistics and reveals that the LBXII distribution yields a good adjustment for the hockey players data. It has the lowest values for all statistics, thus indicating a competitive alternative to the classical W, EW and other BXII generalizations and special models.

The three estimated densities with lower values for the goodness-of-fit statistics and the histogram of the data are given in Figure 6.4. It agrees with what was discussed in the descriptive summary and the results in Table 6.4. Thus, the LBXII model is very competitive with the other fitted distributions and provides a better adjustment for the current data.

Table 6.3: The MLEs of the model parameters and their standard errors for hockey players data.

	c	d	s	a	b
BBXII	0.6639	0.1238	6.1887	12.3249	7.0895
	(0.0459)	(0.0087)	(0.9562)	(0.7336)	(0.5711)
KwBXII	5.3659	0.0240	2.9941	8.1383	3.5522
	(0.4790)	(0.0021)	(0.4481)	(0.4410)	(0.2546)
	c	d	s	λ	
LBXII	0.4665	0.1676	5.0797	13.8981	
	(0.0183)	(0.0069)	(1.5760)	(0.4894)	
BXII	7.7501	0.0093	3.2532		
	(0.8155)	(0.0010)	(0.4724)		
	λ	α	β		
EW	1.5782	0.0411	8.5690		
	(0.2285)	(0.0059)	(1.1925)		
W	10.4164	0.0683			
	(1.3816)	(0.0018)			
	c	m			
LL	0.1296	12.6759			
	(0.0040)	(1.7147)			

Table 6.4: Goodness-of-fit statistics for the fitted models to the hockey players data.

	AIC	CAIC	BIC	HQIC	KS
BBXII	23764.9605	23765.0452	23787.8149	23773.7870	0.3836
KwBXII	23954.2392	23954.3239	23977.0936	23963.0656	0.4249
LBXII	**22660.5691**	**22660.6256**	**22678.8527**	**22667.6303**	**0.1957**
BXII	25640.6040	25640.6378	25654.3166	25645.8999	0.5767
EW	25032.8320	25032.8658	25046.5446	25038.1279	0.6477
W	26436.5128	26436.5297	26445.6546	26440.0434	0.8768
LL	26192.7199	26192.7368	26201.8617	26196.2505	0.7987

6.6.2 Individual payroll income

Table 6.5 provides a descriptive summary of the individual payroll income data. For these data, the mean and median are close and the SD is higher. We also note large values for the skewness and kurtosis coefficients. The amplitude is 134,900 for these data. Just like for the first data set, the descriptive statistics indicate that the payroll income may follow a power law distribution with a right-skew tail.

Tables 6.6 and 6.7 present the MLEs with their standard errors and the goodness-of-fit statistics, respectively. These results are obtained for the LBXII distribution and six competitive

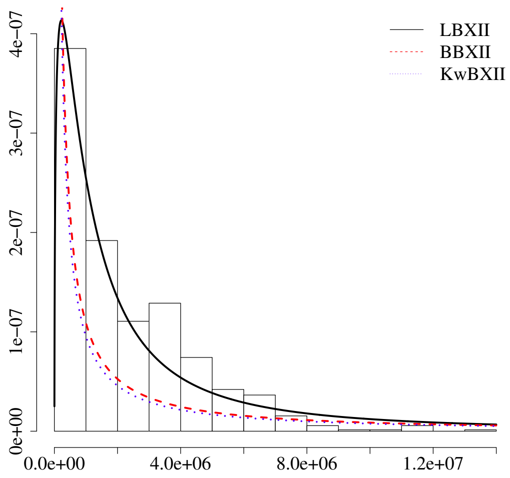


Figure 6.4: Histogram and estimated densities of the LBXII, BBXII and KwBXII models for hockey players data.

Table 6.5: Descriptive statistics for payroll income data.

Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
16714.67	16200.00	9218.184	2.62	19.29	100.00	135000.00

models. The parameter estimates are significant for all fitted models, and the LBXII distribution exhibits the lower values for all goodness-of-fit statistics. Similarly to the first empirical example, the LBXII model shows up as a competitive alternative to the other fitted models.

Figure 6.5 displays the histogram and some plots of the estimated densities for the three most competitive models according to the goodness-of-fit statistics to the payroll income data. These plots are in agreement with the results in Table 6.7. Similarly to the first data set, the LBXII distribution can be used effectively to provide better fits than other considered income distributions for these data and it is a very competitive alternative to the W and EW models.

6.7 Concluding remarks

We introduce the four-parameter *logistic Burr XII* (LBXII) distribution. It can have decreasing and upside-down bathtub hazard functions and can be considered for modeling income distributions, among other applications. We demonstrate that the LBXII density function is an

Table 6.6: The MLEs of the model parameters and their standard errors for payroll income data.

	c	d	s	a	b
BBXII	1.8301	0.0876	8.4148	18.3223	8.3320
	(0.0614)	(0.0029)	(0.3872)	(0.4026)	0.283278
KwBXII	5.3594	0.0354	3.8626	8.828	5.4490
	(0.0987)	(0.0007)	(0.2147)	(0.2225)	(0.1341)
	c	d	s	λ	
LBXII	1.1934	0.1057	5.3612	14.1480	
	(0.0310)	(0.0027)	(0.3478)	(0.1928)	
BXII	1.8585	0.0812	20.4626		
	(0.1622)	(0.0071)	(0.8067)		
	λ	α	β		
EW	2.7063	0.0408	11.2836		
	(0.1329)	(0.0020)	(0.4695)		
W	16.8256	0.1169			
	(0.6817)	(0.0013)			
	c	m			
LL	0.2512	32.2073			
	(0.0032)	(1.2346)			

Table 6.7: Goodness-of-fit statistics for the fitted models to the payroll income data.

	AIC	CAIC	BIC	HQIC	KS
BBXII	112144.5514	112144.5634	112177.1613	112155.9779	0.2825
KwBXII	114423.2534	114423.2654	114455.8633	114434.6799	0.3159
LBXII	**107006.6913**	**107006.6992**	**107032.7792**	**107015.8325**	**0.2010**
BXII	125013.2791	125013.2839	125032.8450	125020.1350	0.5063
EW	122416.3162	122416.3210	122435.8822	122423.1721	0.5575
W	131889.9303	131889.9327	131902.9743	131894.5009	0.8148
LL	129319.9984	129320.0008	129333.0424	129324.5690	0.7323

infinite linear combination of BXII densities. Thus, some mathematical properties of the new distribution are obtained using this result, such as the ordinary and incomplete moments and generating function. We also determine the quantile function for the LBXII distribution, which is useful to obtain any quantiles of interest, to simulate LBXII random variables and provide some alternative expressions for the skewness and kurtosis. We estimate the model parameters by the maximum likelihood method, and a simulation study is provided by a Monte Carlo experiment. We present two applications to illustrate the potentiality of the LBXII distribution for modeling income data. Both data sets exhibit characteristics of a power law distribution, which is very common in income data sets. We note that the LBXII distribution has a good adjustment in

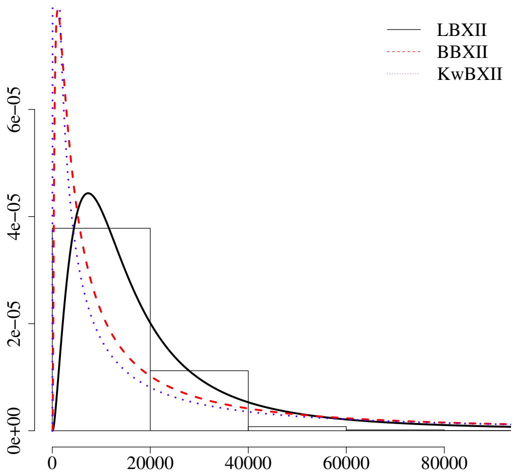


Figure 6.5: Histogram and estimated densities of the LBXII, BBXII and KwBXII models for payroll income data.

both cases, thus being a competitive model to the classical Weibull distribution, exponentiated Weibull model, other BXII generalizations and special models. Finally, the LBXII model may provide an attractive alternative to describe and understand the income distribution behavior.

Chapter 7

Final conclusion and future works

In this thesis we proposed four new continuous distributions by taking the three-parameter Burr XII model as baseline in some generated families, which are defined as special cases of the transformed-transform method. This parameter induction have been attracted researches because it allows adding flexibility to the baseline model and improving the quality of fit. In this context, our purpose is to provide Burr XII generalizations that may be useful to model situations characterized by power law behavior.

This feature is very common in income data and can also be observed in some lifetime examples. Therefore, for each introduced distribution, we considered applications to real data sets in order to illustrate their potentiality for fit these kind of data. The new models have proved to be very appropiate in these situations, but they could also be applied in other areas.

In Chapter 2, we presented a brief survey on the Burr XII distribution and some of its generalizations already introduced in the literature. We discussed some mathematical properties of the Burr XII distribution and realized that they have been used to obtain the properties of its generalizations from linear representations.

In Chapter 3, the Zografos-Balakrishnan Burr XII distribution is defined from the generated family pioneerd by Zografos and Balakrishnan (2009). In Chapter 4, the Ristić-Balakrishnan Burr XII distribution is obtained from the generated family proposed by Ristić and Balakrishnan (2012). Chapter 5 introduces the Weibull Burr XII distribution, which is a special model of the *Weibull-G* family. Finally, the logistic Burr XII model is proposed by taking the three-parameter

Burr XII distribution as baseline in the *logistic-X* family. We study some mathematical properties of these new distributions, estimate their parameters by maximum likelihood method, perform simulation studies to investigate some of their asymptotic properties and illustrate their potentiality by means of applications to real data sets.

Next, we list some future topics to be investigated:

- Cordeiro *et al.* (2014a) defined the exponentiated half-logistic family. Thus, a future research line is to study the exponentiated half-logistic Burr XII distribution, obtained by inserting the Burr XII model as baseline in this family. This extended model can be other alternative for modeling power law tailed data.

- Silva *et al.* (2008) considered the class of location-scale models for introduce a regression model based on the Burr XII distribution. They defined $Y = \log(T)$, where T is a BXII(c, d, s) random variable and considered the reparametrization $c = 1/\sigma$ and $s = \exp(\mu)$. Hence, the log-Burr XII regression model is given by

  $$Y = \mu + \sigma Z,$$

  where the random variable $Z = \frac{Y-\mu}{\sigma}$ and the scale parameter μ depends on the $(n \times p)$ matrix of explanatory variables X, this is, $\mu_i = \boldsymbol{x_i^T}\boldsymbol{\beta}$, where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)^T$, $(-\infty < \beta_j < \infty, j = 1, \cdots, p)$ is a vector of unknown parameters. Following a similar approach, we can introduce regression models based on the Burr XII generalizations proposed in this thesis.

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