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Testing inference in heteroskedastic linear regressions: a comparison of two alternative approaches

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INARA FRANCOYSE DE SOUZA PEREIRA

TESTING INFERENCE IN HETEROSKEDASTIC LINEAR REGRESSIONS: A COMPARISON OF TWO ALTERNATIVE APPROACHES

Dissertação apresentada ao Programa de Pós-Graduação em Estatística da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Estatística.

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A educação tem raízes amargas,
mas os seus frutos são doces.
Aristóteles

Abstract

We consider the issue of performing testing inferences on the parameters that index the linear regression model under heteroskedasticity of unknown form. Quasi-t test statistics use asymptotically correct standard errors obtained from heteroskedasticity-consistent covariance matrix estimators. An alternative approach involves making an assumption about the functional form of the response variances and jointly modeling mean and dispersion effects. In this dissertation we compare the accuracy of testing inferences made using the two approaches. We consider several different quasi-t tests and also z tests performed after generalized least squares estimation which was carried out using three different estimation strategies. Our numerical evaluations were performed using different models, different sample sizes, and different heteroskedasticity strengths. The numerical evidence shows that some quasi-t tests are considerably less size distorted in small samples than the tests carried out after the jointly modeling mean and dispersion effects. Finally, we present and discuss two empirical applications.

Keywords: Generalized least squares. Heteroskedasticity. Linear regression. Ordinary least squares. Quasi-t test. z test.

Resumo

Na presente dissertação nós consideramos a realização de inferências por teste de hipótese sobre os parâmetros que indexam o modelo linear de regressão sob heteroscedasticidade de forma desconhecida. As estatísticas de teste quasi-t empregam erros-padrão assintoticamente corretos oriundos de estimadores consistentes da matriz de covariância do estimador de mínimos quadrados ordinários dos parâmetros de regressão. Um enfoque alternativo envolve a modelagem das variâncias das respostas, ou seja, a modelagem conjunta de efeitos médios e de dispersão. Nós comparamos os dois enfoques através de várias simulações de Monte Carlo. Consideramos vários testes quasi-t e testes z realizados após estimação por mínimos quadrados generalizados realizada através de três enfoques distintos. Nossas avaliações numéricas foram realizadas utilizando diferentes modelos, tamanhos de amostra e graus de heteroscedasticidade. A evidência numérica indica que os testes quasi-t tendem a apresentar distorções de tamanho consideravelmente menores em pequenas amostras do que os testes realizados após a modelagem conjunta dos efeitos médios e de dispersão. Por fim, apresentamos e discutimos duas aplicações empíricas.

Palavras-chave: Mínimos quadrados generalizados. Heteroscedasticidade. Regressão linear. Mínimos quadrados ordinários. Teste quasi-t. Teste z.

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1 Introduction

1.1 Initial considerations

The linear regression model is commonly used in empirical analyses in a variety of fields. It assumes that the mean of the variable of interest (response, dependent variable) equals a linear predictor that involves regression coefficients and a set of independent variables (covariates, regressors). Alternatively, the response is taken to be the sum of the linear predictor (systematic component) and a random, unobservable error. Parameter estimation is typically carried out by ordinary least squares (which requires no distributional assumptions) from a sample of n observations. It is oftentimes assumed that all n errors (or, equivalently, that all n responses) share the same variance. Such an assumption is known as *homoskedasticity*. Under constant error variances and normality, exact testing inferences can be carried out using standard t and F tests. The homoskedasticity assumption, nonetheless, is commonly violated when working with cross-sectional datasets. When that happens, the ordinary least square estimator of the vector of regression coefficients remains unbiased, consistent, and asymptotically normally distributed. Its usual covariance matrix estimator, however, becomes biased and is not consistent under unequal error variances. The common practice is to use a heteroskedasticity-consistent covariance matrix estimator, i.e., an estimator for the variance of the vector of ordinary least regression coefficients estimators that is consistent under both homoskedasticity and heteroskedasticity of unknown form. We can then obtain asymptotically correct standard errors and use them to construct quasi-t test statistics that are, under the null hypothesis, asymptotically distributed as standard normal. Hence, without the need to assume that the errors are normally distributed, the practitioner can perform testing inference on the regression parameters.

An alternative approach is to jointly model the response mean and variance, and then perform z tests. Parameter estimation is performed by estimated (feasible) generalized least squares, i.e., using generalized least squares and replacing the unknown variances with the corresponding estimates. This approach entails the extra burden of modeling dispersion effects in addition to mean effects. The most commonly model assumes that heteroskedasticity is multiplicative, as proposed by Harvey (1976). The multiplicative functional form guarantees that all variance estimates are positive.

The two testing strategies listed above use standard errors obtained from different covariance matrix estimators: quasi-t test statistics use standard errors obtained from a heteroskedasticity-robust covariance matrix estimators whereas in the z test the standard error is obtained from an estimate of the estimated generalized least squares covariance matrix.

Atkinson, Riani, and Torti (2016) have recently compared the covariance matrix estimators obtained using the two approaches. They show that covariance matrix estimation is more accurate when performed via estimated generalized least squares even when the skedastic function is misspecified. Their focus lies in the accuracy of variance and covariance estimates. According to Simonoff (1993), nonetheless, since estimated variances are used primarily for performing inferences, comparisons involving different variance estimators should be related to their intended use, such as the empirical coverages of associated confidence intervals and the true nominal sizes of associated tests. This is the approach we shall pursue in this dissertation. In what follows, we shall address the following question: Are z testing inferences made after estimated generalized least squares estimation more accurate than those based on quasi-t tests? An appealing feature of the latter is that it does not require assumptions about the skedastic function.

In order to motivate the analysis in the remainder of our dissertation, consider the data analyzed by Cribari-Neto (2004). The variable of interest (y) is per capita spending on public schools in the U.S. (49 states and Washington, D.C.; sample size: 50 observations). It is expected that such per capita spending grow with per capita income (x), i.e., richer states are expected to spend more on public schools than less developed states on a per capita basis. We postulate two different relationships, namely: quadratic and linear. In order to distinguish between them, we write the model as $y_i = \beta_1 + \beta_2 x_i + \beta_3 x_i^2 + \epsilon_i$, $i = 1, \ldots, 50$, where ϵ_i is a zero mean random error, and test $\mathcal{H}_0 : \beta_3 = 0$ (linear) against $\mathcal{H}_1 : \beta_3 \neq 0$ (quadratic). Inspection of the data indicates that there is a very atypical data point, namely: Alaska. Additionally, there appears to be heteroskedasticity. A practitioner may be interested in knowing whether the best testing strategy involves the use of a quasi-t test, whose test statistic is based on a heteroskedasticity-robust standard error, or the use of a z test carried out after estimating the model parameters by estimated generalized least squares. Notice that the latter approach requires one to model not only mean effects but also dispersion effects. Interestingly, the two approaches (i.e., the z test and the best performing quasi-t tests) may yield different inferences at the 10% significance level. It will also be seen that z testing inferences are heavily dependent on how the dispersion effects are modeled. We shall return to this application in Chapter 5.

1.2 Organization of dissertation

The dissertation unfolds as follows. There are six chapters, including this introduction. In Chapter 2 we briefly present the linear regression model and the heteroskedasticity-consistent covariance matrix estimators that will be used in the following chapters. Chapter 3 is devoted to estimated generalized least squares estimation of the parameters that index the linear regression model. The numerical evidence on the finite sample performances of the different tests are presented in Chapter 4. We report results on the sizes and powers of the tests, i.e., we examine their performances under both the null and alternative hypotheses. The data generating processes we consider include heteroskedastic and homoskedastic model structures, balanced and leveraged data, and also normal and nonnormal random errors. In Chapter 5 we present and discuss two empirical applications. One of them is the application briefly introduced above. Finally, some concluding remarks are offered in Chapter 6.

1.3 Computing platforms

The simulations were carried out using the Ox matrix programming language (version 7.2). The Ox programming language is an object-oriented matrix programming language developed by Jurgen Doornik and it is freely available for academic usage at `http://www.doornik.com`. The appendix of this dissertation contains two programs written in Ox that were used to obtain some of the numerical results we present. In particular, it contains Ox code for a size simulation and for a power simulation. For more information about the Ox matrix programming language, see Doornik (2009). The figure was produced using the R statistical computing environment (version 3.2.4). R is a free software environment for statistical computing and graphics. For details, see R Core Team (2016). The typesetting environment chosen was LaTeX (Lamport 1986). LaTeX is a document preparation system developed based on TeX, which was developed by Donald Knuth. It is freely available.

2 Model and estimators

The linear regression model can be written as

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{2.1}$$

where $\boldsymbol{y}$ is an $n \times 1$ vector of responses (observations on the dependent variable), X is an $n \times p$ matrix of observations on p covariates ($\text{rank}(X) = p < n$), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is a p-vector of unknown regression parameters and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$ is an n-vector of random errors.

The following assumptions are usually made:

A1 Model (2.1) is correctly specified;

A2 $\mathbb{E}(\epsilon_i) = 0, i = 1, \ldots, n$;

A3 $\text{var}(\epsilon_i) = \mathbb{E}(\epsilon_i^2) = \sigma^2 \; \forall \; i \; (0 < \sigma^2 < \infty)$;

A4 $\mathbb{E}(\epsilon_i \epsilon_j) = 0 \; \forall \; i \neq j$;

A5 $\lim_{n\to\infty} n^{-1} X'X = Q$, where Q is a positive definite matrix.

The ordinary least squares estimator (OLSE) of $\boldsymbol{\beta}$ is obtained by minimizing the sum of squared errors:

$$\boldsymbol{\epsilon}'\boldsymbol{\epsilon} = (\boldsymbol{y} - X\boldsymbol{\beta})'(\boldsymbol{y} - X\boldsymbol{\beta}).$$

It can be expressed in closed-form as

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (X'X)^{-1}X'\boldsymbol{y} \\
&= (X'X)^{-1}X'(X\boldsymbol{\beta} + \boldsymbol{\epsilon}) \\
&= (X'X)^{-1}X'X\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon} \\
&= \boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon}.
\end{aligned}$$

Under Assumptions A1 and A2, $\hat{\boldsymbol{\beta}}$ is unbiased for $\boldsymbol{\beta}$, $\mathbb{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} \; \forall \; \boldsymbol{\beta} \in \mathbb{R}^k$ as we can see below:

$$\begin{aligned}
\mathbb{E}(\hat{\boldsymbol{\beta}}) &= \mathbb{E}[\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon}] \\
&= \boldsymbol{\beta} + (X'X)^{-1}X'\mathbb{E}(\boldsymbol{\epsilon}) \\
&= \boldsymbol{\beta}.
\end{aligned}$$

It is important to note that it is not necessary to assume homoskedasticity to establish the unbiasedness of the OLSE.

Under Assumptions A1, A2 and A4, the covariance matrix of $\boldsymbol{\epsilon}$ is $\Phi = \text{diag}\{\sigma_1^2, \ldots, \sigma_n^2\}$. It then follows that under homoskedasticity (Assumption A3), $\Phi = \sigma^2 I_n$, where I_n is the $n \times n$ identity matrix. Under Assumptions A1, A2 and A4, the covariance matrix of $\hat{\boldsymbol{\beta}}$ can be expressed as

$$\begin{aligned}\Psi &= \mathbb{E}[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'] \\ &= \mathbb{E}[(\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon} - \boldsymbol{\beta})(\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon} - \boldsymbol{\beta})'] \\ &= \mathbb{E}[(X'X)^{-1}X'\boldsymbol{\epsilon}\boldsymbol{\epsilon}'X(X'X)^{-1}] \\ &= (X'X)^{-1}X'\Phi X(X'X)^{-1}.\end{aligned}$$

Φ is the error covariance matrix, which can also be written as $\sigma^2\Omega$, where σ^2 is a positive scalar. When Assumption A3 holds (i.e., under homoskedasticity), $\Phi = \sigma^2 I_n$ and then $\Psi = \sigma^2(X'X)^{-1}$, which can be easily estimated as $\hat{\sigma}^2(X'X)^{-1}$, where $\hat{\sigma}^2 = \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}/(n-p)$ and $\hat{\boldsymbol{\epsilon}} = \boldsymbol{y} - X\hat{\boldsymbol{\beta}}$ are the residuals. Under Assumptions A1 through A4, $\hat{\boldsymbol{\beta}}$ is the best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$ (Gauss-Markov Theorem). This result implies that under homoskedasticity the OLSE has smaller variability than any other estimator of $\boldsymbol{\beta}$ that is both linear and unbiased. Under A1, A2 and A5 (thus, without assuming homoskedasticity), $\hat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$, i.e., $\hat{\boldsymbol{\beta}}$ converges in probability to $\boldsymbol{\beta}$, which will be denoted by $\text{plim}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$, where plim denotes probability limit. In order to prove this result, we use the fact that if $g(\cdot)$ is a continuous function and z_n is a sequence of random vectors, then

$$\text{plim}(g(z_n)) = g(\text{plim}(z_n)),$$

provided that $\text{plim}(z_n)$ exists. Under the Assumptions A1 and A5, we have

$$\begin{aligned}\text{plim}(\hat{\boldsymbol{\beta}}) &= \text{plim}(\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon}) \\ &= \boldsymbol{\beta} + \text{plim}(X'X)^{-1}\,\text{plim}(X'\boldsymbol{\epsilon}) \\ &= \boldsymbol{\beta} + \text{plim}(n^{-1}X'X)^{-1}\,\text{plim}(n^{-1}X'\boldsymbol{\epsilon}) \\ &= \boldsymbol{\beta} + (\text{plim }\, n^{-1}X'X)^{-1}\,\text{plim}(n^{-1}X'\boldsymbol{\epsilon}) \\ &= \boldsymbol{\beta} + Q^{-1}\,\text{plim}(n^{-1}X'\boldsymbol{\epsilon}).\end{aligned}$$

Note that $X'\boldsymbol{\epsilon} = \sum_{i=1}^n \boldsymbol{x}_i\epsilon_i$, where $\boldsymbol{x}_i$ is the ith line of X (as a column vector). It follows from the Law of Large Numbers that $n^{-1}X'\boldsymbol{\epsilon}$ converges in probability to $\mathbb{E}(\boldsymbol{x}_i\epsilon_i)$, which by Assumption A2, equals 0. That is, $\text{plim}(n^{-1}X'\boldsymbol{\epsilon}) = 0$. Therefore,

$$\text{plim}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}.$$

It is important to emphasize that this result holds regardless of whether Assumption A3 is satisfied. When Assumption A3 or Assumption A4 is not valid, the OLSE of $\boldsymbol{\beta}$ is no longer efficient.

When the errors are heteroskedastic but Φ is known (which rarely happens), one can use the generalized least squares estimator (GLSE), which is given by $\hat{\boldsymbol{\beta}}_G = (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}\boldsymbol{y}$. Its covariance matrix is $\text{cov}(\hat{\boldsymbol{\beta}}_G) = \sigma^2(X'\Omega^{-1}X)^{-1}$, as shown below.

The initial idea is to transform the model given in (2.1) using P, a matrix of dimension $n \times n$ which satisfies

$$P\Omega P' = I_n.$$

Since Ω is positive-definite, P always exists. Notice that

$$\Omega^{-1} = P'P.$$

We can use P to transform Model (2.1) as follows:

$$P\boldsymbol{y} = PX\boldsymbol{\beta} + P\boldsymbol{\epsilon}.$$

That is, we have

$$\boldsymbol{y}^* = X^*\boldsymbol{\beta} + \boldsymbol{\epsilon}^*,$$

where $\boldsymbol{y}^* = P\boldsymbol{y}$, $X^* = PX$ and $\boldsymbol{\epsilon}^* = P\boldsymbol{\epsilon}$. The vector of transformed errors $\boldsymbol{\epsilon}^*$ has mean zero:

$$\mathbb{E}(\boldsymbol{\epsilon}^*) = \mathbb{E}(P\boldsymbol{\epsilon}) = P\mathbb{E}(\boldsymbol{\epsilon}) = 0.$$

That is, Assumption A2 is satisfied. The covariance matrix of the vector of transformed erros can be easily obtained:

$$\mathbb{E}(\boldsymbol{\epsilon}^*\boldsymbol{\epsilon}^{*'}) = \mathbb{E}(P\boldsymbol{\epsilon}\boldsymbol{\epsilon}'P') = P\mathbb{E}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}')P' = \sigma^2 P\Omega P' = \sigma^2 I_n.$$

Thus, Assumptions A2 through A4 hold for the transformed model. Minimization of the sum of squared errors yields the following estimator of $\boldsymbol{\beta}$:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_G &= (X^{*'}X^*)^{-1}X^{*'}\boldsymbol{y}^* \\ &= (X'P'PX)^{-1}X'P'P\boldsymbol{y} \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\boldsymbol{y} \\ &= (X'\sigma^{-2}\Omega^{-1}X)^{-1}X'\sigma^{-2}\Omega^{-1}\boldsymbol{y} \\ &= (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}\boldsymbol{y}.\end{aligned}$$

Note that $\hat{\boldsymbol{\beta}}_G$ is unbiased for $\boldsymbol{\beta}$:

$$\begin{aligned}\mathbb{E}(\hat{\boldsymbol{\beta}}_G) &= (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}\mathbb{E}(\boldsymbol{y}) \\ &= (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}X\boldsymbol{\beta} \\ &= \boldsymbol{\beta}.\end{aligned}$$

The covariance matrix of $\hat{\boldsymbol{\beta}}_G$ is

$$\text{cov}(\hat{\boldsymbol{\beta}}_G) = \sigma^2(X^{*'}X^*)^{-1} = (X'\Phi^{-1}X)^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}.$$

The GLSE is efficient in the sense of the Gauss-Markov Theorem.

Since Φ is typically unknown, $\hat{\beta}_G$ cannot be computed. A possible solution is to assume a functional form for the n variances, and then estimate Φ. The estimated (or feasible) generalized least squares (EGLS) estimator is obtained by replacing Φ by such an estimate. It is given by

$$\hat{\hat{\beta}} = (X'\hat{\Phi}^{-1}X)^{-1}X'\hat{\Phi}^{-1}\boldsymbol{y}.$$

Direct estimation of Φ is problematic, since this matrix contains n unknown variances. A possible solution is to postulate a model for the variances, and then estimate the parameters of such a model. The most commonly used model for the error variances is the multiplicative model proposed by Harvey (1976). See the next chapter for further details.

A more commonly used approach involves basing inferences on $\hat{\beta}$ coupled with a consistent estimator for its covariance matrix, i.e., with a covariance matrix estimator that is consistent under both homoskedasticity and heteroskedasticity of unknown form. The most well known consistent covariance estimator is known as HC0 and was introduced by White (1980):

$$\text{HC0} = (X'X)^{-1}X'\hat{\Phi}_0X(X'X)^{-1},$$

where $\hat{\Phi}_0 = \text{diag}\{\hat{\epsilon}_1^2, \ldots, \hat{\epsilon}_n^2\}$.

It has been shown that HC0 can be considerably biased in finite samples, especially when the data contain high leverage data points; see, e.g., Chesher and Jewitt (1987). In particular, it tends to 'optimistic', i.e., it tends to underestimate the true variances. As a consequence, quasi-t tests whose statistic use HC0 standard errors tend to be liberal, i.e., such tests tend to overreject the null hypothesis when such a hypothesis is true.

MacKinnon and White (1985) proposed two alternative covariance matrix estimators. They both include finite sample corrections. When all errors share the same variance, it can be shown that

$$\mathbb{E}(\hat{\epsilon}_i{}^2) = (1 - h_i)\sigma^2, \tag{2.2}$$

where h_i is the ith diagonal element of $H = X(X'X)^{-1}X'$, the 'hat matrix'. It is important to note the h_i's are the leverage measures of the different observations. Building up on previous results by Horn, Horn, and Duncan (1975), MacKinnon and White (1985) introduced the estimator known as HC2. It is given by

$$\text{HC2} = (X'X)^{-1}X'\hat{\Phi}_2X(X'X)^{-1},$$

where $\hat{\Phi}_2 = \text{diag}\{\hat{\epsilon}_1^2/(1 - h_1), \ldots, \hat{\epsilon}_n^2/(1 - h_n)\}$. Using (2.2) it is possible to show that HC2 is unbiased under homoskedasticity.

The authors have also introduced a jackknife covariance matrix estimator. It can be approximated by the following estimator, which is known as HC3:

$$\text{HC3} = (X'X)^{-1}X'\hat{\Phi}_3X(X'X)^{-1},$$

where $\hat{\Phi}_3 = \text{diag}\{\hat{\epsilon}_1^2/(1 - h_1)^2, \ldots, \hat{\epsilon}_n^2/(1 - h_n)^2\}$.

The HC4 heteroskedasticity-consistent covariance matrix estimator was proposed by Cribari-Neto (2004). It is given by

$$\text{HC4} = (X'X)^{-1}X'\hat{\Phi}_4X(X'X)^{-1},$$

where $\hat{\Phi}_4 = \mathrm{diag}\{\hat{\epsilon}_1^2/(1-h_1)^{\delta_1}, \ldots, \hat{\epsilon}_n^2/(1-h_n)^{\delta_n}\}$, $\delta_i = \min\{4, h_i/\bar{h}\}$ and $\bar{h} = n^{-1}\sum_{i=1}^n h_i$, i.e., $\bar{h}$ is the mean leverage. Notice that the exponent of $(1-h_i)$ is not constant; it equals the ratio between the ith leverage measure and the mean leverage up to a truncation constant.

A variant of the HC4 estimator was introduced by Cribari-Neto and Silva (2011). It is known as HC4m and is given by

$$\mathrm{HC4m} = (X'X)^{-1}X'\hat{\Phi}_{4m}X(X'X)^{-1},$$

where $\hat{\Phi}_{4m} = \mathrm{diag}\{\hat{\epsilon}_1^2/(1-h_1)^{\theta_1}, \ldots, \hat{\epsilon}_n^2/(1-h_n)^{\theta_n}\}$. Here, $\theta_i = \min\{\nu_1, h_i/\bar{h}\} + \min\{\nu_2, h_i/\bar{h}\}$, where ν_1 and ν_2 are positive real constants. Based on numerical evidence, the authors proposed using $\nu_1 = 1.0$ and $\nu_2 = 1.5$. For details, see Cribari-Neto and Silva (2011).

The above heteroskedasticity-robust covariance matrix estimators can be used for interval estimation and testing inferences. Heteroskedasticity-robust interval estimation was addressed by Cribari-Neto and Lima (2009). Heteroskedasticity-robust testing inferences were considered by several authors; see, e.g., Cribari-Neto and Lima (2010) and Long and Ervin (2000).

3 Estimated generalized least squares

As noted in the previous chapter, an alternative approach involves modeling the error variances. Consider the following linear model:

$$y_i = \boldsymbol{x}_i'\boldsymbol{\beta} + \sigma_i \epsilon_i, \quad i = 1, \ldots, n,$$

where $\boldsymbol{x}_i$ is a $p \times 1$ vector of observations on the independent variables and the errors are assumed to be standard normally distributed. Following Harvey (1976), the ith error variance is modeled as

$$\sigma_i^2 = \text{var}(y_i) = \exp\{\boldsymbol{z}_i'\gamma\} = \exp\{\gamma_1 + \gamma_2 z_{i2} + \cdots + \gamma_s z_{is}\}, \tag{3.1}$$

where γ is an $s \times 1$ vector of parameters and $\boldsymbol{z}_i'$ denotes the ith row of Z, an $n \times s$ matrix that contains observations on the dispersion covariates which are usually, though not necessarily, related to the mean regressors. This model is known as the multiplicative heteroskedastic linear regression model.

The above model was recently revisited by Atkinson, Riani, and Torti (2016). For a given γ, they estimate of $\boldsymbol{\beta}$ by weighted least squares. The weights used in the estimation of $\boldsymbol{\beta}$ are specified as

$$w_i = 1/\sigma_i^2.$$

The estimator of $\boldsymbol{\beta}$ can thus be written as

$$\tilde{\boldsymbol{\beta}} = (X'WX)^{-1}X'W\boldsymbol{y},$$

where $W = \text{diag}\{w_1, \ldots, w_n\}$ is the $n \times n$ weight matrix. Alternatively, $\tilde{\boldsymbol{\beta}}$ can be computed by regressing $W^{1/2}\boldsymbol{y}$ on $W^{1/2}X$ using ordinary least squares. Finally, γ can be estimated by maximum likelihood making use of the normality assumption.

Notice that the ith weight is now given by $\tilde{w}_i = 1/\tilde{\sigma}_i^2$, where $\tilde{\sigma}_i^2 = \exp(\boldsymbol{z}_i'\tilde{\gamma})$. Let $\tilde{W}$ be the n-dimensional diagonal matrix that contains the estimated weights. The estimated covariance matrix of $\tilde{\boldsymbol{\beta}}$ is $(X'\tilde{W}X)^{-1}$.

For a given $\boldsymbol{\beta}$, the parameter vector γ can be estimated by maximizing the following log-likelihood function:

$$\begin{aligned}
\ell(\gamma) &= \log\left\{\prod_{i=1}^{n}\left[(2\pi\sigma_i^2)^{-1/2}\exp\left(-\frac{1}{2}\frac{(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2}{\sigma_i^2}\right)\right]\right\} \\
&= -\frac{1}{2}\sum_{i=1}^{n}\left\{\log(2\pi) + \log(\sigma_i^2) + (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2/\sigma_i^2\right\}
\end{aligned}$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\left\{\log(2\pi) + \boldsymbol{z}_i'\boldsymbol{\gamma} + (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2/\exp(\boldsymbol{z}_i'\boldsymbol{\gamma})\right\}.$$

The authors present a scoring algorithm that can be used to jointly estimate the model parameters. The score vector with respect to $\boldsymbol{\gamma}$ is

$$S(\boldsymbol{\gamma}) = \frac{\partial \ell(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}} = \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{z}_i\left[\frac{(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2}{\exp(\boldsymbol{z}_i'\boldsymbol{\gamma})} - 1\right].$$

The observed information for $\boldsymbol{\gamma}$ is

$$K(\boldsymbol{\gamma}) = -\frac{\partial^2 \ell(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\gamma}'} = \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{z}_i\boldsymbol{z}_i'\left[\frac{(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2}{\exp(\boldsymbol{z}_i'\boldsymbol{\gamma})}\right].$$

The expected information is thus

$$I(\boldsymbol{\gamma}) = \mathbb{E}[K(\boldsymbol{\gamma})] = \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{z}_i\boldsymbol{z}_i'.$$

Parameter estimation can now be performed using an iterative scoring algorithm. The estimate of $\boldsymbol{\gamma}$ in the $(k+1)$th iteration is given by

$$\boldsymbol{\gamma}_{k+1} = \boldsymbol{\gamma}_k + \delta I(\boldsymbol{\gamma}_k)^{-1}S(\boldsymbol{\gamma}_k) = \boldsymbol{\gamma}_k + \delta(Z'Z)^{-1}\sum_{i=1}^{n}\boldsymbol{z}_i\left[\frac{(y_i - \boldsymbol{x}_i'\boldsymbol{\beta}_k)^2}{\exp(\boldsymbol{z}_i'\boldsymbol{\gamma}_k)} - 1\right],$$

where $0 < \delta < 1$ is the step length. Given $\boldsymbol{\gamma}_{k+1}$, the parameter vector $\boldsymbol{\beta}$ is re-estimated by weighted least squares. The algorithm stops when convergence is reached.

We shall also consider two alternative estimation approaches, as outlined by Harvey (1976), namely: (i) a two-step procedure and (ii) maximum likelihood estimation. Notice that, the variance function in (3.1) implies that

$$\Phi = \begin{bmatrix} \exp(\boldsymbol{z}_1'\boldsymbol{\gamma}) & 0 & 0 & \dots & 0 \\ 0 & \exp(\boldsymbol{z}_2'\boldsymbol{\gamma}) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \exp(\boldsymbol{z}_n'\boldsymbol{\gamma}) \end{bmatrix}.$$

In the two-step procedure, we estimate $\boldsymbol{\gamma}$ from

$$\log(\hat{\epsilon}_i{}^2) = \boldsymbol{z}_i'\boldsymbol{\gamma} + u_i,$$

where $u_i = \log(\hat{\epsilon}_i^2/\sigma_i^2)$. Let $\boldsymbol{q} = (\log(\hat{\epsilon}_1^2), \dots, \log(\hat{\epsilon}_n^2))'$. The estimator of γ is

$$\hat{\boldsymbol{\gamma}} = (Z'Z)^{-1}Z'\boldsymbol{q}.$$

Using $\hat{\boldsymbol{\gamma}}$, we obtain an estimator for $\boldsymbol{\beta}$ as follows:

$$\hat{\hat{\boldsymbol{\beta}}} = (X'\hat{\Phi}^{-1}X)^{-1}X'\hat{\Phi}^{-1}\boldsymbol{y},$$

where $\hat{\Phi} = \text{diag}\{\exp(\boldsymbol{z}_1'\hat{\gamma}), \ldots, \exp(\boldsymbol{z}_n'\hat{\gamma})\}$.

Harvey (1976) showed that the first component of $\hat{\boldsymbol{\gamma}}$, $\hat{\gamma}_1$, is an inconsistent estimator of γ_1, the first component of $\boldsymbol{\gamma}$; in particular, $\text{plim}(\hat{\gamma}_1) = \gamma_1 - 1.2704$. The remaining $s-1$ elements of $\hat{\boldsymbol{\gamma}}$ are, nonetheless, consistent for the corresponding parameters in $\boldsymbol{\gamma}$. Since γ_1 merely introduces a proportionality factor in (3.1), $\hat{\hat{\boldsymbol{\beta}}}$ is consistent for $\boldsymbol{\beta}$.

The third and final approach uses maximum likelihood estimation. The log-likelihood function

$$\ell(\boldsymbol{\beta}, \gamma) = -\frac{1}{2} \sum_{i=1}^{n} \{\log(2\pi) + \boldsymbol{z}_i'\gamma + (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2 / \exp(\boldsymbol{z}_i'\gamma)\}$$

is maximized with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. The estimators asymptotic variance-covariance matrix is given by the inverse of Fisher's information matrix:

$$\begin{bmatrix} (\sum_{i=1}^{n} \sigma_i^{-2} \boldsymbol{x}_i \boldsymbol{x}_i')^{-1} & 0 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ 0 & 2(\sum_{i=1}^{n} \boldsymbol{z}_i \boldsymbol{z}_i')^{-1} \end{bmatrix}.$$

In the tables that follow the acronyms EGLS_1, EGLS_2 and EGLS_3 refer to, respectively, maximum likelihood estimation, the estimation procedure outlined by Atkinson, Riani, and Torti (2016) and the two-step estimation procedure.

4 Numerical evaluation

4.1 Simulation setup

In what follows we shall numerically evaluate the finite sample performances of quasi-t and z testing inferences in linear regressions. Recall that z test statistics use estimated generalized least squares standard errors whereas quasi-t test statistics use heteroskedasticity-robust standard errors. All reported results are based on 10,000 Monte Carlo simulations and were obtained using the Ox matrix programming language (Doornik 2009). The sample sizes are $n = 50, 100, 150, 200$. The covariate values for $n = 50$ were replicated twice, three times and four times when $n = 100, 150, 200$, respectively. This was done so that the strength of heteroskedasticity, which is measured by $\lambda = \max(\sigma_i^2)/\min(\sigma_i^2)$, remains constant as the sample size increases. The values of the dispersion parameters (i.e., the components of γ) in the three experiments were selected so that $\lambda \approx 51$ and 101, $\lambda \approx 6$ and 26 and $\lambda = 1$ and ≈ 3, respectively. The simulations thus cover different heteroskedasticity strengths. Following Atkinson, Riani, and Torti (2016), regardless of the true skedastic function, estimated generalized least squares estimation is carried out using

$$\sigma_i^2 = \exp\Big(\sum_{j=1}^{p} \gamma_j x_{ij}\Big).$$

In the models used in our simulations, $x_{i1} = 1\ \forall i$.

The interest lies in testing the null hypothesis $\mathcal{H}_0 : \beta_j = \beta_j^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \beta_j \neq \beta_j^{(0)}$, for some j in $\{1, \ldots, p\}$, where $\beta_j^{(0)}$ is a given scalar. The quasi-t test statistic is

$$\tau = \frac{\hat{\beta}_j - \beta_j^{(0)}}{\sqrt{\widehat{\mathrm{var}}(\hat{\beta}_j)}},$$

where $\widehat{\mathrm{var}}(\hat{\beta}_j)$ denotes the estimated variance of $\hat{\beta}_j$ obtained from a heteroskedasticity-consistent covariance matrix estimator. Under the null hypothesis, τ is asymptotically distributed as $\mathcal{N}(0, 1)$. The null hypothesis is rejected if $|\tau| > z_{1-\alpha/2}$, where α is the test significance level and $z_{1-\alpha/2}$ denotes the $1 - \alpha/2$ standard normal upper quantile. We consider quasi-t tests based on statistics that use standard errors obtained from HC0, HC2, HC3, HC4 and HC4m. We also consider testing inferences based on the following test statistic:

$$\tau_g = \frac{\tilde{\beta}_{jg} - \beta_j^{(0)}}{\sqrt{\widehat{\text{var}}(\tilde{\beta}_{jg})}},$$

where $\widehat{\text{var}}(\tilde{\beta}_{jg})$ is obtained from the estimated covariance matrix of $\tilde{\boldsymbol{\beta}}_g$, $g = 1, 2, 3$ for the three different estimation approaches discussed in Chapter 3. The rejection rule is the same as in the quasi-t tests, i.e., the null hypothesis is rejected if $|\tau_g| > z_{1-\alpha/2}$. All tests are performed at the 10% and 5% significance levels.

The first numerical evaluation uses the following data generating process:

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \sigma_i \epsilon_i, \quad i = 1, \ldots, n.$$

All covariate values are absolute values of $\mathcal{N}(0, 1)$ random draws. The same covariates are used in the skedastic function. The errors ϵ_i's are generated from $\mathcal{N}(0, 1)$ and from $\chi^2_{(m)}$, $m = 2, 5, 10$; the latter were normalized to have zero mean and unit variance. Data generation is carried out using $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 3$. There is no leverage point. The error variances are

$$\sigma_i^2 = \exp(\gamma + \gamma x_{i2} + \gamma x_{i3} + \gamma x_{i4}).$$

Notice that the skedastic function is equal from that used when performing estimated generalized least squares estimation. We used the following values for γ: 0.935 ($\lambda \approx 51$) and 1.101 ($\lambda \approx 101$).

The second set of Monte Carlo simulations were based on the model

$$y_i = \beta_1 + \beta_2 x_i + \sigma_i \epsilon_i, \quad i = 1, \ldots, n.$$

The covariate values are selected as equally spaced points in $[0, 1]$. The largest covariate value (1.0) is then replaced by $a = 1.0, 2.5, 3.5, 5.0$. When $a = 1.0$, there is no leverage point. When $a > 1$, the data are leveraged. The errors ϵ_i's are randomly generated from the standard normal distribution. Data generation is carried out using $\beta_1 = \beta_2 = 3$. The skedastic function is

$$\sigma_i^2 = \exp(\gamma + \gamma z_i).$$

The values of $z_1, \ldots, z_n$ are selected as squared t_5 random draws. Notice that the skedastic function used for performing estimated generalized least squares estimation differs from the true skedastic function. We use the following values of γ: 0.200 ($\lambda \approx 6$) and 0.393 ($\lambda \approx 26$).

The third Monte Carlo experiment uses

$$y_i = \beta_1 + \beta_2 x_i + \beta_3 x_i^2 + \sigma_i \epsilon_i, \quad i = 1, \ldots, n.$$

The values of x_i are those in the empirical application presented in Section 5.2. The data contain three leverage points ($n = 50$). The errors ϵ_i's are obtained as standard normal random draws. Data generation is performed using $\beta_1 = -150.868$, $\beta_2 = 688.806$ and $\beta_3 = 0$. We carried out simulations under homoskedasticity ($\lambda = 1$) and heteroskedasticity ($\lambda > 1$). The skedastic function is

$$\sigma_i^2 = 3700$$

when the error variances are equal and

$$\sigma_i^2 = 4\eta_i,$$

where $\eta_i = \beta_1 + \beta_2 x_i + \beta_3 x_i^2$, under unequal error variances.

4.2 Size simulations

Tables 4.1 through 4.5 present the tests null rejection rates (entries are percentages). We consider quasi-t tests (Chapter 2) and also three estimated generalized least squares-based tests (Chapter 3). Consider, at the outset, Tables 4.1 ($\lambda \approx 51$) and 4.2 ($\lambda \approx 101$) which contain results from the first simulation experiment. We test $\mathcal{H}_0 : \beta_3 = 3$ against $\mathcal{H}_1 : \beta_3 \neq 3$, and present results for normal and nonnormal (chi-squared distributed) errors. The results in Tables 4.1 and 4.2 lead to important conclusions. First, overall, the EGLS tests are the worst performing tests, especially when the sample size is not large ($n = 50$ and $n = 100$). Note that the null rejection rates of the three estimated generalized least squares-based tests are very similar. For instance, when $n = 50$ and under normal errors, such rejection rates at the 10% significance level are close to 19% (in both tables). Under highly asymmetrical errors (χ_2^2 distributed errors) the EGLS tests are even more liberal, their null rejection rates becoming approximately equal to 24% for EGLS_1 and 21% for EGLS_2 and EGLS_3. Second, the performances of the HC0, HC2, HC3, HC4 and HC4m quasi-t tests show some deterioration as the error distribution becomes more asymmetric; such a deterioration is not, however, as pronounced as that of the EGLS tests. For instance, in Table 4.1, when $n = 50$ and at the 5% significance level, the HC0 (HC2) [HC4] null rejection rates for normal and χ_2^2 errors are 10.67% and 12.22% (8.33% and 10.18%) [6.38% and 8.34%], respectively. Third, overall, all tests become more accurate as the sample size increases.

We now move to the second simulation experiment. The interest lies in testing $\mathcal{H}_0 : \beta_2 = 3$ against $\mathcal{H}_1 : \beta_2 \neq 3$. As before, all tests are carried out at the 10% and 5% significance levels. Recall that the regression model contains a single covariate, that for the smallest sample ($n = 50$) its values are selected as an equally spaced sequence of points from 0 to 1 (i.e., in $[0, 1]$), and that the largest covariate value (1.0) is then replaced by $a = 1.0, 2.5, 3.5, 5.0$. We thus aim to evaluate the impact of increased maximal data leverage on the tests finite sample performances. The tests null rejection rates can be found in Tables 4.3 ($\lambda \approx 6$) and 4.4 ($\lambda \approx 26$). The results we report lead to interesting conclusions. First, the HC0 test becomes much more liberal as the value of a increases, i.e., under progressively more intense maximal leverage, especially when the sample size is not large ($n = 50$ and $n = 100$). For instance, when $n = 50$ and at the 10% nominal level, we see in Table 4.3 that the HC0 null rejection rate goes from 12.12% to 31.02% when the maximal covariate value goes from 1.0 to 5.0. Second, the EGLS_1 and EGLS_2 tests also become progressively more liberal in small samples as the data become more leveraged. Their null rejection rate ($n = 50$, $\alpha = 0.10$, Table 4.4) jump from approximately 18% to approximately 23% when the value of a is increased from 1.0 to 5.0. Even with larger samples the tests size distortions are substantial. For instance, when $n = 200$ and $a = 2.5$, the tests null rejection rates are approximately equal to 15%. Third, the HC3, HC4 and HC4m tests tend to become more conservative under leveraged data, especially when the sample size

Table 4.1 Null rejection rates (%), first simulation experiment; $\lambda \approx 51$.

	n	50		100		150		200	
Errors		5%	10%	5%	10%	5%	10%	5%	10%
$\mathcal{N}(0,1)$	$EGLS_1$	12.34	19.09	7.82	13.73	6.92	12.56	6.58	11.89
	$EGLS_2$	12.19	18.85	7.59	13.35	6.82	12.37	6.52	11.78
	$EGLS_3$	12.03	18.38	8.29	14.11	7.32	12.82	6.73	12.05
	HC0	10.67	16.99	8.09	14.03	7.15	12.75	6.70	12.33
	HC2	8.33	13.99	7.19	12.53	6.53	11.85	6.20	11.47
	HC3	6.24	11.44	6.11	11.21	5.97	11.14	5.67	10.73
	HC4	6.38	11.47	6.09	11.10	5.96	11.08	5.67	10.67
	HC4m	5.66	10.50	5.65	10.68	5.68	10.75	5.55	10.41
$\chi^2_{(10)}$	$EGLS_1$	13.32	19.99	8.21	14.62	7.52	13.12	7.20	12.83
	$EGLS_2$	12.90	19.37	7.76	13.87	7.28	12.69	6.96	12.48
	$EGLS_3$	12.60	18.85	8.89	14.25	8.03	13.52	7.61	13.11
	HC0	10.98	17.07	8.47	14.01	7.26	13.10	6.84	11.96
	HC2	8.75	14.24	7.57	12.68	6.74	12.18	6.46	11.25
	HC3	6.82	11.70	6.58	11.33	6.19	11.30	5.98	10.67
	HC4	6.90	11.77	6.58	11.32	6.16	11.28	6.00	10.61
	HC4m	6.21	10.79	6.28	10.95	5.92	10.95	5.82	10.38
$\chi^2_{(5)}$	$EGLS_1$	14.18	20.87	9.85	15.88	8.01	13.89	6.88	12.43
	$EGLS_2$	13.63	20.00	9.09	14.65	7.49	13.07	6.63	11.98
	$EGLS_3$	12.88	19.06	10.14	16.19	8.45	14.55	7.64	13.23
	HC0	11.20	17.50	8.36	14.16	7.65	13.23	7.02	12.42
	HC2	9.02	14.72	7.43	12.94	7.02	12.33	6.62	11.71
	HC3	7.26	12.15	6.61	11.64	6.48	11.50	6.13	11.00
	HC4	7.20	12.32	6.70	11.66	6.40	11.41	6.13	10.96
	HC4m	6.58	11.28	6.35	11.21	6.19	11.02	5.88	10.73
$\chi^2_{(2)}$	$EGLS_1$	17.31	23.54	11.05	17.28	9.59	14.97	8.28	14.10
	$EGLS_2$	15.93	21.81	9.72	15.29	8.71	13.58	7.57	13.03
	$EGLS_3$	14.52	21.02	11.70	17.61	10.60	16.16	9.53	15.23
	HC0	12.22	18.47	9.37	15.04	8.11	13.86	7.42	12.21
	HC2	10.18	15.62	8.46	13.84	7.45	12.88	6.90	11.61
	HC3	8.16	13.19	7.52	12.57	6.75	12.04	6.53	11.05
	HC4	8.34	13.37	7.61	12.54	6.82	11.95	6.55	11.03
	HC4m	7.45	12.37	7.21	12.15	6.49	11.71	6.41	10.80

Table 4.2 Null rejection rates (%), first simulation experiment; $\lambda \approx 101$.

	n	50		100		150		200	
Errors		5%	10%	5%	10%	5%	10%	5%	10%
$\mathcal{N}(0,1)$	$EGLS_1$	12.17	19.22	7.82	13.66	6.91	12.61	6.50	11.72
	$EGLS_2$	11.81	18.64	7.27	12.66	6.65	12.04	6.37	11.42
	$EGLS_3$	11.61	17.99	8.33	13.68	7.02	12.59	6.61	11.84
	HC0	10.80	17.13	8.09	14.07	7.30	12.90	6.71	12.20
	HC2	8.42	14.07	7.11	12.59	6.61	12.18	6.20	11.50
	HC3	6.34	11.43	6.20	11.23	6.00	11.32	5.70	10.76
	HC4	6.32	11.26	6.13	11.20	6.00	11.23	5.70	10.67
	HC4m	5.72	10.41	5.77	10.60	5.84	10.86	5.53	10.45
$\chi^2_{(10)}$	$EGLS_1$	13.35	19.80	8.24	14.51	7.45	13.02	7.22	12.92
	$EGLS_2$	12.62	18.73	7.38	13.01	6.96	12.18	6.70	12.08
	$EGLS_3$	12.35	18.32	8.73	14.20	7.80	13.46	7.38	12.99
	HC0	11.12	17.46	8.67	14.33	7.46	12.94	6.86	12.01
	HC2	8.81	14.46	7.59	12.74	6.78	12.08	6.35	11.34
	HC3	6.85	11.80	6.75	11.38	6.28	11.25	5.88	10.68
	HC4	6.88	11.85	6.63	11.34	6.22	11.17	5.87	10.60
	HC4m	6.21	10.80	6.31	10.93	6.09	10.89	5.78	10.39
$\chi^2_{(5)}$	$EGLS_1$	14.30	20.97	9.93	15.78	7.88	13.80	6.95	12.18
	$EGLS_2$	13.26	19.47	8.73	13.84	7.11	12.48	6.49	11.41
	$EGLS_3$	12.76	18.83	10.02	15.89	8.36	14.61	7.50	13.01
	HC0	11.53	17.95	8.47	14.40	7.71	13.36	7.19	12.56
	HC2	9.37	15.08	7.57	13.07	7.15	12.43	6.78	11.77
	HC3	7.31	12.34	6.66	11.76	6.57	11.51	6.30	11.08
	HC4	7.31	12.31	6.68	11.69	6.51	11.40	6.21	11.02
	HC4m	6.68	11.45	6.44	11.26	6.27	11.06	5.99	10.75
$\chi^2_{(2)}$	$EGLS_1$	17.04	23.40	11.06	17.34	9.61	14.93	8.42	14.10
	$EGLS_2$	15.09	20.79	9.05	14.36	8.21	12.80	7.25	12.33
	$EGLS_3$	14.69	20.99	11.51	17.37	10.32	16.27	9.81	15.51
	HC0	12.64	19.13	9.76	15.22	8.41	14.04	7.60	12.57
	HC2	10.66	16.11	8.74	14.16	7.78	13.17	7.21	12.00
	HC3	8.67	13.59	7.87	13.00	7.12	12.28	6.79	11.38
	HC4	8.79	13.60	7.88	12.92	7.13	12.30	6.78	11.37
	HC4m	7.79	12.69	7.49	12.52	6.88	11.90	6.67	11.15

is not large. To illustrate that, consider $n = 50$ and $\alpha = 0.10$ in Table 4.4. For $a = 1.0$, the three tests null rejection rates are 9.81%, 10.34% and 9.43%, respectively; when $a = 5.0$, the corresponding figures are 3.16%, 0.47% and 2.05%. The HC4 is particularly affected by the extreme leverage. Fourth, the tests size distortions tend to decrease as n increases.

The tests null rejection rates obtained in the third simulation scenario are presented in Table 4.5 (entries are percentages). The interest lies in testing $\mathcal{H}_0 : \beta_3 = 0$ against $\mathcal{H}_1 : \beta_3 \neq 0$. Recall that we are testing a linear versus a quadratic functional form. We present results obtained under homoskedasticity ($\lambda = 1.0$, $\sigma^2 = 3700$) and mild heteroskedasticity ($\lambda \approx 2.5$, $\sigma_i^2 = 4\eta_i$). The first conclusion we draw from the figures in Table 4.5 is that the EGLS tests are again considerably liberal (especially EGLS_1 and EGLS_2 in small samples), even when the sample size is large ($n = 200$). For instance, under homoskedasticity (heteroskedasticity) and at the 10% significance level the null rejection rates of the of the EGLS_1 test when for $n = 50$ and $n = 200$ are, respectively, 36.53% and 14.42% (37.78% and 14.89%). Second, the HC0 and HC2 tests are again liberal, even when $n = 200$. Third, the HC4 test is once again conservative. Fourth, overall, the HC3 and HC4m tests display good control of the type I error frequency. For example, under heteroskedasticity and when $n = 100$, the HC3 and HC4m null rejection rates at the 10% significance level are, respectively, 11.15% and 9.91%.

4.3 Power simulations

We shall now focus on the tests power, i.e., on the tests ability to detect that the null hypothesis is false when it is indeed false. We performed Monte Carlo simulations using the same scenarios as in the size simulations. Data generation, however, is now carried out under the alternative hypothesis. Since some tests are considerably liberal, we compare the powers of size-corrected tests. At the outset, we consider the setup used in the first set of size simulations. We test $\mathcal{H}_0 : \beta_3 = 3$ but the true parameter value is taken to be 6.0, i.e., data generation is performed using $\beta_3 = 6.0$. Tables 4.6 and 4.7 contain the tests nonnull rejection rates (entries are percentages). The EGLS tests are more powerful than the quasi-t tests regardless of the error distribution and sample size. We also note that the powers of all test increase with the sample size, as expected.

Simulation results for the second scenario are presented in Tables 4.8 and 4.9 (entries are percentages). Recall that the maximal leverage increases with a. We test $\mathcal{H}_0 : \beta_2 = 3$ against a two-sided alternative hypothesis, and data generation is carried out using $\beta_2 = 4.1$. The numerical results reported in the two tables show that the quasi-t tests are slightly less powerful than the EGLS_1 and EGLS_2 tests when $a = 1.0$, i.e., when the data do not contain a leverage point. For instance, when $n = 150$ and at the 10% significance level the powers of the EGLS_1 and EGLS_2 tests are approximately 39% whereas the powers of the quasi-t tests are around 37% (Table 4.9). As the data become more leveraged (i.e., as the value of a increases), however, some quasi-t tests become progressively more powerful than the EGLS tests. Take, for instance, $n = 100$, $a = 5.0$ and $\alpha = 0.10$ (Table 4.9). The EGLS_1, EGLS_2 and EGLS_3 tests nonnull rejection rates are 45.08%, 45.14% and 80.77%, respectively; the corresponding figure for the HC0 (HC3) [HC4m] test is 99.30% (97.50%) [96.50%]. It is also noteworthy that the HC4 test is typically the least powerful quasi-t test when the sample size is small ($n = 50$). Additionally, in small samples and under strongly leveraged data, the EGLS_3 test is more powerful than the EGLS_1 and EGLS_2 tests. When $n \geq 100$, all quasi-t tests are nearly equally powerful.

Table 4.3 Null rejection rates (%), second simulation experiment; $\lambda \approx 6$.

	n	50		100		150		200	
a		5%	10%	5%	10%	5%	10%	5%	10%
1.0	$EGLS_1$	8.42	14.42	7.40	13.46	7.68	13.60	7.28	13.22
	$EGLS_2$	8.42	14.42	7.40	13.46	7.68	13.60	7.28	13.22
	$EGLS_3$	9.93	16.26	8.81	15.11	9.04	15.53	8.71	15.07
	HC0	6.64	12.12	5.76	11.37	5.98	11.26	5.77	10.73
	HC2	5.98	11.07	5.46	10.81	5.73	10.91	5.62	10.53
	HC3	5.45	10.17	5.11	10.19	5.52	10.60	5.42	10.27
	HC4	5.70	10.58	5.30	10.47	5.59	10.74	5.52	10.41
	HC4m	5.22	9.80	4.98	10.00	5.38	10.49	5.31	10.20
2.5	$EGLS_1$	11.11	16.93	6.51	11.63	5.73	11.08	5.10	10.28
	$EGLS_2$	11.10	16.92	6.51	11.63	5.73	11.08	5.10	10.28
	$EGLS_3$	13.75	20.03	9.50	15.27	8.19	14.01	7.31	12.94
	HC0	8.87	15.26	6.96	12.58	6.82	12.01	6.34	11.85
	HC2	6.87	11.64	5.79	11.01	6.00	10.95	5.72	11.10
	HC3	4.95	8.66	4.75	9.38	5.08	9.79	5.17	10.22
	HC4	2.64	4.89	3.36	6.73	3.94	8.30	4.41	8.70
	HC4m	4.16	7.62	4.39	8.68	4.73	9.32	4.90	9.90
3.5	$EGLS_1$	19.21	25.15	8.36	13.44	5.95	10.48	4.65	9.54
	$EGLS_2$	19.26	25.20	8.36	13.43	5.95	10.46	4.65	9.55
	$EGLS_3$	17.13	24.14	11.33	16.72	8.60	14.00	7.64	13.23
	HC0	13.84	20.84	9.68	16.22	8.45	14.53	8.01	13.79
	HC2	8.74	13.89	7.45	12.95	7.04	12.29	6.97	12.00
	HC3	4.95	8.09	5.62	9.90	5.65	10.25	5.91	10.53
	HC4	1.79	2.88	3.07	5.43	3.48	6.80	4.01	7.89
	HC4m	3.84	6.29	4.91	8.37	5.07	9.27	5.49	9.69
5.0	$EGLS_1$	34.82	41.11	13.47	18.87	7.94	12.30	6.00	10.09
	$EGLS_2$	35.28	41.52	13.49	18.89	7.94	12.30	5.99	10.10
	$EGLS_3$	22.31	28.98	13.04	17.89	9.88	14.37	8.91	13.72
	HC0	23.49	31.02	15.16	21.86	11.99	18.12	10.78	16.62
	HC2	11.87	16.91	11.00	16.81	9.19	14.73	8.84	14.09
	HC3	5.06	7.35	7.89	11.99	6.96	11.56	6.91	11.62
	HC4	0.81	1.27	3.40	5.59	3.60	6.43	4.12	7.35
	HC4m	3.24	4.55	6.32	10.07	6.00	10.08	6.17	10.57

Table 4.4 Null rejection rates (%), second simulation experiment; $\lambda \approx 26$.

	n	50		100		150		200	
a		5%	10%	5%	10%	5%	10%	5%	10%
1.0	$EGLS_1$	10.72	17.81	10.30	17.05	9.90	16.91	9.56	16.69
	$EGLS_2$	10.72	17.81	10.30	17.05	9.90	16.91	9.56	16.69
	$EGLS_3$	15.94	24.01	16.43	24.37	16.91	25.02	16.36	24.51
	HC0	6.60	11.89	5.73	11.53	5.88	11.61	5.45	11.23
	HC2	5.73	10.76	5.40	11.15	5.63	11.20	5.29	11.06
	HC3	5.06	9.81	4.98	10.56	5.44	10.85	5.17	10.81
	HC4	5.35	10.34	5.24	10.84	5.53	10.99	5.25	10.92
	HC4m	4.83	9.43	4.83	10.42	5.35	10.71	5.09	10.69
2.5	$EGLS_1$	9.97	16.58	8.78	15.15	8.63	15.38	8.48	15.03
	$EGLS_2$	9.97	16.58	8.78	15.15	8.63	15.38	8.48	15.03
	$EGLS_3$	16.20	24.18	14.39	22.00	13.66	21.14	13.21	20.87
	HC0	4.68	9.74	4.19	9.34	4.67	9.95	4.83	10.22
	HC2	3.10	6.51	3.20	7.97	3.98	8.91	4.25	9.50
	HC3	1.97	4.30	2.53	6.32	3.39	7.94	3.84	8.90
	HC4	1.00	2.14	1.51	4.19	2.39	6.41	3.27	7.58
	HC4m	1.69	3.61	2.17	5.74	3.12	7.54	3.76	8.64
3.5	$EGLS_1$	11.78	17.88	8.74	14.74	8.41	14.92	8.17	14.32
	$EGLS_2$	11.78	17.89	8.74	14.74	8.40	14.92	8.17	14.32
	$EGLS_3$	17.55	25.52	13.70	20.55	11.44	18.07	10.63	17.35
	HC0	6.47	11.99	5.33	10.54	5.22	10.45	5.27	10.72
	HC2	3.27	6.61	3.60	8.09	4.07	8.75	4.41	9.45
	HC3	1.47	3.22	2.38	5.65	3.21	7.11	3.65	8.08
	HC4	0.44	1.00	1.23	2.68	1.64	4.33	2.32	5.90
	HC4m	1.04	2.38	2.06	4.65	2.82	6.42	3.32	7.56
5.0	$EGLS_1$	16.91	23.11	9.22	15.09	8.27	14.01	7.74	13.16
	$EGLS_2$	16.99	23.17	9.22	15.07	8.27	14.00	7.73	13.16
	$EGLS_3$	18.34	25.39	12.00	17.86	8.86	14.41	7.81	12.97
	HC0	12.36	19.10	8.82	15.20	7.83	13.42	7.34	12.92
	HC2	5.09	8.33	6.03	10.80	5.78	10.66	5.89	10.95
	HC3	1.96	3.16	3.89	7.14	4.11	8.06	4.64	8.93
	HC4	0.24	0.47	1.43	2.93	2.03	4.02	2.55	5.46
	HC4m	1.26	2.05	3.18	5.80	3.30	7.05	3.97	8.08

Table 4.5 Null rejection rates (%), third simulation experiment; homoskedasticity and heteroskedasticity.

	n	50		100		150		200	
λ		5%	10%	5%	10%	5%	10%	5%	10%
1.00	$EGLS_1$	29.36	36.53	12.58	19.21	10.00	15.95	8.40	14.42
	$EGLS_2$	29.33	36.55	12.58	19.21	9.99	15.95	8.40	14.42
	$EGLS_3$	15.11	21.49	10.24	15.91	9.18	14.48	8.33	13.25
	HC0	13.90	20.62	9.16	15.12	8.51	14.13	7.35	12.80
	HC2	9.51	15.09	7.11	12.43	7.17	12.33	6.33	11.53
	HC3	5.79	9.46	5.34	9.83	6.01	10.79	5.56	10.14
	HC4	1.98	3.53	2.94	5.83	4.33	7.91	4.06	8.02
	HC4m	4.48	7.49	4.54	8.72	5.49	9.94	5.19	9.58
2.44	$EGLS_1$	30.90	37.78	13.32	20.33	10.64	16.28	8.76	14.89
	$EGLS_2$	30.82	37.77	13.32	20.33	10.64	16.28	8.76	14.89
	$EGLS_3$	16.63	23.44	11.25	16.85	10.13	15.24	8.66	14.16
	HC0	17.55	24.63	11.00	17.63	9.80	15.59	8.37	14.35
	HC2	11.83	18.26	8.63	14.26	8.25	13.55	7.07	12.43
	HC3	7.35	11.45	6.40	11.15	6.82	11.51	6.10	11.01
	HC4	2.38	4.15	3.48	6.55	4.70	8.11	4.19	8.17
	HC4m	5.69	9.07	5.59	9.91	6.29	10.55	5.73	10.15

The tests nonnonull rejection rates for the third simulation scenario are presented in Table 4.10 (entries are percentages). We test $\mathcal{H}_0 : \beta_3 = 0$ against a two-sided alternative hypothesis and the true parameter value is taken to be 1100. (Recall that the values of the other two regression parameters are $\beta_1 = -150.868$ and $\beta_2 = 688.806$.) Again, HC4 is the least powerful test when the sample size is small. When $n \geq 100$, all tests are nearly equally powerful.

4.4 Variability

In the previous sections we focused on testing inferences. It should be noted, however, that the variances of the ordinary and estimated generalized least squares estimators of the parameters that index the linear regression models can be considerably distinct. The estimators of such variances can also behave quite differently in finite samples. In order to show that, we consider the second simulation experiment used in the size and power simulations and present in Table 4.11 the true variances of the two estimators of β_2 for $a = 1.0, 5.0$; recall that when $a = 5.0$ the data are substantially leveraged. The sample sizes are $n = 50, 200$. It is noteworthy that the $EGLS_2$ estimator of the regression slope parameter is more accurate than the OLS estimator. The difference between the two variances is considerably large when $\lambda \approx 25$ and $a = 1.0$ (no leverage point in the data): for $n = 50$, $\mathrm{var}(\hat{\beta}_2) = 1.4085$ and $\mathrm{var}(\tilde{\beta}_2) = 0.5842$, a difference of almost 140%. Nonetheless, such a difference nearly vanishes as the data become substantially leveraged: when $\lambda \approx 25$, $n = 50$ and $a = 5.0$, $\mathrm{var}(\hat{\beta}_2) = 0.0884$ and $\mathrm{var}(\tilde{\beta}_2) = 0.0665$. Notice that under strong leverage the true variances are considerably smaller than under well balanced data, i.e., the OLS and EGLS estimators of β_2 fluctuate considerably less. Additionally, the two estimators become approximately equally accurate.

Table 4.6 Nonnull rejection rates (%), first simulation experiment; $\lambda \approx 51$.

	n	50		100		150		200	
Errors		5%	10%	5%	10%	5%	10%	5%	10%
$\mathcal{N}(0,1)$	$EGLS_1$	45.89	59.65	86.77	92.33	97.40	98.79	99.58	99.80
	$EGLS_2$	45.05	58.97	84.62	90.03	96.23	97.63	98.99	99.21
	$EGLS_3$	34.53	50.65	80.22	88.67	95.77	97.85	99.17	99.68
	HC0	32.47	45.41	64.86	73.61	81.54	88.95	93.61	96.15
	HC2	31.97	45.08	64.62	73.45	81.48	88.86	93.51	96.14
	HC3	31.34	44.62	64.41	73.31	81.30	88.84	93.47	96.13
	HC4	30.92	43.90	64.08	73.08	81.19	88.82	93.44	96.09
	HC4m	30.83	44.35	64.29	73.17	81.16	88.87	93.44	96.13
$\chi^2_{(10)}$	$EGLS_1$	38.56	53.94	86.24	92.80	97.71	98.95	99.76	99.91
	$EGLS_2$	37.55	52.54	82.25	88.60	94.83	96.01	97.89	98.02
	$EGLS_3$	27.95	43.90	79.29	88.74	95.68	98.28	99.48	99.81
	HC0	26.95	39.54	62.21	74.72	85.03	91.85	95.16	97.85
	HC2	26.41	38.69	61.82	74.50	84.97	91.83	95.14	97.84
	HC3	25.88	37.98	61.41	74.46	84.90	91.78	95.12	97.83
	HC4	25.03	37.17	60.57	74.39	84.73	91.72	95.03	97.80
	HC4m	25.22	37.51	60.85	74.43	84.85	91.76	95.03	97.81
$\chi^2_{(5)}$	$EGLS_1$	33.37	52.28	85.73	92.54	98.22	99.38	99.81	99.95
	$EGLS_2$	31.61	49.54	80.11	86.36	93.46	94.53	96.21	96.35
	$EGLS_3$	26.70	45.22	79.70	89.19	96.66	98.65	99.57	99.89
	HC0	23.83	38.71	59.84	75.08	84.45	93.21	96.12	98.57
	HC2	23.34	38.22	58.93	74.81	84.40	93.16	96.09	98.58
	HC3	22.87	37.39	58.77	74.66	84.14	93.12	96.03	98.55
	HC4	22.24	36.11	58.62	74.56	84.07	93.09	96.07	98.58
	HC4m	22.49	36.55	58.52	74.64	84.01	93.12	96.02	98.55
$\chi^2_{(2)}$	$EGLS_1$	20.00	42.93	79.62	92.18	97.98	99.48	99.80	99.95
	$EGLS_2$	19.11	41.47	72.74	82.86	89.72	90.94	92.86	92.97
	$EGLS_3$	19.06	38.51	74.19	88.29	96.32	98.96	99.65	99.88
	HC0	21.81	36.60	58.44	78.84	88.24	95.61	96.61	99.17
	HC2	20.86	35.18	57.78	78.35	88.01	95.47	96.61	99.16
	HC3	19.90	34.40	56.87	78.13	87.87	95.40	96.54	99.16
	HC4	18.78	33.23	56.18	77.55	87.71	95.36	96.50	99.14
	HC4m	19.25	33.87	56.37	77.84	87.76	95.38	96.50	99.16

Table 4.7 Nonnull rejection rates (%), first simulation experiment; $\lambda \approx 101$.

	n	50		100		150		200	
Errors		5%	10%	5%	10%	5%	10%	5%	10%
$\mathcal{N}(0,1)$	$EGLS_1$	26.81	38.69	62.38	73.43	83.08	89.51	92.79	96.12
	$EGLS_2$	25.95	37.24	58.39	68.97	80.32	86.45	90.89	94.25
	$EGLS_3$	18.97	30.58	54.33	66.85	77.72	86.25	90.77	94.90
	HC0	16.22	25.51	34.86	45.01	48.24	60.18	64.03	73.24
	HC2	16.05	25.13	34.83	45.00	48.09	60.19	64.00	73.24
	HC3	15.82	24.79	34.36	44.76	48.06	60.23	63.93	73.27
	HC4	15.65	24.71	34.24	44.53	48.03	60.07	63.89	73.23
	HC4m	15.76	24.55	34.17	44.67	48.00	60.15	63.88	73.27
$\chi^2_{(10)}$	$EGLS_1$	19.11	31.23	56.49	71.10	81.15	88.95	93.28	96.19
	$EGLS_2$	18.63	29.91	53.17	65.95	76.44	83.74	89.53	92.17
	$EGLS_3$	12.93	23.80	47.36	62.67	76.02	85.82	90.32	94.51
	HC0	11.60	19.13	26.04	38.98	45.25	59.10	62.22	74.32
	HC2	11.40	18.66	25.77	38.89	45.22	58.91	62.08	74.34
	HC3	10.95	18.29	25.56	38.86	45.07	58.90	61.80	74.32
	HC4	10.63	18.02	25.10	38.78	44.81	58.80	61.64	74.22
	HC4m	10.81	18.13	25.49	38.95	44.91	58.77	61.88	74.28
$\chi^2_{(5)}$	$EGLS_1$	14.08	28.12	55.53	68.84	82.36	89.93	93.69	97.00
	$EGLS_2$	13.07	26.36	49.20	60.96	75.21	82.39	87.29	90.45
	$EGLS_3$	10.57	21.91	47.13	62.91	75.60	85.43	90.31	95.66
	HC0	9.16	16.18	22.20	36.94	41.37	59.34	62.19	75.68
	HC2	8.95	15.74	21.71	36.44	41.06	59.19	62.09	75.71
	HC3	8.85	15.32	21.47	36.20	40.37	58.90	61.96	75.68
	HC4	8.23	14.93	21.14	35.86	39.91	58.78	62.03	75.64
	HC4m	8.39	15.03	21.47	35.82	40.05	58.87	62.07	75.66
$\chi^2_{(2)}$	$EGLS_1$	6.34	19.10	41.88	65.11	77.61	88.60	92.66	96.53
	$EGLS_2$	6.04	18.06	37.82	56.39	68.02	77.38	83.09	86.02
	$EGLS_3$	5.73	15.74	32.92	55.54	68.99	83.42	88.37	94.98
	HC0	5.80	12.34	16.82	34.64	39.82	59.45	56.42	77.04
	HC2	5.43	11.85	16.45	34.32	39.34	59.17	56.19	76.84
	HC3	5.18	11.43	16.42	33.94	38.94	58.83	55.78	76.62
	HC4	4.98	11.09	15.63	33.84	38.69	58.50	55.40	76.50
	HC4m	4.93	11.04	15.92	33.90	38.67	58.50	55.47	76.52

Table 4.8 Nonnull rejection rates (%), second simulation experiment; $\lambda \approx 6$.

	n	50		100		150		200	
a		5%	10%	5%	10%	5%	10%	5%	10%
	$EGLS_1$	22.62	34.23	47.75	59.59	67.13	76.52	78.93	87.26
	$EGLS_2$	22.62	34.23	47.75	59.58	67.13	76.52	78.93	87.26
	$EGLS_3$	20.86	31.57	44.44	56.92	64.96	74.41	78.74	86.23
	HC0	22.31	33.21	46.94	59.10	66.60	75.62	79.95	86.25
1.0	HC2	22.38	33.15	46.94	59.18	66.61	75.62	79.95	86.26
	HC3	22.41	33.15	46.93	59.18	66.60	75.62	79.94	86.25
	HC4	22.39	33.14	46.84	59.21	66.60	75.64	79.94	86.24
	HC4m	22.40	33.16	46.90	59.23	66.60	75.62	79.93	86.25
	$EGLS_1$	25.38	42.28	68.49	78.44	87.46	91.86	95.30	97.39
	$EGLS_2$	25.38	42.28	68.49	78.44	87.46	91.86	95.30	97.39
	$EGLS_3$	27.73	42.23	65.97	77.50	87.15	92.46	95.79	97.79
	HC0	51.01	62.13	84.03	90.23	95.64	97.66	98.85	99.53
2.5	HC2	47.79	59.07	82.81	89.47	95.31	97.51	98.84	99.50
	HC3	43.06	54.45	80.76	88.66	94.95	97.32	98.74	99.47
	HC4	32.17	42.53	76.20	85.28	93.55	96.73	98.57	99.34
	HC4m	41.00	51.61	80.01	88.02	94.64	97.22	98.70	99.43
	$EGLS_1$	21.07	30.78	74.12	83.79	92.48	95.67	97.91	98.81
	$EGLS_2$	21.02	30.75	74.16	83.81	92.49	95.68	97.91	98.81
	$EGLS_3$	26.90	47.07	74.06	85.72	94.24	97.23	98.60	99.38
	HC0	73.61	83.11	96.05	98.27	99.57	99.83	99.99	100.00
3.5	HC2	62.24	73.76	94.07	97.17	99.40	99.80	99.97	100.00
	HC3	47.73	59.95	91.19	95.81	98.93	99.74	99.96	100.00
	HC4	24.94	34.80	81.64	90.48	98.06	99.40	99.92	99.98
	HC4m	41.28	51.96	88.93	94.80	98.77	99.66	99.96	100.00
	$EGLS_1$	41.01	45.25	66.76	83.94	94.59	97.09	98.71	99.27
	$EGLS_2$	41.96	46.44	66.82	83.98	94.67	97.11	98.72	99.27
	$EGLS_3$	43.89	61.13	76.91	91.51	97.12	99.18	99.62	99.90
	HC0	93.47	96.71	99.30	99.84	99.99	100.00	100.00	100.00
5.0	HC2	74.15	82.47	97.94	99.44	99.97	100.00	100.00	100.00
	HC3	47.59	59.37	95.83	98.49	99.94	99.99	100.00	100.00
	HC4	20.31	33.75	86.83	94.42	99.76	99.97	100.00	100.00
	HC4m	36.97	49.60	93.75	97.64	99.91	99.99	100.00	100.00

Table 4.9 Nonnull rejection rates (%), second simulation experiment; $\lambda \approx 26$.

	n	50		100		150		200	
a		5%	10%	5%	10%	5%	10%	5%	10%
	$EGLS_1$	10.52	17.15	19.61	29.15	29.52	39.01	37.91	48.87
	$EGLS_2$	10.52	17.15	19.61	29.15	29.52	39.01	37.90	48.87
	$EGLS_3$	9.64	16.10	18.93	27.61	26.91	38.02	36.96	48.73
	HC0	10.25	16.82	19.04	27.89	27.11	36.59	35.40	46.71
1.0	HC2	10.23	16.92	19.08	27.90	27.08	36.57	35.40	46.68
	HC3	10.19	16.85	19.12	27.90	27.07	36.55	35.40	46.67
	HC4	10.27	16.78	19.14	27.86	27.05	36.54	35.42	46.66
	HC4m	10.30	16.81	19.12	27.86	27.04	36.53	35.39	46.64
	$EGLS_1$	12.79	21.50	25.01	34.58	33.35	45.09	44.58	55.22
	$EGLS_2$	12.79	21.50	25.01	34.58	33.34	45.09	44.58	55.22
	$EGLS_3$	15.39	24.77	33.59	44.04	47.51	59.18	62.71	71.72
	HC0	32.63	42.59	57.70	66.40	73.36	79.90	83.74	88.90
2.5	HC2	31.12	42.02	57.66	66.83	73.86	80.03	83.83	88.81
	HC3	29.46	40.23	57.84	66.67	73.77	80.01	83.81	88.86
	HC4	22.81	32.14	57.64	65.91	73.99	80.23	83.61	88.93
	HC4m	28.18	38.63	58.19	66.72	73.63	80.10	83.88	88.84
	$EGLS_1$	14.07	23.14	29.06	38.31	37.56	49.12	48.40	58.29
	$EGLS_2$	13.99	23.12	29.08	38.31	37.56	49.13	48.42	58.30
	$EGLS_3$	21.76	34.18	50.32	61.75	68.55	77.91	83.48	88.63
	HC0	54.57	66.24	84.99	89.99	95.13	97.20	98.32	99.27
3.5	HC2	48.96	60.04	83.92	89.12	94.74	96.90	98.25	99.16
	HC3	39.46	49.46	81.66	87.70	94.35	96.50	98.21	99.09
	HC4	21.28	30.32	74.19	82.51	92.70	95.64	97.91	98.90
	HC4m	34.25	44.32	80.30	86.61	94.10	96.36	98.17	99.06
	$EGLS_1$	16.36	22.13	35.22	45.08	45.25	55.74	55.72	65.21
	$EGLS_2$	16.78	22.23	35.20	45.14	45.33	55.79	55.77	65.26
	$EGLS_3$	32.27	48.61	68.55	80.77	90.30	94.25	96.91	98.25
	HC0	83.10	89.31	98.19	99.30	99.88	99.99	100.00	100.00
5.0	HC2	63.53	73.10	96.81	98.56	99.86	99.94	100.00	100.00
	HC3	39.93	51.41	94.49	97.50	99.73	99.91	100.00	100.00
	HC4	17.27	28.42	84.89	93.28	99.24	99.82	99.99	100.00
	HC4m	31.60	42.36	92.81	96.50	99.67	99.90	99.99	100.00

Table 4.10 Nonnull rejection rates (%), third simulation experiment; homoskedasticity and heteroskedasticity.

	n	50		100		150		200	
λ		5%	10%	5%	10%	5%	10%	5%	10%
1.00	EGLS_1	11.77	23.75	55.29	68.80	78.17	86.82	91.59	95.83
	EGLS_2	11.83	23.75	55.29	68.76	78.20	86.83	91.59	95.83
	EGLS_3	23.07	33.98	49.81	64.15	70.90	81.51	87.37	93.40
	HC0	34.39	46.31	65.83	76.44	81.85	89.21	93.41	96.61
	HC2	30.53	43.25	64.53	75.02	80.69	88.62	93.11	96.39
	HC3	26.86	38.03	62.19	72.90	79.24	87.76	92.64	96.19
	HC4	18.09	26.04	54.73	66.98	75.09	85.48	91.64	95.39
	HC4m	24.72	34.40	60.09	71.76	78.43	87.20	92.40	96.07
2.44	EGLS_1	23.84	40.59	84.30	92.65	97.74	99.16	99.81	99.93
	EGLS_2	23.99	40.83	84.31	92.65	97.74	99.16	99.81	99.93
	EGLS_3	41.48	56.27	79.63	89.52	95.38	98.30	99.45	99.88
	HC0	60.56	72.30	90.71	94.95	97.95	99.16	99.80	99.93
	HC2	54.47	66.00	88.25	93.38	97.37	98.96	99.74	99.90
	HC3	45.38	57.37	85.23	91.39	96.43	98.68	99.71	99.89
	HC4	28.07	37.05	76.10	85.98	93.65	97.68	99.53	99.83
	HC4m	40.44	52.17	83.38	90.38	95.90	98.44	99.66	99.87

Table 4.12 contains the mean values of the different variance estimators. First, note that the HC0 estimator is very optimistic when the data are leveraged, i.e., it substantially underestimates the true variance. Second, HC2 is the least biased heteroskedasticity-consistent estimator. Third, HC4 and HC4m can be quite positively biased, especially HC4; that is, they tend to overestimate the true variance, more so when the data are leveraged and the sample size is small ($n = 50$).

A valid question that is commonly posed is: To what extent is it worthwhile to base inferences on the OLSE coupled with heteroskedasticity-robust standard errors under unequal error variances instead of incorporating a dispersion submodel to the regression model and then carrying out estimation and performing inferences using EGLS? The results in Table 4.11 show that the EGLS estimator of the regression parameters can be considerably more accurate than the OLS estimator, especially when the data are not leveraged. The results in Table 4.12 show that the standard, baseline heteroskedasticity-robust standard error (HC0) tends to be largely negatively biased when the data are leveraged, thus resulting in liberal quasi-t testing inferences. Better control of the type I error frequency is achieved by using a positively biased heteroskedasticity-robust standard error, such as HC3 and HC4m.

We then arrive at the following relevant question: How should estimated variances be evaluated? Simonoff (1993) argues that since they are primarily used for performing inferences, a comparison of different variance estimators should involve their intended use. In this dissertation, we focus on the accuracy of testing inferences based on statistics that employ different standard errors.

Table 4.11 True variances of $\hat{\beta}_2$ and $\tilde{\beta}_2$.

λ	a	n	var($\hat{\beta}_2$)	var($\tilde{\beta}_2$)
5.16	1.0	50	0.4930	0.3938
		200	0.1233	0.0985
	5.0	50	0.0566	0.0534
		200	0.0141	0.0133
25.18	1.0	50	1.4085	0.5842
		200	0.3521	0.1461
	5.0	50	0.0884	0.0665
		200	0.0221	0.0166

Table 4.12 Mean estimated variances of $\hat{\beta}_2$ and $\tilde{\beta}_2$.

λ	a	n	HC0	HC2	HC3	HC4	HC4m	$EGLS_2$
5.16	1.0	50	0.4638	0.4917	0.5215	0.5071	0.5329	0.4003
		200	0.1213	0.1231	0.1248	0.1240	0.1255	0.1052
	5.0	50	0.0265	0.0860	0.4411	15.3550	1.0580	0.1294
		200	0.0121	0.0144	0.0172	0.0252	0.0188	0.0299
25.18	1.0	50	1.3171	1.3977	1.4835	1.4414	1.5178	0.8616
		200	0.3461	0.3511	0.3563	0.3537	0.3582	0.2311
	5.0	50	0.0705	0.2396	1.2501	43.7030	3.0062	0.6396
		200	0.0209	0.0239	0.0278	0.0385	0.0299	0.1889

5 Empirical applications

In what follows we shall present and discuss two empirical applications. Testing inferences shall be performed using z and quasi-t tests. Unless stated otherwise, the variables used in the skedastic function for EGLS estimation are the same as those used to model the mean response.

5.1 Man-hours for manning installations data

The dependent variable (y) is the number of monthly man-hours for manning installations in the U.S. Navy in Bachelor Office Quarters, and the independent variables are average daily occupancy (x_2) and number of building wings (x_3). The source of the data is Myers (1990, Table 5.2, p. 218). The data contain 25 observations on the response and on the covariates. Observations 22 and 23 appear to be leverage data points: $h_{22} = 0.720$ and $h_{23} = 0.877$, whereas $3p/n = 0.360$. The regression model we use is

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i, \tag{5.1}$$

$i = 1, \ldots, 25$.

Table 5.1 contains some descriptive statistics on the response variable (y) and on the independents variables. We note that the distribution of y seems to be positively asymmetrically distributed. The response minimal and maximal values are 164.38 and 8266.77, respectively, and its average value equals 2109.39.

Table 5.1 Descriptive Statistics, complete data; man-hours for manning installations data.

Statistics	y	x_2	x_3
Minimum	164.38	2.00	1.00
Maximum	8266.77	811.08	58.00
Mean	2109.39	117.16	11.12
Median	1845.89	95.00	9.00
Variance	3787883.00	28203.08	145.03
Standard deviation	1946.25	167.94	12.04
Skewness	1.46	3.06	2.49
Kurtosis	2.21	9.92	7.26

We tested for the presence of heteroskedasticity using the test proposed by Koenker (1981). The null hypothesis of equal error variances was rejected at the 5% significance level. Hence, there is evidence that the data are heteroskedastic.

Next, the parameters of Model (5.1) were estimated by ordinary and estimated generalized least squares. The OLS point estimates are $\hat{\beta}_1 = 618.551$, $\hat{\beta}_2 = 4.045$ and $\hat{\beta}_3 = 91.455$; the EGLS$_1$ point estimates are $\tilde{\tilde{\beta}}_1 = 125.598$, $\tilde{\tilde{\beta}}_2 = 22.570$ and $\tilde{\tilde{\beta}}_3 = -6.954$; the EGLS$_2$ point estimates are $\tilde{\beta}_1 = 125.580$, $\tilde{\beta}_2 = 22.570$ and $\tilde{\beta}_3 = -6.951$; the EGLS$_3$ point estimates are $\hat{\hat{\beta}}_1 = 132.358$, $\hat{\hat{\beta}}_2 = 18.778$ and $\hat{\hat{\beta}}_3 = 22.207$. It is noteworthy that the OLS parameter estimates are considerably different than those obtained with the three EGLS strategies. We also note that the EGLS$_1$ and EGLS$_2$ estimates of β_3 are quite different from the corresponding EGLS$_3$ estimate; the first two are negative wehereas the latter is postive. Table 5.2 presents the standard errors of the three parameter estimates (se$_i$, $i = 1, \ldots, 3$) obtained using the different approaches. The largest standard errors are those obtained using the HC4 covariance matrix estimator followed by HC4m. Overall, the smallest standard errors are EGLS$_1$ and EGLS$_2$.

Table 5.2 Standard errors, complete data; man-hours for manning installations data.

	se$_1$	se$_2$	se$_3$
EGLS$_1$	81.47	1.87	12.01
EGLS$_2$	81.47	1.87	12.01
EGLS$_3$	119.86	2.73	27.12
HC0	236.44	2.76	28.92
HC2	372.38	7.11	59.12
HC3	741.83	19.90	136.66
HC4	4448.37	161.34	926.48
HC4m	1115.32	33.53	215.30
OLS	345.48	1.71	23.85

Consider the test of $\mathcal{H}_0 : \beta_3 = 0$ against $\mathcal{H}_1 : \beta_3 \neq 0$. The interest lies in determining whether x_3 should be removed from the regression model given that x_2 is already in the model. Table 5.3 contains the tests p-values for the complete data and for the data without the two leverage points. When all observations are used, the only tests that yield rejection of the null hypothesis at the usual significance levels are the test whose statistic uses OLS standard error and the HC0 test. It is interesting to note that when the two leverage data points are removed from the data all tests yield p-values in excess of 0.5, thus indicating that the evidence against the null hypothesis is quite small. Table 5.4 contains the parameter estimates we obtained after the withdrawal of the identified leverage points from the data. Note that now all estimates of β_3 are positive. Table 5.5 contains the standard errors obtained after the withdrawal of the identified leverage points. Note that there is now less discrepancy in these quantities among the estimators used. The HC4 and HC4m standard errors are still the largest ones and the EGLS standard errors are typically the smallest ones.

We note that rejection of the null hypothesis is driven by only two atypical data points. When such observations are not in the data, the EGLS$_1$, EGLS$_2$, EGLS$_3$, HC2, HC3, HC4 and HC4m tests lead to the same conclusion, namely: the covariate x_3 can be safely removed from the regression model. Indeed, the coefficient of determination for the regression with the incomplete data equals 0.959477; when x_3 is removed from the model R^2 remains nearly unchanged: 0.959463. This is evidence that, as long as the two leverage points are not in the

Table 5.3 Tests p-values; man-hours for manning installations data.

complete data, n=25		incomplete data, n=23	
Test	p-value	Test	p-value
$EGLS_1$	0.56266	$EGLS_1$	0.81127
$EGLS_2$	0.56288	$EGLS_2$	0.81218
$EGLS_3$	0.41285	$EGLS_3$	0.53989
HC0	0.00157	HC0	0.92456
HC2	0.12185	HC2	0.93266
HC3	0.50337	HC3	0.94074
HC4	0.92137	HC4	0.94626
HC4m	0.67100	HC4m	0.94358
OLS	0.00013	OLS	0.93363

Table 5.4 Parameter estimates, incomplete data; man-hours for manning installations data.

	β_1	β_2	β_3
OLS	56.538	22.802	1.666
$EGLS_1$	85.837	22.348	1.851
$EGLS_2$	85.860	22.354	1.824
$EGLS_3$	41.784	22.770	4.601

Table 5.5 Standard errors, incomplete data; man-hours for manning installations data.

	se_1	se_2	se_3
$EGLS_1$	63.16	1.65	7.75
$EGLS_2$	62.86	1.65	7.67
$EGLS_3$	74.47	1.16	7.51
HC0	88.84	1.39	17.59
HC2	97.08	1.53	19.71
HC3	107.61	1.74	22.40
HC4	119.23	2.55	24.71
HC4m	110.91	1.85	23.53
OLS	131.74	1.80	20.00

dataset, the second covariate does not significantly add to the fit. The HC2 through HC4m quasi-t tests and the EGLS tests deliver the correct inference in both scenarios. Such inference is not driven by only two (atypical) data points. The HC0 and OLS testing inferences, in contrast, are driven by only two observations: when the two atypical points are in the data the HC0 and OLS tests lead the practitioner to incorrectly conclude that x_3 must be in the model.

5.2 Per capita spending on public education

The data we shall now use were previously analyzed by Cribari-Neto (2004). The source of the data is the U.S. Department of Commerce. The response (y) is per capita spending on public schools and the covariates, x and x^2, are per capita income by state in 1979 in the United States and its square; income is scaled by 10^{-4}. The observation relative to Wisconsin is missing, and that for Washington, D.C. is included. Thus, $n = 50$. The regression model is

$$y_i = \beta_1 + \beta_2 x_i + \beta_3 x_i^2 + \epsilon_i,$$

$i = 1, \ldots, 50$.

Table 5.6 contains some descriptive statistics on the response variable (y) and on the independent variable (x). We note that there is positive asymmetry in the dependent variable (y). Additionally, the its minimal and maximal values are 259 and 821, respectively.

Table 5.6 Descriptive Statistics, complete data; per capita spending on public education data.

Statistics	y	x
Minimum	259.00	0.57
Maximum	821.00	1.08
Mean	373.26	0.76
Median	354.00	0.76
Variance	8940.32	0.01
Standard deviation	94.55	0.10
Skewness	2.16	0.66
Kurtosis	8.08	0.46

The Koenker (1981) test suggests that there is heteroskedasticity. The OLS parameter estimates are $\hat{\beta}_1 = 834.254$, $\hat{\beta}_2 = -1837.172$ and $\hat{\beta}_3 = 1588.499$; the EGLS$_1$ estimates are $\tilde{\tilde{\beta}}_1 = 464.508$, $\tilde{\tilde{\beta}}_2 = -845.283$ and $\tilde{\tilde{\beta}}_3 = 933.734$; the EGLS$_2$ estimates are $\tilde{\beta}_1 = 464.514$, $\tilde{\beta}_2 = -845.299$ and $\tilde{\beta}_3 = 933.744$; the EGLS$_3$ estimates are $\hat{\hat{\beta}}_1 = 486.345$, $\hat{\hat{\beta}}_2 = -906.319$ and $\hat{\hat{\beta}}_3 = 975.892$. Table 5.7 contains the standard errors of the three parameter estimates obtained using the different approaches. Again, the largest standard errors are those obtained using the HC4 covariance matrix estimator followed by HC4m. The smallest standard errors are obtained by OLS followed by the EGLS ones.

The interest lies in testing a linear versus a quadratic functional form, i.e., in testing $\mathcal{H}_0 : \beta_3 = 0$ against $\mathcal{H}_1 : \beta_3 \neq 0$. Table 5.8 contains the tests p-values. Notice that the OLS and HC0 tests reject the null hypothesis (in favor of a quadratic model specification) at the 10% significance level. The data are plotted in Figure 5.1. It is clear that Alaska is a leverage and atypical data point. Indeed, its leverage value (h_{Alaska}) equals 0.651 which considerably exceeds three times the mean leverage ($3p/n = 0.180$). Considering this reference value (0.180), two other observations are singled out as leverage points, namely: Washington, D.C. ($h_{\text{D.C.}} = 0.208$) and Mississippi ($h_{\text{Mississippi}} = 0.200$). We shall, however, follow Cribari-Neto (2004) and focus on the impact of the most atypical observation (Alaska) on the resulting inferences. We removed Alaska from the data, estimated the parameters and performed the OLS, EGLS and quasi-t tests again. We report the tests p-values in Table 5.8 ('incomplete data'). It is noteworthy that when Alaska is not in the data, except for EGLS$_1$ and EGLS$_2$, the tests p-values are large, and hence the null hypothesis is not rejected. The null hypothesis is marginally rejected at the 10% nominal level, however, by the EGLS$_1$ and EGLS$_2$ tests. Hence, rejection of the null hypothesis by the

OLS and HC0 tests with the complete data seems to be driven by a single atypical observation. In contrast, one does not reject the null hypothesis at the usual significance levels when using the HC2, HC3, HC4 and HC4m tests even when the data contain a strong leverage point, i.e., such testing inference is not substantially affected by an atypical observation. We note that the coefficient of determination (R^2) for the regression with the incomplete data equals 0.501079; when x^2 is removed from the model it remains nearly unchanged: 0.498841. This can be viewed as evidence that, as long as Alaska is not in the data, squared per capita income (x^2) does not significantly add to the fit. The EGLS_3 z test and the HC2 through HC4m quasi-t tests deliver the correct inference in both scenarios. For instance, the HC3 (HC4) [HC4m] p-values computed using the complete and incomplete datasets are 0.4262 and 0.7747 (0.7722 and 0.8914) [0.5340 and 0.8096], respectively. The corresponding EGLS_1 (EGLS_2) [EGLS_3] p-values are 0.2082 and 0.0968 (0.2082 and 0.0971) [0.2336 and 0.2269]; notice the reduction in p-values when Alaska is removed from the data which contrasts to the sizable increased in the quasi-t tests p-values. Table 5.9 contains the parameter estimates obtained after the withdrawal of Alaska from the data. Table 5.10 contains the standard errors obtained after the withdrawal of the outlying observation. Note that even though such figure become smaller, there is still considerable discrepancy among them. The HC4 and HC4m standard errors are still the largest ones and the EGLS standard errors are the smallest ones.

Table 5.7 Standard errors, complete data; per capita spending on public education data.

	se_1	se_2	se_3
EGLS_1	421.78	1123.43	741.90
EGLS_2	421.77	1123.41	741.89
EGLS_3	455.78	1226.69	819.24
HC0	461.57	1244.67	830.90
HC2	689.37	1868.52	1251.33
HC3	1096.00	2977.71	1996.45
HC4	3007.83	8181.65	5487.08
HC4m	1401.03	3808.81	2554.33
OLS	327.46	829.43	519.34

It is interesting to note that the EGLS tests yielded conflicting inferences when the incomplete data were used: the EGLS_1 and EGLS_2 z tests rejected the null hypothesis (in favor of the quadratic specification) at the 10% significance level whereas the null hypothesis was not rejected by the EGLS_3 z test. Since the atypical observation (Alaska) is not in the data, one would expect the tests to not reject the null hypothesis under evaluation. The rejection of $\mathcal{H}_0$ by two EGLS tests when the outlier-free data were used was not expected. In order to further investigate that, we redid the analysis using a different skedastic function, namely:

$$\sigma_i^2 = \exp(\gamma_1 + \gamma_2 x_i);$$

i.e., income squared, which is used as a mean regressor, is no longer used a dispersion regressor. The EGLS tests p-values computed using both complete and incomplete data using the new

Table 5.8 Tests p-values; per capita spending on public education data.

complete data, $n = 50$		incomplete data, $n = 49$	
Test	p-value	Test	p-value
$EGLS_1$	0.2082	$EGLS_1$	0.0968
$EGLS_2$	0.2082	$EGLS_2$	0.0971
$EGLS_3$	0.2336	$EGLS_3$	0.2269
HC0	0.0559	HC0	0.6157
HC2	0.2043	HC2	0.6954
HC3	0.4262	HC3	0.7747
HC4	0.7722	HC4	0.8914
HC4m	0.5340	HC4m	0.8096
OLS	0.0022	OLS	0.6496

Table 5.9 Parameter estimates, incomplete data; per capita spending on public education data.

	β_1	β_2	β_3
OLS	-208.724	1000.016	-314.051
$EGLS_1$	-426.827	1625.700	-752.356
$EGLS_2$	-426.675	1625.271	-752.061
$EGLS_3$	-362.640	1450.299	-635.319

Table 5.10 Standard errors, incomplete data; per capita spending on public education data.

	se_1	se_2	se_3
$EGLS_1$	268.82	704.97	453.13
$EGLS_2$	268.89	705.18	453.27
$EGLS_3$	305.18	808.19	525.71
HC0	345.36	935.68	625.66
HC2	437.03	1191.57	802.07
HC3	591.91	1621.86	1097.14
HC4	1228.88	3383.72	2299.70
HC4m	700.56	1923.43	1303.74
OLS	405.27	1064.06	691.30

skedastic function are presented in Table 5.11. At the outset, note that the EGLS tests now deliver the same inferences in both scenarios; there is no longer contradiction between them. It is also noteworthy that all three z tests now seem to deliver the correct inference when the incomplete data are used: the null hypothesis is not rejected (the three p-values are quite large), i.e., there is no evidence against the linear specification. However, when the complete data are used (Alaska is included in the data), the three EGLS tests yield rejection of the null hypothesis at the 10% significance level. Such inference (rejection of the linear specification in favor of the quadratic specification) seems to be driven by a single, atypical data point.

This empirical application shows that testing inferences reached after EGLS parameter estimation can be highly dependent on how the response variances are modeled. It also shows

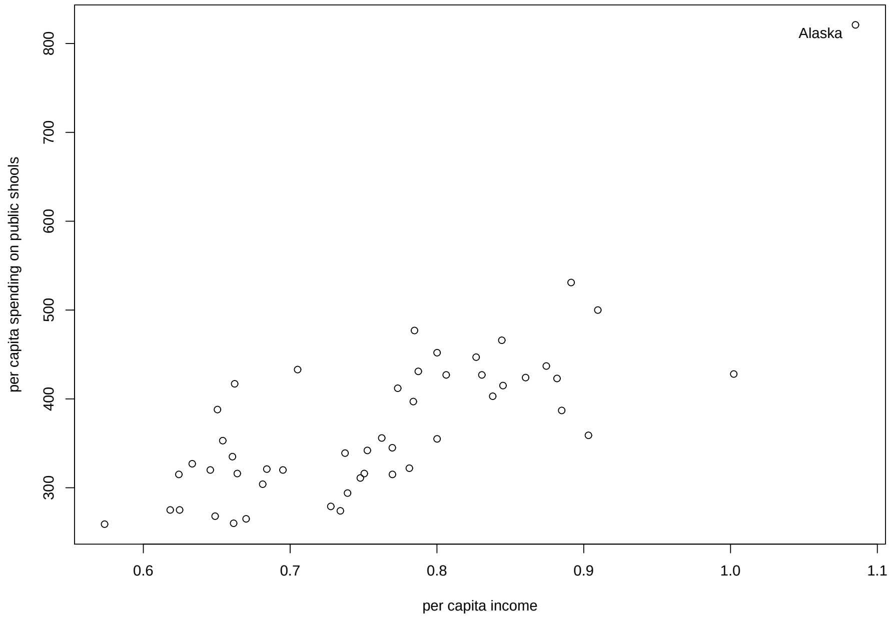

Figure 5.1 Per capita spending on public schools vs. per capita income (×10, 000).

Table 5.11 Tests p-values; per capita spending on public education data; alternative skedastic function.

complete data, $n = 50$		incomplete data, $n = 49$	
Test	p-value	Test	p-value
EGLS_1	0.0738	EGLS_1	0.8733
EGLS_2	0.0738	EGLS_2	0.8733
EGLS_3	0.0706	EGLS_3	0.9954

that EGLS testing inferences can be quite affected by a single, atypical observation. The same happened, we saw earlier, with the HC0 quasi-t test. The other quasi-t tests, however, proved reliable even under highly leveraged data.

6 Concluding remarks

The linear regression model is commonly used in empirical analyses in many fields. Oftentimes such analyses employ cross sectional data and there is heteroskedasticity, i.e., the error variances are not the same for all observations in the sample. Practitioners frequently want to perform testing inferences on the model parameters, e.g., to decide which covariates should be included in the model and which covariates should be removed from it. Broadly speaking, two different testing strategies can be employed. The first involves using quasi-t test statistics that are based on standard errors obtained from a heteroskedasticity-consistent covariance matrix estimator. These standard errors are asymptotically correct under both homoskedasticity and heteroskedasticity of unknown form. The second strategy requires expanding the model to model not only the mean effects but also the dispersion effects. In other words, it is necessary to specify a submodel for the response mean and a separate submodel for the response variances. The likelihood of model misspecification is thus increased. Which testing strategy should be preferred? We seeked to provide this question with an answer. Some authors in the literature focused on the accuracy of estimated variances and covariances of alternative parameter estimators, namely: ordinary least squares and estimated generalized least squares. According to Simonoff (1993), however, since estimated variances are mainly used for performing inferences, a comparison of different variance estimators should involve their intended use, such as the empirical coverages of associated confidence intervals and the sizes of associated tests. This is the approach we pursued. We compared the accuracy of testing inferences made using quasi-t (ordinary least squares estimation coupled with asymptotically correct standard errors) and z (estimated generalized least squares) tests.

Our numerical evaluations were performed using different models, different sample sizes, and different heteroskedasticity strengths. We evaluated the finite sample performances of several quasi-t tests (i.e., of tests based on different heteroskedasticity-robust standard errors) and also those of standard z tests based on estimated generalized least squares estimation. The main difference between the two approaches lies in the fact that the former only requires modeling of mean effects whereas the latter requires the practitioner to model both mean and dispersion effects. We also numerically evaluated the accuracy of point estimates obtained using the two estimation approaches. Our numerical results showed that point estimates obtained by estimated generalized least squares can be considerably more accurate than ordinary least squares point estimates when the data do not include leverage points, but the difference in accuracy becomes much smaller under leveraged data. As noted earlier, nonetheless, our chief interest lies in testing inferences. The numerical evidence we reported showed that overall the best performing quasi-t tests are those based on HC3 and HC4m heteroskedasticity-robust standard errors. It was also shown that such tests typically display better control of the type I error frequency than z tests carried out after joint mean and dispersion modeling, especially under leveraged data. For instance, in an extreme case of data containing a very atypical data point in one of our simulations, when the sample contained only 50 observations and the tests were carried out at the 5% significance level, the EGLS_2 test null rejection rate exceeded 35% (i.e., it was more than seven times larger than the test significance level); the HC3 and HC4m null rejection rates were 5.06% and 3.24%, respectively.

We have also presented and discussed two empirical applications. In one them (the empirical applications mentioned in the dissertation introduction), the interest lied in distinguishing between a linear and a quadratic model specification. In this application, the most reliable quasi-t tests delivered what seems to be the correct inference, regardless of whether the data contain a leverage point which is also an outlier. In contrast, EGLS testing inferences proved to be highly dependent on how the responses variances are modeled and were also quite sensitive to a single, atypical observation (Alaska). This highlights an advantage of quasi-t tests: they do not require the practitioner to model dispersion effects.

In future work, we shall address the following issues:

- We shall consider alternative covariance matrix estimators that are consistent under both homoskedasticity and heteroskedasticity, such as HC5 (Cribari-Neto, Souza, and Vasconcellos 2007) and HC5m (Li et al. 2016).
- We shall consider testing inferences on a vector of parameters rather than on a single model parameter.

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Appendix A - Size simulation

```
/*****************************************************************
DESCRIPTION: Monte Carlo simulation of quasi-t and z tests under
heteroskedasticity. The choice of sample sizes are: 50, 100, 150,
200. Multiple regression. Covariates generated from |N(0,1)| and
errors: normal and chi-square with 2, 5, 10 degrees of freedom.

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*****************************************************************/

#include<oxstd.h>
#include<oxprob.h>
#import <maximize>

const decl samplesize = 1; /* choice 1 (n=50), 2 (n=100), 3 (n=150)
or 4 (n=200) */

static decl svx_1, svx_2, svx_3, s_vy;

floglik(const vP, const adFunc, const avScore, const amHess){

/* log-likelihood function */
adFunc[0] = -0.5*sumc(log(2*3.141593) + (vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3) + ((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)).^2) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));

if (avScore){
(avScore[0])[0] = -0.5*sumc(1 - ((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)).^2) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[1] = -0.5*sumc(svx_1 - (svx_1.*((s_vy - (vP[4] +
```

```
vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[2] = -0.5*sumc(svx_2 - (svx_2.*((s_vy - (vP[4] +
vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[3] = -0.5*sumc(svx_3 - (svx_3.*((s_vy - (vP[4] +
vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[4] = sumc((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[5] = sumc(svx_1.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[6] = sumc(svx_2.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[7] = sumc(svx_3.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
}

if ( isnan(adFunc[0]) || isdotinf(adFunc[0]) )
return 0;
else
return 1;
}

main(){

decl i, s2, X, error, beta, betahat, residual2, gamma, sigmai2,
sigmai, lambda, P, H, h, weight2, weight3, weight4, g, g4, quant,
matrixW, vp, dfunc, ir, nrep, temp, eta, exectime, cfailure, ztest,
cvz1, cvz5, cvz10, alphachapeu, phi, covm3,covm2, What, betahat2,
ztest2, result, count, HC0, HC2, HC3, HC4, ztest3, ztest4, ztest5,
teste, ztest7, betahathat,si2, weights, Ww, yw, Xw, olsnumerator,
Pt, matrixtemp, hmean, g4m1, g4m2, g4m, weight4m, HC4m, ztest6, I,
I2, S, temp4, Psi, temp2, gammatil, temp3, ztest1, cfailure2,
betahat3, betaw, counter;

decl nobs = 50; /* number of base observations (replicate if
greater than 50) */

nrep=10000; /* number of Monte Carlo simulations */
```

```
beta=<3.0;3.0;3.0;3.0>;  /*true beta values */

gamma=<0.935;0.935;0.935;0.935>;   /* true gamma values */
//gamma=<1.101;1.101;1.101;1.101>; /* true gamma values */

cfailure = 0;   /* failure counter (maximum likelihood) */
cfailure2 = 0;  /* failure counter (Atkinson) */

cvz1 = quann(0.995);  /* critical value 1% */
cvz5 = quann(0.975);  /* critical value 5% */
cvz10 = quann(0.950); /* critical value 10% */

quant = zeros(8,3);  /* quantiles */
result = zeros(8,3); /* to save null rejection rates */

ranseed("MWC_52"); /* pseudo-random number generator */
ranseed(1994);     /* generator seed */

exectime = timer(); /* start counting time */

X=1~fabs(rann(nobs, 3)); /* matrix of regressors */

/* if necessary, replicate the covariates values (for n> 50) */
if(samplesize == 1){
X = X;
nobs = nobs;
}
else if(samplesize == 2){
X = X|X;
nobs *= 2;
}
else if(samplesize == 3){
X = X|X|X;
nobs *= 3;
}
else{
X = X|X|X|X;
nobs *= 4;
}

eta = X*beta; /* linear predictor */

P = invertsym(X'X)*X'; /* matrix P */
```

```
Pt = P';                /* transposed of matrix P */
H = X*P;    /* matrix H (hat matrix) */
h = (diagonal(H))';    /* leverage measures */
weight2 = 1.0 ./ (1.0-h);
weight3 = 1.0 ./ ((1.0-h) .^ 2);
g = (nobs/4) * h;
g4 = g .> 4 .? 4 .: g;
weight4 = 1.0 ./ ((1.0-h) .^ g4);
g4m1 = g .> 1.0 .? 1.0 .: g;
g4m2 = g .> 1.5 .? 1.5 .: g;
g4m = g4m1 + g4m2;
weight4m = 1.0 ./ ((1.0-h) .^ g4m);

svx_1 = X[][1]; /* covariate 1 */
svx_2 = X[][2]; /* covariate 2 */
svx_3 = X[][3]; /* covariate 3 */

sigmai2 = exp(X*gamma);

sigmai = (sigmai2).^0.5; /* standard errors */

lambda = double(maxc(sigmai2)/minc(sigmai2)); /* degree of
heteroskedasticity */

betahat = zeros(4, nrep);    /* matrix to save OLS estimates */
betahat2 = zeros(4, nrep);   /* matrix to save EGLS1 estimates */
betahat3 = zeros(4, nrep);   /* matrix to save EGLS2 estimates */
betahathat = zeros(4, nrep); /* matrix to save EGLS3 estimates */

/* vectors used to save test statistics */
ztest = zeros(1, nrep);  /* EGLS1 */
ztest1 = zeros(1, nrep); /* EGLS2 */
ztest2 = zeros(1, nrep); /* EGLS3 */
ztest3 = zeros(1, nrep); /* HC0 */
ztest4 = zeros(1, nrep); /* HC2 */
ztest5 = zeros(1, nrep); /* HC3 */
ztest6 = zeros(1, nrep); /* HC4 */
ztest7 = zeros(1, nrep); /* HC4m */

hmean = meanc(h);             /* mean leverage */
count = sumc( h .> 3*hmean ); /* number of leverage points */

/* following: Monte Carlo Loop */
for(i=0; i<nrep; i++){
```

```
error = rann(nobs, 1);                                /* normal errors */
//error = (ranchi(nobs, 1, 10) - 10)/(sqrt(20)); /* chi-square errors
(10) */
//error = (ranchi(nobs, 1, 5) - 5)/(sqrt(10));   /* chi-square errors
(5) */
//error = (ranchi(nobs, 1, 2) - 2)/(sqrt(4));    /* chi-square errors
(2) */

s_vy= eta + sigmai.*error; /* response vector */

olsc(s_vy, X, &temp); /* regression estimation */

betahat[][i] = temp; /* OLS estimates */

residual2 = (s_vy-X*temp).^2;

matrixtemp = residual2 .* Pt; /* matrix to be used in HCs */

HC0 = P * matrixtemp;                 /* HC0 */
HC2 = P * (matrixtemp .* weight2);    /* HC2 */
HC3 = P * (matrixtemp .* weight3);    /* HC3 */
HC4 = P * (matrixtemp .* weight4);    /* HC4 */
HC4m =  P * (matrixtemp .* weight4m); /* HC4m */

/* following: EGLS2 estimation procedure */
temp2 = P * (nobs*residual2 ./ (sumc(residual2) - 1)); /*
starting values for gamma */

S = zeros(4,1); /* score vector */

counter=0;

do{

++counter;

if( counter <= 1000 ){

S[0][0] = - sumc(1 - ((s_vy - X*temp).^2) ./ (exp(X*temp2)));
S[1][0] = sumc(X[][1] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));
S[2][0] = sumc(X[][2] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));
S[3][0] = sumc(X[][3] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));
```

```
gammatil = temp2 + invertsym(X'X) * S; /* iterative scoring algorithm */

temp3 = temp;
temp4 = temp2;

si2 = exp(X*gammatil);  /* sigma2 hat */

weights = (1 ./ si2);

Ww=diag(sqrt(weights));
yw=Ww*s_vy;
Xw=Ww*X;

olsc(yw, Xw, &betaw); /* weighted regression */

temp = betaw;
temp2 = gammatil;

} /* end of if */

else{
++cfailure2;
break;
}

} while((norm((temp|temp2) - (temp3|temp4))^2 / norm((temp3|temp4))^2)
> (10^(-8)));

if(counter==1001){
i--;
continue;
}

betahat3[][i] = betaw; /* EGLS2 estimates */

What = diag(weights);         /* matrix of weights */
covm2 = invertsym(X'*What*X); /* EGLS2 covariance matrix */

/* following: Harvey estimation procedure (two-steps) */
olsc(log(residual2), X, &alphachapeu); /* regression estimation */
alphachapeu[0] += 1.2704;
phi=diag(exp(X*alphachapeu));
betahathat[][i] = invertsym(X'*invertsym(phi)*X)*
(X'*invertsym(phi)*s_vy);
```

```
covm3 = invertsym(X'*invertsym(phi)*X); /* EGLS3 covariance matrix */

I = zeros(8,8); /* inverse of Fisher's information matrix */

/* following: Harvey estimation procedure (maximum likelihood) */
vp = <5.0; 5.0; 5.0; 5.0; 5.0; 5.0; 5.0; 5.0>; /* starting values */
ir = MaxBFGS(floglik, &vp, &dfunc, 0, FALSE);  /* BFGS with
analytical gradient */

/* convergence check */
if( ir == MAX_CONV || ir == MAX_WEAK_CONV ){

I[0][0] = sumc(1 ./ exp(vp[0] + vp[1]*svx_1 + vp[2]*svx_2 +
vp[3]*svx_3));
I[1][0] = I[0][1] = sumc(svx_1 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[2][0] = I[0][2] = sumc(svx_2 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][0] = I[0][3] = sumc(svx_3 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[4][0] = I[0][4] = I[5][0] = I[0][5] = I[6][0] = I[0][6] =
I[7][0] = I[0][7] = I[4][1] = I[1][4] = I[5][1] = I[1][5] =
I[6][1] = I[1][6] = I[7][1] = I[1][7] = I[4][2] = I[2][4] =
I[5][2] = I[2][5] = I[6][2] = I[2][6] = I[7][2] = I[2][7] =
I[4][3] = I[3][4] = I[5][3] = I[3][5] = I[6][3] = I[3][6] =
I[7][3] = I[3][7] = 0;
I[1][1] = sumc((svx_1.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[2][1] = I[1][2] = sumc((svx_1 .* svx_2) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[3][1] = I[1][3] = sumc((svx_1 .* svx_3) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[2][2] = sumc((svx_2.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][3] = sumc((svx_3.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][2] = I[2][3] = sumc((svx_2 .* svx_3) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[4][4] = nobs/2;
I[5][5] = 0.5*sumc(svx_1.^2);
I[5][4] = I[4][5] = 0.5*sumc(svx_1);
I[6][4] = I[4][6] = 0.5*sumc(svx_2);
I[7][4] = I[4][7] = 0.5*sumc(svx_3);
I[6][6] = 0.5*sumc(svx_2.^2);
```

```
I[6][5] = I[5][6] = 0.5*sumc(svx_1 .* svx_2);
I[7][5] = I[5][7] = 0.5*sumc(svx_1 .* svx_3);
I[7][6] = I[6][7] = 0.5*sumc(svx_2 .* svx_3);
I[7][7] = 0.5*sumc(svx_3.^2);

I2 = invertsym(I);

betahat2[][i] = vp[4:]; /* estimates of maximum likelihood */

/* EGLS test statistics */
ztest[0][i] = (betahat2[2][i] - beta[2]) / sqrt(I2[2][2]);
/* EGLS1 */
ztest1[0][i] = (betahat3[2][i] - beta[2]) / sqrt(covm2[2][2]);
/* EGLS2 */
ztest2[0][i] = (betahathat[2][i] - beta[2]) / sqrt(covm3[2][2]);
/* EGLS3 */

/* quasi-t test statistics */
olsnumerator = betahat[2][i] - beta[2];
ztest3[0][i] = olsnumerator / sqrt(HC0[2][2]);  /* HC0 */
ztest4[0][i] = olsnumerator / sqrt(HC2[2][2]);  /* HC2 */
ztest5[0][i] = olsnumerator / sqrt(HC3[2][2]);  /* HC3 */
ztest6[0][i] = olsnumerator / sqrt(HC4[2][2]);  /* HC4 */
ztest7[0][i] = olsnumerator / sqrt(HC4m[2][2]); /* HC4m */

} /* end of if */

else{
++cfailure;
--i;
} /* end of else */
}     /* end of the Monte Carlo loop */

result[0][0] = sumr(fabs(ztest[0][]) .> cvz1)/nrep*100;  /* EGLS1 1% */
result[1][0] = sumr(fabs(ztest1[0][]) .> cvz1)/nrep*100; /* EGLS2 1% */
result[2][0] = sumr(fabs(ztest2[0][]) .> cvz1)/nrep*100; /* EGLS3 1% */
result[3][0] = sumr(fabs(ztest3[0][]) .> cvz1)/nrep*100; /* HC0 1% */
result[4][0] = sumr(fabs(ztest4[0][]) .> cvz1)/nrep*100; /* HC2 1% */
result[5][0] = sumr(fabs(ztest5[0][]) .> cvz1)/nrep*100; /* HC3 1% */
result[6][0] = sumr(fabs(ztest6[0][]) .> cvz1)/nrep*100; /* HC4 1% */
result[7][0] = sumr(fabs(ztest7[0][]) .> cvz1)/nrep*100; /* HC4m 1% */

result[0][1] = sumr(fabs(ztest[0][]) .> cvz5)/nrep*100;  /* EGLS1 5% */
result[1][1] = sumr(fabs(ztest1[0][]) .> cvz5)/nrep*100; /* EGLS2 5% */
```

```
result[2][1] = sumr(fabs(ztest2[0][]) .> cvz5)/nrep*100; /* EGLS3 5% */
result[3][1] = sumr(fabs(ztest3[0][]) .> cvz5)/nrep*100; /* HC0 5% */
result[4][1] = sumr(fabs(ztest4[0][]) .> cvz5)/nrep*100; /* HC2 5% */
result[5][1] = sumr(fabs(ztest5[0][]) .> cvz5)/nrep*100; /* HC3 5% */
result[6][1] = sumr(fabs(ztest6[0][]) .> cvz5)/nrep*100; /* HC4 5% */
result[7][1] = sumr(fabs(ztest7[0][]) .> cvz5)/nrep*100; /* HC4m 5% */

result[0][2] = sumr(fabs(ztest[0][]) .> cvz10)/nrep*100;  /* EGLS1 10% */
result[1][2] = sumr(fabs(ztest1[0][]) .> cvz10)/nrep*100; /* EGLS2 10% */
result[2][2] = sumr(fabs(ztest2[0][]) .> cvz10)/nrep*100; /* EGLS3 10% */
result[3][2] = sumr(fabs(ztest3[0][]) .> cvz10)/nrep*100; /* HC0 10% */
result[4][2] = sumr(fabs(ztest4[0][]) .> cvz10)/nrep*100; /* HC2 10% */
result[5][2] = sumr(fabs(ztest5[0][]) .> cvz10)/nrep*100; /* HC3 10% */
result[6][2] = sumr(fabs(ztest6[0][]) .> cvz10)/nrep*100; /* HC4 10% */
result[7][2] = sumr(fabs(ztest7[0][]) .> cvz10)/nrep*100; /* HC4m 10% */

/* The quantiles of the different test statistics are calculated
and used as exact critical values in the power simulations */

quant[0][0] = quantiler(fabs(ztest[0][]), 0.995);
quant[1][0] = quantiler(fabs(ztest1[0][]), 0.995);
quant[2][0] = quantiler(fabs(ztest2[0][]), 0.995);
quant[3][0] = quantiler(fabs(ztest3[0][]), 0.995);
quant[4][0] = quantiler(fabs(ztest4[0][]), 0.995);
quant[5][0] = quantiler(fabs(ztest5[0][]), 0.995);
quant[6][0] = quantiler(fabs(ztest6[0][]), 0.995);
quant[7][0] = quantiler(fabs(ztest7[0][]), 0.995);

quant[0][1] = quantiler(fabs(ztest[0][]), 0.975);
quant[1][1] = quantiler(fabs(ztest1[0][]), 0.975);
quant[2][1] = quantiler(fabs(ztest2[0][]), 0.975);
quant[3][1] = quantiler(fabs(ztest3[0][]), 0.975);
quant[4][1] = quantiler(fabs(ztest4[0][]), 0.975);
quant[5][1] = quantiler(fabs(ztest5[0][]), 0.975);
quant[6][1] = quantiler(fabs(ztest6[0][]), 0.975);
quant[7][1] = quantiler(fabs(ztest7[0][]), 0.975);

quant[0][2] = quantiler(fabs(ztest[0][]), 0.950);
quant[1][2] = quantiler(fabs(ztest1[0][]), 0.950);
quant[2][2] = quantiler(fabs(ztest2[0][]), 0.950);
quant[3][2] = quantiler(fabs(ztest3[0][]), 0.950);
quant[4][2] = quantiler(fabs(ztest4[0][]), 0.950);
quant[5][2] = quantiler(fabs(ztest5[0][]), 0.950);
quant[6][2] = quantiler(fabs(ztest6[0][]), 0.950);
```

```
quant[7][2] = quantiler(fabs(ztest7[0][]), 0.950);

savemat("v1.mat", quant);

/* printing of basic information */
print("\n");
println( "\t\t OX PROGRAM: ", oxfilename(0) );
println( "\t\t OX VERSION: ", oxversion() );
println( "\t\t NUM. REPLICATIONS: ", nrep );
println( "\t\t NUM. OBSERVATIONS: ", nobs );
println( "\t\t DATE: ", date() );
println( "\t\t TIME: ", time() );

/* printing results */
println("\nLAMBDA: ", lambda);
println("\nTRUE GAMMA VALUES: ", gamma);
println("\nMAXIMAL LEVERAGE: ", double(maxc(h)));
println("\nTHRESHOLD (3*p/n): ", double(3*hmean));
println("\nNUM. OBSERVATIONS > 3*p/n: ", "%2.0f", double(count));
println("\nNULL REJECTION RATES FOR n=", nobs, ":\n", "%c",
{"1%", "5%", "10%"}, "%r", {"EGLS 1", "EGLS 2", "EGLS 3",
"HC0", "HC2", "HC3", "HC4", "HC4m"}, "%8.2f", result);

println("NUMBER OF CONVERGENCE FAILURES EGLS1: ", cfailure);

println("NUMBER OF CONVERGENCE FAILURES EGLS2: ", cfailure2);

println("\nEXECUTION TIME: ", timespan(exectime)); /* runtime */

print("\n");

}
```

Appendix B - Power simulation

```
/*************************************************************
DESCRIPTION: Monte Carlo simulation of quasi-t and z tests under
heteroskedasticity. The choice of sample sizes are: 50, 100, 150,
200. Multiple regression. Covariates generated from |N(0,1)| and
errors: normal and chi-square with 2, 5, 10 degrees of freedom.

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#include<oxstd.h>
#include<oxprob.h>
#import <maximize>

const decl samplesize = 1; /* choice 1 (n=50), 2 (n=100), 3 (n=150)
or 4 (n=200) */

static decl svx_1, svx_2, svx_3, s_vy;

floglik(const vP, const adFunc, const avScore, const amHess){

/* log-likelihood function */
adFunc[0] = -0.5*sumc(log(2*3.141593) + (vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3) + ((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)).^2) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));

if (avScore){
(avScore[0])[0] = -0.5*sumc(1 - ((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)).^2) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[1] = -0.5*sumc(svx_1 - (svx_1.*((s_vy - (vP[4] +
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vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[2] = -0.5*sumc(svx_2 - (svx_2.*((s_vy - (vP[4] +
vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[3] = -0.5*sumc(svx_3 - (svx_3.*((s_vy - (vP[4] +
vP[5]*svx_1 + vP[6]*svx_2 + vP[7]*svx_3)).^2)) ./ (exp(vP[0] +
vP[1]*svx_1 + vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[4] = sumc((s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[5] = sumc(svx_1.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[6] = sumc(svx_2.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
(avScore[0])[7] = sumc(svx_3.*(s_vy - (vP[4] + vP[5]*svx_1 +
vP[6]*svx_2 + vP[7]*svx_3)) ./ (exp(vP[0] + vP[1]*svx_1 +
vP[2]*svx_2 + vP[3]*svx_3)));
}

if ( isnan(adFunc[0]) || isdotinf(adFunc[0]) )
return 0;
else
return 1;
}

main(){

decl i, s2, X, error, beta, betahat, residual2, gamma, sigmai2,
sigmai, lambda, P, H, h, weight2, weight3, weight4, g, g4, matrixW,
vp, dfunc, ir, nrep, temp, eta, exectime, cfailure, ztest, cvz1,
cvz5, cvz10, alphachapeu, phi, covm3,covm2, What, betahat2, ztest2,
result, count, HC0, HC2, HC3, HC4, ztest3, ztest4, ztest5, ztest7,
betahathat, si2, weights, Ww, yw, Xw, olsnumerator, Pt, matrixtemp,
hmean, g4m1, g4m2, g4m, weight4m, HC4m, ztest6, I, I2, S, temp4,
Psi, temp2, gammatil, temp3, ztest1, cfailure2, betahat3, betaw,
mat1, mat2, mat3, mat4, mat5, mat6, mat7, mat8, mat9, mat10, mat11,
mat12, mat13, mat14, mat15, mat16, mat17, mat18, mat19, mat20, mat21,
mat22, mat23, mat24, mat25, mat26, mat27, mat28, mat29, mat30, mat31,
mat32;

/* quantiles obtained in the size simulations */
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mat1 = loadmat("v1.mat");
mat2 = loadmat("v2.mat");
mat3 = loadmat("v3.mat");
mat4 = loadmat("v4.mat");
mat5 = loadmat("v5.mat");
mat6 = loadmat("v6.mat");
mat7 = loadmat("v7.mat");
mat8 = loadmat("v8.mat");
mat9 = loadmat("v9.mat");
mat10 = loadmat("v10.mat");
mat11 = loadmat("v11.mat");
mat12 = loadmat("v12.mat");
mat13 = loadmat("v13.mat");
mat14 = loadmat("v14.mat");
mat15 = loadmat("v15.mat");
mat16 = loadmat("v16.mat");
mat17 = loadmat("v17.mat");
mat18 = loadmat("v18.mat");
mat19 = loadmat("v19.mat");
mat20 = loadmat("v20.mat");
mat21 = loadmat("v21.mat");
mat22 = loadmat("v22.mat");
mat23 = loadmat("v23.mat");
mat24 = loadmat("v24.mat");
mat25 = loadmat("v25.mat");
mat26 = loadmat("v26.mat");
mat27 = loadmat("v27.mat");
mat28 = loadmat("v28.mat");
mat29 = loadmat("v29.mat");
mat30 = loadmat("v30.mat");
mat31 = loadmat("v31.mat");
mat32 = loadmat("v32.mat");

decl nobs = 50; /* number of base observations (replicate
if greater than 50) */

nrep=10000; /* number of Monte Carlo simulations */

beta=<3.0;3.0;6.0;3.0>; /* true beta values */

gamma=<0.935;0.935;0.935;0.935>;   /* true gamma values */
//gamma=<1.101;1.101;1.101;1.101>; /* true gamma values */

cfailure = 0;   /* failure counter (maximum likelihood) */
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cfailure2 = 0;  /* failure counter (Atkinson) */

cvz1 = quann(0.995);  /* critical value 1% */
cvz5 = quann(0.975);  /* critical value 5% */
cvz10 = quann(0.950); /* critical value 10% */

result = zeros(8,3); /* to save nonnull rejection rates */

ranseed("MWC_52"); /* pseudo-random number generator */
ranseed(1994);     /* generator seed */

exectime = timer(); /* start counting time */

X=1~fabs(rann(nobs, 3)); /* matrix of regressors */

/* if necessary, replicate the covariates values (for n> 50) */
if(samplesize == 1){
X = X;
nobs = nobs;
}
else if(samplesize == 2){
X = X|X;
nobs *= 2;
}
else if(samplesize == 3){
X = X|X|X;
nobs *= 3;
}
else{
X = X|X|X|X;
nobs *= 4;
}

eta = X*beta; /* linear predictor */

P = invertsym(X'X)*X'; /* matrix P */
Pt = P';                /* transposed of matrix P */
H = X*P; /* matrix H (hat matrix) */
h = (diagonal(H))'; /* leverage measures */
weight2 = 1.0 ./ (1.0-h);
weight3 = 1.0 ./ ((1.0-h) .^ 2);
g = (nobs/4) * h;
g4 = g .> 4 .? 4 .: g;
weight4 = 1.0 ./ ((1.0-h) .^ g4);
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g4m1 = g .> 1.0 .? 1.0 .: g;
g4m2 = g .> 1.5 .? 1.5 .: g;
g4m = g4m1 + g4m2;
weight4m = 1.0 ./ ((1.0-h) .^ g4m);

svx_1 = X[][1]; /* covariate 1 */
svx_2 = X[][2]; /* covariate 2 */
svx_3 = X[][3]; /* covariate 3 */

sigmai2 = exp(X*gamma);

sigmai = (sigmai2).^0.5; /* standard errors */

lambda = double(maxc(sigmai2)/minc(sigmai2)); /* degree
of heteroskedasticity */

betahat = zeros(4, nrep);    /* matrix to save OLS estimates */
betahat2 = zeros(4, nrep);   /* matrix to save EGLS1 estimates */
betahat3 = zeros(4, nrep);   /* matrix to save EGLS2 estimates */
betahathat = zeros(4, nrep); /* matrix to save EGLS3 estimates */

/* vectors used to save test statistics */
ztest = zeros(1, nrep);  /* EGLS1 */
ztest1 = zeros(1, nrep); /* EGLS2 */
ztest2 = zeros(1, nrep); /* EGLS3 */
ztest3 = zeros(1, nrep); /* HC0 */
ztest4 = zeros(1, nrep); /* HC2 */
ztest5 = zeros(1, nrep); /* HC3 */
ztest6 = zeros(1, nrep); /* HC4 */
ztest7 = zeros(1, nrep); /* HC4m */

hmean = meanc(h);             /* mean leverage */
count = sumc( h .> 3*hmean ); /* number of leverage points */

/* following: Monte Carlo Loop */
for(i=0; i<nrep; i++){

error = rann(nobs, 1);                           /* normal errors */
//error = (ranchi(nobs, 1, 10) - 10)/(sqrt(20)); /* chi-square errors
(10) */
//error = (ranchi(nobs, 1, 5) - 5)/(sqrt(10));   /* chi-square errors
(5) */
//error = (ranchi(nobs, 1, 2) - 2)/(sqrt(4));    /* chi-square errors
(2) */
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s_vy= eta + sigmai.*error; /* response vector */

olsc(s_vy, X, &temp); /* regression estimation */

betahat[][i] = temp; /* OLS estimates */

residual2 = (s_vy-X*temp).^2;

matrixtemp = residual2 .* Pt; /* matrix to be used in HCs */

HC0 = P * matrixtemp;                  /* HC0 */
HC2 = P * (matrixtemp .* weight2);     /* HC2 */
HC3 = P * (matrixtemp .* weight3);     /* HC3 */
HC4 = P * (matrixtemp .* weight4);     /* HC4 */
HC4m =  P * (matrixtemp .* weight4m); /* HC4m */

/* following: EGLS2 estimation procedure */
temp2 = P * (nobs*residual2 ./ (sumc(residual2) - 1)); /*
starting values for gamma */

S = zeros(4,1); /* score vector */

decl contador;
contador=0;

do{

++contador;

if( contador <= 1000 ){

S[0][0] = - sumc(1 - ((s_vy - X*temp).^2) ./ (exp(X*temp2)));
S[1][0] = sumc(X[][1] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));
S[2][0] = sumc(X[][2] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));
S[3][0] = sumc(X[][3] .* (((s_vy - X*temp).^2) ./ (exp(X*temp2)) - 1));

gammatil = temp2 + invertsym(X'X) * S; /* iterative scoring algorithm */

temp3 = temp;
temp4 = temp2;

si2 = exp(X*gammatil);  /* sigma2 hat */
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weights = (1 ./ si2);

Ww=diag(sqrt(weights));
yw=Ww*s_vy;
Xw=Ww*X;

olsc(yw, Xw, &betaw); /* weighted regression */

temp = betaw;
temp2 = gammatil;

} /* end of if */

else{
++cfailure2;
break;
}

} while((norm((temp|temp2) - (temp3|temp4))^2 / norm((temp3|temp4))^2)
> (10^(-8)));

if(contador==1001){
i--;
continue;
}

betahat3[][i] = betaw; /* EGLS2 estimates */

What = diag(weights);          /* matrix of weights */
covm2 = invertsym(X'*What*X); /* EGLS2 covariance matrix */

/* following: Harvey estimation procedure (two-steps) */
olsc(log(residual2), X, &alphachapeu);  /* regression estimation */
alphachapeu[0] += 1.2704;
phi=diag(exp(X*alphachapeu));
betahathat[][i] = invertsym(X'*invertsym(phi)*X)*
(X'*invertsym(phi)*s_vy);
covm3 = invertsym(X'*invertsym(phi)*X); /* EGLS3 covariance matrix */

I = zeros(8,8); /* inverse of Fisher's information matrix */

/* following: Harvey estimation procedure (maximum likelihood) */
vp = <5.0; 5.0; 5.0; 5.0; 5.0; 5.0; 5.0; 5.0>; /* starting values */
ir = MaxBFGS(floglik, &vp, &dfunc, 0, FALSE);  /* BFGS with
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analytical gradient */

/* convergence check */
if( ir == MAX_CONV || ir == MAX_WEAK_CONV ){

I[0][0] = sumc(1 ./ exp(vp[0] + vp[1]*svx_1 + vp[2]*svx_2 +
vp[3]*svx_3));
I[1][0] = I[0][1] = sumc(svx_1 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[2][0] = I[0][2] = sumc(svx_2 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][0] = I[0][3] = sumc(svx_3 ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[4][0] = I[0][4] = I[5][0] = I[0][5] = I[6][0] = I[0][6] =
I[7][0] = I[0][7] = I[4][1] = I[1][4] = I[5][1] = I[1][5] =
I[6][1] = I[1][6] = I[7][1] = I[1][7] = I[4][2] = I[2][4] =
I[5][2] = I[2][5] = I[6][2] = I[2][6] = I[7][2] = I[2][7] =
I[4][3] = I[3][4] = I[5][3] = I[3][5] = I[6][3] = I[3][6] =
I[7][3] = I[3][7] = 0;
I[1][1] = sumc((svx_1.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[2][1] = I[1][2] = sumc((svx_1 .* svx_2) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[3][1] = I[1][3] = sumc((svx_1 .* svx_3) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[2][2] = sumc((svx_2.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][3] = sumc((svx_3.^2) ./ exp(vp[0] + vp[1]*svx_1 +
vp[2]*svx_2 + vp[3]*svx_3));
I[3][2] = I[2][3] = sumc((svx_2 .* svx_3) ./ exp(vp[0] +
vp[1]*svx_1 + vp[2]*svx_2 + vp[3]*svx_3));
I[4][4] = nobs/2;
I[5][5] = 0.5*sumc(svx_1.^2);
I[5][4] = I[4][5] = 0.5*sumc(svx_1);
I[6][4] = I[4][6] = 0.5*sumc(svx_2);
I[7][4] = I[4][7] = 0.5*sumc(svx_3);
I[6][6] = 0.5*sumc(svx_2.^2);
I[6][5] = I[5][6] = 0.5*sumc(svx_1 .* svx_2);
I[7][5] = I[5][7] = 0.5*sumc(svx_1 .* svx_3);
I[7][6] = I[6][7] = 0.5*sumc(svx_2 .* svx_3);
I[7][7] = 0.5*sumc(svx_3.^2);

I2 = invertsym(I);
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betahat2[][i] = vp[4:]; /* estimates of maximum likelihood */

/* EGLS test statistics */
ztest[0][i] = (betahat2[2][i] - beta[1]) / sqrt(I2[2][2]);
/* EGLS1 */
ztest1[0][i] = (betahat3[2][i] - beta[1]) / sqrt(covm2[2][2]);
/* EGLS2 */
ztest2[0][i] = (betahathat[2][i] - beta[1]) / sqrt(covm3[2][2]);
/* EGLS3 */

/* quasi-t test statistics */
olsnumerator = betahat[2][i] - beta[1];
ztest3[0][i] = olsnumerator / sqrt(HC0[2][2]);  /* HC0 */
ztest4[0][i] = olsnumerator / sqrt(HC2[2][2]);  /* HC2 */
ztest5[0][i] = olsnumerator / sqrt(HC3[2][2]);  /* HC3 */
ztest6[0][i] = olsnumerator / sqrt(HC4[2][2]);  /* HC4 */
ztest7[0][i] = olsnumerator / sqrt(HC4m[2][2]); /* HC4m */

} /* end of if */

else{
++cfailure;
--i;
} /* end of else */
}     /* end of the Monte Carlo loop */

result[0][0] = sumr(fabs(ztest[0][]) .> mat1[0][0])/nrep*100;
/* EGLS1 1% */
result[1][0] = sumr(fabs(ztest1[0][]) .> mat1[1][0])/nrep*100;
/* EGLS2 1% */
result[2][0] = sumr(fabs(ztest2[0][]) .> mat1[2][0])/nrep*100;
/* EGLS3 1% */
result[3][0] = sumr(fabs(ztest3[0][]) .> mat1[3][0])/nrep*100;
/* HC0 1% */
result[4][0] = sumr(fabs(ztest4[0][]) .> mat1[4][0])/nrep*100;
/* HC2 1% */
result[5][0] = sumr(fabs(ztest5[0][]) .> mat1[5][0])/nrep*100;
/* HC3 1% */
result[6][0] = sumr(fabs(ztest6[0][]) .> mat1[6][0])/nrep*100;
/* HC4 1% */
result[7][0] = sumr(fabs(ztest7[0][]) .> mat1[7][0])/nrep*100;
/* HC4m 1% */

result[0][1] = sumr(fabs(ztest[0][]) .> mat1[0][1])/nrep*100;
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/* EGLS1 5% */
result[1][1] = sumr(fabs(ztest1[0][]) .> mat1[1][1])/nrep*100;
/* EGLS2 5% */
result[2][1] = sumr(fabs(ztest2[0][]) .> mat1[2][1])/nrep*100;
/* EGLS3 5% */
result[3][1] = sumr(fabs(ztest3[0][]) .> mat1[3][1])/nrep*100;
/* HC0 5% */
result[4][1] = sumr(fabs(ztest4[0][]) .> mat1[4][1])/nrep*100;
/* HC2 5% */
result[5][1] = sumr(fabs(ztest5[0][]) .> mat1[5][1])/nrep*100;
/* HC3 5% */
result[6][1] = sumr(fabs(ztest6[0][]) .> mat1[6][1])/nrep*100;
/* HC4 5% */
result[7][1] = sumr(fabs(ztest7[0][]) .> mat1[7][1])/nrep*100;
/* HC4m 5% */

result[0][2] = sumr(fabs(ztest[0][]) .> mat1[0][2])/nrep*100;
/* EGLS1 10% */
result[1][2] = sumr(fabs(ztest1[0][]) .> mat1[1][2])/nrep*100;
/* EGLS2 10% */
result[2][2] = sumr(fabs(ztest2[0][]) .> mat1[2][2])/nrep*100;
/* EGLS3 10% */
result[3][2] = sumr(fabs(ztest3[0][]) .> mat1[3][2])/nrep*100;
/* HC0 10% */
result[4][2] = sumr(fabs(ztest4[0][]) .> mat1[4][2])/nrep*100;
/* HC2 10% */
result[5][2] = sumr(fabs(ztest5[0][]) .> mat1[5][2])/nrep*100;
/* HC3 10% */
result[6][2] = sumr(fabs(ztest6[0][]) .> mat1[6][2])/nrep*100;
/* HC4 10% */
result[7][2] = sumr(fabs(ztest7[0][]) .> mat1[7][2])/nrep*100;
/* HC4m 10% */

/* printing of basic information */
print("\n");
println( "\t\t OX PROGRAM: ", oxfilename(0) );
println( "\t\t OX VERSION: ", oxversion() );
println( "\t\t NUM. REPLICATIONS: ", nrep );
println( "\t\t NUM. OBSERVATIONS: ", nobs );
println( "\t\t DATE: ", date() );
println( "\t\t TIME: ", time() );

/* printing results */
println("\nLAMBDA: ", lambda);
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println("\nTRUE GAMMA VALUES: ", gamma);
println("\nMAXIMAL LEVERAGE: ", double(maxc(h)));
println("\nTHRESHOLD (3*p/n): ", double(3*hmean));
println("\nNUM. OBSERVATIONS > 3*p/n: ", "%2.0f", double(count));
println("\nNONNULL REJECTION RATES FOR n=", nobs, ":\n", "%c",
{"1%", "5%", "10%"}, "%r", {"EGLS 1", "EGLS 2", "EGLS 3", "HC0",
"HC2", "HC3", "HC4", "HC4m"}, "%8.2f", result);

println("NUMBER OF CONVERGENCE FAILURES EGLS1: ", cfailure);

println("NUMBER OF CONVERGENCE FAILURES EGLS2: ", cfailure2);

println("\nEXECUTION TIME: ", timespan(exectime)); /* runtime */

print("\n");

}
```