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CIRCUIT THEORY VIA ALGEBRAIC TOPOLOGY

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por

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Abstract

We are proposing a new formulation of circuit theory, taking in consideration its physical distribution in the space. For doing this we will use some concepts of the algebraic topology. Names as Hermann Weyl and Steve Smale did important contributions showing these connections between the theory of circuits and the theory of algebraic topology. In this work, we will go to consider an electrical circuit as a graph or as a one-dimensional complex, where the domain of the boundary operator ∂ is the vector space C_1 generated by the branches (wires of the circuit) and its codomain is the vector space C_0 generated by the nodes. In chapter 3, the Kirchhoff 's current law will be reformulate to the concise formula $\partial \mathbf{I} = \mathbf{0}$ and the Kirchhoff 's potential law will be reformulate to the concise formula $V = -\mathrm{d}\phi$, where $\mathrm{d} : C^0 \to C^1$ is the coboundary map. The methods of mesh-current and node-potential are also discussed in this chapter, as well as a conclusive analysis of the existence and uniqueness of solutions for the electric circuit equations too is realized. In chapter 4 we will study some alternative methods for solving electric circuit equations. The Weyl's method makes use of orthogonal projection operators and this method is summarized by the formula $\pi = \sigma(sZ\sigma)^{-1}sZ$. The Kirchhoff's method uses graph theory to find the values of voltages and electric currents and will be given by $p_\lambda = R^{-1}\sum_{\mathrm{T}} Q_{\mathrm{T}} p_{\mathrm{T}}$. The Green's reciprocity theorem exposes symmetries for some resistive circuits. In chapter 5, we will treat circuits where their branches have at most a battery in series with a capacitor. Here, the Gauss' Law will be reformulated to $\partial Q = -\boldsymbol{\rho}$, and the Poisson's equation will be reformulated to $-\partial C \mathrm{d}\boldsymbol{\phi} = -\boldsymbol{\rho}$. In this chapter, we too study the Dirichlet problem, ending with the study of Green's functions.

Key-words: electric circuits and algebraic topology. electric circuits.smale.mathematical physics. electromagnetism.

Resumo

Estamos propondo uma nova formulação da teoria dos circuitos, levando em consideração a sua distribuição física no espaço. Para fazer isto, usaremos alguns conceitos da topologia algébrica. Nomes como Hermann Weyl e Steve Smale fizeram importantes contribuições mostrando essas conexões entre a teoria dos circuitos e a da topologia algébrica. Neste trabalho, nós consideraremos um circuito elétrico como um grafo ou um complexo unidimensional, onde o domínio do operador fronteira ∂ é o espaço vetorial C_1 gerado pelos ramos (fios do circuito), e o seu codomínio é o espaço vetorial C_0 gerado pelos nós. No capítulo 3, a lei das correntes de Kirchhoff será reformulada para a fórmula concisa $\partial \mathbf{I} = \mathbf{0}$ e a lei das voltagens de Kirchhoff será reformulada para a fórmula concisa $V = -\mathrm{d}\phi$, onde $\mathrm{d} : C^0 \to C^1$ é a aplicação cofronteira. Os métodos da corrente na malha e do potencial nos nós são também discutidos neste capítulo, bem como uma análise conclusiva da existência e unicidade das soluções para as equações dos circuitos elétricos é também realizada. No capítulo 4, estudaremos alguns métodos alternativos para resolver equações de circuitos elétricos. O método de Weyl faz uso de operadores para projeção ortogonal e este método resume-se a fórmula $\pi = \sigma(sZ\sigma)^{-1}sZ$. O método de Kirchhoff usa a teoria de grafos para encontrar os valores de tensões e correntes elétricas e será dado por $p_\lambda = R^{-1}\sum_{\mathrm{T}} Q_{\mathrm{T}} p_{\mathrm{T}}$. O teorema da reciprocidade de Green expõe simetrias para alguns circuitos resistivos. No capítulo 5, vamos tratar circuitos onde seus ramos têm no máximo uma bateria em série com um capacitor. Aqui, a Lei de Gauss será reformulada para $\partial Q = -\rho$, e a equação de Poisson será reformulada para $-\partial C \mathrm{d}\boldsymbol{\phi} = -\boldsymbol{\rho}$. Neste capítulo, nós também estudamos o problema de Dirichlet, terminando com o estudo das funções de Green.

Palavras-chaves: circuitos elétricos e topologia algébrica. circuitos elétricos. smale. física-matemática. eletromagnetismo.

Contents

1 INTRODUCTION

The circuit theory is an approximation of the theory of electromagnetism, where the interest is almost all concentrated in terms of what happens along the wires and nodes of the circuit. The biggest benefit of the development of any theory is, as you might expect, make predictions about the values of its main parameters. In our case, the fundamental quantities in an electrical circuit are: electric charge(Q), energy and electric potential ($\boldsymbol{\phi}$), and thereafter electric currents (I), voltages (V) and electric power (Pot). So this will always be the first goal to be achieved when we thinking about a circuit theory. What we are proposing here is a new formulation of circuit theory, making further consideration about its shape, i.e., doing better observation of how the wires are interconnected. We also intend from the behavior of each branch of the circuit, try to make generalizations about the behavior into the entire circuit. The mathematical tools used in this work are: linear algebra, graph theory and algebraic topology.

Well known names worked in the grounds of the vision of circuit theory via algebraic topology. We will mention some of these names. Kirchhoff, as we will see soon in this work, made important considerations about the topology of the circuit. Maxwell considered that the topology could have an important role in the formulation of electromagnetic boundary value problems, although it has not been exploited by him. But it was Weyl with its articles (WEYL, 1923) and (WEYL, 1924) that not only established the connection between circuit theory and algebraic topology, but also helped to justify the own theory of algebraic topology (still discredited among mathematicians), called by him of *Combinatorial analyzes situs*. After this, Steve Smale with his article (SMALE, 1972) makes important aplications this new perspective to study dynamical systems associated with electrical circuits.

Now, we will briefly comment about the chapters of this work. **Chapter 2** is devoted to a brief review of topics of linear algebra that are essential and indispensable to the proper understanding on whole text. The main references were (BAMBERG; STERNBERG, 1988), (LANG, 1987) and (HOFFMAN; KUNZE, 1971).

Chapter 3 is where we introduce the concepts of graph theory and algebraic topology in the study of electrical circuits, i.e., this chapter is essential for understanding the following chapters. Branches and nodes, for example, constitute an one-dimensional complex, a fundamental concept in topology. With this will conquer a new perspective to study a circuit.

Chapter 4 is devoted exclusively to resistive circuits and present more two methods to solve the equations of an electrical circuit. The first of these methods was thought by Weyl and makes use of orthogonal projections. It is a creative way to associate geometric notions to solve the equations of electric circuits. The other method was created by Kirchhoff and uses graph theory (specifically maximal trees) to find the Weyl's orthogonal projection. It is extremely elegant the way that Kirchhoff thought graph theory in the study of eletrical circuits. We conclude this chapter with the Green's reciprocity theorem, where is possible to find symmetries in resistive circuits which apparently has not any symmetry.

Chapter 5 deals with capacitive circuits. The most interesting aspect of this study is precisely the notion of discretization of electrostatics. Here we find the discrete versions of Gauss' Law, Poisson equation, Laplace equation and Dirichlet problem , concluding the chapter with a discrete version of Green's functions. At this point opens up the future possibility of adapting this new theory for the continuous version, thereby covering the whole electromagnetism, not just the fraction destined to circuit theory.

Our work has (BAMBERG; STERNBERG, 1990) as the main reference, but several contributions were made, or in mathematics organization of ideas discussed, or in mathematical demonstrations where there were only categorical statements, or adding new definitions and propositions. As examples of this, we cite the following contributions to the Chapter 2: generalization of the operator ∂ (page 22), the diagram on page 33, helping the overall view of the matter, the definition 3.73, the lemma 3.74, corollaries 3.53, 3.54, 3.69, 3.71, theorem 3.70 (demonstration omitted from the book). In Chapter 3 we mention the contributions of lemmas 4.15, 4.17, corollary 4.31 and propositions 4.4, 4.6, 4.14 and 4.32. No Chapter 4 there were several contributions: remarks 5.25, 5.33, definitions 5.30, 5.34, lemmas 5.2, 5.5, 5.6,5.23, corollaries 5.9, 5.32, 5.35, proposition 5.16 (full demonstration), 5.20 (full demonstration), 5.26, 5.31, theorem 5.8 and finally an adjustment in the sign of Green's second formula.

2 A BRIEF REVIEW OF LINEAR ALGEBRA

2.1 Vector Space

Definition 2.1. A **vector space** over a field K is a set V on which are defined two operations

$$\begin{array}{rcl} + : \; V \times V & \to & V \\ (\mathbf{u}, \mathbf{v}) & \mapsto & \mathbf{u} + \mathbf{v} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} . : \; K \times V & \to & V \\ (c, \mathbf{v}) & \mapsto & c.\mathbf{v} \end{array}$$

called *addition* and *scalar multiplication*, satisfying the following properties:

$$\begin{array}{ll} \text{(A1)} & (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}), \ \forall \ \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \\ \text{(A2)} & \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}, \ \forall \ \mathbf{u}, \mathbf{v} \in V, \\ \text{(A3)} & \exists \ \mathbf{0} \in V \mid \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}, \ \forall \ \mathbf{v} \in V, \\ \text{(A4)} & \forall \ \mathbf{u} \in V, \ \exists \ \mathbf{v} \in V \mid \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{0}, \\ \text{(M1)} & 1.\mathbf{v} = \mathbf{v}, \ 1 \in K, \ \forall \ \mathbf{v} \in V, \\ \text{(M2)} & (bc).\mathbf{v} = b.(c.\mathbf{v}), \ \forall \ b, c \in K, \ \forall \ \mathbf{v} \in V, \\ \text{(M3)} & a.(\mathbf{u} + \mathbf{v}) = a.\mathbf{u} + a.\mathbf{v}, \ \forall \ a \in K, \ \forall \ \mathbf{u}, \mathbf{v} \in V, \\ \text{(M4)} & (b + c).\mathbf{v} = b.\mathbf{v} + c.\mathbf{v}, \ \forall \ b, c \in K, \ \forall \ \mathbf{v} \in V. \end{array} \tag{2.1}$$

Definition 2.2. A subset W of V is a **vector subspace** of V if:

(i) $W \neq \emptyset$,

(ii) $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$,

(iii) $\alpha \in K, \ \mathbf{u} \in W \Rightarrow \alpha.\mathbf{u} \in W$.

Remark 2.3. We will work only with finite dimensional vector spaces.

2.2 Linear Transformations

Definition 2.4. A **linear map**(or **linear transformation**) from V to W is a function $T : V \to W$ with the following properties

(i) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$,

(ii) $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$.

Definition 2.5. Let $T : V \to W$ be a linear map. The **kernel of** T, denoted by $ker\ T$, is defined by:

$$ker\ T := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}.$$

Definition 2.6. Let $T : V \to W$ be a linear map. The **range of** T, denoted by $Im\ T$, is defined by:

$$Im\ T := \{T(\mathbf{v}) \mid \mathbf{v} \in V\}.$$

Theorem 2.7. *Let $T : V \to W$ be a linear map. Then:*

$$\dim V = \dim kerT + \dim ImT \tag{2.2}$$

Proof. Vide (LANG, 1987). □

2.3 Quotient space

Definition 2.8. Let V be a vector space and let W be a subspace of V. Given a vector $\mathbf{v} \in V$, we define $\overline{\mathbf{v}}$ as being the set:

$$\overline{\mathbf{v}} := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\} \tag{2.3}$$

$\overline{\mathbf{v}}$ is called **equivalence class of v module W**.

Definition 2.9. Let us define the following operations between equivalence classes and scalars:

$$\begin{cases} \overline{\mathbf{v}} + \overline{\mathbf{w}} & := & \overline{\mathbf{v} + \mathbf{w}} \\ c.\overline{\mathbf{v}} & := & \overline{c.\mathbf{v}} \end{cases} , \ \forall\ c \in \mathbb{R}$$

Remark 2.10. The above operations are well defined, that is, they don't depend on the choice of the class representatives.

Lemma 2.11. *The set of equivalence classes equipped with the above operations determine a vector space. This space is called **quotient space of V in W**, and is denoted by $\mathbf{V}/\mathbf{W}$.*

Proof. Just check the axioms 2.1. We will check one of these axioms.
Commutativity:
$\overline{\mathbf{v}} + \overline{\mathbf{w}} = \overline{\mathbf{v} + \mathbf{w}} = \overline{\mathbf{w} + \mathbf{v}} = \overline{\mathbf{w}} + \overline{\mathbf{v}}$. □

Theorem 2.12. *If $T : V \to W$ is a linear map, then:*

$$V/ker(T) \cong Im(T)$$

Proof. Vide (HOFFMAN; KUNZE, 1971). □

Proposition 2.13. *dim* V/W = *dim* V − *dim* W.

Proof. Let $\beta_W = \{\mathbf{w}_1, \dots, \mathbf{w}_s\}$ be a basis of W. Completing this basis, we have $\beta_V = \{\mathbf{w}_1, \dots, \mathbf{w}_s, \mathbf{u}_1, \dots, \mathbf{u}_t\}$ a basis of V. To complete the proof, we need to prove that $\{\overline{\mathbf{u}}_1, \dots, \overline{\mathbf{u}}_t\}$ is a basis of V/W.
Let $\overline{\mathbf{v}} \in V/W$, then:

$$\overline{\mathbf{v}} = \overline{\alpha_1\mathbf{w}_1 + \cdots + \alpha_s\mathbf{w}_s + \gamma_1\mathbf{u}_1 + \cdots + \gamma_t\mathbf{u}_t} =$$

$$\alpha_1\overline{\mathbf{w}}_1 + \cdots + \alpha_s\overline{\mathbf{w}}_s + \gamma_1\overline{\mathbf{u}}_1 + \cdots + \gamma_t\overline{\mathbf{u}}_t = \gamma_1\overline{\mathbf{u}}_1 + \cdots + \gamma_t\overline{\mathbf{u}}_t$$

Therefore, $\{\overline{\mathbf{u}}_1, \dots, \overline{\mathbf{u}}_t\}$ is a generator set of V/W.

Now, we will show that $\{\overline{\mathbf{u}}_1, \dots, \overline{\mathbf{u}}_t\}$ is linearly independent.

$$\alpha_1\overline{\mathbf{u}}_1 + \cdots + \alpha_t\overline{\mathbf{u}}_t = \overline{\mathbf{0}} = \overline{\alpha_1\mathbf{u}_1 + \cdots + \alpha_t\mathbf{u}_t}$$

$$\Rightarrow \alpha_1\mathbf{u}_1 + \cdots + \alpha_t\mathbf{u}_t = \xi_1\mathbf{w}_1 + \cdots + \xi_s\mathbf{w}_s$$

$$\Rightarrow \alpha_1\mathbf{u}_1 + \cdots + \alpha_t\mathbf{u}_t - \xi_1\mathbf{w}_1 - \cdots - \xi_s\mathbf{w}_s = 0$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_t = \xi_1 = \cdots = \xi_s = 0$$

. □

2.4 Direct sums

First, let's define direct sums of two subspaces.

Definition 2.14. Let W_1, W_2 be subspaces of V. When $W_1 \bigcap W_2 = \{\mathbf{0}\}$, we say that the sum $W = W_1 + W_2$ is **direct**, or W is a direct sum of W_1 and W_2 , and write $W = W_1 \oplus W_2$.

Now, we wish to consider direct sums of several subspaces. For this, we need a concept of independence of subspaces.

Definition 2.15. Let $W_1, \dots, W_k$ subspaces of a vector space V. We say that $W_1, \dots, W_k$ are **independent** if:

$$\mathbf{v}_1 + \cdots + \mathbf{v}_k = \mathbf{0},\ \mathbf{v_i} \in W_i,\ \text{for } i = 1, \dots, k \Rightarrow \mathbf{v_i} = \mathbf{0}, \quad \text{for } i = 1, \dots, k$$

.

Theorem 2.16. *Let V be a vector space over a field F. Let $W_1, \dots, W_k$ be subspaces of V and let $W = W_1 + \cdots + W_k$. The following conditions are equivalent.*

(i) $W_1, \dots, W_k$ *are independent.*

(ii) Each vector $\mathbf{v} \in W$ *is written uniquely as*

$$\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$$

where $\mathbf{v}_i \in W_i$*, for* $i = 1, \ldots, k$*.*

(iii) For each $j, 2 \leq j \leq k$*, we have* $W_j \bigcap (W_1 + \cdots + W_{j-1}) = \{\mathbf{0}\}$*.*

Proof. (i)⇒(ii). Let $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$ and $\mathbf{v} = \mathbf{w}_1 + \cdots + \mathbf{w}_k$, where $\mathbf{v}_j, \mathbf{w}_j \in W_j$, for $j = 1, \ldots, k$, then:

$$\begin{aligned} \mathbf{v}_1 + \cdots + \mathbf{v}_k &= \mathbf{w}_1 + \cdots + \mathbf{w}_k \\ (\mathbf{v}_1 - \mathbf{w}_1) + \cdots + (\mathbf{v}_k - \mathbf{w}_k) &= \mathbf{0} \end{aligned}$$

From (i), we have $(\mathbf{v}_i - \mathbf{w}_i) = \mathbf{0} \Rightarrow \mathbf{v}_i = \mathbf{w}_i$, for $i = 1, \ldots, k$.
(ii)⇒(iii). If $\mathbf{w}_j = \mathbf{w}_1 + \cdots + \mathbf{w}_{j-1}$, with $\mathbf{w}_i \in W_i$, $\forall i$, we have $\mathbf{0} + \cdots + \mathbf{0} + \mathbf{w}_j = \mathbf{w}_1 + \cdots + \mathbf{w}_{j-1} + \mathbf{0}$. Therefore, by (ii), we have that $\mathbf{w}_1 = \cdots = \mathbf{w}_{j-1} = \mathbf{w}_j = \mathbf{0}$.
(iii)⇒(i).Let $\mathbf{w}_1 + \cdots + \mathbf{w}_k = \mathbf{0}$ and consider j the largest integer such that $\mathbf{w}_j \neq \mathbf{0}$. Then:

$$\mathbf{w}_j = -\mathbf{w}_1 - \cdots - \mathbf{w}_{j-1}$$

By (iii), we have $\mathbf{w}_j = \mathbf{0}$, which is a contradiction. □

Definition 2.17. If one (and hence all) of the three conditions of theorem 2.16 hold for $W_1, \ldots, W_k$, we say that the sum $W = W_1 + \cdots + W_k$ is **direct** or W **is the direct sum of** $W_1 \ldots W_k$, and we write $W = W_1 \oplus \cdots \oplus W_k$.

Theorem 2.18. *Let* V *be a vector space over the field* F *and let* $W_1, \ldots, W_k$ *be vector subspaces of* V*. The following two statements are equivalent:*

(i) $V = W_1 \oplus \cdots \oplus W_k$*.*

(ii) If $\mathcal{B}_i$ *is a basis of* W_i*, for* $i = 1, \ldots, k$*, then* $\mathcal{B}$*, such that* $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$*, is a basis of* V*.*

Proof. (i)⇒ (ii). Let $\mathcal{B}_i = \{u_{i1}, \ldots, u_{id_i}\}$ be a basis of W_i, for $i = 1, \ldots, k$. We will show that $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$ is a basis of V.
First, the vectors of $\mathcal{B}$ generate V since any $\mathbf{v} \in V \Rightarrow \mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$, with $\mathbf{v}_i = \sum_{j=1}^{d_i} c_{ij}\mathbf{u}_{ij}$, $\forall i$. Therefore, $\mathbf{v} = \sum_{n=1}^{k}\sum_{j=1}^{d_n} c_{nj}\mathbf{u}_{nj}$
Second, the vectors are L.I. because $\sum_{n=1}^{k}\sum_{j=1}^{d_n} c_{nj}\mathbf{u}_{nj} = \mathbf{0}$, and how sum is direct, we have $\sum_{j=1}^{d_i} c_{ij}\mathbf{u}_{ij} = \mathbf{0}$, for $1 \leq i \leq k \Rightarrow c_{ij} = 0$, for $1 \leq i \leq k$ and $1 \leq j \leq d_i$.

(ii)$\Rightarrow$(i).First, as $\mathcal{B}$ is a basis, your vectors generate V. Then:

$$V = W_1 + \cdots + W_k$$

To prove that the sum is direct, just observe the definition (2.17) and use the item (i) of Theorem 2.16 , because, since the vectors of $\mathcal{B}$ are L.I, we have as consequence that:

$$\sum_{i=1}^{k} \mathbf{v_i} = \mathbf{0} \Rightarrow \mathbf{v_i} = \mathbf{0},\ i = 1, \ldots, k$$

□

Corollary 2.19. *If $V = W_1 \oplus \cdots \oplus W_k$, then dim V= dim W_1+... + dim W_k.*

Proof. It is a direct consequence of Theorem 2.18, item(ii) . □

2.5 Orthogonal Complement

Definition 2.20. Consider a vector space V equipped with an inner product $<,>$ and consider a non-empty subset S of V. Then:

$$S^{\perp_{\text{compl}}} := \{\mathbf{v} \in V \mid \mathbf{v} \text{ is orthogonal to all vectors of } S\} \tag{2.4}$$

Lemma 2.21. *$S^{\perp_{\text{compl}}}$ is a vector subspace of V, even if S is not a subspace of V.*

Proof. Just check the conditions of definition (2.2). □

Proposition 2.22. *If S is a vector subspace of V, then:*

$$V = S \oplus S^{\perp_{\text{compl}}} \tag{2.5}$$

*and $S^{\perp_{\text{compl}}}$ is called **orthogonal complement** of S.*

Proof. The sum is direct since $S \cap S^{\perp_{\text{compl}}} = \mathbf{0}$. In fact, let $\mathbf{v} \in S \cap S^{\perp_{\text{compl}}}$. So $< \mathbf{v}, \mathbf{v} >= 0 \Rightarrow \mathbf{v} = \mathbf{0}$. Now let $\mathcal{B}_S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be an orthogonal basis of S. Completing this basis and using the Gram-Schmidt method, we find $\mathcal{B}_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{u}_1, \ldots, \mathbf{u_s}\}$ an orthogonal basis of V. To conclude the proof, we show that $\mathcal{B}_{S^{\perp_{\text{compl}}}} = \{\mathbf{u}_1, \ldots, \mathbf{u}_s\}$ is a basis of $S^{\perp_{\text{compl}}}$. In fact, let $\mathbf{w} \in S^{\perp_{\text{compl}}}$, then $< \mathbf{w}, \mathbf{v}_i >= 0$ to $i = 1, \ldots, r$. Then:

$$\mathbf{w} = 0.\mathbf{v}_1 + \cdots + 0.\mathbf{v}_r + \alpha_1.\mathbf{u}_1 + \cdots + \alpha_s.\mathbf{u}_s \tag{2.6}$$

With (2.6), we prove that $\mathcal{B}_{S^{\perp_{\text{compl}}}}$ is a generating set. That is a linearly independent set follows immediately, since $\mathcal{B}_{S^{\perp_{\text{compl}}}} \subset \mathcal{B}_V$. □

2.6 Projections

Definition 2.23. Let V be a vector space and $P : V \to V$ a linear operator. P is a **projection** of V onto $Im\ P$ if $P \circ P = P$.

Definition 2.24. Let V be a vector space with inner product. The projection $P : V \to V$ of V on W is called **orthogonal** if the kernel of P is the orthogonal complement of W, i.e., $ker\ P = W^{\perp_{\text{compl}}}$.

Proposition 2.25. *The orthogonal projection $P : V \to V$ of V on W exists and is unique.*

Proof. Let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be an orthogonal basis of the subspace W. Knowing that $V = W \oplus W^{\perp_{\text{compl}}}$, then:

$$P(\mathbf{v}) = \sum_{i=1}^{k} \frac{1}{\|\mathbf{w}_i\|^2} < \mathbf{v}, \mathbf{w}_i > \mathbf{w}_i, \quad \forall\ \mathbf{v} \in V. \tag{2.7}$$

□

2.7 Linear Functionals

2.7.1 Dual Vector Space

Definition 2.26. If V is a vector space over a field $\mathbb{R}$, a **linear functional** on V is a linear transformation of V in $\mathbb{R}$.

Definition 2.27. The space of all linear functionals of V will be denoted by V^* and called the **dual space of** V.

Proposition 2.28. *The set V^*, with the following operations*

$$(\alpha + \beta)(\mathbf{v}) := \alpha(\mathbf{v}) + \beta(\mathbf{v}),\ \forall\ \mathbf{v} \in V$$

$$(c.\alpha)(\mathbf{v}) := c\alpha(\mathbf{v}),\ \forall c \in \mathbb{R},\ \forall\ \mathbf{v} \in V$$

with $\alpha, \beta \in V^*$, is a vector space over $\mathbb{R}$.

Proof. Just check the axioms in definition (2.1) that define a vector space. Let's check only one, and leave the rest as an exercise.

- Additive inverse element: $\forall\ \alpha \in V^*, \exists - \alpha \in V^*$ such that

$$(\alpha - \alpha)(\mathbf{v}) = \alpha(\mathbf{v}) - \alpha(\mathbf{v}) = 0 = -\alpha(\mathbf{v}) + \alpha(\mathbf{v}) = (-\alpha + \alpha)(\mathbf{v}),\ \forall\ \mathbf{v} \in V.$$

□

Definition 2.29. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of the vector space V. The linear functional $\mathbf{v}^i : V \to \mathbb{R}$ is defined as follows:

$$\mathbf{v}^i(\mathbf{v}_j) = \begin{cases} 1 \quad , \quad \text{if } i = j, \\ \\ 0 \quad , \quad \text{if } i \neq j. \end{cases}$$

Theorem 2.30. *If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of the vector space V, then $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ is a basis of the dual vector space V^*.*

Proof. Let us first show that $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ is a generator set of V^*. Let $f \in V^*$. Then we have:

$$\begin{cases} f(\mathbf{v}_1) & = & \kappa_1 \\ & \vdots & \\ f(\mathbf{v}_n) & = & \kappa_n \end{cases}$$

$\forall\ \mathbf{v} \in V,\ \exists\ \alpha_i \in \mathbb{R}$ such that $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$. Then, we have:

$$f(\mathbf{v}) = f\left(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i\right) = \sum_{i=1}^{n} \alpha_i f(\mathbf{v}_i).$$

Therefore:

$$f(\mathbf{v}) = \alpha_1\kappa_1 + \cdots + \alpha_n\kappa_n. \tag{2.8}$$

We claim that:

$$f = \kappa_1\mathbf{v}^1 + \cdots + \kappa_n\mathbf{v}^n. \tag{2.9}$$

Indeed: $(\kappa_1\mathbf{v}^1 + \cdots + \kappa_n\mathbf{v}^n)(\mathbf{v}) = \kappa_1\mathbf{v}^1(\mathbf{v}) + \cdots + \kappa_n\mathbf{v}^n(\mathbf{v}) = \alpha_1\kappa_1 + \cdots + \alpha_n\kappa_n$.
Then by (2.8), we have that the identity (2.9) is true. Now, it suffices to show that $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ is a linearly independent set. Indeed, from:

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}^i = \mathbf{0}$$

we have:

$$(\sum_{i=1}^{n} \alpha_i \mathbf{v}^i)(\mathbf{v}_j) = \alpha_j = \mathbf{0}(\mathbf{v}_j) = 0, \text{ for } j = 1, \ldots, n.$$

□

Corollary 2.31. *dim* V = *dim* V^*

Proof. Direct consequence. □

2.7.2 New Notation

We now introduce a new notation that will appear somewhat strange at first, but will prove suggestive when the transition from the discrete case to the continuous case is made, using, for example, the Stokes' theorem. In this work, we will only deal with the discrete case, nevertheless we introduce this notation, to ensure a future perspective.

Let $\mathbf{v} \in V$ such that its coordinates in some basis $\mathcal{B}_V$ is $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and let $\alpha \in V^*$ such that its coordinates in the dual basis $\mathcal{B}_{V^*}$ is $\alpha = (\alpha^1, \dots, \alpha^n)$. Then:

$$\alpha(\mathbf{v}) = \int_{\mathbf{v}} \alpha = \alpha^1 \mathbf{v}_1 + \cdots + \alpha^n \mathbf{v}_n. \tag{2.10}$$

Remark 2.32. Obviously in the equation (2.10) it doesn't appear an integral as usual. It's just a way of represent the calculation of the linear functional on its associated vector space.

2.8 Annihilator

Definition 2.33. Let W be a subspace of V. The **annihilator** of W is a subset $W^\perp$ of V^* formed by linear functions $\alpha \in V^*$ such that

$$\alpha(\mathbf{w}) = 0, \ \forall \ \mathbf{w} \in W$$

Lemma 2.34. *$W^\perp$ is a vector subspace of V^*.*

Proof. Let us verify the conditions of definition (2.2).

(i) $W^\perp \neq \emptyset$ because $\mathbf{0} \in W^\perp$(zero function).

(ii) $\alpha, \beta \in W^\perp$. Then

$$(\alpha + \beta)(\mathbf{w}) = \alpha(\mathbf{w}) + \beta(\mathbf{w}) = 0, \ \forall \ \mathbf{w} \in W$$

Therefore $\alpha + \beta \in W^\perp$.

(iii) $c \in \mathbb{R}, \ \alpha \in W^\perp$ Then:

$$(c.\alpha)(\mathbf{w}) = c.\alpha(\mathbf{w}) = c.0 = 0, \ \forall \ \mathbf{w} \in W$$

Therefore $c.\alpha \in W^\perp$.

□

Proposition 2.35. *$dim\ V = dim\ W + dim\ W^\perp$.*

Proof. Let $\beta_W = \{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be a basis of W. Completing this base, we have $\beta_V = \{\mathbf{w}_1, \dots, \mathbf{w}_r, \mathbf{u}_1, \dots, \mathbf{u}_s\}$ is a basis of V. By theorem 2.30, we have $\beta_{V^*} = \{\mathbf{w}^1, \dots, \mathbf{w}^r, \mathbf{u}^1, \dots, \mathbf{u}^s\}$ is a basis of V^*.To conclude the proof, just show that the set $\beta_{W^\perp} = \{\mathbf{u}^1, \dots, \mathbf{u}^s\}$ form a basis of the annihilator $W^\perp$.

First, we will check that the set $\beta_{W^\perp}$ is a generator set of $W^\perp$. Let $f \in W^\perp$, then $f(\mathbf{w}_i) = 0$, to $\forall\ \mathbf{w}_i \in \beta_W$, i.e., f nullifies all elements of basis of W. Therefore:

$$f = 0.\mathbf{w}^1 + \cdots + 0.\mathbf{w}^r + \alpha_1.\mathbf{u}^1 + \cdots + \alpha_s.\mathbf{u}^s. \tag{2.11}$$

The verification that the set is linearly independent is immediate, since $\beta_{W^\perp} \subset \beta_{V^*}$. □

2.9 Adjoint transformation

2.9.1 A and A^*

Definition 2.36. Let V and W be vector spaces, and let $A : V \to W$ be a linear transformation. We define the **adjoint** of A as being the function $A^* : W^* \to V^*$ such that:

$$A^*(\alpha) = \alpha \circ A, \quad \forall\ \alpha \in W^* \tag{2.12}$$

Lemma 2.37. *The adjoint $A^* : W^* \to V^*$ defined above is a linear transformation.*

Proof. For $\beta_1, \beta_2 \in W^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, we have:

$$A^*(\alpha_1\beta_1 + \alpha_2\beta_2) = (\alpha_1\beta_1 + \alpha_2\beta_2) \circ A = \alpha_1(\beta_1 \circ A) + \alpha_2(\beta_2 \circ A) = \alpha_1 A^*(\beta_1) + \alpha_2 A^*(\beta_2).$$

□

Theorem 2.38. *The matrices of $A : V \to W$ and $A^* : W^* \to V^*$ (with respect to a certain basis of V and W, and the corresponding dual bases of V and W) are the transpose of each other.*

Proof. Let $\beta_V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, $\beta_W = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ basis of V and W , respectively, and let $\beta_{V^*} = \{\mathbf{v}^1, \dots, \mathbf{v}^m\}$, $\beta_{W^*} = \{\mathbf{w}^1, \dots, \mathbf{w}^n\}$ be the corresponding dual basis of V^* and W^*. We have:

$$A(\mathbf{v}_j) = \sum_{i=1}^{n} a_{ij}\mathbf{w}_i, \quad \forall\ j = 1, \dots, m \tag{2.13}$$

The scalars a_{ij} are the entries of the matrix A.
Note that:

$$a_{ij} = \mathbf{w}^i(A(\mathbf{v}_j)) \tag{2.14}$$

Let us now consider the matrix's representation of A^*:

$$A^*(\mathbf{w}^l) = \sum_{k=1}^{m} b_{kl}\mathbf{v}^k, \quad \forall l = 1, \dots, n \tag{2.15}$$

The scalars b_{kl} represent the entries of the matrix A^*.
Applying $A^*(\mathbf{w}^l)$ in the vector $\mathbf{v}_j$, , and using (2.15), (2.12) and (2.14), we find:

$$A^*(\mathbf{w}^l)(\mathbf{v}_j) = b_{jl} = \mathbf{w}^l(A(\mathbf{v}_j)) = a_{lj}$$

Therefore:

$$b_{jl} = a_{lj}. \tag{2.16}$$

□

2.9.2 Maps i and i^*

Consider W a vector subspace of the vector space V.

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, leftrightarrow, "\simeq"'] & V \arrow[d, leftrightarrow, "\simeq"] \\ W^* \cong V^*/W^{\perp} & V^* \arrow[l, "i^*"'] \end{tikzcd}$$

i and i^* are adjoint maps.

Definition 2.39. For every $\mathbf{f} \in V^*$ and for all $\mathbf{v} \in V$, we have:

$$(\mathbf{f} + W^{\perp})(\mathbf{v}) := \{\mathbf{f}(\mathbf{v}) + \mathbf{g}(\mathbf{v}) \mid \mathbf{g} \in W^{\perp}\}$$

Proposition 2.40. *Let* $\begin{array}{rcl} i : W & \to & V \\ \mathbf{w} & \mapsto & \mathbf{w} \end{array}$ *be the inclusion map and* $\begin{array}{rcl} \boldsymbol{\pi} : V^* & \to & V^*/W^{\perp} \\ \mathbf{v} & \mapsto & \overline{\mathbf{v}} \end{array}$ *the projection map. Then* i *and* $\boldsymbol{\pi}$ *are adjoint maps, i.e.,* $i^* = \boldsymbol{\pi}$*.*

Proof. $\forall\ \boldsymbol{\Psi} \in V^*$ and $\forall\ \mathbf{w} \in W$, by definition 2.39 we have:

$$\boldsymbol{\Psi} \circ i(\mathbf{w}) = \boldsymbol{\Psi}(\mathbf{w}) = (\boldsymbol{\Psi} + W^{\perp})(\mathbf{w}) = \boldsymbol{\pi}(\boldsymbol{\Psi})(\mathbf{w})$$

Therefore, we have:

$$\boldsymbol{\Psi} \circ i = \boldsymbol{\pi}(\boldsymbol{\Psi}).$$

□

Proposition 2.41. Let $\begin{array}{rcl} i : W & \to & V \\ \mathbf{w} & \mapsto & \mathbf{w} \end{array}$ the inclusion map and $i^* : V^* \to V^*/W^{\perp}$ its adjoint transformation. Then i^* is the projection map.

Proof. For $\forall\ w \in W$ and $\forall\ \alpha \in V^*$, we have:

$$i^*(\alpha)(\mathbf{w}) = \alpha \circ i(\mathbf{w}) = \alpha(\mathbf{w})$$

On the other hand, by definition 2.39, we have:

$$i^*(\alpha)(\mathbf{w}) = (\beta + W^\perp)(\mathbf{w}) = \beta(\mathbf{w}),\ \text{for some } \beta \in V^*.$$

Therefore $\alpha(\mathbf{w}) = \beta(\mathbf{w}),\ \forall \mathbf{w} \in W \Rightarrow \beta - \alpha \in W^\perp \Rightarrow \beta = \alpha + W^\perp$. □

2.10 Some Important Results

Proposition 2.42. $W^\perp \equiv (V/W)^*$.

Proof. Just build an isomorphism between $W^\perp$ and V/W^*. For this, consider the following bases:

$$\begin{aligned}
\beta_W &= \{\mathbf{w}_1, \ldots, \mathbf{w}_s\} && (2.17)\\
\beta_V &= \{\mathbf{w}_1, \ldots, \mathbf{w}_s, \mathbf{u}_1, \ldots, \mathbf{u}_t\} && (2.18)\\
\beta_{V/W} &= \{\overline{\mathbf{u}}_1, \ldots, \overline{\mathbf{u}}_t\} && (2.19)\\
\beta_{W^*} &= \{\mathbf{w}^1, \ldots, \mathbf{w}^s\} && (2.20)\\
\beta_{V^*} &= \{\mathbf{w}^1, \ldots, \mathbf{w}^s, \mathbf{u}^1, \ldots, \mathbf{u}^t\} && (2.21)\\
\beta_{(V/W)^*} &= \{\overline{\mathbf{u}}^1, \ldots, \overline{\mathbf{u}}^t\} && (2.22)\\
\beta_{W^\perp} &= \{\mathbf{u}^1, \ldots, \mathbf{u}^t\} && (2.23)\\
\beta_{V^*/W^\perp} &= \{\overline{\mathbf{w}}^1, \ldots, \overline{\mathbf{w}}^s\} && (2.24)
\end{aligned}$$

Now, define the linear transformation:

$$\begin{aligned}
\psi : W^\perp &\rightarrow (V/W)^*\\
\mathbf{u}^j &\mapsto \overline{\mathbf{u}}^j
\end{aligned}$$

This linear transformation takes the basis (2.23) to the basis (2.22) . Moreover, by corollary 2.31, by proposition 2.13 and by proposition 2.35, we have dim $W^\perp$=dim $(V/W)^*$. Therefore, ψ is an isomorphism. □

Proposition 2.43. $V^*/W^\perp \equiv W^*$.

Proof. Similarly, define the linear transformation:

$$\begin{aligned}
\psi : W^* &\rightarrow (V^*/W^\perp)\\
\mathbf{w}^j &\mapsto \overline{\mathbf{w}}^j
\end{aligned}$$

This linear transformation takes the basis (2.20) on basis (2.24). Moreover, by corollary 2.31, by proposition 2.13 and by proposition 2.35, we have dim W^*=dim $V^*/W^\perp$. Therefore, ψ is an isomorphism. □

Proposition 2.44. *Let $A^* : W^* \to V^*$ be an adjoint transformation of $A : V \to W$. Then the following equalities are valid:*

$$Im(A)^{\perp} = ker(A^*), \tag{2.25}$$

$$ker(A)^{\perp} = Im(A^*). \tag{2.26}$$

In particular, if A is injective, then A^* is surjective, and conversely, if A^* is injective, A is surjective.

Proof. Let us prove the equality (2.25).

If $\beta \in ker(A^*)$, we have:

$$\beta \circ A(\mathbf{v}) = A^*(\beta)(\mathbf{v}) = \mathbf{0}(\mathbf{v}) = 0, \quad \forall\ \mathbf{v} \in V$$

So $ker(A^*) \subset Im(A)^{\perp}$. Now consider $\alpha \in Im(A)^{\perp}$, then:

$$\alpha(A(\mathbf{v})) = 0, \quad \forall\ \mathbf{v} \in V \Rightarrow (A^*\alpha)(\mathbf{v}) = 0, \quad \forall\ \mathbf{v} \in V \Rightarrow A^*\alpha = \mathbf{0} \Rightarrow \alpha \in ker(A^*).$$

Therefore $Im(A)^{\perp} \subset ker(A^*)$
Let us now prove the equality (2.26).

Let $\beta \in Im(A^*)$, then $\exists\ \alpha \in W^*$ such that $\beta = A^*(\alpha)$. Therefore, $\forall\ \mathbf{v} \in ker(A)$, we have:

$$\beta(\mathbf{v}) = A^*(\alpha)(\mathbf{v}) = \alpha(A(\mathbf{v})) = \alpha(\mathbf{0}) = 0 \Rightarrow \beta \in ker(A)^{\perp}$$

So $Im(A^*) \subset ker(A)^{\perp}$.

We will now show the inverse inclusion. Let $\mathcal{B}_{kerA} = \{\mathbf{u}_1, \dots, \mathbf{u}_r\}$. . Completing this basis, we have $\mathcal{B}_V = \{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{v}_1, \dots, \mathbf{v}_s\}$. For $f \in ker(A)^{\perp}$ and $\forall\ \alpha \in W^*$, we have:

$$(A^*\alpha)(\mathbf{u}_j) = \alpha(A(\mathbf{u}_j)) = \alpha(\mathbf{0}) = 0 = f(\mathbf{u}_j), \quad \forall\ \mathbf{u}_j \in \mathcal{B}_{ker\ A} \tag{2.27}$$

As a consequence of Theorem 2.2, we have that $\{A(\mathbf{v}_1), \dots, A(\mathbf{v}_s)\}$ is a L.I. set. So choosing $\gamma \in W^*$ such that

$$\gamma(A(\mathbf{v}_i)) = f(\mathbf{v}_i), \quad i = 1, \dots, s \tag{2.28}$$

we have, for (2.27) and (2.28), that :

$$A^*(\gamma) \equiv f.$$

Therefore, $ker(A)^{\perp} \subset Im(A^*)$. □

3 ALGEBRAIC TOPOLOGY IN THE ANALYSIS OF ELECTRICAL CIRCUITS

Introduction

In this chapter, we present an unorthodox approach to the analysis of electrical circuits using tools from algebraic topology and graph theory . An electrical circuit is seen as a graph as well as a one-dimensional complex, where the domain of the boundary operator is the vector space generated by the branches (wires of the circuit) and its codomain is the vector space generated by the nodes. An electric circuit has homology and cohomology groups. Kirchhoff 's laws have a concise and elegant formulation through the boundary and coboundary maps. The methods of mesh-current and node-potential are also discussed, as well as a conclusive analysis of the existence and uniqueness of solutions for the circuit equations. This approach allows us to analyze electrical circuits by making effective considerations about its shape.

3.1 Elements of Graph Theory

Definition 3.1. A **branch** is a line segment or oriented arc. The ends of the branch are called **nodes**.

Definition 3.2. A **path** is a sequence of nodes and branches such that the end node of a branch is the starting node of the next branch.

Definition 3.3. A path is **closed** if the starting node of its first branch coincide with the ending node of its last branch.

Definition 3.4. A path is **simple** if all elements of the succession of nodes and branches are distinct, that is, they are "covered" only once (except the first and last nodes of a closed path).

Definition 3.5. A **mesh** is a simple closed path.

Definition 3.6. The collection formed by branches and nodes is called a **one-dimensional complex**.

We will treat the circuits as graphs, as we see in figure 1.

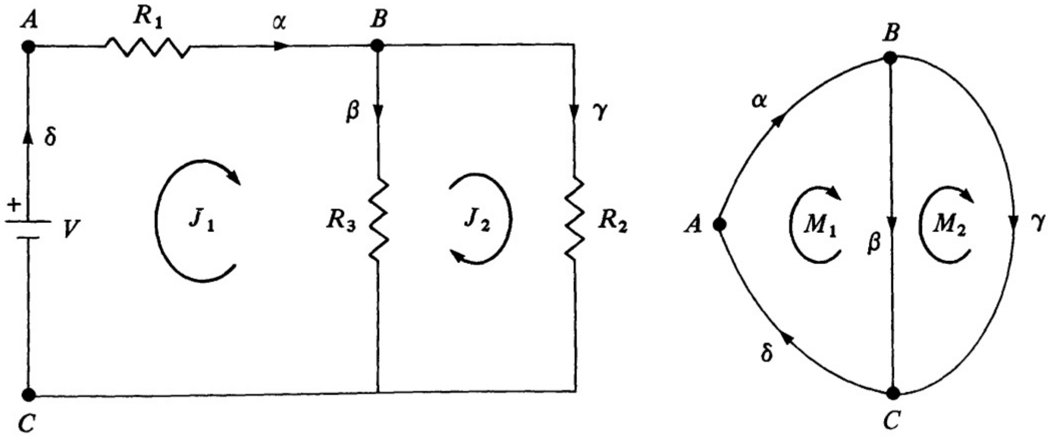


Figure 1 – Real circuit versus Topological circuit.

3.2 Structure of Vector Space

Consider a circuit with the following set of branches $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_n\}$. We represent each branch as follows:

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (1, 0, \dots, 0), \\ \boldsymbol{\alpha}_2 &= (0, 1, \dots, 0), \\ &\vdots \\ \boldsymbol{\alpha}_n &= (0, 0, \dots, 1). \end{aligned}$$

With this representation, we can generate a vector space whose canonical basis are the branches of the circuit. This vector space will be denoted by C_1.

Definition 3.7. The vector space C_1 is called **space of the one-chains**.

Remark 3.8. As a consequence of the above, we have dim C_1 = number of branches of the circuit.

Likewise, consider the set $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_M\}$ of nodes of the circuit. We represent each node as follows:

$$\begin{aligned} \mathbf{A}_1 &= (1, 0, \dots, 0), \\ \mathbf{A}_2 &= (0, 1, \dots, 0), \\ &\vdots \\ \mathbf{A}_m &= (0, 0, \dots, 1). \end{aligned}$$

With this representation, we can generate a vector space whose canonical basis are the nodes of the circuit. This vector space will be denoted by C_0.

Definition 3.9. The vector space C_0 is called **space of the zero-chains**.

Remark 3.10. As a consequence of the above, we have dim C_0 = number of nodes in the circuit.

Example 3.11. Consider the circuit of Figure 2. From the above, we have:

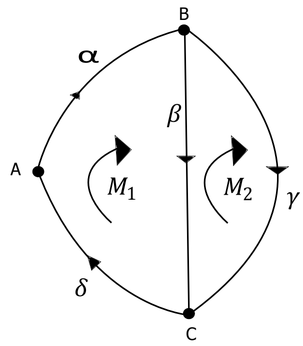


Figure 2 – Circuit's graph.

$\boldsymbol{\alpha} = (1,0,0,0), \boldsymbol{\beta} = (0,1,0,0), \boldsymbol{\gamma} = (0,0,1,0), \boldsymbol{\delta} = (0,0,0,1)$.
The set $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$ forms a basis of C_1.
dim $C_1 = 4$.

$\mathbf{A} = (1,0,0), \mathbf{B} = (0,1,0), \mathbf{C} = (0,0,1)$
The set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ forms a basis of C_0
dim $C_0 = 3$

In this new approach, we may, for example, represent the current vector I $\in C_1$ and the potential vector $\boldsymbol{\phi} \in C_0$ as follows:
$\mathbf{I} = (I_\alpha, I_\beta, I_\gamma, I_\delta) = I_\alpha\boldsymbol{\alpha} + I_\beta\boldsymbol{\beta} + I_\gamma\boldsymbol{\gamma} + I_\delta\boldsymbol{\delta}$,
$\boldsymbol{\phi} = (\phi_A, \phi_B, \phi_C) = \phi_A\ \mathbf{A} + \phi_B\ \mathbf{B} + \phi_C\ \mathbf{C}$.
We can also represent the meshes M_1 and M_2 as follows:
$\mathbf{M_1} = (1,1,0,1) = 1\boldsymbol{\alpha} + 1\boldsymbol{\beta} + 0\boldsymbol{\gamma} + 1\boldsymbol{\delta}$,
$\mathbf{M_2} = (0,-1,1,0) = 0\boldsymbol{\alpha} - 1\boldsymbol{\beta} + 1\boldsymbol{\gamma} + 0\boldsymbol{\delta}$.

Remark 3.12. Note that meshes can only have as coordinates ±1 or zero. This is a consequence of the fact that the meshes are simple paths.

3.3 Boundary and Coboundary Maps

Definition 3.13. We define the **boundary map** $\partial : C_1 \longrightarrow C_0$ as the linear transformation such that:

$$\partial\kappa = B - A. \tag{3.1}$$

where κ is any branch of the circuit and A and B are, respectively, the start node and end node of this branch.

Generalizing the concept for any $\mathbf{K} = (\kappa_\alpha, \kappa_\beta, \kappa_\gamma, \dots) \in C_1$, we have:

$$\partial\mathbf{K} = \mathbf{L},$$

where $\mathbf{L} = (L_A, L_B, L_C, \dots) \in C_0$, where, for example, L_A is equal to:

$$L_A = \underbrace{(\kappa_{\delta_1} + \cdots + \kappa_{\delta_l})}_{\text{go to A}} - \underbrace{(\kappa_{\varepsilon_1} + \cdots + \kappa_{\varepsilon_t})}_{\text{leave A}} \tag{3.2}$$

To show how compute (3.2) from (3.1), consider the following sets:

$$\{\delta_1, \dots, \delta_l\}, \quad \text{set of all branches that have A as a final node.} \tag{3.3}$$

$$\{\epsilon_1, \dots, \epsilon_t\}, \quad \text{set of all branches that have A as a initial node.} \tag{3.4}$$

Then:

$$\partial(\kappa_{\delta_1}\delta_1 + \cdots + \kappa_{\delta_l}\delta_l + \kappa_{\epsilon_1}\epsilon_1 + \cdots + \kappa_{\epsilon_t}\epsilon_t) = \sum_{i=1}^{l} \kappa_{\delta_i}\partial(\delta_i) + \sum_{j=1}^{t} \kappa_{\epsilon_j}\partial(\epsilon_j).$$

And, from (3.1), (3.3) and (3.4), we have:

$$\sum_{i=1}^{l} \kappa_{\delta_i}\partial(\delta_i) + \sum_{j=1}^{t} \kappa_{\epsilon_j}\partial(\epsilon_j) = \sum_{i=1}^{l} \kappa_{\delta_i}(A - B_{\delta_i}) + \sum_{j=1}^{t} \kappa_{\epsilon_j}(B_{\epsilon_j} - A) \tag{3.5}$$

where B_{δ_i} is the inicial node of the branch δ_i and B_{ϵ_j} is the final node of the branch ϵ_j.

Then, isolating the node A in (3.5), we find (3.2).

Example 3.14. Calculating L_A in the following circuit's node, we have:

$$L_A = \kappa_\rho - \kappa_\alpha - \kappa_\eta.$$

Definition 3.15. We denote by Z_1 the **kernel of the boundary map** ∂. The elements of Z_1 are called **cycles**.

$$\mathbf{K} \in Z_1 \Rightarrow \partial \mathbf{K} = \mathbf{0},$$
$$Z_1 \subset C_1.$$

Example 3.16. Every mesh is a cycle, but not every cycle is a mesh. This example emphasizes this fact.

We have that $\mathbf{M_1} = (1, 1, 0, 1)$ and $\mathbf{M_2} = (0, -1, 1, 0)$

- $\mathbf{M_1}, \mathbf{M_2}, \mathbf{I} \in Z_1$, that is, are cycles.
- $\mathbf{M_1}$ and $\mathbf{M_2}$ are meshes, however $\mathbf{I}$ isn't a mesh.

Definition 3.17. We denote by B_0 the image of the boundary map ∂. We call B_0 the **space of boundaries**: $\partial C_1 \equiv B_0 \subset C_0$.

Definition 3.18. We denote by C^1 the dual space of the one-chains, also called **one-cochains**.

Definition 3.19. We denote by C^0 the dual space of the zero-chains, also called **zero-cochains**.

Definition 3.20. The linear transformation $\mathrm{d} : C^0 \to C^1$ is called **coboundary map**.The map d is, by definition, adjoint to the map ∂ and, therefore, $\forall \phi \in C^0$, we have:

$$\mathrm{d}\phi = \phi\partial.$$

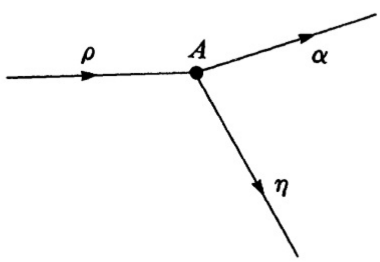
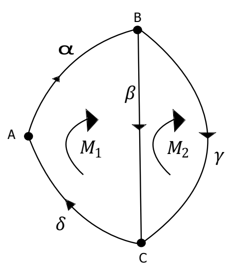
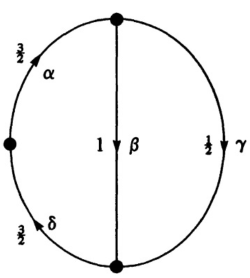


Figure 3 – Node A of the circuit





$\partial\mathbf{M_1} = \mathbf{0}, \partial\mathbf{M_2} = \mathbf{0}$ $\qquad$ $\partial\mathbf{I} = \mathbf{0}$, **onde** $\mathbf{I} = (\frac{3}{2}, 1, \frac{1}{2}, \frac{3}{2})$

Remark 3.21. Let $\boldsymbol{\phi} \in C^0$ and $\mathbf{K} \in C_1$. Using the notation (2.10), we have:

$$\int_{\mathbf{K}} \mathrm{d}\boldsymbol{\phi} = \int_{\partial \mathbf{K}} \boldsymbol{\phi}. \tag{3.6}$$

Definition 3.22. We denote by Z^0 the **kernel of the coboundary map** d and we denote by B^1 the **space of coboundaries**, which is the image of d.

Remark 3.23. Soon we will give a physical meaning to the subspace Z^0 and B^1.

Proposition 3.24. *Z^0 is the annihilator of B_0.*

Proof. Follows immediately from proposition (2.44). □

Proposition 3.25. *B^1 is the annihilator of Z_1.*

Proof. Follows immediately from proposition 2.44. □

3.4 Homology and Cohomology Groups

Definition 3.26. The **homology group** H_0 is defined by: $H_0 = Z_0/B_0(= C_0/B_0)$.

Proposition 3.27. *The subspace Z^0 is isomorphic to the dual of the homology group H_0.*

Proof. Follows immediately from propositions (2.42) and (3.24). □

Definition 3.28. A topological space is called **path-connected** if for any point we get a path connecting it to any other point of the space.

Definition 3.29. A **connected component** of a topological space is a maximal path-connected subspace.

Lemma 3.30. *The dimension of the homology group H_0 is equal to the number of connected components of the circuit.*

Proof. Let $\mathbf{A}$ and $\mathbf{D}$ be any two nodes of a connected component of the circuit. Thus there exists a path, represented by the 1-chain $\mathbf{P} \in C_1$, such that $\partial \mathbf{P} = \mathbf{D} - \mathbf{A}$. Then $\mathbf{D} - \mathbf{A} \in B_0$ for any two nodes $\mathbf{A}$ and $\mathbf{D}$ of the connected component.

Let $\mathbf{L} \in C_0$ be a 0-chain. Then we have:

$$\mathbf{L} = \sum_{N=A,B,C,\ldots} L_N \mathbf{N} = (L_A, L_B, \ldots).$$

We can rewrite the above equation as follows:

$$\mathbf{L} = \sum L_N[\mathbf{A} + (\mathbf{N} - \mathbf{A})] = \sum L_N \mathbf{A} + \sum_{N \neq A} L_N(\mathbf{N} - \mathbf{A}).$$

If we are in a connected component, then $\mathbf{N}-\mathbf{A}\in B_0$, for $N=A,B,\dots$
Then we have $\mathbf{L}-(\sum L_N)\mathbf{A}=\sum_{N\neq A}L_N(\mathbf{N}-\mathbf{A})\Rightarrow \mathbf{L}-(\sum L_N)\mathbf{A}\in B_0$. Therefore, $\mathbf{L}$ and $(\sum L_N)\mathbf{A}$ belong to the same equivalent class of the quotient space $H_0=C_0/B_0$. Consequently, for a connected component, every 0-chain $\mathbf{L}$ is in the same equivalence class of some multiple of $\mathbf{A}$, that is, $\overline{\mathbf{L}}=\alpha\overline{\mathbf{A}}$. Then $\overline{\mathbf{A}}$ generates all classes of the quotient space. Therefore, for a connected component, $\dim H_0=1$.

Remark 3.31. Since $\mathbf{A}$ doesn't belong to the subspace B_0 ($\mathbf{A}\notin\partial C_1$), we have $\overline{\mathbf{A}}\neq\overline{\mathbf{0}}$.

Now we generalize the result to several connected components. Consider a general complex. Let $\mathbf{A}$ be a node of this complex. Let us use the following procedure. Consider all branches that have $\mathbf{A}$ as a boundary point. Now join all other nodes that are in the opposite border of these branches and repeat the argument in relation to these new nodes, thereby generating new branches and nodes. Keep repeating this procedure a finite number of times, until all the connected component is covered.

Let $S_0(\mathbf{A})$ be the set formed by the node $\mathbf{A}$ and all nodes found in the previous paragraph. Now let $S_1(\mathbf{A})$ be the set of all branches involved in the previous procedure. If a node $\mathbf{B}\notin S_0(\mathbf{A})$, this implies that there is no path connecting the nodes $\mathbf{A}$ and $\mathbf{B}$. Therefore the node $\mathbf{B}$ belongs to another connected component of the complex in question. Using the method shown above, we also will find the sets $S_0(\mathbf{B})$ and $S_1(\mathbf{B})$, and so on. Therefore, following this algorithm, we find the disjoint sets $S_0(\mathbf{A}),S_0(\mathbf{B}),\dots$ and the disjoint sets $S_1(\mathbf{A}),S_1(\mathbf{B}),\dots$.

The set of branches and nodes as canonical bases of C_1 and C_0, respectively. Therefore, we have:

$$C_1=C_1(\mathbf{A})\oplus C_1(\mathbf{B})\oplus\dots$$

$$C_0=C_0(\mathbf{A})\oplus C_0(\mathbf{B})\oplus\dots$$

such that $\partial C_1(\mathbf{A})=B_0(\mathbf{A})\subset C_0(\mathbf{A}),\partial C_1(\mathbf{B})=B_0(\mathbf{B})\subset C_0(\mathbf{B}),\dots$. With this, we have:

$$H_0=\frac{C_0}{B_0}=\frac{C_0(\mathbf{A})\oplus C_0(\mathbf{B})\oplus\dots}{B_0(\mathbf{A})\oplus B_0(\mathbf{B})\oplus\dots}$$

$$H_0=(C_0(\mathbf{A})\oplus C_0(\mathbf{B})\oplus\dots)+(B_0(\mathbf{A})\oplus B_0(\mathbf{B})\oplus\dots)$$

$$H_0=(C_0(\mathbf{A})+B_0(\mathbf{A}))\oplus(C_0(\mathbf{B})+B_0(\mathbf{B}))\oplus\dots$$

Therefore:

$$H_0=H_0(\mathbf{A})\oplus H_0(\mathbf{B})\oplus\dots$$

So, by corollary 2.19, we have:

$$\dim\, H_0 = \dim\, H_0(\mathbf{A}) + \dim\, H_0(\mathbf{B}) + \ldots$$

Thus dim H_0 = number of connected components. □

The next theorem will need the following definitions

Definition 3.32. A **tree** is a connected complex without meshes.

Definition 3.33. Let S be a connected complex. A **maximal tree of S** is a tree that contains all nodes of S.

Definition 3.34. We say that the meshes $\mathbf{M}_1, \ldots, \mathbf{M}_k$ are linearly independent if and only if:

$$\sum_{i=1}^{k} \alpha_i \mathbf{M}_i = \mathbf{0} \Rightarrow \alpha_i = 0,\ \forall i.$$

Theorem 3.35. *There is a basis of the subspace Z_1 that consists only of meshes.*

Proof. Initially, we will analyze the simplest case, in which the complex is a tree. First, observe that in any tree there is a node belonging to only one branch (one extremity of the tree). Starting from this node, and traversing the tree, you realize that every new branch is covered accompanied by a new node. Therefore, we find the following relation for any tree:

$$n = r + 1 \tag{3.7}$$

where n = number of nodes and r = number of branches.
As we are in a connected component, by lemma 3.30, we have:

$$\dim\, H_0 = 1 \tag{3.8}$$

Then dim C_0 - dim B_0 = 1. So:

$$\dim\, B_0 = n - 1 \tag{3.9}$$

But, by the *Kernel-Range Theorem* in relation to the boundary operator ∂, we have:

$$\dim\, Z_0 = r - \dim\, B_0. \tag{3.10}$$

Then, using equations (3.7), (3.9) and (3.10), we conclude that:

$$\dim\, Z_0 = 0. \tag{3.11}$$

Therefore, the theorem is proven on the case when the connected complex is a tree. We now consider the more general case, where we have a complex (not necessarily a tree) in a connected component. In this general case, we have:

$$n \leq r + 1 \tag{3.12}$$

As we are in a connected component, the identity (3.8) is still correct. By equations (3.9) and (3.10), we find:

$$\dim Z_1 = r + 1 - n \tag{3.13}$$

Therefore, if we find $r + 1 - n$ independent meshes, we find a basis of Z_1 consisting only of meshes, thus completing the proof. To assist us in achieving this goal, we develop a method which consists of three fundamental points:

(i) Certainly some branches will need to be removed from the complex

(ii) The withdrawals of these branches should eliminate all the meshes of the complex, without, however, eliminating its nodes . Therefore, we will have a maximal tree at the end of the process .

(iii) Each branch removed must belong to only one mesh, that is, this branch cannot be shared by other meshes .

Remark 3.36. Let us denote the set of removed branches by $\overline{T}$.

The first question is: how many branches are there in $\overline{T}$? To answer this question, just use the relation (3.7), which is valid for any tree, and calculate the removed branches as follows:

$$r - (n - 1) = r + 1 - n$$

Note that this number is equal to the dimension of Z_1 (see (3.13)). As each branch taken is associated with only one mesh, then we identify the following meshes $\{M_i\}_{i=1}^{n+1-r}$ with these branches. As $M_i \in Z_1,\ \forall i$, then we have just to show that these meshes are linearly independent.

In fact, they are L.I., because each mesh M_i is associated with a branch $r_i \in \overline{T}$ that, on the other hand, belongs exclusively to this mesh. Therefore:

$$\sum_{i=1}^{n+1-r} \alpha_i \mathbf{M}_i = \mathbf{0} \Rightarrow \alpha_i = 0,\ \forall i.$$

□

Remark 3.37. In the proof of Theorem 3.35, it is enough we work with only one connected component since the connected components are independent of each other. So to find the result in the entire complex, just repeat the argument in each connected component of the complex.

Remark 3.38. As a consequence of Theorem 3.35, we have that $\mathbf{I} \in Z_1 \Rightarrow \mathbf{I} = \sum J_i \mathbf{M_i}$, where the meshes $\mathbf{M_i}$, with $i = 1, 2, \ldots$, are linearly independent and form a basis of Z_1 . The scalars J_i represent the currents of these meshes.

Definition 3.39. The **homology group** H_1 is defined by: $H_1 = Z_1/B_1(\simeq Z_1)$.

Remark 3.40. Physically, the homology group H_1 will represent the mesh currents.

Definition 3.41. The **cohomology group** H^1 is defined by: $H^1 = Z^1/B^1(= C^1/B^1)$.

Remark 3.42. Physically, the cohomology group H^1 represents the voltages in the meshes.

Remark 3.43. The cohomology group is the dual of the homology group.

3.5 Kirchhoff's Laws

We will now mention the classic versions of the Kirchhoff laws, accompanied by a new formulation, using the boundary and coboundary operators.

3.5.1 Kirchhoff's Current Law

Classic Version: In a node, the sum of intensities of currents arriving is equal to the sum of intensities of currents leaving. This law expresses the **conservation of charges**.

Reformulation: If $\mathbf{I}$ is a 1-chain representing the distribution of currents in an electrical circuit, then:

$$\partial \mathbf{I} = \mathbf{0}.$$

Remark 3.44. The current vector $\mathbf{I}$ is a cycle, that is, $\mathbf{I} \in Z_1$.

3.5.2 Kirchhoff's Voltage Law

Classic Version: There is a function ϕ, called **electrostatic potential**, such that the **voltage** across each branch of the circuit will be the difference of ϕ applied to the initial and final nodes.

Example 3.45. Let α be a branch with A and B being its start and end nodes, respectively. Then $V^\alpha = \phi(A) - \phi(B)$ is the voltage in this branch.

Reformulation: There is a 0-cochain $\phi : C_0 \to \mathbb{R}$, called potential function, such that:

$$V = -\mathrm{d}\boldsymbol{\phi} \tag{3.14}$$

where $V \in C^1$ represents the voltage in all branches of the circuit.

Example 3.46. Returning to the example above, we have:
$V^{\boldsymbol{\alpha}} = -\mathrm{d}\boldsymbol{\phi}(\boldsymbol{\alpha}) = -\boldsymbol{\phi}\partial(\boldsymbol{\alpha}) = -\boldsymbol{\phi}(B - A) = \boldsymbol{\phi}(A) - \boldsymbol{\phi}(B)$

Remark 3.47. The equation (3.14) provides a physical meaning to the subspace B^1. This subspace is therefore **the space of the voltages of the circuit**.

Remark 3.48. For any mesh $\mathbf{M} \in Z_1$, with $V = -\mathrm{d}\boldsymbol{\phi}$, , for some $\boldsymbol{\phi} \in C^0$, we have:

$$\int_{\mathbf{M}} \mathbf{V} = -\int_{\mathbf{M}} \mathrm{d}\boldsymbol{\phi} = -\int_{\partial\mathbf{M}} \boldsymbol{\phi} = 0.$$

Therefore, the sum of the voltages of the branches of each mesh in a circuit is zero. This is the second version of the Kirchhoff's voltage law.

Remark 3.49. The remark 3.48 can also be explained by the fact that $\mathbf{V} \in B^1$, which is the annihilator of Z_1. So $\mathbf{V}$ vanishes for any mesh $\mathbf{M}$.

3.6 Electric Power

Definition 3.50. We define the **power** in a branch $\boldsymbol{\alpha}$ by the following formula:

$$\mathrm{Pot}_{\boldsymbol{\alpha}} = V^{\boldsymbol{\alpha}}.I_{\boldsymbol{\alpha}} \tag{3.15}$$

Theorem 3.51 (Tellegen's theorem)**.** *The total power dissipated in a resistive circuit is zero.*

Proof. Let $\mathbf{I} \in C_1$ the current of the circuit and $V \in C^1$ its voltage. Then:

$$\mathrm{Pot} = \sum V^{\alpha} I_{\alpha} = \int_{\mathbf{I}} \mathbf{V} = -\int_{\mathbf{I}} d\boldsymbol{\phi} = -\int_{\partial\mathbf{I}} \boldsymbol{\phi} = 0.$$

□

3.7 Restricted Coboundary Map

Proposition 3.52. *The subspace Z^0 represents the space of the potentials that are constant along each connected component of the circuit. Equivalently, Z^0 is the space of potentials such that the voltages on the branches is null.*

Proof. $\boldsymbol{\phi} \in Z^0 \Rightarrow \mathrm{d}\boldsymbol{\phi} = \mathbf{0}$
Let $\partial\mathbf{K}$ be an arbitrary element of B_0. Then:

$$\int_{\partial\mathbf{K}} \boldsymbol{\phi} = \int_{\mathbf{K}} \mathrm{d}\boldsymbol{\phi} = \mathbf{0}$$

As $\mathbf{K}$ is arbitrary, particularly the result is true for every branch of the circuit. □

Corollary 3.53. *The operator* d *of a circuit with a single connected component with a ground node is injective.*

Proof. The ground node will always have zero potential. Therefore the set Z^0 of the constant potentials can only contain the zero vector. □

Corollary 3.54. *dim Z^0 = number of connected components.*

Proof. Direct consequence of proposition 3.27 and lemma 3.30. □

Definition 3.55. $P^0 = C^0/Z^0$ is called **restricted potential space**.

Proposition 3.56. $P^0 \approx B_0^*$.

Proof. Direct consequence of propositions (2.43) and (3.24). □

By isomorphism theorem, the application $\mathrm{d} : C^0 \to C^1$ induces an *injective* map $[\mathrm{d}] : P^0 \to C^1$ given by:

$$[\mathrm{d}]\overline{\boldsymbol{\phi}} = \mathrm{d}\boldsymbol{\phi}.$$

Definition 3.57. [d] is called **restricted coboundary map**.

Remark 3.58. The voltage $V = -[\mathrm{d}]\overline{\boldsymbol{\phi}}$ associated with the potential $\overline{\boldsymbol{\phi}} \in P^0$ is *uniquely* determined (injectivity of [d]).

The map [d] induces the map $[\partial]$ such that $[\partial]$ and [d] are adjoint transformations, that is:

$$[\mathrm{d}]\overline{\boldsymbol{\phi}} = \overline{\boldsymbol{\phi}} \circ [\partial].$$

Definition 3.59. $[\partial] : C_1 \to B_0$ is called **restricted boundary map**.

Example 3.60. Consider figure 4. We have:

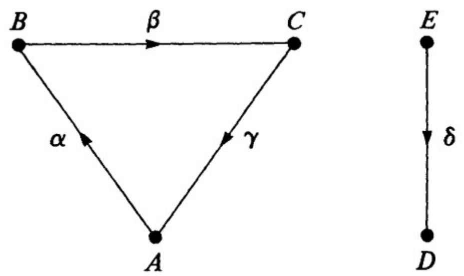


Figure 4 – Boundary and coboundary operators.

Basis of $C^0 : \{A^*, B^*, C^*, D^*, E^*\}$.
Basis of $C^1 : \{\alpha^*, \beta^*, \gamma^*, \delta^*\}$. Then:

$$\mathrm{d}(A^*) = -1\alpha^* + 0\beta^* + 1\gamma^* + 0\delta^*,$$

$$\mathrm{d}(B^*) = 1\alpha^* - 1\beta^* + 0\gamma^* + 0\delta^*,$$

$$\mathrm{d}(C^*) = 0\alpha^* + 1\beta^* - 1\gamma^* + 0\delta^*,$$

$$\mathrm{d}(D^*) = 0\alpha^* + 0\beta^* + 0\gamma^* + 1\delta^*,$$

$$\mathrm{d}(E^*) = 0\alpha^* + 0\beta^* + 0\gamma^* - 1\delta^*.$$

Therefore, coboundary and boundary operators are equal to:

$$\mathrm{d} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \partial = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now, by corollary 3.54, we have:

$$\dim Z^0 = 2 \quad \text{and} \quad \dim P^0 = \dim C^0 - \dim Z^0 = 3$$

Let **A** and **D** be the ground nodes of the circuit. Note that $\{\overline{\mathbf{B}}^*, \overline{\mathbf{C}}^*, \overline{\mathbf{E}}^*\}$ is L.I., so is a basis of P^0. Then:

$$[\mathrm{d}]\overline{\mathbf{B}}^* = 1\alpha^* - 1\beta^* + 0\gamma^* + 0\delta^*$$

$$[\mathrm{d}]\overline{\mathbf{C}}^* = 0\alpha^* + 1\beta^* - 1\gamma^* + 0\delta^*$$

$$[\mathrm{d}]\overline{\mathbf{E}}^* = 0\alpha^* + 0\beta^* + 0\gamma^* - 1\delta^*$$

Therefore, the restricted coboundary and boundary maps are:

$$[\mathrm{d}] = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad [\partial] = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.16}$$

Remark 3.61. By (3.16), notice that the method to find [d] is eliminating the columns of ground nodes to the matrix of the map d. And the method to find $[\partial]$ is eliminating the lines of ground nodes to the matrix of the map ∂.

Remark 3.62. We have $[\partial] : C_1 \to B_0$. But here B_0 is modified because the lines relating to ground nodes were removed.

3.8 Special Maps: σ and s

The space H_1 is a copy of the subspace Z_1. Think H_1 as an abstract vector space, independent of C_1, representing the space of currents of meshes.

Definition 3.63. The **inclusion** map $\sigma : H_1 \to C_1$ is the map which identifies H_1 with the subspace Z_1 of C_1. Physically, σ converts the mesh currents in currents on the branches.

Definition 3.64. The map $s : C^1 \to H^1$ is the adjoint map of σ.

Proposition 3.65. *The map $s : C^1 \to H^1$ converts voltages in the branches in voltages on the meshes.*

Proof. We observe that each column j of the matrix of σ with respect to the canonical bases is equal to the coordinates of the mesh $\mathbf{M}_j$, which, on the other hand, belongs to the base of Z_1. As the matrix of the operator s is the transpose of the matrix σ, we have that each line of the matrix of the operator s correspond to the coordinates of one mesh of Z_1. Then, the result follows immediately, just by multiplying s by voltage vector $\mathbf{V}$. □

Proposition 3.66. *The map $s : C^1 \to H^1$ represents the canonical projection of C^1 on C^1/B^1.*

Proof. Direct consequence of Proposition 2.41. □

Example 3.67. From the figure 5, we obtain the following results:

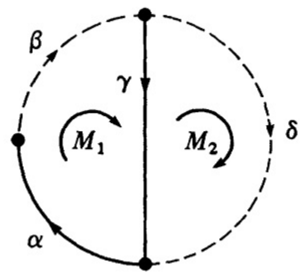


Figure 5 – Meshes and branches

For $J_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $J_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have:

$$\begin{aligned} \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \alpha + \beta + \gamma, \\ \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= -\gamma + \delta. \end{aligned}$$

Therefore:

$$\sigma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{3.17}$$

For the mesh current J_1, J_2, we find the following branch's current.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} J_1 \\ J_1 \\ J_1 - J_2 \\ j_2 \end{bmatrix} .$$

By (3.17), we have:

$$s = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} .$$

Considering $\mathbf{V} = \begin{bmatrix} V^\alpha \\ V^\beta \\ V^\gamma \\ V^\delta \end{bmatrix}$, we find the following tensions in the meshes:

$$s.\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} . \begin{bmatrix} V^\alpha \\ V^\beta \\ V^\gamma \\ V^\delta \end{bmatrix} = \begin{bmatrix} V^\alpha + V^\beta + V^\gamma \\ -V^\gamma + V^\delta \end{bmatrix}$$

The following diagram summarizes the operators and spaces that were seen with their respective relationships.

$$\begin{tikzcd} H_1 \equiv Z_1 \arrow[r, "\sigma"] \arrow[d, "\cong"'] & C_1 \arrow[r, "{[\partial]}"] \arrow[d, "R"'] & B_0 = \mathrm{Im}(\partial) \cong C_1/Z_1 \arrow[r, "i"] \arrow[d, "\cong"'] & H_0 = C_0/B_0 \arrow[d, "\cong"'] \\ H^1 = C^1/B^1 & C^1 \arrow[l, "s"] & P^0 = C^0/Z^0 \arrow[l, "{[\mathrm{d}]}"] & Z^0 \arrow[l, "i^*"] \end{tikzcd}$$

3.9 The Maxwell Methods

3.9.1 Ohm's Law

Considering the figure 6, we have:

Ohm's Law for a Specific Branch of the Circuit:

$$V^\alpha - W^\alpha = z_\alpha(\mathrm{I}_\alpha - K_\alpha).$$

For the entire circuit, we write:

$$\mathbf{V} - \mathbf{W} = Z(\mathbf{I} - \mathbf{K})$$

Observe now that Z is a diagonal matrix whose entries are the resistances, and $\mathbf{I}, \mathbf{V}, \mathbf{W}, \mathbf{K}$ are column matrices.

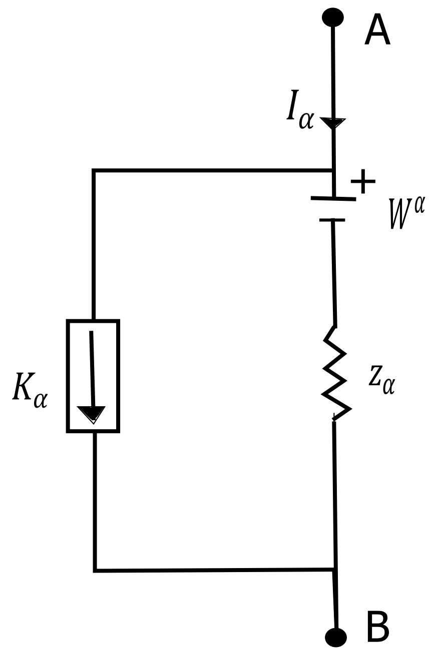


Figure 6 – Branch α of the circuit.

Joining the Ohm's law with Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL), we obtain the following system:

$$\begin{cases} \mathbf{V}-\mathbf{W} &= Z(\mathbf{I}-\mathbf{K}) & \text{(Ohm's Law)} \\ \mathbf{I} &= \sigma(\mathbf{J}) & \text{(KCL)} \\ \mathbf{V} &= -[\mathrm{d}]\overline{\boldsymbol{\phi}} & \text{(KVL)} \end{cases} \tag{3.18}$$

At this time, we will present two methods for solve equations 3.18.

3.9.2 Maxwell's Mesh-Current Method

Consider the system ((3.18)), where the values of $\mathbf{K}, \mathbf{W}$ and Z are provided. We want to find $\mathbf{I}$ and $\mathbf{V}$.

Applying the operator s, we have:

$$s(\mathbf{V}-\mathbf{W}) = \underbrace{s(\mathbf{V})}_{=0} - s(\mathbf{W}) = -s(\mathbf{W}) = s(Z(\mathbf{I}-\mathbf{K})) = sZ\mathbf{I} - sZ\mathbf{K}.$$

We have $s(\mathbf{V}) = 0$ because $\mathbf{V} = -[d]\overline{\phi}$, then $\mathbf{V}$ respects the Kirchhoff voltage law, i.e., the sum of the voltages in the meshes is equal to zero.

$$\begin{aligned} sZ\mathbf{I} &= sZ\mathbf{K} - s\mathbf{W} \\ \Rightarrow sZ\sigma(\mathbf{J}) &= s(Z\mathbf{K}-\mathbf{W}) \qquad (3.19) \\ \Rightarrow \mathbf{J} &= (sZ\sigma)^{-1}s(Z\mathbf{K}-\mathbf{W}). \qquad (3.20) \end{aligned}$$

Equation (3.20) will be valid if the operator $sZ\sigma$ is invertible. The next theorem makes an important statement about this.

Theorem 3.68. ***In resistive circuits****, the map $sZ\sigma$ is invertible.*

Proof. We need to show that the linear map $sZ\sigma\colon H_1 \to H^1$ is a bijection. For that, we need to show that it is injective (since $\dim H_1 = \dim H^1$).

Consider the nonzero vector $\mathbf{J} \in H_1$. We want to show that $sZ\sigma(\mathbf{J}) \neq \mathbf{0}$. As s is the adjoint of σ, we have:

$$[(sZ\sigma)\mathbf{J}](\mathbf{J}) = s(Z\mathbf{I})(\mathbf{J}) = (Z\mathbf{I})\sigma(\mathbf{J}) = (Z\mathbf{I})\mathbf{I} = \int_I Z\mathbf{I} = \sum_\gamma Z_\gamma I_\gamma^2.$$

Since $Z_\gamma > 0$(resistive circuit) and $\mathbf{I} \neq \mathbf{0}$ (because σ is injective and $\mathbf{J} \neq \mathbf{0}$), we have $\sum_\gamma Z_\gamma I_\gamma^2 > 0 \Rightarrow [(sZ\sigma)\mathbf{J}](\mathbf{J}) > 0 \Rightarrow (sZ\sigma)(\mathbf{J}) \neq \mathbf{0}$. □

Corollary 3.69. *For resistive circuits, the solution of the system* (3.18) ***exists and is unique****, that is, there is only a single pair* $\mathbf{I}$ *and* $\mathbf{V}$ *satisfying the system.*

Proof. The result follows from theorem 3.68 applied to the equation (3.19). Once $\mathbf{J}$ is determined $\mathbf{I}$ is computed using (KCL) and $\mathbf{V}$ is found using Ohm's law. □

3.9.3 Maxwell's Node-Potential Method

Consider the system ((3.18)), where the values of $\mathbf{K}, \mathbf{W}$ and Z are provided. We want to find $\mathbf{I}$ and $\mathbf{V}$.

$$\begin{aligned}
\mathbf{V} - \mathbf{W} &= Z(\mathbf{I} - \mathbf{K}) \\
Z^{-1}(\mathbf{V} - \mathbf{W}) &= \mathbf{I} - \mathbf{K} \\
[\partial]\, Z^{-1}(-[\mathrm{d}]\overline{\boldsymbol{\phi}} - \mathbf{W}) &= -\,[\partial]\,\mathbf{K} \\
[\partial]\, Z^{-1}[\mathrm{d}]\; \overline{\boldsymbol{\phi}} &= [\partial]\,(\mathbf{K} - Z^{-1}\mathbf{W}) \qquad (3.21)\\
\overline{\boldsymbol{\phi}} &= ([\partial]\, Z^{-1}[\mathrm{d}])^{-1}\,[\partial]\,(\mathbf{K} - Z^{-1}\mathbf{W}). \qquad (3.22)
\end{aligned}$$

The equation (3.22) will be valid if the operator $[\partial]\, Z^{-1}[\mathrm{d}]$ is invertible. The next theorem makes an important statement about this.

Theorem 3.70. ***In resistive circuits****, the map $[\partial]Z^{-1}[\mathrm{d}]$ is invertible.*

Proof. We have $[\partial]Z^{-1}[\mathrm{d}] : P^0 \to B_0$. Since $\dim\, P^0 = \dim\, B_0$, it suffices to show the injectivity of $[\partial]Z^{-1}[\mathrm{d}]$.

If $\overline{\boldsymbol{\phi}} \neq \mathbf{0}$, then:

$$\overline{\boldsymbol{\phi}}[([\partial]Z^{-1}[\mathrm{d}])\overline{\boldsymbol{\phi}}] = [\mathrm{d}]\overline{\boldsymbol{\phi}}(Z^{-1}[\mathrm{d}]\overline{\boldsymbol{\phi}}) = \mathbf{V}Z^{-1}\mathbf{V} = \int_{Z^{-1}\mathbf{V}} \mathbf{V} = \sum_\gamma \frac{(V^\gamma)^2}{Z_\gamma}.$$

Since $Z_\gamma > 0$ (resistive circuit) and $\mathbf{V} \neq \mathbf{0}$ (because [d] is injective and $\overline{\boldsymbol{\phi}} \neq \mathbf{0}$), we have $\dfrac{(V^\gamma)^2}{Z_\gamma} > 0 \Rightarrow \overline{\boldsymbol{\phi}}[([\partial]Z^{-1}[\mathrm{d}])\overline{\boldsymbol{\phi}}] > 0 \Rightarrow ([\partial]Z^{-1}[\mathrm{d}])(\overline{\boldsymbol{\phi}}) \neq \mathbf{0}$, para $\boldsymbol{\phi} \neq \mathbf{0}$. □

Corollary 3.71. *For resistive circuits, the solution of the system* (3.18) ***exists and is unique****, that is, there is only a single pair* $\boldsymbol{I}$ *and* $\boldsymbol{V}$ *satisfying the system.*

Proof. The result follows from the theorem 3.70 applied to the equation (3.21). □

3.10 RLC Circuits

More generally, in a circuit whose branches are resistors, capacitors or inductors, Z remains a diagonal matrix $r \times r$ (where r is the number of branches of the circuit), and its entries are the impedances of the branches, which may be functions of the frequency $f = \frac{\omega}{2\pi}$. In this case, *not always* the matrices $sZ\sigma$ and $[\partial]Z^{-1}[\mathrm{d}]$ will be invertible.

Recall that:

- Capacitor impedance= $-\frac{i}{\omega C}$
- Inductor impedance= $i\omega L$

There exist m linearly independent meshes, then the matrix $sZ\sigma$ will have order m and its determinant is given by the polynomial $D(\omega)$, whose degree is at most m. The operator $sZ\sigma$ is not invertible if $|D(\omega)| = 0$, which means that there can be at most m values of ω such that $sZ\sigma$ is not invertible. Similarly, the operator $[\partial]Z^{-1}[\mathrm{d}]$, of order $r - m$, will have at most $r - m$ values of ω such that the modulus of the determinant is zero, ie, $|D_1(\omega)| = 0$.

The resonant frequencies are defined in two ways:

Definition 3.72. If ω is such that the determinant $D(\omega)$ of $sZ\sigma$ is zero or the determinant $D_1(\omega)$ of $[\partial]Z^{-1}[\mathrm{d}]$ is zero, then $f = \frac{\omega}{2\pi}$ is called a **resonance frequency** of the system.

Definition 3.73. If for the operators $sZ\sigma$ and $[\partial]Z^{-1}[\mathrm{d}]$ the determinants $|D(\omega)| \neq 0$ and $|D_1(\omega)| \neq 0$ for all ω, then $f = \frac{\omega}{2\pi}$ will be **resonance frequency** of the system if ω generate in the circuit the *maximum value* of the mesh current J.

Lemma 3.74. *The roots of the determinants of operators* $sZ\sigma$ *and* $[\partial]Z^{-1}[\mathrm{d}]$ *are the same.*

Proof. This fact is a consequence of non-uniqueness (or nonexistence) of solutions of the equation

$$\mathbf{V} - \mathbf{W} = Z(\mathbf{I} - \mathbf{K}) \tag{3.23}$$

that is, if by the mesh-current method we find more than one solution (or even no solution) to the equation (3.23), then this result must be confirmed also by the node potential method, and vice versa. Therefore:

$$\det(sZ\sigma) = D(\omega) = 0 \Leftrightarrow \det([\partial]Z^{-1}[\mathrm{d}]) = D_1(\omega) = 0,$$

i.e., the polynomials D and D_1 have the same roots. $\square$

For the case where $|D(\omega)| = 0$, the equation

$$(sZ\sigma)\mathbf{J} = \mathbf{0} \tag{3.24}$$

will have nontrivial solutions.

Definition 3.75. The non-trivial solutions of the equation (3.24) are called **normal modes** of the system.

Example 3.76. In this example, we will find the resonant frequencies and some normal modes of the system. Looking at the figure 7, we have:

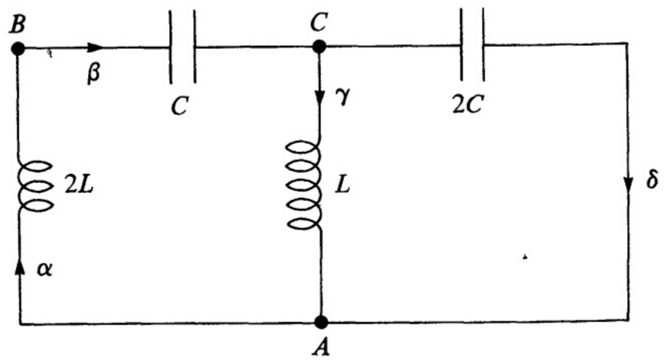


Figure 7 – Resonance frequencies and normal modes.

$$\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \quad , \quad s = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Also we have:

$$Z = i \begin{pmatrix} 2\omega L & 0 & 0 & 0 \\ 0 & -\frac{1}{\omega C} & 0 & 0 \\ 0 & 0 & \omega L & 0 \\ 0 & 0 & 0 & -\frac{1}{2\omega C} \end{pmatrix}$$

Multiplying the matrices, we have:

$$sZ\sigma = i \begin{pmatrix} 3\omega L - (\omega C)^{-1} & -\omega L \\ -\omega L & \omega L - (2\omega C)^{-1} \end{pmatrix}$$

Therefore, we find the determinant:

$$\mathrm{Det}(sZ\sigma) = -2\omega^2 L^2 + \frac{5L}{2C} - \frac{1}{2\omega^2 C^2}.$$

Setting $\mathrm{Det}(sZ\sigma) = 0$, we find the following angular frequencies:

$$\omega = \frac{1}{\sqrt{LC}} \quad , \qquad \omega = \frac{1}{2\sqrt{LC}}.$$

Therefore, the resonance frequencies of the system are equal to:

$$f = \frac{1}{2\pi\sqrt{LC}} \quad , \qquad f = \frac{1}{4\pi\sqrt{LC}}.$$

To find the corresponding normal modes, we solve the equation $(sZ\sigma)\mathbf{J} = \mathbf{0}$ for each resonance frequency.For example, to $\omega = \frac{1}{\sqrt{LC}}$ we have:

$$sZ\sigma = i\begin{pmatrix} 2\sqrt{\frac{L}{C}} & -\sqrt{\frac{L}{C}} \\ -\sqrt{\frac{L}{C}} & \frac{1}{2}\sqrt{\frac{L}{C}} \end{pmatrix}$$

and a normal mode to $(sZ\sigma)\mathbf{J} = \mathbf{0}$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which means that the current in the second mesh is twice greater than the current in the first mesh. Similarly, setting $\omega = \frac{1}{2\sqrt{LC}}$, we have:

$$sZ\sigma = i\begin{pmatrix} -\frac{1}{2}\sqrt{\frac{L}{C}} & -\frac{1}{2}\sqrt{\frac{L}{C}} \\ -\frac{1}{2}\sqrt{\frac{L}{C}} & -\frac{1}{2}\sqrt{\frac{L}{C}} \end{pmatrix}$$

and a normal mode to $(sZ\sigma)\mathbf{J} = \mathbf{0}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which means that J_1 and J_2 has the same amplitude, but opposite phase.

Example 3.77. Find the resonance frequency and the maximum mesh current. From figure 8, we have:

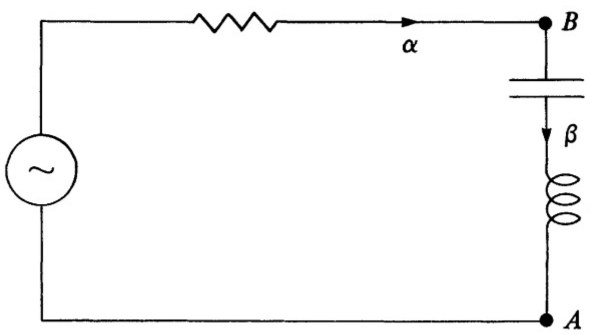


Figure 8 – RLC circuit.

$$\sigma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} R & 0 \\ 0 & i(\omega L - \frac{1}{\omega C}) \end{pmatrix}.$$

Therefore, we have:

$$sZ\sigma = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & i(\omega L - \frac{1}{\omega C}) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (R + i(\omega L - \frac{1}{\omega C})).$$

Then, knowing that $\mathbf{W} = \begin{pmatrix} Ee^{i\omega t} \\ 0 \end{pmatrix}$, from equation 3.20 we have:

$$J = \frac{1}{R + i(\omega L - \frac{1}{\omega C})} Ee^{i\omega t}$$

The mesh current clearly has its peak value when $\omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$. Therefore, the resonance frequency is equal to:

$$f = \frac{1}{2\pi\sqrt{LC}}$$

and the peak value of the mesh current is:

$$|J| = \frac{E}{R}$$

4 METHOD OF ORTHOGONAL PROJECTION

Introduction

In this chapter we will restrict our study only to resistive circuits and we will present two alternative methods for solving circuit equations. The Weyl's method makes use of orthogonal projection operators while the Kirchhoff's method uses graph theory to find the values of voltages and electric currents. The Green's reciprocity theorem exposes some symmetries for some resistive circuits.

4.1 Weyl's Method of Orthogonal Projection

4.1.1 Weyl's Method

As in chapter 2, let $\alpha, \beta, \gamma, \ldots$ denote the branches of a resistive circuit.

Definition 4.1. Let Z be the diagonal matrix of resistors. The **inner product** $(,)_Z : C_1 \times C_1 \to \mathbb{R}$ is defined by:

$$(\mathbf{I}, \mathbf{I}')_Z = \int_{\mathbf{I}} Z\mathbf{I}' = r_\alpha I_\alpha I'_\alpha + r_\beta I_\beta I'_\beta + \ldots \tag{4.1}$$

Remark 4.2. As the entries of Z are all positive, we have that the inner product above is positive definite.

Remark 4.3. Let us denote by π the orthogonal projection of C_1 on Z_1.

The method of Weyl's projection consists of the following equations:

$$\begin{cases} \mathbf{I} &= \pi(\mathbf{K} - Z^{-1}\mathbf{W}), \\ \mathbf{V} &= Z(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}). \end{cases} \tag{4.2}$$

Proposition 4.4. *The equations* (4.2) *are equivalent to the equations* (3.18) *for electrical circuits.*

Proof. Firstly we show that (4.2) $\Rightarrow$ (3.18).

(i) By definition of projection on Z_1, we have that $\mathbf{I} \in Z_1$. Then:

$$\mathbf{I} = \sigma(\mathbf{J}) \tag{4.3}$$

(ii) We have:

$$\pi\left((\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W})\right) = (\pi^2 - \pi)(\mathbf{K} - Z^{-1}\mathbf{W}) = (\pi - \pi)(\mathbf{K} - Z^{-1}\mathbf{W}) = \mathbf{0}.$$

Therefore:

$$(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) \in \ker(\pi) \Rightarrow (\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) \in Z_1^{\perp\text{compl}}$$

$$\Rightarrow (\mathbf{I}', (\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}))_Z = 0 \ \forall \ \mathbf{I}' \in Z_1$$
$$\Rightarrow \int_{\mathbf{I}'} Z(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) = \int_{\mathbf{I}'} V = 0, \ \forall \ \mathbf{I}' \in Z_1$$

So

$$\mathbf{V} \in Z_1^{\perp} = B^1. \tag{4.4}$$

Therefore:

$$\mathbf{V} = -\mathrm{d}\boldsymbol{\phi}, \ \text{for some } \boldsymbol{\phi} \in C^0. \tag{4.5}$$

(iii)

$$\mathbf{V} = Z(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) = Z(\pi[(\mathbf{K} - Z^{-1}\mathbf{W})] - (\mathbf{K} - Z^{-1}\mathbf{W}))$$
$$= Z(\mathbf{I} - \mathbf{K} + Z^{-1}\mathbf{W}) = \mathbf{W} + Z(\mathbf{I} - \mathbf{K})$$

Therefore:

$$\mathbf{V} - \mathbf{W} = Z(\mathbf{I} - \mathbf{K}) \tag{4.6}$$

The equations (4.3), (4.5) and (4.6) are the equations forming system (3.18).

Next we show that (3.18) $\Rightarrow$ (4.2).

(i) We have:

$$\mathbf{V} - \mathbf{W} = Z(\mathbf{I} - \mathbf{K}) \Rightarrow \mathbf{V} = \mathbf{W} + Z(\mathbf{I} - \mathbf{K}).$$

Therefore:

$$\mathbf{V} = Z(Z^{-1}\mathbf{W} - \mathbf{K} + \mathbf{I}). \tag{4.7}$$

Since

$$V = -[\mathrm{d}]\boldsymbol{\phi},$$

then

$$\int_{\mathbf{I}'} V = 0, \ \forall \ \mathbf{I}' \in Z_1,$$

so

$$\int_{I'} Z(Z^{-1}\mathbf{W} - \mathbf{K} + \mathbf{I}) = 0, \ \forall \ \mathbf{I}' \in Z_1.$$

Therefore:

$$(\mathbf{I}', Z^{-1}\mathbf{W} - \mathbf{K} + \mathbf{I})_Z = 0, \ \forall \ \mathbf{I}' \in Z_1, \tag{4.8}$$

hence:

$$Z^{-1}\mathbf{W} - \mathbf{K} + \mathbf{I} \in Z_1^{\perp\text{compl}} \Rightarrow (\mathbf{K} - Z^{-1}\mathbf{W}) - \mathbf{I} \in Z_1^{\perp\text{compl}}$$

$$\Rightarrow (\mathbf{K} - Z^{-1}\mathbf{W}) - \mathbf{I} \in \ker(\pi).$$

From

$$\mathbf{K} - Z^{-1}\mathbf{W} = (\mathbf{K} - Z^{-1}\mathbf{W} - \mathbf{I}) + \mathbf{I} \tag{4.9}$$

we have:

$$\pi(\mathbf{K} - Z^{-1}\mathbf{W}) = \pi(\mathbf{K} - Z^{-1}\mathbf{W} - \mathbf{I}) + \pi(\mathbf{I}) = \mathbf{I}, \tag{4.10}$$

therefore:

$$\mathbf{I} = \pi(\mathbf{K} - Z^{-1}\mathbf{W}). \tag{4.11}$$

(ii) From the equations (4.7) and (4.11), we have:

$$\mathbf{V} = Z(Z^{-1}\mathbf{W} - \mathbf{K} + \pi(\mathbf{K} - Z^{-1}\mathbf{W})) = Z(\mathbf{K} - Z^{-1}\mathbf{W})(\pi - \mathrm{I})$$

that is:

$$\mathbf{V} = Z(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) \tag{4.12}$$

The equations (4.11) and (4.12) recover the system (4.2). □

Example 4.5. From figure 9, we have:

$$\mathbf{K} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}; \quad \mathbf{W} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}; \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

The space C_1 has dimension two, with canonical basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, and the space Z_1 has dimension one and a basis equal to $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. To find the Weyl's orthogonal

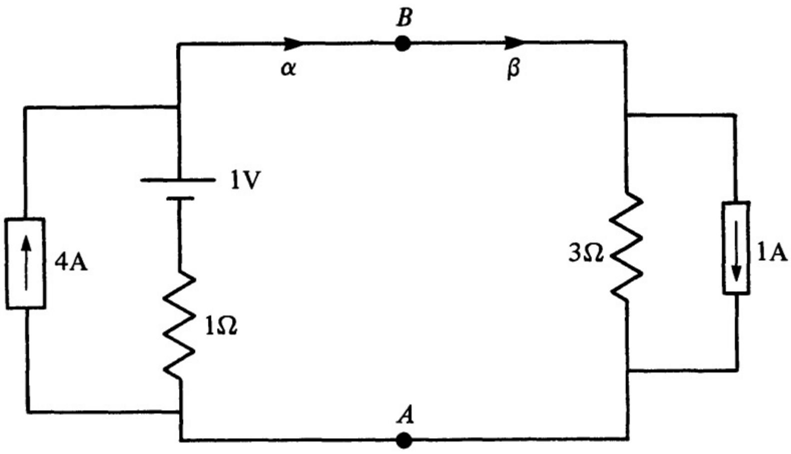


Figure 9 – Weyl's projection.

projection, we will normalize the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)_Z = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 4$$

Then, an orthonormal basis of Z_1 is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Therefore, by formula (2.7), we have:

$$\pi(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right)_Z \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \tag{4.13}$$

$$\pi(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right)_Z \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \end{pmatrix} \tag{4.14}$$

With this, we have:

$$\pi = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}. \tag{4.15}$$

As long as

$$\mathbf{K} - Z^{-1}\mathbf{W} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{4.16}$$

we get

$$\mathbf{I} = \pi(\mathbf{K} - Z^{-1}\mathbf{W}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \tag{4.17}$$

Still using the Weyl's orthogonal projection, we have:

$$\mathbf{V} = Z(\pi - \mathrm{I})(\mathbf{K} - Z^{-1}\mathbf{W}) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}. \tag{4.18}$$

4.1.2 Explicit Expression for the Weyl's Orthogonal Projection

According to Maxwell's Mesh-Current Method, we have:

$$\mathbf{J} = (sZ\sigma)^{-1}s(Z\mathbf{K} - \mathbf{W}) \tag{4.19}$$

Applying σ in the equation (4.19), we have:

$$\mathbf{I} = \sigma(sZ\sigma)^{-1}s(Z\mathbf{K} - \mathbf{W}). \tag{4.20}$$

Therefore:

$$\mathbf{I} = \sigma(sZ\sigma)^{-1}sZ(\mathbf{K} - Z^{-1}\mathbf{W}) \tag{4.21}$$

Comparing the equation (4.21) with the Weyl's formula $\mathbf{I} = \pi(\mathbf{K} - Z^{-1}\mathbf{W})$, we can intuit the outcome of the next proposition.

Proposition 4.6. *The Weyl's orthogonal projection operator can be explicitly given by* $\pi = \sigma(sZ\sigma)^{-1}sZ$.

Proof. We need to show that $\pi = \sigma(sZ\sigma)^{-1}sZ$ is an orthogonal projection of C_1 in Z_1.

(i) $\mathrm{Im}(\pi) = Z_1$.
- $\mathrm{Im}(\pi) \subset Z_1$. Obvious, since σ is an injection of Z_1 in C_1.
- $Z_1 \subset \mathrm{Im}(\pi)$, because for all $\mathbf{I} \in Z_1, \exists\, \xi \in [Z^{-1}s^{-1}(sZ\sigma)\sigma^{-1}(\mathbf{I})] \subset C_1 \mid \pi(\xi) = \mathbf{I}$.

Remark 4.7. Since s is not injective, for $\overline{\mathbf{u}} \in H^1$, we have:

$$s^{-1}(\overline{\mathbf{u}}) = \{\mathbf{v} \in C^1 \mid s(\mathbf{v}) = \overline{\mathbf{u}}\}.$$

(ii) $\pi : C_1 \to Z_1$ satisfies $\pi^2 = \pi$. In fact, we have:

$$\pi^2 = (\sigma \underbrace{(sZ\sigma)^{-1}sZ)(\sigma}_{\mathrm{Id}}(sZ\sigma)^{-1}sZ) = \sigma(sZ\sigma)^{-1}sZ = \pi. \tag{4.22}$$

(iii) $\ker(\pi) = Z_1^{\perp \mathrm{compl}}$.
- $\ker(\pi) \subset Z_1^{\perp \mathrm{compl}}$. Indeed, $\forall\, \mathbf{I} \in Z_1$ and $\mathbf{U} \in \ker(\pi)$, we have:

$$(\mathbf{I}, \mathbf{U})_Z = (\pi\mathbf{I}, \mathbf{U})_Z = (\sigma(sZ\sigma)^{-1}sZ(\mathbf{I}), \mathbf{U})_Z = (\sigma(sZ\sigma)^{-1}sZ(\mathbf{I}), Z\mathbf{U}). \tag{4.23}$$

As the matrix of $\sigma(sZ\sigma)^{-1}s$ is symmetric, it is a self-adjoint transformation, so for (4.23) we have:

$$(Z\mathbf{I}, \sigma(sZ\sigma)^{-1}sZ(\mathbf{U})) = (\mathbf{I}, 0)_Z = 0, \quad \forall\, \mathbf{I} \in Z_1.$$

Then $\mathbf{U} \in Z_1^{\perp \mathrm{compl}}$
- $Z_1^{\perp \mathrm{compl}} \subset \ker(\pi)$. Indeed, let $\mathbf{U} \in Z_1^{\perp \mathrm{compl}}$. Then, $\forall\, \mathbf{I} \in Z_1$, we have:

$$(\mathbf{I}, \mathbf{U})_Z = 0 \Rightarrow (\pi\mathbf{I}, \mathbf{U})_Z = (\sigma(sZ\sigma)^{-1}sZ(\mathbf{I}), Z\mathbf{U}). \tag{4.24}$$

Since $\sigma(sZ\sigma)^{-1}s$ is a self-adjoint transformation, for (4.24) we have:

$$(Z\mathbf{I}, \sigma(sZ\sigma)^{-1}sZ(\mathbf{U})) = (\mathbf{I}, \sigma(sZ\sigma)^{-1}sZ(\mathbf{U}))_Z = 0, \ \forall\, \mathbf{I} \in Z_1 \tag{4.25}$$

Since equation (4.25) is true for $\forall\, \mathbf{I} \in Z_1$, it is particularly true for $\mathbf{I} = \sigma(sZ\sigma)^{-1}sZ(\mathbf{U})$. Then:

$$(\sigma(sZ\sigma)^{-1}sZ(\mathbf{U}), \sigma(sZ\sigma)^{-1}sZ(\mathbf{U}))_Z = 0 \Rightarrow \sigma(sZ\sigma)^{-1}sZ(\mathbf{U}) = 0 \Rightarrow \mathbf{U} \in \ker(\pi). \tag{4.26}$$

□

4.2 Kirchhoff's Method

Definition 4.8. Suppose we have a connected complex and let T be a maximal tree in this complex. We define the linear operator p_{T} as follows:

$$p_{\mathrm{T}}(\alpha) = \begin{cases} 0 \quad , & \text{if} \quad \alpha \in \mathrm{T} \\ \\ M_\alpha \quad , & \text{if} \quad \alpha \notin \mathrm{T} \end{cases} \tag{4.27}$$

where M_α is a mesh containing the branch α.

Remark 4.9. M_α is an element of the basis of Z_1.

Remark 4.10. M_α is the only mesh that contains the branch α.

Remark 4.11. $p_{\mathrm{T}}(M_\alpha) = M_\alpha$ because the branch α is present in the mesh M_α, , while the other branches of M_α belong to the maximal tree.

Remark 4.12. The sense of the mesh M_α is chosen so that it contains "$+\alpha$" and not "$-\alpha$".

Example 4.13. Find the matrix of the linear operator p_{T}. See the figure 10.

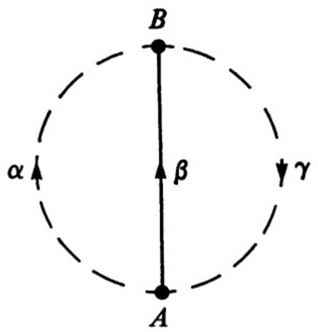


Figure 10 – Maximal tree

$$\begin{aligned} p_{\mathrm{T}}(\alpha) &= \alpha - \beta \\ p_{\mathrm{T}}(\beta) &= 0 \\ p_{\mathrm{T}}(\gamma) &= \beta + \gamma \end{aligned}$$

Then:

$$[p_{\mathrm{T}}] = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.14. *$p_{\mathrm{T}} : C_1 \to C_1$ is a projection over Z_1.*

Proof. • First, we will show that $p_{\mathrm{T}}(J) = J,\ \forall\ J \in Z_1$. Indeed, if $J \in Z_1$, then $J = \sum \alpha_i M_i$. Then by remark 4.11, we have:

$$p_{\mathrm{T}}(J) = \sum \alpha_i p_{\mathrm{T}}(M_i) = \sum \alpha_i M_i = J$$

• Im $p_{\mathrm{T}} = Z_1$.
Im $p_{\mathrm{T}} \subset Z_1$ (obvious).
$Z_1 \subset$ Im p_{T}. In fact, since $p_{\mathrm{T}}(J) = J,\ \forall\ J \in Z_1$.
• $p_{\mathrm{T}}^2 = p_{\mathrm{T}}$. Indeed, $\forall\ A \in C_1$, we have:

$$p_{\mathrm{T}}^2(A) = p_{\mathrm{T}}(p_{\mathrm{T}}(A)).$$

Since $p_{\mathrm{T}}(A) \in Z_1$, then:

$$p_{\mathrm{T}}(p_{\mathrm{T}}(A)) = p_{\mathrm{T}}(A).$$

□

Lemma 4.15. *If the projection is self-adjoint, then it is orthogonal.*

Proof. $\alpha \in \ker p_{\mathrm{T}}$, then, $\forall\ \beta \in Z_1$, we have:

$$0 = (p_{\mathrm{T}}(\alpha), \beta) = (\alpha, p_{\mathrm{T}}(\beta)) = (\alpha, \beta)$$

So $\alpha \in Z_1^{\perp\text{compl}}$. Therefore $\ker p_{\mathrm{T}} \subset Z_1^{\perp\text{compl}}$.
Now let $\alpha \in Z_1^{\perp\text{compl}}$. Then, $\forall\ \beta \in Z_1$, we have:

$$0 = (\alpha, \beta) = (\alpha, p_{\mathrm{T}}(\beta)) = (p_{\mathrm{T}}(\alpha), \beta).$$

In particular, choosing $\beta = p_{\mathrm{T}}(\alpha)$, we have:

$$(p_{\mathrm{T}}(\alpha), p_{\mathrm{T}}(\alpha)) = 0 \Rightarrow p_{\mathrm{T}}(\alpha) = \mathbf{0} \Rightarrow \alpha \in \ker p_{\mathrm{T}}.$$

Therefore $Z_1^{\perp\text{compl}} \subset \ker p_{\mathrm{T}}$. □

Remark 4.16. The projection p_{T} is not necessarily orthogonal.

The Kirchhoff's method aims to produce a projection in Z_1 which is orthogonal. To this end, by lemma 4.15, just build a projection in Z_1 which is self-adjoint. For this purpose, we introduce the following construction: for each maximal tree T we associate a real number λ_{T}, with $0 \leq \lambda_{\mathrm{T}} \leq 1$, such that $\sum\limits_{\mathrm{T}} \lambda_{\mathrm{T}} = 1$. With this we construct the following operator:

$$p_\lambda = \sum_{\mathrm{T}} \lambda_{\mathrm{T}} p_{\mathrm{T}}. \tag{4.28}$$

where p_{T} is the operator of the definition (4.8).

Lemma 4.17. *p_λ is a projection in Z_1.*

Proof. • $p_\lambda(J) = J, \ \forall \ J \in Z_1$. Indeed, we have:

$$p_\lambda(J) = \sum_{\mathrm{T}} \lambda_{\mathrm{T}} p_{\mathrm{T}}(J) = \sum_{\mathrm{T}} \lambda_{\mathrm{T}} J = (\underbrace{\sum_{\mathrm{T}} \lambda_{\mathrm{T}}}_{1})J = J \tag{4.29}$$

• Im $p_\lambda = Z_1$. Indeed:
Im $p_\lambda \subset Z_1$ (obvious).
$Z_1 \subset$ Im p_λ, because, by equation (4.29), $p_\lambda(J) = J, \ \forall \ J \in Z_1$.
• $p_\lambda^2 = p_\lambda$, because $p_\lambda^2(A) = p_\lambda(\underbrace{p_\lambda(A)}_{\in Z_1}) = p_\lambda(A), \ \forall \ A \in C_1$. □

In general, p_λ is not orthogonal. However, Kirchhoff made a special choice for the coefficients λ_{T} to become p_λ an orthogonal projection.

Definition 4.18. $Q_{\mathrm{T}} := \prod_{\beta \notin \mathrm{T}} r_\beta$, where as before r_β is the electrical resistance in the branch β.

Definition 4.19. $R := \sum_{\mathrm{T}} Q_{\mathrm{T}}$.

Definition 4.20. $\lambda_{\mathrm{T}} := \frac{Q_{\mathrm{T}}}{R}$.

Theorem 4.21. *The operator $p_\lambda : C_1 \to C_1$ given by $p_\lambda = R^{-1} \sum_{\mathrm{T}} Q_{\mathrm{T}} p_{\mathrm{T}}$ is an orthogonal projection onto Z_1.*

Proof. By Lemma 4.17 we have that p_λ is a projection in Z_1. Just show that $Rp_\lambda = \sum_{\mathrm{T}} Q_{\mathrm{T}} p_{\mathrm{T}}$ is a self-adjoint projection and therefore, by lemma 4.15, p_λ will be an orthogonal projection. As the branches of the circuit form a basis of C_1, it is sufficient to show that, for any pair α and β of branches, we have:

$$\sum_{\mathrm{T}} Q_{\mathrm{T}} (p_{\mathrm{T}}\alpha, \beta)_Z = \sum_{\mathrm{T}} Q_{\mathrm{T}} (\alpha, p_{\mathrm{T}}\beta)_Z \tag{4.30}$$

So with α and β fixed and summing over all maximal trees of complex, three cases can happen.

1^{st} case:A maximal tree T with $\alpha, \beta \in \mathrm{T}$. See Figure 11. Then:

$$(p_{\mathrm{T}}(\alpha), \beta)_Z = (\alpha, p_{\mathrm{T}}(\beta))_Z = 0 \tag{4.31}$$

Therefore , the factor on the left side of (4.31) is equal to the factor on the right side .

2^{nd} case: A maximal tree T with $\alpha, \beta \notin \mathrm{T}$. See the figure 12. Then:
$(p_{\mathrm{T}}(\alpha), \beta)_Z = 0$ because the branch β does not belong to the mesh $M_\alpha = p_{\mathrm{T}}(\alpha)$, and $(\alpha, p_{\mathrm{T}}(\beta))_Z = 0$ for the same reason explained above. So we have:

$$(p_{\mathrm{T}}(\alpha), \beta)_Z = (\alpha, p_{\mathrm{T}}(\beta))_Z \tag{4.32}$$

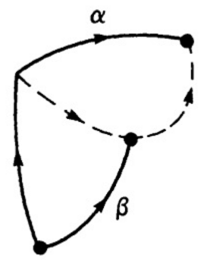


Figure 11 – 1^{st} case

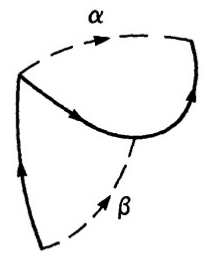


Figure 12 – 2^{nd} case

3^{rd} case: The third case is more complicated. We will have a maximal tree T for which only one of the branches belong to T. For example, $\beta \in \mathrm{T}$ and $\alpha \notin \mathrm{T}$. See Figure 13. Consider two possibilities. First, if $\beta \notin M_\alpha = p_{\mathrm{T}}(\alpha)$, then:

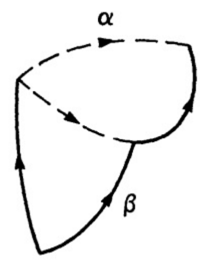


Figure 13 – 3^{rd} case

$$(p_{\mathrm{T}}(\alpha), \beta)_Z = (\alpha, p_{\mathrm{T}}(\beta))_Z = 0 \tag{4.33}$$

Therefore, the factor on the left side of (4.33) is equal to the factor on the right side. Now we have to work a bit more on the second possibility. Consider that $\beta \in M_\alpha = p_{\mathrm{T}}(\alpha)$, then:

$$(p_{\mathrm{T}}(\alpha), \beta)_Z = \pm r_\beta \ \text{ e } \ (\alpha, p_{\mathrm{T}}(\beta))_Z = 0 \tag{4.34}$$

where the positive sign occurs when α and β occur with the same sign in M_α. Otherwise, the negative sign occurs. Note that, in this case, *there will always be a unique maximal tree* T' formed **only** permuting α for β, i.e., if before $\alpha \notin \mathrm{T}$ and $\beta \in \mathrm{T}$, we now have $\alpha \in \mathrm{T}'$ and $\beta \notin \mathrm{T}'$(keeping the rest intact). Therefore, we have:

$$p_{\mathrm{T}'}(\beta) = \pm p_{\mathrm{T}}(\alpha)$$

and:

$$(\alpha, p_{\mathrm{T}'}(\beta))_Z = \pm r_\alpha \text{ e } (p_{\mathrm{T}'}(\alpha), \beta)_Z = 0 \tag{4.35}$$

Also note that:

$$\begin{cases} Q_{\mathrm{T}} &= r_\alpha \cdots \hat{r}_\beta \cdots , \\ Q_{\mathrm{T}'} &= \hat{r}_\alpha \cdots r_\beta \cdots , \end{cases} \tag{4.36}$$

where hats indicate the removal of the factor.

Therefore, by (4.34), (4.35) and (4.36), we have:

$$Q_{\mathrm{T}}(p_{\mathrm{T}}(\alpha), \beta)_Z + Q_{\mathrm{T}'}(p_{\mathrm{T}'}(\alpha), \beta)_Z = Q_{\mathrm{T}}(\alpha, p_{\mathrm{T}}(\beta))_Z + Q_{\mathrm{T}'}(\alpha, p_{\mathrm{T}'}(\beta))_Z \tag{4.37}$$

Therefore, by (4.31), (4.32), (4.33) and (4.37), summing over all maximum trees T, we have:

$$\sum_{\mathrm{T}} Q_{\mathrm{T}}(p_{\mathrm{T}}(\alpha), \beta)_Z = \sum_{\mathrm{T}} Q_{\mathrm{T}}(\alpha, p_{\mathrm{T}}(\beta))_Z. \tag{4.38}$$

□

Corollary 4.22. *The Weyl's orthogonal projection operator has the following explicit formula:*

$$\pi = R^{-1} \sum_{\mathrm{T}} Q_{\mathrm{T}} p_{\mathrm{T}}.$$

Proof. Comes from theorem 4.21 and the uniqueness of the orthogonal projection operator on the subspace Z_1. □

Example 4.23. In figure 14 there are three maximal trees. The first maximal tree T_1 is formed only by the branch α, the second T_2 is formed by branch β, and the third T_3 by branch γ.

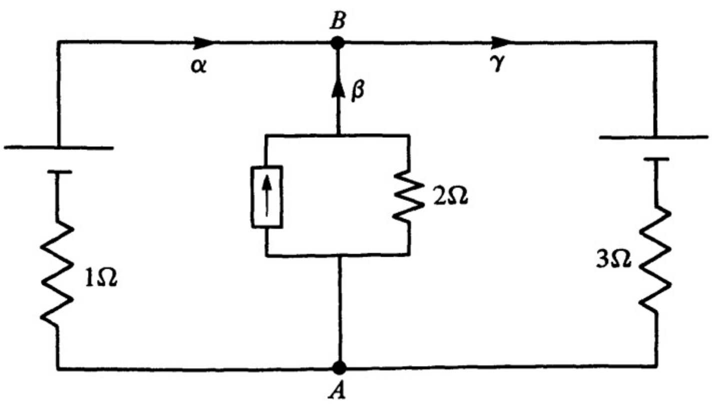


Figure 14 – Kirchhoff's method

For T_1, we have:

$$p_{\mathrm{T}_1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_{\mathrm{T}_1} = r_\beta . r_\gamma = 6$$

For T_2, we have:

$$p_{\mathrm{T}_2} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_{\mathrm{T}_1} = r_\alpha . r_\gamma = 3$$

For T_3, we have:

$$p_{\mathrm{T}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_{\mathrm{T}_1} = r_\alpha . r_\beta = 2$$

Then $R = Q_{\mathrm{T}_1} + Q_{\mathrm{T}_2} + Q_{\mathrm{T}_3} = 11$. Since

$$\pi = \frac{1}{R} \sum_{i=1}^{3} Q_{\mathrm{T}_i} p_{\mathrm{T}_i}$$

$$\pi = \frac{1}{11} [6.p_{\mathrm{T}_1} + 3.p_{\mathrm{T}_2} + 2.p_{\mathrm{T}_3}]$$

Therefore:

$$\pi = \frac{1}{11} \begin{pmatrix} 5 & -6 & 6 \\ -3 & 8 & 3 \\ 2 & 2 & 9 \end{pmatrix}$$

where π is the orthogonal projection over Z_1. The vectors

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

form a basis for the subspace Z_1 and since π is a projection over Z_1, we have:

$$\pi \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \pi \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Notice now the operator

$$\mathrm{I}_d - \pi = \frac{1}{11} \begin{pmatrix} 6 & 6 & -6 \\ 3 & 3 & -3 \\ -2 & -2 & 2 \end{pmatrix}$$

The matrix $\mathrm{I}_d - \pi$ projects onto the orthogonal subspace (orthogonal complement with respect to Z_1). As the dimension of this subspace is 1, the basis of the orthogonal complement of Z_1 consists of a single vector. We have:

$$\frac{1}{11}\begin{pmatrix} 6 & 6 & -6 \\ 3 & 3 & -3 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 11 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}.$$

Then $\left\{\begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}\right\}$ is a basis of the orthogonal complement of Z_1.

Remark 4.24. Note that the vector

$$Z\begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

physically represents a voltage that follows the Kirchhoff's voltage law because in relation to the usual inner product, we have:

$$(\begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}) = (\begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}) = 0.$$

4.3 Green's Reciprocity Theorem

Consider a circuit in which except for two branches α and β, all other branches contain only resistors. The branches α and β may contain current sources and/or voltage sources and/or resistors. For a specific choice of sources for α and β, we find the current vector $\mathbf{I}$ and the voltage vector $\mathbf{V}$ satisfying Kirchhoff's laws. Modifying the sources of the branches α and β, we find a new pair of solution $\hat{\mathbf{I}}, \hat{\mathbf{V}}$ for the current and voltage vectors, respectively, both satisfying the Kirchhoff's laws. Then:

$$\mathbf{I}, \hat{\mathbf{I}} \in Z_1, \quad \mathbf{V}, \hat{\mathbf{V}} \in B^1. \tag{4.39}$$

Therefore, by (4.39), we have:

$$\int_{\hat{\mathbf{I}}} \mathbf{V} = \int_{\mathbf{I}} \hat{\mathbf{V}} = 0$$

Consequently:

$$\sum_{\substack{\text{all} \\ \text{branches}}} V^\gamma \hat{I}_\gamma = \sum_{\substack{\text{all} \\ \text{branches}}} \hat{V}^\gamma I_\gamma. \tag{4.40}$$

Isolating the terms associated with the branches α and β from (4.40), we have:

$$V^\alpha \hat{I}_\alpha + V^\beta \hat{I}_\beta + \sum_{\substack{\text{other} \\ \text{branches}}} V^\gamma \hat{I}_\gamma = \hat{V}^\alpha I_\alpha + \hat{V}^\beta I_\beta + \sum_{\substack{\text{other} \\ \text{branches}}} \hat{V}^\gamma I_\gamma. \tag{4.41}$$

For the branches different from α and β, we have:

$$V^\gamma = r_\gamma I_\gamma \text{ and } \hat{V}^\gamma = r_\gamma \hat{I}_\gamma.$$

Then:

$$\sum_{\substack{\text{other}\\\text{branches}}} V^\gamma \hat{I}_\gamma = \sum_{\substack{\text{other}\\\text{branches}}} r_\gamma I_\gamma \hat{I}_\gamma = \sum_{\substack{\text{other}\\\text{branches}}} r_\gamma \hat{I}_\gamma I_\gamma = \sum_{\substack{\text{other}\\\text{branches}}} \hat{V}^\gamma I_\gamma \tag{4.42}$$

Therefore, by (4.41) and (4.42), we have:

$$V^\alpha \hat{I}_\alpha + V^\beta \hat{I}_\beta = \hat{V}^\alpha I_\alpha + \hat{V}^\beta I_\beta \tag{4.43}$$

With this, we demonstrated the following theorem.

Theorem 4.25 (*Green's reciprocity theorem*)**.** *For a resistive circuit where, except for two branches α and β (which may contain, in addition to resistors, voltage sources and/or current sources), all other branches contain only resistors, we have:*

$$\hat{V}^\alpha I_\alpha + \hat{V}^\beta I_\beta = V^\alpha \hat{I}_\alpha + V^\beta \hat{I}_\beta \tag{4.44}$$

Now consider a purely resistive circuit (no voltage or current sources). There are two ways to add branches to this circuit:

1) Soldering entry (parallel branch). See the figure 15.

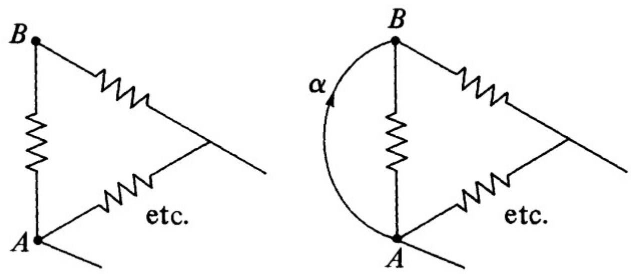


Figure 15 – Soldering entry.

Remark 4.26. This method doesn't add new nodes.

2) Pliers entry (branch in series). See the figure 16.

Remark 4.27. This method adds new nodes.

Remark 4.28. In both cases, the added branch may contain a short circuit, a voltage source, a current source or a resistor. Let's analyze the following cases for a purely resistive circuit, where two branches α and β were added using either of two methods previously discussed.

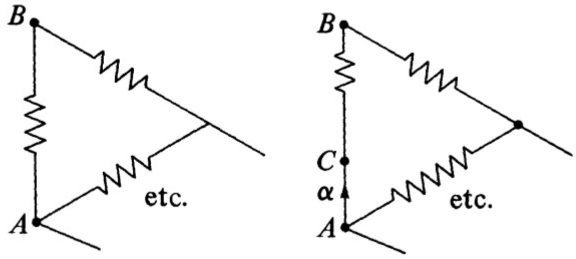


Figure 16 – Pliers entry.

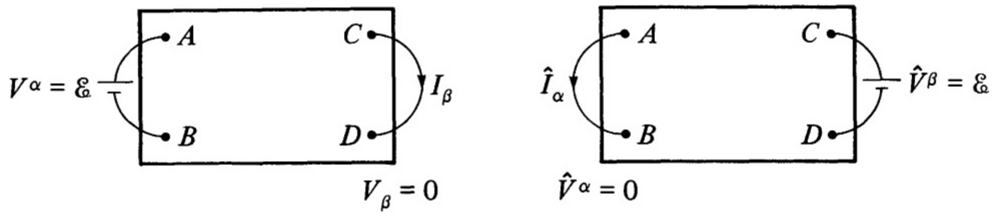


Figure 17 – Symmetry between applied voltage and resulting current

1^{st} case: Insert a battery whose voltage is $\mathcal{E}$ in α and measure the short-circuit current of I_β in β. Then connect the same battery in β and measure the short-circuit current of $\hat{I}_\alpha$ in α. See figure 17. By Green's reciprocity theorem, we have:

$$\hat{V}^\alpha I_\alpha + \hat{V}^\beta I_\beta = V^\alpha \hat{I}_\alpha + V^\beta \hat{I}_\beta$$

Since $\hat{V}^\alpha = V^\beta = 0$ (short-circuit), then:

$$\hat{V}^\beta I_\beta = V^\alpha \hat{I}_\alpha$$

$$\mathcal{E} I_\beta = \mathcal{E} \hat{I}_\alpha$$

Therefore:

$$I_\beta = \hat{I}_\alpha \tag{4.45}$$

Remark 4.29. Although the circuit itself doesn't need to have symmetry properties, the relationship between applied voltage and resulting current is symmetric.

2^{nd} case: Insert a current source J in α and measure the voltage V^β for the open circuit in the branch β. Now connect the same current source j in the branch β and measure the voltage $\hat{V}^\alpha$ for the open circuit in the branch α. . See the figure 18.

By Green's reciprocity theorem, we have:

$$\hat{V}^\alpha I_\alpha + \hat{V}^\beta I_\beta = V^\alpha \hat{I}_\alpha + V^\beta \hat{I}_\beta$$

Since $I_\beta = \hat{I}_\alpha = 0$, we have:

$$\hat{V}^\alpha I_\alpha = V^\beta \hat{I}_\beta$$

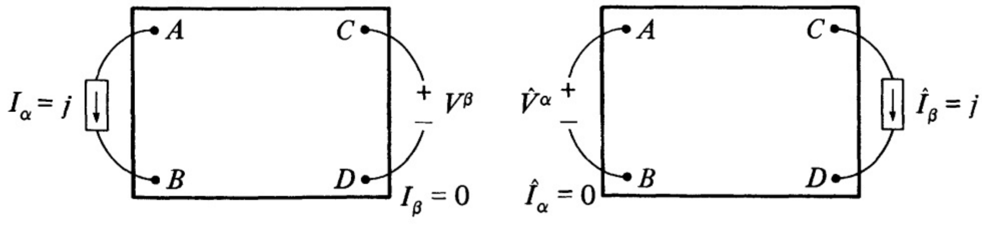


Figure 18 – Symmetry between the current source and resulting voltage.

$$\hat{V}^\alpha J = V^\beta J$$

Therefore:

$$\hat{V}^\alpha = V^\beta \tag{4.46}$$

Remark 4.30. Although the circuit itself doesn't need to have symmetry properties, the relationship between the current applied and the resulting voltage is symmetric.

More generally, the circuit can be a n-ports resistive circuit, i.e., a circuit where we can connect "n" devices. If current sources $I_\alpha, I_\beta, \dots$ are connected to the various ports, the resulting voltages $V^\alpha, V^\beta, \dots$ will be dependent on the current according some linear relationship of the type:

$$\mathbf{V} = R\mathbf{I}. \tag{4.47}$$

where R is a matrix of order n.

Corollary 4.31. *R is a symmetric matrix.*

Proof. Putting up a current source at the port μ and leaving the $n-1$ other ports open, we have:

$$\begin{pmatrix} \vdots \\ V^\lambda \\ \vdots \end{pmatrix} = R. \begin{pmatrix} 0 \\ \vdots \\ J_\mu \\ \vdots \\ 0 \end{pmatrix} \Rightarrow V^\lambda = R_{\lambda\mu} J_\mu \tag{4.48}$$

Now putting a current source at the door λ and letting $n-1$ other ports open, we have:

$$\begin{pmatrix} \vdots \\ \hat{V}^\mu \\ \vdots \end{pmatrix} = R. \begin{pmatrix} 0 \\ \vdots \\ \hat{J}_\lambda \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \hat{V}^\mu = R_{\mu\lambda} \hat{J}_\lambda \tag{4.49}$$

Making $J_\mu = \hat{J}_\lambda = J$, from Green's reciprocity theorem and, hence, by equation (4.46), we have:

$$V^\lambda = \hat{V}^\mu$$

$$\Rightarrow R_{\lambda\mu}\ J = R_{\mu\lambda}\ J$$

$$\Rightarrow R_{\lambda\mu} = R_{\mu\lambda}. \tag{4.50}$$

□

Proposition 4.32. *Relations of the type* $\mathbf{I} = G\mathbf{V}$ *or* $\mathbf{V} = R\mathbf{I}$*, with* G, R *symmetric matrices, represent a generalization of Green's theorem.*

Proof. Let $\mathbf{I} = G\mathbf{V}$, with G a symmetric matrix of order n. Therefore G is a self-adjoint operator. Then:

$$(\mathbf{V}, G\hat{\mathbf{V}}) = (G\mathbf{V}, \hat{\mathbf{V}}).$$

Therefore:

$$V^{\alpha_1}\hat{I}_{\alpha_1} + \cdots + V^{\alpha_n}\hat{I}_{\alpha_n} = \hat{V}^{\alpha_1}I_{\alpha_1} + \cdots + \hat{V}^{\alpha_n}I_{\alpha_n} \tag{4.51}$$

The proof for the case $\mathbf{V} = R\mathbf{I}$, with R symmetric matrix, is analogous. □

The Green's reciprocity theorem can also be derived as a consequence of the mesh-current solution. In the case where there are no current sources, so that $\mathbf{K} = \mathbf{0}$, we have:

$$\mathbf{J} = (sZ\sigma)^{-1}s(-\mathbf{W}).$$

Then:

$$I = \sigma\mathbf{J} = -\sigma(sZ\sigma)^{-1}s(\mathbf{W}).$$

Since Z is a symmetric matrix and s is the transpose matrix of σ, we have:

$$G = -\sigma(sZ\sigma)^{-1}s$$

is a symmetric matrix. Then:

$$\mathbf{I} = G\mathbf{W}. \tag{4.52}$$

So, by proposition 4.32, with the equation (4.52) we find the generalization of Green's theorem.

Similarly we can start from the node-potential solution,with the vector voltage source $\mathbf{W} = \mathbf{0}$, then:

$$\boldsymbol{\phi} = ([\partial]Z^{-1}[d])^{-1}[\partial]\mathbf{K}.$$

Then:

$$V = -[\mathrm{d}]\boldsymbol{\phi} = -[\mathrm{d}]([\partial]Z^{-1}[\mathrm{d}])^{-1}[\partial]\mathbf{K}.$$

Since $[\mathrm{d}]$ is the transpose of $[\partial]$, the matrix

$$R = -[\mathrm{d}]([\partial]Z^{-1}[\mathrm{d}])^{-1}[\partial]$$

is symmetrical. Therefore we have:

$$\mathbf{V} = R\mathbf{K} \tag{4.53}$$

Therefore, by proposition 4.32, the equation (4.53) represents the Green's generalized theorem.

Remark 4.33. Relations between $\mathbf{V}$ and $\mathbf{W}$, and between $\mathbf{I}$ and $\mathbf{K}$ generally are not described by symmetric matrices.

5 CAPACITIVE NETWORKS

Introduction

In this chapter we will treat circuits where their branches have at most a battery in series with a capacitor. When the battery is triggered, charges will accumulate on the plates of the capacitor until eventually a steady state is reached. In this state, no current will be flowing. At this time, we are interested in discovering the charge on nodes and capacitors, and the voltage on the branches. To achieve this goal, in this chapter we will study the resolution of the Poisson equation and the Dirichlet problem, ending with the study of Green's functions. The importance of studying capacitive circuits lies in the fact that within the electromagnetism, the study of these circuits is equivalent to the study of a discretization of electrostatics.

5.1 Sign's Conventions

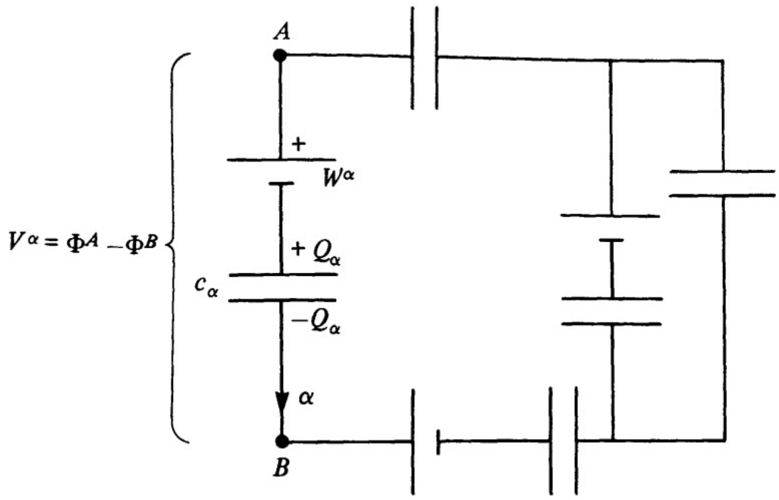


Figure 19 – Sign's conventions.

$\mathbf{V}^\alpha$ and $\mathbf{W}^\alpha$: positive V^α and W^α refer to drop of potential when the branch is traversed in the sense defined by the arrow.

$\mathbf{Q}_\alpha$: the sign of Q_α is defined by the equation: $Q_\alpha = C_\alpha(V_\alpha - W_\alpha)$, where $C_\alpha > 0$.

Plates of a capacitor: in the sense defined by the arrow, the *first* plate of the capacitor will be positive (observe the figure 19). In this case, your charge will have the same sign as Q_α (charge of capacitor). The negative plate will have opposite sign to the sign of Q_α.

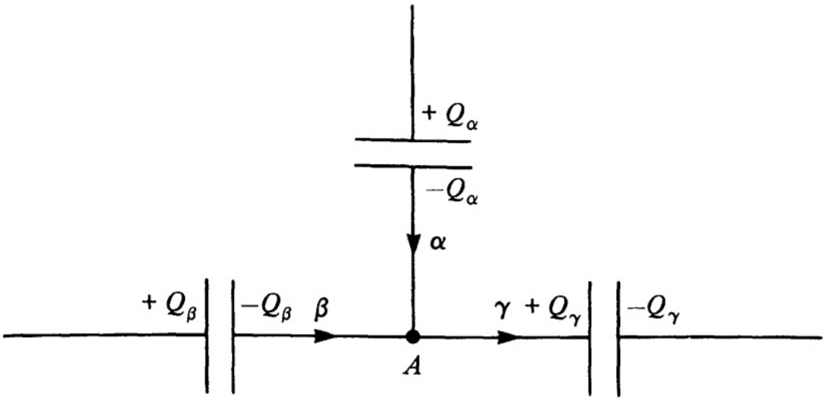


Figure 20 – Charge at the node A.

$\boldsymbol{\rho}$: is a zero-chain that represents the total charge on each node. Each node is connected to the positive and\or negative plates the some capacitors. Therefore, on each node, to find the resulting charge, we will use the following convention:

$$\rho_A = \underbrace{\sum Q_i}_{\text{positive plates}} - \underbrace{\sum Q_j}_{\text{negative plates}} \tag{5.1}$$

where Q_i, Q_j are, for example, the charges of the capacitors i, j.

Example 5.1. Observing the figure 20, we have:

$$\rho_A = -Q_\alpha - Q_\beta + Q_\gamma \tag{5.2}$$

5.2 Some Analogies with Resistive Networks

In capacitive circuits, we have a vector $\mathbf{Q} \in C_1$, where its coordinates represent the charges of capacitors in their respective branches.

$$\mathbf{Q} = (Q_\alpha, Q_\beta, Q_\gamma, \dots). \tag{5.3}$$

The vectors $\mathbf{V}, \mathbf{W} \in C^1$, as before, represent respectively the voltages and voltage sources in each branch.

The matrix C is a diagonal matrix where its inputs are the capacitances of each branch of the circuit. Therefore $C : C^1 \to C_1$ will represent an isomorphism between the spaces C^1 and C_1.

From the figure 19, we deduce the following general equation *for each branch* of the capacitive circuit:

$$V^\alpha - W^\alpha = \frac{Q_\alpha}{C_\alpha}. \tag{5.4}$$

Generalizing the equation (5.4) for all branches of the circuit, we get the following matrix equation:

$$\mathbf{V} - \mathbf{W} = C^{-1}\mathbf{Q}. \tag{5.5}$$

Lemma 5.2. *Let* $\mathbf{Q} \in C_1$ *be the vector representing the charges of the capacitors, and let* $\boldsymbol{\rho} \in C_0$ *be the charges at the nodes. Then:*

$$\partial Q = -\boldsymbol{\rho}. \tag{5.6}$$

The equation (5.6) *is known as* ***Gauss' Law****.*

Proof. Analyzing equation (5.1) and noting that the negative plate of the capacitor is always located close to the end node of the branch, whereas the positive plate of the capacitor is located next the starting node, we can infer that the charges on the negative plate are always *going* toward the end node, while the charges on the positive plate are always *leaving* of the initial node. Then:

$$\rho_A = \underbrace{\overset{\text{positive plates}}{\sum Q_i}}_{\text{leave to A}} \quad - \quad \underbrace{\overset{\text{negative plates}}{\sum Q_j}}_{\text{go to A}} \tag{5.7}$$

Clearly the relation on the right-hand side of equation (5.7) is equal to the relation of the right-hand side of equation (3.2), but with reversed sign. □

Remark 5.3. As a consequence of the fact that the sum of the charges of the two plates of a capacitor is equal to zero, we have:

$$\sum_{\text{all nodes}} \rho_A = 0 \tag{5.8}$$

At this time, we will make some considerations. Firstly, let the voltage $\mathbf{V}$ satisfy Kirchhoff's voltage law, i.e.:

$$V = [\mathrm{d}]\boldsymbol{\phi}. \tag{5.9}$$

Now suppose that initially the capacitors of the circuit are discharged. Then, after charging them, by the conservation of charges, we have $\partial\mathbf{Q} = \mathbf{0}$. Therefore, in analogy with what was done in the mesh-current method, we introduce the *mesh charges*, described by the vector $\mathbf{P} \in Z_1$, such that:

$$\mathbf{Q} = \sigma(\mathbf{P}). \tag{5.10}$$

So replace $\mathbf{I}$ for $\mathbf{Q}$, $\mathbf{J}$ for $\mathbf{P}$, Z for C^{-1} and making $\mathbf{K} = \mathbf{0}$ (since there are no current sources in our capacitive circuit), we observe that the equations (5.5), (5.9), (5.10) are completely analogous to those equations found in the analysis of resistive circuits. Therefore:

$$\mathbf{P} = (s\mathbf{C}^{-1}\sigma)^{-1}(-s\mathbf{W}), \tag{5.11}$$

$$\boldsymbol{\phi} = -([\partial]C[\mathrm{d}])^{-1}[\partial]C\mathbf{W}. \tag{5.12}$$

Remark 5.4. The same considerations about *existence and uniqueness* of charge $\mathbf{Q}$ and voltage $\mathbf{V}$ are also valid in this new context, because the matrix C has only positive entries.

5.3 Poisson's Equation

Until the end of this chapter, we will assume that the capacitive circuits do not have voltage sources, i.e., $\mathbf{W} = \mathbf{0}$. Therefore, manipulating the equations

$$\mathbf{Q} = C\mathbf{V}, \quad \mathbf{V} = -\mathrm{d}\boldsymbol{\phi} \text{ and } \partial\mathbf{Q} = -\boldsymbol{\rho}$$

we deduce that:

$$-\partial C\mathrm{d}\boldsymbol{\phi} = -\boldsymbol{\rho} \tag{5.13}$$

The equation (5.13) is known as **Poisson's equation**. The operator $-\partial C\mathrm{d}$ is called *Laplacian* and is denoted by Δ. With this, we have $\Delta : C^0 \to C_0$ and the equation (5.13) can be rewritten as

$$\Delta\boldsymbol{\phi} = -\boldsymbol{\rho}. \tag{5.14}$$

Lemma 5.5. *The Laplacian Δ is a symmetric operator.*

Proof. Indeed, since $\partial = (\mathrm{d})^{\mathrm{T}}$ for the matrices of the operators ∂ and d with respect to canonical basis, we have:

$$\Delta^{\mathrm{T}} = (-\partial C\mathrm{d})^{\mathrm{T}} = -\mathrm{d}^{\mathrm{T}}C^{\mathrm{T}}\partial^{\mathrm{T}} = -\partial C\mathrm{d} = \Delta.$$

□

Lemma 5.6. *Let $\mathbf{A}$ be a node of the circuit and let $\mathbf{u} \in C^0$, then:*

$$(\Delta\mathbf{u})(\mathbf{A}) = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})) \tag{5.15}$$

where we are summing over all the branches $\boldsymbol{\alpha}$ such that $\partial\boldsymbol{\alpha} = \pm(\mathbf{B} - \mathbf{A})$ for some $\mathbf{B} \in C_0$. The charge on node A is given by:

$$\rho_A = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{A}) - \mathbf{u}(\mathbf{B})) \tag{5.16}$$

Proof. Let $\boldsymbol{\alpha}$ be a branch with $\mathbf{A}$ as one of its nodes, and $\mathbf{B}$ as the other. Then:

$$\mathrm{d}\mathbf{u}(\boldsymbol{\alpha}) = \mathbf{u}\partial(\boldsymbol{\alpha}) = \pm\mathbf{u}(\mathbf{B} - \mathbf{A}) = \pm(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})). \tag{5.17}$$

The formula of the functional $\mathrm{d}\mathbf{u}$ summing *only* over the branches α that have $\mathbf{A}$ as one of its nodes is equal to:

$$\mathrm{d}\mathbf{u} = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} \lambda_\alpha\boldsymbol{\alpha}^* \tag{5.18}$$

where $\boldsymbol{\alpha}^*$ is an element of the canonical basis of the dual space C^1. By (5.17) and (5.18), we have:

$$\mathrm{d}\mathbf{u} = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} \pm(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))\boldsymbol{\alpha}^* \tag{5.19}$$

Multiplying (5.19) by the matrix of the capacitances C, we have:

$$C\mathrm{d}\mathbf{u} = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} \pm(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))C_\alpha\boldsymbol{\alpha} \tag{5.20}$$

Applying in (5.20) the operator $-\partial$, we have:

$$-\partial C\mathrm{d}\mathbf{u} = -\sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} \pm(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))C_\alpha\partial\boldsymbol{\alpha}.$$

Then:

$$-\partial C\mathrm{d}\mathbf{u} = -\sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} \pm(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))C_\alpha(\pm(\mathbf{B}-\mathbf{A})). \tag{5.21}$$

Isolating the node $\mathbf{A}$, we have the following result :

$$-\partial C\mathrm{d}\mathbf{u}(A) = \left(\sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))\right)A. \tag{5.22}$$

Therefore:

$$(\Delta\mathbf{u})(\mathbf{A}) = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})). \tag{5.23}$$

To calculate the *charge* on node $\mathbf{A}$ by (5.14) and (5.23), we have:

$$\rho_A = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{A}) - \mathbf{u}(\mathbf{B})). \tag{5.24}$$

□

Now observe that, in particular, $\mathbf{u}$ satisfies the *Laplace's equation*

$$\Delta\mathbf{u} = \mathbf{0}. \tag{5.25}$$

if, and only if, for each node $\mathbf{A}$, we have:

$$\mathbf{u}(\mathbf{A}) = \frac{1}{\sum C_\alpha}\sum C_\alpha\mathbf{u}(\mathbf{B}) \tag{5.26}$$

summed over all branches $\boldsymbol{\alpha}$ which has the node $\mathbf{A}$ at one end. Therefore, Laplace's equation tells us that the potential in each node is the weighted average of the potentials at nearest neighbor nodes, with the weight being given by capacitances .

Definition 5.7. Let $[\mathrm{d}] : P^0 \to C^1$ be the restricted coboundary map and let $[\partial] : C_1 \to B_0$ be the restricted boundary map. Then we define the *restricted Laplacian* as:

$$[\Delta] = -[\partial]C[\mathrm{d}].$$

Theorem 5.8. *The operator* $[\Delta]$ *is invertible.*

Proof. Analogous to the proof of theorem 3.70, just replacing Z^{-1} by C. $\square$

Corollary 5.9. *For a circuit with ground node,* $\Delta = [\Delta]$.

Proof. For a circuit with ground node, d is injective. Therefore $\mathrm{d} = [\mathrm{d}], \partial = [\partial]$ and $\Delta = [\Delta]$. $\square$

Let $\boldsymbol{\psi} \in P^0$ and consider the *Poisson's restricted equation*:

$$[\Delta]\boldsymbol{\psi} = -\boldsymbol{\rho}. \tag{5.27}$$

The solution of Poisson's restricted equation is given by:

$$\boldsymbol{\psi} = -[\Delta]^{-1}\boldsymbol{\rho}. \tag{5.28}$$

Remark 5.10. From equations (5.27) and (5.28), in agreement with what was previously seen, the rows corresponding to the *circuit's ground* of $\boldsymbol{\psi} \in P^0$ and $\boldsymbol{\rho} \in B_0$ are **eliminated**.

The diagram of the figure 21 relates the domains of the Laplacian and restricted Laplacian.

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[rrrrr, bend right, "-\Delta"'] & P^0 \arrow[r, "{[\mathrm{d}]}"] & C^1 \arrow[r, "C"] & C_1 \arrow[r, "{[\partial]}"] & B_0 \arrow[r] & C_0
\end{tikzcd}$$

Figure 21 – Diagram of the restricted Laplacian.

Example 5.11. Consider figure 22. 4 units of charge are at node **B** and 1 unit at node **C**. The node **A** is the ground of the circuit. Find the potential of **B** and **C** and the charge on the node **A**(as units, use microfarads for capacitance, microcoulombs for charge and volts for potential). Analyzing the figure 22, we have:

$$[\partial] = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad [\mathrm{d}] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then:

$$[\partial]C[\mathrm{d}] = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$$

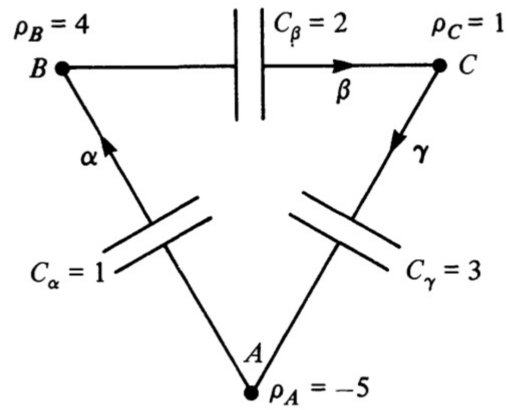


Figure 22 – Poisson's Equation (example 5.11).

Therefore, the solution of Poisson's equation is :

$$([\partial]C[\mathrm{d}])^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}, \quad \boldsymbol{\psi} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then we have $\boldsymbol{\phi}^B = 2\ V$ and $\boldsymbol{\phi}^C = 1\ V$. Then $V^\alpha = -2\ V$ and $V^\gamma = 1\ V$. Using $\mathbf{Q} = C\mathbf{V}$, we find that $Q_\alpha = -2\ \mu C$ and $Q_\gamma = 3\ \mu C$. As

$$\boldsymbol{\rho}_A = Q_\alpha - Q_\gamma \tag{5.29}$$

then $\boldsymbol{\rho}_A = -5\ \mu C$.

5.4 Boundary and Interior Nodes

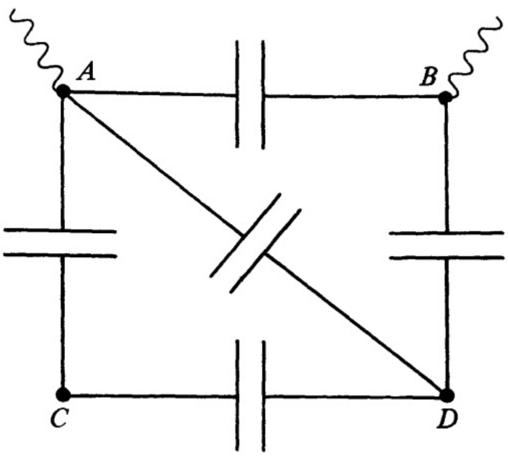


Figure 23 – Boundary nodes and interior nodes.

Imagine a circuit of capacitors, with no battery, like in figure 23. We subdivide the nodes into two types: **boundary nodes and interior nodes**. Boundary nodes, as **A** and **B**, are connected to external sources. These external sources maintain their potential in a specific value. Interior nodes, such as **C** and **D**, do not connect to any external source, but only with the others nodes of the circuit. With this, we have a decomposition of the space C_0 (charges of the nodes) and C^0 (potentials of the nodes).

The space C_0 has the following decomposition:

$$C_0 = C_0^{\text{bound}} \oplus C_0^{\text{int}} \tag{5.30}$$

where C_0^{bound} consists of all zero-chains where the only *nonzero* coordinates are related to the boundary nodes, and C_0^{int} consists of all zero-chains where the only *nonzero* coordinates are related to the interior nodes.

Similarly, the space C^0 will have the following decomposition:

$$C^0 = C^0_{\text{bound}} \oplus C^0_{\text{int}} \tag{5.31}$$

where C^0_{bound} consists of all linear functionals that vanish on C_0^{int}. Analogously, C^0_{int} consists of all linear functionals that vanish on C_0^{bound}.

Example 5.12. In figure 23, we have that **A** and **B** are boundary nodes, while **C** and **D** are interior nodes. Then by (5.30), we can decompose ρ in a unique way as the sum of an element of C_0^{bound} and an element of C_0^{int}.

$$\boldsymbol{\rho} = \boldsymbol{\rho}^{\text{bound}} + \boldsymbol{\rho}^{\text{int}} \tag{5.32}$$

i.e.,

$$\begin{pmatrix} \rho_A \\ \rho_B \\ \rho_C \\ \rho_D \end{pmatrix} = \begin{pmatrix} \rho_A \\ \rho_B \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \rho_C \\ \rho_D \end{pmatrix}. \tag{5.33}$$

On the other hand, the potential $\boldsymbol{\phi}$ can be uniquely decomposed as the sum of an element of C^0_{bound} with an element of C^0_{int}.

$$\boldsymbol{\phi} = \boldsymbol{\phi}_{\text{bound}} + \boldsymbol{\phi}_{\text{int}} \tag{5.34}$$

i.e.,

$$\begin{pmatrix} \phi^A \\ \phi^B \\ \phi^C \\ \phi^D \end{pmatrix} = \begin{pmatrix} \phi^A \\ \phi^B \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \phi^C \\ \phi^D \end{pmatrix}. \tag{5.35}$$

In a general problem, the potential of each boundary node is specified, and the charge of each interior node will also be provided. With this, we want to find the charge of boundary nodes and the potential of interior nodes. This general problem can be expressed as the superposition of two simpler problems:

1. **Dirichlet Problem**:
   Data provided: the charges of interior nodes are all zero, ie, $\boldsymbol{\rho}^{\text{int}} = 0$ and the potential $\boldsymbol{\phi}_{\text{bound}}$ of the boundary nodes is provided.
   We need to find: the potentials of the interior nodes $\boldsymbol{\phi}_{\text{int}}$ and the charges of the boundary nodes $\boldsymbol{\rho}^{\text{bound}}$.

   To find $\boldsymbol{\phi}_{\text{int}}$, we use the equation:

   $$\Delta(\boldsymbol{\phi}_{\text{int}} + \boldsymbol{\phi}_{\text{bound}}) = \mathbf{0} \quad \text{at all interior nodes.} \tag{5.36}$$

   Now to find $\boldsymbol{\rho}^{\text{bound}}$, we use:

   $$\Delta(\boldsymbol{\phi}_{\text{int}} + \boldsymbol{\phi}_{\text{bound}}) = -\boldsymbol{\rho}^{\text{bound}}. \tag{5.37}$$

2. **Poisson equation problem**:
   Data provided: the potentials of the boundary nodes are all nulls, ie, $\boldsymbol{\phi}_{\text{bound}} = \mathbf{0}$, and the charges of the interior nodes $\boldsymbol{\rho}^{\text{int}}$ are provided.
   We need to find: the potentials of the interior nodes $\boldsymbol{\phi}_{\text{int}}$ and the charges of the boundary nodes $\boldsymbol{\rho}^{\text{bound}}$.

   To find $\boldsymbol{\phi}_{\text{int}}$, we use the equation:

   $$\Delta(\boldsymbol{\phi}_{\text{int}}) = -\boldsymbol{\rho}^{\text{int}} \quad \text{at all interior nodes.} \tag{5.38}$$

   Now, to find $\boldsymbol{\rho}^{\text{bound}}$, we use:

   $$\Delta(\boldsymbol{\phi}_{\text{int}}) = -\boldsymbol{\rho}^{\text{bound}} \quad \text{at all boundary nodes.} \tag{5.39}$$

5.5 Decomposition of C^1

Definition 5.13. For $C : C^1 \to C_1$, we define the **inner product** $(,)_C : C^1 \times C^1 \to \mathbb{R}$ as follows:

$$(\mathbf{V}, \hat{\mathbf{V}})_C = \int_{C\mathbf{V}} \hat{V} = \sum_{\text{all branches}} C_\alpha V^\alpha \hat{V}^\alpha \tag{5.40}$$

Remark 5.14. The inner product (5.40) is positive definite because the matrix C is diagonal, and its diagonal entries are positive.

Remark 5.15. We can represent the total energy stored in the capacitors as follows:

$$\frac{1}{2}(V, V)_C = \sum \frac{1}{2} C_\alpha (V^\alpha)^2 \tag{5.41}$$

Proposition 5.16. *The space C^1 can be decomposed in the following orthogonal direct sum:*

$$C^1 = \mathrm{d}(C^0) \oplus C^{-1} Z_1 \tag{5.42}$$

Proof. We must show that $C^{-1}Z_1$ is the orthogonal complement of $\mathrm{d}(C^0)$. First, we have $C^{-1}Z_1 \subset [\mathrm{d}(C^0)]^{\perp_{\mathrm{compl}}}$. In fact, for $\mathbf{V} \in \mathrm{d}(C^0)$ and for some $\mathbf{I} \in Z_1$, $C^{-1}\mathbf{I} \in C^{-1}Z_1$, we have:

$$(\mathbf{V}, C^{-1}\mathbf{I})_C = \int_{C(C^{-1}\mathbf{I})} \mathbf{V} = \int_{\mathbf{I}} \mathbf{V} = 0 \tag{5.43}$$

To prove that $C^{-1}Z_1 = [\mathrm{d}(C^0)]^{\perp_{\mathrm{compl}}}$, we will show that $\dim(C^{-1}Z_1) = \dim[\mathrm{d}(C^0)]^{\perp_{\mathrm{compl}}}$. By (2.43), we know that:

$$C^1/\mathrm{d}C^0 \approx Z_1^* \tag{5.44}$$

$$\Rightarrow \dim\, C^1 - \ \dim\, \mathrm{d}C^0 = \ \dim\, Z_1^* = \ \dim\, Z_1$$

$$\Rightarrow \dim\, C^1 = \ \dim\, \mathrm{d}C^0 + \ \dim\, Z_1 \tag{5.45}$$

We show that $\dim\, C^{-1}Z_1 = \ \dim\, Z_1$. In fact, since

$$C^{-1}_{|Z_1} : Z_1 \to \ \mathrm{Im}\, C^{-1}Z_1 \tag{5.46}$$

is an isomorphism. □

Definition 5.17. The subspace D^1 is, by definition, the orthogonal complement of $\mathrm{d}C^0_{\mathrm{int}}$ with respect to the space $\mathrm{d}C^0$.

As a consequence of the definition (5.17), we have:

$$\mathrm{d}(C^0) = \mathrm{d}C^0_{\mathrm{int}} \oplus D^1. \tag{5.47}$$

By (5.42) e (5.47), we have:

$$C^1 = D^1 \oplus \mathrm{d}C^0_{\mathrm{int}} \oplus C^{-1}Z_1. \tag{5.48}$$

The identity (5.48) is a decomposition of C^1 into three mutually orthogonal subspaces.

Remark 5.18. For the next lemma, we will use the notation: m = number of meshes, n_b = number of boundary nodes and n_i = number of interior nodes.

Lemma 5.19.

$$\dim\, D^1 = n_b - 1. \tag{5.49}$$

Proof. (i) By (5.46), we know that $\dim\, C^{-1}Z_1 = \ \dim\, Z_1 = m$.
(ii) If there is *at least* a boundary node, then $\mathrm{d} : C^0_{\mathrm{int}} \to C^1$ is injective since, as we have seen in proposition 3.52, Z^0 is the space of constant potentials. With at least one boundary node, there is at least one coordinate with null value. Therefore, the only

constant potential we may have will be the constant vector $\mathbf{0}$. Therefore, by theorem 2.2, we have $\dim \mathrm{d}C^0_{\text{int}} = \dim C^0_{\text{int}} = n_i$.
(iii) We know that $\dim C^1 = n - 1 + m$, i.e.:

$$\dim C^1 = n_i + n_b - 1 + m \tag{5.50}$$

By equation (5.48), we have:

$$\dim C^1 = \dim D^1 + \dim C^0_{\text{int}} + \dim C^{-1}Z_1 \tag{5.51}$$

By equations (5.50) and (5.51), united to the results (i) and (ii), we have:

$$\dim D^1 = n_b - 1 \tag{5.52}$$

□

Proposition 5.20. $\mathrm{d}\boldsymbol{\phi} \in D^1$ *if and only if* $\boldsymbol{\phi}$ *is a solution of the Dirichlet problem.*

Proof. Since $D^1 \subset \mathrm{d}C^0$, then if $\mathbf{V} \in D^1$ we have $\mathbf{V} = \mathrm{d}\boldsymbol{\phi}$ and $\mathbf{V} \perp \mathrm{d}C^0_{\text{int}}$. Then:

$$0 = (\mathbf{V}, \mathrm{d}\boldsymbol{\phi}_{\text{int}})_C = \int_{C\mathbf{V}} \mathrm{d}\boldsymbol{\phi}_{\text{int}} = \int_{\partial C\mathbf{V}} \boldsymbol{\phi}_{\text{int}}, \quad \forall \boldsymbol{\phi}_{\text{int}} \in C^0_{\text{int}}. \tag{5.53}$$

As for any interior node, we can find a function $\boldsymbol{\phi}_{\text{int}}$ that not annuls only in this node, then by (5.53), we have $\partial C\mathbf{V} \in C^{\text{bound}}_0$. Substituting $V = d\boldsymbol{\phi}$, we have

$$\Delta\boldsymbol{\phi} = 0, \quad \text{at all interior nodes,} \tag{5.54}$$

Therefore, $\boldsymbol{\phi}$ is a solution of the Dirichlet problem.

Conversely, if $\boldsymbol{\phi} \in C^0$ is a solution of the Dirichlet problem, then:

$$\partial C \mathrm{d}\boldsymbol{\phi} = \mathbf{0}, \quad \text{at all interior nodes.} \tag{5.55}$$

Considering $V = \mathrm{d}\boldsymbol{\phi}$, we have:

$$0 = \int_{\partial C\mathrm{d}\boldsymbol{\phi}} \boldsymbol{\phi}_{\text{int}} = \int_{\partial C\mathbf{V}} \boldsymbol{\phi}_{\text{int}} = \int_{C\mathbf{V}} \mathrm{d}\boldsymbol{\phi}_{\text{int}} = (\mathrm{d}\boldsymbol{\phi}_{\text{int}}, \mathbf{V})_C, \quad \forall \boldsymbol{\phi}_{\text{int}} \in C^0_{\text{int}}. \tag{5.56}$$

Therefore, by definition 5.17, we have $V \in D^1$. □

Example 5.21. The capacitive circuit of the figure 24, where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are boundary nodes (where $\mathbf{A}$ is the ground node), while $\mathbf{D}$ is an interior node. We know that $C^1 = \mathrm{d}C^0_{\text{int}} \oplus D^1 \oplus C^{-1}Z_1$. We want to find the bases of the subspaces $\mathrm{d}C^0_{\text{int}},\ D^1,\ C^{-1}Z_1$. From figure 24, we find the following matrices:

$$\partial = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \mathrm{d} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.57}$$

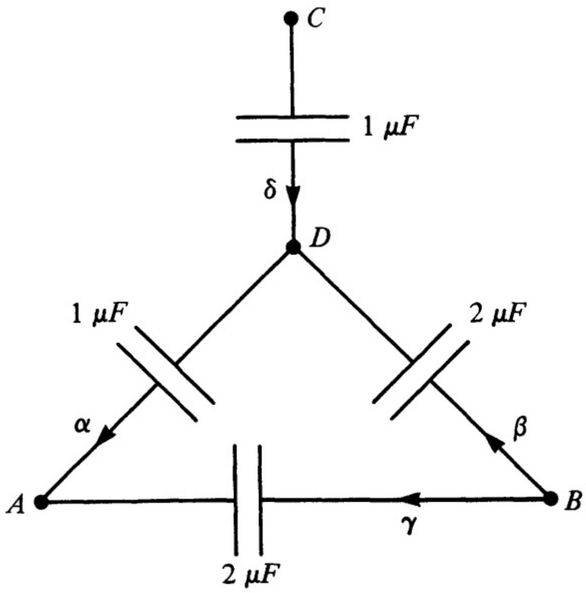


Figure 24 – Capacitive circuit.

As we have 4 branches, then dim $C^1 = 4$. As we only have one mesh, dim $Z_1 = 1 \Rightarrow$ dim $C^{-1}Z_1 = 1$. . Thus, for example, an element of the basis of $C^{-1}Z_1$ is:

$$C^{-1}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \tag{5.58}$$

Remembering that as the circuit has a ground node in **A**, this implies that d is an injective operator (consequence of the corollary 3.53). Therefore, as we have only one interior node, then dim $\mathrm{d}C^0_{\mathrm{int}} = 1$. So just find a vector $\mathrm{d}C^0_{\mathrm{int}}$ to find a base. So, for $\boldsymbol{\phi}_{\mathrm{int}} = (0,0,0,1)$, we have:

$$\mathrm{d}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{5.59}$$

By decomposition (5.48), , we conclude that dim $D^1 = 2$. As $\Delta = -\partial C\mathrm{d}$, then, from the matrices (5.57), we find:

$$\Delta = \begin{pmatrix} -3 & 2 & 0 & 1 \\ 2 & -4 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & -4 \end{pmatrix}. \tag{5.60}$$

Choosing two linearly independent potential-vectors $(0,1,0,\boldsymbol{\phi}_D)$, $(0,0,1,\boldsymbol{\phi}_D)$ and solving the two following Dirichlet problems, we find the following potential for the interior node:

$$\Delta.\begin{pmatrix} 0 \\ 1 \\ 0 \\ \boldsymbol{\phi}_D \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{\phi}_D = \frac{1}{2}. \tag{5.61}$$

Therefore, we have the solution $\boldsymbol{\phi} = (0, 1, 0, \frac{1}{2})$.

$$\Delta.\begin{pmatrix} 0 \\ 0 \\ 1 \\ \boldsymbol{\phi}_D \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{\phi}_D = \frac{1}{4}. \tag{5.62}$$

Therefore, we have the solution $\boldsymbol{\phi} = (0, 0, 1, \frac{1}{4})$.

With this, we find the following two linearly independent vectors in D^1:

$$\mathrm{d}\boldsymbol{\phi} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}, \tag{5.63}$$

$$\mathrm{d}\boldsymbol{\phi} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 0 \\ -\frac{3}{4} \end{pmatrix}. \tag{5.64}$$

So after multiplication by suitable scalars, we find the following basis for the subspace D^1:

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -3 \end{pmatrix} \right\}. \tag{5.65}$$

5.6 Solution of the Boundary-value Problem by Weyl's Method of Orthogonal Projection

5.6.1 Poisson's Equation

Suppose that a charge $\boldsymbol{\rho}$ is specified for all interior nodes. We want to find out the solution of Poisson's equation given this interior charge. For this, we build a distribution of voltage $\hat{\mathbf{V}}$ such that $-\partial C\hat{\mathbf{V}} = \boldsymbol{\rho}$, at all interior nodes. Since $\hat{\mathbf{V}} \in C^1$, by (5.48), we have:

$$\hat{\mathbf{V}} = \mathbf{V} + \mathbf{U} + \mathbf{W} \tag{5.66}$$

where $\mathbf{V} \in \mathrm{d}C^0_{\text{int}},\ \mathbf{U} \in D^1,\ \mathbf{W} \in C^{-1}Z_1$. Note that $\partial C(\mathbf{U}+\mathbf{W}) = \partial C(\mathbf{U}) + \partial C(\mathbf{W}) = \mathbf{0}^{\text{int}}$ at all interior nodes, because $\partial C(\mathbf{U}) = \mathbf{0}^{\text{int}}$ at all interior nodes (Dirichlet's problem), and, since $\mathbf{W} \in C^{-1}Z_1,\ \exists\, \mathbf{I} \in Z_1$ such that $\mathbf{W} = C^{-1}\mathbf{I}$. So $\partial C(\mathbf{W}) = \partial C C^{-1}\mathbf{I} = \partial \mathbf{I} = \mathbf{0}$. Therefore, we have:

$$-\partial C\hat{\mathbf{V}} = -\partial C\mathbf{V} = \boldsymbol{\rho}, \quad \text{at all interior nodes.} \tag{5.67}$$

If we denote by π the orthogonal projection of C^1 on $\mathrm{d}C^0_{\text{int}}$, we have:

$$\mathbf{V} = \pi\hat{\mathbf{V}}. \tag{5.68}$$

Now let $\boldsymbol{\phi}_{\text{int}}$, such that $V = -\mathrm{d}\boldsymbol{\phi}_{\text{int}}$. Then by (5.67), we have $\boldsymbol{\phi}_{\text{int}}$ is the solution of the Poisson equation.

5.6.2 Dirichlet's Problem

We want to find out the solution of the Dirichlet problem for a specific value of the potentials at the boundary nodes. Let us denote by $\hat{\boldsymbol{\phi}}$ the potential that is equal to zero for the interior nodes and equal to a specified value at the boundary nodes, imposed by the Dirichlet problem. With this, let $\hat{\mathbf{V}} = -\mathrm{d}\hat{\boldsymbol{\phi}}$. So, since π is the orthogonal projection of C^1 on $\mathrm{d}C^0_{\text{int}}$, we have:

$$\hat{\mathbf{V}} = \pi\hat{\mathbf{V}} + (1-\pi)\hat{\mathbf{V}}. \tag{5.69}$$

Since $\hat{\mathbf{V}} \in \mathrm{d}C^0$, then by (5.47), we have $(1-\pi)\hat{\mathbf{V}} \in D^1$. Therefore, $\pi\hat{\mathbf{V}} = -\mathrm{d}\boldsymbol{\psi}$, with $\boldsymbol{\psi} \in C^0_{\text{int}}$, and $(1-\pi)\hat{\mathbf{V}} = -\mathrm{d}\boldsymbol{\phi}$, with $\Delta\boldsymbol{\phi} = \mathbf{0}$, at all interior nodes. Then:

$$-\mathrm{d}\hat{\boldsymbol{\phi}} = -\mathrm{d}\boldsymbol{\psi} - \mathrm{d}\boldsymbol{\phi} = -\mathrm{d}(\boldsymbol{\psi} - \boldsymbol{\phi}). \tag{5.70}$$

Considering that the circuit has a ground, then d is injective. Thus, for $\boldsymbol{\psi} \in C^0_{\text{int}}$, we have $\hat{\boldsymbol{\phi}} = \boldsymbol{\phi}$ for all boundary nodes. Therefore, $\boldsymbol{\phi}$ is the desired solution to the Dirichlet problem.

Example 5.22. In this example, we will calculate the Weyl's projection and then we will solve the Poisson equation and the Dirichlet problem. First, let's look at the previous example, the equation (5.59), and note that a possible basis of $\mathrm{d}C^0_{\text{int}}$ is:

$$\mathbf{U} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}. \tag{5.71}$$

Finding the norm of $\mathbf{U}$, we have:

$$\|\mathbf{U}\| = \sqrt{(\mathbf{U}, \mathbf{U})_C} = 2. \tag{5.72}$$

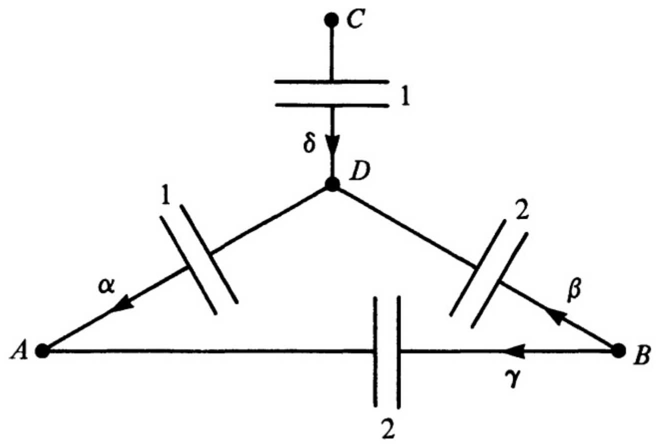


Figure 25 – Weyl's projection method.

Then, for any $\mathbf{V} \in C^1$, we have:

$$\pi\mathbf{V} = \frac{1}{4}(\mathbf{V}, \mathbf{U})_C \mathbf{U} \tag{5.73}$$

We know that $\boldsymbol{\pi} : C^1 \to \mathrm{d}C^0_{\text{int}}$. So to find the matrix of $\boldsymbol{\pi}$, we apply $\boldsymbol{\pi}$ to the vectors of the basis $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

With this, we find:

$$\boldsymbol{\pi} = \frac{1}{4}\begin{pmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (1-\boldsymbol{\pi}) = \frac{1}{4}\begin{pmatrix} 3 & 2 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 1 & -2 & 0 & 3 \end{pmatrix}. \tag{5.74}$$

Look at figure 25 and suppose that $\boldsymbol{\rho}_D = 1$. We want to solve the Poisson equation. One possibility is to have $\hat{Q}_\alpha = \hat{Q}_\beta = \hat{Q}_\gamma = 0, \hat{Q}_\delta = -1$, then $\mathbf{V} = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$. Then $\boldsymbol{\pi}\hat{\mathbf{V}} = \frac{1}{4}\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$. Then, since $\mathbf{A}$ is a ground, we have:

$$\boldsymbol{\phi} = \begin{pmatrix}0\\0\\0\\\frac{1}{4}\end{pmatrix} \quad \text{and} \quad \boldsymbol{\rho} = \frac{1}{4}\begin{pmatrix}-1\\-2\\-1\\4\end{pmatrix}. \tag{5.75}$$

as solution of Poisson's equation.

We now will solve the Dirichlet problem assuming that to the figure 25, we have $\boldsymbol{\phi}^A = 0, \boldsymbol{\phi}^B = 5, \boldsymbol{\phi}^C = 6$. Let's build $\hat{\boldsymbol{\phi}}$, such that $\hat{\boldsymbol{\phi}} = \boldsymbol{\phi}$ for all boundary nodes and $\hat{\boldsymbol{\phi}}^D = 0$.Then we have:

$$\hat{\mathbf{V}} = -d\hat{\boldsymbol{\phi}} = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 6 \end{pmatrix}. \tag{5.76}$$

So, we have: $\mathbf{V} = (1-\pi)\hat{\mathbf{V}} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 1 & -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 5 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \\ 2 \end{pmatrix}.$

With this, we have the solution of the Dirichlet problem $\boldsymbol{\phi}$, such that $V = -d\boldsymbol{\phi}$, is equal to:

$$\boldsymbol{\phi} = \begin{pmatrix} 0 \\ 5 \\ 6 \\ 4 \end{pmatrix}.$$

5.7 Green's Functions

Before starting this section, we will demonstrate the following lemma.

Lemma 5.23. *For a capacitive circuit with ground node, let* $\Delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be the Laplace operator write in block form with* A *a invertible matrix of order equal to* $\dim C_0^{\text{int}}$*. Then the solution of the Poisson equation problem exist and is unique.*

Proof. The existence of the solution is guaranteed by the corollary 5.9 and theorem 5.8. To prove uniqueness, consider the following Poisson equation problem:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_{\text{int}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\rho}^{\text{int}} \\ -\boldsymbol{\rho}^{\text{bound}} \end{pmatrix}$$

Then:

$$\begin{aligned} A\boldsymbol{\phi}_{\text{int}} &= -\boldsymbol{\rho}^{\text{int}} \\ C\boldsymbol{\phi}_{\text{int}} &= -\boldsymbol{\rho}^{\text{bound}} \end{aligned}$$

As the block A is invertible, then:

$$\boldsymbol{\phi}_{\text{int}} = -A^{-1}\boldsymbol{\rho}^{\text{int}}.$$

Then $\boldsymbol{\phi}_{\text{int}}$ is uniquely determined and hence $\boldsymbol{\rho}^{\text{bound}}$ is also uniquely determined. □

Therefore, to ensure the uniqueness of solutions of the Poisson equation problem, we will consider throughout this section that the capacitive circuit under study has the Laplace operator with block A invertible.

Definition 5.24. The map $\mathbf{G} : C_0^{\text{int}} \to C_{\text{int}}^0$ which gives to the charge distribution ρ (restricted to the interior nodes) the potential $\boldsymbol{\phi}_{\text{int}}$ which solves the Poisson equation is called the **Green's operator**.

Remark 5.25. The entries of the operator's matrix $\mathbf{G}$ are denoted by $G(\mathbf{A}, \mathbf{B})$ (line B, column A), with $\mathbf{A}, \mathbf{B} \in C_0$ nodes of the circuit. When $\mathbf{A}$ is a boundary node, we have $G(\mathbf{A}, \mathbf{B}) = 0$.

Proposition 5.26. *If* $\mathbf{A}$ *is an interior node, then* $G(\mathbf{A}, \mathbf{B})$ *is equal to the potential at node* $\mathbf{B}$ *result from solving the Poisson equation for a configuration of a unit charge at interior node* $\mathbf{A}$ *and charge zero at the remaining interior nodes.*

Proof. As the operator $\mathbf{G}$ solves the Poisson equation for a given configuration of charges on interior nodes, then multiplying the line $\mathbf{B}$ of the matrix of the operator $\mathbf{G}$ by the charge $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ of the interior node $\mathbf{A}$, we will have:

$$\boldsymbol{\phi}_B = \mathbf{G} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = G(\mathbf{A}, \mathbf{B}). \tag{5.77}$$

□

Remark 5.27. For every boundary node $\mathbf{B}$, we have $G(\mathbf{A}, \mathbf{B}) = 0$.

Definition 5.28. The entry $G(\mathbf{A}, \mathbf{B})$ of the matrix $\mathbf{G}$, with $\mathbf{A}, \mathbf{B} \in C_0$ nodes of the circuit, is a function of two variables called **Green's function**.

Remark 5.29. When the Green's function is considered as the first variable $\mathbf{A}$ fixed, with $\mathbf{A} \in C_0^{\text{int}}$, then we have:

$$G(\mathbf{A}, \cdot) \in C_{\text{int}}^0. \tag{5.78}$$

Of the remark 5.29, we have:

$$\Delta G(\mathbf{A}, \cdot) = \begin{cases} -1 & , \text{ if } G(\mathbf{A}, \mathbf{A}). \\ \\ 0 & , \text{ if } G(\mathbf{A}, \mathbf{B}), B \neq A \text{ e } B \in C_0^{\text{int}}. \end{cases} \tag{5.79}$$

Definition 5.30. Let π_2 be the orthogonal projection in the subspace C_0^{int}. Then:

$$\Delta_2 := \pi_2 \circ \Delta_{|_{C^0_{\mathrm{int}}}}. \tag{5.80}$$

As a consequence of the equation (5.80), we have:

$$\Delta_2 : C^0_{\mathrm{int}} \to C_0^{\mathrm{int}}. \tag{5.81}$$

Proposition 5.31. *The operator $\Delta_2 : C^0_{\mathrm{int}} \to C_0^{\mathrm{int}}$ is an isomorphism between the spaces C^0_{int} and C_0^{int}.*

Proof. By lemma 5.23, we know that in a circuit that has at least one boundary node(in this context, it is similar to have a ground node), each $\boldsymbol{p}^{\mathrm{int}}$ is associated with *only one* $\boldsymbol{\phi}_{\mathrm{int}}$ (and vice versa). Therefore Δ_2 is a bijection. □

Corollary 5.32. $\mathbf{G} = (-\Delta_2)^{-1}$.

Proof. Since Δ_2 is an isomorphism, we have:

$$-\Delta_2 \boldsymbol{\phi}_{\mathrm{int}} = \boldsymbol{\rho}_{\mathrm{int}} \Leftrightarrow (-\Delta_2)^{-1} \boldsymbol{\rho}_{\mathrm{int}} = \boldsymbol{\phi}_{\mathrm{int}}.$$

Therefore, by definition 5.24, we have $\mathbf{G} = (-\Delta_2)^{-1}$. Then:

$$-\Delta_2 \boldsymbol{\phi}_{\mathrm{int}} = \boldsymbol{\rho}_{\mathrm{int}} \Leftrightarrow \mathbf{G} \boldsymbol{\rho}_{\mathrm{int}} = \boldsymbol{\phi}_{\mathrm{int}}. \tag{5.82}$$

□

Remark 5.33. The corollary 5.32 ensures the existence of the Green's operator **G**.

Now, reformulating the definition (5.24), we have:

Definition 5.34. The map $\mathbf{G} : C_0^{\mathrm{int}} \to C^0_{\mathrm{int}}$, such that $\mathbf{G} = (-\Delta_2)^{-1}$, is called **Green's operator**.

Corollary 5.35. **G** *is a symmetric operator.*

Proof. We need just show that $\Delta_2^{\mathrm{T}} = \Delta_2$. Indeed, as the matrix of the operator π_2 is a diagonal matrix, and how, by lemma 5.5, the Laplacian is symmetric, we have:

$$\Delta_2^{\mathrm{T}} = (\pi_2 \circ \Delta_{|_{C^0_{\mathrm{int}}}})^{\mathrm{T}} = \Delta^{\mathrm{T}}_{|_{C^0_{\mathrm{int}}}} \circ \pi_2^{\mathrm{T}} = \Delta^{\mathrm{T}}_{|_{C^0_{\mathrm{int}}}} \circ \pi_2 = \Delta_2.$$

□

By (5.82), for each node **B**, we have:

$$\mathbf{u}(\mathbf{B}) = \sum \boldsymbol{\rho}(\mathbf{A}) G(\mathbf{A}, \mathbf{B}), \tag{5.83}$$

such that the sum of (5.83) extends to all interior nodes $\mathbf{A}$.

We can use the Green function to solve the Dirichlet problem. To achieve this purpose, we will demonstrate a certain identity.

Note that for $\mathbf{u}, \mathbf{v} \in C^0$ and for every node $\mathbf{A} \in C_0$, we have:

$$-\sum_{\text{all } \mathbf{A}} \mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) = \sum_{\text{all } \mathbf{A}} \mathbf{u}(\mathbf{A})\partial C\mathrm{d}\mathbf{v}(\mathbf{A})$$

$$= \int_{\partial C\mathrm{d}\mathbf{v}} \mathbf{u} = \int_{C\mathrm{d}\mathbf{v}} \mathrm{d}\mathbf{u} = (\mathrm{d}\mathbf{u}, \mathrm{d}\mathbf{v})_C = (\mathrm{d}\mathbf{v}, \mathrm{d}\mathbf{u})_C = \int_{C\mathrm{d}\mathbf{u}} \mathrm{d}\mathbf{v} = \int_{\partial C\mathrm{d}\mathbf{u}} \mathbf{v}$$

$$= \sum_{\text{all } \mathbf{A}} \mathbf{v}(\mathbf{A})\partial C\mathrm{d}\mathbf{u}(\mathbf{A}) = -\sum_{\text{all } \mathbf{A}} \mathbf{v}(\mathbf{A})\Delta\mathbf{u}(\mathbf{A}).$$

Therefore:

$$\sum_{\text{all } \mathbf{A}} \mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) = \sum_{\text{all } \mathbf{A}} \Delta\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{A}). \tag{5.84}$$

Spliting the sum of two parts, one on the boundary nodes and the other on the interior nodes, we have:

$$\sum_{\substack{\text{interior} \\ \text{nodes } \mathbf{A}}} [\mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) - \Delta\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{A})] = -\sum_{\substack{\text{boundary} \\ \text{nodes } \mathbf{B}}} [\mathbf{u}(\mathbf{B})\Delta\mathbf{v}(\mathbf{B}) - \Delta\mathbf{u}(\mathbf{B})\mathbf{v}(\mathbf{B})] \tag{5.85}$$

The identity (5.85) is called **Green's formula**.

Let us now choose any two interior nodes $\mathbf{A}_1$ and $\mathbf{A}_2$ and set $\mathbf{u}, \mathbf{v} \in C^0_{\text{int}}$ as follows:

$$\mathbf{u} = G(\mathbf{A}_1, \cdot) \text{ and } \mathbf{v} = G(\mathbf{A}_2, \cdot)$$

Using $\mathbf{u}, \mathbf{v}$ in the Green's formula (5.85), we have that the right side is zero because $\mathbf{u}(\mathbf{B}) = \mathbf{v}(\mathbf{B}) = 0$ for every boundary node $\mathbf{B}$. Also note that:

$$\Delta\mathbf{u}(\mathbf{A}) = \begin{cases} -1 \quad , & \text{if } \mathbf{A} = \mathbf{A}_1, \\ \\ \;\;0 \quad , & \text{if } \mathbf{A} \neq \mathbf{A}_1 \text{ and } \mathbf{A} \in C_0^{\text{int}}, \end{cases}$$

and

$$\Delta\mathbf{v}(\mathbf{A}) = \begin{cases} -1 \quad , & \text{if } \mathbf{A} = \mathbf{A}_2, \\ \\ \;\;0 \quad , & \text{if } \mathbf{A} \neq \mathbf{A}_2 \text{ and } \mathbf{A} \in C_0^{\text{int}}. \end{cases}$$

Therefore, the Green's formula (5.85) becomes:

$$-\mathbf{u}(\mathbf{A}_2) + \mathbf{v}(\mathbf{A}_1) = 0$$

$$\Rightarrow \mathbf{u}(\mathbf{A}_2) = \mathbf{v}(\mathbf{A}_1).$$

Therefore:

$$G(\mathbf{A}_1, \mathbf{A}_2) = G(\mathbf{A}_2, \mathbf{A}_1). \tag{5.86}$$

Remark 5.36. In agreement with the corollary 5.35, the identity (5.86) shows once again that the matrix of the operator $\mathbf{G}$ is symmetric.

If we write $\mathbf{u} = \boldsymbol{\phi}, \Delta\mathbf{u} = -\boldsymbol{\rho}, \mathbf{v} = \hat{\boldsymbol{\phi}}, \Delta\mathbf{v} = -\hat{\boldsymbol{\rho}},$, the Green's formula (5.85) becomes:

$$\sum_{\substack{\text{interior}\\ \text{nodes } \mathbf{C}}} (\boldsymbol{\rho}_C \hat{\boldsymbol{\phi}}^C - \hat{\boldsymbol{\rho}}_C \boldsymbol{\phi}^C) = \sum_{\substack{\text{boundary}\\ \text{nodes } \mathbf{B}}} (\hat{\boldsymbol{\rho}}_B \boldsymbol{\phi}^B - \boldsymbol{\rho}_B \hat{\boldsymbol{\phi}}^B) \tag{5.87}$$

With this, we realized that the Green's formula is nothing more than the version for capacitive circuits of Green's reciprocity theorem 4.51.

Now suppose that $\hat{\boldsymbol{\phi}}$ is a solution of the Dirichlet problem, then $\hat{\boldsymbol{\rho}} = \mathbf{0}$ at the interior nodes. Let $\boldsymbol{\phi} = G(\mathbf{A}, \cdot)$ in (5.87), with $\mathbf{A}$ an interior node. Then we have:

$$\sum_{\substack{\text{interior}\\ \text{nodes } \mathbf{C}}} (-\Delta G(\mathbf{A},\mathbf{C})\hat{\boldsymbol{\phi}}^C - \hat{\boldsymbol{\rho}}_C G(\mathbf{A},\mathbf{C})) = \sum_{\substack{\text{boundary}\\ \text{nodes } \mathbf{B}}} (\hat{\boldsymbol{\rho}}_B G(\mathbf{A},\mathbf{B}) + \Delta G(\mathbf{A},\mathbf{B})\hat{\boldsymbol{\phi}}^B). \tag{5.88}$$

Since $-\Delta G(\mathbf{A},\mathbf{C}) = 1$ if $\mathbf{C} = \mathbf{A}$ and 0 for the others interior nodes, since $\hat{\boldsymbol{\rho}}_C = 0$ at all interior nodes and how $G(\mathbf{A},\mathbf{B}) = 0$ for all $\mathbf{B}$ boundary node, (5.88) becomes:

$$\hat{\boldsymbol{\phi}}^A = \sum_{\substack{\text{boundary}\\ \text{nodes } \mathbf{B}}} \Delta G(\mathbf{A},\mathbf{B})\hat{\boldsymbol{\phi}}^B \tag{5.89}$$

with $\hat{\boldsymbol{\phi}}^A$ being the potential of Dirichlet problem's solution for each interior node $\mathbf{A}$.

The matrix $(\Delta G(\mathbf{A},\mathbf{B}))_{\dim C^0_{\text{int}} \times \dim C^0_{\text{bound}}} : C^0_{\text{bound}} \to C^0_{\text{int}}$ is called **Poisson Kernel**.

Example 5.37. Return to the example (5.22), we have, from (5.75), that:

$$G(\mathbf{D},\mathbf{D}) = \frac{1}{4},\ \Delta G(\mathbf{D},\mathbf{A}) = \frac{1}{4},\ \Delta G(\mathbf{D},\mathbf{B}) = \frac{1}{2},\ \Delta G(\mathbf{D},\mathbf{C}) = \frac{1}{4}.$$

Then, $\boldsymbol{\phi}^D$ (solution of the Dirichlet's problem) is given by:

$$\boldsymbol{\phi}^D = \Delta G(\mathbf{D},\mathbf{A})\boldsymbol{\phi}^A + \Delta G(\mathbf{D},\mathbf{B})\boldsymbol{\phi}^B + \Delta G(\mathbf{D},\mathbf{C})\boldsymbol{\phi}^C.$$

Therefore:

$$\boldsymbol{\phi}^D = \frac{1}{4}\boldsymbol{\phi}^A + \frac{1}{2}\boldsymbol{\phi}^B + \frac{1}{4}\boldsymbol{\phi}^C.$$

There is another version for the Green formula , called Green's second formula. Let's prove it.

Let $\mathbf{A}$ be a boundary node. We know from lemma 5.6 that:

$$(\Delta\mathbf{u})(\mathbf{A}) = \sum_{\boldsymbol{\alpha}:\partial\boldsymbol{\alpha} = \pm(\mathbf{B}-\mathbf{A})} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})) \tag{5.90}$$

summed over all nodes $\mathbf{B}$ that are in the neighborhood of $\mathbf{A}$.

In (5.90), we can divide the right side of the equality as follows:

$$(\Delta\mathbf{u})(\mathbf{A}) = \sum_{\substack{\mathbf{A} \text{ in boundary}\\ \boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})\\ \mathbf{B} \text{ an interior node}}} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})) + \sum_{\substack{\mathbf{A} \text{ in boundary}\\ \boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A})\\ \mathbf{B} \text{ a boundary node}}} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})). \tag{5.91}$$

Let us denote the second term of the sum (5.91) for $\Delta^{\text{bound}}\mathbf{u}(\mathbf{A})$, ie:

$$(\Delta\mathbf{u})(\mathbf{A}) = \left(\sum_{\substack{\mathbf{A}\text{ in boundary} \\ \boldsymbol{\alpha}:\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A}) \\ \mathbf{B}\text{ an interior node}}} C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A})) \right) + \Delta^{\text{bound}}\mathbf{u}(\mathbf{A}). \tag{5.92}$$

We can think of the boundary nodes with all branches of the circuit connecting two boundary nodes as a circuit by itself. Then Δ^{bound} will be the Laplace operator of this subcircuit. As this subcircuit doesn't have interior nodes, by Green's formula (5.85), we have:

$$\sum_{\text{boundary }\mathbf{A}} (\mathbf{u}(A)\Delta^{\text{bound}}\mathbf{v}(A) - \Delta^{\text{bound}}\mathbf{u}(A)\mathbf{v}(A)) = 0. \tag{5.93}$$

Therefore, from (5.92) and (5.93), we have:

$$\sum_{\text{boundary }\mathbf{A}} (\mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) - \Delta\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{A})) =$$

$$= \sum_{\text{boundary }\mathbf{A}} \sum_{\substack{\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A}) \\ \mathbf{B}\text{ in interior}}} (\mathbf{u}(\mathbf{A})C_\alpha(\mathbf{v}(\mathbf{B}) - \mathbf{v}(\mathbf{A})) - C_\alpha(\mathbf{u}(\mathbf{B}) - \mathbf{u}(\mathbf{A}))\mathbf{v}(\mathbf{A})) =$$

$$= \sum_{\text{boundary }\mathbf{A}} \sum_{\substack{\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A}) \\ \mathbf{B}\text{ in interior}}} C_\alpha(\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{B}) - \mathbf{u}(\mathbf{B})\mathbf{v}(\mathbf{A})).$$

So:

$$\sum_{\text{boundary }\mathbf{A}} (\mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) - \Delta\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{A})) = \sum_{\text{boundary }\mathbf{A}} \sum_{\substack{\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A}) \\ \mathbf{B}\text{ in interior}}} C_\alpha(\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{B}) - \mathbf{u}(\mathbf{B})\mathbf{v}(\mathbf{A})). \tag{5.94}$$

Replacing (5.94) on Green's formula (5.85), we find:

$$\sum_{\mathbf{A}\text{ in interior}} (\mathbf{u}(\mathbf{A})\Delta\mathbf{v}(\mathbf{A}) - \Delta\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{A})) = - \sum_{\substack{\mathbf{A}\text{ in} \\ \text{boundary}}} \sum_{\substack{\partial\boldsymbol{\alpha}=\pm(\mathbf{B}-\mathbf{A}) \\ \mathbf{B}\text{ in interior}}} C_\alpha(\mathbf{u}(\mathbf{A})\mathbf{v}(\mathbf{B}) - \mathbf{u}(\mathbf{B})\mathbf{v}(\mathbf{A})) \tag{5.95}$$

The identity (5.95) is called **Green's second formula**.

5.8 Green's Reciprocity Theorem in Electrostatics

Remark 5.38. In this section some results for systems of charged conductors are listed without proof, merely in order to show some generalizations of the theory previously studied for capacitive systems.

As a slight generalization of capacitive networks, we may consider a system of charged conductors, each of which has a well-defined charge $\boldsymbol{\rho}$ and a well-defined potential $\boldsymbol{\phi}$. The total charges on each of the various conductors may be described in terms of a vector $\boldsymbol{\rho} = \begin{pmatrix} \rho_A \\ \rho_B \\ \vdots \end{pmatrix}$ in a space we may call C_0, while the potentials form a vector $\boldsymbol{\phi} = \begin{pmatrix} \phi^A \\ \phi^B \\ \vdots \end{pmatrix}$ in its dual space C^0.

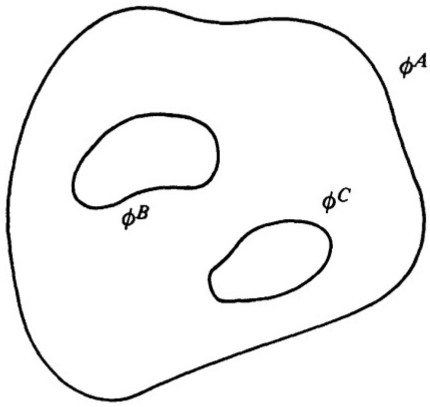


Figure 26 – Conductors.

Remark 5.39. The stored electrostatic energy E on a system of conductors can be given by:

$$E = \frac{1}{2}\int_{\boldsymbol{\rho}} \boldsymbol{\phi} = \frac{1}{2}\sum \rho_A \phi^A.$$

The total conductor charges $\boldsymbol{\rho}$ may be expressed in terms of the potentials $\boldsymbol{\phi}$ by a Laplace operator Δ, so that:

$$\boldsymbol{\rho} = -\Delta\boldsymbol{\phi}.$$

Remark 5.40. : In the physics literature, $-\Delta$ is usually called *the matrix of capacitance coefficients* and the inverse of the matrix $-\Delta$ is called *the matrix of potential coefficients.*

In the current context, the operator Δ depends on the shape of the conductors, their distribution in space and on fundamental constants of electrostatics. Generally the calculation of Δ is extremely difficult. But in some cases this calculation is simple, as in the following example, which deals with concentric spheres.

Example 5.41. Consider in figure 27 a system of two concentric spheres, with radii r_A and r_B In Gaussian units, for $\rho_A = 1$ and $\rho_B = 0$, we find the potentials $\phi^A = \frac{1}{r_A}$, $\phi^B = \frac{1}{r_B}$, while, if $\rho_A = 0$ and $\rho_B = 1$, we find the potentials $\phi^A = \phi^B = \frac{1}{r_B}$. Then, we have:

$$-\Delta^{-1} = \begin{pmatrix} 1/r_A & 1/r_B \\ 1/r_B & 1/r_B \end{pmatrix}$$

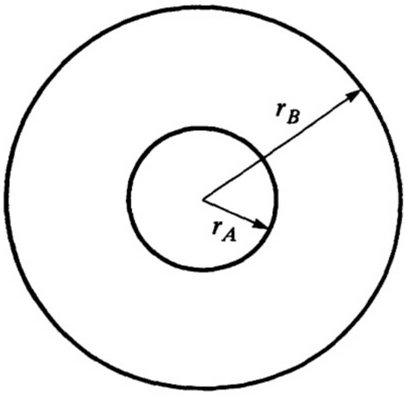


Figure 27 – Two concentric spheres.

This matrix permits us to calculate the potential of the two spheres for an arbitrary charge distribution. Its inverse gives the Laplace operator:

$$-\Delta = \frac{r_B}{r_B - r_A} \begin{pmatrix} r_A & -r_A \\ -r_A & r_B \end{pmatrix}$$

This matrix determines the charges on the two spheres for specified potentials. For the case $\phi_A = 1, \phi_B = 0$ it gives:

$$\begin{pmatrix} \rho_A \\ \rho_B \end{pmatrix} = \frac{r_A r_B}{r_B - r_A} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

i.e., there are equal and opposite charges of magnitude $\frac{r_A r_B}{r_B - r_A}$ on the two spheres. This quantity $\frac{r_A r_B}{r_B - r_A}$ is called the capacitance of the pair of spheres.

Example 5.42. For any number of concentric spheres, the reasoning is the same. For example, consider the three concentric spheres of figure 28. Then:

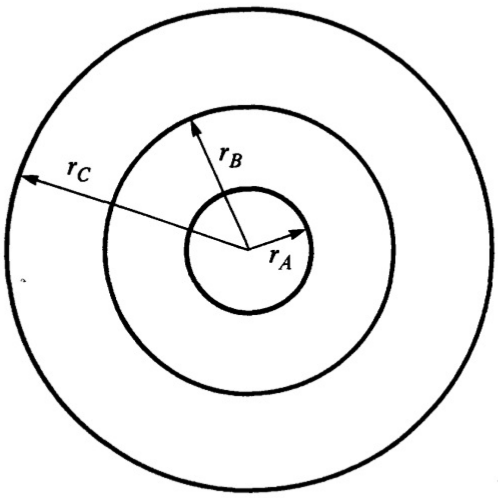


Figure 28 – Three concentric spheres.

$$-\Delta^{-1} = \begin{pmatrix} 1/r_A & 1/r_B & 1/r_C \\ 1/r_B & 1/r_B & 1/r_C \\ 1/r_C & 1/r_C & 1/r_C \end{pmatrix}.$$

In the same way we use for capacitive circuits, we may also use the *Green's reciprocity theorem* for a system of conductors. So let $(\boldsymbol{\rho}, \boldsymbol{\phi})$ and $(\boldsymbol{\rho}', \boldsymbol{\phi}')$ are two settings for charge and potential of a system of conductors, so Green's reciprocity theorem states that:

$$\int_{\boldsymbol{\rho}'} \boldsymbol{\phi} = \int_{\boldsymbol{\rho}} \boldsymbol{\phi}'.$$

Lemma 5.43. Δ *is a self-adjoint operator.*

Proof. Let $\boldsymbol{\rho} = -\Delta\boldsymbol{\phi}$, $\boldsymbol{\rho}' = -\Delta\boldsymbol{\phi}'$. Then

$$\int_{-\Delta\boldsymbol{\phi}'} \boldsymbol{\phi} = \int_{-\Delta\boldsymbol{\phi}} \boldsymbol{\phi}',$$

i.e.,

$$(\Delta\boldsymbol{\phi}', \boldsymbol{\phi}) = (\boldsymbol{\phi}', \Delta\boldsymbol{\phi}).$$

□

The same way as was done in the study of capacitive circuits, here we can also classify some conductors as boundary conductors whose potential may be established by connecting batteries to them, while others are inner conductors whose charge may be specified. And in the identical way, we will have the Poisson equation problem and the Dirichlet problem.

For a system of conductors, let $(\boldsymbol{\rho}', \boldsymbol{\phi}')$ be the charge and potential for the Poisson equation problem and let $(\boldsymbol{\rho}, \boldsymbol{\phi})$ be the charge and potential to the Dirichlet problem. By Green's reciprocity theorem, we have:

$$\sum_{\text{interior}} \rho'_A \phi^A + \sum_{\text{boundary}} \rho'_B \phi^B = \sum_{\text{interior}} \rho_A \phi'^A + \sum_{\text{boundary}} \rho_B \phi'^B$$

But given that $\phi' = 0$ on the boundary(Poisson) and $\rho = 0$ in the interior(Dirichlet), we get:

$$\sum_{\text{interior}} \rho'_A \phi^A = -\sum_{\text{boundary}} \rho'_B \phi^B \tag{5.96}$$

Suppose now that $\rho' = -\Delta G(\mathbf{C}, .)$ where G is the Green function and $\mathbf{C}$ an interior node. With this, the equation (5.96) will be equal to:

$$\phi^C = \sum_{\text{boundary}} \Delta G(\mathbf{C}, \mathbf{B})\phi^B \tag{5.97}$$

This is the Green's function solution to the Dirichlet problem e the matrix $[\Delta G(\mathbf{C}, \mathbf{B})]$ is the Poisson Kernel. Observe that the equation (5.97) is similar to the equation (5.89) to the capacitive circuits.

Example 5.44. Consider the system of three large parallel conducting planes shown in figure 29. We regard A and C as boundary conductors, B as an interior conductor without charge and we consider that in this geometry the potential in the plane B is a linear function of the position between the plates A and C. Then:

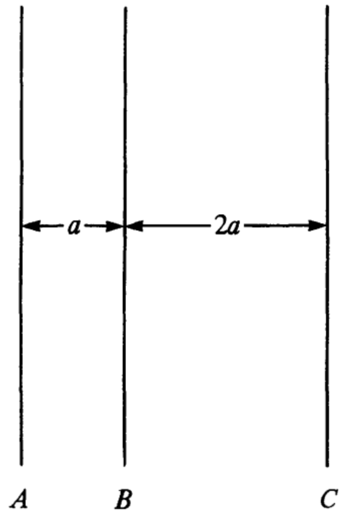


Figure 29 – Parallel conducting planes.

$$\phi^B = \phi^A + \frac{1}{3}(\phi^C - \phi^A).$$

So:

$$\phi^B = \frac{2}{3}\phi^A + \frac{1}{3}\phi^C.$$

This is a solution to Dirichlet's problem.

Now consider Poisson's equation, with charge ρ'_B on the middle plane, $\phi'^A = \phi'^C = 0$. By reciprocity theorem,

$$\rho'_B\phi^B = -\rho'_A\phi^A - \rho'_C\phi^C.$$

But $\phi^B = \frac{2}{3}\phi^A + \frac{1}{3}\phi^C$, then:

$$\rho'_B\frac{2}{3}\phi^A + \rho'_B\frac{1}{3}\phi^C = -\rho'_A\phi^A - \rho'_C\phi^C.$$

Since ϕ^A and ϕ^C are linearly independent, we have:

$$\rho'_A = -\frac{2}{3}\rho'_B,\ \rho'_C = -\frac{1}{3}\rho'_B. \tag{5.98}$$

The equations above show the induced charge in the planes A and C.

6 SUMMARY AND PERSPECTIVES

In this work we introduce the analysis of electrical circuits made with strong considerations about its shape. The graph theory and algebraic topology were used to make possible this goal. For example, we saw how the boundary and coboundary maps act in the electrical circuit. We also saw that a circuit have homology (H_0, H_1) and cohomology (H^1) groups. We defined the vector space C_0, C_1 of nodes and branches, respectively, and also their dual spaces C^0 and C^1. We defined the vector subspaces of cycles Z_1 and of boundaries B_0, as well as the dual subspace Z^0 and B^1, providing a physical meaning to them. Through Maxwell's Mesh-Current Method and Maxwell's Node-Potential Method, we analyze the existence and uniqueness of the Kirchhoff equations for resistive electrical circuits and also for more general circuits (RLC circuits).

We also worked with a geometric method, conceived by Weyl, introducing equations that are equivalent (in resistive circuits) to Kirchhoff's equations for electrical circuits. Kirchhoff, on the other hand, contributed with an alternative way (based on graph theory) to discover the Weyl's orthogonal projection. Green's Reciprocity Theorem allows us to find some symmetries for the circuit.

We learned how to work with capacitive circuits, introducing the discrete versions of the Gauss' law, Dirichlet problem, Laplace and Poisson equations. We discuss how to decompose the space C^1 in a direct sum, and define the space D^1, relating this space with the solutions of the Dirichlet problem. The decomposition of the space C^1 also enabled us to know a new geometric method for solving the Dirichlet and Poisson problems, using orthogonal projection. We finished our study with the Green's functions, the Green's formula (which in turn is equivalent to the Green's reciprocity theorem for capacitive networks), Poisson Kernel and Green's second formula.

As future perspectives, we intend relate capacitive circuits with the general theory of electrostatics explaining, for example, how the star operator plays the same role to the capacitance matrix. More generally, we intend very soon develop, using the exterior differential calculus, a continuous version for the study of electrical circuits, thus formulating a new perspective to electromagnetism's study.

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