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AN EMPIRICAL EVALUATION OF STRUCTURAL CHANGES IN QUANTILE AUTOREGRESSIVE MODELS

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Dissertação apresentada ao Programa de Pós-Graduação em Estatística do Centro de Ciências Exatas e da Natureza da Universidade Federal de Pernambuco, como requisito parcial à obtenção do título de mestre em Estatística.

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Dedico este trabalho a minha família.

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“A ship is safe in harbor, but that’s not what ships are for.”

(Shedd. John A. 1928)

ABSTRACT

This work proposes an evaluation on a subgradient test for structural change and on the usual coverage tests to evaluate Value at Risk (VaR) estimates, obtained by quantile regression. In an initial analysis, exchange-traded funds returns were evaluated during the United States subprime mortgage crisis. This task was performed with aid of a subgradient test for structural change (Qu), which allows us to evaluate whether the parameter values remain stable throughout the series and in a generalized moments method based duration test (GMM) for coverage evaluation. The empirical results shown break dates in the 5%-quantiles few days before the Lehman Brothers bankruptcy event. Motivated by the empirical results, simulation studies using heteroscedastic autoregressive processes were performed under different scenarios with and without structural breaks. The simulation studies show that the structural change test is capable of detecting breaks quite accurately. However, the usual VaR coverage tests are conservative.

Keywords: Subprime crisis. Quantile Regression. Structure change. Value-at-Risk. Linear autoregressive conditional heteroskedasticity. Subgradient test.

RESUMO

Este trabalho propõe uma avaliação em um teste subgradiente para mudança estrutural e dos testes de cobertura usuais para avaliação das estimativas de Valor em Risco (VaR), obtidas por regressão quantílica. Em uma análise inicial, retornos de exchange-traded funds foram avaliados durante a crise do subprime nos Estados Unidos. Esta tarefa foi realizada com ajuda do teste de quebra estrutural subgradiante (Qu), que permite avaliar se os valores dos parametros permanecem estáveis durante toda a série e o método dos momentos generalizados baseados na duração (GMM) para a avaliação da cobertura. Os resultados empíricos mostram datas de quebra no quantil 5% poucos dias antes do evento da falência do Lehman Brothers. Motivados pelos resultados empíricos obtidos, estudos de simulação utilizando processos autoregressivos heteroscedásticos foram realizados sob diferentes cenários com e sem quebras estruturais,. Os estudos de simulação evidenciam que o teste mudança estrutural é capaz de detectar quebras com bastante precisão. Entretanto, os teste usuais de cobertura do VaR mostram-se conservativos.

Palavras-chave: Crise do subprime. Regressão quantílica. Mudança estrutural. Valor em Risco. Modelos autoregressivos heteroscedásticos. Teste subgradiente.

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

τ	Quantile
t	Time
α	Autoregressive cofficients
ω	Variance multiplier

CONTENTS

1 INTRODUCTION

Quantile regression can be related to Value-at-Risk (VaR) when trying to estimate this risk measure by using a conditional statistical structure. Focusing on it, the quantile autoregression approach proposed by Koenker (KOENKER; ZHAO, 1996) is a pioneer work in literature. To explore VaR and quantile autoregression seems to be an interesting approach in applied finance.

As an example of VaR's usefulness, we can cite the stock market crash on Wall Street in 1987 and other crisis such as the Subprime mortgage that have attracted a great deal of attention among investors, researchers and practitioners. In this sense, VaR is a statistical measure associated with extreme quantiles which can be used to evaluate extreme price movements in financial data and, in the financial markets. In addition, the quantile autoregression can improve the VaR estimation. To investigate extreme price movements is very important in the risk management area.

Another analysis that can be related with extremal quantiles and, perhaps exemplified as financial crisis, is the structural change evaluation. In this sense, we can highlight the statistical procedure proposed by QU(2008) which propose to investigate structural changes using quantile regression.

The goal of this work is, in an applied setting, to explore VaR estimation based on quantile autoregression and under structural changes. To that end, some initial applications were done using exchange traded funds during the United States subprime mortgage crisis. In addition, simulations were performed in order to evaluate the nominal level and the power of some coverage tests proposed in the literature, and to analyze the behavior of the QU's test.

The rest of the work is organized in four more Chapters which include the Literature review (Chapter 2), methodology and empirical motivation (Chapter 3), simulation analysis (Chapter 4) and conclusion (Chapter 5).

2 LITERATURE REVIEW

2.1 CONDITIONAL HETEROSCEDASTIC MODELS AND QUANTILE REGRESSION

Conditional heteroscedastic models and quantile regression are two distinct statistical concepts which their foundations were written and published in less than 5 years from each other.

The first concept (conditional heteroscedastic models) is a class of models which its purpose is establish a conditional variance to explain the heteroskadasticity often presented in a time series. In the financial time series context, the term 'volatility' is interchangeable with 'conditional variance'. The pioneer model from this class was introduced by ENGLE(1982) and was named ARCH model, acronym for autoregressive conditional heteroskedasticity. A few years later, Engle's model was extended by BOLLERSLEV(1986) who introduced the GARCH modeling, abbreviation of generalized ARCH model. The GARCH models generalize the ARCH equation in a similar manner as an ARMA (Autoregressive Moving Average) model (BOX *et al.*, 2015) extends an AR (Autoregressive) model. Since Bollerslev in 1986, several extentions of GARCH models have been proposed by researchers, including the EGARCH model (NELSON, 1991), the linear GARCH model of Taylor (TAYLOR, 2008) and many others.

The second concept (quantile regression) was introduced by KOENKER; BASSETT(1978) and it concerns a relation establishment between the τ-quantile of a response variable and explanatory variables, where $\tau \in (0,1)$. Inspired in the ideia of conditional mean from classic linear regression models, quantile regression seeks to model a conditional quantile. In Koenker's work this relation was shown to be linear, but it might be easily extended as nonlinear as well (KOENKER; PARK, 1996).

Futhermore, quantile regression has been used in time series analysis and, the development of ARCH and GARCH models estimated by quantile regression methods has been undertaken, the former developments by Koenker (KOENKER; ZHAO, 1996) and the latter by Lee (LEE; NOH, 2013).

Studying quantiles might be useful in many situations, especially as exploratory tool for skewed distributions, like household wages (NGUYEN *et al.*, 2007) or financial returns (ROCKAFELLAR; URYASEV, 2002).

Regards to financial returns, the GARCH model seems computationally more convenient than the linear GARCH model, but linear GARCH may be more appropriate for modeling financial returns. As it was once noted, (DUFFIE; PAN, 1997) the maximum likelihood estimation of the quadratic form associated to GARCH model has a potential disadvantage of being overly sensitivity to extreme returns. The linear GARCH structure is less sensitive to extreme returns, but it is more difficult to handle mathematically. In addition, the linear GARCH approach is well suited for quantile estimation because of its linear formulation.

According to KOENKER; ZHAO(1996), it was clear that misspecification of the form of the conditional distribution used to define the likelihood estimation in ARCH and GARCH models can create serious problems for parameter estimation and conditional prediction intervals. This motivation led them to investigate methods that are not so sensitive to the normality assumption, so usual when using ARCH and GARCH models. In particular, quantile regressions methods used to estimate ARCH and GARCH models have a valuable property for this case: It does not assume any particular distribution, just some mild conditions.

2.2 VALUE-AT-RISK

Value-at-Risk (VaR) traces its roots to the infamous financial disasters of early 1990s that had engulfed many large financial institutions, as stated by JORION(2000). After those events, regulators were forced to act against the poor supervision and turned the VaR as a quantifying measure aimed to improve the market risk management.

Formally, the VaR measure proposes to show the worst expected loss over a given horizon at a given confidence. VaR is usually estimated by standard statistical techniques to assess the market risk exposure by providing a single number which summarizes this type of risk. For example, a financial institution could say that the daily VaR of its trading portfolio is $2 millions at 99% confidence level, in other words, there is only 1% chance, under normal market conditions, of a loss greater than $2 millions.

There are many approaches to evaluate VaR, but it is natural to evaluate it by using quantile regression. According to GAGLIANONE *et al.*(2011), due to its capability of conditional distribution exploration with distribution-free assumption and also for its capability to be used to estimate ARCH or GARCH models, which are widely used in financial time series analysis. According to Xiao (XIAO; GUO; LAM, 2015), VaRs estimated by quantile regression and GARCH are able to improve the estimation results, under normal market conditions.

A honorable mention which also uses standard quantile regression techniques and was explicitly developed for predicting VaR is the CAViaR model (ENGLE; MANGANELLI, 2004). According to this approach, the returns follow special cases of a GARCH process and the regressors are latent and dependent on unknown parameters. The estimation of CAViaR uses nonlinear quantile regression techniques that are not directly applicable for the proposal of our work. For this reason, and by simplicity, we focused on ARCH/GARCH modeling estimated by quantile regression.

2.3 STRUCTURAL CHANGES

A structural change is generally defined when a series abruptly changes at a certain point in the time. This behavior can involve a change in the mean or the other parameters of the stochastic process corresponding to the series. The ability to detect when the structure of the time series changes might give discernment into the modeling problem and structural changes tests are helpful to determine when and whether there is a significant change in the data.

HAMILTON(1990) argue that many of the major exogenous economic events that influence financial series are shocks such as the doubling of oil prices experienced over the past decades. These events can be considered as episodes with an identifiable duration in which the response of economic series might be expected to have a noteworthy difference from that seen outside these periods. Several studies about structural changes focus merely on variation in the conditional mean while, under diverse number of contexts, structural changes in the conditional distribution or in conditional quantiles are the ones of key importance.

QU(2008) proposed testing procedures for structural change in conditional quantile(s) with unknown timing. More precisely, he adopted the methodology of quantile regression and proposes two types of test statistics for structural change occurring in a specific quantile. The test is based on sequentially weighted empirical subgradients. It has excellent size properties even in small samples, local power under parameter's changes whose regressors has mean equal to zero and does not require estimating any nuisance parameters, only requires estimating the model under the null hypothesis. The main idea is that if there is a structural change, then the parameter estimates obtained imposing the null hypothesis will not be close to the true values for at least one subset of the sample. As a result, the estimated residuals will persistently fall below or above the true quantile, forcing the subgradient to take a large value.

According to QU(2008), this test is directly related to the CUSUM tests (PAGE, 1954) where quantities derived from ordinary least squares procedures or recursive residuals are explored, however he highlights that CUSUM fails against parameter change whose regressor has mean zero and does require estimate a parameter which is not the immediate interest, but must be accounted in the analysis of the parameters of interest, then suffering of non-monotonic power.

3 METHODOLOGY AND EMPIRICAL MOTIVATION

In the first part of this chapter (Section 3.1) we present a brief description of the methods we used to evaluate VaR in our empirical investigation. The second part (Section 3.2) presents an empirical analysis using quantile autoregression to estimate VaR of some Exchange Trade Funds (ETFs) and an application of QU's test to evaluate structural changes in the analyzed time series. This analysis motivates the simulations presented in Chapter 4.

3.1 METHODS

3.1.1 Bollerslev's and Taylor's GARCH model

We say $\{\varepsilon_t\}$ follows a GARCH model, with parameters (p,q) (BOLLERSLEV(1986)), if it can be expressed as

$$\varepsilon_t = \sigma_t v_t, \tag{3.1}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{3.2}$$

where $\{v_t\}$ is a sequence of independent and identically distributed random variables with zero mean and variances equal to one; $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)'$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_q)'$ contain the parameters that model the conditional variance σ_t^2 ($\alpha_0 > 0$, $\alpha_i \geq 0$ for all $i \in \{1,2,\ldots,p\}$ and $\beta_i \geq 0$ for all $j \in \{1,2,\ldots,q\}$). When using GARCH modeling, it is common to associate the term 'volatility' with σ_t^2. If $\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1$, $\{\varepsilon_t\}$ is weakly stationary.

A modified version of the GARCH model is the linear GARCH proposed by Taylor. If $\{\varepsilon_t\}$ follows a linear GARCH model with parameters (p,q), then it is defined as

$$\varepsilon_t = \sigma_t v_t, \tag{3.3}$$

$$\sigma_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i |\varepsilon_t| + \sum_{j=1}^{q} \beta_j \sigma_{t-j}. \tag{3.4}$$

GARCH models include ARCH models as a special case whenever its parameter $q = 0$. In other words, when $\beta = 0$. In our study, we used linear ARCH models which were estimated by quantile regression.

3.1.2 Quantile Regression

Let ε be a random variable with distribution function $F(\varepsilon)$. The τth quantile ($\tau \in (0,1)$) is defined as

$$F^{-1}(\tau) = \inf\{\varepsilon \in \mathbb{R} : F(\varepsilon) \geq \tau\}. \tag{3.5}$$

KOENKER; BASSETT(1978) proposed the loss function $\rho_\tau(u) = u(\tau - \mathbb{1}(u < 0))$, where $\mathbb{1}(x < 0)$ is an indicator function and $F^{-1}(\tau)$ is the τ-quantile. When using $\rho_\tau(u)$ and the empirical distribution function in the expected value $E(\rho_\tau(\varepsilon - \xi))$, it is possible show that the τth sample quantile ($\hat{\xi}$) is obtained by

$$\hat{\xi} = \min_{\xi \in \mathbb{R}} \sum_{i=1}^{n} \rho_\tau(\varepsilon_i - \xi), \tag{3.6}$$

where n is the sample size and ε_i is the ith observation in the sample data (from ε).

Quantile regression deals with a conditional quantile, thus, if $\xi'_{\varepsilon|x}(\tau) = x'\beta_\tau$, then the quantile ξ is conditional to the vector of regressors x. Applying a systematic structure $\xi_{\varepsilon|x}(\tau) = x'\beta_\tau$, the vector β_τ (3.6) can be estimated by (KOENKER; CHESHER; JACKSON, 2005)

$$\hat{\beta}_\tau = \min_{\beta \in \mathbb{R}} \sum_{i=1}^{n} \rho_\tau(\varepsilon_i - x_i\beta). \tag{3.7}$$

3.1.3 Linear ARCH models estimated by quantile regression

The linear ARCH model (equation (3.4)) and quantile regression (Equation (3.7)) can be used in GARCH modeling (see KOENKER; XIAO (2006)). Let $\{\varepsilon_t\}$ follow a linear ARCH(p) process (a special case of the Equation (3.4)). Given the information set $\mathscr{F}_{t-1}$ representing all results of $\{\varepsilon_t\}$ from the first period until $t-1$, the conditional quantile function can be written as

$$Q_\varepsilon(\tau|\mathscr{F}_{t-1}) = \left(\alpha_0 + \sum_{i=1}^{p} \alpha_i|\varepsilon_{t-i}|\right) F_v^{-1}(\tau), \tag{3.8}$$

where $F_v^{-1}(\tau)$ denotes the τth quantile of the innovations $\{v_t\}$ with an unknown distribution and $\mathbb{E}(v_t^2) = 1$, where $\mathbb{E}(\cdot)$ denotes the expected value. The restriction $\mathbb{E}(v_t^2) = 1$ is used as manner to achive $\mathbb{E}(\varepsilon_t^2) = \sigma_t^2$, that is say σ_t^2 represents ε_t variance, but this restriction can be relaxed (DROST; KLAASSEN, 1997). Thus, another form to represent the linear ARCH process is

$$\varepsilon_t = \left(1 + \sum_{i=1}^{p} \gamma_i|\varepsilon_{t-i}|\right) \sqrt{\omega_0} v_t, \tag{3.9}$$

where $\gamma_i = \frac{\alpha_i}{\alpha_0}$ and $\sqrt{\omega_0} = \alpha_0$ are the new coefficients, $\{\sqrt{\omega_0} v_t\}$ represents a new innovation, with $\mathbb{E}(v_t^2) < \infty$ (not necessarily equal to one). Thus, a new conditional quantile function of $\{\varepsilon_t\}$ can be obtained

$$Q_{\varepsilon^*}(\tau|\mathscr{F}_{t-1}) = \left(1 + \sum_{i=1}^{p} \gamma_i |\varepsilon_{t-i}|\right) F_{v^*}^{-1}(\tau), \tag{3.10}$$

where $F_v^{-1}(\tau)$ denotes the τth quantile of the new innovations $\{\sqrt{\omega_0} v_t\}$ with an unknown distribution and $\mathbb{E}(|\sqrt{\omega_0} v_t|^2) < \infty$. The use of a linear form of ARCH model suggests the computation of the regression quantiles by stantard linear programming techniques which are more efficient when comparing with others non-linear approaches (see KOENKER; ZHAO (1996)). Throughout this work, the model in (3.10) is frequently used to estimate VaR.

3.1.4 Value at Risk (VaR)

Value-at-Risk measures the maximum loss that can be expected, at a particular significance level, over a given trading horizon (JORION, 2000; GAGLIANONE *et al.*, 2011). Let ε_t be the return of a portfolio at t-period, and $\tau \in (0,1)$ the VaR's significance level. Thus, VaR (VaR_τ^t) is obtained by solving the equation

$$\Pr(\varepsilon_t \leq -\text{VaR}_\tau^t | \mathscr{F}_{t-1}) = \tau \tag{3.11}$$

where $\mathscr{F}_{t-1}$ is the information set available at time $t-1$ and $\Pr(\cdot)$ denotes some probability. From (3.11), $-\text{VaR}_t^\tau$ is the τ-th conditional quantile of the returns. Based on the information available up to $(t-1)$-period, we can use $-\text{VaR}_\tau^t$ as an one-step ahead prediction for the τ-th quantile of the returns. Equation (3.11) can be rewritten as

$$-\text{VaR}_\tau^t = F_{\varepsilon_t|\mathscr{F}_{t-1}}^{-1}(\tau) = \inf\{\varepsilon : F_{\varepsilon_t|\mathscr{F}_{t-1}}(\varepsilon) \geq \tau\}. \tag{3.12}$$

3.1.5 Qu's test for structural changes

The Qu's test (QU, 2008) seeks to determine whether the coefficients of a linear quantile regression remain the same over the data range. For the accomplishment of this task he proposed the following inferential procedure:

Consider the random variable ε_{t_i}, where the subscript indicates that process was observed in t_i-period. Suppose that the τth conditional quantile of ε_{t_i} can be written as a linear function

$$Q_{\varepsilon_{t_i}}(\tau|x_{t_i}) = x_{t_i}\beta_{t_i\tau}, \tag{3.13}$$

where x_{t_i} is the vector of explanatory variables corresponding to the τth quantile of ε_{t_i} and β_{t_i} is the corresponding vector of coefficients. The response of ε_{t_i} to x_{t_i} is different from that from ε_{t_j} to x_{t_j} if and only if $\beta_{t_i\tau} \neq \beta_{t_j\tau}$, $i \neq j$, $\tau \in (0,1)$. The hypotheses of Qu's test are given bellow

$$\mathscr{H}_0 : \beta_{t_i\tau} = \beta_{0\tau} \quad \text{for all } i, \tag{3.14}$$

$$\mathscr{H}_1 : \beta_{t_i\tau} = \begin{cases} \beta_{1\tau} & \text{for } t_i \in \{t_1, t_2, \ldots, t_k\}, \\ \tilde{\beta}_{1\tau} & \text{for } t_i \in \{t_{k+1}, t_{k+2}, \ldots, t_n\}, \end{cases} \tag{3.15}$$

where t_k denotes the break point, and n is the sample size of the analyzed time series.

Qu's test evaluates the subgradient using a subsample, that uses the sort data from the beginning up to $t = \lfloor \lambda n \rfloor$, $\lambda \in [0,1]$, where $\lfloor . \rfloor$ denotes a floor function. It is important to note that in optimization problems such as in Equation (3.6) the objective function is not differentiable and this is the reason why the subgradient is used instead of the gradient. Subgradient generalize the derivative in cases of convex functions which are not differentiable. The subgradient statistic proposed by Qu is defined by (QU, 2008)

$$S_n(\lambda, \tau, \vartheta) = n^{1/2} \sum_{t=t_1}^{\lfloor \lambda n \rfloor} x_t \psi_\tau(\varepsilon_t - x_t' \vartheta), \tag{3.16}$$

where ϑ is some estimate for β_τ and $\psi_\tau(u) = \mathbb{1}_{(u \leq 0)} - \tau$. Under the null hypothesis, $\psi_\tau(\varepsilon_i - x_{t_i}' \beta_{0\tau})$ is a sequence of independent binary random variables with mean zero and variance $\tau(1-\tau)$. In this sense, it can be considered as a pivot quantity to make the decisions about the rejection/nonrejection of the null hypothesis. Let $X = (x_{t_1}', \ldots, x_{t_n}')'$ be the matrix of explanatory variable, and define $H_{\lambda,n}(\beta_{0\tau}) \triangleq (n^{-1} X'X)^{-1/2} S_n(\lambda, \tau, \beta_{0\tau})$. QU (2008) concludes that, under a few assumptions, $H_{\lambda,n}(\beta_{0\tau})$ converges to a limiting distribution that is nuisance parameter free. In other words, $H_{\lambda,n}(\theta_{0\tau}) \xrightarrow{d} \mathscr{N}(0, \lambda^2 \tau(1-\tau))$. Replacing $\beta_{0\tau}$ by the quantile regression estimate, we get

$$H_{\lambda,n}(\widehat{\beta}_{0\tau}) = (X'X)^{-1/2} \sum_{t=t_1}^{\lfloor \lambda n \rfloor} x_t \psi_\tau(\varepsilon_t - x_t' \widehat{\beta}_{0\tau}) \tag{3.17}$$

According to QU (2008), under the null hypothesis, $H_{\lambda,n}(\hat{\beta}_\tau)$ converges to a non degenerate distribution, whereas under alternative hypothesis it diverges for some λ. Since that the true break point is unknown, it is necessary to investigate over all the possiblities. In addition, also according to QU (2008), if $H_{\lambda,n}(\hat{\beta}_{0\tau})$ is multiplied by λ (i.e, $\lambda H_{1,n}(\hat{\beta}_{0\tau})$) it often yields better finite sample results. In this sense, QU (2008) defines the following (alternative) statistic:

$$SQ_\tau \triangleq \sup_{\lambda \in [0,1]} \left| \left| (\tau(1-\tau))^{-1/2} \left[H_{\lambda,n}(\hat{\beta}_\tau) - \lambda H_{1,n}(\hat{\beta}_\tau) \right] \right| \right|_\infty \tag{3.18}$$

where $||.||_\infty$ is the uniform norm (RUDIN, 2006). Under the null hypothesis, the preceding statistic (Equation (3.18)) converges to a p-vector of independent Brownian bridge processes (REVUZ; YOR, 2013) on $[0,1]$. The critical values for the SQ_τ test were obtained via simulations by Qu in his aforecited paper.

3.1.6 Coverage tests

Coverage tests are widely used to evaluate VaR estimate. It is not enough to know if a certain VaR model produces a plausible percentage of VaR violations[1], it is necessary to evaluate if the VaR violations are independent of each other as stated by CHRISTOFFERSEN (1998). Tests that simply check the expected rate of violations are called unconditional coverage tests, and the others seeking a more sophisticated approach regarding clusters and violation moments are called conditional coverage tests.

Let $\mathbb{I}_{1-\tau}$ be the indicator function which is equal to 1 when there is a VaR$_\tau$ violation. The CHRISTOFFERSEN's null hypothesis (independence of VaR violations) is $\mathscr{H}_0 : \Pi = \Pi_\alpha = \begin{bmatrix} \tau & 1-\tau \\ \tau & 1-\tau \end{bmatrix}$. It is tested against $\Pi = \begin{bmatrix} \pi_{01} & 1-\pi_{01} \\ \pi_{11} & 1-\pi_{11} \end{bmatrix}$, where $\pi_{ij} = \Pr(\mathbb{I}^t_{1-\tau} = j | \mathbb{I}^{t-1}_{1-\tau} = i)$.The GMM test (CANDELON *et al.*, 2010) is based on an orthonormal polynomial M_{j+1} corresponding to a geometric distribution in a form of the number of trials needed to get one success with probability $s \in (0,1)$. This polynomial M_{j+1} is given $(\forall d \in \mathbb{N}^+)$ by

$$M_{j+1}(d;s) = \frac{(1-s)(2j+1)+s(j-d+1)}{(j+1)\sqrt{1-s}} M_j(d;s) - \frac{j}{j+1} M_{j-1}(d;s), \tag{3.19}$$

where d is the the duration between two VaR violations (supposed to be geometrically distributed under null hypothesis), $j \in \mathbb{N}$, $M_0(d;s) = 1$, and $M_{-1}(d;s) = 0$. If the geometric distribution is the true distribution of d with a success probability s, then the expected value of $M_j(d;s)$ is zero for all j.

Let us denote $\{d_i\}_{i=1}^N$ as a sequence of N durations between violations. The null hypothesis of conditional coverage GMM test is

$$\mathscr{H}_0 : E(M(d_i;s)) = 0, \tag{3.20}$$

where M denotes a $(p,1)$ vector of M_j for $j = 1,\ldots,p$. Under some regularity conditions, and under null hypothesis, the GMM's test statistic converges to a chi-square distribution as described

[1] Consider ε_t some return at t-period. A VaR violation occurs if the observed return exceeds the VaR's estimate, i.e., if $\varepsilon_t > \text{VaR}^t_\tau$.

bellow

$$J_{cc} = \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d_i;s) \right)^{\top} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} M(d_i;s) \right) \xrightarrow{d} \chi_{(\ell)}, \tag{3.21}$$

where ℓ is the number of orthornormal polynomials used as moment conditions (see (CANDELON *et al.*, 2010))

3.2 AN EMPIRICAL MOTIVATION

Exchange Traded Funds (ETFs) have become larger in popularity over the last few years. In Brazil, which is considered an important emerging economy, ETFs first appeared in 2004 (BM&FBOVESPA; BOVESPA, 2008). Around the World, ETFs have experienced a significant increase in terms of assets and number of financial products. A reason that can justify this behavior is the fact that investors have been learned their mechanisms and have been able to know about these kind of financial products (ABNER, 2013).

The principal advantage of trading an ETFs is that, as a stock index, it can replicate the performance of a benchmark. Therefore, an ETF can be considered as an investment strategy easy to manager and benefited by lower management fees. Summing up, ETFs represent an interesting investment option with some advantages:

- They are more liquid than mutual funds since they can be traded on an intraday basis.
- The composition of an ETF is completely known.
- Arbitrage opportunities.
- Tax-efficient when compared to mutual funds.

In the literature about ETFs, empirical studies have been performed in order to assess the efficiency of this kind of investment option. The United States, for example, represents almost 70% of the ETF market in terms of assets under management, although the number of exchange traded products in the U.S. accounts for only 30% of the world total (ABNER, 2013).

U.S. market is somehow concentrated at the top 5 largest ETFs which represent 22.7% of the market in the United States. Leaders on the ETF market include SPDR, iShares and Vanguard. The following subsections explain the ETFs explored in the empirical analysis section (Section 3.3).

3.2.1 SPDR S & P 500 ETF - SPY

The SPDR S&P 500 ETF trades under the symbol ‘SPY’. The target behind this ETF is to give an investment vehicle that at least roughly produces returns in line with the S&P 500 Index before expenses. SPY is consistently one of the highest volume trading having attributes that are intended to accomplish goals on U.S. exchanges. Average volume is typically over 60 million shares, although that does fluctuate over time. Many investors and hedge funds use SPY because it represents the S&P 500 index - a basket of 500 major U.S. companies. Each asset in SPY index must have positive earnings in the most recent quarter, and also over the last four quarters. Investors focusing on SPY aim to invest in a wide range of large U.S. companies only by buying a single share.

3.2.2 Vanguard Information Technology ETF - VGT

The VGT tracks the performance of the MSCI US IMI Information Technology 25/50 Index. The index tracks the returns of companies belong to the information technology sector. The fund includes stocks of companies that serve the electronics and computer industries, and companies that manufacture products based on the latest applied science in this sector. Its top holdings include Apple Inc., Microsoft Corp., Facebook Inc. and Google’s parent company Alphabet Inc.

The fund is market-cap-weighted, which means that larger holdings have a greater influence on the fund’s performance as their market capitalization increases. The fund uses a passive indexing approach that seeks to fully replicate the performance of the benchmark, or if regulatory guidelines require it, the fund will use a sampling strategy.

3.2.3 Industrial Select Sector SPDR ETF - XLI

XLI provides investors exposure to prices and yield performance of publicly traded equity securities of companies in the Industrial Select Sector Index. Under normal market conditions, the fund generally invests at least 95% of its total assets in the securities comprising the index. The index includes securities of companies from the following industries: aerospace and defense; industrial conglomerates; marine; transportation infrastructure; machinery; road and rail; air freight and logistics; commercial services and supplies; etc.

In general, the performance in the industrial goods sector is largely driven by supply and demand for building construction in the residential, commercial and industrial real estate segments, as well as the demand for manufactured products, showing not high levels of volatility.

3.2.4 Consumer Staples Select Sector SPDR ETF - XLP

The XLP tracks the performance of the Consumer Staples Select Sector Index. Seeking to track the performance of the index, the fund employs a replication strategy. It generally invests at least 95% of its total assets in the securities comprising the index. The index includes securities of companies from the following industries: food and staples retailing; household products; food products; beverages; tobacco; and personal products.

Due to their low volatility, consumer staples stocks are considered to play a key role in defensive strategies, and intuitively will be the hardest return to archive any break detection.

3.3 EMPIRICAL ANALYSIS

The main point of this section is to use the quantile regression methods discussed in Sections 3.1.2,3.1.3, 3.1.5, and 3.1.6 in order to evaluate VaR measure corresponding to the four ETFs discussed in sections 3.2.1 to 3.2.4. The period of analysis corresponds to the daily returns from July 2007 to June 2009 (from 7/03/2007 to 6/29/2009, 502 trading days). This two-year period was chosen to include the subprime mortgage crisis in the United States.

3.3.1 Modelling procedure

In an initial study, we used quantile ARCH models to estimate 5% and 1% VaRs corresponding to the ETFs stock indices. After that we investigated some possible break points in the return series in the analyzed period. The Augmented Dickey–Fuller (ADF) (DICKEY; FULLER, 1981) and McLeod-Li (MCLEOD; LI, 1983) tests were done, however it is important to note that the ADF test is biased towards nonrejection of nul hypothesis (unit root) rejection (see PERRON (1990) pointed out. Partial autocorrelation function (PACF) and autocorrelation function (ACF) were plotted for the regular and absolute returns. This procedure was done to get insights about the orders of the ARCH processes. Finally, ten fitted models were evaluated. To help on choosing the models, AIC (AKAIKE, 1974), pseudo R^2, GMM's p-values, Christoffersen's p-values, Qu's statistics were computed. It is important to note that the pseudo R^2 we used is based on the quantile regression loss function (see KOENKER; CHESHER; JACKSON (2005))

3.3.2 Modelling ETFs

Serial correlations in return series are not significant because they are not predictable (see, e.g., ISSLER (1999)). On the other hand, when considering absolute (or squared) returns, it is possible to evaluate the predictability of conditional variance by plotting Autocorrelation (ACF) and Partial Autocorrelation (PACF) functions. The plots in Figures 1 to 2 show only a few significant values for autocorrelations and partial autocorrelations. The plots corresponding to the absolute returns show a completely different scenario. These plots revel some visual evidence of the predictability of variances in all ETFs, which justifies the use of GARCH modeling. In addition, after implementing ADF test, we did not find any evidence of unity roots. As expected, McLeod-Li's test detected serial correlation between absolute returns in all ETFs returns.

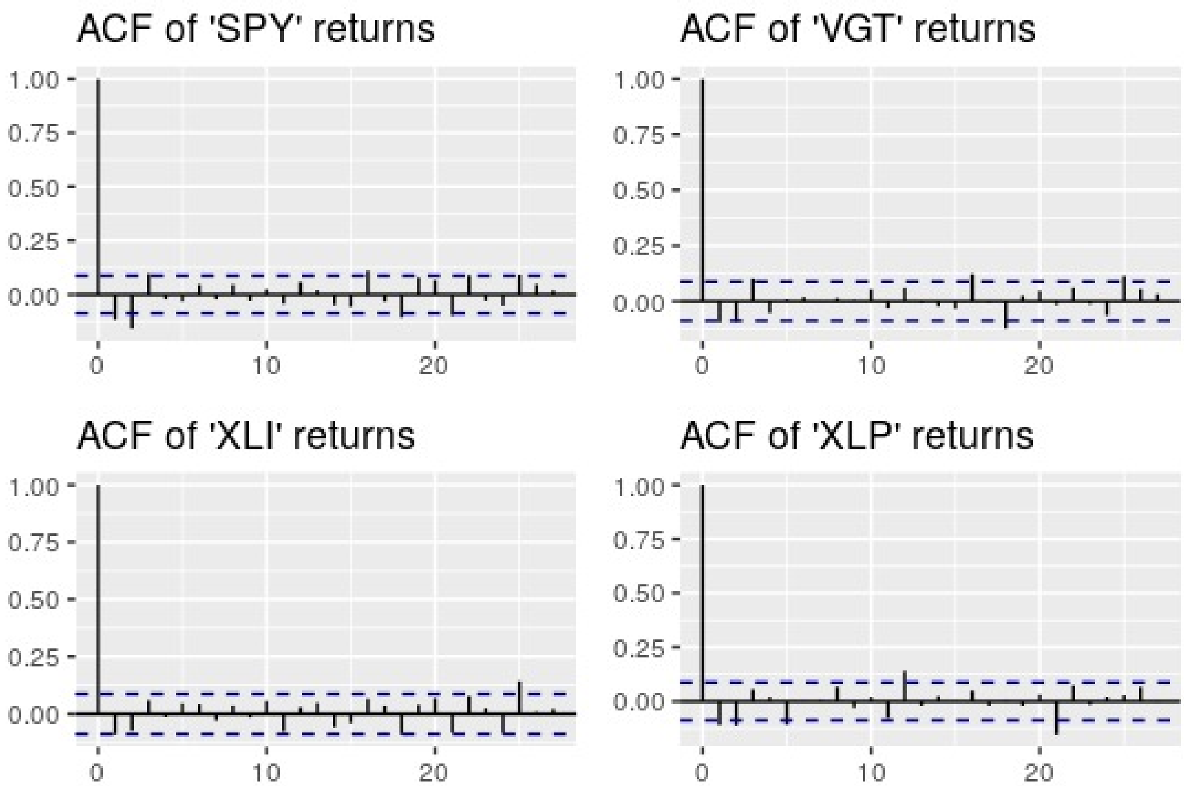

Figure 1 – ACF of returns for all ETFs.

As a second step, focusing on τ-quantiles ($\tau = 1\%, 5\%$), we fitted ten ARCH(p) models with orders (p) from 1 to 10. AIC and pseudo R^2 (called R^1 by KOENKER; MACHADO(1999)) were computed. GMM's and Christoffersen's tests were implemented to help with the model selection.

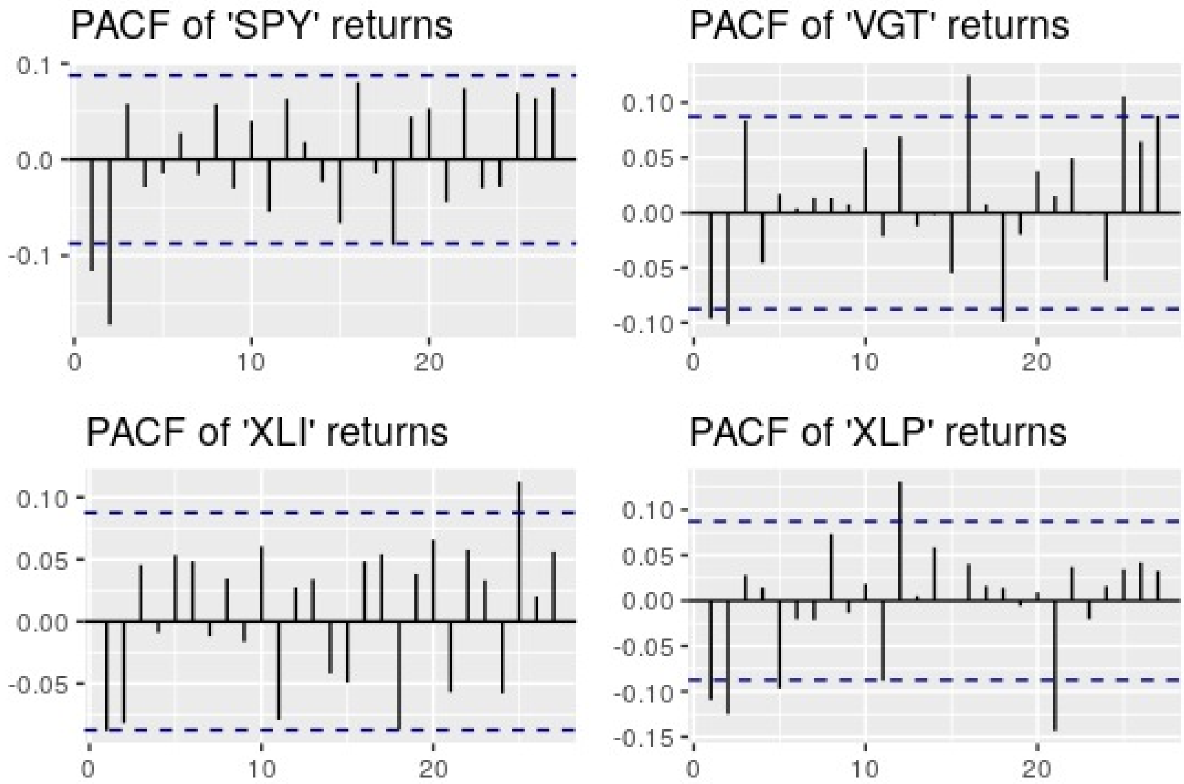

Figure 2 – PACF of returns for all ETFs.

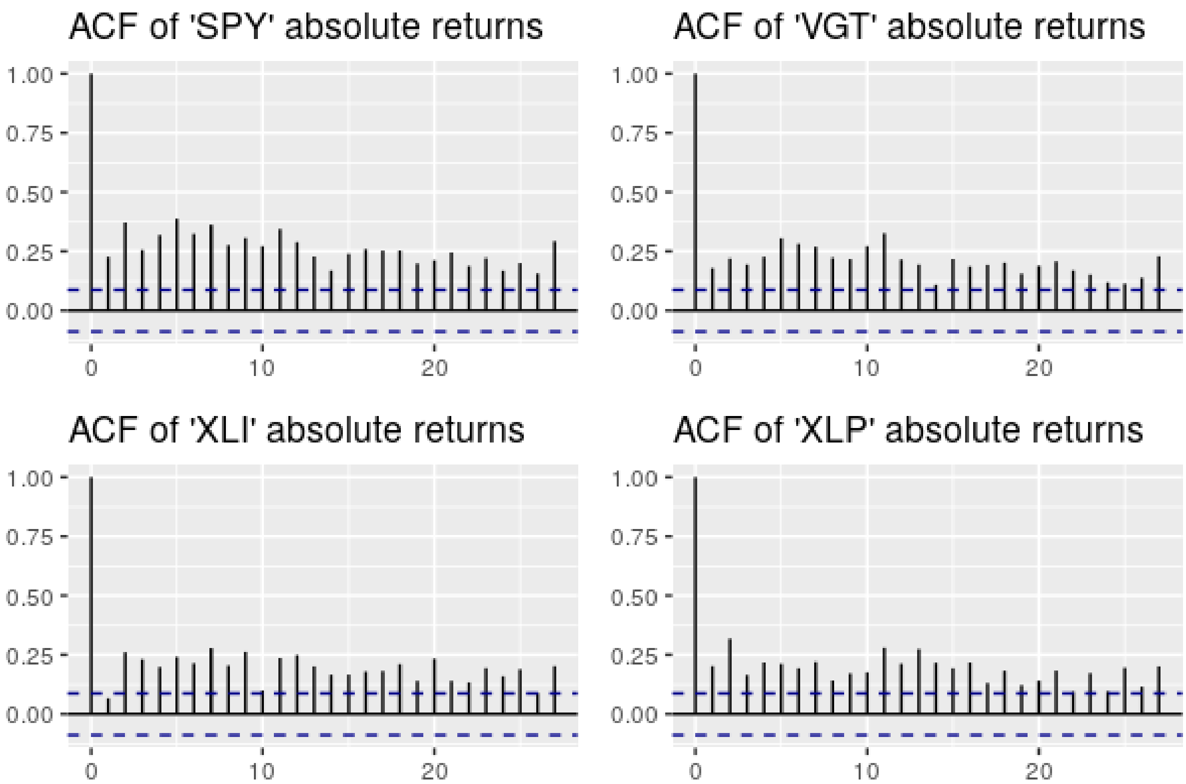

Figure 3 – ACF of absolute returns for all ETFs.

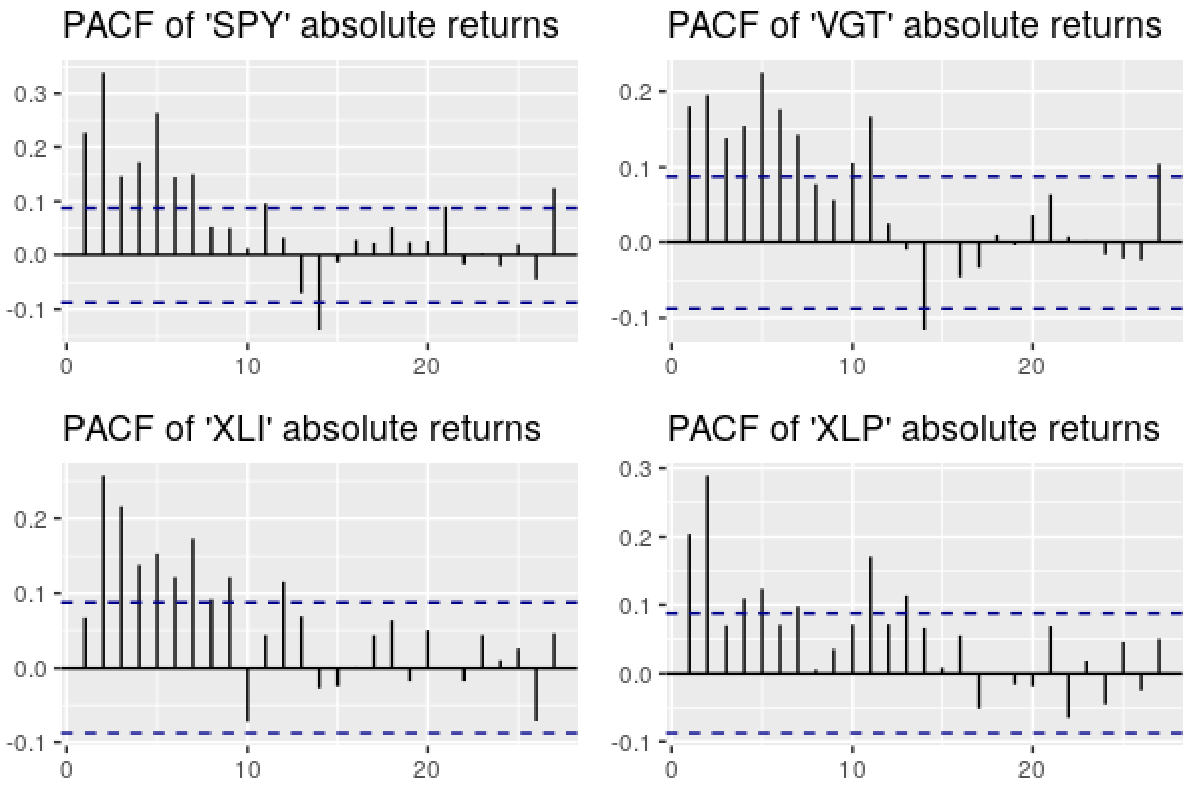

Figure 4 – PACF of absolute returns for all ETFs.

3.3.3 Model selection

In an initial analysis, eighty VaR models (ten for each ETF in both quantiles, 5% and 1%) were fitted using quantile linear ARCH modeling, and the τ-quantiles were estimated. For each ETF, we considered $\tau = 5\%$ and 1% by using the different ARCH orders $(p = 1, \ldots, 10)$. The model selection started observing the *AIC* and pseudo R^2 measures, presented in Tables 1 and 2. The AIC (Table 1) selects for both quantiles $(\tau = 5\%, 1\%)$ high order quantile linear ARCHs, none lower than six and some, especially when $\tau = 1\%$, reaching ten. A similar behavior occurs when using pseudo R^2 values, reaching a stable and less increasing state at higher orders. It is important to note that AIC and the pseudo R^2 are not sufficient to evaluate VaR violations. This is the reason we did not focus on these measures as the main criteria to qualify the goodness of fit of the VaR estimates. In this sense, the widely used coverage tests were analyzed, this is shown in Subsection 3.3.3.1.

3.3.3.1 Coverage tests

The results from the Christoffersen's test are shown in Tables 3 and 4. As can be seen, considering a nominal level of 5% and $\tau = 5\%$, the null hypothesis were not rejected for

ARCH	$\tau = 5\%$				$\tau = 1\%$			
order	SPY	VGT	XLI	XLP	SPY	VGT	XLI	XLP
1	-1888	-1959	-1895	-2437	-1444	-1609	-1553	-1996
2	-1978	-2010	-1952	-2452	-1615	-1691	-1694	-2169
3	-1973	-2012	-1973	-2450	-1632	-1711	-1735	-2170
4	-2062	-2055	-2019	-2460	-1803	-1750	-1799	-2165
5	-2081	-2053	-2027	-2482	-1881	-1787	-1794	-2221
6	-2098	-2088	-2031	-2475	-1912	-1915	-1803	-2222
7	-2123	-2095	-2029	-2477	-1963	-1978	-1866	-2315
8	-2121	-2102	-2024	-2486	-2010	-2020	-1864	-2331
9	-2114	-2096	-2020	-2479	-2008	-2014	-1914	-2337
10	-2113	-2092	-2013	-2485	-2008	-2009	-1921	-2341

Table 1 – AIC values for the fitted models

ARCH	$\tau = 5\%$				$\tau = 1\%$			
order	SPY	VGT	XLI	XLP	SPY	VGT	XLI	XLP
1	0.048	0.028	0.002	0.051	0.027	0.017	0.012	0.008
2	0.136	0.084	0.064	0.073	0.185	0.102	0.148	0.172
3	0.139	0.093	0.091	0.080	0.205	0.126	0.188	0.180
4	0.219	0.139	0.138	0.097	0.335	0.166	0.245	0.183
5	0.240	0.144	0.153	0.125	0.390	0.202	0.247	0.234
6	0.259	0.180	0.163	0.127	0.414	0.304	0.259	0.241
7	0.283	0.193	0.168	0.136	0.447	0.353	0.310	0.315
8	0.288	0.206	0.171	0.152	0.477	0.384	0.314	0.332
9	0.289	0.207	0.174	0.153	0.481	0.386	0.353	0.342
10	0.294	0.210	0.175	0.166	0.485	0.388	0.363	0.350

Table 2 – pseudo R^2 values for the fitted models

almost all ETFs data when considering the ARCH order $p = 1$. When $\tau = 1\%$ (and $p = 1$), in all cases, the null hypotheses were not rejected. For $\tau = 5\%$, when $p = 2$ or 3, the null hypotheses were rejected only for SPY (CT_{uc} and CT_{cc}, when $p = 2$, and CT_{cc}, for $p = 3$) and XLI (CT_{uc}, for $p = 3$). Regarding to Christoffersen's coverage test, in the next chapter, we show simulation evidences that Christoffersen's test is a permissive test and we recommend to consider this question in the model selection.

ARCH	SPY		VGT		XLI		XLP	
order	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}
1	0.054	0.045	0.239	0.064	0.828	0.966	0.429	0.013
2	0.021	0.028	0.235	0.064	0.168	0.386	0.117	0.018
3	0.051	0.044	0.830	0.135	0.021	0.067	0.114	0.058
4	0.001	0.003	0.012	0.020	0.004	0.016	0.411	0.056
5	0.000	0.000	0.012	0.020	0.002	0.009	0.019	0.011
6	0.000	0.000	0.030	0.066	0.019	0.062	0.002	0.001
7	0.001	0.000	0.007	0.006	0.002	0.006	0.004	0.001
8	0.004	0.005	0.001	0.002	0.002	0.006	0.000	0.000
9	0.001	0.000	0.006	0.006	0.017	0.058	0.001	0.001
10	0.006	0.019	0.017	0.003	0.001	0.003	0.000	0.000

Table 3 – P-values of Christoffersen's test when $\tau = 5\%$. CT_{uc} and CT_{cc} denotes unconditional and conditional tests, respectively.

The results of the GMM test are shown in Tables 5 and 6. As can be seen, the

ARCH order	SPY		VGT		XLI		XLP	
	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}	CT_{uc}	CT_{cc}
1	0.666	0.142	0.048	0.059	0.329	0.610	0.996	0.951
2	0.048	0.059	0.048	0.059	0.215	0.407	0.008	0.016
3	0.020	0.032	0.008	0.002	0.003	0.008	0.008	0.016
4	0.000	0.000	0.000	0.000	0.000	0.000	0.007	0.016
5	0.000	0.000	0.000	0.000	0.000	0.000	0.019	0.004
6	0.000	0.000	0.001	0.001	0.001	0.003	0.001	0.001
7	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.015
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 4 – P-values of Christoffersen's test when $\tau = 1\%$. CT_{uc} and CT_{cc} denotes unconditional and conditional tests, respectively.

null hypotheses were rejected in almost all cases (unconditional and conditional tests). An interesting exception is when $\tau = 1\%$ and $p = 1$ (Table 6) ARCH(1). In this case, the p-values corresponding to XLI and XLP suggest not rejection of the null hypotheses for all coverage tests. It is important to note that, the aim of coverage tests is to feedback whether VaR are well-estimated or not. Thus, coverage tests precedes other model criteria as AIC when fitting VaR models and, because of that, lower order models should be further evaluated.

ARCH order	SPY			VGT		
	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}
1	0.000	0.000	0.000	0.014	0.000	0.000
2	0.040	0.000	0.000	0.014	0.000	0.000
3	0.046	0.000	0.000	0.094	0.000	0.001
4	0.004	0.000	0.000	0.005	0.000	0.000
5	0.001	0.000	0.000	0.002	0.000	0.000
6	0.001	0.000	0.000	0.004	0.000	0.000
7	0.004	0.000	0.000	0.001	0.000	0.000
8	0.014	0.000	0.000	0.006	0.000	0.000
9	0.006	0.000	0.000	0.001	0.000	0.000
10	0.017	0.000	0.000	0.038	0.000	0.000
ARCH order	XLI			XLP		
	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}
1	0.047	0.057	0.142	0.027	0.009	0.002
2	0.000	0.000	0.000	0.013	0.007	0.007
3	0.002	0.000	0.000	0.001	0.000	0.002
4	0.000	0.000	0.000	0.003	0.004	0.013
5	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000
7	0.009	0.000	0.000	0.000	0.000	0.000
8	0.009	0.000	0.000	0.000	0.000	0.000
9	0.010	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000

Table 5 – P-values for GMM test when $\tau = 5\%$. GMM_{uc} denotes the unconditional test. GMM_{cc_3} and GMM_{cc_5} correspond to the conditional tests with $\ell = 3$ and 5, respectively.

ARCH order	SPY			VGT		
	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}
1	0.040	0.021	0.010	0.006	0.001	0.000
2	0.007	0.002	0.000	0.004	0.000	0.000
3	0.006	0.001	0.000	0.008	0.002	0.000
4	0.000	0.000	0.000	0.002	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.003	0.000	0.000
7	0.000	0.000	0.000	0.003	0.001	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000
ARCH order	XLI			XLP		
	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}	GMM_{uc}	GMM_{cc_3}	GMM_{cc_5}
1	0.447	0.722	0.944	0.080	0.068	0.081
2	0.032	0.023	0.032	0.009	0.004	0.005
3	0.002	0.000	0.000	0.005	0.001	0.000
4	0.000	0.000	0.000	0.003	0.000	0.000
5	0.000	0.000	0.000	0.016	0.008	0.003
6	0.001	0.000	0.000	0.002	0.000	0.000
7	0.001	0.000	0.000	0.009	0.004	0.005
8	0.000	0.000	0.000	0.002	0.000	0.000
9	0.000	0.000	0.000	0.002	0.000	0.000
10	0.000	0.000	0.000	0.001	0.000	0.000

Table 6 – P-values for GMM test when $\tau = 1\%$. GMM_{uc} denotes the unconditional test. GMM_{cc_3} and GMM_{cc_5} correspond to the conditional tests with $\ell = 3$ and 5, respectively.

3.3.3.2 Graphical analysis

Values-at-Risk measured by quantile linear ARCH model ranging from 1 to 3 were selected for sake of some visual inspection of the original series and its fitted models. Red lines are related to 1%-quantiles and orange lines to 5%-quantiles. In general, all ETFs in evaluation often shown higher violation sequences than expected at the end of 2008 and few violations or none at all at the period preceding the end of 2008. This can be seen in Figures 5 to 8. In each Figure, the black line is the returns, the orange line is the $VaR_{5\%}$, the red line represent $VaR_{1\%}$. In addition, a dashed horizontal blue line at 5% represents a daily hypothetical loss limit in an ETF investiment. According to Figures 5 to 8, in terms of VaR estimates, SPY seems to be the most aggressive index. In the opposite direction, we can highlight the XLP index. It is important to note that in Figure 7, the VaR fitted with a ARCH(1) in XLI series have some serious problems because the clear cross-quantile at critical period in the end of 2008. This violates the basic principle that distribution functions and their associated inverse functions should be monotone increased.

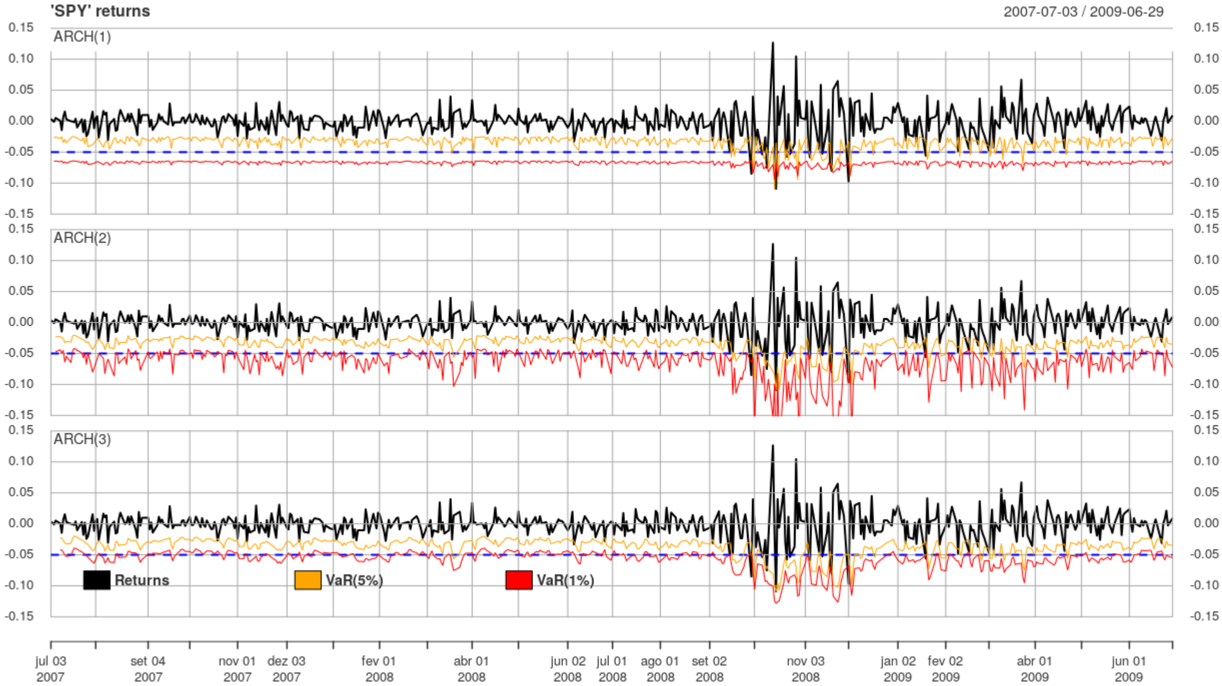

Figure 5 – SPY returns and its $VaR_{5\%}$ and $VaR_{1\%}$ fitted as quantile linear ARCH of order 1 to 3, top-bottom.

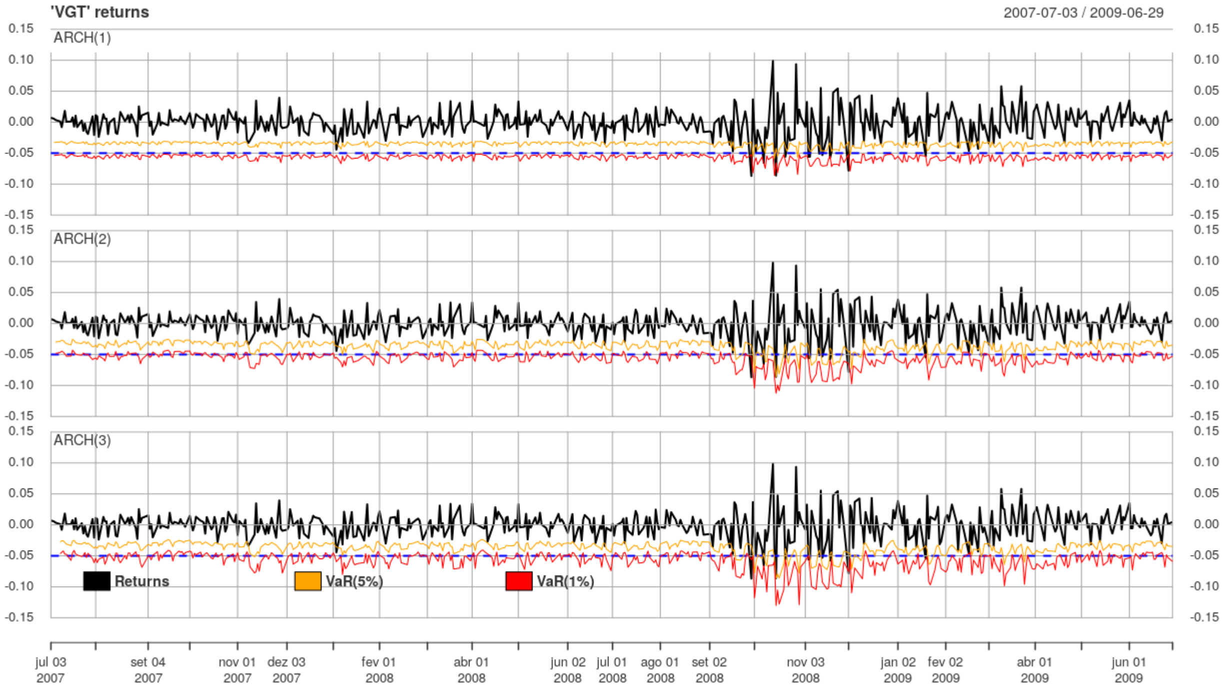

Figure 6 – VGT returns and its $VaR_{5\%}$ and $VaR_{1\%}$ fitted as quantile linear ARCH of order 1 to 3, top-bottom.

3.3.4 Testing for structural change

As mentioned before (Section 3.3), our period of analysis involves the subprime crisis in United States. In this sense, it is important to evaluate structural changes on the 5% and

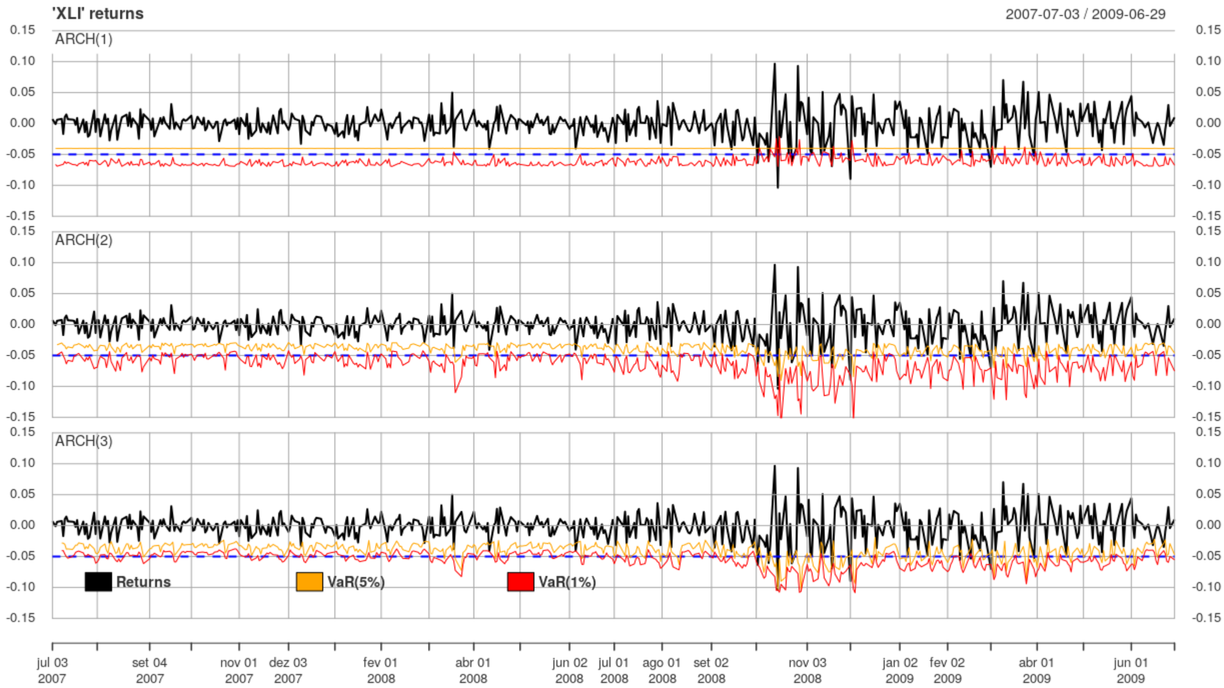

Figure 7 – XLI returns and its $VaR_{5\%}$ and $VaR_{1\%}$ fitted as quantile linear ARCH of order 1 to 3, top-bottom.

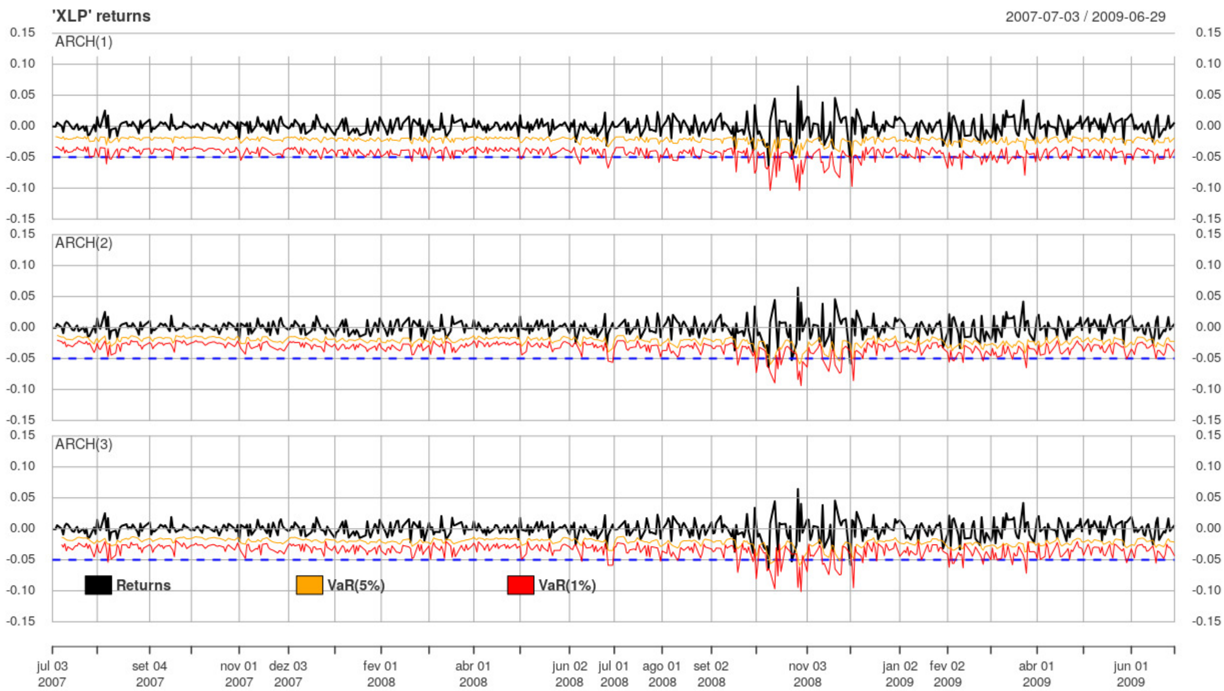

Figure 8 – XLP returns and its $VaR_{5\%}$ and $VaR_{1\%}$ fitted as quantile linear ARCH of order 1 to 3, top-bottom.

1%- quantiles. For this task Qu's test was implemented and the results are shown in Tables 7 and 8. In the tables, the critical values, the test statistics and the probable structure change date are shown according to the corresponding quantile ARCH model. As can be seen, when choosing a small ARCH order (e.g., $p < 4$), 5%- quantiles presented more break signs when comparing with 1%-quantiles. When $\tau = 5\%$, the quantile linear ARCH models order ranging from 1 to 4 detect breaks in SPY, VGT and XLI. XLP behaved similar, but only when $p = 1, 2$. Noteworthy, XLI also shows breaks when using models ranging from 7 to 10. When $\tau = 1\%$, the quantile linear ARCH models presented breaks in 29 times, out of 40 models, and confirmed in all first order models used despite have a trend towards high order models to show those breaks.

Most importantly, it is notable especially when $\tau = 5\%$ the break dates are directly related to September 15th, 2008. This was the date of Lehman Brothers' bankruptcy. It was the largest bankruptcy filing in United States history. Noteworthy that SPY, the most traditional ETF showed here, discovered break signs, when quantile $\tau = 5\%$, just 7 trading days before the major event. This is a sign of the utility of this tool in financial markets applications. In addition, XLI and XLP, both markets famous for being relatively stable also detected a break point near September 15th, 2008.

ARCH	Crit.	SPY		VGT		XLI		XLP	
order	value	SQ_τ	Break at	SQ_τ	Break at	SQ_τ	Break at	SQ_τ	Break at
1	1.329	2.947	03-09-08	3.140	12-09-08	2.810	12-09-08	2.218	16-09-08
2	1.453	2.239	03-09-08	2.339	25-08-08	2.677	03-09-08	2.136	16-09-08
3	1.517	2.206	03-09-08	2.326	12-09-08	2.134	03-09-08	1.436	16-09-08
4	1.569	2.154	03-09-08	1.796	03-09-08	1.621	23-07-08	1.521	16-09-08
5	1.601	1.763	05-06-08	1.524	03-09-08	1.512	06-08-08	1.484	29-04-08
6	1.628	1.304	03-09-08	1.083	08-07-08	1.623	06-08-08	1.485	29-04-08
7	1.650	0.619	24-12-08	1.264	15-10-08	1.952	03-09-08	1.594	16-09-08
8	1.655	1.296	16-09-08	1.380	26-09-08	2.081	03-09-08	0.889	19-05-08
9	1.684	1.167	08-09-08	1.078	05-06-08	1.967	03-09-08	1.627	02-10-08
10	1.695	0.926	08-09-08	1.312	06-11-08	2.191	03-09-08	1.335	16-09-08

Table 7 – Qu's test values for the fitted models at quantile $\tau = 5\%$

ARCH	Crit.	SPY		VGT		XLI		XLP	
order	value	SQ_τ	Break at	SQ_τ	Break at	SQ_τ	Break at	SQ_τ	Break at
1	1.329	1.672	26-09-08	1.681	26-09-08	1.421	01-10-08	1.366	16-09-08
2	1.453	1.399	26-09-08	1.693	26-09-08	0.857	01-10-08	1.067	15-09-08
3	1.517	0.856	26-09-08	1.964	26-09-08	1.796	01-10-08	1.195	16-09-08
4	1.569	1.509	26-09-08	0.740	07-10-08	1.791	01-10-08	1.677	05-06-08
5	1.601	1.892	05-06-08	1.669	26-09-08	1.512	14-10-08	1.469	05-06-08
6	1.628	1.692	05-06-08	0.874	07-01-09	1.722	09-02-09	1.859	02-10-08
7	1.650	1.881	05-06-08	3.320	02-10-08	1.824	13-10-08	1.890	05-06-08
8	1.655	2.741	12-09-08	1.863	07-11-08	1.729	14-10-08	0.992	08-10-08
9	1.684	2.201	14-10-08	2.042	27-01-09	2.563	08-10-08	2.318	18-11-08
10	1.695	2.682	16-09-08	2.999	12-02-09	2.661	08-10-08	2.276	17-11-08

Table 8 – Qu's test values for the fitted models at quantile $\tau = 1\%$

As a conclusion of our empirical analysis we can highlight that the model selection was harsh and unfruitful when considering models without structural break in financial crisis. For this reason, a structural change analysis was proposed. The Qu subgradient test had similar break detection on both investigated quantiles ($\tau = 1\%, 5\%$), but the accuracy when those breaks occurred seemed to be more interesting for the less extreme quantile ($\tau = 5\%$). The test showed a good performance in general on both quantiles while trying to see the financial crisis period as a structural break. This fact was more noticeable when $\tau = 5\%$. After making the structural change analysis we can see how it influenced the coverage tests results. In fact, we found evidence about structural changes in all ETFs we analyzed and, after comparing this insight with the results from Cristofersen's and GMM's coverage tests, we were motivated to make a simulation study involving these statistical procedures. The analysis are shown in Chapter 4.

4 SIMULATION ANALYSIS

In the simulation study, ten different models were considered, each one with the parameter structure changing over time. The objective is to achieve insights about the behavior of some tests which were used before to model VaR in ETFs. Let ε_t be a modified linear ARCH(2) model, defined as

$$\varepsilon_t = \begin{cases} (1+\alpha_{1,0}|\varepsilon_{t-1}|+\alpha_{2,0}|\varepsilon_{t-2}|)v_t\sqrt{\omega_0} & \text{, when } t \in [0, \lfloor\lambda_1 n\rfloor) \\ (1+\alpha_{1,1}|\varepsilon_{t-1}|+\alpha_{2,1}|\varepsilon_{t-2}|)v_t\sqrt{\omega_1} & \text{, when } t \in [\lfloor\lambda_1 n\rfloor, \lfloor\lambda_2 n\rfloor) \\ \vdots & \\ (1+\alpha_{1,k}|\varepsilon_{t-1}|+\alpha_{2,k}|\varepsilon_{t-2}|)v_t\sqrt{\omega_k} & \text{, when } t \in [\lfloor\lambda_k n\rfloor, n] \end{cases} \quad (4.1)$$

where $t \in \{0,1,...,n\}$, $0 < \lambda_1 < \ldots < \lambda_k < 1$ and $v_t\sqrt{\omega_i} \sim \mathcal{N}(0,\omega_i) : i \in \{0,1,\ldots,k\}$.

The random process in Equation 4.1 describes a reparametrized linear ARCH(2) process with k breaks in parameters $(\alpha_{1,i},\alpha_{1,i},\omega_i) : i \in \{0,1,\ldots,k\}$. Noteworthy, $\lambda_j : j \in \{1,\ldots,k\}$ represents the relative position of the j-th break across the time window (T), it is not the index t itself at the break moment. Equation (4.1) is equivalent to (3.9) by including a structural change (represented by $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k)$). This modified ARCH process was used in our simulation study.

4.1 SIMULATED MODELS

Two thousand Monte Carlo replications from 10 different modified linear ARCH(2) processes were generated in order to evaluate Qu's test as well as the GMM's and Cristoffersen's coverage tests. It was used R programming language and environment for statistical computing (R Core Team, 2018) and coded *set.seed(42)* in Mersenne-Twister pseudorandom number generator as starting point for these proccesses generated. Each simulated model was planned to achieve different perspectives of the tests used in the empirical analysis. In addition, we intended to simulate a return series as seen in previous chapter but also under different break circumstances. Figure 9 shows a particular case when simulating a model behaving like the SPY stock index.

Table 9 shows λ-values (indicating break point positions), the values of parameters $(\alpha_{1,i},\alpha_{2,i},\omega_i)$ (used after the break). The table contains all information necessary to understand the processes generated according to Equation (4.1). Note that λ is initially equal to zero. This means that the following parameters are the starting parameters (according to the generated process). The choice of each α was based on LEE; NOH (2013). A brief description of models

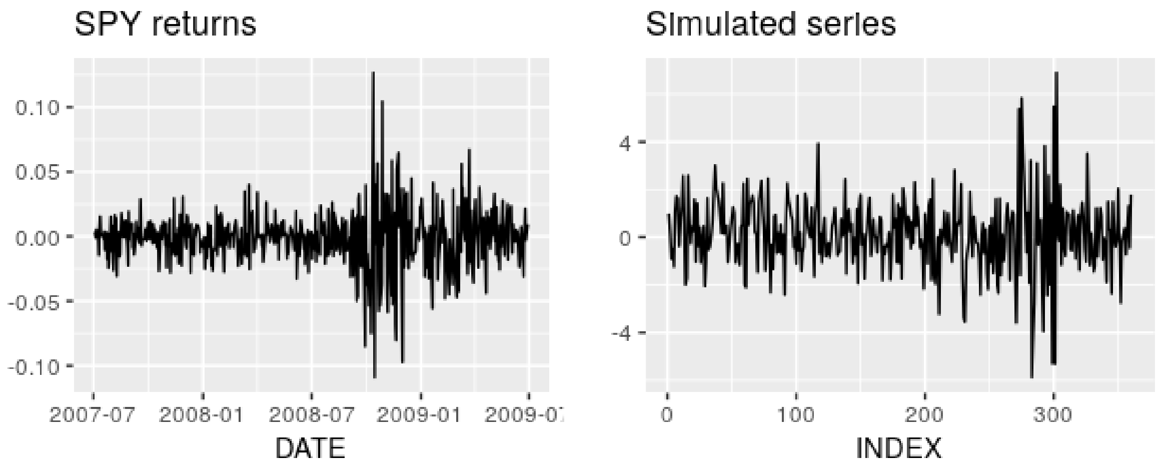

Figure 9 – SPY returns(left) and simulation process(right).

(from A to J) are listed bellow

Model	λ	α-parameters	ω
A	0	(0.15, 0.06)	1
B	0	(0.15, 0.0375)	1
	.25	(0.35, 0.1400)	2
C	0	(0.15, 0.0375)	1
	.5	(0.35, 0.1400)	2
D	0	(0.15, 0.0375)	1
	.75	(0.35, 0.1400)	2
E	0	(0.15, 0.0375)	1
	.5	(0.35, 0.1400)	1
F	0	(0.15, 0.06)	1
	.25	(0.35, 0.14)	2
	.35	(0.15, 0.06)	1

Model	λ	α-parameters	ω
G	0	(0.15, 0.06)	1
	.75	(0.35, 0.14)	2
	.85	(0.15, 0.06)	1
H	0	(0.15, 0.0375)	1
	.4	(0.35, 0.1400)	2
	.6	(0.15, 0.0375)	3
I	0	(0.15, 0.0375)	1
	.25	(0.35, 0.1400)	2
	.75	(0.15, 0.0375)	1
J	0	(0.15, 0.0375)	1
	.25	(0.35, 0.1400)	2
	.5	(0.15, 0.0375)	1
	.75	(0.35, 0.1400)	2

Table 9 – Simulated processes

- Model A is a simple linear ARCH(2), used mainly to evaluate the nominal size of the tests;
- Models B, C, D were conceived to check a single break at different points over the time;
- Model E aimed to check the impact of only changing the structure of conditional volatility;
- Model F and G were proposed to evaluate the behavior of the tests when a brake with clusters of volatility happens;
- Model H stands for a more progressive break;
- Model I has a long length cluster;
- Model J had three breaks and formed interchangeable clusters.

All generated process were fitted by using a linear ARCH(2) model estimated by quantile regression (see equation (3.10)). Models B, C, D represent similar models, but under different break periods.

4.2 SIMULATION RESULTS

The simulation results are summarized in Tables 10 through 21 and Figures 10 and 11. The discussion of the results is made in next Subsections.

4.2.1 Coverage tests

After making a statistical investigation on all simulated data we concluded that the presence of structural changes did not cause violation of the null hypothesis of the coverage tests. Because of that, our analysis focused on investigation by considering the statistics under null hypothesis. The results when considering the CHRISTOFFERSEN's tests (Tables 10 to 13) display null rejection rates much smaller than the nominal levels (5% and 1%). It suggests very conservative tests (with or without structural breaks).

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	One break					
B	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
C	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
E	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Two breaks					
F	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
G	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
H	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
I	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Three breaks					
J	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 10 – Percentage of unconditional Christoffersen test statistics under 5% p-value (rejected tests)

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	One break					
B	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
C	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
E	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Two breaks					
F	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
G	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
H	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
I	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Three breaks					
J	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 11 – Percentage of unconditional Christoffersen's test statistics under 1% p-value (rejected tests)

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0020	0.0055	0.0015	0.0010	0.0015	0.0015
	One break					
B	0.0015	0.0065	0.0010	0.0020	0.0045	0.0015
C	0.0015	0.0065	0.0030	0.0000	0.0030	0.0015
D	0.0015	0.0085	0.0025	0.0005	0.0045	0.0010
E	0.0015	0.0050	0.0000	0.0015	0.0035	0.0025
	Two breaks					
F	0.0030	0.0120	0.0025	0.0020	0.0030	0.0055
G	0.0040	0.0070	0.0015	0.0020	0.0045	0.0045
H	0.0015	0.0035	0.0020	0.0005	0.0055	0.0010
I	0.0050	0.0070	0.0020	0.0015	0.0050	0.0025
	Three breaks					
J	0.0025	0.0060	0.0025	0.0020	0.0035	0.0020

Table 12 – Percentage of conditional Christoffersen's test statistics under 5% p-value (rejected tests)

Tables 14 through 17 show the results GMM tests. The unconditional test (Tables 14 and 15) is very conservative in almost all cases. However, there were some scenarios

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0005	0.0005	0.0005	0.0000	0.0000
	One break					
B	0.0000	0.0010	0.0000	0.0010	0.0000	0.0000
C	0.0000	0.0005	0.0000	0.0000	0.0005	0.0000
D	0.0000	0.0005	0.0000	0.0005	0.0000	0.0000
E	0.0005	0.0000	0.0000	0.0005	0.0000	0.0005
	Two breaks					
F	0.0005	0.0000	0.0005	0.0005	0.0005	0.0005
G	0.0000	0.0000	0.0000	0.0020	0.0000	0.0005
H	0.0000	0.0000	0.0000	0.0005	0.0000	0.0000
I	0.0000	0.0010	0.0000	0.0010	0.0000	0.0005
	Three breaks					
J	0.0005	0.0010	0.0005	0.0010	0.0005	0.0000

Table 13 – Percentage of conditional Christoffersen's test statistics under 1% p-value (rejected tests)

in which the null rejection rates were much larger than the nominal levels: (1) in Table 14, models H ($n = 360, 720, 1080$), I ($n = 360, 720$) and B ($n = 1080$), for $\tau = 5\%$, and models D ($n = 720, 1080$), C, H and I ($n = 1080$), for $\tau = 1\%$; (2) in Table 15, model H ($n = 720, 1080$), for $\tau = 5\%$.

The conditional test was realized by considering the order ℓ equal to 3 and 5 (Tables 16 and 17, and, 18 and 19, respectively). In general, the test seems to be conservative. However, in some scenarios it presented a very liberal behavior: (1) in Table 16, models C, H, I, J ($n = 360, 720, 1080$), and B ($n = 1080$), for $\tau = 5\%$, and, D ($n = 720, 1080$), for $\tau = 1\%$; (2) in Table 17, models C, H, I, J ($n = 720, 1080$), and B ($n = 1080$), for $\tau = 5\%$; (3) in Table 18, models C, H, J ($n = 360, 720, 1080$), and B, D, I ($n = 720, 1080$), for $\tau = 5\%$, and, D, F, G ($n = 720, 1080$), for $\tau = 1\%$; (4) in Table 19, models C ($n = 360, 720, 1080$), and B, D, H, J ($n = 720, 1080$), for $\tau = 5\%$, and, D, G ($n = 1080$), for $\tau = 1\%$.

Models	No breaks					
	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0005	0.0015	0.0000	0.0005	0.0000	0.0020
	One break					
B	0.0090	0.0010	0.0625	0.0035	0.2665	0.0065
C	0.0790	0.0015	0.0460	0.0540	0.0190	0.2285
D	0.0005	0.0080	0.0000	0.1665	0.0000	0.1920
E	0.0010	0.0020	0.0005	0.0065	0.0005	0.0085
	Two breaks					
F	0.0005	0.0045	0.0000	0.0305	0.0000	0.0260
G	0.0005	0.0030	0.0005	0.0185	0.0000	0.0150
H	0.1765	0.0010	0.3980	0.0320	0.3540	0.0955
I	0.1020	0.0020	0.1080	0.0545	0.0800	0.2340
	Three breaks					
J	0.0010	0.0015	0.0005	0.0035	0.0095	0.0000

Table 14 – Null rejection rates in unconditional GMM test and 5% nominal level.

Models	No breaks					
	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	One break					
B	0.0005	0.0000	0.0000	0.0000	0.0010	0.0000
C	0.0035	0.0000	0.0185	0.0000	0.0045	0.0005
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0360
E	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Two breaks					
F	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
G	0.0000	0.0000	0.0000	0.0005	0.0000	0.0015
H	0.0035	0.0000	0.1420	0.0000	0.2500	0.0015
I	0.0015	0.0000	0.0220	0.0000	0.0215	0.0015
	Three breaks					
J	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 15 – Null rejection rates in unconditional GMM test and 1% nominal level.

Models	No breaks					
	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0015	0.0040	0.0000	0.0040	0.0000
	One break					
B	0.0010	0.0010	0.0595	0.0010	0.2230	0.0005
C	0.0675	0.0025	0.3070	0.0140	0.4685	0.0375
D	0.0125	0.0120	0.0650	0.1030	0.0885	0.1865
E	0.0005	0.0015	0.0200	0.0030	0.0320	0.0025
	Two breaks					
F	0.0015	0.0095	0.0125	0.0310	0.0165	0.0375
G	0.0015	0.0095	0.0075	0.0325	0.0180	0.0395
H	0.0180	0.0015	0.5090	0.0070	0.7890	0.0145
I	0.0155	0.0020	0.1130	0.0100	0.3260	0.0360
	Three breaks					
J	0.0005	0.0030	0.1875	0.0060	0.3625	0.0120

Table 16 – Null rejection rates in conditional GMM test with $\ell = 3$ and 5% nominal level.

Models	No breaks					
	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0000	0.0000	0.0030	0.0000	0.0030	0.0000
	One break					
B	0.0000	0.0000	0.0485	0.0000	0.1915	0.0000
C	0.0410	0.0000	0.2595	0.0005	0.4290	0.0015
D	0.0090	0.0000	0.0485	0.0050	0.0640	0.0415
E	0.0005	0.0000	0.0170	0.0000	0.0260	0.0000
	Two breaks					
F	0.0000	0.0000	0.0075	0.0010	0.0110	0.0075
G	0.0010	0.0000	0.0035	0.0015	0.0130	0.0055
H	0.0060	0.0000	0.2480	0.0000	0.6280	0.0005
I	0.0005	0.0000	0.0575	0.0005	0.2630	0.0010
	Three breaks					
J	0.0005	0.0000	0.1585	0.0000	0.3210	0.0000

Table 17 – Null rejection rates in conditional GMM test with $\ell = 3$ and 1% nominal level.

4.2.2 Qu's structure change test

For the Qu's test we analyzed the rates of null hypothesis rejection (no breaks) and the power of the test (one, two and three breaks). The results are shown in Tables 20 (5%

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0050	0.0040	0.0170	0.0040	0.0155	0.0030
	One break					
B	0.0080	0.0035	0.1095	0.0015	0.2765	0.0045
C	0.1930	0.0121	0.5815	0.0225	0.7980	0.0515
D	0.0885	0.0541	0.2270	0.2280	0.3230	0.4595
E	0.0190	0.0060	0.0565	0.0065	0.0800	0.0205
	Two breaks					
F	0.0260	0.0256	0.0480	0.0850	0.0655	0.1585
G	0.0235	0.0286	0.0460	0.1010	0.0685	0.1795
H	0.0865	0.0121	0.4685	0.0085	0.8715	0.0160
I	0.0185	0.0101	0.3295	0.0215	0.6535	0.0320
	Three breaks					
J	0.0980	0.0110	0.4865	0.0300	0.7545	0.0675

Table 18 – Null rejection rates in conditional GMM test with $\ell = 5$ and 5% nominal level.

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0020	0.0005	0.0095	0.0015	0.0090	0.0000
	One break					
B	0.0025	0.0010	0.0850	0.0010	0.2410	0.0000
C	0.1410	0.0000	0.4715	0.0015	0.7165	0.0100
D	0.0485	0.0045	0.1535	0.0675	0.2320	0.2800
E	0.0085	0.0010	0.0350	0.0015	0.0510	0.0080
	Two breaks					
F	0.0105	0.0080	0.0300	0.0285	0.0350	0.0670
G	0.0135	0.0075	0.0210	0.0350	0.0375	0.0955
H	0.0530	0.0010	0.3535	0.0010	0.6320	0.0025
I	0.0035	0.0000	0.2100	0.0015	0.5360	0.0050
	Three breaks					
J	0.0385	0.0020	0.3865	0.0075	0.6605	0.0155

Table 19 – Null rejection rates in conditional GMM test with $\ell = 5$ and 1% nominal level.

nominal level) and 21 (1% nominal level). The rates of null hypothesis rejection presented greater than the nominal levels when $\tau = 1\%$ and $n = 360, 720$. In the other cases, the rates were close to nominal levels. The power of the test was high in many scenarios. However, considering 1% nominal level (Table 21), the simulations show very small powers for some scenarios in models B ($\tau = 1\%, n = 360, 720$), E ($\tau = 5\%, n = 360, 720$; $\tau = 1\%, n = 1080$), F ($\tau = 5\%, n = 360, 720$), G ($\tau = 5\%, n = 360$), H ($\tau = 1\%, n = 360$), I ($\tau = 5\%, n = 360, 720$; $\tau = 1\%, n = 360, 720, 1080$), and J ($\tau = 5\%, n = 360$; $\tau = 1\%, n = 720, 1080$).

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0525	0.1710	0.0510	0.1075	0.0620	0.0785
	One break					
B	0.4070	0.1300	0.9085	0.1085	0.9875	0.1630
C	0.7570	0.3000	0.9760	0.7125	0.9995	0.9585
D	0.4170	0.4590	0.7475	0.7005	0.9105	0.8440
E	0.1325	0.2135	0.2000	0.1700	0.2960	0.1875
	Two breaks					
F	0.1640	0.2885	0.2270	0.2625	0.3520	0.2825
G	0.2060	0.3645	0.3200	0.3400	0.5065	0.3925
H	0.9485	0.2480	1.0000	0.6185	1.0000	0.9830
I	0.0510	0.1025	0.4055	0.0405	0.7015	0.0310
	Three breaks					
J	0.1480	0.1865	0.3995	0.1685	0.6570	0.1845

Table 20 – Null rejection rates in Qu's test results. 5% nominal level was considered.

	No breaks					
Models	n = 360		n = 720		n = 1080	
	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$	$\tau = 5\%$	$\tau = 1\%$
A	0.0125	0.0805	0.0075	0.0325	0.0115	0.0175
	One break					
B	0.1050	0.0675	0.7630	0.0360	0.9625	0.0420
C	0.5715	0.1140	0.9285	0.3255	0.9975	0.8270
D	0.1970	0.2910	0.4950	0.4825	0.7360	0.7065
E	0.0470	0.1165	0.0605	0.0605	0.1155	0.0670
	Two breaks					
F	0.0530	0.1835	0.0830	0.1345	0.1455	0.1255
G	0.0800	0.2390	0.1345	0.1865	0.2555	0.1965
H	0.8560	0.0975	0.9960	0.2400	1.0000	0.6985
I	0.0060	0.0475	0.1260	0.0110	0.4130	0.0080
	Three breaks					
J	0.0530	0.1030	0.1665	0.0670	0.4020	0.0705

Table 21 – Null rejection rates in Qu's test results. 1% nominal level was considered.

Figures 10 (models B to E) and 11 (models F to J) show simulated process (left charts) and the corresponding λ-histograms (right charts). In Figure 10, model E presented a more volatile process and the corresponding λ-histogram identified the true brake point very well. This fact also happened for models B, C and D. In Figure 11, models F and G, λ-histogram identified the second brake point. The λ-histogram of model H identified the first break point. The two breaks were well identified in Model I and the λ-histogram identified only two points in model J.

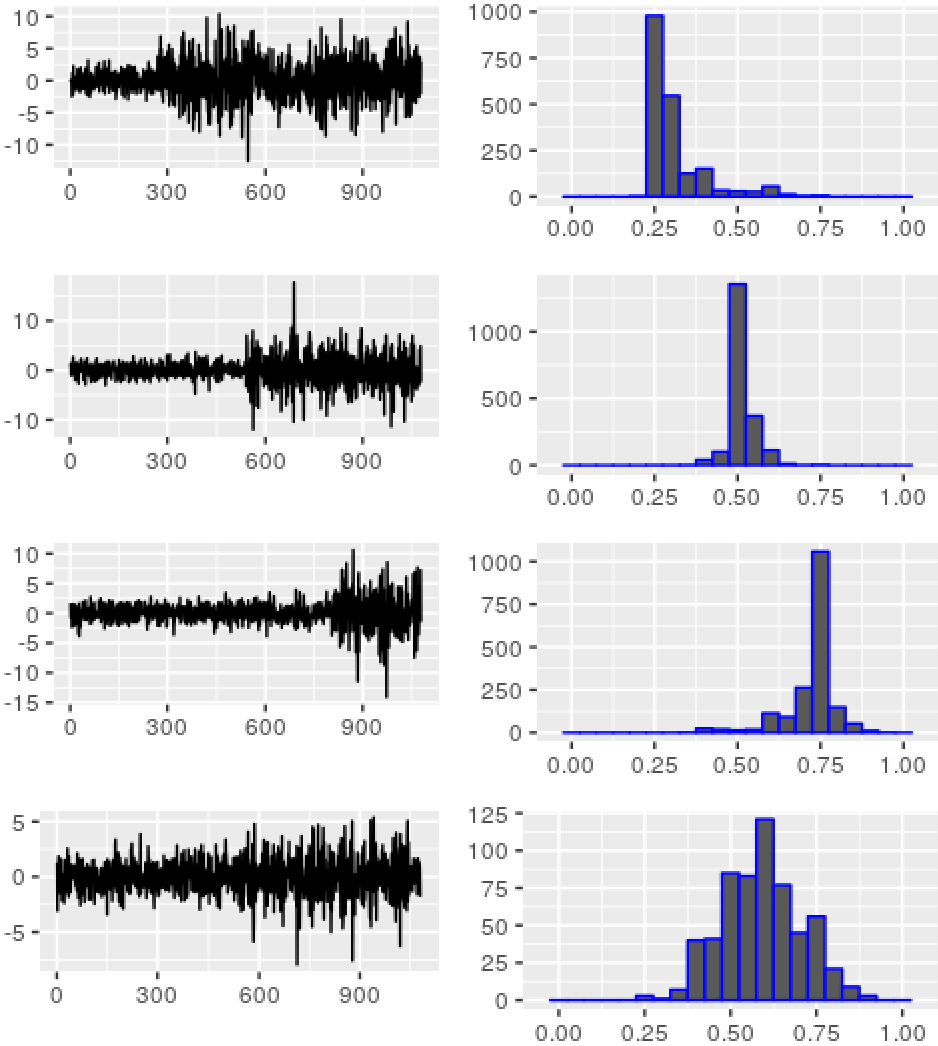

Figure 10 – Single break models (B to E), top-down. A single model simulation(left) and the histogram of λ relative position of its break points when it is rejected Qu's test hypothesis given $\tau = 5\%$ and $n = 1080$(right).

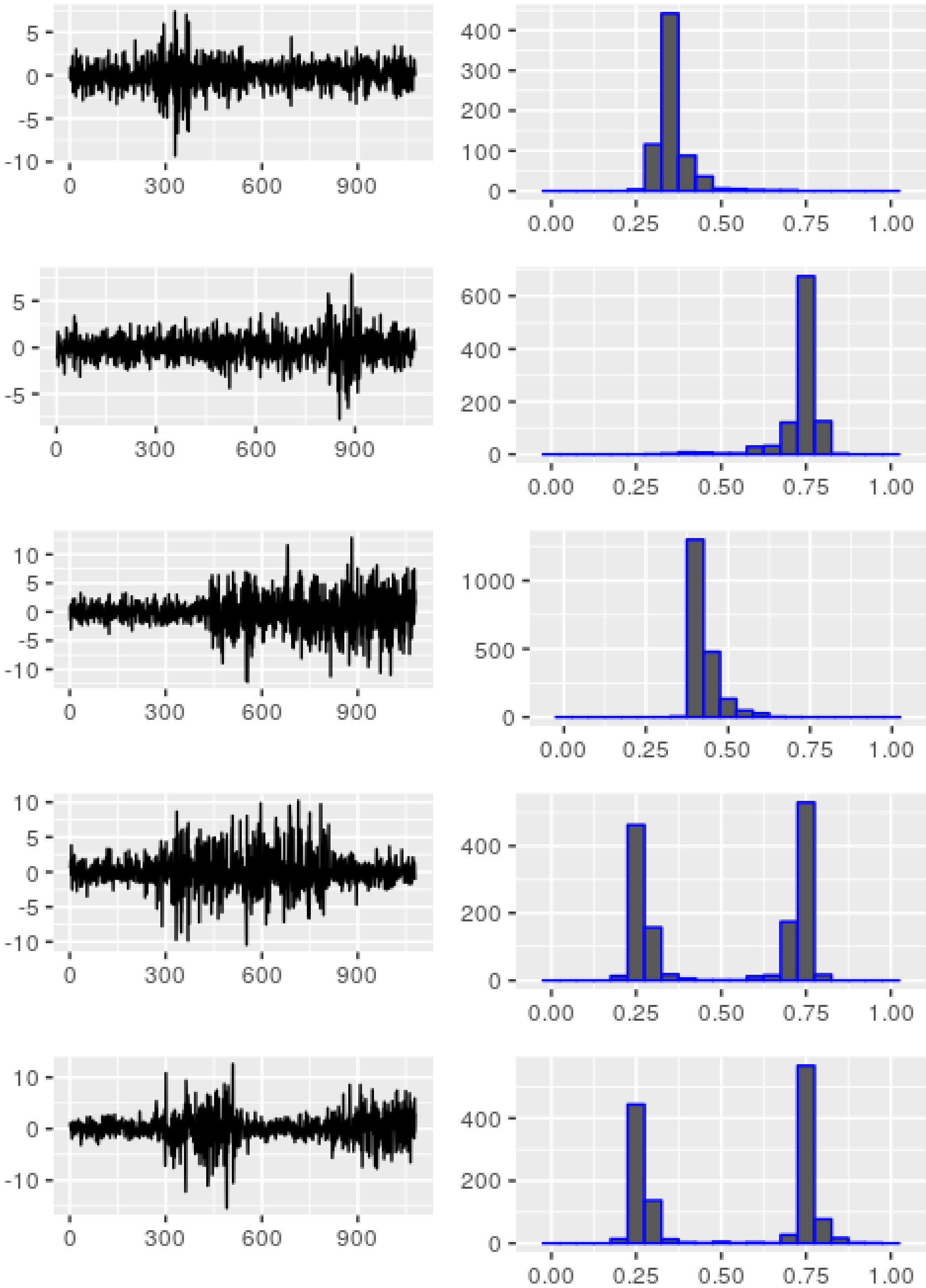

Figure 11 – Multi break models (F to J), top-down. A single model simulation(left) and the histogram of λ relative position of its break points when it is rejected Qu's test hypothesis given $\tau = 5\%$ and $n = 1080$(right).

5 CONCLUSION

Quantile regression provides a framework to cope with Value-at-Risk methodology. In this sense, this is a robust method, with distribution free, so valuable when dealing with financial time series. Regarding the applied part of this work, where we used many different ETFs returns during the Subprime mortgage crisis, our investigation concludes in favor to Qu's test capacity to detect structural changes in real data. However, this empirical study did not conclude positively from the widely used coverage tests (CHRISTOFFERSEN, 1998; CANDELON *et al.*, 2010). This motivated our simulation study.

In the simulation analysis we considered two thousand Monte Carlo replications from ten different linear ARCH(2) processes which represent several simulation scenarios focusing on designing simulated data behaving similar to the ETFs and presenting structural break points. We estimated VaR at 1% and 5% levels and analyzed the null rejections rates of the coverage tests as well as the power of QU's test. Between the main conclusions we highlight that the coverage tests behaved as very conservatives. Qu's structural change test performed a notable power in almost all model proposed during the simulation. However it was noted that its power might be severely downgraded in some kinds of series structures like those showing clusters, so common in financial time series. Anyway, in our simulations, Qu's test retained yet a significant power in these cluster situations. It was also noted in the multiple break models with similar traits a tendency of detect break in the middlemost break.

As a future work we suggest a theoretical development on Christorfesen's and GMM's null statistics in order to incorporate adjusts to make them dealing with structural breaks, improving null rejection rates of these tests.

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