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JOÃO PAULO CAVALCANTE

EIGENVALUE FOR ANGULAR TEUKOLSKY EQUATION VIA ACCESSORY PARAMETER FOR PAINLEVÉ V

Recife
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Dissertation presented to the graduation program of the Physics Department of Universidade Federal de Pernambuco as part of the duties to obtain the degree of Master of Philosophy in Physics.

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Abstract

The purpose of this dissertation is to present an alternative way to compute the eigenvalues for spheroidal harmonics, in view of its applications to arbitrary spin quasi-normal frequencies of Kerr black hole. The alternative is based on the relation between the connection problem of the angular Teukolsky Master Equation (TME) and the dependence of the Painlevé V transcendent on monodromy data. The latter has an expansion in terms of irregular conformal blocks, uncovered by the AGT correspondence, which can in principle be used for explicit calculations. The isomonodromic deformations in the angular TME is translated to two conditions on the Painlevé V transcendent which are solved to find the expansion of the accessory parameter of the angular TME and consequently the first terms of the expansion of the eigenvalue ${}_s\lambda_{lm}$.

Keywords: Teukolsky master equation. Isomonodromic deformations. Function τ_V transcendent. Accessory parameter. Eigenvalue ${}_s\lambda_{lm}$.

Resumo

A proposta desta dissertação é apresentar uma forma alternativa para o cálculo dos autovalores para harmônicos esferoidais, em vista de sua aplicação em modos quase-normais no buraco negro de Kerr. Essa forma alternativa é baseada na relação entre problema de conexão da parte angular da equação Master de Teukolsky (TME) e a dependência da função τ_V transcendente de Painlevé V sobre *monodromy data*. Essa última tem a expansão em termos de blocos conformes de primeiro tipo, descoberto pela correspondência AGT, que pode, em princípio, ser usado explicitamente. As deformations isomonodrômicas na parte angular da TME é transladado em duas condições para a função τ_V que são resolvidas de modo a encontrar a expansão do parâmetro acessório da parte angular da TME e consequentemente os primeiros termos da expansão do autovalor ${}_s\lambda_{lm}$.

Palavras-chave: Equação Master de Teukolsky. Deformações isomonodrômicas. Função transcendente τ_V. Parâmetro acessório. Autovalor ${}_s\lambda_{lm}$.

Contents

1 Introduction

Conformal field theory is a quantum field theory on a Euclidean two-dimensional space, that is invariant under local conformal transformations. In contrast to other types of quantum field theories, two-dimensional conformal field theories have an infinite number of symmetries, where in some cases can be used to solve problems directly. Based on this number of symmetries, conformal field theory has been intensively studied in the last three decades [2, 3, 4]. An attractive property of this theory is that any correlation function can be written as a linear combination of conformal blocks, where conformal blocks are functions determined by conformal symmetries and labeled by representations of the Lie algebra. In this theory the Lie algebra is called Virasoro algebra, which appears as a central extension of the Witt algebra with the central extension depending explicitly on the central charge c [5]. The value of c is crucial in 2d conformal field theory, and it is related with the Weyl anomaly or trace anomaly of the energy-momentum tensor [6]. The importance of the values of c appears directly, for example, when we describe via conformal field theory a system with free bosons or fermions, where c takes the values $c = 1$ and $c = 1/2$, respectively [7]. As a function labeled by the Virasoro algebra, the conformal blocks also depend explicit on the value of c; therefore, to write the correlation function is crucial to define c.

Still, in the discussion of to write the correlation functions in terms of the conformal blocks, Alday, Gaiotto, and Tachikawa recently discovered in [8] a relevant relation between conformal field theory in 2d and $\mathcal{N} = 2$ four-dimensions super-symmetric gauge theory, commonly referred to as AGT correspondence. This correspondence brought a new formalism for the conformal block where it is possible to express the conformal block in terms of Nekrasov partition functions [9]. On the light of AGT correspondence, it was recently observed that some correlation functions from conformal field theory, with $c = 1$ in the Virasoro algebra, can be expressed in terms of τ-functions associated to Painlevé equations, to be more precise the Painlevé VI, V, and III [10]. These observations appear even more in AGT correspondence which provides, via Nekrasov partition function, explicit representations for conformal blocks, correlation functions and consequently an exact expansion of the corresponding τ-functions.

There exists in the literature six types of Painlevé equations which are commonly denoted by PVI, PV, PIV, PIII, PII, and PI, such equations have been playing an increasingly important role in mathematical physics, especially in the applications to classical and quantum integrable systems, random matrix theory and now in 2d conformal field theory via τ-functions associated [10, 11]. The most natural mathematical framework for Painlevé equations appears in the theory of isomonodromic deformations of linear systems develop by Jimbo, Miwa, and Ueno [12, 13, 14]. Essentially, this theory is about how to deform certain parameters of linear systems such that its monodromy representation is preserved, this condition conducts to systems of linear partial differential equations whose the integrability conditions leads to Painlevé equations.

The method of isomonodromic deformations is a powerful a tool to associate linear with integrable nonlinear equations and to solve severe problems such as connection problems of nonlinear differential equations, asymptotic properties of the Painlevé equations and accessory parameter in Heun equation. The last example is the start point in this dissertation, where the isomonodromic deformations theory will be used on the confluent Heun equation, in order to find the accessory parameter associated. In the treatment of isomonodromic deformation, the confluent Heun equation can be recast as a linear system which has the same number of singularities of the equation, in this case, two regular singularities and one irregular singularity of rank 1. Where in this system, the investigation of isomonodromic deformations leads to the Painlevé V. How this comes about will be discussed in this dissertation.

The Painlevé V tau function τ_V an isomonodromic invariant in isomonodromic deformations theory is crucial in the calculation of the accessory parameter expansion. Furthermore, the results from AGT correspondence are also useful, given that the τ_V is now expressed in terms of correlation function from 2d conformal field theory in c=1 [11]. With the isomonodromic deformation treatment and the accessory parameter expansion calculated, we find the explicit expansion of the eigenvalues of the angular Teukolsky Master equation (TME) in view of the angular equation is a confluent Heun equation and can be written in terms of a linear system.

This dissertation is divided into three chapters. In Chapter 1, we present a short introduction to conformal field theory in two dimensions, where we define Virasoro algebra, primary operators and the concept of insertion limit on the Riemann sphere. The correlation function between four primary operators is studied as well as Conformal Block (CB) defined by Operator Product Expansion (OPE). With the OPE method limited, the AGT correspondence is used to rewrite the CB in terms of Nekrasov partition functions. Finally, at the end of the chapter, we define Whittaker sates of rank r in the Virasoro algebra and Confluent CB of First Kind as a limit of the conformal block with four primary operators. The limit allows us to write the correlation function between two Whittaker states of rank $r = 0$ and one of rank $r = 1$.

In Chapter 2, we start introducing some general ideas about Painlevé equations and write the explicit form of the Painlevé VI and V. In order to investigate isomonodromic deformation, we define a generic linear system and study the analytic continuation of solutions via Monodromy matrices and Stokes phenomenon. Using the Jimbo, Miwa, and Ueno results, we present the general idea of isomonodromic deformations in the generic system, Schlesinger equations, and the τ-function as an isomonodromic invariant. With the generic system defined, we investigate a nontrivial system with two regular points and one irregular point, which is associated with deformations in the Confluent Heun equation. Furthermore, we study isomonodromy deformation on Painlevé V and defined τ_V associated. We finish this chapter using the AGT correspondence with $c = 1$ to express the τ_V-function in terms of the correlation function between two Whittaker states of rank $r = 0$ and one state of rank $r = 1$ in the Virasoro algebra, defined in Chapter 1.

In Chapter 3, we derivate the two conditions of the τ_V-function: the first condition from the isomonodromic deformation theory and second condition named Toda equation. Using the τ_V expression defined in Chapter 2 and the conditions, we find the first terms of the accessory parameter expansion associated with the confluent Heun equation. With the expansion of the accessory parameter calculated, we express the first seven terms of the expansion for the eigenvalue ${}_s\lambda_{lm}$ of the angular Teukolsky Master equation, where these terms are in agreement with the results found in the literature.

Finally, in Chapter 4 we present a short review of the dissertation and also propose new directions and perspectives for our work.

2 Conformal Block

Conformal Blocks (CB) are special functions determined by conformal symmetry and responsible for building correlation functions in two-dimensional conformal field theory. Such correlations play an essential role in this chapter, to be more precise, we are interested in the correlation function between four primary operators, as well as in the correlation function of three primary operators, which appears as a confluence limit of the correlation function of four primary operators. Therefore, we start this chapter giving a short introduction to 2d conformal field theory and explaining the idea behind conformal blocks. To write explicitly the two correlation functions, we also study how conformal blocks are written in terms of Nekrasov partition functions, using such partition functions we write the conformal block and confluent CB of the first kind. We finish this chapter writing the explicit expression for four operators correlation function and the confluent correlation function in terms of Nekrasov partition functions.

Conformal Field Theory

The first formalism about Conformal Field Theory (CFT) in two dimensions appeared in the seminal paper by A. Belavin, A. Poliakov and A. Zamolodchikov [15], since then, many applications are made in different areas like condensed matter, string theory, and black holes.

The approach for studying CFT is somewhat different from the usual approach for quantum field theory. Because instead of starting with a classical action for the fields and quantizing them via the canonical quantization or the path integral method, one employs the symmetries of the theory. The procedure which uses symmetries is called nowadays conformal bootstrap, with which it is possible to define and in certain cases even solve the theory just by exploiting the consequence of the symmetries. In 2d CFT, the bootstrap procedure can be used directly, since there is an infinite number of generators.

The exciting idea is that by studying CFT, it is possible to understand some statistical systems since in the statistical point of view, conformal field theory describes the critical behavior of some systems at second order phase transitions. For example, the two-dimensional Ising Model has two phases, disordered and ordered phases, which are associated with high and low temperature, respectively. These two phases are related to each other by a second order phase transition at the self-dual point. At the critical point, the field theory has fluctuations on all lengths such that the scale invariance appears, wherein some systems such scale invariance leads to the complete conformal symmetry. The relation between scale invariance phenomenon and conformal invariance in $D = 2$ dimension possess an intrinsic link arising from the renormalization group [16]. In black hole physics, we also have an example, the 3D BTZ black hole named after Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli, where from AdS/CFT conjecture leads to 2D CFT on the boundary [17].

2.1 Conformal Field Theory in d=2

We start this subsection by introducing conformal transformations and determining the conditions for conformal invariance. We also present the Virasoro algebra as an extension of the Witt algebra and define primary operators. Furthermore, we calculate the correlation function of two and three primary operators explicitly.

Conformal Transformations and Condition for Conformal Invariance

Let us consider a two-dimensional flat space with local transformations which preserve the angle of intersection between two curves, Figure 2.1. Such transformations are called

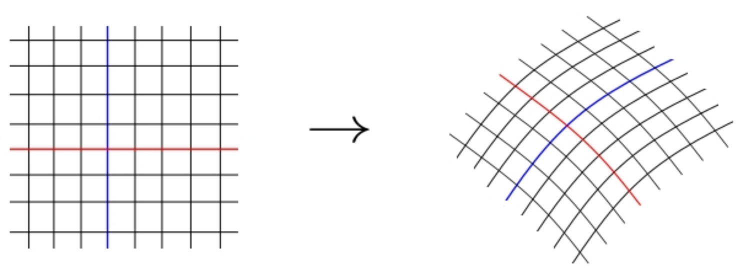

Figure 2.1: Conformal Transformation

conformal transformations and by these transformations the metric changes in the following form

$$g'_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)g_{\mu\nu}, \tag{2.1}$$

with $g_{\mu\nu}$ the Euclidian metric.

In direction to prove the angular invariance, let us use the definition (2.1) and the angle definition which is given by

$$\cos\,(\alpha(x_0)) := \frac{u^{\rho}(x)w^{\sigma}(x)}{||u||||w||}g_{\rho\sigma}(x)\bigg|_{x=x_0}, \tag{2.2}$$

with the norm of the vectors $||u|| = \sqrt{u^\nu u^\mu g_{\nu\mu}}$ and $||w|| = \sqrt{w^\nu w^\mu g_{\nu\mu}}$. By a conformal transformation $x \to x'(x)$ the angle between the lines is written as

$$\cos\,(\alpha'(x_0)) = \frac{u'^\rho(x'(x)) w'^\sigma(x'(x))}{||u'||||w'||} g'_{\rho\sigma}(x'(x))\Big|_{x=x_0}. \tag{2.3}$$

Using the transformation of the vector

$$u'^\rho(x'(x)) = \frac{\partial x'^\rho}{\partial x^\mu} u^\mu(x), \tag{2.4}$$

we have

$$\cos\,(\alpha'(x_0)) = \frac{u^\mu(x) w^\nu(x)}{||u||||w||} \frac{1}{\Lambda(x)} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x'(x))\Big|_{x=x_0}, \tag{2.5}$$

with the equation (2.1), we obtain

$$\begin{aligned} \cos\,(\alpha'(x_0)) &= \frac{u^\mu(x) w^\mu(x)}{||u||||w||} g_{\mu\nu}(x)\Big|_{x=x_0} \\ &= \cos\,(\alpha(x_0)), \end{aligned} \tag{2.6}$$

which means that the angle is invariant by conformal transformation, $\alpha'(x_0) = \alpha(x_0)$.

Now, in order to investigate the conditions associated with conformal transformations, let us consider an infinitesimal coordinate transformation given by

$$x'^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2). \tag{2.7}$$

Replacing in the left-hand side of (2.1), we find

$$g_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = g_{\mu\nu} + (\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu) + \mathcal{O}(\epsilon^2). \tag{2.8}$$

where the second term must be proportional to the metric, thus

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = K(x) g_{\mu\nu}, \tag{2.9}$$

with $K(x)$ some function of x. By taking the trace in the equation above and replacing $K(x)$ in the same equation, we get

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = (\partial\,.\,\epsilon) g_{\mu\nu}, \qquad \partial\,.\,\epsilon = \partial^\sigma \epsilon^\rho g_{\rho\sigma} = \partial^\sigma \epsilon_\sigma. \tag{2.10}$$

From the relation above, we can obtain the Cauchy-Riemann equations by using complex variables. It is well-known that on the Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$:

$$\begin{aligned} z &= x_0 + i x_1, \quad \epsilon = \epsilon_0 + i\epsilon_1, \quad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \\ \bar{z} &= x_0 - i x_1, \quad \bar{\epsilon} = \epsilon_0 - i\epsilon_1, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \end{aligned} \tag{2.11}$$

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \quad \partial_0\epsilon_1 = -\partial_1\epsilon_0, \tag{2.12}$$

where we observe from the equations above that $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$ are holomorphic and antiholomorphic functions. Furthermore, the most general conformal transformation in the complex plane will be written as $f(z) = z + \epsilon(z)$. Thus, via conformal transformation $z \to f(z)$ the metric will change in the following form

$$d^2s = \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dz d\bar{z}, \tag{2.13}$$

where $\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} = \left|\frac{\partial f}{\partial z}\right|^2$ is the scale factor.

2.1.1 Witt Algebra

The generators of conformal transformations, which can be found using (2.10), compose a Lie algebra which in two dimensions is called Witt algebra. To obtain the commutation relations associated with such algebra, let us consider $\epsilon(z)$ a meromorphic function as an expansion in Laurent series such that (2.7) becomes

$$z' = z + \epsilon(z) = z - \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \quad \text{and} \quad \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} - \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \tag{2.14}$$

with ϵ_n, $\bar{\epsilon}_n \in \mathbb{C}$. From (2.14), we identify the generators corresponding to a transformation for a particular n as,

$$L_n = -z^{n+1}\partial_z, \qquad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{2.15}$$

given that $n \in \mathbb{Z}$, we have thus an infinite number of generators in two dimensions. Where the Lie brackets associated with Witt algebra are given by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n}, \\ [L_m, \bar{L}_n] &= 0. \end{aligned} \tag{2.16}$$

The first commutation relation can be proved by considering the act of the L_m and L_n in a holomorphic function, $h(z)$,

$$\begin{aligned} [L_m, L_n]h(z) &= (L_m L_n - L_n L_m)h(z) \\ &= z^{m+1}\partial_z(z^{n+1}\partial_z h(z)) - z^{n+1}\partial_z(z^{m+1}\partial_z h(z)) \\ &= (n+1)z^{m+n+1}\partial_z h(z) - (m+1)z^{m+n+1}\partial_z h(z)) \\ &= -(m-n)z^{m+n+1}\partial_z h(z) \\ &= (m-n)L_{m+n}h(z) \end{aligned}$$

The second commutation in (2.16) can be obtained considering the antiholomorphic function $\bar{h}(\bar{z})$. In the next pages, we are not going to consider generators from the antiholomorphic part $\{\bar{L}_n\}$, since the treatment is analogue.

Global Conformal Transformation

In the part of the Witt algebra generated by $\{L_n\}$, the generators L_n are not defined everywhere in the complex plane. In particular, when $z = 0$ in the equation (2.15), the generator will depend on the value of n to be well defined, the same problem is observed at infinity. In order to understand and consider the point at infinity, let us take the conformal compactification of $\mathbb{R}^2$, in this case the Riemann sphere, $S^2 \simeq \mathbb{C} \cup \{\infty\}$. Now, on the Riemann sphere, we can deal with $z = 0$ and $z = \infty$:
At $z = 0$, we find that L_n is well defined only for $n \geq -1$,

$$L_n = -z^{n+1}\partial_z, \qquad n \geq -1.$$

To investigate the behavior of L_n at $z = \infty$, we need perform the change of variable $z = -\frac{1}{w}$, and study the limit $w \to 0$. Thus, we obtain

$$L_n = -\left(-\frac{1}{w}\right)^{n-1}\partial_w \qquad n \leq 1$$

where $\partial_z = (-w)^2\partial_w$. Therefore, we have the following constraints,

$$L_n = \begin{cases} n \geq -1, & z = 0, \\ n \leq 1, & z = \infty, \end{cases} \tag{2.17}$$

where from these constraints we obtain the generators associated to a subalgebra in the Witt algebra generated by the set $\{L_{-1}, L_0, L_1\}$, with these generators responsible for the global conformal transformations. From (2.14) and (2.15), it is possible to check directly that L_{-1}, L_0, and L_1 are interpreted as translation, dilation, rotation and special conformal transformation (SCT),

$$\begin{aligned} L_{-1} = -\partial_z \quad &\Longrightarrow \quad z' = z - \epsilon_{-1} \quad : \quad Translation \\ L_0 = -z\partial_z \quad &\Longrightarrow \quad z' = (1-\epsilon_0)z \;\; : \;\; \begin{cases} Dilation, & \epsilon_0 \text{ is } \mathbb{R} \\ Rotation, & \epsilon_0 \text{ is } \mathbb{C} \end{cases} \\ L_1 = -z^2\partial_z \quad &\Longrightarrow \quad z' = z - \epsilon_1 z^2 \quad : \quad SCT. \end{aligned}$$

In summary, the set of operators generate transformations on the Riemann sphere of the following form

$$z \mapsto f(z) = \frac{az+b}{cz+d}, \qquad \text{with} \qquad a, b, c, d \in \mathbb{C}, \tag{2.18}$$

where the transformation is invertible when $ad - bc \neq 0$. By considering $ad - bc = 1$, we can infer that the conformal group on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$ is the Möbius group, $SL(2, \mathbb{C})/\mathbb{Z}_2$. The quotient by $\mathbb{Z}_2$ is due to the fact that (2.18) is unaffected by taking all of a, b, c, and d to minus themselves. In the $SL(2,\mathbb{C})$ language, the conformal transformations associated with the operators $\{L_{-1}, L_0, L_1\}$ are given by [18]

$$\begin{aligned} Translation : \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \quad Rotations : \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \\ Dilation : \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad SCT : \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \end{aligned} \tag{2.19}$$

where $B = a_1 + ia_2$, $C = b_1 - ib_2$ and λ, $\theta \in \mathbb{R}$.

2.1.2 Virasoro Algebra

The generators of the local conformal transformations which obey the Witt algebra (2.16) are related with classical generators. The quantum version of these generators obeys an identical algebra the Virasoro algebra, except for a central charge c that appears as an extension of the Witt algebra. The central charge is perhaps the most important number which characterizes the CFT. The c is intrinsically related to Weyl anomaly which appears when we try to quantize the Witt algebra on the Riemann sphere. Essentially, this anomaly is related to the trace of the energy-momentum tensor, where classically the trace is zero; however, in the quantization on a curved background, the trace is proportional to c. We are not going to prove all mathematical definitions, more details in [7, 4, 15, 6]. In this dissertation, we assume directly that the Lie bracket in the Virasoro algebra is defined by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.20}$$

and the explicit deduction of the Virasoro algebra can be seen in [5]. Since Virasoro algebra is the quantum version associated with Witt algebra, we can build the spectrum in CFT by using L_n, analogous to quantum theory, more details about extension of the Witt algebra in [6, 18, 7].

To build such a spectrum is necessary to define quasi-primary and primary operators in the Virasoro algebra. Both are local operators on the Riemann sphere, but only the primary operators are annihilated by the lowering generators. In the point of view of representation theory, such operators are the lowest dimension operators in the conformal algebra. All other operators in the representation are called descendants and are obtained by acting on the primary operators with the raising generators. Such definitions will become apparent in the next pages.

2.1.3 Quasi-Primary Operators

In two dimensions the operator $\mathcal{O}_\Delta(z)$ is defined as quasi-primary if by conformal transformation $z \mapsto f(z)$ the operator changes to the following way

$$\mathcal{O}(z) \mapsto \mathcal{O}'(z) = \left(\frac{\partial f}{\partial z}\right)^{\Delta} \mathcal{O}(f(z)), \tag{2.21}$$

where Δ is the holomorphic conformal dimension associated to $\mathcal{O}(z)$. From (2.21), we can also define a primary operator, where if an operator changes as the equation above for *any* conformal transformation, it is a primary operator. Essentially, every primary operator is quasi-primary operator, but the reverse is not true, for example, the energy-tensor in 2D CFT is a quasi-primary, however, via conformal transformation do not change as (2.21) [3, 6].

Let us investigate how a primary operator $\mathcal{O}(z)$ changes under an infinitesimal conformal transformation, $z \mapsto f(z)$. Replacing $f(z) = z + \epsilon(z)$ with $\epsilon(z) \ll 1$ in (2.21), the following quantities are obtained

$$\begin{aligned} \left(\frac{\partial f}{\partial z}\right)^{\Delta} &= 1 + \Delta \partial_z \epsilon(z) + \mathcal{O}(\epsilon^2), \\ \mathcal{O}(z + \epsilon(z)) &= \mathcal{O}(z) + \epsilon(z)\partial_z \mathcal{O}(z) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.22}$$

We have directly

$$\delta_\epsilon \mathcal{O}(z) = \big(\Delta \partial_z \epsilon(z) + \epsilon(z)\partial_z\big)\mathcal{O}(z), \tag{2.23}$$

since $\epsilon(z)$ is a Laurent series and it is well-known from quantum theory that, a small variation in a field is associated with commutation between a conserved charge and a field, we can thus use the same idea to find the commutation relation related with the equation (2.23). In this case, we have an infinite number of conserved charges or conformal charges written as the sum of all generators [3],

$$Q_\epsilon = -\sum_{n\in\mathbb{Z}} \epsilon_n L_n. \tag{2.24}$$

Therefore, using $\epsilon(z)$ and Q_ϵ, we find the general commutation relation,

$$[L_n, \mathcal{O}_\Delta(z)] = z^n\big(z\partial_z + (n+1)\Delta\big)\mathcal{O}_\Delta(z). \tag{2.25}$$

From the equation above, we find how the global generators act on the primary operator $\mathcal{O}_\Delta(z)$. To do that, let us consider $n = 0, \pm 1$

$$\begin{aligned} [L_{-1}, \mathcal{O}_\Delta(z)] &= \partial_z \mathcal{O}_\Delta(z), \\ [L_0, \mathcal{O}_\Delta(z)] &= (z\partial_z + \Delta)\mathcal{O}_\Delta(z), \\ [L_1, \mathcal{O}_\Delta(z)] &= \big(z^2\partial_z + 2z\Delta\big)\mathcal{O}_\Delta(z), \end{aligned} \tag{2.26}$$

where, as it was explained previously we can see explicitly in the relations above how the global operators act on $\mathcal{O}_\Delta(z)$, with the commutation relations representing translation, dilation, rotation, and special conformal transformation. It is possible to set the position of the operator $\mathcal{O}_\Delta(z)$ at $z = 0$ which means consider a reference point where the operator is not translated. Therefore,

$$\begin{aligned} [L_{-1}, \mathcal{O}_\Delta(0)] &= \partial_z \mathcal{O}_\Delta(0), \\ [L_0, \mathcal{O}_\Delta(0)] &= \Delta \mathcal{O}_\Delta(0), \\ [L_1, \mathcal{O}_\Delta(0)] &= 0. \end{aligned} \tag{2.27}$$

In the commutations above L_1 annihilate the operator $\mathcal{O}_\Delta(0)$ and L_{-1} changes the position of $\mathcal{O}_\Delta(0)$ by translation, thus, $\mathcal{O}_\Delta(0)$ is a primary operator with L_1 and L_{-1} representing the lowering and raising generator respectively, in representation theory.

We have written lots of commutation relations, and carefully pointed out the interpretation of the global generators. Now we will introduce the idea of states in conformal field theory. In general, the conformal dimension of any operator can be related to quantum numbers, such connection allow us to define the operator-state correspondence, denote by $\mathcal{O}_\Delta \leftrightarrow |\Delta\rangle$ [19, 6]. In such correspondence, each insertion of an operator on the Riemann sphere are represented by states as

$$\lim_{z\to 0} \mathcal{O}_\Delta(z)\,|0\rangle = |\Delta\rangle\,, \tag{2.28}$$

where we consider the insertion of $\mathcal{O}_\Delta$ at $z = 0$. The vacuum state $|0\rangle$ is characterized by no operator insertion on the sphere and satisfies the following conditions,

$$L_m\,|0\rangle = 0, \quad m \geq -1, \quad \langle 0|\,L_m = 0, \quad m \leq 1,$$

where we also have $L_{\pm 1,0}\,|0\rangle = 0$, [15, 18].

Using the insertion limit, we can derive the following relations from the last two commutations in (2.27)

$$\begin{aligned} L_0\,|\Delta\rangle &= \Delta\,|\Delta\rangle\,, \\ L_n\,|\Delta\rangle &= 0,\ \ n \geq 1. \end{aligned} \tag{2.29}$$

Also, we define in the dual case

$$\begin{aligned} \langle\Delta|\,L_0 &= \Delta\,\langle\Delta|\,, \\ \langle\Delta|\,L_n &= 0,\ \ n \leq 1, \end{aligned} \tag{2.30}$$

with $L_{-n} = L_n^\dagger$.

All operators which satisfy the relations above are classified as primary operators with conformal dimension Δ. The first commutation in (2.27) is related to descendants operators that are built from $|\Delta\rangle$ and are commonly created using L_n with $n \leq -1$ or in the dual case by $n \geq 1$.

Let us give an example of how to calculate the conformal dimension of a descendant operator. In this example, we are going to consider the first descendant operator defined by

$$\mathcal{O}_\Delta^{(-1)}(z) = L_{-1}\mathcal{O}_\Delta(0), \tag{2.31}$$

the equation above can be written as

$$\mathcal{O}_\Delta^{(-1)}(z)\,|0\rangle = L_{-1}\mathcal{O}_\Delta(z)\,|0\rangle \qquad \rightarrow \qquad |\Delta_1\rangle = L_{-1}\,|\Delta\rangle\,. \tag{2.32}$$

To compute the conformal dimension of $|\Delta_1\rangle$, we apply L_0 and use the commutation relations (2.20) and (2.29),

$$\begin{aligned} L_0\,|\Delta_1\rangle &= L_0 L_{-1}\,|\Delta\rangle\,, \\ &= ([L_0, L_{-1}] + L_{-1}L_0)\,|\Delta\rangle\,, \\ &= (\Delta+1)L_{-1}\,|\Delta\rangle\,, \\ &= (\Delta+1)\,|\Delta_1\rangle\,, \end{aligned} \tag{2.33}$$

so the holomorphic dimension is $\Delta + 1$.

In general, we can construct the states from $|\Delta\rangle$ and its descendants by using (2.26),

Operator	Dimension
$\vdots$	
$L_{-1}L_{-1}L_{-1}\,|\Delta\rangle\,,\; L_{-2}L_{-1}\,|\Delta\rangle\,, L_{-3}\,|\Delta\rangle$	$\Delta + 3$
$L_{-1}L_{-1}\,|\Delta\rangle\,,\; L_{-2}\,|\Delta\rangle$	$\Delta + 2$
$L_{-1}\,|\Delta\rangle$	$\Delta + 1$
$|\Delta\rangle$	$\Delta.$

(2.34)

Using partitions it is possible to generalize (2.33) labeling the descendants as $L_{-\lambda}\,|\Delta\rangle = L_{-\lambda_N}...L_{-\lambda_1}\,|\Delta\rangle$ with $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0\}$ and conformal dimension $\Delta_\lambda = \Delta + |\lambda|$, $|\lambda| = \sum_i^N \lambda_i$. The partitions formalism can be identified by Young diagram. Since the conformal blocks are functions of Young diagrams, they play a crucial role in this dissertation, hence we are going to fix some notations.

Young Diagram

A set of all Young tableau will be denoted by $\mathbb{Y}$. For $\lambda \in \mathbb{Y}$, we denote the transposed tableau as λ' with λ_i, and λ'_j the number of boxes in ith row and jth column of λ, with $|\lambda|$ the total number of boxes. Given a box $(i,j) \in \lambda$, we define the arm-length $A_{\mathbb{Y}}(\bullet)$, the leg-lenght $L_{\mathbb{Y}}(\circ)$, and the hook length $h_\lambda(i,j)$ as,

$$\begin{aligned} A_{\mathbb{Y}}(\bullet) &= \lambda_i - j, \\ L_{\mathbb{Y}}(\circ) &= \lambda'_j - i, \\ h_\lambda(i,j) &= A_{\mathbb{Y}}(\bullet) + L_{\mathbb{Y}}(\circ) + 1. \end{aligned} \tag{2.35}$$

An example,

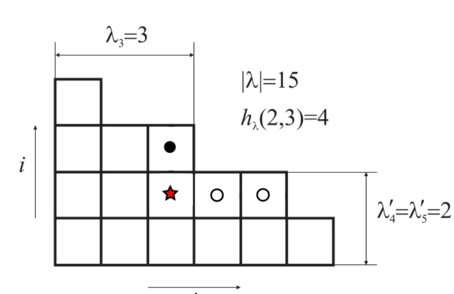

Figure 2.2: Young tableau related to partition $\lambda = \{6, 5, 3, 1\}$.

which represents a partition $\lambda = \{6, 5, 3, 1\}$ with λ_3 the number of boxes in the row, $\lambda'_{4,5}$ the number of boxes in the column, and $h_\lambda(2,3)$ the hook length of the box with the red star. To calculate the hook length, we add the arm-length $A_\lambda(\bullet) = 1$, leg-length

$L_\lambda(\circ) = 2$, and one, $h_\lambda(2,3) = 1 + 2 + 1 = 4$. We also have the size of the diagram given by $|\lambda| = 15$.

2.1.4 Correlation Functions

In terms of the holomorphic part and using the relation (2.21), a general correlation function on the Riemann sphere can be defined by

$$\langle \mathcal{O}_{\Delta_1}(z_1)...\mathcal{O}_{\Delta_p}(z_p)\rangle = \prod_{i=1}^{p} \left(\frac{\partial f}{\partial z}\right)^{\Delta_i}_{z=z_i} \langle \mathcal{O}_{\Delta_1}(f(z_1))...\mathcal{O}_{\Delta_p}(f(z_p))\rangle \, . \tag{2.36}$$

Using this definition, we can use the conformal bootstrap to find correlation functions between primary operators in conformal field theory. In particular, the calculation of the two- and three-point correlation function are obtained straightforwardly. The four-point function can be constructed by using anharmonic ratios defined in two dimensions as

$$w(z_1, z_2, z_3, z_4) = \frac{|z_1 - z_2||z_3 - z_4|}{|z_1 - z_3||z_2 - z_4|}, \quad v(z_1, z_2, z_3, z_4) = \frac{|z_1 - z_2||z_3 - z_4|}{|z_2 - z_3||z_1 - z_4|}. \tag{2.37}$$

These ratios are invariant under global transformations, and we can perform such transformations to set $z_4 = \infty$, $z_3 = 1$, $z_2 = z$, and $z_1 = 0$, where the dependence in z does not fix the explicit form of the four-point correlation function and leads to the definition of the conformal block. The explicit form of the conformal block will be explained in the following section. The general calculation of the p-point functions with $p > 4$ is complicated and may have an arbitrary dependence on these ratios, i.e., not fixed by global conformal symmetry.

Two-point function

A two-point function is defined as

$$\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\rangle = g(z_1, z_2), \tag{2.38}$$

the goal here, it is to use the bootstrap procedure to find the general form of $g(z_1, z_2)$. The invariance under translations, $f(z) = z + a$, and rotation, $f(z) = cz$ with $c \in \mathbb{C}$ and $|c| = 1$, requires from (2.21) that $g(z_1, z_2)$ must to be of the form $g(z_1, z_2) = g(|z_1 - z_2|)$. The invariance under rescaling $f(z) = \lambda z$, implies that

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\rangle &\to \langle \lambda^{\Delta_1+\Delta_2}\mathcal{O}_{\Delta_1}(\lambda z_1)\mathcal{O}_{\Delta_2}(\lambda z_2)\rangle \\ &= \lambda^{\Delta_1+\Delta_2} g(\lambda|z_1 - z_2|) = g(|z_1 - z_2|), \end{aligned} \tag{2.39}$$

from which, we conclude

$$g(|z_1 - z_2|) = \frac{C_{12}}{|z_1 - z_2|^{\Delta_1+\Delta_2}}, \tag{2.40}$$

where C_{12} is a normalization constant. The last symmetry is associated to special conformal transformation. Let us consider the inversion $f(z) = \frac{-1}{z}$,

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\rangle &\to \langle \frac{1}{z_1^{2\Delta_1}}\mathcal{O}_{\Delta_1}\left(\frac{-1}{z_1}\right)\frac{1}{z_2^{2\Delta_2}}\mathcal{O}_{\Delta_2}\left(\frac{-1}{z_2}\right)\rangle \\ &= \frac{1}{z_1^{2\Delta_1}}\frac{1}{z_2^{2\Delta_2}}\frac{C_{12}}{|\frac{-1}{z_2}+\frac{1}{z_1}|^{\Delta_1+\Delta_2}},\end{aligned} \tag{2.41}$$

which is satisfied if $\Delta_1 = \Delta_2 = \Delta$. The general form is written as

$$\langle \mathcal{O}_{\Delta_i}(z_i)\mathcal{O}_{\Delta_j}(z_j)\rangle = \frac{\delta_{ij}}{|z_1-z_2|^{\Delta_i+\Delta_j}} = \begin{cases}\frac{1}{|z_1-z_2|^{2\Delta}} & i=j,\\ 0 & i\neq j,\end{cases} \tag{2.42}$$

where we are considering C_{12} to be δ_{ij}.

Three-point function

For three-point function, we have

$$\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_3}(z_3)\rangle = h(z_1,z_2,z_3). \tag{2.43}$$

Using the same argument from two-point function, the translation and rotation symmetries lead to

$$\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_3}(z_3)\rangle = h(|z_{12}|,|z_{23}|,|z_{13}|), \tag{2.44}$$

with $|z_{ij}| = |z_i - z_j|$. From rescaling symmetry

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_3}(z_3)\rangle &= \langle \lambda^{\Delta_1}\mathcal{O}_{\Delta_1}(\lambda z_1)\lambda^{\Delta_2}\mathcal{O}_{\Delta_2}(\lambda z_2)\lambda^{\Delta_3}\mathcal{O}_{\Delta_3}(\lambda z_3)\rangle \\ &= \lambda^{\Delta_1+\Delta_2+\Delta_3}h(\lambda|z_{12}|,\lambda|z_{23}|,\lambda|z_{13}|).\end{aligned} \tag{2.45}$$

Such results allow us to write one possible solution as

$$h(|z_{12}|,|z_{23}|,|z_{13}|) = \frac{C_{123}}{|z_{12}|^a|z_{23}|^b|z_{13}|^c}, \tag{2.46}$$

with $a+b+c = \Delta_1+\Delta_2+\Delta_3$ and C_{123} is again some normalization constant. From the SCT, we obtain more one constraint

$$\frac{1}{z_1^{2\Delta_1}z_2^{2\Delta_2}z_3^{2\Delta_3}}\frac{(z_1z_2)^a(z_2z_3)^b(z_1z_3)^c}{|z_{12}|^a|z_{23}|^b|z_{13}|^c} = \frac{1}{|z_{12}|^a|z_{23}|^b|z_{13}|^c}. \tag{2.47}$$

Using the first constraint in the equation above lead to

$$a = \Delta_1+\Delta_2-\Delta_3,\ b = \Delta_2+\Delta_3-\Delta_1,\ c = \Delta_1+\Delta_3-\Delta_2, \tag{2.48}$$

by replacing these results in (2.47), we have

$$\langle \mathcal{O}_{\Delta_1}(z_1)\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_3}(z_3)\rangle = \frac{C_{123}}{|z_{12}|^{\Delta_{123}}|z_{23}|^{\Delta_{231}}|z_{13}|^{\Delta_{132}}}, \tag{2.49}$$

with $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

2.1.5 Operator Product Expansion - OPE

It is well-known from quantum theory the importance of the short distance expansion of a product of two fields [20]. In the CFT approach, such expansion can be obtained from the generic form of two- and three-point functions, which allows us to extract the explicit form of the OPE between two primary operators in terms of descendants. The main idea is to suppose the existence of a complete set of local descendants operators $\{\mathcal{O}_{\Delta}^{(-\lambda)}(z_1)\}$ in the theory, then the completeness of this set is equivalent to the OPE,

$$\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_1}(z_1) = \sum_{\Delta_\lambda} \frac{C_{\Delta_2\Delta_1}^{\Delta_\lambda}}{z_{12}^{\Delta_2+\Delta_1-\Delta_\lambda}} \mathcal{O}_{\Delta}^{(-\lambda)}(z_1). \tag{2.50}$$

where $z_{12} := z_2 - z_1$ and $\Delta_\lambda = \{\Delta, \lambda\}$ is a multi-index with Δ labels the primary operators while λ - the Young diagram $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0\}$ labels their descendants, with the conformal dimension $\Delta_\lambda = \Delta + |\lambda|$ and the number of boxes $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_N$. Observe that from the OPE it is possible to recover the two-point function from the equation above by considering $\mathcal{O}_{\Delta}^{(-\lambda)}(z_1 = 0)$ to be the identity with $\Delta_\lambda = 0$

Four-Point function

Using the OPE expression, we can compute the general form of the four-point function. Therefore, from (2.50), the correlation function between four primary operators can be written as

$$\langle \mathcal{O}_{\Delta_4}(z_4)\mathcal{O}_{\Delta_3}(z_3)\mathcal{O}_{\Delta_2}(z_2)\mathcal{O}_{\Delta_1}(z_1)\rangle = \sum_{\Delta_\lambda} \frac{C_{\Delta_2\Delta_1}^{\Delta_\lambda}}{z_{12}^{\Delta_2+\Delta_1-\Delta_\lambda}} \langle \mathcal{O}_{\Delta_4}(z_4)\mathcal{O}_{\Delta_3}(z_3)L_{-\lambda}\mathcal{O}_{\Delta}(z_1)\rangle . \tag{2.51}$$

The order of each operator is important because it can lead to different types of four-point functions. In the literature, there are three types of correlations which are associated with the channels s, u and t, as we have in quantum theory [7, 20]. In this dissertation, we are interested in the four-point function related to the s-channel. Therefore, using global symmetry we fix $z_4 = \infty$, $z_3 = 1$, $z_2 = z$, and $z_1 = 0$.

$$\langle \mathcal{O}_{\Delta_\infty}(\infty)\mathcal{O}_{\Delta_3}(1)\mathcal{O}_{\Delta_2}(z)\mathcal{O}_{\Delta_1}(0)\rangle = \sum_{\Delta_\lambda} \frac{C_{\Delta_2\Delta_1}^{\Delta_\lambda}}{z^{\Delta_2+\Delta_1-\Delta_\lambda}} \langle \mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)L_{-\lambda}\mathcal{O}_{\Delta}(0)\rangle . \tag{2.52}$$

It is well-known from the last section, equation (2.49), that the structure constant C_{123} carries information about the correlation function between three primary operators. The constant $C_{\Delta_2\Delta_1}^{\Delta_\lambda}$ shares the same idea, representing the correlation function of two primaries and one descendant. In order to find the explicit form of (2.52), we have to compute $C_{\Delta_2\Delta_1}^{\Delta_\lambda}$, therefore, let us consider the general correlation between three descendants, where to treat with $C^{\Delta_\lambda}_{\Delta_2\Delta_1}$ we fix the three operators at 0, 1 and ∞. Thus, we have

$$\begin{aligned}\langle L_{-\mu}\mathcal{O}_{\Delta_i}(\infty)L_{-\nu}\mathcal{O}_{\Delta_j}(1)L_{-\eta}\mathcal{O}_{\Delta_k}(0)\rangle &= \langle L_{-\mu}\mathcal{O}_{\Delta_i}(\infty)(L_{-\nu}\mathcal{O}_{\Delta_j}(1)L_{-\eta}\mathcal{O}_{\Delta_k}(0))\rangle \\ &= \sum_{\Delta_\lambda} C^{\Delta_\lambda}_{\Delta_\nu\Delta_\eta}\,\langle L_{-\mu}\mathcal{O}_{\Delta_i}(\infty)L_{-\lambda}\mathcal{O}_{\Delta}(0)\rangle \\ &= \sum_{\Delta_\lambda} C^{\Delta_\lambda}_{\Delta_\nu\Delta_\eta} Q_{\mu\lambda}(\Delta_i,\Delta)\end{aligned} \tag{2.53}$$

where we used the OPE between $L_{-\nu}\mathcal{O}_{\Delta_j}(1)L_{-\eta}\mathcal{O}_{\Delta_k}(0)$ to write the second line. Also, we define the conformal dimensions $\Delta_\eta = \Delta + |\eta|$, $\Delta_\lambda = \Delta_k + |\lambda|$, and $\Delta_\nu = \Delta_j + |\nu|$, where again $|\eta|$, $|\lambda|$ and $|\nu|$ represent the total number of boxes in the Young diagrams. To find the structure constant in (2.52), it is possible to invert the equation (2.53) to write as [21]

$$C^{\Delta_\lambda}_{\Delta_\nu\Delta_\eta} = \sum_{\Delta_\mu}[Q_{\mu\lambda}(\Delta_i,\Delta)]^{-1}\,\langle L_{-\mu}\mathcal{O}_{\Delta_i}(\infty)L_{-\nu}\mathcal{O}_{\Delta_j}(1)L_{-\eta}\mathcal{O}_{\Delta_k}(0)\rangle\,. \tag{2.54}$$

To ensure the normalization between $\mathcal{O}_{\Delta_i}(\infty)$ and $\mathcal{O}_{\Delta}(0)$ and to eliminate any divergence in $C^{\Delta_\lambda}_{\Delta_\nu\Delta_\eta}$, we set $\Delta_i = \Delta$, as result $Q_{\mu\lambda}(\Delta)$, the Kac-Shapovalov matrix, has a block-diagonal structure $Q_{\mu\lambda}(\Delta) \sim \delta_{|\mu|,|\lambda|}$. Thus,

$$Q_{\mu\lambda}(\Delta) = \langle L_{-\mu}\mathcal{O}_{\Delta}(\infty)L_{-\lambda}\mathcal{O}_{\Delta}(0)\rangle\,, \tag{2.55}$$

where each element of $Q_{\mu\lambda}(\Delta)$ can be computed algebraically as the matrix element of descendants states,

$$Q_{\mu\lambda}(\Delta) = \langle\Delta|\,L_{\mu_1}...L_{\mu_M}L_{-\lambda_1}...L_{-\lambda_N}\,|\Delta\rangle\,.$$

Since we are interested in $C^{\Delta_\lambda}_{\Delta_1\Delta_2}$ from (2.54) we obtain

$$C^{\Delta_\lambda}_{\Delta_2\Delta_1} = \sum_{\Delta_\mu}[Q_{\mu\lambda}(\Delta)]^{-1}\,\langle L_{-\mu}\mathcal{O}_{\Delta}(\infty)\mathcal{O}_{\Delta_2}(1)\mathcal{O}_{\Delta_1}(0)\rangle\,. \tag{2.56}$$

Replacing at (2.52) we have,

$$\langle\mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)\mathcal{O}_{\Delta_2}(z)\mathcal{O}_{\Delta_1}(0)\rangle = z^{-\Delta_2-\Delta_1}\sum_{\Delta_\lambda,\Delta_\mu} z^{\Delta_\lambda}\Gamma^{\Delta_\lambda}_{\Delta_4\Delta_3}[Q_{\mu\lambda}(\Delta)]^{-1}\Gamma^{\Delta_\mu}_{\Delta_2\Delta_1}, \tag{2.57}$$

where we define

$$\begin{aligned}\Gamma^{\Delta_\lambda}_{\Delta_4\Delta_3} &= \langle\mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)L_{-\lambda}\mathcal{O}_{\Delta}(0)\rangle\,, \\ \Gamma^{\Delta_\mu}_{\Delta_2\Delta_1} &= \langle L_{-\mu}\mathcal{O}_{\Delta}(\infty)\mathcal{O}_{\Delta_2}(1)\mathcal{O}_{\Delta_1}(0)\rangle\,.\end{aligned} \tag{2.58}$$

Using the Virasoro commutation relations we can compute the explicit form of the correlations $\Gamma^{\Delta_\lambda}_{\Delta_4\Delta_3}$ and $\Gamma^{\Delta_\mu}_{\Delta_2\Delta_1}$, after some algebra - for more details, see [21]. We arrive at

$$\begin{aligned}\Gamma^{\Delta_\lambda}_{\Delta_4\Delta_3} &= \langle\mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)L_{-\lambda}\mathcal{O}_{\Delta}(0)\rangle = \langle\mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)\mathcal{O}_{\Delta}(0)\rangle\,\gamma^{\Delta_\lambda}_{\Delta_4\Delta_3}, \\ \Gamma^{\Delta_\mu}_{\Delta_2\Delta_1} &= \langle L_{-\mu}\mathcal{O}_{\Delta}(\infty)\mathcal{O}_{\Delta_2}(1)\mathcal{O}_{\Delta_1}(0)\rangle = \langle\mathcal{O}_{\Delta}(\infty)\mathcal{O}_{\Delta_2}(1)\mathcal{O}_{\Delta_1}(0)\rangle\,\gamma^{\Delta_\mu}_{\Delta_2\Delta_1},\end{aligned} \tag{2.59}$$

with

$$\gamma^{\Delta_\lambda}_{\Delta_4\Delta_3} = \prod_{l=1}^{N}(\Delta - \Delta_4 + \lambda_l\Delta_3 + \sum_{n=1}^{l-1}\lambda_n), \qquad \gamma^{\Delta_\mu}_{\Delta_2\Delta_1} = \prod_{p=1}^{M}(\Delta - \Delta_1 + \mu_p\Delta_2 + \sum_{m=1}^{p-1}\mu_m), \quad (2.60)$$

where again $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0\}$ and $\mu = \{\mu_1 \geq \mu_2 \geq ... \geq \mu_M \geq 0\}$. The expressions in (2.59) express the correlation functions with only one descendant with the correlation function without descendant, where we will denote the correlations without descendant as $C^{\Delta}_{\Delta_4\Delta_3}$ and $C^{\Delta}_{\Delta_2\Delta_1}$.

Now we have enough information to express the four-point function in 2d CFT, using the above results the four-point function is written as

$$\langle \mathcal{O}_{\Delta_4}(\infty)\mathcal{O}_{\Delta_3}(1)\mathcal{O}_{\Delta_2}(z)\mathcal{O}_{\Delta_1}(0)\rangle = \sum_{\Delta} C^{\Delta}_{\Delta_4\Delta_3}C^{\Delta}_{\Delta_2\Delta_1} z^{\Delta-\Delta_1-\Delta_2}\mathcal{F}_c(\Delta_4,\Delta_3,\Delta_2,\Delta_1,\Delta;z), \quad (2.61)$$

where the function $\mathcal{F}_c(\Delta_4,\Delta_3,\Delta_2,\Delta_1,\Delta;z)$ is called conformal block, a power series in z with coefficients depending on four conformal dimensions related to operators and one conformal dimension Δ associated to intermediate operator. The explicit form of the conformal block is the following

$$\mathcal{F}_c(\Delta_4,\Delta_3,\Delta_2,\Delta_1,\Delta;z) = \sum_{\lambda,\mu\in\mathbb{Y}} \gamma^{\Delta_\lambda}_{\Delta_4,\Delta_3}[Q_{\mu\lambda}(\Delta)]^{-1}\gamma^{\Delta_\mu}_{\Delta_2,\Delta_1} z^{|\lambda|}, \quad (2.62)$$

with the Kac-Shapovalov matrix given by a block-diagonal structure. From (2.59) and (2.60), the first three terms of the conformal block are

$$\begin{aligned}\mathcal{F}_c(\Delta_4,\Delta_3,\Delta_2,\Delta_1,\Delta;z) &= 1 + \frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_4+\Delta_3)}{2\Delta}z + \\ &+\Bigg[\frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)(\Delta-\Delta_4+\Delta_3)(\Delta-\Delta_4+\Delta_3+1)}{2\Delta(1+2\Delta)} + \\ &+\frac{(1+2\Delta)\big(\Delta_1+\Delta_2+\frac{\Delta(\Delta-1)-3(\Delta_1-\Delta_2)^2}{1+2\Delta}\big)\big(\Delta_4+\Delta_3+\frac{\Delta(\Delta-1)-3(\Delta_4-\Delta_3)^2}{1+2\Delta}\big)}{(1-4\Delta)^2+(c-1)(1+2\Delta)}\Bigg]\frac{z^2}{2} + \ldots \, .\end{aligned} \quad (2.63)$$

The exact calculation can be seen in [21]. The real computation of (2.62) becomes complicated at higher levels. In the direction of solving this problem, a big step was made in the paper of Luis F. Alday, Davide Gaiotto and Yuji Tachikawa [8], where they conjectured an expression for correlation functions on the Riemann surface of genus g and n punctures as Nekrasov partition function of a certain $\mathcal{N} = 2$ Super Conformal Gauge Theory in four dimensions.

In this dissertation, we take $g = 0$, where all calculations are related to the Rieman sphere. Also, we are interested in the four-point function that is associated with four punctures, $n = 4$. In next section, we are going to treat how the conformal block is written in terms of the Luis F. Alday, Davide Gaiotto and Yuji Tachikawa correspondence or AGT correspondence, and also about Nekrasov partition function.

2.2 Conformal Block via Instanton Partition Functions

In the AGT correspondence paper [8] was revealed a deep connection between 2d CFT and $\mathcal{N}=2$ SCFT gauge theory. This correspondence turned out to be very important for the 2d CFT. In particular, in their paper Alday, Gaiotto and Tachikawa gave an explicit combinatorial formula for the expansion of the conformal blocks in terms of Nekrasov partition function [9]. Based on this paper, Alba, Fateev, Litvinov, and Tarnopolskiy(AFLT) [22] studied the origin of such expansion for the conformal block, from a CFT point of view. They considered the algebra $\mathcal{A} = \mathrm{Vir} \otimes \mathcal{H}$ which is the tensor product of mutually commuting Virasoro and Heisenberg algebras and discovered the special orthogonal basis of states in the highest weight representations of $\mathcal{A}$. Such discovery allowed to compute the conformal block expansion in agreement with the expansion proposed in the AGT correspondence. Therefore, in order to write the explicit form of the conformal block and given that is according with the AGT correspondence, we are going to consider the definitions from AFLT paper, as well as the definitions from A. Belavin and V. Belavin paper, where the explicit form for the Nekrasov partition functions was written - see Appendix A in [23].

We know that the algebra $\mathcal{A}$ is defined by the tensor product between the Virasoro and Heisenberg algebras. In this new algebra, we have the following commutation relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \\ [a_n, a_m] &= \frac{n}{2}\delta_{m+n,0}, \quad [L_n, a_m] = 0. \end{aligned} \tag{2.64}$$

where in $\mathcal{A}$ the primary operator carries information from the both algebras and we can denote, for example, by $|\Delta, \alpha\rangle = |\Delta\rangle \otimes |\alpha\rangle$ [22].

In the $\mathcal{A}$ algebra the Virasoro conformal block is denoted as $\mathcal{B}_{Vir}(t)$, where we choose $z = t$. Using the operator product expansion, $\mathcal{B}_{Vir}(t)$ assumes the form of a power series

$$\mathcal{B}_{Vir}(t) = \sum_{N=0}^{\infty} \langle N; \Delta_4, \Delta_3 | \Delta_2, \Delta_1; N\rangle \, t^N, \tag{2.65}$$

where $\Delta_{1,2,3,4}$ are the conformal dimensions of each operator inserted on the Riemann sphere. We also have the vector $|N; \Delta_4, \Delta_3\rangle$ defined as a linear combination of Nth level descendants $L_{-\lambda}|\Delta\rangle$, with $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0\}$ [23]. The Heisenberg conformal block is also a function of t and it is defined by

$$\mathcal{B}_{\mathcal{H}}(t) = (1-t)^{2(\frac{\epsilon}{2}+\hat{\theta}_2)(\frac{\epsilon}{2}+\hat{\theta}_3)}. \tag{2.66}$$

In this case, the conformal block is built using the vectors $|N, \alpha\rangle$ criated by the action of a_n, which satisfies the commutation relation $[a_n, a_m] = \frac{n}{2}\delta_{m+n,0}$. On the level N the state $|N, \alpha\rangle$ obeys the following recursive relation, $a_n |N, \alpha\rangle = \alpha\, |N-n, \alpha\rangle$ [23].

The computation of the equations (2.65) and (2.66) is complicated, and we will not explain here, it is not the goal. The derivation of the equation (2.66) in terms of combinatorial expansion can be seen in the paper by Marshakov, Mironov, and Morozov [21]. For the equation (2.65) we consider the results from the Appendix [23] and use the definition from AFLT [22]. Therefore, we introduce the conformal block in the $\mathcal{A}$ algebra as

$$\mathcal{B}(t) = \mathcal{B}_{\mathcal{H}}(t)\mathcal{B}_{Vir}(t), \tag{2.67}$$

where, in terms of AGT correspondence, the Heisenberg block corresponds to U(1) factor [8], in the gauge group in $\mathcal{N} = 2$ SCFT, and the Virasoro block is built in terms of Young diagrams defined in the last section.

Now we come to a surprising observation, Alday, Gaiotto, and Tachikawa checked explicitly in [8] that the conformal block $\mathcal{B}(t)$ is exactly the conformal block of the Virasoro algebra (2.62) with central c for four operators of dimensions $\Delta_{1,2,3,4}$ inserted at 0, t, 1, ∞ respectively, and with an intermadiate state in the s-channel whose dimension is Δ. Where the conformal dimensions $\Delta_{1,2,3,4}$ and Δ are given by

$$\Delta_i = \frac{\hat{\theta}_i(2\epsilon - \hat{\theta}_i)}{4\epsilon_1\epsilon_2}, \quad i = 1, 2, 3, 4, \qquad \Delta = \frac{\alpha(\epsilon - \alpha)}{\epsilon_1\epsilon_2}, \quad \alpha = \frac{\epsilon}{2} + \frac{\hat{\sigma}}{2}, \tag{2.68}$$

and the central charge in the Virasoro algebra:

$$c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}, \quad \epsilon = \epsilon_1 + \epsilon_2. \tag{2.69}$$

where $\epsilon_{1,2}$ are deformation parameters in the gauge theory [1].

To give an idea about insertions on the sphere, and explain how these set of conformal dimensions are organized in the s-channel, let us consider the following pictures. The Figure (2.4) is frequently used in the literature to represent the four-point function related to s-channel and help us to understand the order of each operator, as well as the localization of the intermediate state. Furthermore, as a theory on the Riemann sphere, we can use the Figure 2.3 as a representation of the insertion of four punctures or $\mathcal{O}_\Delta$ operators on the sphere [8].

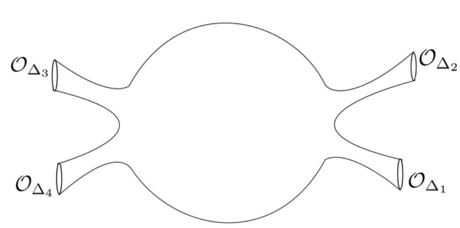

Figure 2.3: Riemann sphere with 4 punctures

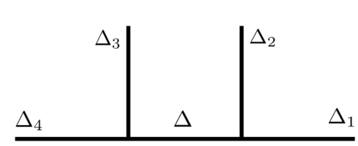

Figure 2.4: Four-point function for s-channel

[1] The deformation parameters appear as deformations in the gauge theory and are directly connected with the central charge defined in the SCFT on the Riemann sphere, see Subsection 3.2 and Section 6 [8]

From [23], the term $\langle N;\Delta_4,\Delta_3|\Delta_2,\Delta_1;N\rangle$ in (2.65) depends on the Nekrasov partition functions and it is given by

$$\langle N;\Delta_4,\Delta_3|\Delta_2,\Delta_1;N\rangle = \sum_{\substack{\lambda,\mu\in\mathbb{Y}\\ |\lambda|+|\mu|=N}} \frac{\mathcal{Z}(\vec{a},\mu_1)\mathcal{Z}(\vec{a},\mu_2)\bar{\mathcal{Z}}(\vec{a},\mu_3)\bar{\mathcal{Z}}(\vec{a},\mu_4)}{\mathcal{Z}(\hat{\sigma})\bar{\mathcal{Z}}(\hat{\sigma})}, \tag{2.70}$$

with

$$\begin{gathered}\mu_1 = \frac{\epsilon}{2}+\frac{\hat{\theta}_3}{2}-\frac{\hat{\theta}_4}{2},\ \mu_2 = \frac{\epsilon}{2}-\frac{\hat{\theta}_1}{2}+\frac{\hat{\theta}_2}{2},\\ \mu_3 = -\frac{\epsilon}{2}+\frac{\hat{\theta}_1}{2}+\frac{\hat{\theta}_2}{2},\ \mu_4 = -\frac{\epsilon}{2}+\frac{\hat{\theta}_3}{2}+\frac{\hat{\theta}_4}{2},\end{gathered} \tag{2.71}$$

$$\mathcal{Z}(\vec{a},\mu_i) = \prod_{\lambda\in\mathbb{Y}}(\phi(a_1)-\mu_i+\epsilon)\prod_{\mu\in\mathbb{Y}}(\phi(a_2)-\mu_i+\epsilon)),\quad i=1,2,$$

$$\bar{\mathcal{Z}}(\vec{a},\mu_j) = \prod_{\lambda\in\mathbb{Y}}(\phi(a_1)-\mu_j))\prod_{\mu\in\mathbb{Y}}(\phi(a_2)-\mu_j)),\quad j=3,4,$$

$$\begin{aligned}\mathcal{Z}(\hat{\sigma}) = &\prod_{\lambda\in\mathbb{Y}}[-\epsilon_1 A_\lambda(\bullet)+\epsilon_2(L_\lambda(\circ)+1)][\hat{\sigma}-\epsilon_1 A_\mu(\bullet)+\epsilon_2(L_\mu(\circ)+1)]\times\\ &\prod_{\mu\in\mathbb{Y}}[-\epsilon_1 A_\mu(\bullet)+\epsilon_2(L_\mu(\circ)+1)][-\hat{\sigma}-\epsilon_1 A_\lambda(\bullet)+\epsilon_2(L_\lambda(\circ)+1)],\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{Z}}(\hat{\sigma}) = &\prod_{\lambda\in\mathbb{Y}}[\epsilon+\epsilon_1 A_\lambda(\bullet)-\epsilon_2(L_\lambda(\circ)+1)][\epsilon-\hat{\sigma}+\epsilon_1 A_\mu(\bullet)-\epsilon_2(L_\lambda(\circ)+1)]\times\\ &\prod_{\mu\in\mathbb{Y}}[\epsilon+\epsilon_1 A_\mu(\bullet)-\epsilon_2(L_\mu(\circ)+1)][\epsilon+\hat{\sigma}+\epsilon_1 A_\lambda(\bullet)-\epsilon_2(L_\mu(\circ)+1)],\end{aligned} \tag{2.72}$$

$$\phi(a_i) = a_i+\epsilon_1(i-1)+\epsilon_2(j-1),$$

where from the gauge theory $\vec{a}=(a_1,a_2)=(\hat{\sigma},-\hat{\sigma})$ is the adjoint vacuum expectation value (VEV) of $U(2)$, $\mu_{1,2}$ are eingenvalues of mass of two hypermultiplet in the fundamental $\mathcal{Z}(\vec{a},\mu_i)$, and $\mu_{3,4}$ are those of the anti-fundamental $\bar{\mathcal{Z}}(\vec{a},\mu_j)$. The last two partitions $\mathcal{Z}(\sigma)$, and $\bar{\mathcal{Z}}(\sigma)$ are related to the adjoint hypermultiplet, furthermore, the terms $A_\mu(\bullet)$ and $L_\mu(\circ)$ was defined in (2.35). Here we are interested in the explicit expression of the conformal block, therefore, we eliminate any discussion of group theory and supersymmetry in this dissertation, more details of the definitions above can be seen in Appendix B of [8]. Therefore, replacing (2.70) in (2.65) and considering the definitions (2.71) and (2.72), we have[2]

$$\mathcal{B}_{Vir}(t) = \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\hat{\theta}_4,\hat{\theta}_3,\hat{\theta}_2,\hat{\theta}_1,\hat{\sigma})t^{|\lambda|+|\mu|}, \tag{2.73}$$

[2]The "hat" in the parameters does not mean operators. It is just a strange choice of parameters.

where we defined

$$\mathcal{B}_{\lambda,\mu}(\hat{\theta}_4,\hat{\theta}_3,\hat{\theta}_2,\hat{\theta}_1,\hat{\sigma}) = \frac{\mathcal{Z}(\hat{\sigma},\mu_1)\mathcal{Z}(\hat{\sigma},\mu_2)\bar{\mathcal{Z}}(\hat{\sigma},\mu_3)\bar{\mathcal{Z}}(\hat{\sigma},\mu_4)}{Z(\hat{\sigma})\bar{Z}(\hat{\sigma})}, \tag{2.74}$$

$$\mathcal{Z}(\hat{\sigma},\mu_1) = \prod_{\lambda\in\mathbb{Y}}(\frac{\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)+\frac{\hat{\theta}_4}{2}-\frac{\hat{\theta}_3}{2})\prod_{\mu\in\mathbb{Y}}(\frac{-\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)+\frac{\hat{\theta}_4}{2}-\frac{\hat{\theta}_3}{2}),$$

$$\mathcal{Z}(\sigma,\mu_2) = \prod_{\lambda\in\mathbb{Y}}(\frac{\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_2}{2}+\frac{\hat{\theta}_1}{2})\prod_{\mu\in\mathbb{Y}}(\frac{-\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_2}{2}+\frac{\hat{\theta}_1}{2}),$$

$$\bar{\mathcal{Z}}(\hat{\sigma},\mu_3) = \prod_{\lambda\in\mathbb{Y}}(\frac{\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_1}{2}-\frac{\hat{\theta}_2}{2})\prod_{\mu\in\mathbb{Y}}(\frac{-\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_1}{2}-\frac{\hat{\theta}_2}{2}),$$

$$\bar{\mathcal{Z}}(\hat{\sigma},\mu_4) = \prod_{\lambda\in\mathbb{Y}}(\frac{\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_3}{2}-\frac{\hat{\theta}_4}{2})\prod_{\mu\in\mathbb{Y}}(\frac{-\hat{\sigma}}{2}+\epsilon_{ij}(\epsilon_1,\epsilon_2)-\frac{\hat{\theta}_3}{2}-\frac{\hat{\theta}_4}{2}),$$

with $\mathcal{Z}(\hat{\sigma})$ and $\bar{\mathcal{Z}}(\hat{\sigma})$ defined in (2.72), and $\epsilon_{ij}(\epsilon_1,\epsilon_2) = \epsilon_1(i-\frac{1}{2})+\epsilon_2(j-\frac{1}{2})$. Now we know that, via AGT correspondence, the equations (2.62) and (2.67) are the same with the central charge gives by (2.69), thus, we finally find the four-point correlation function where it is necessary to replace the equation (2.73) in (2.67). However, to connect this chapter with Chapter 2 and use the τ function, we have to set the value of the central charge in the CFT.

2.2.1 *Conformal Field Theory for $c = 1$*

It was shown in the papers [10] and [11] that correlation functions in 2D CFT can be interpreted in terms of generic τ_{VI} and τ_V-functions, when the central charge takes the value $c = 1$. Such interpretation was initially observed in the series of papers by Sato, Miwa and Jimbo [24]-[25] , where it was shown that isomonodromic deformations in linear systems admitted an explicit form in terms of correlation functions of local operators or *monodromy fields* [29]. As an isomonodromic invariant, the τ-function became a useful tool to study correlation functions in conformal field theory with any applications in quantum integral systems and black holes.

In order to explore this idea, let us set $c = 1$ in the four-point correlation function. The discussions about τ_V and τ_{VI}-function, as well as the associated Painlevé VI and Painlevé V, will be left for Chapter 2. Essentially, the idea here is just set the value of the central charge to build the conformal block and also the confluent CB of the first kind that we are going to explain in the next section. Thus, in this case, the conformal dimensions in (2.68) are replaced by

$$\Delta = \frac{\hat{\sigma}^2}{4}, \qquad \Delta_i = \frac{\hat{\theta}_i^2}{4}, \qquad i = 0, t, 1, \infty, \tag{2.75}$$

where, in order to fix the notation with the next chapters, we replace $\hat{\theta}_1 = \hat{\theta}_0$, $\hat{\theta}_2 = \hat{\theta}_t$, $\hat{\theta}_3 = \hat{\theta}_1$, and $\hat{\theta}_4 = \hat{\theta}_\infty$. When $c = 1$, we conclude that $\epsilon_1 = -\epsilon_2$. We can choose $\epsilon_1 = -\epsilon_2 = 1$ which is analogue to take $\hbar = 1$. Thus, the conformal block in the $\mathcal{A}$ algebra will be defined by

$$\mathcal{B}(t) := (1-t)^{2\hat{\theta}_t\hat{\theta}_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) t^{|\lambda|+|\mu|}, \tag{2.76}$$

with

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}(\hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) = &\prod_{\lambda\in\mathbb{Y}} \frac{[(\hat{\sigma}+2(i-j)+\hat{\theta}_t)^2-\hat{\theta}_0^2][(\hat{\sigma}+2(i-j)+\hat{\theta}_1)^2-\hat{\theta}_\infty^2]}{16h_\lambda^2(i,j)[\lambda'_j+\mu_i-i-j+1+\hat{\sigma}]^2} \\ &\prod_{\mu\in\mathbb{Y}} \frac{[(-\hat{\sigma}+2(i-j)+\hat{\theta}_t)^2-\hat{\theta}_0^2][(-\hat{\sigma}+2(i-j)+\hat{\theta}_1)^2-\hat{\theta}_\infty^2]}{16h_\mu^2(i,j)[\mu'_j+\lambda_i-i-j+1-\hat{\sigma}]^2},\end{aligned} \tag{2.77}$$

where we identify the hook lenghts $h_\lambda(i,j)$ and $h_\mu(i,j)$ defined in (2.35).

As we can see in (2.61), the sum in the four-point function is under Δ's that are associated to descendant operators in the intermediate channel $\Delta = \hat{\sigma}^2/4$, Figure 2.4. Since we can define a tower of states in the intermediate channel, it is possible to define $\Delta = (\frac{\hat{\sigma}}{2}+n)^2$, where, instead of to sum in Δ, we pass to sum in n with $n \in \mathbb{Z}$. Thus, the four-point function defined in (2.61) will take the following form

$$\begin{aligned}&\langle \mathcal{O}_{\Delta_\infty}(\infty)\mathcal{O}_{\Delta_1}(1)\mathcal{O}_{\Delta_t}(t)\mathcal{O}_{\Delta_0}(0)\rangle \\ &\qquad = \sum_n C(\{\Delta_i\}, \frac{\hat{\sigma}}{2}+n) t^{(\frac{\hat{\sigma}}{2}+n)^2-\Delta_0-\Delta_t} \mathcal{B}(\{\hat{\theta}_i\}, \hat{\sigma}+2n; t).\end{aligned} \tag{2.78}$$

where we define $\{\hat{\theta}_i\} = (\hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t, \hat{\theta}_0)$, $\{\Delta_i\} = (\Delta_\infty, \Delta_1, \Delta_t, \Delta_0)$ and

$$C(\{\Delta_i\}, \frac{\hat{\sigma}}{2}+n) = C_{\Delta_\infty\Delta_1}^{(\hat{\sigma}/2+n)^2} C_{\Delta_0\Delta_t}^{(\hat{\sigma}/2+n)^2}. \tag{2.79}$$

2.3 Confluent Conformal Block

In this section, we are going to use the last result for the four-point function, as well as the explicit expression for the conformal block to study the three-point function as a limit from the four-point function. Furthremore, we derive the exact form for the Confluent Conformal Block of the First Kind.

The AGT correspondence has triggered the study of confluent CB [30, 31], where one class of them, relevant in this dissertation, corresponds to the decoupling of one of $\mu_{1,2,3,4}$ in the gauge theory. This decoupling consists of sending one of the $\mu_{1,2,3,4}$ to infinity, analogous to a coalescence process that appears for example between the Hypergeometric differential equation and the Confluent Hypergeometric equation, where this latter is obtained rescaling the independent variable and sending two singular points to infinity in the first equation, where the two singularities coalesce into one point at infinity. In this case, to send $\mu_{1,2,3,4}$ to infinity we send two θ's to infinity, and also rescale the equation (2.78). By a quick analysis, we can observe that there exist other types of limits (2.71) that can lead to different types of confluent CB [11, 10]. However, in this dissertation we consider the confluent limit on the s-channel Figure (2.4) associated with the Confluent Conformal Block of First Kind, that corresponds to send $\mu_4 \to \infty$, therefore, we will send $\hat{\theta}_1$ and $\hat{\theta}_\infty$ to infinity in (2.71). Where the limit can be represented by the Figure (2.5), where $\mathcal{O}^{[1]}_{\Lambda_1,\Lambda_2}$ will be explained.

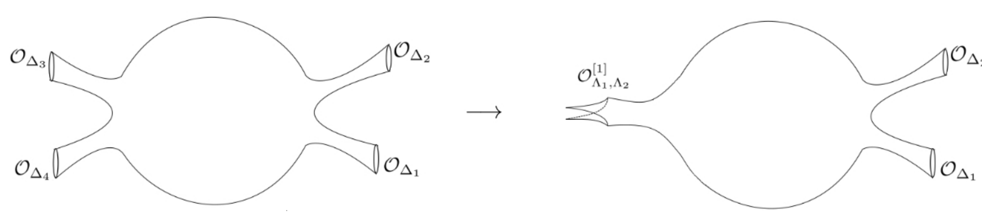

Figure 2.5: Confluent Limit.

In the CFT, the limit is linked with Whittaker states of rank 1 represented by $\mathcal{O}^{[1]}_{\Lambda_1,\Lambda_2}$, these states were initially introduced by Bonelli, Maruyoshi and Tanzini in [32] and the existence of these states in the Virasoro algebra was verified in [33, 34]. Since then, the confluent versions of conformal blocks have been studied in the context of 4D SCFT, via AGT correspondence, as well as the relation with quantum Painlevé equations [10, 11].

In this section, we consider the results and definitions of Whittaker state, and write the confluent conformal block of the first kind, [31, 11]. The Whittaker states do not satisfy the relations (2.29) and (2.30) and in order to consider these states, a general Virasoro algebra $Vir_c^{[r]}$ is built with the same commutations (2.20). However, the action of L_n in states changes as [35]

$$\begin{aligned} &L_n \left|\Lambda\right\rangle = \Lambda_n \left|\Lambda\right\rangle \qquad &(n = r, r+1, ..., 2r), \\ &L_n \left|\Lambda\right\rangle = 0 \qquad &(n > 2r). \end{aligned} \tag{2.80}$$

In the dual case,

$$\begin{aligned} &\left\langle\Lambda\right| L_n = \Lambda_n \left\langle\Lambda\right| \qquad &(n = -r, -r-1, ..., -2r), \\ &\left\langle\Lambda\right| L_n = 0 \qquad &(n < -2r), \end{aligned} \tag{2.81}$$

with $\Lambda = (\Lambda_r, \Lambda_{r+1}...\Lambda_{2r})$ and r is the rank. It is not difficult to verify that $r = 0$, we recover the relations (2.29) and (2.30) with $\left|\Lambda\right\rangle = \left|\Delta\right\rangle$. In the case $r = 1$, we denote $\left|\Lambda\right\rangle = \left|\Lambda_1, \Lambda_2\right\rangle$, with $\left|\Lambda_1, \Lambda_2\right\rangle$ representing the Whittaker state of rank 1 with

$$L_1 \left|\Lambda_1, \Lambda_2\right\rangle = \Lambda_1 \left|\Lambda_1, \Lambda_2\right\rangle, \qquad L_2 \left|\Lambda_1, \Lambda_2\right\rangle = \Lambda_2 \left|\Lambda_1, \Lambda_2\right\rangle,$$

$$L_n \left|\Lambda_1, \Lambda_2\right\rangle = 0 \quad (n > 2).$$

As in Figure (2.4), the diagram associated with the confluent limit in the four-point function for s-channel is given by the figure below:

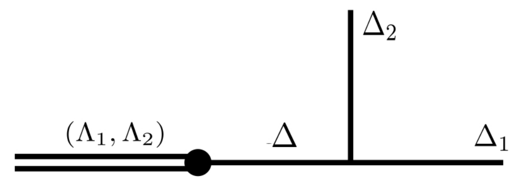

Figure 2.6: Confluent three-point function for s-channel.

Where the double line in the diagram representing the Whittaker state and the dot, meaning the connection between states. It will be convenient to define as $\Lambda_1 = \hat{\theta}_*$ and $\Lambda_2 = \frac{1}{4}$ [11]. Where, essentially, the value of Λ_2 carries no special meaning, and it could be made arbitrary since we can rescale using t and $\hat{\theta}_*$.

Returning to the limit $\mu_4 \to \infty$: $\hat{\theta}_1$, $\hat{\theta}_\infty$ and t are rescaled by

$$\hat{\theta}_1 = \frac{\delta + \hat{\theta}^*}{2}, \quad \hat{\theta}_\infty = \frac{\delta - \hat{\theta}^*}{2}, \quad t \to \frac{t}{\delta}, \quad \delta \to \infty, \tag{2.82}$$

where replacing $\hat{\theta}_1$ and $\hat{\theta}_\infty$ in (2.76), we write the confluent CB of first kind as

$$\mathcal{D}(t) := \lim_{\delta\to\infty} \left(1 - \frac{t}{\delta}\right)^{2\hat{\theta}_t(\frac{\delta+\hat{\theta}^*}{2})} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\frac{\delta - \hat{\theta}^*}{2}, \frac{\delta + \hat{\theta}^*}{2}, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}\right) \frac{t^{|\lambda|+|\mu|}}{\delta^{|\lambda|+|\mu|}} \tag{2.83}$$

or

$$\mathcal{D}(t) := e^{-\hat{\theta}_t t} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{D}_{\lambda,\mu}(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) t^{|\lambda|+|\mu|}, \tag{2.84}$$

with

$$\mathcal{D}_{\lambda,\mu}(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) = \lim_{\delta\to\infty} \mathcal{B}_{\lambda,\mu}\left(\frac{\delta + \hat{\theta}^*}{2}, \frac{\delta - \hat{\theta}^*}{2}, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}\right) \frac{1}{\delta^{|\lambda|+|\mu|}}. \tag{2.85}$$

Using the equation (2.77), we have

$$\begin{aligned} \mathcal{D}_{\lambda,\mu}(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) = \prod_{\lambda\in\mathbb{Y}} \frac{\big(\hat{\sigma} + 2(i-j) + \hat{\theta}_*\big)\big((\hat{\sigma} + 2(i-j) + \hat{\theta}_t)^2 - \hat{\theta}_0^2\big)}{8h_\lambda^2(i,j)\big(\lambda'_j + \mu_i - i - j + 1 + \hat{\sigma}\big)^2} \times \\ \prod_{\mu\in\mathbb{Y}} \frac{\big(-\hat{\sigma} + 2(i-j) + \hat{\theta}_*\big)\big((-\hat{\sigma} + 2(i-j) + \hat{\theta}_t)^2 - \hat{\theta}_0^2\big)}{8h_\mu^2(i,j)\big(\mu'_j + \lambda_i - i - j + 1 - \hat{\sigma}\big)^2}. \end{aligned} \tag{2.86}$$

Observe that, in terms of partitions, it is easy to verify that the number of terms in the numerator reduces from eight to six.

Analogous to (2.78), the confluent three-point function with two operators of rank 0 and one Whittaker operator of rank $r = 1$ is written as

$$\langle \mathcal{O}^{[1]}_{\Lambda_1,\Lambda_2}(\infty)\mathcal{O}_{\Delta_t}(t)\mathcal{O}_{\Delta_0}(0)\rangle = \sum_n C(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \frac{\hat{\sigma}}{2} + n) t^{(\frac{\hat{\sigma}}{2}+n)^2-\hat{\theta}_0^2-\hat{\theta}_t^2} \mathcal{D}(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_*, \hat{\sigma} + 2n; t) \tag{2.87}$$

with, $(\Lambda_1, \Lambda_2) = (\hat{\theta}_*, \frac{1}{4})$ and

$$C(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \frac{\hat{\sigma}}{2} + n) = C^{(\hat{\sigma}/2+n)^2}_{(\hat{\theta}_*,1/4)} C^{(\hat{\sigma}/2+n)^2}_{\frac{\hat{\theta}_0^2}{4}\ \frac{\hat{\theta}_t^2}{4}}. \tag{2.88}$$

In this chapter, we gave an introduction to conformal field theory and finished with the exact expression for the four-point and confluent three-point correlation functions, where to find such expressions we had to use the AGT correspondence to express the conformal block in terms of Nekrasov partition function. The expression for the confluent three-point function (2.87) plays a crucial role in this dissertation given that is linked with τ_V-function when c=1, such connection will be explored in the next chapters. In the next chapter, we also define explicitly the Painlevé VI and V, that we mentioned in Section 1.2.

3 Fifth Painlevé equation

In this chapter, we start defining Painlevé transcendents as solutions of nonlinear Painlevé equations and explaining how the Painlevé VI and Painlevé V are connected by a specific confluent limit. To understand explicitly such equations, it is necessary to define isomonodromic deformations theory in linear systems, thus, we present the general idea about it, as well as Monodromy matrices and Stokes phenomenon. We also define the Schlesinger equations and the generic τ-function as an isomonodromic invariant and using a generic linear system, we study the isomonodromic deformations of the linear system associated with Painlevé V, where this allows us to define the isomonodromic invariant τ_V-function. Lastly, based in the paper by Lisovyy, Nagoya, and Roussillon [11], we connect the explicit expression for τ_V-function with the confluent three-point function that was defined in the first chapter by setting the value of the central charge in the 2d conformal field theory to be one, $c = 1$.

3.1 Painlevé Transcendents

Painlevé transcendents are solutions to certain nonlinear second-order ordinary differential equations in the complex plane. Such solutions are widely recognized as important special functions with a broad range of applications including classical and quantum integrable models, 2d Ising model, random matrix theory and black hole physics [36, 37, 38]. Many aspects of Painlevé equations, such as their analytic, geometric properties and asymptotic problems have been extensively studied in the last four decades [39, 40, 41].

In the literature, the Painlevé transcendents are necessarily solutions of a set of nonlinear second-order equations denominated Painlevé equations. It is well-known that there exist six Painlevé equations which are connected by limits and commonly denoted using roman numbers with PI, PII, PIII, PIV, PV, and PVI with each equation has the following form,

$$\frac{d^2q}{dz^2} = F\left(z, q, \frac{dq}{dz}\right) \tag{3.1}$$

where F is meromorphic in q and rational in $\frac{dq}{dz}$.

The fifth Painlevé equation can be obtained by the confluent limit on Painlevé VI, where the limit is obtained by the confluence of two singular points and scaling transformations on the PVI parameters [10]. Therefore, in the Painlevé V, we have

$$\text{Painlevé VI} \xrightarrow[\text{limit}]{\text{confluent}} \text{Painlevé V},$$

where the limit is taken by replacing $(z, q, \alpha, \beta, \gamma, \delta) \to (1 + \epsilon z, q, \alpha, \beta, \epsilon^{-1}\gamma - \epsilon^{-2}\delta, \epsilon^{-2}\delta)$ in the first equation below and taking the limit $\epsilon \to 0$ [42],

Painlevé VI:

$$\begin{aligned}\frac{d^2q}{dz^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{dq}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{dq}{dz} \\ + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{q^2} + \gamma\frac{z-1}{(q-1)^2} + \delta\frac{z(z-1)}{(q-z)^2}\right],\end{aligned} \tag{3.2}$$

Painlevé V:

$$\frac{dq^2}{dz^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{(q-1)^2}{z^2}\left(\alpha q + \frac{\beta}{q}\right) + \gamma\frac{q}{z} + \delta\frac{q(q+1)}{q-1}. \tag{3.3}$$

where α, β, γ, and δ are complex constants. For generic parameters, the solutions of (3.3) have critical points at $z = 0, \infty$. We also have singular behavior when $q(z_0) = 0, 1, \infty$ and we say the points 0 and 1 are regular and appears as a simple pole in the right-hand side of (3.3), however, $z = \infty$ is an irregular point and does not appear as a simple pole, where such irregularity arises as a consequence of the confluent limit.

As we know, the solutions of Painlevé equations are the Painlevé transcendents. Such solutions can not be expressed in terms of elementary functions, for example, exponential functions, logarithms, and hyperbolic functions or in terms of other special functions: Bessel, gamma, and hypergeometric functions. It means that there is no simple representation of these functions. Although the Painlevé transcendents do not share properties with elementary and special functions, a remarkable result was first presented in the 1980s by Jimbo [43], where, using the theory of isomonodromic deformations and monodromy data for linear system, it was possible to define Painlevé transcendents in terms of connection formulas from the linear system [13, 14, 12]. In the next sections, we briefly review definitions of Fuschian systems, monodromy matrices and theory of isomonodromic deformations for the nontrivial case leading to Painlevé V.

3.2 Linear System

Before defining the linear system, we start this section defining a second order ordinary differential equation, which essentially will help us to understand how to build the systems. We also give an example of the linear equation the hypergeometric equation, furthermore, in order to introduce the idea of the limit of confluence between linear equations, we present the confluent hypergeometric equation. Lastly, we define the general linear system with n regular singularities and one irregular singular point at infinity.

Ordinary Differential Equation

A second-order differential equation in the complex plane is defined by

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y = 0, \tag{3.4}$$

with

$$p(z) = \sum_{i=1}^{n-1} \frac{A_i}{z - z_i}, \qquad q(z) = \sum_{i=1}^{n-1} \left(\frac{B_i}{(z - z_i)^2} + \frac{C_i}{z - z_i} \right), \tag{3.5}$$

where $n - 1$ counts the number of insertions in the complex plane. In this dissertation, it is extremally important to classify the nature of each insertion. Therefore, we will consider the classification by Poincaré rank, r [44, 45], where $r = 0$ represents the regular singularity and $r \geq 1$ the irregular. Here, we will work with the cases $r = 0$ and $r = 1$.

Initially, we define a short test for $r = 0$, then treat with $r = 1$. Therefore, for $r = 0$ the singularity satisfies the following condition:

- The singularity of rank r=0 is regular (or called Fuchsian singularity) if either $p(z)$ or $q(z)$ diverges as $z \to z_i$, but $(z - z_i)^{r+1}p(z)$ and $(z - z_i)^{2r+2}q(z)$ remain finite as $z \to z_i$. The limit finite means that the functions $(z - z_i)^{r+1}p(z)$ and $(z - z_i)^{2r+2}q(z)$ are analytics at $z = z_i$. However, if these conditions are not satisfied the singularity is irregular with rank $r \geq 1$.

To check if the singularity is irregular of rank $r = 1$, we have to supose that $p(z)$ and $q(z)$ in (3.4) have poles of high order [46]. Thus, in this case, we write $\bar{p}(z)$ and $\bar{q}(z)$ as

$$\begin{aligned} \bar{p}(z) &= \sum_{i=1}^{n-1} \left(\frac{D_i}{(z - z_i)^2} + \frac{E_i}{z - z_i} \right), \\ \bar{q}(z) &= \sum_{i=1}^{n-1} \left(\frac{F_i}{(z - z_i)^4} + \frac{G_i}{(z - z_i)^3} + \frac{H_i}{(z - z_i)^2} + \frac{I_i}{z - z_i} \right). \end{aligned} \tag{3.6}$$

The idea is again analog, we check the value of the rank by considering the condition:

- The singularity of rank r=1 is irregular if either $\bar{p}(z)$ or $\bar{q}(z)$ diverges as $z \to z_i$, but $(z - z_i)^{r+1}p(z)$ and $(z - z_i)^{2r+2}\bar{q}(z)$ remain finite as $z \to z_i$. Note that, it is easy to realize that the two limits

$$\lim_{z \to z_i} (z - z_i)^{r+1} \left(\frac{D_i}{(z - z_i)^2} + \frac{E_i}{z - z_i} \right),$$

$$\lim_{z \to z_i} (z - z_i)^{2r+2} \left(\frac{F_i}{(z - z_i)^4} + \frac{G_i}{(z - z_i)^3} + \frac{H_i}{(z - z_i)^2} + \frac{I_i}{z - z_i} \right),$$

are finite if $r = 1$. Where again we have that the productors $(z - z_i)^{r+1}p(z)$ and $(z - z_i)^{2r+2}\bar{q}(z)$ are analytics at $z = z_i$. If these conditions are not satisfied the singularity is irregular with some rank $r > 1$.

For more general case $r > 1$, we can use the same idea, however, it is necessary to realize that the poles in the funções in (3.4) are linked directly with the value of r since the poles of high order in the polynomials lead to the real value of r.

Riemann ODE and Confluent Limit

One trivial example of ODE which will be useful to treat it is the Riemann differential equation given by three regular singularities $n = 3$, where using Möbius transformation in (3.4) we fix the singularities at 0, 1 and ∞,

$$\frac{d^2y}{dz^2} + \left(\frac{A_1}{z} + \frac{A_2}{z-1}\right)\frac{dy}{dz} + \left[\frac{B_1}{z^2} + \frac{B_2}{(z-1)^2} + \frac{C_1}{z} + \frac{C_2}{z-1}\right]y(z) = 0. \tag{3.7}$$

The constants above can be found by using pairs of characteristic exponents relative to the behavior of each pair of solutions around the singular points 0, 1, and ∞, respectively. Therefore, we have from [47]

$$z(1-z)\frac{d^2y}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{dy}{dz} - \alpha\beta\ y(z) = 0, \tag{3.8}$$

with the following pair of characteristic exponents

$$z = 0: \ (0, 1-\gamma) \qquad z = 1: \ (0, \gamma-\alpha-\beta): \qquad z = \infty: \ (\alpha, \beta).$$

The equation above is in the form of the canonical Hypergeometric equation, whose solutions are known as Hypergeometric functions, with the pair of solutions given by

$$\begin{aligned}
z =& 0 \ : \begin{cases} {}_2F_1(\alpha, \beta, \gamma; z) \\ z^{1-\gamma}{}_2F_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z), \end{cases} \\
z =& 1 \ : \begin{cases} {}_2F_1(\alpha, \beta, \alpha+\beta+1-\gamma; 1-z) \\ (1-z)^{\gamma-\beta-\alpha}{}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z), \end{cases} \\
z =& \infty \ : \begin{cases} z^{-\alpha}{}_2F_1(\alpha, \alpha-\gamma+1, \alpha-\beta+1; 1/z) \\ z^{-\beta}{}_2F_1(\beta, \beta-\gamma+1, \beta-\alpha+1; 1/z). \end{cases}
\end{aligned} \tag{3.9}$$

The exact calculations of the solutions via Frobenius methods, limits in the equation, and review of the theory can be seen in [48, 47].

Other types of differential equations are built from the Hypergeometric equation by taking specific limits. One of these limits, it is responsible for the Confluent Hypergeometric differential equation, where, in the hypergeometric equation the limit is made by replacing z by z/b, sending $b \to \infty$, and subsequently replacing the parameter c by b. In effect, the regular singularity at $z = 1$ in the hypergeometric differential equation coalesces into an irregular singularity at ∞ [49]. Thus, we obtain

$$zy''(z) + (b-z)y'(z) - ay(z) = 0. \tag{3.10}$$

The equation above has one regular singularity at the origin and one irregular singularity at infinity with Poincaré rank $r = 1$, and the solutions of this equation are the Confluent Hypergeometric functions [49].

To deal with isomonodromic deformations, and compute the monodromy matrices, the second-order differential equation (3.4) is commonly written in terms of a linear system given by

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z), \tag{3.11}$$

where $\Phi(z)$, $A(z)$ are 2×2 matrices. Each line of $\Phi(z)$ satisfies a second-order differential equation analogous to (3.4), where, via Frobenius method we can use the solutions of these equations to build the fundamental solutions of the system. Where the solutions are linearly independent if their Wronskian, $W(\Phi(z); z) = \det\Phi(z)$, does not vanish identically. A trivial example, it is the Hypergeometric system where there are three solutions matrices around 0, 1, and ∞, that we denote as $\Phi^{(0)}(z)$, $\Phi^{(1)}(z)$, and $\Phi^{(\infty)}(z)$ respectively with the elements of the fundamental solutions given by Hypergeometric functions as (3.9). We can also build the system related to the Confluent Hypergeometric equation (3.10) by using the Confluent Hypergeometric functions. Note that, in this case, the system has two singularities one regular at $z = 0$ and one irregular at $z = \infty$ of rank $r = 1$.

Now let us take the general linear system with n regular singularities and one irregular at infinity with $r = 1$. Such generalization will become clear in the Painlevé V section. Therefore, we have the following system [42]

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z), \qquad A(z) = \sum_{i=1}^{n} \frac{A_i}{(z - z_i)} + A_\infty, \tag{3.12}$$

with A_i associated to the regular point z_i and A_∞ to the irregular point with $r = 1$ at infinity. The general expression of the fundamental solution matrix $\Phi^{(i)}(z)$ around each point is written as

$$\Phi^{(i)}(z) = G_{(i)}\left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j^{(i)} \epsilon_{(i)}^j\right)\exp\left(\sum_{j=-r_i}^{-1} \frac{1}{j} A_j^{(i)} \epsilon_{(i)}^j + A_0^{(i)} \log(\epsilon_{(i)})\right) \tag{3.13}$$

with $G_{(i)}$, $\Phi_j^{(i)}$, $A_j^{(i)}$, and $A_0^{(i)}$ constant matrices, $r_\infty = 1$ and $r_i = 0$, $i = 1, ..., n$. ϵ_i define as

$$\epsilon_{(i)} = \begin{cases} z - z_{(i)} & i = 1, ..., n \\ 1/z, & i = \infty. \end{cases}$$

As an excellent review of the definitions above, Robert Conte's book can be helpful [42].

3.3 Solutions of Linear System and Monodromy Matrix

Using the definitions above, let us study the properties of the solutions around each singularity, as well as the analytic continuation of solutions via monodromy matrix.

3.3.1 Solutions Around Regular Points

Initially, we consider just the regular points in the system, the treatment at $z=\infty$ will be left to the next section. Around each regular point, the fundamental solution is written as

$$\Phi^{(i)}(z) = G_{(i)}\left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j^{(i)} \epsilon_{(i)}\right)\epsilon_{(i)}^{A_0^{(i)}}, \qquad i = 1, 2, ..., n \tag{3.14}$$

where the term $\epsilon_{(i)}^{A_0^{(i)}}$ is responsible for the multivaluedness with $A_0^{(i)}$ controling the branching of $\Phi^{(i)}(z)$. To describe the multivaluedness of the fundamental solution $\Phi^{(i)}(z)$, we can consider the projective plane $\mathbb{P}^1$ around the singular points and a universal covering $\bar{\mathbb{P}}^1$, such that the covering map is defined by $\pi : \bar{\mathbb{P}}^1 \to \mathbb{P}^1$ with $\Phi(z)$ single-value on $\bar{\mathbb{P}}^1$. Thus, let consider γ a path in $\bar{\mathbb{P}}^1$, starting at a point z on the Riemann sphere and ending at z_γ, such that $\pi(z) = \pi(z_\gamma)$. The fundamental matrix $\Phi(z_\gamma)$ at z_γ is single-valued and still satisfies (3.12), this implies that there exists a nonsingular constant matrix M_γ with

$$\Phi(z_\gamma) = \Phi(z)M_\gamma. \tag{3.15}$$

As we can see, each path γ around the singularities is related to M_γ, where the mapping $\gamma \to M_\gamma$ defines the monodromy representation associated with the differential system (3.12). Therefore, the n regular singularity fixed at each z_i is related to γ_i with

$$\Phi^{(i)}(z_{\gamma_i}) = \Phi^{(i)}(z_i)M_i \qquad i = 1, 2, ..., n, \tag{3.16}$$

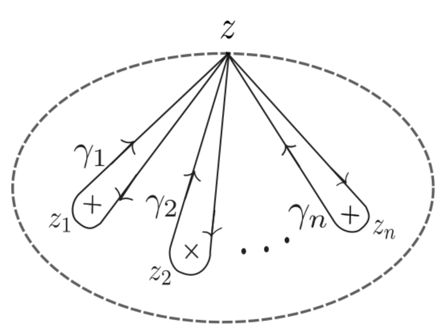

Figure 3.1: Paths on the Riemann Sphere.

where the dashed path represents the path around the point at infinity and γ_i related with each M_i. The set of M_i and the monodromy at infinity form the monodromy group of (3.12).

In the system, we also define that the matrices $A_1, ..., A_n$ can be diagonalizable if there are a set of invertible matrices G_i such that

$$A_0^{(i)} = G_{(i)}^{-1} A_i G_{(i)}. \tag{3.17}$$

The behavior around singularities plays a central role in the system and help us to find the explicit form of M_i in (3.16). We know that the behavior around each regular point is given by (3.14) where replacing $\epsilon_{(i)}$ we get

$$\Phi^{(i)}(z) = G_{(i)}\big[\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j^{(i)}(z - z_i)^j\big](z - z_i)^{A_0^{(i)}}, \tag{3.18}$$

where $\Phi_j^{(i)}$ is a constant matrix. To determine such constants, we need to expand $A(z)$ around z_i, where it is convenient to define

$$A(z) = G_i \sum_{j=0}^{\infty} A_j^{(i)}(z - z_i)^{j-1} G_i^{-1}, \quad i = 1, ..., n. \tag{3.19}$$

Using the relation above and replacing in (3.18), it is possible to prove that the eigenvalues of $A_{(i)}$'s are in modulo distinct of nonzero integers, where this assumption eliminates any logarithmic behavior. The eigenvalues are roots of the indicial equation in the Frobenius method which leads two independent solutions and it is well-known that, if the roots are repeated or differ by an integer, the second solution has a logarithmic behavior, therefore, to eliminate such behavior the eigenvalues must not be separated by nonzero integer (counting zero) [50].

From (3.16), we can also observe that any solution of (3.12) around a regular point can be written as

$$\Phi(z) = \Phi^{(i)}(z)C_{(i)}, \qquad i = 1, ..., n, \tag{3.20}$$

with $C_{(i)}$ some invertible constant matrix. To find the explicit form of the monodromy matrices in (3.15), let us consider the transformation $z \to e^{2i\pi}z$ around z_i, such transformation leads to

$$\Phi^{(i)}(e^{2i\pi}(z - z_{(i)}) + z_{(i)}) = \Phi^{(i)}(z)e^{2i\pi A_0^{(i)}}, \tag{3.21}$$

then, using the equation (3.20), we define the monodromy matrix M_i as

$$M_i = C_i^{-1} e^{2i\pi A_0^{(i)}} C_i. \tag{3.22}$$

3.3.2 Solutions Around Irregular Point

Now we are going to investigate the solution at infinity where the behavior of the fundamental solution is a little complicated since it is proportional to exponential (3.13) with such exponential influencing directly in the definition of the monodromy matrix. We know from (3.13) that the solution around infinity is given by

$$\Phi^{(\infty)}(z) = \left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j^{(\infty)} z^{-j}\right) z^{-A_0^{(\infty)}} e^{-A_{-1}^{(\infty)} z}, \tag{3.23}$$

where $A_0^{(\infty)}$ and $A_{-1}^{(\infty)}$ are diagonal and given in terms of the eigenvalues of each matrix. The term $z^{-A_0^{(\infty)}}$ is related to the multivaluedness of the solution $\Phi^{(\infty)}(z)$, on the other hand, the second term $e^{-A_{-1}^{(\infty)}z}$ is responsible for the growth and decay of the solution at infinity, and also for the Stokes phenomenon.

Stokes Phenomenon

The exponential term plays a constraint in each solution of the system at infinity, where the solutions are restricted by Stokes lines $\mathcal{L}$ in the complex plane and the behavior of the solutions can differ in different sectors [51], see Figure 3.2.

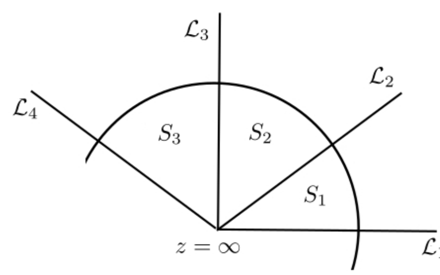

Figure 3.2: Stokes lines

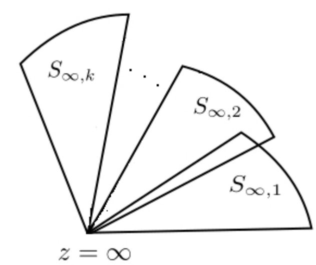

Figure 3.3: Stokes sectors

The sectors are defined by [52]

$$\mathcal{S}_k = \left\{ z \in \mathbb{C}, -\frac{1}{2}\pi + (k-2)\pi < \text{Arg}(z) < \frac{3}{2}\pi + (k-2)\pi \right\}, \quad k \in \mathbb{Z}. \tag{3.24}$$

where k labels the sector, see Figure 3.2. Let us consider a solution in the first sector $\Phi_{k=1}^{(\infty)}(z)$, inside this sector $\Phi^{(\infty)}(z)$ is not defined entirely, and other sectors are necessary to build all solution. To take this into account, let us consider that there exists a set of sectors in which a unique $\Phi_k(z)$ behaves like the general solution (3.23),

$$\Phi^{(\infty)}(z) \sim \Phi_k^{(\infty)}(z) \quad \text{in } \mathcal{S}_k. \tag{3.25}$$

Using the argument that the fundamental matrices Φ_{k+1} and Φ_k have the same asymptotic expansion in $\mathcal{S}_k$ (3.25) and using the Stokes matrices, we can connect the solution by

$$\Phi_{k+1}^{(\infty)}(z) = \Phi_k^{(\infty)}(z) S_k, \tag{3.26}$$

see Figure (3.3), where

$$S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix} \quad S_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1} & 1 \end{pmatrix}, \tag{3.27}$$

and the parameters s_{2k} and s_{2k+1} are Stokes multipliers [52].

A striking consequence of (3.26), it is that the equation (3.25) can be written as

$$\Phi^{(\infty)}(z) \sim \Phi_0^{(\infty)} S_k S_{k-1} \ldots S_1 \quad \text{in } \mathcal{S}_k. \tag{3.28}$$

In this case, the covering map $\pi(z) \to \pi(z_\gamma)$ send the solutions from a sector $\mathcal{S}_k$ to $\mathcal{S}_{k+2}$ and vice versa:

$$\Phi_{k+2}^{(\infty)}(e^{2i\pi}z) = \Phi_k^{(\infty)}(z)e^{2i\pi A_0^{(\infty)}}. \tag{3.29}$$

From (3.26), we recover the k-sector by

$$\Phi_k^{(\infty)}(e^{-2i\pi}z) = \Phi_k^{(\infty)}(z)S_kS_{k+1}e^{-2i\pi A_0^{(\infty)}}. \tag{3.30}$$

Comparing with (3.15), we identify the monodromy matrix

$$M_k^{(\infty)} = S_kS_{k+1}e^{-2i\pi A_0^{(\infty)}} \tag{3.31}$$

where by recurrence the monodromy associated to the next sector can be expressed as

$$M_{k+1}^{(\infty)} = S_k^{-1}M_k^{(\infty)}S_k. \tag{3.32}$$

Replacing (3.31) in (3.32) we obtain the monodromy matrix in terms of Stokes's matrices and exponential of $A_0^{(\infty)}$, thus

$$M_{k+1}^{(\infty)} = S_{k+1}e^{-2i\pi A_0^{(\infty)}}S_k. \tag{3.33}$$

Here we will choose the first sector $k = 1$, such that the matrix at infinity will be defined as

$$M_\infty = M_2^{(\infty)} = S_2e^{-2i\pi A_0^{(\infty)}}S_1, \tag{3.34}$$

where such choice will be helpful in the next chapter.

The linear system (3.12) has $n+1$ monodromy matrices where it is always possible to choose γ's in such a way that the product $\gamma_1\gamma_2...\gamma_n\gamma_\infty$ is homotopic to a point z on the Riemann sphere, with the monodromies matrices satisfying the following constraint:

$$M_1...M_nM_\infty = \mathbb{I}, \tag{3.35}$$

where the deformation is defined to be isomonodromic, if only if, it leaves invariant all the matrices.

Isomonodromic Deformations Theory

The study of isomonodromic deformations in the linear system is based on considering, $\Phi(z)$ and $A(z)$ depending on the point a, with the point a belonging to the set z_i, $i = 1, ..., n$. Where such point has the interpretation of the gauge parameter in the isomonodromic deformation theory. For a short review on isomonodromic deformations, see [13, 14]. To reinforce the concept behind deformations in the linear system with just one irregular point at infinity and define the general idea about the isomonodromic invariant τ-function, let us consider the following theorem.

Theorem 1[13] *The deformation equations of the linear system given by*

$$\begin{aligned} \frac{\partial}{\partial z}\Phi(z,a) =& A(z,a)\Phi(z,a) \\ A(z,a) =& \sum_{i=1}^{n}\frac{A_i(a)}{(z-z_i)} + A_\infty, \end{aligned} \tag{3.36}$$

are isomonodromic if only if $\Phi(z,a)$ satisfies the following linear partial differential equation

$$\frac{\partial}{\partial a}\Phi(z,a) = -\frac{A_a(a)}{z-a}\Phi(z,a) \tag{3.37}$$

that is equivalent to a completely integral system of nonlinear differential equations for A_i and A_a given by

$$\frac{\partial A_i}{\partial a} = \frac{[A_a, A_i]}{a-z_i} \qquad \frac{\partial A_a}{\partial a} = \sum_{\substack{j=1\\ j\neq a}}^{n}\frac{[A_j, A_a]}{a-z_i} + [A_\infty, A_a]. \tag{3.38}$$

The equations above are known it as *Schlesinger equations* and can be proved by considering the commutation relation $\partial_z\partial_a\Phi(z,a) = \partial_a\partial_z\Phi(z,a)$ between (3.36) and (3.37), see [42]. The Schlesinger equations above means that deformations which preserve the monodromy of a generic linear system are governed by the integrable systems of partial differential equations (PDE).

Isomonodromic τ-function

The Schlesinger equations are commonly written in terms of the Jimbo-Miwa-Ueno isomonodromic τ-function which plays a central role in deformations theory [13]

$$d\ \log\tau = \sum_{\substack{j=1\\ j\neq a}}^{n}\mathrm{Tr}(A_iA_a)\frac{da}{a-z_i} + \mathrm{Tr}(A_\infty A_a)da, \tag{3.39}$$

with the 1-form satisfying

$$d(d\ \log\tau) = 0. \tag{3.40}$$

The equation (3.40) has a form of conservation law, where in this case the conservation is related to flow of isomonodromy in the linear system. Therefore, we define H to represent such conservation quantity:

$$H = \frac{d}{da}\log(\tau(a)) = \sum_{\substack{j=1\\ j\neq a}}^{n}\frac{\mathrm{Tr}(A_iA_a)}{a-z_i} + \mathrm{Tr}(A_\infty A_a). \tag{3.41}$$

3.4 Isomonodromic Deformation Problem for PV

To study isomonodromic deformations which leads to Painlevé V the start point is to consider a system with two regular singularities and one irregular singularity, thus, we will fix the two regular singularities at $z = 0$ and $z = t$ and the irregular at $z = \infty$. Taking $a = t$, the PDEs (3.36),(3.37) are written as

$$\frac{\partial}{\partial z}\Phi(z,t) = A(z,t)\Phi(z,t), \qquad A(z,t) = A_\infty + \frac{A_0}{z} + \frac{A_t}{z-t}, \tag{3.42}$$

$$\frac{\partial}{\partial t}\Phi(z,t) = -\frac{A_t}{z-t}\Phi(z,t), \tag{3.43}$$

and from (3.38), the Schlesinger equation are given by

$$\frac{\partial}{\partial t}A_0 = -\frac{1}{t}[A_0, A_t], \qquad \frac{\partial}{\partial t}A_t = \frac{1}{t}[A_0, A_t] + [A_\infty, A_t]. \tag{3.44}$$

The fundamental matrix $\Phi(z)$ is given by

$$\Phi(z) = \begin{pmatrix} y_{(1)}(z) & y_{(2)}(z) \\ u_{(1)}(z) & u_{(2)}(z) \end{pmatrix}, \tag{3.45}$$

where we can work with (3.42) in order to find a second-order differential equation for $y_{(i)}(z)$ with $i = 1, 2$. The second line obeys an analogue differential equation with $u_{(i)}(z)$ related to $y_{(i)}(z)$ by differentiation and multiplication by a rational function

$$u_{(i)}(z) = \frac{1}{A_{12}(z)}\left(\frac{dy_{(i)}}{dz} - A_{11}(z)y_{(i)}(z)\right), \tag{3.46}$$

with $A_{11}(z)$ and $A_{12}(z)$ elements of $A(z)$. Thus, the first line satisfy

$$\begin{gathered} y''_{(i)}(z) + p(z)y'_{(i)}(z) + q(z)y_{(i)}(z) = 0, \\ p(z) = -\mathrm{Tr}(A) - \partial_z \log(A_{12}), \quad q(z) = \det(A) - A'_{11} + A_{11}\partial_z\log(A_{12}). \end{gathered} \tag{3.47}$$

Using basis change, we take A_∞ to be diagonal, which leads to the assumption that A_{12} vanishes as $\mathcal{O}(z^{-2})$, when $z \to \infty$. Therefore, from (3.42) $A_{12}(z)$ is written as

$$A_{12}(z) = \frac{k(z-\lambda)}{z(z-t)}, \quad k = a_{12}^{(0)} + a_{12}^{(t)}, \quad \lambda = \frac{t a_{12}^{(0)}}{a_{12}^{(0)} + a_{12}^{(t)}}. \tag{3.48}$$

Now substituting in (3.47):

$$\begin{aligned} y''_{(i)}(z) - &\left(\mathrm{Tr}A_\infty + \frac{\mathrm{Tr}A_0 - 1}{z} + \frac{\mathrm{Tr}A_t - 1}{z-t} - \frac{1}{z-\lambda}\right)y'_{(i)}(z) \\ &+\left(\det A_\infty + \frac{\det A_0}{z^2} + \frac{\det A_t}{(z-t)^2} + \frac{c_0}{z} + \frac{c_t}{z-t} + \frac{\mu}{z-\lambda}\right)y_{(i)}(z) = 0, \end{aligned} \tag{3.49}$$

with

$$
\begin{aligned}
c_0 &= \mathrm{Tr}A_\infty \mathrm{Tr}A_0 - \mathrm{Tr}(A_\infty A_0) + \frac{1}{t}\mathrm{Tr}(A_0 A_t) - \frac{1}{t}\mathrm{Tr}A_0 \mathrm{Tr}A_t \\
&\quad - a_{11}^{(\infty)} - \frac{1}{\lambda}a_{11}^{(0)} + \frac{1}{t}(a_{11}^{(0)} + a_{11}^{(t)}), \\
c_t &= \mathrm{Tr}A_\infty \mathrm{Tr}A_t - \mathrm{Tr}(A_\infty A_t) - \frac{1}{t}\mathrm{Tr}(A_0 A_t) + \frac{1}{t}\mathrm{Tr}A_0 \mathrm{Tr}A_t \\
&\quad - a_{11}^{(\infty)} - \frac{1}{\lambda - t}a_{11}^{(t)} - \frac{1}{t}(a_{11}^{(0)} + a_{11}^{(t)}), \\
\mu &= a^{(\infty)} + \frac{1}{\lambda}a_{11}^{(0)} + \frac{1}{z-t}a_{11}^{(t)},
\end{aligned}
\tag{3.50}
$$

where the terms c_t, c_0 and μ obey,

$$
c_0 + c_t + \mu = \mathrm{Tr}A_\infty(\mathrm{Tr}A_0 + \mathrm{Tr}A_t) - \mathrm{Tr}(A_\infty(A_0 + A_t)) - a_{11}^{(\infty)}.
$$

The parameter c_t is called accessory parameter and plays an important role in isomonodromy deformations, since it is related to τ_V [53]. We also remark that such accessory parameter does not appear in deformed systems with $n \leq 3$, for example, the hypergeometric system and confluent hypergeometric system. However, for systems with a value of $n \geq 4$, as well as confluent limits of such systems, the accessory parameter appears naturally [54, 45]. In the next chapter, we will work in the linear system associated with the confluent Heun equation to find c_t by using τ_V. It must be clear by now that the study of such system leads to the associated Painlevé V.

The constant λ defined in (3.48) appears as a singularity in (3.49). This singularity is classified as apparent with indicial exponents 0 and 2 with trivial monodromy, $M_\lambda = \mathbb{I}$. From (3.40), we identify H in c_t as

$$
\begin{aligned}
c_t &= \mathrm{Tr}A_\infty \mathrm{Tr}A_t + \frac{1}{t}\mathrm{Tr}A_0 \mathrm{Tr}A_t - H - a_{11}^{(\infty)} - \frac{1}{\lambda - t}a_{11}^{(t)} - \frac{1}{t}(a_{11}^{(0)} + a_{11}^{(t)}), \\
H &= \mathrm{Tr}(A_\infty A_t) + \frac{1}{t}\mathrm{Tr}(A_0 A_t).
\end{aligned}
\tag{3.51}
$$

In order to write c_t as function of λ, μ and t and using the absence of logarithmic behavior at $z = \lambda$, we can expand $p(z)$, $q(z)$ around λ, where by residue the only relevant terms are

$$
p(z) = -\frac{1}{z-\lambda} + p_1 + p_2(z-\lambda) + \ldots, \quad q(z) = \frac{\mu}{z-\lambda} + q_1 + q_2(z-\lambda) + \ldots, \tag{3.52}
$$

$$
\begin{aligned}
p_1 &= -\mathrm{Tr}A_\infty + \frac{1 - \mathrm{Tr}A_0}{\lambda} + \frac{1 - \mathrm{Tr}A_1}{\lambda - t}, \\
\mu &= \frac{a_{11}^{(0)}}{\lambda} + \frac{a_{11}^{(t)}}{\lambda - t}, \\
q_1 &= \det A_\infty + \frac{\det A_0}{\lambda^2} + \frac{\det A_t}{(\lambda - t)^2} + \frac{c_0}{\lambda} + \frac{c_t}{\lambda - t}.
\end{aligned}
\tag{3.53}
$$

Now, by expanding $y_{(i)}(z)$ in Frobenius series

$$y_{(i)}(z) = \sum_{n=0}^{\infty} a_n (z-\lambda)^{\alpha+n},$$

we can find the indicial equation that solve any logarithmic behavior. Without mentioning all calculation, the indicial equation is obtained as $(\mu + p_1)\mu + q_1 = 0$. Substituting in (3.53):

$$\begin{aligned} &\mu^2 - \left(\mathrm{Tr}A_\infty + \frac{\mathrm{Tr}A_0}{\lambda} + \frac{\mathrm{Tr}A_1 - 1}{\lambda - t}\right)\mu + \det A_\infty + \frac{\det A_0}{\lambda^2} + \frac{\det A_t}{(\lambda-t)^2} \\ &+ \frac{\mathrm{Tr}A_\infty(\mathrm{Tr}A_0 + \mathrm{Tr}A_t) - \mathrm{Tr}(A_\infty(A_0+A_t)) - a_{11}^{(\infty)}}{\lambda} + \frac{tc_t}{\lambda(\lambda-t)} = 0, \end{aligned} \tag{3.54}$$

which leads to

$$\begin{aligned} c_t(\lambda,\mu,t) = &-\frac{\lambda(\lambda-t)}{t}\Bigg[\mu^2 - \left(\mathrm{Tr}A_\infty + \frac{\mathrm{Tr}A_0}{\lambda} + \frac{\mathrm{Tr}A_t - 1}{\lambda - t}\right)\mu + \det A_\infty \\ &+ \frac{\det A_0}{\lambda^2} + \frac{\det A_t}{(\lambda-t)^2} + \frac{\mathrm{Tr}A_\infty(\mathrm{Tr}A_0 + \mathrm{Tr}A_t) - \mathrm{Tr}(A_\infty(A_0+A_t)) - a_{11}^{(\infty)}}{\lambda}\Bigg]. \end{aligned} \tag{3.55}$$

Now let us fix the notation between Chapter 1 and Chapter 2 by choosing the following parametrizations to A_0, A_t and A_∞

$$\begin{aligned} &\mathrm{Tr}A_\infty = 0, \quad \mathrm{Tr}A_t = \hat{\theta}_t, \quad \mathrm{Tr}A_0 = \hat{\theta}_0, \quad \mathrm{Tr}(A_\infty(A_0+A_t)) = -\frac{\hat{\theta}_\infty}{2}, \\ &\det A_0 = \det A_t = 0, \quad \det A_\infty = \det\frac{1}{2}\sigma_3 = -\frac{1}{4}. \end{aligned} \tag{3.56}$$

where the parametrizations will become clear in the next section. Using the parametrizations, we write $a_{11}^{(0)}$ and $a_{11}^{(t)}$ in terms of λ, μ and t as

$$\begin{aligned} a_{11}^{(0)} &= -\frac{\lambda(\lambda-t)}{t}\left[\mu - \frac{1}{2} - \frac{\hat{\theta}_0 + \hat{\theta}_t - \hat{\theta}_\infty}{2(\lambda-t)}\right], \\ a_{11}^{(t)} &= \frac{\lambda(\lambda-t)}{t}\left[\mu - \frac{1}{2} - \frac{\hat{\theta}_0 + \hat{\theta}_t - \hat{\theta}_\infty}{2\lambda}\right], \end{aligned} \tag{3.57}$$

where replacing the parametrizations in (3.54) and (3.51), we finally get c_t

$$c_t = -\frac{\lambda(\lambda-t)}{t}\left[\mu^2 - \left(\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_t}{\lambda-t}\right)\mu - \frac{1}{4} + \frac{\hat{\theta}_\infty}{2\lambda} + \frac{\hat{\theta}_0\hat{\theta}_t}{\lambda-t}\right]. \tag{3.58}$$

Here, λ and μ have the interpretation of canonical variables, in the system, with Poisson bracket defined by

$$\frac{\partial\lambda}{\partial t} = \{\lambda, c_t\}, \quad \frac{\partial\mu}{\partial t} = \{\mu, c_t\}. \tag{3.59}$$

In the Poisson bracket above, c_t describes the evolution of λ as a function of the gauge parameter t, where the equation of motion associated to λ is the non-linear Painlevé V associated to the Schlesinger equations (3.44):

$$\begin{aligned}\frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} \\ + \frac{(\lambda-1)^2}{t^2}\left(\frac{\hat{\theta}_0^2}{2}t - \frac{\hat{\theta}_t^2}{2t}\right) - (\hat{\theta}_\infty + 1)\frac{\lambda}{t} - \frac{\lambda(\lambda+1)}{2(\lambda-1)},\end{aligned} \tag{3.60}$$

with $\lambda(t)$ as a meromorphic function on the universal covering $\mathbb{P}^1 \setminus \{0, \infty\}$. Using the parametrization of A_0, A_t and A_∞ we write the equation (3.49) as,

$$y''_{(i)}(z) + \left(\frac{1-\hat{\theta}_0}{z} + \frac{1-\hat{\theta}_t}{z-t} - \frac{1}{z-\lambda}\right)y'_{(i)}(z) + \left(-\frac{1}{4} + \frac{c_0}{z} + \frac{c_t}{z-t} + \frac{\mu}{z-\lambda}\right)y_{(i)}(z). \tag{3.61}$$

where, as the consequence of the nature of the singularity λ, the equation (3.61) is the deformed confluent Heun equation. Furthermore, the c_t defined in (3.58) describes the evolution of the apparent singularity λ by (3.60).

As we know, confluent Heun equation is associated with Painlevé V, as well as with isomonodromic flow H. Slavyanov studied such relation in [41], where the relationship between the Heun class of second-order linear equations and the Painlevé equations arose as an interesting connection. Also, Carneiro Da Cunha and Novaes in [53] used Painlevé V associated with confluent Heun equation in Kerr scattering problem. We want to emphasize that, in terms of Heun class, the Confluent Heun equations mentioned above arise when two or more of the regular singularities merge to form an irregular singularity. That is analogous to the derivation of the confluent hypergeometric equation from the hypergeometric equation explained in the subsection Fuchsian ODE.

We have defined H as the isomonodromic flow in (3.41), therefore, from equation (3.58), we find explicitly

$$\frac{d}{dt}\log(\tau_V(\{\hat{\theta}_i\}, t)) = -\frac{\lambda(\lambda-t)}{t}\left[\mu^2 - \left(\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_t}{\lambda - t}\right)\mu + \frac{\hat{\theta}_\infty}{2\lambda} - \frac{1}{4}\right] - \frac{\hat{\theta}_0\hat{\theta}_t}{t} \tag{3.62}$$

with $\{\hat{\theta}_i\} = (\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty)$. The equation above will be useful in the next chapter where, necessarily, it will give the condition to recover the confluent Heun equation from deformed equation (3.61).

3.5 Conformal field theory of Painlevé V

From the AGT correspondence, and based in the papers: Gamayun, Iorgov, and Lisovyy [29], where conformal field theory for $c = 1$ was discussed in the context of isomonodromic deformations theory for Painlevé VI, and also in Lisovyy, Nagoya, and Roussillon [11] where an analogous discussion was made in the context of Painlevé V. We can write

$\tau_V(\{\hat{\theta}_i\}, t)$ in terms of correlation function in 2d CFT, with $c = 1$, thus, the formalism of partitions defined in Chapter 1 can be used to write the explicit form of $\tau_V(t)$.

As we know, the linear system (3.42) associated with Painlevé V has two regular singular points fixed at 0 and t and one irregular singularity of rank one at infinity. The correlation function associated with this type of system is a three-point function with two operators of rank 0 satisfying (2.30) and one Whittaker operator of rank $r = 1$ satisfying (2.80) - see the definitions in [11]. Note that, the three-point function mentioned, it was explained in the Confluent Conformal Block section, equation (2.87). Therefore, from AGT correspondence, the $\tau_V(t)$ is given by

$$\tau_V(t) = \langle \mathcal{O}^{[1]}_{\Lambda_1,\Lambda_2}(\infty)\mathcal{O}_{\Delta_t}(t)\mathcal{O}_{\Delta_0}(0)\rangle\Big|_{c=1}, \tag{3.63}$$

where using the parametrizations defined (3.56), the conformal dimension related to each operator is given by

$$\begin{aligned} \Delta_0 = \frac{1}{4}\mathrm{Tr}A_0^2 = \frac{\hat{\theta}_0^2}{4}, \quad \Delta_t = \frac{1}{4}\mathrm{Tr}A_t^2 = \frac{\hat{\theta}_t^2}{4}, \\ (\Lambda_1, \Lambda_2) = \big(\mathrm{Tr}(A_\infty(A_0 + A_t)), \frac{1}{4}\big) = \big(-\hat{\theta}_\infty, \frac{1}{4}\big). \end{aligned} \tag{3.64}$$

Comparing with Chapter 1, it is easy to notice that $\hat{\theta}_*$ was replaced by $-\hat{\theta}_\infty$, where the signal is consequence of the parametrizations of A_0, A_t, and A_∞. Thus, $\tau_V(t)$ written as

$$\tau_V(t) = \sum_n C(\hat{\theta}_\infty, \hat{\theta}_t, \hat{\theta}_0, \frac{\hat{\sigma}}{2} + n) t^{\frac{1}{4}(\hat{\sigma}+2n)^2 - \frac{1}{4}(\hat{\theta}_0^2 + \hat{\theta}_t^2)} \mathcal{D}\big(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty, \hat{\sigma} + 2n; t\big), \tag{3.65}$$

with

$$\begin{aligned} \mathcal{D}(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty, \hat{\sigma}; t) &= e^{-\hat{\theta}_t t} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{D}_{\lambda,\mu}(\hat{\theta}_*, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) t^{|\lambda|+|\mu|}, \\ \mathcal{D}_{\lambda,\mu}(\hat{\theta}_\infty, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma}) &= \prod_{\lambda\in\mathbb{Y}} \frac{\big(\hat{\sigma} + 2(i-j) - \hat{\theta}_\infty\big)\big((\hat{\sigma} + 2(i-j) + \hat{\theta}_t)^2 - \hat{\theta}_0^2\big)}{8h_\lambda^2(i,j)\big(\lambda'_j + \mu_i - i - j + 1 + \hat{\sigma}\big)^2} \\ &\quad \prod_{\mu\in\mathbb{Y}} \frac{\big(-\hat{\sigma} + 2(i-j) - \hat{\theta}_\infty\big)\big((-\hat{\sigma} + 2(i-j) + \hat{\theta}_t)^2 - \hat{\theta}_0^2\big)}{8h_\mu^2(i,j)\big(\mu'_j + \lambda_i - i - j + 1 - \hat{\sigma}\big)^2}, \\ C(\hat{\theta}_\infty, \hat{\theta}_t, \hat{\theta}_0, \frac{\hat{\sigma}}{2} + n) &= C^{(\hat{\sigma}/2+n)^2}_{(-\hat{\theta}_\infty, \frac{1}{4})} C^{(\hat{\sigma}/2+n)^2}_{\frac{\hat{\theta}_0^2}{4}\ \frac{\hat{\theta}_t^2}{4}}. \end{aligned} \tag{3.66}$$

The structure constant above is the normalization constant of the correlation function with each structure in the right-hand side, counting the contributions of each part in the diagram, Figure 3.4. Where the first part is related to the correlation function between one state of rank 0 and one Whittaker state of rank 1, and in the second part, we have a correlation function with three operators of rank 0.

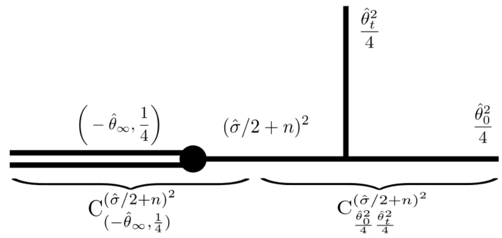

Figure 3.4: Confluent three-point function for s-channel.

The computation of each structure using 2d CFT is complicated when n is large, which means consider more contribution in the intermediate channel $\Delta = (\frac{\sigma}{2} + n)^2$. Therefore, instead of CFT we can determine the explicit form of the structure constant from the Jimbo asymptotic formula [43], that express the asymptotic behavior of PV tau function in terms of monodromy. Hence, the structure $C(\hat{\theta}_\infty, \hat{\theta}_t, \hat{\theta}_0, \hat{\sigma})$ is expressed in terms of G-Barnes functions and the parameter s_V,

$$C(\{\hat{\theta}\}, \tilde{\sigma}) = s_V^n \prod_{\epsilon=\pm} \frac{G(1+\frac{1}{2}(\epsilon\tilde{\sigma} - \hat{\theta}_\infty))G(1+\frac{1}{2}(\hat{\theta}_t + \hat{\theta}_0 + \epsilon\tilde{\sigma}))G(1+\frac{1}{2}(\hat{\theta}_t - \hat{\theta}_0 + \epsilon\tilde{\sigma}))}{G(1+\epsilon\tilde{\sigma})}. \tag{3.67}$$

with $\tilde{\sigma} = \hat{\sigma} + 2n$ and G-Barnes function satisfying $G(1+z) = \Gamma(z)G(z)$. Where s_V is interpreted in terms of linear systems where the diagram Figure 3.4 is separated into two parts with the two-point correlation representing a Confluent Hypergeometric system and the three-point function a Hypergeometric system with s_V gluing the systems [11].

Now we finally have the explicit expression for $\tau_V(t)$, which together with (3.61) and (3.62) will help us to solve the accessory parameter as well as the eigenvalue for the angular Teukolsky Master equation.

4 Kerr Black Hole Application

After building the τ_V as a function of the confluent CB of the first kind, we finally arrive at the chapter in which it is explained an alternative way to find the expansion of the eigenvalue for the angular Teukolsky Master equation (TME). There are several ways available [55, 56, 57] to compute the expansion for the angular eigenvalue, here we use the isomonodromic deformations theory as a new alternative to find the explicit expansion. To do that, we work with the angular TME. The treatment of the radial Teukolsky equation, as well as the study of quasi-normal modes, are subjects for future work.

In this chapter, we start by explaining how the Angular Teukolsky Master equation is derived, then with the angular equation defined, we use the isomonodromic deformations theory described in Chapter 2, as well as the first and second conditions of the τ_V function relevant in the calculation of the accessory parameter expansion. We also derive the Toda equation necessary to define the exact form of the second condition of τ_V. Lastly, we use the accessory parameter expansion to find the first seven coefficients of the expansion of the angular eigenvalue.

4.1 Teukolsky Master Equation

Teukolsky Master Equation (TME) appears when we consider the Newman-Penrose (NP) formalism to investigate perturbations of spin-s fields in the Kerr metric, where in terms of NP formalism the TME represents first-order perturbations of the Einstein field equation [58]. Essentially, the NP formalism consists in to project the metric onto a compost basis of 4 null vectors **l**, **n**, **m** and $\bar{\mathbf{m}}$, commonly called null tetrads, with **l**, **n** real vectors and **m**, $\bar{\mathbf{m}}$ complex vectors. In such formalism, the metric is decomposed as

$$g^{\mu\nu} = -l^\mu n^\nu - n^\mu l^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu, \tag{4.1}$$

with the null tetrads satisfying the orthogonality conditions $l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0$, and the normalization conditions given by $l^\mu n_\mu = -1, m^\mu \bar{m}_\mu = 1$ [59].

To study perturbations in Kerr by using NP formalism, we are going to consider the metric in Boyer-Lindquist coordinates

$$\begin{aligned} ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\phi + \frac{\Sigma}{\Delta}d^2r + \Sigma d^2\theta \\ -\sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \end{aligned} \tag{4.2}$$

where the rotation parameter a is the angular momentum per unit mass, $a = J/M$, and the function Δ and Σ are given by

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \qquad \Sigma = r^2 + a^2\cos^2\theta. \tag{4.3}$$

Without showing all steps, we assume directly from Teukolsky's paper the explicit form of the Teukolsky master equation. The derivation of TME and a review of the Newman-Penrose formalism can be seen in [60] and [59], respectively. Thus,

$$\begin{aligned} &\left[\frac{(r^2+a^2)}{\Delta} - a\sin^2\theta\right]\frac{\partial^2\Psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\Psi}{\partial\phi^2} \\ &-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\Psi}{\partial\phi} \\ &-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\Psi}{\partial t} + (s^2\cot^2\theta - s)\Psi = 0, \end{aligned} \tag{4.4}$$

where we are considering the vacuum case, with the energy tensor equal to zero, $T = 0$. The spin-weight field parameter s takes the values 0, ±1, ±2 for outgoing scalar, electromagnetic, and gravitational fields, respectively, and values $s = \pm1/2, \pm3/2$ for fermionic perturbations - to review perturbations of spin-s fields in the Kerr metric we recommend [61, 62, 63, 64]. The equation above can be separated by writing the field $\Psi(x^\mu)$ as $\Psi(x^\mu) = e^{-i\omega t}e^{im\phi}R(r)S(\theta)$, where replacing in the equation we obtain the radial and angular Teukolsky Master equation related to $R(r)$ and $S(\theta)$, respectively

4.2 Spin-Weighted Spheroidal Harmonics

From (4.4), the angular TME is the spin-weighted spheroidal harmonics equation, which has appeared in a variety of physical and mathematical problems, even before the black hole application [55, 57]:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dS}{d\theta}\right] + \left[a^2\omega^2\sin^2\theta - 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + s + \lambda\right]S(\theta) = 0. \tag{4.5}$$

where we identify the separation constant λ and also the poles in $\theta = 0$ and $\theta = \pi$. Changing variable $x = \cos\theta$, the differential equation becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dS}{dx}\right] + \left[a^2\omega^2(1-x^2) - 2a\omega sx - \frac{(m+sx)^2}{(1-x^2)} + s + \lambda\right]S(x) = 0, \tag{4.6}$$

where to use the isomonodromic formalism, let us take

$$y(z) = (1+x)^{\hat{\theta}_{z_0}/2}(1-x)^{\hat{\theta}_0/2}S(x), \qquad z = -2a\omega(1-x). \tag{4.7}$$

The variable change brings the differential equation into the canonical confluent Heun form

$$\frac{d^2y}{dz^2} + \left[\frac{1-\hat{\theta}_0}{z} + \frac{1-\hat{\theta}_{z_0}}{z-z_0}\right]\frac{dy}{dz} + \left[-\frac{1}{4} + \frac{\hat{\theta}_\infty}{2z} - \frac{z_0 c_{z_0}}{z(z-z_0)}\right]y(z) = 0, \tag{4.8}$$

with

$$z_0 = -4a\omega, \qquad z_0 c_{z_0} = \lambda + 2a\omega s + a^2\omega^2. \tag{4.9}$$

and monodromy parameters defined by

$$\hat{\theta}_0 = -m - s, \qquad \hat{\theta}_{z_0} = m - s, \qquad \hat{\theta}_\infty = 2s. \tag{4.10}$$

We also identify the accessory parameter $z_0 c_{z_0}$ in the confluent Heun equation. The differential equation (4.8) has 3 singular points: two regular at $z = 0$ and $z = z_0$ and an irregular singular point of Poincaré rank 1 at $z = \infty$. As we know series expansions for the solutions $y(z)$ at the regular points can be obtained from the Frobenius method. We also understand that the point at infinity is trickier, because the solutions present the *Stokes phenomenon*, and as it was explained, the convergence of the solutions is conditional to sectors of the complex plane, see Figure 3.2.

In order to define, in the next pages, the quantization condition between the regular points, we will use solutions in (4.8) which are regular at both points 0 and z_0:

$$y(z) = \begin{cases} z^0(1+\mathcal{O}(z)) & z \to 0, \\ (z-z_0)^0(1+\mathcal{O}(z-z_0)) & z \to z_0, \end{cases} \tag{4.11}$$

which will place a restriction on the value of λ and eliminate divergent solutions.

It is well-known, from Chapter 2, that the confluent Heun equation (4.8) can be cast as a first order matrix equation:

$$\frac{d\Phi}{dz} = A(z)\Phi(z) = \left(\frac{1}{2}\sigma_3 + \frac{A_0}{z} + \frac{A_t}{z-t}\right)\Phi(z), \tag{4.12}$$

with $\Phi(z)$ defined as in (3.45) and A_∞ defined to be $\frac{1}{2}\sigma_3$.

Following the definitions of the Chapter 2, the fundamental solution around the regular points $z = 0$ and $z = t$ has the same form of (3.18), where we write it as

$$\Phi^{(0)}(z) = G_0\big[\mathbb{I} + \sum_{j=1}^{\infty}\Phi_j^{(0)}z^j\big]z^{A_0^{(0)}}, \tag{4.13}$$

$$\Phi^{(t)}(z) = G_t\big[\mathbb{I} + \sum_{j=1}^{\infty}\Phi_j^{(t)}(z-t)^j\big](z-t)^{A_0^{(t)}}, \tag{4.14}$$

with $A_0^{(0)}$ and $A_0^{(t)}$ defined as (3.17)

$$A_0^{(0)} = G_0^{-1} A_0 G_0, \qquad A_0^{(t)} = G_t^{-1} A_t G_t. \tag{4.15}$$

Since there are many parametrizations for A_0 and A_t, we are going to consider a parametrization such that the matrices $A_0^{(0)}$ and $A_t^{(0)}$ around each singularity are defined by

$$A_0^{(0)} = \frac{1}{2}\hat{\theta}_0 \sigma_3, \qquad A_t^{(0)} = \frac{1}{2}\hat{\theta}_t \sigma_3, \tag{4.16}$$

where $\hat{\theta}_0$, $\hat{\theta}_t$ are eingevalues of $A_0^{(0)}$, $A_t^{(0)}$, respectively, and they are related to the trace of A_0 and A_t in (4.12). The monodromy associated with the analytic continuation of the fundamental solution $\Phi(z)$ around 0 and t is

$$\begin{aligned} \Phi(ze^{2\pi i}) &= \Phi(z) M_0, \\ \Phi((z-t)e^{2\pi i} + t) &= \Phi(z) M_t, \end{aligned} \tag{4.17}$$

with the monodromy matrices given by

$$M_0 = C_0^{-1} e^{\pi i \hat{\theta}_0 \sigma_3} C_0, \qquad M_t = C_t^{-1} e^{\pi i \hat{\theta}_t \sigma_3} C_t. \tag{4.18}$$

Around the irregular singular point, $z = \infty$, the asymptotic solution is defined in sectors defined in (3.24), thus, from (3.23) and taking (3.25) the solution on the k-sector is written as,

$$\Phi_k^{(\infty)}(z) = \left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j^{(\infty)} z^{-j}\right) \exp\left[\frac{1}{2}\sigma_3 z + \frac{1}{2}((\hat{\theta}_0 + \hat{\theta}_t)\mathbb{I} - \hat{\theta}_\infty \sigma_3) \log z\right], \; z \in \mathcal{S}_k, \tag{4.19}$$

where from (3.23), we defined $A_0^{(\infty)} = -\frac{1}{2}\sigma_3$, $A_{-1}^{(\infty)} = -\frac{1}{2}((\hat{\theta}_0 + \hat{\theta}_t)\mathbb{I} - \hat{\theta}_\infty \sigma_3)$, and $\hat{\theta}_i$'s are defined as (3.56),

$$\hat{\theta}_0 = \mathrm{Tr} A_0, \qquad \hat{\theta}_t = \mathrm{Tr} A_t, \qquad \hat{\theta}_\infty = -\mathrm{Tr}\left[\sigma_3 (A_0 + A_t)\right] \tag{4.20}$$

with

$$\det A_0 = \det A_t = 0, \quad \det A_\infty = \det \frac{1}{2}\sigma_3 = -\frac{1}{4}.$$

It is customary to define the monodromy at $z = \infty$ in the sector $k = 1$, such that using the equations (3.30),(3.31) and (3.32), the monodromy around infinity is defined by

$$\Phi(e^{-2\pi i} z) = \Phi(z) M_\infty, \tag{4.21}$$

with the monodromy matrix M_∞ written as [52]

$$M_\infty = S_2 e^{-i\pi \hat{\theta}_\infty \sigma_3} S_1, \tag{4.22}$$

and the three matrices around each point satisfying the relation,

$$M_\infty M_t M_0 = \mathbb{I}. \tag{4.23}$$

It will be convenient to define the trace of M_∞ as an independent parameter [43],

$$2\cos\pi\hat{\sigma} = \mathrm{Tr}M_\infty = 2\cos\pi\hat{\theta}_\infty + s_1 s_2 e^{-i\pi\hat{\theta}_\infty}. \tag{4.24}$$

Furthermore, in order to facilitate the notation, we define the parameter $\hat{\rho}$ which represents all monodromy data associated to the system (4.12),

$$\hat{\rho} = \{\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty; s_1, s_2\}. \tag{4.25}$$

with s_1 and s_2 representing the Stokes multipliers.

4.2.1 Connection matrix and the quantization condition

Here we are interested in solutions between $z = 0$ and $z = t$, which are connected by a connection matrix, allowing us to find the quantization condition (4.11) in terms of monodromy data. The linear system related to (4.11) is the Hypergeometric system written as

$$\frac{d}{dz}\Phi(z) = \left(\frac{A_0}{z} + \frac{A_t}{z-t}\right)\Phi(z), \tag{4.26}$$

where in this system the fundamental solution at $z = 0$ can be built by using the Frobenius method in the first line of $\Phi(z)$ and then using (3.46) to find the second line. In a short way, from (4.13), we have

$$\Phi^{(0)}(z) = G_0\left(\mathbb{I} + \mathcal{O}(z)\right) z^{\frac{1}{2}\hat{\theta}_0\sigma_3}. \tag{4.27}$$

with $A_0^{(0)} = \frac{1}{2}\hat{\theta}_0\sigma_3$. We can find a basis where the monodromy matrix M_0 is diagonal with

$$\Phi^{(0)}(ze^{2\pi i}) = \Phi^{(0)}(z)M_0, \tag{4.28}$$

therefore, in this basis of solutions the monodromy M_0 is given by $M_0 = e^{i\pi\hat{\theta}_0\sigma_3}$.

The monodromy around $z = t$ defined as (4.17) is also diagonal in this basis, but the matrix C_t is now related to the *connection matrix* between the fundamental solutions $\Phi^{(t)}(z)$ and $\Phi^{(0)}(z)$:

$$\Phi^{(t)}((z-t)e^{2\pi i} + t) = \Phi^{(0)}(z)C_{t0}^{-1}e^{\pi i\hat{\theta}_t\sigma_3}C_{t0}, \tag{4.29}$$

with the constant connection matrix defined by $C_{t0} = \Phi^{(t)}(z)^{-1}\Phi^{(0)}(z)$. Now if the parameters in the matrix system (4.26) are such that the conditions (4.11) are satisfied, we can prove that the connection matrix C_{t0} is either lower triangular or upper triangular.

As a consequence, the product between the monodromy matrix at $z = 0$, and $z = t$, will satisfy the following equation

$$\mathrm{Tr} M_0 M_t = 2\cos\pi(\hat{\theta}_0 + \hat{\theta}_t). \tag{4.30}$$

Let us now prove that the lower and upper triangular condition in the connection matrix leads to the equation (4.30). To do this, let us take a generic Fuchsian system. Such a system is analogous to the Hypergeometric system (4.26) and written as

$$\frac{\partial\Psi(z)}{\partial z} = \left(\frac{B_0}{z} + \frac{B_t}{z-t}\right)\Psi(z). \tag{4.31}$$

The system above has the solutions $\Psi^{(0)}(z)$ and $\Psi^{(t)}(z)$ around $z = 0$ and $z = t$, respectively, thus the generic monodromies around these points can be found via

$$\begin{aligned} \Psi^{(0)}(e^{2\pi i}z) =& \Psi^{(0)}(z)\mathcal{M}_0 \\ \Psi^{(t)}(e^{2\pi i}(z-t)+t) =& \Psi^{(t)}(z-t)\mathcal{M}_t. \end{aligned} \tag{4.32}$$

Via conjugation we can find a generic base whose monodromy matrices can be written in terms of a diagonal matrix. Thus, let us consider

$$\mathcal{M}_0 = C_0 D_0 C_0^{-1}, \qquad \mathcal{M}_t = C_t D_t C_t^{-1} \tag{4.33}$$

where D_0 and D_t are diagonal matrices. In that way, we choose the matrices D_0 and D_t to be in the following form

$$D_0 = \begin{pmatrix} e^{i\pi\eta_0} & 0 \\ 0 & e^{-i\pi\eta_0} \end{pmatrix} \qquad D_t = \begin{pmatrix} e^{i\pi\eta_t} & 0 \\ 0 & e^{-i\pi\eta_t} \end{pmatrix} \tag{4.34}$$

with the matrices satisfying the convenient conditions $\mathrm{Tr}(D_0) = 2\mathrm{cos}(\pi\eta_0)$, $\mathrm{Tr}(D_t) = 2\mathrm{cos}(\pi\eta_t)$, $\det D_0 = 1$, and $\det D_t = 1$. The trace of the product of M_0 and M_t is, therefore, written as

$$\begin{aligned} \mathrm{Tr}(\mathcal{M}_0\mathcal{M}_t) &= \mathrm{Tr}(C_0 D_0 C_0^{-1} C_t D_t C_t^{-1}) \\ &= \mathrm{Tr}(C_t^{-1} C_0 D_0 C_0^{-1} C_t D_t), \end{aligned} \tag{4.35}$$

where we used the cyclic property of the trace in the second line. The product of matrix $C_t^{-1}C_0$ in the trace has the interpretation of connection matrix between the fundamental solutions $\Psi^{(0)}(z)$ and $\Psi^{(t)}(z)$ and we define as $\bar{C}_{t0} = C_t^{-1}C_0$. Therefore, let us take the generic connection matrix as

$$\bar{C}_{t0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.36}$$

with the constants a, b, c and d functions of η_0 and η_t and $\det\bar{C}_{t0} \neq 0$. Replacing the matrix above in the trace equation and using (4.34) we have,

$$\mathrm{Tr}(\mathcal{M}_0\mathcal{M}_t) = \frac{2ad\,\cos(p\pi(\eta_0+\eta_t)) + 2bc\,\cos(p\pi(\eta_0-\eta_t))}{ad-bc}. \tag{4.37}$$

Given that, the $\bar{C}_{t0}$ can be lower triangular or upper triangular, this implies that b or c must be zero. Thus,

$$\mathrm{Tr}\mathcal{M}_0\mathcal{M}_t = 2\cos p\pi(\eta_0+\eta_t), \tag{4.38}$$

where from a generic case the relation above has the same form of (4.30).

Returning to the quantization discussion, the condition (4.30) is satisfied when the separation constant λ corresponds to the angular eigenvalue in the equation (4.8). Using the property (4.23) and the definition of $\hat{\sigma}$ from (4.24), we arrive at

$$\hat{\sigma}(\lambda) = \hat{\theta}_0 + \hat{\theta}_t + 2j, \qquad j \in \mathbb{Z}, \tag{4.39}$$

where we explicit the dependence on λ, but in fact, $\hat{\sigma}$ depends on all parameters in (4.8).

The condition (4.39) does not provide a full solution of the system, and we need to beware of the values of $\hat{\theta}_i$ since some $\hat{\theta}_i$ are related to non-normalizable solutions of the differential equation (4.8). Essentially, in the next section, we are going to use the quantization condition, and it will be clear from the context which values of $\hat{\theta}_i$ lead to the normalizable solutions.

4.2.2 τ-function and Painlevé V system

In this section and the next, we are going to define two conditions for the τ_V-function to help in the calculation of the accessory parameter.

Now we are ready to return to section 2.4, where we studied the isomonodromic deformation in Painlevé V. The first idea again is to see the parameter t in (4.12) as gauge parameter in the space of flat holomorphic connections $A(z,t)$. In this case, the system will give a deformation equation associated to the first line in the fundamental matrix solution, and to recover the differential equation (4.8), we take t to z_0. The advantage of this deformation stems from the fact that we can yet translate conditions, such as the quantization condition (4.11) in terms of gauge-invariant properties of (4.10), where in this case is saved in the monodromy data, $\hat{\rho}$.

We know from the Chapter 2 that the system must obey

$$\begin{aligned} \frac{\partial\Phi}{\partial z} = A(z,t)\Phi(z,t) = &\left(\frac{1}{2}\sigma_3 + \frac{A_0}{z} + \frac{A_t}{z-t}\right)\Phi(z,t), \\ \frac{\partial\Phi}{\partial t} = &-\frac{A_t}{z-t}\Phi(z,t), \end{aligned} \tag{4.40}$$

where the condition of the mixed derivative $\partial_z\partial_t\Phi = \partial_t\partial_z\Phi$, requires that A_0 and A_t satisfy the *Schlesinger equations*:

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0], \qquad \frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_t, A_0] - \frac{1}{2}[A_t, \sigma_3], \tag{4.41}$$

whose solution gives a one-parameter family of matrix systems with different values of t, but the same $\hat{\rho}$. From section 2.4, the equation (3.61) can be written as

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y = 0, \tag{4.42}$$

$$p(z) = \frac{1-\hat{\theta}_0}{z} + \frac{1-\hat{\theta}_t}{z-t} - \frac{1}{z-\lambda}, \qquad q(z) = -\frac{1}{4} + \frac{\hat{\theta}_\infty - 1}{2z} - \frac{tc_t}{z(z-t)} + \frac{\lambda\mu}{z(z-\lambda)}, \tag{4.43}$$

where the equation (3.54) is rewritten in the following form,

$$\mu^2 - \left[\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_t - 1}{\lambda - t}\right]\mu + \frac{\hat{\theta}_\infty - 1}{2\lambda} - \frac{tc_t}{\lambda(\lambda-t)} = \frac{1}{4}. \tag{4.44}$$

The equation above allows us to write c_t as function of λ, μ and t. As a result of the indicial equation, the algebraic condition (4.44) tells us that the singularity at $z = \lambda$ in (4.42) is apparent without logarithmic behavior. Then the monodromy matrix around $z = \lambda$ is trivial.

In order to recover the confluent Heun equation (4.8) and take into account the family of isomonodromic connections, it is necessary to consider the following choices,

$$\hat{\theta}_0 \to \hat{\theta}_0, \qquad \hat{\theta}_{t_0} \to \hat{\theta}_{t_0} - 1, \qquad \hat{\theta}_\infty \to \hat{\theta}_\infty + 1, \qquad \lambda(t_0) = t_0, \qquad \mu(t_0) = -\frac{c_{t_0}}{\hat{\theta}_{t_0} - 1}. \tag{4.45}$$

These choices bring the deformed confluent Heun equation (4.42) to (4.8). From (4.24), it is not difficult to show that $\hat{\sigma}$ has the following shift, $\hat{\sigma} \to \hat{\sigma} - 1$. These conditions are more conveniently written in terms of the τ_V-function defined in (3.41), where to leave the τ_V depending only on terms of monodromy data $\hat{\rho}$, we must take $\hat{A}_0 = A_0 - \frac{1}{2}\hat{\theta}_0\mathbb{I}$ and $\hat{A}_t = A_t - \frac{1}{2}\hat{\theta}_t\mathbb{I}$:

$$\frac{d}{dt}\log\tau_V(\hat{\rho};t) = \frac{1}{2}\mathrm{Tr}\sigma_3(A_t - \frac{1}{2}\hat{\theta}_t\mathbb{I}) + \frac{1}{t}\mathrm{Tr}(A_0 - \tfrac{1}{2}\hat{\theta}_0\mathbb{I})(A_t - \tfrac{1}{2}\hat{\theta}_t\mathbb{I}), \tag{4.46}$$

where the condition above is according with the Jimbo-Miwa-Ueno (JMU) τ_V-function defined in [43, 10]. Therefore, (4.45) leads to

$$\frac{d}{dt}\log\tau_V(\hat{\rho};t) = c_t + \frac{\hat{\theta}_0\hat{\theta}_t}{2t}, \qquad \frac{d}{dt}t\frac{d}{dt}\log\tau_V(\hat{\rho};t) + \frac{\hat{\theta}_t}{2} = 0. \tag{4.47}$$

The second condition (4.47) stems from the second derivative of the τ_V function, calculated using the Schlesinger equations and imposing (4.45). The left-hand side can be related through the *Toda equation* [65] to a product of τ_V-functions, such an equation will be proved soon. Thus,

$$\frac{d}{dt}t\frac{d}{dt}\log\tau_V(\hat{\rho};t) + \frac{\hat{\theta}_t}{2} = K_V\frac{\tau_V(\hat{\rho}^+;t)\tau_V(\hat{\rho}^-;t)}{\tau_V^2(\hat{\rho};t)}, \tag{4.48}$$

where K_V is independent of t and the $\hat{\rho}^\pm$ are related to $\hat{\rho}$ by the simple shifts,

$$\hat{\rho}^\pm = \{\hat{\theta}_0, \hat{\theta}_t \pm 1, \hat{\sigma} \pm 1, \hat{\theta}_\infty \mp 1; s_1, s_2\}. \tag{4.49}$$

Miwa's theorem [66] tells us that τ_V defined by (4.46) is analytic in t except at the critical points $t = 0$ and $t = \infty$. Therefore, either $\tau_V^+(\hat{\rho}^+; t)$ or $\tau_V^-(\hat{\rho}^-; t)$ has to vanish in (4.48).

The proof of (4.48) is a little laborious and straightforward: Initially, we consider a basis of solutions where A_t is diagonal,

$$\frac{\partial \Phi(z)}{\partial z}[\Phi(z)]^{-1} = A(z) = A_\infty + \frac{A_0}{z} + \frac{1}{z-t}\begin{pmatrix} \hat{\alpha}_t & 0 \\ 0 & \hat{\beta}_t \end{pmatrix}. \tag{4.50}$$

From the fundamental solution $\Phi(z)$, we define the derived solutions that are represented by

$$\Phi^+(z) = L^+(z)\Phi(z) = \begin{pmatrix} 1 & 0 \\ p^+ & 1 \end{pmatrix}\begin{pmatrix} z-t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q^+ \\ 0 & 1 \end{pmatrix}\Phi(z), \tag{4.51}$$

$$\Phi^-(z) = L^-(z)\Phi(z) = \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (z-t)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q^- & 1 \end{pmatrix}\Phi(z), \tag{4.52}$$

here we dropped the t dependence. From (4.50), $\Phi^\pm(z)$ satisfy

$$\frac{\partial \Phi^\pm}{\partial z}[\Phi^\pm(z)]^{-1} = A_\infty^\pm + \frac{A_0^\pm}{z} + \frac{1}{z-t}\begin{pmatrix} \hat{\alpha}_t \pm 1 & 0 \\ 0 & \hat{\beta}_t \end{pmatrix}. \tag{4.53}$$

Note that, $\Phi^\pm(z)$ is related to the possible shifts represented in (4.49). In terms of monodromy it is clear that the monodromy data of $\Phi^\pm(z)$ are related to that of $\Phi(z)$ by (4.51) and (4.52). Given $\Phi^\pm(z)$, one can establish the Toda equation (4.48) by comparing the corresponding expressions for each τ_V-function (4.46), and choosing $p^\pm$ and $q^\pm$ in order to keep the form of the new connection, defined through (4.12), mantain the partial fraction form at $z = t$ and $z = \infty$. The parameters $p^\pm$ and $q^\pm$ are determined from requirement that the transformation of A_∞ does not include term proportional to z and that the transformation of A_t is still diagonal. Let us start to work with $\Phi^+(z)$, replacing (4.51) in (4.53) we find

$$\frac{\partial \Phi^+}{\partial z}[\Phi^+(z)]^{-1} = \frac{\partial L(z)^+}{\partial z}(L^+(z))^{-1} + L^+(z)A(z)(L^+(z))^{-1}. \tag{4.54}$$

Defining the matrices A_∞^+ and A_0^+ as function of A_∞ and A_0 respectively,

$$A_i^+ = \begin{pmatrix} 1 & q^+ \\ 0 & 1 \end{pmatrix} A_i \begin{pmatrix} 1 & -q^+ \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & \tilde{d}_i \end{pmatrix}, \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad i = \infty, 0. \tag{4.55}$$

such that,

$$\begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & \tilde{d}_i \end{pmatrix} = \begin{pmatrix} a_i + q^+ c_i & b_i - (a_i - d_i)q^+ - c_i(q^+)^2 \\ c_i & d_i - q^+ c_i \end{pmatrix}, \tag{4.56}$$

after some algebra in the right-hand side of (4.54), we compare the equations (4.53) and (4.54) in order to find how the matrices A_∞^+, A_0^+, and A_t^+ are written. Thus,

$$A_\infty^+ = \begin{pmatrix} 1 & 0 \\ p^+ & 1 \end{pmatrix}\begin{pmatrix} \tilde{a}_\infty & 0 \\ 0 & \tilde{d}_\infty \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p^+ & 1 \end{pmatrix} - (z-t)\tilde{b}_\infty \begin{pmatrix} p^+ & -1 \\ (p^+)^2 & -p^+ \end{pmatrix} + \frac{1}{z-t}\begin{pmatrix} 0 & 0 \\ \tilde{c}_\infty & 0 \end{pmatrix}, \tag{4.57}$$

$$\frac{1}{z}A_0^+ = \frac{1}{z}\begin{pmatrix}1 & 0\\ p^+ & 1\end{pmatrix}\begin{pmatrix}\tilde{a}_0 & -\tilde{b}_0 t\\ -\frac{\tilde{c}_0}{t} & \tilde{d}_0\end{pmatrix}\begin{pmatrix}1 & 0\\ -p^+ & 1\end{pmatrix} - \tilde{b}_0\begin{pmatrix}p^+ & -1\\ (p^+)^2 & -p^+\end{pmatrix} + \frac{1}{z-t}\begin{pmatrix}0 & 0\\ \frac{\tilde{c}_0}{t} & 0\end{pmatrix}, \tag{4.58}$$

$$\frac{1}{z-t}A_t^+ = \frac{1}{z-t}\begin{pmatrix}\hat{\alpha}_t & 0\\ 0 & \hat{\beta}_t\end{pmatrix} + q^+(\hat{\alpha}_t - \hat{\beta}_t)\begin{pmatrix}p^+ & -1\\ (p^+)^2 & -p^+\end{pmatrix} + \frac{1}{z-t}\begin{pmatrix}0 & 0\\ p^+(\hat{\alpha}_t - \hat{\beta}_t) & 0\end{pmatrix}. \tag{4.59}$$

Adding these three terms above with the first term in (4.54) given by

$$\frac{\partial L^+(z)}{\partial z}L^+(z) = \frac{1}{z-t}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{z-t}\begin{pmatrix}0 & 0\\ p^+ & 0\end{pmatrix}, \tag{4.60}$$

we find the constraint that the extra terms must satisfy to recover the system. Therefore, from A_∞^+, the term proportional to z must vanish which implies $\tilde{b}_\infty = 0$, the extra terms proportional to $(z-t)^{-1}$ also must be canceled, then the last terms in A_∞^+, A_0^+, A_t^+, and the equation (4.60) lead to following constraint, $p^+(\hat{\alpha}_t - \hat{\beta}_t + 1) = -\tilde{c}_0/t - \tilde{c}_\infty$. Using the first contraint we find the explicit form of q^+:

$$q^+ = \frac{\hat{\alpha}_\infty - a_\infty}{c_\infty}, \qquad q^+ = -\frac{\hat{\beta}_\infty - d_\infty}{c_\infty}, \tag{4.61}$$

where $\hat{\alpha}_\infty$ and $\hat{\beta}_\infty$ are eingevalues of A_∞. Replacing q^+ in the definition (4.56) we have $\tilde{a}_\infty = \hat{\alpha}_\infty$, $\tilde{d}_\infty = \hat{\beta}_\infty$, and $\tilde{c}_\infty = c_\infty$. Therefore, A_t will keep the form in (4.53) with the matrices A_0^+ and A_t^+ written as

$$A_0^+ = \begin{pmatrix}1 & 0\\ p^+ & 1\end{pmatrix}\begin{pmatrix}\tilde{a}_0 & -\tilde{b}_0 t\\ -\tilde{c}_0/t & \tilde{d}_0\end{pmatrix}\begin{pmatrix}1 & 0\\ -p^+ & 1\end{pmatrix}, \tag{4.62}$$

$$A_\infty^+ = \begin{pmatrix}1 & 0\\ p^+ & 1\end{pmatrix}\begin{pmatrix}\hat{\alpha}_\infty & \tilde{b}_0 - q^+(\hat{\alpha}_t - \hat{\beta}_t)\\ 0 & \hat{\beta}_\infty\end{pmatrix}\begin{pmatrix}1 & 0\\ -p^+ & 1\end{pmatrix}. \tag{4.63}$$

The calculation for decreasing the value of $\hat{\alpha}_t$ is entirely analogous. Starting from the original linear system, we have

$$\Phi^-(z) = L^-(z)\Phi(z) = \begin{pmatrix}1 & p^-\\ 0 & 1\end{pmatrix}\begin{pmatrix}(z-t)^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ q^- & 1\end{pmatrix}\Phi(z), \tag{4.64}$$

that satisfies,

$$\frac{\partial \Phi^-}{\partial z}[\Phi^-(z)]^{-1} = A^- + \frac{A_0^-}{z} + \frac{1}{z-t}\begin{pmatrix}\hat{\alpha}_t - 1 & 0\\ 0 & \hat{\beta}_t\end{pmatrix}. \tag{4.65}$$

Again we define the matrices A_∞^- and A_0^- as function of A_∞ and A_0, respectively

$$A_i^- = \begin{pmatrix}1 & 0\\ q^- & 1\end{pmatrix}A_i\begin{pmatrix}1 & 0\\ -q^- & 1\end{pmatrix} = \begin{pmatrix}\tilde{a}_i & \tilde{b}_i\\ \tilde{c}_i & \tilde{d}_i\end{pmatrix}, \qquad i = \infty, 0. \tag{4.66}$$

$$\begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & \tilde{d}_i \end{pmatrix} = \begin{pmatrix} a_i - q^- c_i & b_i \\ c_i + (a_i - d_i)q^- - c_i(q^-)^2 & d_i + q^- c_i \end{pmatrix}. \tag{4.67}$$

In this case, the matrices A_∞^-, A_t^-, A_0^-, and the extra term are given directly by

$$A_\infty^- = \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a}_\infty & 0 \\ 0 & \tilde{d}_\infty \end{pmatrix} \begin{pmatrix} 1 & -p^- \\ 0 & 1 \end{pmatrix} - (z-t)\tilde{c}_\infty \begin{pmatrix} -p^- & (p^-)^2 \\ -1 & p^- \end{pmatrix} + \frac{\tilde{b}_\infty}{z-t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.68}$$

$$\frac{A_t^-}{z-t} = \frac{1}{z-t} \begin{pmatrix} \hat{\alpha}_t & 0 \\ 0 & \hat{\beta}_t \end{pmatrix} - q^-(\hat{\alpha}_t - \hat{\beta}_t) \begin{pmatrix} -p^- & (p^-)^2 \\ -1 & p^- \end{pmatrix} + \frac{1}{z-t} \begin{pmatrix} 0 & -p^-(\hat{\alpha}_t - \hat{\beta}_t) \\ 0 & 0 \end{pmatrix}, \tag{4.69}$$

$$\frac{A_0^-}{z} = \frac{1}{z} \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a}_0 & -\tilde{b}_0/t \\ -\tilde{c}_0 t & \tilde{b}_0 \end{pmatrix} \begin{pmatrix} 1 & -p^- \\ 0 & 1 \end{pmatrix} - \tilde{c}_0 \begin{pmatrix} -p^- & (p^-)^2 \\ -1 & p^- \end{pmatrix} + \frac{\tilde{b}_0/t}{z-t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.70}$$

$$\frac{\partial L^-}{\partial z}[L^-]^{-1} = \frac{1}{z-t} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{z-t} \begin{pmatrix} 0 & p^- \\ 0 & 0 \end{pmatrix}. \tag{4.71}$$

Since the term $(z-t)$ must vanish to keep the form of the system, we must take $\tilde{c}_\infty = 0$ which leads to the analogous result for q^+:

$$q^- = -\frac{\hat{\alpha}_\infty - a_\infty}{b_\infty}, \qquad q^- = \frac{\hat{\beta}_\infty - d_\infty}{b_\infty}, \tag{4.72}$$

as well as the constraint $p^-(\hat{\alpha}_t - \hat{\beta}_t - 1) = -\tilde{b}_0/t - \tilde{b}_\infty$, that cancel the terms $(z-t)^{-1}$. Thus, the matrix A_0^- and A_t^- are expressed as

$$A_0^- = \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a}_0 & -\tilde{b}_0/t \\ -\tilde{c}_0 t & \tilde{b}_0 \end{pmatrix} \begin{pmatrix} 1 & -p^- \\ 0 & 1 \end{pmatrix}, \tag{4.73}$$

$$A_\infty^- \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_\infty & 0 \\ \tilde{c}_0 + q^-(\hat{\alpha}_t - \hat{\beta}_t) & \hat{\beta}_\infty \end{pmatrix} \begin{pmatrix} 1 & -p^- \\ 0 & 1 \end{pmatrix}. \tag{4.74}$$

Now we are ready to find the Toda equation. Using the definition of isomonodromy flow (3.41) we write the Hamiltonians of this new system as

$$H^\pm = \frac{d}{dt}\log\tau_V^\pm(t) = \mathrm{Tr}A_\infty^\pm A_t^\pm + \frac{1}{t}\mathrm{Tr}A_0^\pm A_t^\pm. \tag{4.75}$$

Here, replacing the matrices to the system $\Phi^\pm(z)$ and using $q^\pm$, we find the following Hamiltonians:

$$H^+ = \hat{\alpha}_\infty + \frac{1}{t}a_0 + \frac{1}{t}c_0 q^+ + \hat{\alpha}_t\left(a_\infty + \frac{a_0}{t}\right) + \hat{\beta}_t\left(d_\infty + \frac{d_0}{t}\right), \tag{4.76}$$

$$H^{-} = -\hat{\alpha}_\infty - \frac{1}{t}a_0 + \frac{1}{t}b_0 q^{-} + \hat{\alpha}_t\left(a_\infty + \frac{a_0}{t}\right) + \hat{\beta}_t\left(d_\infty + \frac{d_0}{t}\right). \tag{4.77}$$

It is not difficult to prove that the last two terms in the equations above is related to the isomonodromy flow of the original system. Thus, we write the equations as

$$\begin{aligned} H^{+} - H &= \hat{\alpha}_\infty + \frac{1}{t}a_0 + \frac{1}{t}c_0 q^{+}, \\ H^{+} - H &= -\hat{\alpha}_\infty - \frac{1}{t}a_0 + \frac{1}{t}b_0 q^{-}. \end{aligned} \tag{4.78}$$

Let us take the derivate of the Hamiltonian of the original system, which is expressed as

$$H = \frac{d}{dt}\log\tau_V(\hat{\rho}, t) = \mathrm{Tr}A_\infty A_t + \frac{1}{t}\mathrm{Tr}A_0 A_t. \tag{4.79}$$

We have directly

$$\frac{d}{dt}t\frac{d}{dt}\log\tau_V(\hat{\rho}, t) = \mathrm{Tr}A_\infty A_t = a_\infty\hat{\alpha}_t + d_\infty\hat{\beta}_t. \tag{4.80}$$

Where we use the definition of A_t and A_∞ from (4.55) and A_t in (4.50). Taking a second derivate in (4.80) and then using the Schlesinger equations defined in (4.41), we arrive at,

$$\frac{d^2}{dt^2}t\frac{d}{dt}\log\tau_V(\hat{\rho}, t) = \frac{1}{t}\mathrm{Tr}(A_\infty[A_0, A_t]) = -\frac{1}{t}(\hat{\alpha}_t - \hat{\beta}_t)(b_0 c_\infty - c_0 b_\infty). \tag{4.81}$$

Using the equations for q^{+}, q^{-}, (4.75), and (4.78) we find the relations between the second and first derivate on t,

$$\frac{d^2}{dt^2}t\frac{d}{dt}\log\tau_V(\hat{\rho}, t) = (\hat{\alpha}_t - \hat{\beta}_t)\frac{b_\infty c_\infty}{\hat{\alpha}_\infty - a_\infty}\frac{d}{dt}\log\frac{\tau_V^{+}(\hat{\rho}^{+}, t)\tau_V^{-}(\hat{\rho}^{-}, t)}{\tau_V(\hat{\rho}, t)}. \tag{4.82}$$

After a little algebra we can also prove the useful relations

$$\begin{aligned} (\hat{\alpha}_t - \hat{\beta}_t)\frac{b_\infty c_\infty}{\hat{\alpha}_\infty - a_\infty} &= (\hat{\alpha}_t - \hat{\beta}_t)(a_\infty - \hat{\beta}_\infty) \\ (\hat{\alpha}_t - \hat{\beta}_t)(a_\infty - \hat{\beta}_\infty) &= \frac{d}{dt}t\frac{d}{dt}\log\tau_V(\hat{\rho}, t) - (\hat{\alpha}_\infty + \hat{\beta}_\infty)\hat{\beta}_t - \hat{\beta}_\infty(\hat{\alpha}_t - \hat{\beta}_t). \end{aligned} \tag{4.83}$$

where we finally stablish the Toda equation for the τ_V-function of Painlevé V by replacing the equations above in (4.82)and then integrating on t,

$$\frac{d}{dt}t\frac{d}{dt}\log\tau_V(\hat{\rho}, t) - \hat{\alpha}_\infty\hat{\beta}_t - b_\infty\hat{\alpha}_t = K_V\frac{\tau_V^{+}(\hat{\rho}^{+}, t)\tau_V^{-}(\hat{\rho}^{-}, t)}{\tau_V^2(\hat{\rho}, t)} \tag{4.84}$$

with K_V, as it was defined, a constant independent of t.

4.2.3 Convenient parameterization for the linear system

In the previous section, it was convenient to parametrize the linear system in such a way that A_t was diagonal. However, this is not the most common parameterization for such system; rather it is useful to use the following parameterization

$$A_\infty = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \bar{A}_i = \begin{pmatrix} \bar{a}_i & \bar{b}_i \\ \bar{c}_i & \bar{d}_i \end{pmatrix} = \begin{pmatrix} p_i + \hat{\theta}_i & -q_i p_i \\ \frac{1}{q_i}(p_i + \hat{\theta}_i) & -p_i \end{pmatrix}, \quad i = 0, t, \tag{4.85}$$

where the bar reminds us that, in the basis used above, $\bar{A}_t$ is not diagonal. In order to diagonalize it, we use

$$A_t = G_t \begin{pmatrix} \hat{\theta}_t & 0 \\ 0 & 0 \end{pmatrix} G_t^{-1}, \quad G_t = \begin{pmatrix} q_t & 1 \\ 1 & \frac{p_t + \hat{\theta}_t}{q_t p_t} \end{pmatrix}, \tag{4.86}$$

which satisfy the constraint (4.20). The last constraint in (4.20) can be written as

$$p_0 + p_t = -\frac{\hat{\theta}_0 + \hat{\theta}_t + \hat{\theta}_\infty}{2}, \tag{4.87}$$

where we observe that such constraint reduces the number of free parameters to 3. Conjugating with G_t, we have the matrix A_∞ used in the last section,

$$A_\infty = \frac{1}{2\hat{\theta}_t}\begin{pmatrix} \bar{a}_t - \bar{d}_t & 2\bar{c}_t \\ 2\bar{b}_t & -\bar{a}_t + \bar{d}_t \end{pmatrix} \tag{4.88}$$

The matrix A_0 is also computed straightforwardly. The form of A_∞ is enough to stablish the divergence of $\tau^+(\hat{\rho}^+, t)$. Where we have to q^+ and q^-,

$$q^+ = \frac{\hat{\alpha}_\infty - a_\infty}{c_\infty} = \frac{\bar{d}_t}{\bar{b}_t}, \quad q^- = -\frac{\hat{\alpha}_\infty - a_\infty}{b_\infty} = -\frac{\bar{d}_t}{\bar{c}_t}, \tag{4.89}$$

which for the boundary condition $\lambda = t$ or $\bar{b}_t = 0$ means that q^+ will diverge. In terms of entries, we have

$$H^+ - H = \frac{1}{2} + \frac{1}{t}\left(\bar{a}_0 + \frac{\bar{b}_0}{\bar{b}_t}\bar{d}_t\right), \qquad H^- - H = -\frac{1}{2} - \frac{1}{t}\left(\bar{a}_0 + \frac{\bar{c}_0}{\bar{c}_t}\bar{d}_t\right). \tag{4.90}$$

We now introduce the position of the apparent singularity λ and the canonically conjugate parameter μ,

$$\lambda = \frac{t\bar{b}_0}{\bar{b}_0 + \bar{b}_t}, \qquad \mu = \frac{1}{2} + \frac{\bar{a}_0}{\lambda} + \frac{\bar{a}_t}{\lambda - t}. \tag{4.91}$$

Therefore, by replacing λ and μ the equations (4.90) are written as,

$$H^+ - H = \frac{1}{2} + \frac{\lambda}{t}\left(\mu - \frac{1}{2}\right) - \frac{\lambda}{t(\lambda - t)}\hat{\theta}_t \tag{4.92}$$

$$H^{-}-H=-\frac{1}{2}+\frac{(\lambda-t)\big(\mu-\frac{1}{2}\big)-\frac{1}{2}(\hat{\theta}_0+\hat{\theta}_t-\hat{\theta}_\infty)}{\lambda\big(\mu-\frac{1}{2}\big)-\frac{1}{2}(\hat{\theta}_0+\hat{\theta}_t-\hat{\theta}_\infty)}\Big(\frac{\lambda}{t}\big(\mu-\frac{1}{2}\big)-\frac{\hat{\theta}_0}{t}\Big). \tag{4.93}$$

After more algebraic manipulations and using the isomonodromy flow definition, we show that,

$$\frac{d}{dt}\log\frac{\tau_V^{+}(\hat{\rho}^{+};t)}{\tau_V(\hat{\rho};t)}=-\frac{1}{2}-\frac{\lambda}{t}\left(\mu-\frac{1}{2}\right)+\frac{\lambda}{t(\lambda-t)}\hat{\theta}_t \tag{4.94}$$

$$\frac{d}{dt}\log\frac{\tau_V^{-}(\hat{\rho}^{-};t)}{\tau_V(\hat{\rho};t)}=\frac{1}{2}-\frac{(\lambda-t)\left(\mu-\frac{1}{2}\right)-\frac{1}{2}(\hat{\theta}_0+\hat{\theta}_t-\hat{\theta}_\infty)}{\lambda\left(\mu-\frac{1}{2}\right)-\frac{1}{2}(\hat{\theta}_0+\hat{\theta}_t-\hat{\theta}_\infty)}\left(\frac{\lambda}{t}\left(\mu-\frac{1}{2}\right)-\frac{\hat{\theta}_0}{t}\right). \tag{4.95}$$

Given that the first line has a divergent limit $\lambda\to t$, we conclude by taking the integration on t and then taking the limit that we can substitute the second condition in (4.47) by the simpler one

$$\tau_V(\hat{\rho};t_0)=0. \tag{4.96}$$

Where the monodromy data is that of (4.8):

$$\hat{\rho}=\{\hat{\theta}_0,\hat{\theta}_{t_0},\hat{\sigma},\hat{\theta}_\infty;s_1,s_2\}, \tag{4.97}$$

here, we make explicit the dependence in $\hat{\sigma}$ from the quantization condition. Thus, in terms of monodromy data, the first condition in (4.47) is given by

$$c_{t_0}=\frac{d}{dt}\log\tau_V(\hat{\rho}^{-};t_0)-\frac{\hat{\theta}_0(\hat{\theta}_{t_0}-1)}{2t_0}. \tag{4.98}$$

with the shift in $\hat{\rho}^{-}$ defined in (4.49).

4.2.4 Accessory parameter for the confluent Heun equation

Primarily, the accessory parameter c_{t_0} depends on the derivate of τ_V-function, thus, to find the explicit expansion of c_{t_0}, it is necessary to find the root of the JMU τ_V-function in (4.96), then replace the value of this root and considering the shifts defined in $\hat{\rho}^{-}$ inside the logarithm argument.

We know from Chapter 2 that the τ_V-function is written by using the formalism of partition functions with $c=1$ in the CFT, thus τ_V-function is given from (3.65) by

$$\tau_V(\hat{\rho};t)=\sum_n\tilde{C}(\{\hat{\theta}\},\hat{\sigma}+2n)s_V^n t^{\frac{1}{4}(\hat{\sigma}+2n)^2-\frac{1}{4}(\hat{\theta}_0^2+\hat{\theta}_t^2)}\mathcal{D}\big(\hat{\theta}_0,\hat{\theta}_t,\hat{\theta}_\infty,\hat{\sigma}+2n;t\big),$$

$$\tilde{C}(\{\hat{\theta}\},\hat{\sigma})=\prod_{\epsilon=\pm}\frac{G(1+\frac{1}{2}(\epsilon\hat{\sigma}-\hat{\theta}_\infty))G(1+\frac{1}{2}(\hat{\theta}_t+\hat{\theta}_0+\epsilon\hat{\sigma}))G(1+\frac{1}{2}(\hat{\theta}_t-\hat{\theta}_0+\epsilon\hat{\sigma}))}{G(1+\epsilon\hat{\sigma})}, \tag{4.99}$$

with $\{\hat{\theta}\} = \{\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty\}$, G-Barnes function satisfying $G(z+1) = \Gamma(z)G(z)$, and the confluent conformal block of the first kind, $\mathcal{D}\big(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty, \hat{\sigma}; t\big)$ defined in (3.66). Note that, we define $\tilde{C}$ from (3.67) just to keep the explicit dependence of τ_V in terms of s_V.

To analyze the asymptotic behavior and to check with the literature, let us write (4.99) as

$$\tau_V(\hat{\rho}; t) = \tilde{C}(\{\hat{\theta}\}, \hat{\sigma}) t^{\frac{1}{4}(\hat{\sigma}^2 - \hat{\theta}_0^2 - \hat{\theta}_t^2)} e^{-\frac{1}{2}\hat{\theta}_t t} \hat{\tau}_V(\hat{\rho}; t). \tag{4.100}$$

where $\hat{\tau}_V(\hat{\rho}; t)$ is

$$\begin{aligned} \hat{\tau}_V(\hat{\rho}; t) &= \sum_n \bar{C}(\{\hat{\theta}\}, \hat{\sigma} + 2n) s_V^n t^{n\hat{\sigma} + n^2} \mathcal{D}\big(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty, \hat{\sigma} + 2n; t\big), \\ \bar{C}(\{\hat{\theta}\}, \hat{\sigma} + 2n) &= \frac{\tilde{C}(\{\hat{\theta}\}, \hat{\sigma} + 2n)}{\tilde{C}(\{\hat{\theta}\}, \hat{\sigma})}, \end{aligned} \tag{4.101}$$

with $\hat{\tau}_V(\hat{\rho}; t)$ just involving the combinatorial expansion of the confluent CB of first kind and ratios of gammas. The asymptotics of $\hat{\tau}_V(\hat{\rho}; t)$ is the same when we compare with [43] - replacing $\hat{\sigma}$ for σ:

$$\begin{aligned} \hat{\tau}_V(\hat{\rho}; t) = 1 &+ \left(\frac{\hat{\theta}_t}{2} - \frac{\hat{\theta}_\infty}{4} - \frac{\hat{\theta}_\infty(\hat{\theta}_0^2 - \hat{\theta}_t^2)}{4\hat{\sigma}^2} \right) t \\ &+ \frac{(\hat{\sigma} - \hat{\theta}_\infty)((\hat{\sigma} + \hat{\theta}_t)^2 - \hat{\theta}_0^2)}{8\hat{\sigma}^2(\hat{\sigma} - 1)^2} \hat{s}^{-1} t^{1-\hat{\sigma}} + \frac{(\hat{\sigma} + \hat{\theta}_\infty)((\hat{\sigma} - \hat{\theta}_t)^2 - \hat{\theta}_0^2)}{8\hat{\sigma}^2(\hat{\sigma} + 1)^2} \hat{s} t^{1+\hat{\sigma}} + \mathcal{O}(t^2, t^{2 \pm 2\Re\hat{\sigma}}), \end{aligned} \tag{4.102}$$

where the parameter $\hat{s}$ in the equation above is related to s_V by a string of gamma functions

$$\hat{s} = \frac{\Gamma^2(1-\hat{\sigma})\Gamma(1+\frac{1}{2}(\hat{\sigma} - \hat{\theta}_\infty))\Gamma(1+\frac{1}{2}(\hat{\theta}_t + \hat{\theta}_0 + \hat{\sigma}))\Gamma(1+\frac{1}{2}(\hat{\theta}_t - \hat{\theta}_0 + \hat{\sigma}))}{\Gamma^2(1+\hat{\sigma})\Gamma(1-\frac{1}{2}(\hat{\sigma} + \hat{\theta}_\infty))\Gamma(1+\frac{1}{2}(\hat{\theta}_t + \hat{\theta}_0 - \hat{\sigma}))\Gamma(1+\frac{1}{2}(\hat{\theta}_t - \hat{\theta}_0 - \hat{\sigma}))} s_V. \tag{4.103}$$

In order to find explictly the accessory parameter expansion, let us define in (4.101) the variable $X(\hat{\sigma}; t) = s_V t^{\hat{\sigma}}$. $\hat{\tau}_V$ is meromorphic in the variables X and t, then we can use such property and the second condition for τ_V (4.96) in order to invert the function to write X as a function of $\hat{\theta}$'s, $\hat{\sigma}$ and t_0. From (4.96) and (4.100), it is easy to see that to invert the series we just need to work with (4.101). Thus,

$$\hat{\tau}_V(\hat{\rho}; t_0) = \sum_n \bar{C}(\{\hat{\theta}\}, \hat{\sigma} + 2n) X^n(\hat{\sigma}; t_0) t_0^{n^2} \mathcal{D}\big(\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty, \hat{\sigma} + 2n; t_0\big) = 0. \tag{4.104}$$

To find the terms of the accessory parameter expansion is necessary to define the value of n and the number of terms in the confluent CB of the first kind. To simplify the calculations, we express in this dissertation the first three terms in the expansion of c_{t_0}; however, computationally we can get more terms. To compare with the literature, we will use, in the next section, the first seven terms to express the λ expansion. To find these terms, we take $n = 0, \pm1, \pm2, \pm3$ and the confluent CB expansion of first kind goes up to t_0^8. With these choices, we ensure all contributions of t_0 in the expansion. Expanding the terms in (4.104) and in order to write $X(\hat{\sigma}, t_0)$ as a function of the $\hat{\theta}$'s, we can consider $X(\hat{\sigma}; t)$ as an perturbation expansion,

$$X(\hat{\sigma}, t_0) = \sum_{k=0} \chi_k t_0^k \tag{4.105}$$

where it is possible to find each term of $X(\hat{\sigma}, t_0)$ by using the second condition of τ_V. After some algebra, the explicit form of the first three terms are

$$X(\{\theta\}, \hat{\sigma}; t_0) = \frac{(\hat{\sigma} - \hat{\theta}_\infty)((\hat{\sigma} + \hat{\theta}_{t_0})^2 - \hat{\theta}_0^2)}{8\hat{\sigma}^2(\hat{\sigma} - 1)^2} t_0 \left(1 + \chi_1 t_0 + \chi_2 t_0^2 + \mathcal{O}(t_0^3)\right) \tag{4.106a}$$

with,

$$\chi_1 = (\hat{\sigma} - 1)\frac{\hat{\theta}_\infty(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)}{\hat{\sigma}^2(\hat{\sigma} - 2)^2}, \tag{4.106b}$$

and

$$\begin{aligned}\chi_2 = \frac{\hat{\theta}_\infty^2(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^2}{64} &\left(\frac{5}{\hat{\sigma}^4} - \frac{1}{(\hat{\sigma} - 2)^4} - \frac{2}{(\hat{\sigma} - 2)^2} + \frac{2}{\hat{\sigma}(\hat{\sigma} - 2)}\right) \\ &- \frac{(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^2 + 2\hat{\theta}_\infty^2(\hat{\theta}_0^2 + \hat{\theta}_{t_0}^2)}{64}\left(\frac{1}{\hat{\sigma}^2} - \frac{1}{(\hat{\sigma} - 2)^2}\right) \\ + \frac{(1 - \hat{\theta}_\infty^2)((\hat{\theta}_0 - 1)^2 - \hat{\theta}_{t_0}^2)((\hat{\theta}_0 + 1)^2 - \hat{\theta}_{t_0}^2)}{128} &\left(\frac{1}{(\hat{\sigma} + 1)^2} - \frac{1}{(\hat{\sigma} - 3)^2}\right),\end{aligned} \tag{4.106c}$$

where, we just express the first three terms of $X(\{\hat{\theta}\}, \hat{\sigma}, t_0)$.

To use (4.98) and find the accessory parameter, we must shift the monodromy parameters by one unit. A simple calculation using (4.103) yields to

$$X(\hat{\rho}^-; t) = \frac{8\hat{\sigma}^2(\hat{\sigma} - 1)^2}{(\hat{\sigma} - \hat{\theta}_\infty)((\hat{\sigma} + \hat{\theta}_t)^2 - \hat{\theta}_0^2)t} X(\hat{\rho}; t). \tag{4.107}$$

Now by using (4.98)

$$c_{t_0} = \frac{(\hat{\sigma} - 1)^2 - (\hat{\theta}_0 + \hat{\theta}_{t_0} - 1)^2}{4t_0} - \frac{\hat{\theta}_{t_0} - 1}{2} + \frac{d}{dt}\log\hat{\tau}(\hat{\rho}^-; t_0), \tag{4.108}$$

and expanding the $\hat{\tau}_V$ term, we find the expansion for the accessory parameter

$$t_0 c_{t_0} = k_0 + k_1 t_0 + k_2 t_0^2 + k_3 t_0^3 + \ldots k_n t_0^n + \ldots, \tag{4.109a}$$

with the three first terms in the expansion, given by

$$k_0 = \frac{(\hat{\sigma}-1)^2 - (\hat{\theta}_0 + \hat{\theta}_{t_0} - 1)^2}{4}, \tag{4.109b}$$

$$k_1 = -\frac{\hat{\theta}_\infty(\hat{\sigma}(\hat{\sigma}-2) - \hat{\theta}_0^2 + \hat{\theta}_{t_0}^2)}{4\hat{\sigma}(\hat{\sigma}-2)}, \tag{4.109c}$$

$$\begin{aligned} k_2 = \frac{1}{32} + \frac{\hat{\theta}_\infty^2(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^2}{64}\left(\frac{1}{\hat{\sigma}^3} - \frac{1}{(\hat{\sigma}-2)^3}\right) + \frac{(1-\hat{\theta}_\infty^2)(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^2 + 2\hat{\theta}_\infty^2(\hat{\theta}_0^2 + \hat{\theta}_{t_0}^2)}{32\hat{\sigma}(\hat{\sigma}-2)} \\ - \frac{(1-\hat{\theta}_\infty^2)((\hat{\theta}_0 - 1)^2 - \hat{\theta}_{t_0}^2)((\hat{\theta}_0+1)^2 - \hat{\theta}_{t_0}^2)}{32(\hat{\sigma}+1)(\hat{\sigma}-3)}, \end{aligned} \tag{4.109d}$$

$$\begin{aligned} k_3 = \frac{\hat{\theta}_\infty^3(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^3}{256}\left(\frac{1}{(\hat{\sigma}-2)^5} - \frac{1}{\hat{\sigma}^5}\right) + \\ \frac{4(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)^3\hat{\theta}_\infty - \big(5(\hat{\theta}_0^6 - \hat{\theta}_{t_0}^6) + 8\hat{\theta}_{t_0}^4 + 15\hat{\theta}_0^2\hat{\theta}_{t_0}^4 - \hat{\theta}_0^4(8 + 15\hat{\theta}_{t_0}^2)\big)\hat{\theta}_\infty^3}{1024} \\ \left(\frac{1}{\hat{\sigma}^3} - \frac{1}{(\hat{\sigma}-2)^3}\right) + \frac{1}{24576}(\hat{\theta}_{t_0}^2 - \hat{\theta}_0^2)\hat{\theta}_\infty\Bigg(64 + 80\hat{\theta}_\infty^2 + 8\hat{\theta}_{t_0}^2(20 - 29\hat{\theta}_\infty^2) + \\ (\hat{\theta}_0^4 + \hat{\theta}_{t_0}^4)(125\hat{\theta}_\infty^2 - 116) + \hat{\theta}_0^2\big(160 - 232\hat{\theta}_\infty^2 + \hat{\theta}_{t_0}^2(232 - 250\hat{\theta}_\infty^2)\big)\Bigg)\left(\frac{1}{\hat{\sigma}} - \frac{1}{\hat{\sigma}-2}\right) \\ - \frac{((-1+\hat{\theta}_0)^2 - \hat{\theta}_{t_0}^2)(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)((1+\hat{\theta}_0)^2 - \hat{\theta}_{t_0}^2)\hat{\theta}_\infty(1-\hat{\theta}_\infty^2)}{96(3-\hat{\sigma})(1+\hat{\sigma})} \\ + \frac{((-2+\hat{\theta}_0)^2 - \hat{\theta}_{t_0}^2)(\hat{\theta}_0^2 - \hat{\theta}_{t_0}^2)((2+\hat{\theta}_0)^2 - \hat{\theta}_{t_0}^2)\hat{\theta}_\infty(4-\hat{\theta}_\infty^2)}{4096(4-\hat{\sigma})(2+\hat{\sigma})}. \end{aligned} \tag{4.109e}$$

In the calculations above we assumed $\Re\hat{\sigma} > 0$, the corresponding expression for $\Re\hat{\sigma} < 0$ can be obtained by sending $\hat{\sigma} \to -\hat{\sigma}$. The higher order terms become increasingly complicated and we have the structure where the term k_n is a rational function of the monodromy parameters.

It should be emphasized that (4.96) and (4.98) are exact relations, even though their usefulness stems from the ability to compute the τ_V function for Painlevé V efficiently. Miwa's theorem [66] shows that the τ_V function is analytic in the whole complex plane except at $t = 0$ and $t = \infty$. Thus, the expansion (4.99) has an infinite radius of convergence, even if it becomes exponentially hard to compute the higher order coefficients in the expansion, due to their combinatorial nature.

4.2.5 The spheroidal harmonics eigenvalues

Let us now to express the first seven terms of the separation constant by using the expansion of the accessory parameter (4.109a). In angular case, the monodromies parameters and $\hat{\sigma}$ which is associated with the quantization condition (4.39) are written as

$$\hat{\theta}_0 = -m - s, \qquad \hat{\theta}_{t_0} = m - s, \qquad \hat{\theta}_\infty = 2s, \qquad t_0 = -4a\omega, \qquad \hat{\sigma} = -2s + 2j. \quad (4.110)$$

Replacing the parameters and using the equations (4.9), we find the expansion of the angular eigenvalue expansion ${}_s\lambda_{\ell,m}(a\omega)$,

$${}_s\lambda_{\ell,m}(a\omega) = \sum_{n=0}^{\infty} f_n (a\omega)^n, \quad (4.111)$$

where in order to express the dependence on s, l and m, we are denoting the separation constant as ${}_s\lambda_{lm}$. We also define a useful function $h(\ell)$ as

$$h(\ell) = \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell - 1)\ell^3(2\ell + 1)}.$$

Thus, the first seven coefficients are the following

$$\begin{aligned}
f_0 &= (\ell - s)(\ell + s + 1), \\
f_1 &= -\frac{2ms^2}{\ell(\ell+1)}, \\
f_2 &= h(l+1) - h(l) - 1, \\
f_3 &= \frac{2h(l)ms^2}{(l-1)l^2(l+1)} - \frac{2h(l+1)ms^2}{l(l+1)^2(l+2)}, \\
f_4 &= m^2 s^4 \left(\frac{4h(\ell+1)}{\ell^2(\ell+1)^4(\ell+2)^2} - \frac{4h(\ell)}{(\ell-1)^2\ell^4(\ell+1)^2} \right) - \frac{(\ell+2)h(\ell+1)h(\ell+2)}{2(\ell+1)(2\ell+3)} \\
&\quad + \frac{h^2(\ell+1)}{2(\ell+1)} + \frac{h(\ell)h(\ell+1)}{2\ell^2+2\ell} - \frac{h^2(\ell)}{2\ell} + \frac{(\ell-1)h(\ell-1)h(\ell)}{4\ell^2-2}, \\
f_5 &= m^3 s^6 \left(\frac{8h(\ell)}{\ell^6(\ell+1)^3(\ell-1)^3} - \frac{8h(\ell+1)}{\ell^3(\ell+1)^6(\ell+2)^3} \right) + \\
&\quad ms^2 h(\ell) \left(-\frac{h(\ell+1)(7\ell^2+7\ell+4)}{\ell^3(\ell+2)(\ell+1)^3(\ell-1)} - \frac{h(\ell-1)(3\ell-4)}{\ell^3(\ell+1)(2\ell-1)(\ell-2)} \right) + \\
&\quad ms^2 \left(\frac{(3\ell+7)h(\ell+1)h(\ell+2)}{\ell(\ell+1)^3(\ell+3)(2\ell+3)} - \frac{3h^2(\ell+1)}{\ell(\ell+1)^3(\ell+2)} + \frac{3h^2(\ell)}{\ell^3(\ell-1)(\ell+1)} \right),
\end{aligned}$$

$$
\begin{aligned}
f_6 = {} & \frac{16m^4s^8}{\ell^4(\ell+1)^4}\left(\frac{h(\ell+1)}{(\ell+1)^4(\ell+2)^4}-\frac{h(\ell)}{\ell^4(\ell-1)^4}\right)- \\
& \frac{4m^2s^4}{\ell^2(\ell+1)^2}\left(\frac{(3\ell^2+14\ell+17)h(\ell+1)h(\ell+2)}{(\ell+1)^3(\ell+2)(\ell+3)^2(2\ell+3)}-\frac{3h^2(\ell+1)}{(\ell+1)^3(\ell+2)^2}+\frac{3h^2(\ell)}{\ell^3(\ell-1)^2}\right) \\
& +\frac{4m^2s^4}{\ell^2(\ell+1)^2}\left(\frac{(11\ell^4+22\ell^3+31\ell^2+20\ell+6)h(\ell)h(\ell+1)}{\ell^3(\ell-1)^2(\ell+1)^3(\ell+2)^2}\right. \\
& \left.+\frac{(3\ell^2-8\ell+6)h(\ell-1)h(\ell)}{\ell^3(\ell-2)^2(\ell-1)(2\ell-1)}\right)+\frac{h(\ell+1)h(\ell+2)}{4(\ell+1)(2\ell+3)^2}\left(\frac{(\ell+3)h(\ell+3)}{3}\right. \\
& \left.+\frac{\ell+2}{\ell+1}\left((\ell+2)h(\ell+2)-(7\ell+10)h(\ell+1)+\frac{(3\ell^2+2\ell-3)h(\ell)}{\ell}\right)\right) \\
& +\frac{h(\ell)h(\ell+1)}{4\ell^2(\ell+1)^2}\left((2\ell^2+4\ell+3)h(\ell)-(2\ell^2+1)h(\ell+1)\right. \\
& \left.-\frac{(\ell^2-1)(3\ell^2+4\ell-2)h(\ell-1)}{(2\ell-1)^2}\right)+\frac{h^3(\ell+1)}{2(\ell+1)^2}-\frac{h^3(\ell)}{2\ell^2} \\
& +\frac{h(\ell-1)h(\ell)}{4\ell^2(2\ell-1)^2}\left((\ell-1)(7\ell-3)h(\ell)-(\ell-1)^2h(\ell-1)-\frac{\ell(\ell-2)h(\ell-2)}{3}\right)
\end{aligned} \tag{4.112}
$$

as it was explained, the last four terms in the parameter accessory were omitted in the last section; however, these terms are in agreement with [67] - see [68] for a thorough review. Again, to calculate the next terms in (4.111), we need to find more terms in accessory parameter expansion (4.109a). Essentially, we have to consider more terms in the confluent CB of first kind expansion and contributions of n in the intermediate channel. Also, in order to recover the asymptotics, we chose $j = \ell + s + 1$ in (4.39). Where the minimum eigenvalue of ℓ is $|s|$ and the azimuthal momenta are constrained by $|m| \leq \ell$ [60].

5 Conclusion and Perspectives

This dissertation introduced an alternative way to calculate the explicit expansion of the eigenvalue ${}_s\lambda_{lm}$ for the angular Teukolsky Master equation. That is derivated from the Newmann-Penrose formalism in the study of perturbations in the Kerr black hole metric where to find the expansion we had introduced: In the first chapter, conformal field theory in two dimensions, as well as correlation functions between operators where an explicit expression for the confluent three-point function was obtained. Using partition functions formalism from AGT correspondence, we express the conformal block of the first kind. In the second chapter, we also introduced a general idea about deformations in linear systems. The isomonodromic invariant τ_V related to isomonodromic deformations in the linear system with two regular points and one irregular point, and the Schlesinger equations for Painlevé V. Furthermore, the first and second chapters were linked by setting the central charge in c=1 with the τ_V written in terms of confluent CB of the first kind. In the third chapter, we treated with connection problem in the angular TME, where the connection problem was translated to two conditions on the τ_V-function, expressed in terms of confluent CB of the first kind. Lastly, using the two conditions we found the first terms of the accessory parameter, which led to the first terms of the expansion of the eigenvalue ${}_s\lambda_{lm}$ in power of $a\omega$.

Now, we have been working on a paper with these results and in the radial Teukolsky Master equation that will lead to the study of quasinormal modes for gravitational, electromagnetic and scalar perturbations. Furthermore, the expansion of ${}_s\lambda_{lm}$ it is not the only result, the accessory parameter expansion has the interpretation of derivative of the classical conformal block, in the CFT point of view [69]. Such understanding will give directions to understand the classical limit in the central charge, precisely $c \to \infty$. We also have been working in the accessory parameter expansion at infinity, where in this case the τ_V-function depends on the confluent conformal block of the second kind. In terms of perturbation in Kerr, the accessory parameter expansion at infinity leads to the expansion of ${}_s\lambda_{lm}$ in the high-frequency case.

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