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GABRIEL LUZ ALMEIDA

ON SYMMETRIES OF EXACT SOLUTIONS OF EINSTEIN FIELD EQUATIONS

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Dissertation presented to the graduation program of the Physics Department of Universidade Federal de Pernambuco as part of the duties to obtain the degree of Master of Science in Physics.

Concentration area: Theoretical and computational physics.

Supervisor: Prof. Dr. Carlos Alberto Batista da Silva Filho.

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Abstract

In this dissertation, symmetries are studied in the context of general relativity. Initially, we address the problem of separability of the Hamilton-Jacobi equation for the geodesic Hamiltonian, a particular Hamiltonian function constructed from the metric of the spacetime that gives rise to the geodesic equation. In this scenario, the existence of classes of coordinate systems that separate the Hamilton-Jacobi equation, the so-called separability structures, turns out to be intimately connected to the existence of symmetries. In fact, this study leads to the most general form taken by the metric tensor in n dimensions containing $m \leq n$ rank-2 Killing tensors in involution with each other and $r = n - m$ commuting Killing vector fields. The close relationship between the notion of separability structures and the existence of symmetries is manifest in this framework. In particular, we show that the existence of a separability structure enables the complete integrability of the geodesic motion. Then, a study on symmetries from the point of view of the action of continuous group on differential manifolds is conducted. A review on groups, Lie groups and Lie algebras is provided, and a study on spaces admitting the particular case of a separability structure with $m = 2$ is done under the light of these tools. Finally, equipped with all this knowledge, starting with the most general four-dimensional spacetime possessing two commuting Killing vectors and a nontrivial Killing tensor, we analytically integrate Einstein-Yang-Mills equations for a completely arbitrary gauge group. We assume that the gauge field inherits the symmetries of the background and is aligned with the principal null directions of the spacetime. In particular, generalizations of the Kerr-NUT-(A)dS spacetime containing nonabelian gauge fields as source of matter are obtained.

Keywords: Symmetries. Integrability. Einstein-Yang-Mills theory. Exact solutions. Lie groups. Lie algebras.

Resumo

Nesta dissertação, simetrias são estudadas no contexto de relatividade geral. Inicialmente, tratamos do problema da separabilidade da equação de Hamilton-Jacobi para o hamiltoniano geodésico, uma função hamiltoniana específica, construída a partir da métrica do espaço-tempo, que dá origem à equação geodésica. Nesse cenário, a existência de classes de sistemas de coordenadas que separam a equação de Hamilton-Jacobi, as chamadas estruturas de separabilidade, estão intimamente conectadas à existência de simetrias. De fato, esse estudo nos leva à forma mais geral adotada pelo tensor métrico em n dimensões contendo $m \leq n$ tensores de Killing de rank 2 em involução entre si e $r = n - m$ campos vetoriais de Killing que comutam entre si. A relação íntima entre a noção de estruturas de separabilidade com a existência de simetrias é evidente nesse cenário. Em particular, mostramos que a existência de uma estrutura de separabilidade permite a integrabilidade completa do movimento geodésico. Em seguida, um estudo de simetrias do ponto de vista da ação de um grupo contínuo em variedades diferenciáveis é conduzido. Uma revisão de grupos, grupos de Lie e álgebras de Lie é fornecido, e um estudo sobre espaços admitindo uma estrutura de separabilidade com $m = 2$ é feito sob luz desta abordagem. Então, munidos de todo esse conhecimento, partindo do espaço-tempo quadridimensional mais geral possuindo dois vetores de Killing que comutam entre si e um tensor de Killing não-trivial, integramos analiticamente as equações de Einstein-Yang-Mills para um grupo de calibre completamente arbitrário. Consideramos que os campos de calibre herdam as simetrias do espaço-tempo de fundo e estão alinhados com as direções principais nulas do espaço-tempo. Em particular, generalizações da solução de Kerr-NUT-(A)dS contendo campos de calibre não-abelianos como fontes de matéria são obtidas.

Palavras-chave: Simetrias. Integrabilidade. Teoria de Einstein-Yang-Mills. Soluções exatas. Grupos de Lie. Álgebras de Lie.

Contents

1 Introduction

In 1915, Albert Einstein proposed a new theory of gravitation, the well-known theory of general relativity. Interestingly, he did not believe, at first, that exact solutions to his field equation could be attained, due to the highly nonlinear character present in this tensor equation, that comprises a total of ten coupled partial differential equations. To his surprise, however, it was not long after, just a couple of months apart from the publication of his paper, that Karl Schwarzschild published a very simple exact solution, representing the exterior gravitational field of a spherical body. The success in the attainment of this exact solution was possible thanks to the use of symmetries. Since then, symmetries have played a prominent role in the search for exact solutions in the context of general relativity. The use of symmetries is not restricted to this effect though, rather they have also been used as a mechanism to aid in the integration of the geodesic equation. Indeed, symmetries are long known to be connected to the notion of integrability in various fields of physics. In general relativity, symmetries are also important to describe the final stage of the gravitational collapse, since this is expected to result in an axisymmetric gravitational system presenting invariance under time translation. In face of this, the purpose of the present work is to investigate symmetries, showing the existing connection with the notion of integrability.

In this dissertation, we study symmetries first by addressing the problem of the integrability of the geodesic equation. In this scenario, symmetries are manifest in spaces admitting a coordinate system that separates the Hamilton-Jacobi equation for the geodesic Hamiltonian, a particular Hamiltonian function constructed from the metric of the space-time that gives rise to the geodesic equation. We also study symmetries from the point of view of the group action on differentiable manifolds. The relevance of such investigations is confirmed through the attainment of exact solutions in the framework of the Einstein-Yang-Mills theory, having symmetries as starting point.

The outline of the presentation is the following: in chapter 2 we present the theory of separability of the Hamilton-Jacobi equation, aiming at investigating the particular case where the Hamiltonian function is the one build from the metric, defined on the cotangent bundle of Riemannian manifolds. The concept of separability structures is then introduced and shown to be connected to the existence of symmetries and to the integrability of the geodesic equation. Practical applications of all the key concepts introduced along the chapter are then worked out at the end of the chapter through examples drawn from general relativity.

In chapter 3, in order to gain a deeper understanding on the geometric aspects behind the notion of symmetries, we give formal definitions of groups, Lie groups and Lie algebras. The invariant vector fields defining a Lie algebra are shown to play a fundamental role in the action of Lie groups on differentiable manifolds. The particular case where this group action keeps the metric tensor invariant is investigated, showing that its infinitesimal generators are essential in the integration of the geodesic motion. The chapter is finished with a study on the symmetries of the general spaces presenting a separability structure of type $\mathcal{S}_2$.

In chapter 4, starting from the most general n-dimensional spaces endowed with $(n-2)$ commuting Killing vector fields and a nontrivial rank-2 Killing tensor, we intend to obtain exact solutions for the Einstein-Yang-Mills theory in four dimensions. We further assume that these spaces possess a naturally defined null frame and that the gauge fields are subject to the symmetries of the underlying geometry. In order to attain this integration, a review on the Petrov classification, optical scalars, Frobenius theorem and the Goldberg-Sachs theorem is provided. We then solve completely the problem proposed, showing explicit examples of nontrivial solutions to this problem. Finally, coordinate transformations and redefinition of the arbitrary parameters defining the general solution for the case tackled explicitly in this dissertation is performed, the solutions being identified as generalizations of the Kerr-NUT-(A)dS spaces. In particular, it is worth mentioning that the results, in special the exact solutions found in this chapter, are original.

Finally, in chapter 5, we summarize the main results obtained along this dissertation, and then we discuss the perspectives for future investigations.

2 The Theory of Separability of the Hamilton-Jacobi Equation

The problem of integrability of the geodesic equation arises naturally in the study of separability of the Hamilton-Jacobi equation for a particular Hamiltonian function called the geodesic Hamiltonian, defined on the cotangent bundle of Riemannian manifolds. With this in mind, in this chapter we shall introduce the theory of separability of the Hamilton-Jacobi equation in Riemannian geometry, showing the close relationship between the notion of separability structures and the existence of sufficiently many symmetries to completely integrate the geodesic equation. In fact, the complete integrability of the geodesic equation will always hold for spaces admitting a separability structure. Bearing this in mind, the first three sections of this chapter cover the basics of symplectic manifolds, showing how the notion of Hamiltonian mechanics emerges in the framework of differential geometry, followed by sections covering the theory of separability itself. Finally, the chapter is finished with examples drawn from Einstein's theory of general relativity.

2.1 Symplectic Manifolds

The most important structure underlying the theory of separability of the Hamilton-Jacobi equation is the notion of symplectic manifold. For this reason, we give here the basic definition and work out some important properties of such structure.

Let Q be a differentiable manifold of dimension n and (q^i) a local coordinate system defined on it. It is well known that the cotangent bundle of Q, denoted by T^*Q, is also a differentiable manifold, of dimension $2n$. The coordinate system (q^i) gives rise to a natural local chart (q^i, p_i) on T^*Q, where the p_i spanning the cotangent space of Q at q^i are the components or arbitrary 1-forms with respect to the basis (dq^i). In the context of Hamiltonian dynamics, the q^i are interpreted as the position coordinates, while the p_i are the momentum coordinates. These coordinates can be used to define a natural structure on T^*Q called the **canonical symplectic structure** (also called the **canonical symplectic form**), given by

$$\boldsymbol{\omega} = \sum_{i=1}^{n} dp_i \wedge dq^i \,. \tag{2.1}$$

The natural character of this object originates as a consequence of the fact that any coordinate transformation on the q^i gives rise to the same form (2.1), since the momenta 1-forms dp_i transform inversely to the position 1-forms dq^i.

In general terms, a symplectic structure $\boldsymbol{\omega}$ is defined to be any nondegenerate[1] and

[1] By a nondegenerate 2-form $\boldsymbol{\omega}$, we mean any 2-form field satisfying the following: for all vector fields

closed differential 2-form field defined on an even-dimensional manifold M^{2n}. The respective manifold $(M^{2n}, \boldsymbol{\omega})$ is then called **symplectic manifold**. Thus, any differentiable manifold Q gives rise naturally to a symplectic manifold consisting of its cotangent bundle T^*Q together with the symplectic structure $\boldsymbol{\omega}$ defined by equation (2.1). A symplectic manifold constructed this way is denoted by $(T^*Q, \boldsymbol{\omega})$.

Symplectic forms $\boldsymbol{\omega}$ can be used to map vector fields $\boldsymbol{\xi}$ to 1-forms $\boldsymbol{\omega}_{\boldsymbol{\xi}}$ by means of the relation $\boldsymbol{\omega}_{\boldsymbol{\xi}}(\boldsymbol{\eta}) = \boldsymbol{\omega}(\boldsymbol{\eta}, \boldsymbol{\xi})$, which must be valid for all vectors $\boldsymbol{\eta}$. In fact, this mapping is an isomorphism between the tangent and cotangent spaces of the corresponding sympletic manifold $(M^{2n}, \boldsymbol{\omega})$. Thus, denoting by $\boldsymbol{\omega}^{-1}$ the inverse mapping $\boldsymbol{\omega}^{-1} : T^*M^{2n} \to TM^{2n}$, to every function H defined on $(M^{2n}, \boldsymbol{\omega})$ we can associate a vector field $\boldsymbol{\omega}^{-1}(dH)$. Vector fields constructed in this way are called **Hamiltonian vector fields**, the corresponding functions being called **Hamiltonian functions**. This nomenclature is not merely coincidental and can be understood from the following: given local coordinates (q^i, p_i) on the sympletic manifold $(T^*Q, \boldsymbol{\omega})$, the 1-form field dH corresponding to the Hamiltonian function $H(q^i, p_i)$ reads

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \right).$$

Then, using the mapping $\boldsymbol{\omega}^{-1}$ defined from the canonical symplectic form (2.1), this 1-form field is taken to the following Hamiltonian vector field:

$$\boldsymbol{\omega}^{-1}(dH) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{2.2}$$

In particular, the integral curves of this vector field, which are given by the equation $(\dot{q}^i, \dot{p}^i) = \boldsymbol{\omega}^{-1}(dH)$, reduce to the following set of equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{2.3}$$

where the dots above the coordinates q^i and p_i represent derivatives with respect to a parameter τ that parameterizes the curve. We recognize equations (2.3) as being the **Hamilton's canonical equations**. From this, we see the close relationship existing between symplectic manifolds and Hamiltonian dynamics.

2.2 Lie Algebra of Poisson Brackets

A recurring structure appearing in the study of Hamiltonian dynamics is the Poisson bracket. In this section, we show how this important concept arises in the context of differential geometry, and show, in particular, how it is connected to the well-known Lie bracket of vector fields. As we shall see, the set of all Hamiltonian functions equipped with the Poisson bracket operation forms a Lie algebra.

Remember, a Lie algebra is defined to be a vector space closed under a binary operation $[\,,]$, which is bilinear, antisymmetric and satisfies the Jacobi identity. In symplectic manifolds, a simple and important example of Lie algebra is provided by the set of all Hamiltonian vector fields, with the bilinear operation being the usual Lie bracket of vector fields. This case is easily proven to be a Lie algebra since the Lie bracket carries all the necessary properties of the binary operation of a Lie algebra, and, in addition, any

$\boldsymbol{\xi} \neq 0$, there exists another vector $\boldsymbol{\eta}$ such that $\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$ at all points of M^{2n}.

linear combination of Hamiltonian vector fields, say $\boldsymbol{\omega}^{-1}(dG)$ and $\boldsymbol{\omega}^{-1}(dH)$, is also a Hamiltonian vector field: $a\boldsymbol{\omega}^{-1}(dG)+b\boldsymbol{\omega}^{-1}(dH)=\boldsymbol{\omega}^{-1}(d(aG+bH))$, where a and b are constant numbers, while G and H are functions on the cotangent bundle. The closure of this set under the Lie bracket will be proved in this section. In fact, this Lie algebra is a subalgebra of the larger Lie algebra of all vector fields defined on the same manifold.

Another important case of Lie algebra is provided by the vector space of Hamiltonian functions together with the Poisson bracket. To understand this better, consider a symplectic manifold $(M^{2n},\boldsymbol{\omega})$, where M^{2n} is a differentiable manifold of dimension $2n$ and $\boldsymbol{\omega}$ is an arbitrary symplectic form defined on it. Then, to each Hamiltonian function H defined on M^{2n}, we can associate a one-parameter group of diffeomorphisms g_H^τ defined to be the phase flow of the Hamiltonian vector field $\boldsymbol{\omega}^{-1}(dH)$, which means that

$$\left.\frac{d}{d\tau}\right|_{\tau=0} g_H^\tau(x)=\boldsymbol{\omega}^{-1}(dH(x))\,,\quad \text{for all } x \text{ of } M^{2n}\,. \tag{2.4}$$

An interesting property of such Hamiltonian phase flows is that they preserve the symplectic form. Indeed, this result can easily be proved in local coordinates for the canonical symplectic form (2.1) by making use of the Lie derivative.

The **Poisson bracket** $\{F,H\}$ of two Hamiltonian functions F and H is then defined to be the derivative of F in the direction of the phase flow of H:

$$\{F,H\}(x)=\left.\frac{d}{d\tau}\right|_{\tau=0} F(g_H^\tau(x))\,. \tag{2.5}$$

Notice that the Poisson bracket of any two functions defined on M^{2n} is again a function on the same manifold. It follows directly from (2.4) and (2.5) that

$$\{F,H\}(x)=\left.\frac{d}{d\tau}\right|_{\tau=0} F(g_H^\tau(x))=dF(\boldsymbol{\omega}^{-1}(dH(x)))\,. \tag{2.6}$$

In particular, using local coordinates on the right-hand side of this expression, it is easy to verify that the Poisson bracket is antisymmetric $\{F,G\}=-\{G,F\}$ for any two Hamiltonian functions F and G. Moreover, since the mapping $\boldsymbol{\omega}^{-1}$ and the exterior derivative of functions are linear operators, the Poisson bracket is bilinear:

$$\{F,aG+bH\}=a\{F,G\}+b\{F,H\}\,, \tag{2.7}$$

for any constants a and b and Hamiltonian functions F, G and H.

Consider, for instance, the symplectic manifold $(T^*Q,\boldsymbol{\omega})$ with canonical symplectic structure $\boldsymbol{\omega}$, as defined naturally by the n-dimensional manifold Q with local coordinates (q^i). In this case, the Hamiltonian vector field $\boldsymbol{\omega}^{-1}(dH)$ is given by equation (2.2) and, consequently, using expression (2.6), the Poisson bracket of the Hamiltonian functions F and H takes the well-known form:

$$\{F,H\}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}\right)\,. \tag{2.8}$$

Now, in order to see the relation between the Poisson bracket of two functions F and H and the Lie bracket of the corresponding Hamiltonian vector fields $\boldsymbol{\omega}^{-1}(dF)$ and $\boldsymbol{\omega}^{-1}(dH)$, recall that the Lie derivative of a function f in the direction of the vector field

$\boldsymbol{\xi}$, $\mathcal{L}_{\boldsymbol{\xi}} f$, is simply the derivative of f in the direction of the flow of $\boldsymbol{\xi}$. Thus, given two Hamiltonian functions F and G, we have from definition (2.5) that

$$\mathcal{L}_{\left[\boldsymbol{\omega^{-1}}(dH)\right]} G = \left.\frac{d}{d\tau}\right|_{\tau=0} G(g_H^\tau) = \{G, H\}\,. \tag{2.9}$$

Besides this, because the Lie derivative obeys the Leibniz rule of derivation and Hamiltonian phase flows preserve the symplectic form, the following holds:

$$\mathcal{L}_{\left[\boldsymbol{\omega^{-1}}(dH)\right]}(\boldsymbol{\omega^{-1}}(dG)) = \boldsymbol{\omega^{-1}}\left(\mathcal{L}_{\left[\boldsymbol{\omega^{-1}}(dH)\right]} dG\right)\,. \tag{2.10}$$

Recall also that the Lie bracket of two vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is related to the Lie derivative by $[\boldsymbol{\xi}, \boldsymbol{\eta}] = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{\eta}$. Consequently, using (2.10), the Lie bracket of the Hamiltonian vector fields $\boldsymbol{\omega^{-1}}(dF)$ and $\boldsymbol{\omega^{-1}}(dH)$ results in the following relation:

$$\left[\boldsymbol{\omega^{-1}}(dH), \boldsymbol{\omega^{-1}}(dG)\right] = \mathcal{L}_{\left[\boldsymbol{\omega^{-1}}(dH)\right]}(\boldsymbol{\omega^{-1}}(dG)) = \boldsymbol{\omega^{-1}}\left(\mathcal{L}_{\left[\boldsymbol{\omega^{-1}}(dH)\right]} dG\right)\,. \tag{2.11}$$

Finally, from the fact that the exterior derivative commutes with the Lie derivative of differential forms, combining relations (2.9) and (2.11), we are lead to

$$\left[\boldsymbol{\omega^{-1}}(dH), \boldsymbol{\omega^{-1}}(dG)\right] = -\boldsymbol{\omega^{-1}}(d\{H, G\})\,, \tag{2.12}$$

where the antisymmetry of the Poisson bracket was used. This proves the sought relationship: the Hamiltonian vector field resulting from the Poisson bracket of two Hamiltonian functions H and G is equal, up to a minus sign, to the Lie bracket of the corresponding Hamiltonian vector fields $\boldsymbol{\omega^{-1}}(dH)$ and $\boldsymbol{\omega^{-1}}(dG)$. For this reason, it is customary, in this context, to call the negative of the Lie Bracket the Poisson bracket of vector fields. In particular, once we know that the Jacobi identity holds for any three vector fields under the Lie bracket operation, it follows from (2.12) that it also holds for the Poisson bracket of any three Hamiltonian functions, up to a closed 1-form. In special, using the canonical Poisson bracket (2.8), it is easy to verify that the Jacobi identity is identically satisfied, a feature that is actually common to any notion of Poisson brackets. Namely, for any three Hamiltonian functions F, G and H defined on our sympectic manifold, we have:

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0\,. \tag{2.13}$$

In conclusion, since the Poisson bracket is a binary and closed operation on the space of Hamiltonian functions defined on symplectic manifolds, being, in addition, antisymmetric, bilinear and satisfying the Jacobi identity (2.13), it endows the space of Hamiltonian functions with a structure of Lie algebra.

2.3 First Integrals and Killing Tensors

It is widely known that the concept of Lie algebra is intimately connected to the notion of symmetry. In fact, in the framework of Hamiltonian dynamics, the Poisson bracket appears naturally in conserved quantities along the Hamiltonian flow. To see this, consider two Hamiltonian functions F and H defined on the canonical symplectic manifold $(T^*Q, \boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is the canonical symplectic structure (2.1). Moreover, assume that

H describes a particular physical system and that F is nonconstant. Then, the time evolution of the function $F(q^i, p_i)$ along the Hamiltonian flow of H is dictated by

$$\frac{dF}{d\tau} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p^i}\dot{p}^i \right) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} \right) = \{F, H\}, \tag{2.14}$$

where the Hamilton's canonical equations (2.3) have been used in the second equality. It is clear from this that if the Poisson bracket on the right-hand side of this expression vanishes, then F is a conserved quantity along the Hamiltonian phase flow of H. Consequently, due to Noether's theorem, there exists an underlying continuous symmetry in the corresponding physical system described by H.

For the general case where $(M^{2n}, \boldsymbol{\omega})$ is a $2n$-dimensional symplectic manifold with arbitrary symplectic structure $\boldsymbol{\omega}$ defined on it, the Hamiltonian function F is said to be a **first integral** (or **constant of the motion**) with respect to the Hamiltonian function H if the Poisson bracket of these two quantities vanishes,

$$\{F, H\} = 0 \, . \tag{2.15}$$

Any two functions F and H satisfying (2.15) are said to be in **involution**. For this general case, the equivalent expression for equation (2.14) is given by definition (2.5), and the same physical interpretation follows: a first integral is a function of q and p that is conserved along the Hamiltonian phase flow. Notice that, from the property (2.7), any linear combination of first integrals is also a first integral and, from the Jacobi identity (2.13), the Poisson bracket of any two first integrals gives rise also to a first integral. Thus, the set of first integrals is easily seen to form a Lie subalgebra of the larger Lie algebra of Hamiltonian functions, under the Poisson bracket operation.

For the canonical symplectic manifold $(T^*Q, \boldsymbol{\omega})$ with natural coordinates (q^i, p_i), an important class of Hamiltonian functions is provided by the set of all symmetric tensors on the manifold base Q, and defined in the following way. To every rank-r symmetric tensor $\boldsymbol{K}$ of this set, we assign a Hamiltonian function E_K through

$$E_K = \frac{1}{r!} \sum_{i_1=1}^{n} \dots \sum_{i_r=1}^{n} K^{i_1 \dots i_r}(q^i) p_{i_1} \dots p_{i_r} \, . \tag{2.16}$$

Functions defined this way can be used to establish a notion of bracket for symmetric tensors that will turn out to be of great importance in the theory of separability of the Hamilton-Jacobi. In fact, defining functions E_{K_1} and E_{K_2} according to (2.16) corresponding to the symmetric tensors $\boldsymbol{K_1}$, of rank-r, and $\boldsymbol{K_2}$ of, rank-s, respectively, it is not difficult to show that their Poisson bracket can be written as

$$\{E_{K_1}, E_{K_2}\} = - \sum_{i_1}^{n} \dots \sum_{i_{r+s-1}}^{n} \left(\frac{1}{r!s!} \right) p_{i_1} \dots p_{i_{r+s-1}} \times$$
$$\left\{ \sum_{j=1}^{n} \left[r K_1^{j(i_1 \dots i_{r-1}} \partial_j K_2^{i_r \dots i_{r+s-1})} - s K_2^{j(i_1 \dots i_{s-1}} \partial_j K_1^{i_s \dots i_{r+s-1})} \right] \right\},$$

where the ∂_j means derivation with respect to the coordinate q^j. The quantity enclosed by the curly brackets is known as the **Schouten-Nijenhuis (SN) bracket** of the symmetric tensors $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$, and denoted by $[\boldsymbol{K_1}, \boldsymbol{K_2}]$:

$$[\boldsymbol{K_1}, \boldsymbol{K_2}]^{i_1 \dots i_{r+s-1}} = \sum_{j=1}^{n} \left[r K_1^{j(i_1 \dots i_{r-1}} \partial_j K_2^{i_r \dots i_{r+s-1})} - s K_2^{j(i_1 \dots i_{s-1}} \partial_j K_1^{i_s \dots i_{r+s-1})} \right] . \tag{2.17}$$

We immediately notice from this formula that the SN bracket of the two symmetric tensors $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$, respectively of rank r and s, resulted in another symmetric tensor, but this time of rank-$(r+s-1)$. In the case where $[\boldsymbol{K_1}, \boldsymbol{K_2}] = 0$, $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ are said to be in involution, and this is equivalent to $\{E_{K_1}, E_{K_2}\} = 0$. Further important properties of the Schouten-Nijenhius bracket are listed below, [1, 2]:

(i) for manifolds Q endowed with a symmetric (torsion-free) connection $\boldsymbol{\nabla}$, the replacement of the ∂_i appearing in (2.17) by ∇_i leaves the bracket unchanged;

(ii) for vector fields, the SN bracket coincides with the Lie bracket; Similarly, the SN bracket of a vector field $\boldsymbol{X}$ with a rank-2 symmetric tensor $\boldsymbol{K}$ is equal to the Lie derivative of $\boldsymbol{K}$ in the direction of the flow of $\boldsymbol{X}$: $[\boldsymbol{X}, \boldsymbol{K}] = \mathcal{L}_{\boldsymbol{X}} \boldsymbol{K}$;

(iii) for symmetric tensors $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ of same order, the SN bracket becomes antisymmetric: $[\boldsymbol{K_1}, \boldsymbol{K_2}] = -[\boldsymbol{K_2}, \boldsymbol{K_1}]$. In particular, $[\boldsymbol{K_1}, \boldsymbol{K_1}] = 0$;

(iv) for any three symmetric tensors $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ and $\boldsymbol{K_3}$, the following property holds: $[\boldsymbol{K_1} \odot \boldsymbol{K_2}, \boldsymbol{K_3}] = [\boldsymbol{K_1}, \boldsymbol{K_3}] \odot \boldsymbol{K_2} + \boldsymbol{K_1} \odot [\boldsymbol{K_2}, \boldsymbol{K_3}]$, where the symbol $\odot$ represents the symmetrized tensor product;

(v) for any three symmetric tensors $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ and $\boldsymbol{K_3}$, the Jacobi identity $[\boldsymbol{K_1}, [\boldsymbol{K_2}, \boldsymbol{K_3}]] + [\boldsymbol{K_3}, [\boldsymbol{K_1}, \boldsymbol{K_2}]] + [\boldsymbol{K_2}, [\boldsymbol{K_3}, \boldsymbol{K_1}]] = 0$ holds.

In the last two properties, (iv) and (v), the tensors $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ and $\boldsymbol{K_3}$ are of arbitrary and possibly different rank.

A case of particular importance occur when Q is a Riemannian manifold[2] , with metric tensor denoted by $\boldsymbol{g}$: $(Q, \boldsymbol{g})$. In this case, being the metric a symmetric tensor, it follows that relation (2.16) can be used to define a privileged Hamiltonian function on the symplectic manifold $(T^*Q, \boldsymbol{\omega})$ called the **geodesic Hamiltonian function**:

$$H = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} p_i p_j \,, \tag{2.18}$$

where g^{ij} are the components of the metric tensor $\boldsymbol{g}$ in the coordinate system (q^i). In particular, considering a symmetric connection $\boldsymbol{\nabla}$ compatible with the metric $\boldsymbol{g}$, namely $\boldsymbol{\nabla_X g} = 0$ for all vector fields $\boldsymbol{X}$ (or, equivalently, $\nabla_k g_{ij} = 0$), we have that the Poisson bracket of the function E_K associated to a rank-r symmetric tensor $\boldsymbol{K}$ with the geodesic Hamiltonian given by (2.18) results in

$$\{E_K, H\} = \sum_{i_1=1}^{n} \cdots \sum_{i_{r+1}=1}^{n} \left(\frac{1}{r!}\right) \nabla^{(i_1} K^{i_2 \ldots i_{r+1})} p_{i_1} \ldots p_{i_{r+1}} \,, \tag{2.19}$$

where we have made use of the property (i) above to replace ∂_j by ∇_j. Therefore, for the case where the quantity E_K is a first integral with respect to the geodesic Hamiltonian function H, $\{E_K, H\} = 0$, this equation immediately leads to

$$\nabla^{(i_1} K^{i_2 \ldots i_{r+1})} = 0 \,. \tag{2.20}$$

[2]The rigorous definition of a Riemannian manifold is of a differentiable manifold Q endowed with a positive-definite metric tensor $\boldsymbol{g}$. Nevertheless, here we use this nomenclature in the broader sense encompassing metric tensors of arbitrary signature.

In its turn, this equation amounts to saying that the SN bracket of $\boldsymbol{K}$ and $\boldsymbol{g}$ vanishes:

$$\{E_K, H\} = 0 \iff [\boldsymbol{K}, \boldsymbol{g}] = 0\,. \tag{2.21}$$

We recognize equation (2.20) as the defining equation for **Killing Tensors**. Thus, in the context of Hamiltonian dynamics, Killing tensors $\boldsymbol{K}$ arise as homogeneous polynomials in momenta defined on the cotangent bundle of a Riemannian manifold $(Q, \boldsymbol{g})$, which are in involution with the corresponding geodesic Hamiltonian. Equivalently, Killing tensors are symmetric tensors in involution with the metric with respect to the SN bracket, (2.21). In particular, from the property (ii), we immediate see that **Killing vectors** are the special case where $\boldsymbol{K} = \boldsymbol{X}$ is a vector in involution with the metric $\boldsymbol{g}$: $[\boldsymbol{X}, \boldsymbol{g}] = 0$.

Besides providing a better understanding on Killing tensors, the importance of Hamiltonian functions of the type (2.18) in Riemannian geometry resides in the fact that the flow of the corresponding Hamiltonian vector field $\boldsymbol{\omega^{-1}}(dH)$, which is a vector field in T^*Q, is projected onto the geodesics of the Riemannian manifold $(Q, \boldsymbol{g})$. This projection is done by the mapping $\pi_Q : T^*Q \to Q$ that, in natural coordinates (q^i, p_i), takes points (q^i, p_i) on T^*Q to points q^i on the manifold base Q. To see this, remember that the integral curves of the Hamiltonian vector field $\boldsymbol{\omega^{-1}}(dH)$ on the cotangent bundle T^*Q are dictated by the Hamilton's equations (2.3). Thus, considering H to be the geodesic Hamiltonian (2.18), we have:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} g^{ij} p_j \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -\tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (\partial_i g^{jk}) p_j p_k\,. \tag{2.22}$$

Notice that the first of these equations is equivalent to

$$p_i = \sum_{j=1}^{n} g_{ij} \dot{q}^j\,.$$

Hence, we see that both the p_i and $\dot{p}_i$ can be written in terms of the coordinates q^i and $\dot{q}^i$, along with the metric components. Thus, differentiating the first of the equations in (2.22) with respect to the parameter τ, and using the above relations for p_i and $\dot{p}_i$, after some simple manipulations we arrive at the following equation:

$$\frac{d^2 q^i}{d\tau^2} + \sum_{k=1}^{n} \sum_{l=1}^{n} \left[\tfrac{1}{2} \sum_{j=1}^{n} g^{ij} \left(\partial_k g_{lj} + \partial_l g_{kj} - \partial_j g_{lk} \right) \right] \left(\frac{dq^k}{d\tau} \right) \left(\frac{dq^l}{d\tau} \right) = 0\,.$$

We immediately recognize the quantity inside the square brackets as being the Christoffel symbols Γ^i_{kl}, showing therefore that the integral curves of Hamiltonian vector fields associated to the geodesic Hamiltonian projected on the manifold base Q are indeed the geodesics of $(Q, \boldsymbol{g})$. Moreover, we learn that τ is an affine parameter since the coordinates q^i satisfy the geodesic equation

$$\frac{d^2 q^i}{d\tau^2} + \sum_{k=1}^{n} \sum_{l=1}^{n} \Gamma^i_{kl} \left(\frac{dq^k}{d\tau} \right) \left(\frac{dq^l}{d\tau} \right) = 0\,.$$

From this result, it is clear that first integrals relative to geodesic Hamiltonians are quantities that are conserved along the geodesic motion on $(Q, \boldsymbol{g})$.

2.4 The Hamilton-Jacobi Equation and Separation of Variables

It is well known from classical mechanics that the method of Hamilton-Jacobi provides an interesting technique to find general solutions of the Hamilton's equation. In general, however, this method is very complicated and can only be successfully applied when the Hamilton-Jacobi equation is separable, in which case this method becomes a useful computational tool. In this section we outline the main points concerning this method and make the bridge to differential geometry.

Considering local coordinates (q^i, p_i) on the cotangent bundle T^*Q of our usual n-dimensional differentiable manifold Q, Hamiltonian functions $H(q^i, p_i)$ give rise to the following partial differential equation known as the **Hamilton-Jacobi equation**:

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = h\,, \tag{2.23}$$

where h is a constant usually called the **energy**. A **complete integral** of this equation is a solution $W(q^i, a_i)$ depending on n real parameters a_i. These constants are such that we can always choose $a_n = h$, and W is such that the $n \times n$ matrix with components given by $\partial^2 W/\partial q^i \partial a_j$ is invertible everywhere. Namely, $\det(\partial^2 W/\partial q^i \partial a_j) \neq 0$. Once a complete integral $W(q^i, a_i)$ for a Hamiltonian H is known, the solutions for the Hamilton's canonical equations are determined by the following equations:

$$b^i + \frac{\partial W}{\partial a_i} = \delta^i_n \tau \quad \text{and} \quad p_i = \frac{\partial W}{\partial q^i}\,, \qquad \text{for } i = 1, \ldots, n\,, \tag{2.24}$$

with $a_n = h$, the b^i being n further constants, and τ being the time parameter. Then, we can solve this system of $2n$ equations for the components q^i and p_i, to be expressed in terms of the time parameter τ and the $2n$ parameters a_i and b^i, with the latter being determined by initial conditions. As anticipated at the beginning of this section, the most relevant cases in which this method can be successfully applied are the ones in which the complete integral $W(q^i, a_i)$ can be split into n additive terms, each term depending only on one of the variables q^i:

$$W(q^i, a_i) = W_1(q^1, a_i) + W_2(q^2, a_i) + \ldots + W_n(q^n, a_i)\,. \tag{2.25}$$

In general, each of the functions W_i appearing in (2.25) depend on the n parameters a_i. In this case the Hamilton-Jacobi equation (2.23) becomes also **separable**

$$H_i\left(q^i, \frac{dW_i}{dq^i}; a_1, \ldots, a_n\right) = a_i\,, \qquad \text{for } i = 1, \ldots, n\,,$$

as can be seen from direct derivation of (2.23) with respect to q^j, and, hence, each of the functions W_i can always be solved by quadrature. Thus, the Hamilton-Jacobi equation, a partial differential equation in the n variables q^i, becomes a set of n first-order ordinary differential equations, each one for each of the variables q^i, which always has solution. Moreover, for a given Hamiltonian function H, a coordinate system (q^i) defined on some neighborhood of the base manifold Q is said to be a **separable coordinate system** if it allows the existence of a separable complete integral of the form (2.25).

In reference [3], Levi-Civita provides a condition for the existence of a complete integral of the form (2.25) for the Hamilton-Jacobi equation with Hamiltonian given by $H(q^i, p_i)$.

Denoting by ∂_i and ∂^i the derivatives $\partial/\partial q^i$ and $\partial/\partial p_i$, respectively, these conditions are given by the following set of $n(n-1)/2$ equations[3]:

$$\partial^i H \partial^j H \partial_i \partial_j H + \partial_i H \partial_j H \partial^i \partial^j H - \partial^i H \partial_j H \partial_i \partial^j H - \partial^j H \partial_i H \partial_j \partial^i H = 0\,, \tag{2.26}$$

for $i \neq j$. For Hamiltonians of type (2.18), the Hamilton-Jacobi equation becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} \partial_i W \partial_j W = 2h\,,$$

and the above conditions, called the **Levi-Civita conditions**, boil down to

$$\sum_{hkrs} \left(g^{ir} g^{js} \partial_i \partial_j g^{hk} + \tfrac{1}{2} g^{ij} \partial_i g^{hk} \partial_j g^{rs} - g^{is} \partial_i g^{jh} \partial_j g^{kr} - g^{js} \partial_j g^{ih} \partial_i g^{kr} \right) p_h p_k p_r p_s = 0\,, \tag{2.27}$$

where again the indices i and j must be different from each other.

The geometric importance in the separability of the Hamilton-Jacobi equation for geodesic Hamiltonians lies in the fact that it ensures the complete integrability of the geodesic motion. This is because, as we have seen in the previous section, the solutions of the Hamilton's canonical equations (and, hence, the Hamilton-Jacobi equation) for this type of Hamiltonians are the geodesics of the corresponding space. Thus, since separability of the Hamilton-Jacobi equation implies its complete integrability, then we are always given the geodesic motion in the case we have a separable Hamilton-Jacobi equation for the geodesic Hamiltonian function.

2.5 Orthogonal Separable Systems

In this section we consider the simpler and more restrictive case where the separable coordinate system (q^i) respective to the geodesic Hamiltonian function (2.18) is such that the components of the inverse metric g^{ij} are diagonal on it. Namely, $g^{ij} = 0$ for $i \neq j$. In this case, the coordinate system is called **orthogonal separable system**, and the geodesic Hamiltonian takes the simpler form

$$H = \tfrac{1}{2} \sum_{i=1}^{n} g^{ii} p_i^2\,,$$

and the separability conditions (2.27) reduce to the following set of equations:

$$g^{ii} g^{jj} \partial_i \partial_j g^{hh} - g^{ii} \partial_i g^{jj} \partial_j g^{hh} - g^{jj} \partial_j g^{ii} \partial_i g^{hh} = 0\,, \qquad \text{for } i \neq j\,. \tag{2.28}$$

In the case of orthogonal separability, the most general form allowed for the components g^{ii} is given by $g^{ii} = {\phi_{(n)}}^i$, where ${\phi_{(i)}}^j$ are the components of the inverse of a $n \times n$ **Stäckel matrix** $[{\phi_i}^{(j)}]$. A Stäckel matrix $[{\phi_i}^{(j)}]$ is defined to be a regular matrix such that the components of its ith row are functions only of the coordinate q^i: ${\phi_i}^{(j)} = {\phi_i}^{(j)}(q^i)$. Thus, the components of the inverse metric correspond to the last row of the inverse

[3] In order to avoid confusion, throughout this chapter we do not use Einstein summation convention. Instead, we use the summation symbol $\sum$ explicitly, as occurrence of equations like (2.26) are frequent in the theory of separability of the Hamilton-Jacobi equation.

Stäckel matrix $[\phi_{(i)}{}^j]$. Indeed, in this case, the Hamilton-Jacobi equation is given simply by

$$\tfrac{1}{2}\sum_{i=1}^{n} g^{ii}(\partial_i W_i)^2 = h\,.$$

Then, differentiating this equation with respect to a_j and defining

$$\phi_i{}^{(j)} = \frac{\partial W_i}{\partial q^i}\frac{\partial^2 W_i}{\partial a_j \partial q^i} \quad \text{and} \quad c^j = \frac{\partial h}{\partial a_j}\,,$$

noticing that the $\phi_i{}^{(j)}$ are functions only of q^i, we are lead to

$$\sum_{i=1}^{n} g^{ii}\phi_i{}^{(j)} = c^j \quad \Rightarrow \quad g^{ii} = \sum_{i=1}^{n} c^j \phi_{(j)}{}^i\,.$$

Hence, since we can always choose $a_n = h$, without loss of generality, we obtain the desired result. The other $n-1$ rows can be used to build rank-2 symmetric tensors $\boldsymbol{K_i}$ (also diagonal in the coordinates (q^i)), with components given by $(K_i)^{jj} = \phi_{(i)}{}^j$. In this case, the inverse metric is recovered if we set $i = n$. In fact, these quantities are Killing tensors in involution, as the corresponding functions E_{K_i}, defined by (2.16), commute with the Hamiltonian $H = E_{K_n}$, and commute also with each other. To prove this, notice that the defining characteristic of Stäckel matrices can be formulated as $\partial_i \phi_k{}^{(l)} = \delta^i_k \partial_i \phi_i{}^{(l)}$. This equation, in turn, is equivalent to $\delta^i_h \partial_i \phi_k{}^{(l)} = \delta^i_k \partial_i \phi_h{}^{(l)}$. Then, multiplying this latter equation by the expression $\phi_{(l)}{}^j \phi_{(m)}{}^h \phi_{(p)}{}^k$, followed by summation in the indices l, h and k, we arrive at the following important relations valid for Stäckel matrices:

$$\phi_{(m)}{}^i \partial_i \phi_{(p)}{}^j - \phi_{(p)}{}^i \partial_i \phi_{(m)}{}^j = 0\,. \tag{2.29}$$

On the other hand, a direct calculation of the Poisson bracket of the quantities E_{K_m} and E_{K_p} gives the following result:

$$\{E_{K_m}, E_{K_p}\} = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\phi_{(m)}{}^i \partial_i \phi_{(p)}{}^j - \phi_{(p)}{}^i \partial_i \phi_{(m)}{}^j\right] p_i p_j^2 = 0\,,$$

where the last equality holds as a consequence of (2.29), proving, therefore, that the tensors $\boldsymbol{K_i}$ are in involution with each other. In particular, since $E_{K_n} = H$, the functions E_{K_i} are first integrals quadratic in the momenta, confirming that, indeed, the tensors $\boldsymbol{K_i}$ are rank-2 Killing tensors.

2.6 Classes of Separable Systems

In the theory of separability of the Hamilton-Jacobi equation, a very important concept that arises is that of equivalence of separable coordinate systems. This is because, in general, a coordinate transformation does not preserve the separability property of the Hamilton-Jacobi equation. In the case it does, the two separable coordinate systems in question might lead to two different complete integrals. Accordingly, two separable coordinate systems (q^i) and (q'^i) for the same Hamiltonian function H and defined on the same neighborhood on Q are said to be **equivalent** if they give rise to the same complete integral. In other words, these two coordinate systems are equivalent if the complete integral of the Hamilton-Jacobi equation with Hamiltonian H represent the same n-parameter families of real functions on the manifold Q. The simplest scenario where this is the case is provided by a **separated transformation**: $q'^i = q'^i(q^i)$. In this case both systems give rise to the same integral since they generate the same coordinate hypersurfaces.

For separable systems, a coordinate q^i for which the function $\partial_i H/\partial^i H$ is linear in the momenta is called **first class coordinate**, being dubbed **second class coordinate** otherwise. A coordinate which makes $\partial_i H = 0$ is also considered to be of first class and is called, in addition, **ignorable** (or **cyclic**). At this point it is convenient to denote by Latin letters (a, b, c, ...) the indices corresponding to the $m \leq n$ second class coordinates, and by Greek letters (α, β, γ, ...) the $r = n - m$ coordinates of the first class, including ignorable coordinates.

The following important results connect the definitions of first and second class coordinates with the notion of equivalent separable coordinate systems. For the proofs, see for instance [4]:

(i) the number of first class coordinates (and hence, the number of second class coordinates) are equal for any two equivalent separable coordinate systems;

(ii) for every separable coordinate system there exists a family of equivalent separable systems where all the first class coordinates are ignorable.

These two results are general and hold irrespective of the kind of Hamiltonian considered. The following results, on the other hand, are only true, in general, for geodesic Hamiltonian functions:

(iii) the second class coordinates of two equivalent separable coordinate systems are related to each other by a separated transformation, $q'^a = q'^a(q^a)$;

(iv) two equivalent separable systems $(q^i) = (q^a, q^\alpha)$ and $(q'^i) = (q'^a, q'^\alpha)$, the first being such that all first class coordinates are ignorable, are connected to each other by transformations of the type $dq^a = dq'^a$ and $dq^\alpha = \sum_{i=1}^n A^\alpha{}_i(q'^i)dq'^i$, up to a separated transformation;

(v) the second class coordinates (q^a) in a separable coordinate system $(q^i) = (q^a, q^\alpha)$ are orthogonal. Namely, $g^{ab} = 0$ for $a \neq b$.

The collection of all equivalent separable coordinate systems associated to the same Hamiltonian function H is called **separability structure** and is denoted by $\mathcal{S}_r$, with r denoting the number of first class coordinates, which is common to all the members of this collection according to the first of the properties above. Notice that, since this definition is based on the concept of coordinate systems, separability structures are defined only locally on Q.

2.7 The Normal Form

In this section, starting from the properties for equivalent separable coordinate systems listed in the previous section, and considering only geodesic Hamiltonian functions, we seek the most simple form allowed for the components of the corresponding inverse metric g^{ij} in separable coordinates within the same separability structure $\mathcal{S}_r$. To this end, let us consider the separable systems in $\mathcal{S}_r$ such that all first class coordinates are ignorable (property (ii)). Then, since in separable coordinate systems the second class coordinates are orthogonal (property (v)), the Levi-Civita conditions (2.27) for such coordinates reduce to

$$g^{aa}g^{bb}\partial_a\partial_b g^{cc} - g^{aa}\partial_a g^{bb}\partial_b g^{cc} - g^{bb}\partial_b g^{aa}\partial_a g^{cc} = 0\,, \qquad \text{for } a \neq b\,,$$

which is the same conditions for orthogonal systems (2.28). Thus, from the discussion of section 2.5, we conclude that the components of the inverse metric g^{ij} for indices running over the second class coordinates are given by

$$g^{ab} = \delta^{ab}{\phi_{(m)}}^a\,,$$

where, as usual, m is the number of second class coordinates and ${\phi_{(a)}}^b$ are the components of a $m \times m$ inverse Stäckel matrix, ${\phi_{(m)}}^a$ corresponding to the components of its last row.

Further simplifications are still possible, but for this intent, let us define first **isotropic coordinates** q^a to be second class coordinates such that $g^{aa} = 0$. Although this definition is clearly coordinate dependent, the same property holds for all coordinate systems within the same separability structure $\mathcal{S}_r$ for geodesic Hamiltonians. This is true because, in this case, second class coordinates for equivalent separable systems are related to each other by separated transformations (property (iii)), implying that in all these systems we have $g^{aa} = 0$. In particular, this kind of coordinates are never present in strictly Riemannian manifolds, as null coordinates are not allowed to exist in such geometries.

From now on, let us denote by $\mathcal{S}_{r,m_2}$ separability structures containing m_2 isotropic coordinates, leaving the notation $\mathcal{S}_r$ for separability structures without isotropic coordinates. Then, taking into account the previous results, a theorem due to Benenti [4], asserts that in every separability structure $\mathcal{S}_{r,m_2}$ there exists a coordinate system (q^i) such that the first class coordinates (q^α) are all ignorable and, in addition, the metric in this coordinate system takes the following form:

$$[g^{ij}] = \begin{array}{c} \begin{array}{ccc} \quad m_1 & \quad\; m_2 & \;\; r \end{array} \\ \left[\begin{array}{ccc} \delta^{a_1 b_1}{\phi_{(m)}}^{a_1} & 0 & 0 \\ 0 & 0 & g^{a_2\alpha} \\ 0 & g^{\alpha a_2} & g^{\alpha\beta} \end{array}\right] \end{array} \begin{array}{c} m_1 \\ m_2 \\ r \end{array}\,, \tag{2.30}$$

where m_1 is the number of nonisotropic second class coordinates q^{a_1}, $m_2 = m - m_1$ the number of isotropic coordinates q^{a_2}, and r the number of ignorable first class coordinates q^α. The coordinate systems in which this happens are called **normal separable coordinates**, and the respective form taken by the metric tensor (2.30) is called the **normal form**. Besides this, Benenti also proves in [4] that the components $g^{a_2\alpha}$ and $g^{\alpha\beta}$ acquire the following simpler form in these coordinates:

$$g^{a_2\alpha} = \theta^\alpha_{a_2}{\phi_{(m)}}^{a_2} \quad \text{and} \quad g^{\alpha\beta} = \sum_{a=1}^{m} \eta^{\alpha\beta}_a {\phi_{(m)}}^a\,,$$

where $\theta^\alpha_{a_2}$ and $\eta^{\alpha\beta}_a$ are functions only of the coordinate corresponding to their lower index: $\theta^\alpha_{a_2} = \theta^\alpha_{a_2}(q^{a_2})$, $\eta^{\alpha\beta}_a = \eta^{\alpha\beta}_a(q^a)$, and $\eta^{\alpha\beta}_a$ are symmetric in their upper indices.

In summary, in every separable structure $\mathcal{S}_{r,m_2}$ there always exists a coordinate system (q^i) where all its first class coordinates (q^α) are ignorable and the metric components g^{ij} takes the form (2.30), with components $g^{a_2\alpha}$ and $g^{\alpha\beta}$ given by the general form $g^{a_2\alpha} = \theta^\alpha_{a_2}{\phi_{(m)}}^{a_2}$ and $g^{\alpha\beta} = \sum_{a=1}^{m} \eta^{\alpha\beta}_a {\phi_{(m)}}^a$. In this case, the index a_1 runs over the values

$1, \ldots, m_1$; a_2 over $m_1+1, \ldots, m$; and α and β over $m+1, \ldots, n$. Notice that, separability structures having neither second class isotropic coordinates nor first class coordinates are necessarily orthogonal.

The normal form represents the largest number of simplifications we can carry out within a separability structure without affecting the generality. Moreover, it fully characterizes the separability structure, since all the other members of this class can be reached by means of coordinate transformations of kind (iv).

2.7.1 Normal Form, Killing Tensors and Killing Vectors

In the construction and simplification of the normal form (2.30), only the last row of the inverse Stäckel matrix $[{\phi_{(a)}}^b]$ was used. In fact, the other $m-1$ rows can be used to build rank-2 symmetric tensors $\boldsymbol{K_a}$ by the simple replacement ${\phi_{(m)}}^b \to {\phi_{(a)}}^b$ in the components of the normal form, similar to what was done in section 2.5 for orthogonal separable coordinates. In this case, these $m-1$ quantities give rise to second order homogeneous polynomials in the momenta, denoted by E_{K_a}, according to the general formula (2.16):

$$\begin{aligned} E_{K_b} &= \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(K_b)^{ij}p_ip_j \\ &= \tfrac{1}{2}\sum_{a_1}{\phi_{(b)}}^{a_1}p_{a_1}^2 + \sum_{a_2\alpha}{\phi_{(b)}}^{a_2}\theta^{\alpha}_{a_2}p_{a_2}p_{\alpha} + \tfrac{1}{2}\sum_{a\alpha\beta}{\phi_{(b)}}^{a}\eta^{\alpha\beta}_{a}p_{\alpha}p_{\beta}\,, \end{aligned} \tag{2.31}$$

the geodesic Hamiltonian H being recovered by setting $b = m$. Then, evaluating the Poisson bracket of two of these quantities, say E_{K_b} and E_{K_c}, we end up with the following expression:

$$\begin{aligned} \{E_{K_b}, E_{K_c}\} &= \tfrac{1}{2}\sum_{a_1b_1}\chi^{a_1b_1}_{cb}p_{a_1}p_{b_1}^2 + \sum_{a_1a_2\alpha}\chi^{a_1a_2}_{cb}\theta^{\alpha}_{a_2}p_{a_1}p_{a_2}p_{\alpha} + \tfrac{1}{2}\sum_{aa_1\alpha\beta}\chi^{a_1a}_{cb}\eta^{\alpha\beta}_{a}p_{a_1}p_{\alpha}p_{\beta} \\ &+\tfrac{1}{2}\sum_{a_1a_2\alpha}\chi^{a_2a_1}_{cb}\theta^{\alpha}_{a_2}(p_{a_1})^2p_{\alpha} + \sum_{a_2b_2\alpha\beta}\chi^{a_2b_2}_{cb}\theta^{\alpha}_{a_2}\theta^{\beta}_{b_2}p_{b_2}p_{\alpha}p_{\beta} + \tfrac{1}{2}\sum_{aa_2\alpha\beta\gamma}\chi^{a_2a}_{cb}\theta^{\gamma}_{a_2}\eta^{\alpha\beta}_{a}p_{\alpha}p_{\beta}p_{\gamma}\,, \end{aligned}$$

where the functions χ^{ab}_{cd} are defined by

$$\chi^{ab}_{cd} = {\phi_{(c)}}^a\partial_a{\phi_{(d)}}^b - {\phi_{(d)}}^a\partial_a{\phi_{(c)}}^b\,.$$

Notice, however, that the above expression is precisely equal to (2.29) and, hence, vanish identically since $[{\phi_{(a)}}^b]$ is an inverse Stäckel matrix. Therefore, since $E_{K_m} = H$, this proves not only that the quantities E_{K_a} are first integrals, giving rising to rank-2 Killing tensors $\boldsymbol{K_a}$, but also that they are in involution with each other. Moreover, since Stäckel matrices are by definition regular, it follows that the set of tensors $\{\boldsymbol{K_a}\}$, for $a = 1, \ldots, m$, and $\boldsymbol{K_m} = \boldsymbol{g^{-1}}$, with $\boldsymbol{g^{-1}}$ representing the inverse metric tensor, is composed by pointwise independent Killing tensors.

In addition to the m symmetries generated by this set of Killing tensors, the ignorable first class coordinates q^{α} give rise to a set of r independent commuting killing vectors $\{\partial_{\alpha}\}$ adapted to the respective normal coordinates. Then, in total, we have n independent first integrals, m being of second order in the momenta, while r are of first order. The independent nature of the first integrals generated by these Killing tensors and Killing vectors one more time can be seen as a consequence of the regularity of the Stäckel matrix ${\phi_a}^{(b)}$.

The above results can be synthesized in the following statement: *Every n-dimensional Riemannian manifold $(Q, \boldsymbol{g})$ allowing the existence of a coordinate system that separates the Hamilton-Jacobi equation for the corresponding geodesic Hamiltonian H, are naturally equipped with a set of $m = n - r$ rank-2 Killing tensors and r commuting Killing vectors, where r is the number of coordinates q^i such that $\partial_i H/\partial^i H$ is either a first order polynomial in the momenta or vanishes.*

2.8 Intrinsic Characterization of Separability Structures of Type $\mathcal{S}_r$

The intent of the present section is to provide a geometric characterization of separability structures that do not present second class isotropic coordinates. Recall that this assumption does not represent any restriction for strictly Riemannian manifolds, as they naturally cannot have such type of coordinates. In this case, the nonvanishing components of the inverse metric in normal coordinates are given by $g^{aa} = \phi_{(m)}{}^a$ and $g^{\alpha\beta} = \sum_{a=1}^{m} \eta_a^{\alpha\beta} \phi_{(m)}{}^a$, and, accordingly, the nonvanishing components of the Killing tensors $\boldsymbol{K_a}$ related to the second class coordinates q^a are given by $(K_a)^{bb} = \phi_{(a)}{}^b$ and $(K_a)^{\alpha\beta} = \sum_{b=1}^{m} \eta_b^{\alpha\beta} \phi_{(a)}{}^b$.

For a reason that should become clear later on in this section, we shall denote the r independent Killing vectors respective to the ignorable coordinates q^α by $\boldsymbol{X_\alpha}$. Then, using the general form for the metric and Killing tensors in normal coordinates given above, we are able to derive the following properties:

(i) the Killing tensors and Killing vectors, in addition to being in involution with members of the same set, are in involution with each other: $[\boldsymbol{K_a}, \boldsymbol{X_\alpha}] = 0$;

(ii) the coordinate vectors $\boldsymbol{X_a} = \partial_a$ form a set of m common orthogonal eigenvectors of the Killing tensors $\boldsymbol{K_a}$, for all a, which are in involution: $[\boldsymbol{X_a}, \boldsymbol{X_b}] = 0$. In this case, $\boldsymbol{K_a}$ is though of as the matrix with components $(K_a)^i{}_j$;

(iii) the eigenvectors of $\boldsymbol{K_a}$ are orthogonal to the Killing vectors $\boldsymbol{X_\alpha}$ and, moreover, they are in involution with the latter ones: $\boldsymbol{X_a} \cdot \boldsymbol{X_\alpha} = 0$, $[\boldsymbol{X_a}, \boldsymbol{X_\alpha}] = 0$, the latter result following as an obvious consequence of both $\boldsymbol{X_a}$ and $\boldsymbol{X_\alpha}$ being coordinate vectors.

In fact, the properties listed above are necessary conditions for the existence of a separability structure of type $\mathcal{S}_r$ on a Riemannian manifold $(Q, \boldsymbol{g})$, as the normal form is always attainable for any separability structure. Sufficient conditions can also be worked out from integrability equations for the set of Killing tensors $\boldsymbol{K_a}$, ultimately leading to the Levi-Civita conditions (2.27) for the normal form, [7]. The following theorem, due to Benenti [5, 7], summarizes these points and represents the most important result in the theory of separability of the Hamilton-Jacobi regarding its application to differential geometry:

Theorem 1. *An n-dimensional Riemannian manifold $(Q, \boldsymbol{g})$ admits a local separability structure of type $\mathcal{S}_r$ if and only if the following conditions are met:*

(1) there exist r independent commuting Killing vectors $\boldsymbol{X_\alpha}$: $[\boldsymbol{X_\alpha}, \boldsymbol{X_\beta}] = 0$;

(2) there exist $n - r$ independent rank-2 Killing tensors $\boldsymbol{K_a}$ such that $[\boldsymbol{K_a}, \boldsymbol{K_b}] = 0$ and $[\boldsymbol{K_a}, \boldsymbol{X_\alpha}] = 0$;

(3) the Killing tensors $\boldsymbol{K_a}$ have in common $n - r$ orthogonal eigenvectors $\boldsymbol{X_a}$ such that $[\boldsymbol{X_a}, \boldsymbol{X_\alpha}] = 0$, $\boldsymbol{X_a} \cdot \boldsymbol{X_\alpha} = 0$, and $[\boldsymbol{X_a}, \boldsymbol{X_b}] = 0$.

It is worth mentioning that, as remarked at the end of section 2.6, the notion of separability structure is local (as it is based on the notion of coordinate systems) and, therefore, the applicability of the above theorem follows accordingly.

An important consequence of this theorem is that, once both sets of vectors $\{\boldsymbol{X_a}\}$ and $\{\boldsymbol{X_\alpha}\}$ are separately in involution, and, in addition, are constituted by pointwise independent vector fields, according to Frobenius' theorem they generate two distinct foliations of the same manifold. Moreover, due to the orthogonality property $\boldsymbol{X_a} \cdot \boldsymbol{X_\alpha} = 0$, these two foliations are orthogonal. Let us denote by $\{Z_{n-r}\}$ the foliation corresponding to the integrable system $\{\boldsymbol{X_a}\}$, and by $\{W_r\}$ the foliation respective to the system $\{\boldsymbol{X_\alpha}\}$. An important aspect of these foliations is that each integral submanifold of $\{W_r\}$ is flat, which follows from the fact that they are spanned by the abelian r-parameter group of isometries generated by the Killing vectors $\boldsymbol{X_\alpha}$, and each integral submanifold of $\{Z_{n-r}\}$ possesses an orthogonal separable structure $\mathcal{S}_{n-r}$ following as a consequence of the fact that the vectors $\boldsymbol{X_a}$ are orthogonal, and that all the conditions of theorem 1 are trivially satisfied for the restriction of the Killing tensors $\boldsymbol{K_a}$ to the integral submanifolds of $\{Z_{n-r}\}$.

Concerning the uniqueness of the vectors and tensors in theorem 1, it is not difficult to see that any linear combination of the Killing tensors $\boldsymbol{K_a}$, including combinations of the inverse metric $\boldsymbol{g^{-1}}$ and symmetrized products of the Killing vectors $\boldsymbol{X_\alpha}$,

$$\tilde{\boldsymbol{K}}_{\boldsymbol{b}} = \sum_{a=1}^{n-r} c_b{}^a \boldsymbol{K_a} + \sum_{\alpha=m+1}^{n} \sum_{\beta=m+1}^{n} d_b^{\alpha\beta} \boldsymbol{X_\alpha} \odot \boldsymbol{X_\beta} + k_b \boldsymbol{g^{-1}} \,,$$

where $c_b{}^a$, $d_b^{\alpha\beta}$ and k_b are constants, and $\det [c_b{}^a] \neq 0$, gives rise to an equivalent set of Killing tensors $\{\tilde{\boldsymbol{K}}_{\boldsymbol{a}}\}$ satisfying all the conditions of theorem 1, as the old set $\{\boldsymbol{K_a}\}$ did. The properties of the SN bracket listed in section 2.3 turn out to be handy to prove this statement. Taking into account the expression above, it is possible to prove that one of these Killing tensors can always be chosen to be the inverse metric tensor $\boldsymbol{g^{-1}}$, [5]. Similarly, the sets of vectors $\{\boldsymbol{X_a}\}$ and $\{\boldsymbol{X_\alpha}\}$ are not uniquely determined. In fact, transformations of the type

$$\tilde{\boldsymbol{X}}_{\boldsymbol{\alpha}} = \sum_{\beta=m+1}^{n} A_\alpha{}^\beta \boldsymbol{X_\beta} \quad \text{and} \quad \tilde{\boldsymbol{X}}_{\boldsymbol{a}} = f(x^a) \boldsymbol{X_a} \,,$$

where $[A_\alpha{}^\beta]$ is a regular constant matrix and $f(x^a)$ is a function depending solely on the coordinate x^a, preserve all the conditions of theorem 1 and, therefore, the new sets $\{\tilde{\boldsymbol{X}}_{\boldsymbol{a}}\}$ and $\{\tilde{\boldsymbol{X}}_{\boldsymbol{\alpha}}\}$ represent a new equivalent basis of vector fields.

An important question that arises naturally in this context and has important physical relevance is whether the nontrivial Killing tensors can be reduced to symmetrized products of Killing vectors. As it turns out, although the theorems on separability structures do not imply general statements about reducibility of the Killing tensors, in the case of $\mathcal{S}_{n-2}$ structures it is known that if the Killing tensor $\boldsymbol{K}$ is reducible, one of its components must be a Killing vector independent from the $n-2$ Killing vectors $\boldsymbol{X_\alpha}$, [5]. The most important example of a space containing an irreducible Killing tensor is provided by the metric of Kerr. In fact, this Killing tensor was discovered by Carter in his famous paper [8], where it made possible the complete integrability of the geodesic equation. On the other hand, an example of spacetime with reducible Killing tensor is provided by the Schwarzschild solution, as we will see in the examples at the end of this chapter.

2.9 The Canonical Form of Separability Structures of Type $\mathcal{S}_{n-2}$

As it will be exemplified in section 2.10 and throughout chapter 4 of this dissertation (or [9]), the most important spacetimes relevant to general relativity are the four-dimensional spaces endowed with a separability structure of type $\mathcal{S}_{n-2}$. As a matter of fact, there are enough examples in the literature to justify the importance of separability structures of this type. In fact, further four-dimensional examples can be found in [10, 11], and higher-dimensional spacetimes possessing this same separability structure can be checked at [12].

Having said that, let us consider spaces $(Q, \boldsymbol{g})$ admitting a separability structure of type $\mathcal{S}_{n-2}$. In terms of the normal separable coordinates it means that the nonvanishing components of the inverse metric are given by $g^{aa} = {\phi_{(2)}}^a$ and $g^{\alpha\beta} = \sum_{a=1}^{2} \eta_a^{\alpha\beta} {\phi_{(2)}}^a$, where the indices a, b run over the values 1 and 2 and correspond to the second class coordinates x and y, while the indices α, β varies among the values $3, \ldots, n$, and correspond to the ignorable first class coordinates denoted here by $\{\sigma^\alpha\}$. In this case the Stäckel matrix ${\phi_a}^{(b)}$ is 2×2 and, therefore, we can make use the following canonical form for 2×2 Stäckel matrices:

$$[{\phi_a}^{(b)}] = \begin{bmatrix} {\phi_1}^{(1)} & {\phi_1}^{(2)} \\ {\phi_2}^{(1)} & {\phi_2}^{(2)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\psi_1} & \dfrac{\phi_1}{\psi_1} \\ -\dfrac{1}{\psi_2} & \dfrac{\phi_2}{\psi_2} \end{bmatrix}, \tag{2.32}$$

where the functions ψ_1 and ϕ_1 depend on the first of the two second class coordinates, x, while ψ_2 and ϕ_2 depend on the second coordinate, y. Notice that, since the first line of this matrix depends just on x and the second just on y, we confirm that this matrix is indeed a Stäckel matrix. Besides this, its determinant reads $(\phi_1 + \phi_2)/\psi_1\psi_2$ and, hence, the functions ϕ_1 and ϕ_2 cannot be simultaneously vanishing, otherwise this would render this matrix singular. It follows that the canonical form for the inverse of a 2×2 Stäckel matrix is given by

$$[{\phi_{(a)}}^b] = \begin{bmatrix} {\phi_{(1)}}^1 & {\phi_{(1)}}^2 \\ {\phi_{(2)}}^1 & {\phi_{(2)}}^2 \end{bmatrix} = \begin{bmatrix} \dfrac{\phi_2\psi_1}{\phi_1+\phi_2} & \dfrac{-\phi_1\psi_2}{\phi_1+\phi_2} \\ \dfrac{\psi_1}{\phi_1+\phi_2} & \dfrac{\psi_2}{\phi_1+\phi_2} \end{bmatrix}. \tag{2.33}$$

Consequently, since the components of the inverse metric g^{ij} are given in terms of the components of an inverse Stäckel matrix, we can make use of the canonical form (2.33), which immediately yields the following expressions:

$$g^{ab} = \frac{\delta^{ab}\psi_a}{\phi_1+\phi_2} \quad \text{and} \quad g^{\alpha\beta} = \sum_{a=1}^{2} \frac{\eta_a^{\alpha\beta}\psi_a}{\phi_1+\phi_2}.$$

Thus, the inverse metric tensor, denoted by $\boldsymbol{g^{-1}}$, can be written in tensor form as

$$\boldsymbol{g^{-1}} = \frac{1}{\phi_1+\phi_2}\left[\sum_{\alpha\beta}(\eta_1^{\alpha\beta}\psi_1 + \eta_2^{\alpha\beta}\psi_2)\partial_\alpha \otimes \partial_\beta + \psi_1\partial_x \otimes \partial_x + \psi_2\partial_y \otimes \partial_y\right], \tag{2.34}$$

where, recall, functions with subscript 1 are functions of the coordinate x, while the ones with subscript 2 are functions of y. The coordinate vectors ∂_α must be understood as $\partial_{\sigma^\alpha} = \partial/\partial\sigma^\alpha$, where σ^α are the ignorable coordinates. In addition to this general form taken by the inverse metric, remember also that spaces with separability structure of type $\mathcal{S}_{n-2}$ contain a nontrival Killing tensor $\boldsymbol{K}$ with nonvanishing components given by $K^{ab} = \delta^{ab}{\phi_{(1)}}^a$ and $K^{\alpha\beta} = \sum_{a=1}^{2} \eta_a^{\alpha\beta}{\phi_{(1)}}^a$. Thus, similarly to what was done for the components of the inverse metric above, making use of the canonical form (2.33), the nontrivial Killing tensor $\boldsymbol{K}$ attain the following form:

$$\boldsymbol{K} = \frac{1}{\phi_1+\phi_2}\left[\sum_{\alpha\beta}(\eta_1^{\alpha\beta}\phi_2\psi_1 - \eta_2^{\alpha\beta}\phi_1\psi_2)\partial_\alpha\otimes\partial_\beta + \phi_2\psi_1\partial_x\otimes\partial_x - \phi_1\psi_2\partial_y\otimes\partial_y\right] .$$

The discussion held in the previous section on foliations can easily be exemplified in this section. For that, notice that the integrable distributions $\{\boldsymbol{X_a}\}$ and $\{\boldsymbol{X_\alpha}\}$ correspond respectively to the sets $\{\partial_x, \partial_y\}$ and $\{\partial_{\sigma^3}, \ldots, \partial_{\sigma^n}\}$. Then, the foliation $\{Z_2\}$ associated to the first of these integrable distributions is parameterized by the coordinates x and y through $\{x, y, \sigma^3 = \sigma_0^3, \ldots, \sigma^n = \sigma_0^n\}$, for constants $\sigma_0^3, \ldots, \sigma_0^n$. The induced metric in each of the submanifolds Z_2 is given by the restriction of the metric (2.34) to the vectors ∂_x and ∂_y:

$$\boldsymbol{g^{-1}}\big|_{Z_2} = \frac{1}{\phi_1+\phi_2}\left(\psi_1\partial_x\otimes\partial_x + \psi_2\partial_y\otimes\partial_y\right) .$$

This two-dimensional metric is easily seen to be in orthogonal separable coordinates as its form precisely exhibits the normal form for orthogonal separable systems: components given by $g^{ab} = \delta^{ab}{\phi_{(2)}}^a$ and having no first class coordinates.

For the second integrable distribution, $\{\partial_{\sigma^3}, \ldots, \partial_{\sigma^n}\}$, the parameterization given by $\{x = x_0, y = y_0, \sigma^3, \ldots, \sigma^n\}$, with x_0 and y_0 being constants, gives rise to the integral submanifolds W_{n-2} that generate the foliation $\{W_{n-2}\}$. In this case, the induced metric on each of these $(n-2)$-dimensional submanifolds is given by

$$\boldsymbol{g^{-1}}\big|_{W_{n-2}} = \frac{1}{\phi_1+\phi_2}\left[\sum_{\alpha\beta}(\eta_1^{\alpha\beta}\psi_1 + \eta_2^{\alpha\beta}\psi_2)\partial_\alpha\otimes\partial_\beta\right] ,$$

where ϕ_1, ϕ_2, ψ_1, ψ_2, $\eta_1^{\alpha\beta}$ and $\eta_2^{\alpha\beta}$ are now constants for the corresponding functions evaluated at $x = x_0$ and $y = y_0$. Consequently, since all the components of the induced metric are constants, the corresponding induced Riemannian tensor is vanishing, giving rise to a flat foliation, agreeing with the discusion in section 2.8.

2.10 Examples

In the present section, examples of physically relevant space drawn from Einstein's theory of general relativity are used to illustrate the most significant points discussed in the preceding sections on the theory of separability. In fact, the most important spacetimes in general relativity have separability structures, enabling, thus, the full integrability of the geodesic motion. Particular emphasis is given to Kerr spacetime possessing a separability structure of type $\mathcal{S}_2$ and such that the nontrivial Killing tensor gives rise to the quadratic conserved quantity found by Carter in his 1968 paper, [8]. This example will be worked out in chapter 4.

2.10.1 Separability Structures in Spacetimes with Spherical Symmetry

In this first example, four-dimensional spacetimes possessing spherical symmetry are considered. Thereby, let us consider the following sufficiently general metrics:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + g(r)^{-1}(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.35}$$

An important spacetime contained within this class is the Reissner-Nordström solution in an (anti-)de Sitter background, which is attained by setting

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2 \qquad \text{and} \qquad g(r) = \frac{1}{r^2}\,. \tag{2.36}$$

Denoting the coordinates of this metric by $r = q^1$, $\theta = q^2$, $\phi = q^3$ and $t = q^4$, and the corresponding momentum coordinates by $p_r = p_1$, $p_\theta = p_2$, $p_\phi = p_3$ and $p_t = p_4$, the geodesic Hamiltonian associated to the metric (2.35) is given by

$$H = \tfrac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} g^{ij}p_ip_j = \frac{1}{2}\left[f(r)p_r^2 + g(r)p_\theta^2 + \frac{g(r)}{\sin^2\theta}p_\phi^2 - \frac{1}{f(r)}p_t^2\right]. \tag{2.37}$$

From this Hamiltonian, we can promptly write down the corresponding Hamilton-Jacobi equation which, after some simple rearrangement, reads

$$\frac{f(r)}{g(r)}\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{1}{f(r)g(r)}\left(\frac{\partial W}{\partial t}\right)^2 = \frac{2h}{g(r)}\,.$$

The separation of variables for this equation can easily be accomplished by defining separation constants c_1, c_2 and c_3 by

$$c_1^2 = \left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2, \quad c_2 = \frac{\partial W}{\partial t} \quad \text{and} \quad c_3 = \frac{\partial W}{\partial \phi}\,.$$

In this case, we were able to achieve full separation of the Hamilton-Jacobi equation for the Hamiltonian (2.37), with complete integral given by the sum of the functions W_r, W_θ, W_ϕ and W_t, which are given by the following expressions:

$$W_r = \int dr\sqrt{\frac{1}{f(r)}\left[2h + \frac{c_2^2}{f(r)} - c_1^2 g(r)\right]}\,, \quad W_\theta = \int d\theta\sqrt{c_1^2 - \frac{c_3^2}{\sin^2\theta}}\,,$$
$$W_\phi = c_3\phi \quad \text{and} \quad W_t = c_2 t\,.$$

Since the separation of variables in additive form could be attained for the Hamiltonian (2.37), the coordinate system $\{r, \theta, \phi, t\}$ is classified as separable. Furthermore, it is immediate to see that the coordinates ϕ and t are ignorable first class coordinates, since this Hamiltonian does not depend on these coordinates, $\partial_\phi H = \partial_t H = 0$. Besides this, evaluation of the quantities $\partial_r H/\partial^r H$ and $\partial_\theta H/\partial^\theta H$ does not result in linear polynomials in the momenta and, therefore, the coordinates r and θ are of second class. In particular, no isotropic coordinates exist in this separable coordinate system. In fact, the coordinate system $\{r, \theta, \phi, t\}$ is already adapted to normal coordinates as the first class coordinates ϕ and t are both ignorable and the inverse metric is in accordance with the form (2.30):

$$[g^{ij}] = \begin{bmatrix} f(r) & 0 & 0 & 0 \\ 0 & g(r) & 0 & 0 \\ 0 & 0 & g(r)\sin^{-2}\theta & 0 \\ 0 & 0 & 0 & -f(r)^{-1} \end{bmatrix}.$$

Knowing that the coordinates r, θ, ϕ and t are normal separable coordinates, we can work backwards to find the components of the inverse Stäckel matrix ${\phi_{(a)}}^b$ and also find the functions $\eta^{\alpha\beta}_a$. To this end, remember that the components of the inverse metric corresponding to the second class coordinates are given by $g^{ab} = \delta^{ab}{\phi_{(2)}}^a$ and the ones corresponding to the first class ignorable coordinates given by $g^{\alpha\beta} = \sum_{a=1}^{2}\eta^{\alpha\beta}_a{\phi_{(2)}}^a$. From this, the following results can easily be drawn:

$$ {\phi_{(2)}}^1 = f(r)\,, \quad {\phi_{(2)}}^2 = g(r)\,, \quad \eta^{33}_2 = \sin^{-2}\theta\,, \quad \eta^{44}_1 = -f(r)^{-2}\,, $$

the remaining functions $\eta^{\alpha\beta}_a$ being all vanishing: $\eta^{33}_1 = \eta^{44}_2 = 0$ and $\eta^{\alpha\beta}_a = 0$ for $\alpha \neq \beta$. Then, using the canonical form for the inverse Stäckel matrix (2.33), the functions ϕ_1, ϕ_2, ψ_1 and ψ_2 can easily be obtained in terms of $f(r)$ and $g(r)$[4]:

$$ \phi_1 = \frac{1}{g(r)}\,, \quad \phi_2 = 0\,, \quad \psi_1 = \frac{f(r)}{g(r)} \quad \text{and} \quad \psi_2 = 1\,. \tag{2.38} $$

Thus, with these functions, the Stäckel matrix and its inverse reads, respectively,

$$ [{\phi_a}^{(b)}] = \begin{bmatrix} f(r)^{-1}g(r) & f(r)^{-1} \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad [{\phi_{(a)}}^b] = \begin{bmatrix} 0 & -1 \\ f(r) & g(r) \end{bmatrix}\,. $$

Having constructed the Stäckel matrix for the current normal separable coordinates and possessing, in addition, the functions $\eta^{\alpha\beta}_a$, expression (2.31) can be used to build the first integral E_K. In this case, we must set $\theta^{\alpha}_{\mathrm{a}} = 0$ and the index $a_1 = a$ runs over 1 and 2 since there are no isotropic coordinates. Thus, we are lead to

$$ E_K = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(K_b)^{ij}p_ip_j = -\frac{1}{2}\left[p_\theta^2 + (\sin^{-2}\theta)p_\phi^2\right]\,. $$

The corresponding Killing tensor is easily obtained from this expression, and reads:

$$ \boldsymbol{K} = -\left[\partial_\theta\otimes\partial_\theta + (\sin^{-2}\theta)\partial_\phi\otimes\partial_\phi\right]\,. \tag{2.39} $$

As a matter of fact, besides the Killing vectors ∂_ϕ and ∂_t, spaces of type (2.35) are endowed with two additional independent Killing vectors on account of the spherical symmetry, namely

$$ \boldsymbol{Y_1} = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \quad \text{and} \quad \boldsymbol{Y_2} = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi\,. $$

Combining these vectors along with the Killing vector ∂_ϕ in the following way

$$ -\left(\boldsymbol{Y_1}\otimes\boldsymbol{Y_1} + \boldsymbol{Y_2}\otimes\boldsymbol{Y_2} + \partial_\phi\otimes\partial_\phi\right) = -\left[\partial_\theta\otimes\partial_\theta + (\sin^{-2}\theta)\partial_\phi\otimes\partial_\phi\right]\,, $$

we are lead precisely to the Killing tensor (2.39), showing, therefore, that it is reducible. The fact that $\boldsymbol{K}$ is made of symmetrized products of Killing vectors that are independent from the Killing vectors related to the ignorable coordinates ϕ and t is in total accordance with the discussion at the end of section 2.8.

[4]Notice that this is not the general solution for the functions ϕ_1, ϕ_2, ψ_1 and ψ_2. Nevertheless, all these solutions are equivalent, inasmuch as they give rise to Killing tensors differing from each other only up to terms proportional to the metric or, possibly, to symmetrized products of Killing vectors. In present example, the general solution provides a Killing tensor $\tilde{\boldsymbol{K}} = c_1\boldsymbol{K} + c_2\boldsymbol{g^{-1}}$, where c_1 and c_2 are constants and $\boldsymbol{K}$ is the Killing tensor for the particular choice (2.38).

2.10.2 Geodesics on the Schwarzschild Spacetime

In this second example, we intend to illustrate the method of Hamilton-Jacobi to obtain the well-known geodesic equations for the motion on the equatorial plane of the Schwarzschild geometry. To this end, consider spacetimes given by the metric (2.35), with functions $f(r)$ and $g(r)$ given by (2.36) with $e = 0$ and $\Lambda = 0$. In this case, the Hamiltonian function H is given by (2.37), setting $\theta = \pi/2$ and $p_\theta = 0$:

$$H = \tfrac{1}{2}\left[f(r)p_r^2 + g(r)p_\phi^2 - f(r)^{-1}p_t^2\right]. \tag{2.40}$$

Then, following the same procedure as in the previous example, a complete integral for the Hamilton-Jacobi equation with Hamiltonian (2.40) can easily be attained, yielding

$$W = \int dr\sqrt{\frac{1}{f(r)}\left[2h + \frac{a_2^2}{f(r)} - a_1^2 g(r)\right]} + a_1\phi + a_2 t\,.$$

Once we have an expression for the complete integral W, we can make use of equations (2.24) to obtain the general solution for the geodesic motion. For the Hamiltonian (2.40), these equations are given in terms of the following quadratures:

$$\left.\begin{aligned} b^1 + \int \frac{dr}{\sqrt{f(r)\left[2h - a_1^2 g(r)\right] + a_2^2}} &= \tau \\ b^2 - \int dr \frac{a_1 g(r)}{\sqrt{f(r)\left[2h - a_1^2 g(r)\right] + a_2^2}} + \phi &= 0 \\ b^3 + \int dr \frac{a_2}{f(r)\sqrt{f(r)\left[2h - a_1^2 g(r)\right] + a_2^2}} + t &= 0 \end{aligned}\right\} \tag{2.41}$$

with momenta coordinates given by

$$p_r = \sqrt{\frac{1}{f(r)}\left[2h + \frac{a_2^2}{f(r)} - a_1^2 g(r)\right]}\,, \quad p_\phi = a_1 \quad \text{and} \quad p_t = a_2\,. \tag{2.42}$$

Although, in theory, we just need to plug the explicit expressions for $f(r)$ and $g(r)$ into (2.41), solve the integrals and, then, solve the system in order to obtain r, θ and ϕ as functions of the time parameter τ, these integrals cannot be solved analytically for the $f(r)$ and $g(r)$ corresponding to the Schwarzschild geometry. Notice, however, that differentiating the first of the equations in (2.41) with respect to the parameter τ, and using the explicit form for $f(r)$ and $g(r)$, we are lead to

$$\left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{2m}{r}\right)\left(2h - \frac{a_1^2}{r^2}\right) + a_2^2\,. \tag{2.43}$$

This equation is well-known in this context, representing the radial motion, and can be found in any general relativity textbook. See, for instance, equation (5.64) in [13]. In addition to this equation, other important equations are obtained from the last two momenta components in (2.42), p_ϕ and p_t. To this end, recall from section 2.3 that $p_i = \sum_j g_{ij}\dot{q}^j$. Then, from this relation, we obtain:

$$r^2\left(\frac{d\phi}{d\tau}\right) = a_1 \quad \text{and} \quad \left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right) = -a_2\,. \tag{2.44}$$

These are equations for conservation of angular momentum and energy, respectively, and can also be compared with equations (5.62) and (5.61) in [13]. Finally, plugging the constants a_1 and a_2 given in (2.44) into (2.43), we are lead to

$$-\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{d\phi}{d\tau}\right)^2=2h\,.$$

This equation, also present in [13] (equation 5.63), corresponds to the quantity

$$2h=\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}\left(\frac{dq^i}{d\tau}\right)\left(\frac{dq^j}{d\tau}\right)\,,$$

which is constant along the geodesics, conveying the fact that the parameterization respective to τ is, indeed, affine, as it was anticipated at the end of section 2.3.

3 Group Theory and Symmetries

It is well known that the concept of symmetry in physics is intimately connected to the abstract notion of groups. Indeed, groups are widely used in the most varied branches of physics as a tool to implement symmetries. Taking this into account, in this chapter we present the precise definition of groups, providing the most relevant features shared by this important structure. Then, starting from this simple notion, we define what is a Lie group and define the notion of a Lie algebra as a particular set of vector fields. The exponential map then plays an important role, as we can make use of it to define a Lie group, starting from a Lie algebra. These notions happen to be of fundamental importance in the description of symmetries when applied to a differentiable manifold. In fact, we show that the set of diffeomorphisms that keep the metric tensor invariant has origin in a group action, the so-called isometry group, whose generators are the Killing vector fields. These latter, in turn, give rise to the notion of conserved quantities, having wide applications in general relativity. At the end of the chapter, we present a brief analysis of spaces with a separability structure of type $\mathcal{S}_{n-2}$ in light of this framework.

3.1 Introduction to Groups

A **group** G is defined to be a set of elements g endowed with a multiplication operation $\cdot$ that takes any two elements g_1 and g_2 in G to another element $g_1 \cdot g_2$, also in G, such that the following algebraic properties are satisfied:

1. *associativity*: for any three elements g_1, g_2 and g_3 in G, the product $\cdot$ is such that $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$;

2. *identity*: in every group, there exists an element called the **identity**, denoted by e, with the property that for any g in G, we have $e \cdot g = g \cdot e = g$;

3. *inverse*: for every element g in G, there exists an element g^{-1} also in G, called the **inverse** of g, such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Consider, as a first example, the cyclic group $C_n = (e, g, g^2, \cdots, g^n = e)$. In this group, the multiplication operation $\cdot$ is such that $g^m = g \cdot g \cdot \ldots \cdot g$, with g being repeated m times on the right hand side of this equality. From this, it is clear that the inverse element is given by $(g^m)^{-1} = g^{n-m}$. A particular realization of such a group is provided by the set $(1, e^{1(2\pi i/n)}, e^{2(2\pi i/n)}, \ldots, e^{n(2\pi i/n)} = 1)$, with multiplication operation being the usual multiplication of complex numbers. Another simple example is given by the set of integers $\mathbb{Z} = (\ldots, -2, -1, 0, 1, 2, \ldots)$ together with the ordinary sum of real numbers, $G = (\mathbb{Z}, +)$. In this case the identity element is the number 0, and the inverse of any number g being its negative $-g$. A third and important example is provided the set of

2×2 matrices which implement counter clockwise rotations on the plane of an angle θ, given by

$$R_2(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3.1}$$

with the group multiplication being the usual matrix multiplication. In particular, it is easy to verify that $R_2(\theta_1)R_2(\theta_2) = R_2(\theta_1 + \theta_2)$, and that the inverse of an element $R(\theta)$ is given by $R(-\theta)$. Besides this, the identity element of the group is given by the identity matrix itself: $R(0) = I$. This group is called the **special orthogonal group** and is denoted by $SO(2)$. Notice that in all these examples, the order of the multiplication was not important, as the same result could be obtained in either order. Nevertheless, the multiplication operation of a group is not always commutative, i.e., for elements g_1 and g_2 of a group G, we may have $g_1 \cdot g_2 \neq g_2 \cdot g_1$. For the case where every product of elements in a group commutes, the group is said to be an **abelian group**. The groups C_n, $(\mathbb{Z}, +)$ and $SO(2)$ are examples of abelian groups. For a counterexample, consider the group formed by all 2×2 matrices with real entries and determinant equal to one. This group is called the **special linear group** $SL(2, \mathbb{R})$, and can be represented by

$$A(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 \\ x_3 & \frac{1+x_2x_3}{x_1} \end{bmatrix}. \tag{3.2}$$

It is very easy to verify that this set of matrices satisfy the axioms of a group for any choice of real numbers x_1, x_2 and x_3. In particular, performing the multiplication of two of these elements, it is easy to check that $A(x_1, x_2, x_3)A(y_1, y_2, y_3) \neq A(y_1, y_2, y_3)A(x_1, x_2, x_3)$, providing thus an example of non-abelian group.

A **subgroup** H of G is defined as a subset of G that is a group in its own. In other words, for any pair of elements h_1, h_2 of $H \subset G$, the product $h_1 \cdot h_2$ is also in H, and if $h \in H$, so is its inverse h^{-1}. Notice that the identity element e necessarily makes part of any subgroup, and it alone forms a trivial subgroup of G. An example of subgroup is provided by the $SO(2)$ group, which is a subgroup of the $SL(2, \mathbb{R})$ group, as it can be realized by choosing $x_1 = \cos\theta$, $x_2 = -\sin\theta$ and $x_3 = \sin\theta$ in (3.2). Group elements which commutes with every other element in the group also form a subgroup called the **center of the group**.

Given two groups G and G', a mapping $\phi : G \to G'$ that preserves the group product, namely for any two elements $g_1, g_2 \in G$ we have $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$, is called a **homomorphism**. If, in addition, ϕ is one-to-one, then it is said to be an **isomorphism** between G and G'. Two groups G and G' for which there exists such a mapping are said to be **isomorphic** and this correspondence is denoted by $G \simeq G'$. Notice that, for a homomorphism ϕ, $\phi(g) = \phi(e \cdot g) = \phi(e) \cdot \phi(g)$, implying that if e is the identity element of G, then $\phi(e) = e'$ is the identity of G'. Likewise, from $e' = \phi(g \cdot g^{-1}) = \phi(g) \cdot \phi(g^{-1})$, we have that $[\phi(g)]^{-1} = \phi(g^{-1})$. These are important properties of homomorphisms. As an example, consider the group $U(1)$ comprised by the complex numbers $(e^{i\vartheta})$, with ϑ real, and group product being the usual multiplication of complex numbers. This group is isomorphic to the $SO(2)$ group (3.1), $U(1) \simeq SO(2)$, as can easily be seen from the identification $\phi(e^{i\theta}) = R_2(\theta)$.

The subset of G formed by elements which are mapped to the identity e' of G' by means of the homomorphism ϕ is called **kernel** of ϕ: $Ker(\phi) = \{g \in G | \phi(g) = e'\}$. In fact, the kernel of ϕ is a subgroup of G, as can easily be checked through the group axioms. Moreover, from the properties given above for homomorphisms, it is not difficult to see that ϕ is an isomorphism if, and only if, $Ker(\phi) = \{e\}$.

Another important concept that arises in group theory is that of a **conjugate element**. We say that an element $g_1 \in G$ is conjugate to $g_2 \in G$ if there exists another element h, also in G, such that $g_1 = h \cdot g_2 \cdot h^{-1}$. This relation is usually denoted by $g_1 \sim g_2$, and, in fact, it forms an equivalence class:

1. *reflexivity*: $g \sim g$;
2. *symmetry*: if $g_1 \sim g_2$, then $g_2 \sim g_1$;
3. *transitivity*: If $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 \sim g_3$.

One can easily reach the conclusion that each element of a group belongs to one, and only one, equivalence class, and the identity alone forms a class by itself. This definition enables us to define a **conjugate subgroup** H_g to a subgroup H of G, defined as following: for any fixed element g of G, $H_g = \{g \cdot h \cdot g^{-1}; h \in H\}$. An **invariant subgroup** H of G is a subgroup which is equal to all its conjugate subgroups. Namely, if $H = H_g$ for all $g \in G$. In other words, a subgroup H is an invariant subgroup of G if for all elements $h \in H$ and $g \in G$, we have that $g \cdot h \cdot g^{-1} \in H$. We immediately notice that all subgroups of abelian groups are invariant subgroups and that the kernel of a homomorphism ϕ is, as well, an invariant subgroup since $\phi(g \cdot h \cdot g^{-1}) = \phi(g) \cdot \phi(h) \cdot \phi(g)^{-1} = \phi(g) \cdot e' \cdot \phi(g)^{-1} = e'$, for any $h \in Ker(\phi)$ and $g \in G$. Notice also that every group possess at least two trivial invariant subgroups: the subgroup containing the identity element alone, and the whole group itself. A group that contains no invariant subgroup besides the trivial ones is called **simple**, while one that contains at most a non-abelian invariant subgroup is called **semi-simple**.

3.1.1 Physically Important Examples of Groups

In this section, a few general examples of groups which are physically relevant are presented in order to better illustrate the importance of this structure in physics.

General Linear Groups, $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$

The **general linear group** $GL(n, \mathbb{R})$ ($GL(n, \mathbb{C})$) is a matrix group defined by the set of all $n \times n$ invertible matrices R with real (complex) entries. This is an example of non-abelian group. A particular subgroup is the **special linear group** $SL(2, \mathbb{R})$ ($SL(2, \mathbb{C})$), defined by the subset of $n \times n$ matrices A with $\det(A) = 1$. Notice that the special linear group is an invariant subgroup of the general linear group, inasmuch as we have that $\det(RAR^{-1}) = \det(A) = 1$ for any $R \in GL(n, \mathbb{R})$. The $SL(2, \mathbb{C})$ group plays a fundamental role in the theory of spinors in four-dimensional spacetimes.

Orthogonal and Special Orthogonal Groups, $O(n)$ and $SO(n)$

The **orthogonal group** $O(n)$ is a matrix group defined as the subset of matrices $A \in GL(n, \mathbb{R})$ satisfying $AA^T = I = A^T A$. An important subgroup is formed by the subset of matrices A in $O(n)$ with $\det(A) = 1$, called the **special orthogonal group** and denoted by $SO(n)$. The $SO(2)$ and $SO(3)$ groups implement rotations respectively in the two- and three-dimensional Euclidean space. In both relativistic and classical mechanics, systems which are invariant under the action of the $SO(3)$ group possess conserved angular momentum. In quantum mechanics, the irreducible representations of the $SO(3)$ are used in the central force problem to label the stationary states. Besides this, the $SO(3)$ group also plays an important role in general relativity, as this is a symmetry group of spacetimes representing static spherical objects, such as static stars.

Generalized Orthogonal and Lorentz Groups, $O(k, n-k)$ and $O(3,1)$

A possible generalization of the orthogonal group $O(n)$ is the **generalized orthogonal group** $O(k, n-k)$ which is composed by the matrices $A \in GL(n, \mathbb{R})$ which satisfies $O\eta O^T = \eta = O^T \eta O$, where $\eta = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ is a diagonal matrix composed by k plus signs and $n-k$ negative signs. The case $O(3,1)$ is the Lorentz group, which has fundamental importance in describing symmetries in specialrelativity, including in quantum field theory.

Unitary and Special Unitary Groups, $U(n)$ and $SU(n)$

The set of matrices in $GL(n, \mathbb{C})$ satisfying $AA^\dagger = 1 = A^\dagger A$, where the $\dagger$ represents the conjugate transpose operation, forms a group called the **unitary group** and the subgroup comprising the elements with $\det(A) = 1$ is called **special unitary group** and denoted by $SU(n)$. The $SU(n)$ group is fundamental in the theory of elementary particles as it is the symmetry group of the standard model. In particular the $SU(2)$ is part of the symmetry group in the electroweak interaction, which unifies electromagnetism and the weak interactions, and the $SU(3)$ in quantum chromodynamics, the theory of the strong interaction between quarks and gluons. The $SU(n)$ group is an example of a simple group.

Discrete Groups

Not only continuous, by also discrete groups are important in physics. For instance, the cyclic group C_n may represent discrete rotations of an angle multiple of $2\pi/n$, which can be applied in solid state physics. Likewise, the group of translations $T_a(n)$, which takes points x in $\mathbb{R}^n$ to points $x + a$, becomes a discrete group of symmetry in a system of particles evenly spaced in an rectangular grid of width a.

3.1.2 Cosets and Quotient Groups

An important notion that arises in group theory is the concept of cosets. Let H be a subgroup of G with elements $(h_1, h_2, \ldots)$, and g an element of G which is not necessarily in H. Then, the set formed by the elements $(g \cdot h_1, g \cdot h_2, \ldots)$ and denoted by gH is called a **left coset** of H. Likewise, the set $Hg = (h_1 \cdot g, h_2 \cdot g, \ldots)$ is called a **right coset** of H. Notice that sets constructed this way are not subgroups of G if g is not in H, inasmuch as they do not contain the identity element.

An important characteristic of cosets is that two left cosets of the same subgroup are either the same set or do not have any element in common. Therefore, they comprise disjoint subsets of the group [20]. This result follows directly from the rearrangement theorem, which states that for any group elements g_1, g_2 and g_3 of G, we have that $g_1 \cdot g_2 = g_1 \cdot g_3$ implies $g_2 = g_3$ and, as a consequence, for a group with elements $(g_1, g_2, \ldots)$, the set $(g_k \cdot g_1, g_k \cdot g_2, \ldots)$ is just a rearrangement of the group G, for any $g_k \in G$.

Indeed, consider two distinct elements g_1 and g_2 of G which are not in H, and assume that $g_1 \cdot h_i = g_2 \cdot h_j$, namely the cosets $g_1 H$ and $g_2 H$ have at least one element in common. From this, we have that $g_1 = g_2 \cdot h_j \cdot h_i^{-1}$. Then, consider another element h_k of H such that $k \neq i$ and $k \neq j$. In this case we have that $g_1 \cdot h_k = (g_2 \cdot h_j \cdot h_i^{-1}) \cdot h_k = g_2 \cdot (h_j \cdot h_i^{-1} \cdot h_k)$, which is clearly an element of $g_2 H$. Thus, since $h_j \cdot h_i^{-1} \cdot h_k$ spans the whole subgroup H as we vary k throughout all its possible values (due to the rearrangement theorem), it follows that $g_1 H = g_2 H$. Therefore, if two cosets of the same subgroup have at least one element in common, then, in fact, the two cosets coincide completely. As a consequence, all the distinct cosets of this subgroup will split the whole group into disjoint sets.

Now, let H be an invariant subgroup of G with elements $(h_1, h_2, \ldots)$. The multiplication of two cosets $g_1 H$ and $g_2 H$ of this subgroup may be defined as the coset consisting of all products $g_1 \cdot h_i \cdot g_2 \cdot h_j = g_1 \cdot g_2 \cdot h_k$, hence $g_1 H \cdot g_2 H = (g_1 \cdot g_2) H$. The need for H to be an invariant subgroup lies in the fact that the multiplication of cosets given by $g_1 \cdot h_i \cdot g_2 \cdot h_j = g_1 \cdot g_2 \cdot h_k$ would not be well defined, since, in general, $h_k = g_2^{-1} \cdot h_i \cdot g_2 \cdot h_j$ is not an element of H. For invariant subgroups, however, $h_k = (g_2^{-1} \cdot h_i \cdot g_2) \cdot h_j = h_l \cdot h_j \in H$, for some l. In this case, the cosets of H can be thought of as being elements of another group with group multiplication as defined above. The identity element of this group is the subgroup $H = e \cdot H$ itself, and the inverse of gH is given by the coset $g^{-1} H$. Thus, the set of all cosets of an invariant subgroup H of G together with this notion of multiplication forms a group called the **quotient group** of G, which is denoted by G/H.

Example: Symmetric Group S_n

The **symmetric group** S_n (also known as the **permutation group**) is a group with elements p which produce exchanges within a set of n quantities (x_i). For instance, the group element $p_{ij} = (ij)$ exchanges the quantities $x_i \leftrightarrow x_j$, and $p_{ijk} = (ijk)$ makes $x_i \to x_j \to x_k \to x_i$, while all the other members of (x_i) are kept untouched. Notice that, in this notation, $(ij) = (ji)$ and $(ijk) = (jki) = (kij) \neq (ikj)$. The group multiplication $p \cdot p'$ must be understood as the following: we first perform the exchanges corresponding to the group element p', then the ones corresponding to p. As an example, consider $n = 3$. In this case, our set (x_i) has three quantities (x_1, x_2, x_3), while the group S_3 has six elements:

$$p_1 = e\,, \quad p_2 = (12)\,, \quad p_3 = (23)\,, \quad p_4 = (31)\,, \quad p_5 = (123)\,, \quad p_6 = (321)\,.$$

As an illustration of the group multiplication, consider the product $p_4 \cdot p_5$:

$$p_4 \cdot p_5 = (31) \cdot (123) = \begin{array}{ccccc} x_1 & \xrightarrow{(123)} & x_2 & \xrightarrow{(31)} & x_2 \\ x_2 & \xrightarrow{(123)} & x_3 & \xrightarrow{(31)} & x_1 \\ x_3 & \xrightarrow{(123)} & x_1 & \xrightarrow{(31)} & x_3 \end{array} = (12) = p_2\,.$$

Following this procedure, we are able to perform any multiplication we wish. In particular, from the product $p_5 \cdot p_4 = p_3 \neq p_4 \cdot p_5$, we see that this group is non-abelian. In fact, the symmetric group S_3 is the smallest non-abelian group that can be constructed [20]. Two examples of subgroups of S_3 are provided by the sets $H_1 = (p_1, p_5, p_6)$ and $H_2 = (p_1, p_2)$. From this, we can construct the left cosets:

$p_1 H_1 = p_5 H_1 = (p_1, p_5, p_6)$	$p_1 H_2 = p_2 H_2 = (p_1, p_2)$
$p_2 H_1 = p_3 H_1 = p_4 H_1 =$ (p_2, p_3, p_4)	$p_3 H_2 = p_6 H_2 = (p_3, p_6)$
	$p_4 H_2 = p_5 H_2 = (p_4, p_5)$

Notice that both sets of cosets are distinct partitions of the same group S_3. Besides this, it is simple to verify that H_1 is an invariant subgroup of S_3, while H_2 is not. Thus, we can construct the quotient group S_3/H_1, consisting of the left cosets $(p_1 H_1, p_2 H_1)$. In this case, p_1H_1 is the identity element of the group, while p_2H_1 is such that $p_2H_1 \cdot p_2H_1 = (p_2 \cdot p_2)H_1 = p_1H_1$, since $p_2 \cdot p_2 = p_1$. Thus, we see that the quotient S_3/H_1 is isomorphic to the cyclic group C_2, $S_3/H_1 \simeq C_2$.

3.2 Lie Groups

Lie groups arise as a unification of the algebraic concept of groups with the differential-geometric notion of manifolds. The group elements of a Lie group are identified with points in a manifold, endowing the group with a structure of topology. Thus, notions such as *compactness* and *connectedness* become relevant.

More precisely, a group G is defined to be a **Lie group** if, in addition to being a group in the usual sense, it is such that each group element g is associated to a point x in some differentiable manifold M, $g \to g(x)$, and such that both the multiplication and inverse operations, $g(x) \cdot g(y) = g(z)$ and $g(x)^{-1} = g(y)$, are parameterized by differentiable functions $z = f(x, y)$ and $y = h(x)$. The dimension of the Lie group is defined to be the dimension of the underlying manifold.

As an illustration, consider the $SO(2)$ group given by (3.1). We easily see that the manifold that parameterizes this group is the unit circle S^1 with coordinate θ. In this case, since $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$, the multiplication operation is given by the differentiable function $f(\theta_1, \theta_2) = \theta_1 + \theta_2$. Likewise, the inverse is given by $h(\theta) = -\theta$, which is also differentiable. Therefore, the $SO(2)$ group is indeed a Lie group. In particular, it is compact, since S^1 is a compact manifold.

Another example of a Lie group is provided by the special unitary group $SU(2)$. Consider the following representation for its group elements:

$$U = \begin{bmatrix} x^4 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 \end{bmatrix},$$

with $(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2 = 1$, which stems from $\det(U) = 1$. It is easy to check that $UU^\dagger = I = U^\dagger U$. The constraint $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$ is the defining equation for the 3-sphere S^3. Hence, each element of the $SU(2)$ group is identified with some point in this manifold. Choosing the branch $x^4 = +\sqrt{1 - \vec{x}^2}$, where $\vec{x}^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, the variables x^1, x^2 and x^3 become coordinates covering the upper half of S^3. Thus, the relation $g \in SU(2) \leftrightarrow x \in S^3$ is given by

$$g(x_1, x_2, x_3) = \begin{bmatrix} \sqrt{1 - \vec{x}^2} + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & \sqrt{1 - \vec{x}^2} - ix^3 \end{bmatrix}. \tag{3.3}$$

The group multiplication $g(x_1, x_2, x_3) \cdot g(y_1, y_2, x_3) = g(z_1, z_2, z_3)$ provides

$$z^i(x, y) = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon^i{}_{jk} y^j x^k + x^i \sqrt{1 - \vec{y}^2} + y^i \sqrt{1 - \vec{x}^2}\,, \tag{3.4}$$

where $\epsilon^i{}_{jk}$ is the Levi-Civita symbol, and the inverse $g(x^1, x^2, x^3)^{-1} = g(y^1, y^2, y^3)$ gives $y^i(x) = -x^i$, as can be verified. Notice that, both of these functions are differentiable. Therefore, the $SU(2)$ is indeed a Lie group. In fact, most of the Lie groups are matrix groups and, in particular, all the matrix groups presented in 3.1.1 are Lie groups. The example above is particularly interesting since it exemplifies the fact that general coordinate systems are, usually, unable to parameterize the whole group [21]. More on this *covering problem* will be discussed below.

3.3 Lie Algebras

Once Lie groups are manifolds, all the important structures of differential geometry can be applied to them. In particular, there exists a class of vector fields defined on Lie groups with fundamental importance in the description of symmetries. In fact, these vector fields generate the so-called Lie algebra of the group. For this reason, we present here the geometric notion behind the definition of Lie algebras.

Let G be a Lie group. Then, since the multiplication operation is a differentiable map, we are able to define a diffeomorphism on the Lie group G through

$$\begin{aligned} L_g : G &\to G \\ h &\mapsto g \cdot h \,. \end{aligned}$$

This diffeomorphism is called **left translation**[1] and is fundamental to defining Lie algebras, as we will shortly see. Before, though, consider the following properties:

$$L_{g^{-1}} = (L_g)^{-1} \quad \text{and} \quad L_g \circ L_h = L_{g\cdot h}\,,$$

which are valid for all g and h in G, as can be easily proven.

In differential geometry, differentiable maps between manifolds can be used to map vector fields from one manifold to the other. In fact, this is done by means of the **differential map** d. Therefore, the left translation L_g defined above naturally induces a map dL_g of vector fields on the Lie group G to vector fields also on G. As a matter of notation, let us denote by $\boldsymbol{v_g}$ the restriction of the vector field $\boldsymbol{v}$ to the tangent space T_gG. Then, the differential map $dL_g(\boldsymbol{v_h})$ takes the vector $\boldsymbol{v_h} \in T_hG$ to another vector $dL_g(\boldsymbol{v_h}) \in T_{g\cdot h}G$. To understand how this works, let us define a coordinate system (x^i) on an open set of G and denote by $g = g(x)$ an arbitrary group element in these coordinates. Then, defining a vector $\boldsymbol{v_h}$ at T_hG by $\boldsymbol{v_h} = v^i\partial_i$, and fixed group elements g_0 and h respectively by $g_0 = g(x = x_0)$ and $h = g(x = x_1)$, this map is given by the following expression:

$$dL_{g_0}(\boldsymbol{v_h}) = \sum_{i=1}^{n}\sum_{j=1}^{n} v^i \frac{\partial {L_{g_0}}^j}{\partial x^i}\bigg|_{x=x_1} \frac{\partial}{\partial x^j} \quad \in \quad T_{L_{g_0}h}G = T_{g_0\cdot h}G\,. \tag{3.5}$$

In this mapping, the functions ${L_{g_0}}^j = {L_{g_0}}^j(x)$ are the components of the left translation $L_{g_0}g = g_0 \cdot g$ in the coordinates (x^i). The two following properties, valid for any group elements g, h and vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$, are of great importance and will be essential in the definition of Lie algebras:

$$dL_g \circ dL_h = d(L_g \circ L_h) = dL_{g\cdot h} \quad \text{and} \quad dL_g([\boldsymbol{v},\boldsymbol{w}]) = [dL_g(\boldsymbol{v}), dL_g(\boldsymbol{w})]\,. \tag{3.6}$$

Vector fields $\boldsymbol{v}$ satisfying $dL_g(\boldsymbol{v_h}) = \boldsymbol{v_{g\cdot h}}$ for every $g, h \in G$ are called **left invariant vector fields**. In other words, left invariant vector fields are vector fields which are kept unchanged under the application of the differential map dL_g:

$$dL_g(\boldsymbol{v}) = \boldsymbol{v}\,, \qquad \text{for every } g \in G.$$

[1] Right translation can be similarly constructed and share the same properties as the left translation. Indeed, all the general results presented in this section are valid for right translations.

Conversely, given a basis for the tangent space at the identity, equation (3.5) can be used to build vector fields satisfying $dL_g(\boldsymbol{v}) = \boldsymbol{v}$. Namely, we can define vectors $\boldsymbol{v_h} \equiv dL_h(\boldsymbol{v_e}) \in T_hG$, for all $h \in G$. Then, from the first of the properties in (3.6):

$$dL_g(\boldsymbol{v_h}) = dL_g(dL_h(\boldsymbol{v_e})) = (dL_g \circ dL_h)(\boldsymbol{v_e}) = (dL_{g\cdot h})(\boldsymbol{v_e}) = \boldsymbol{v_{g\cdot h}}\,, \tag{3.7}$$

which is by definition a left invariant vector field. It is clear from this construction that every Lie group admits the existence of such class of vector fields.

Notice that since the differential map is linear, any linear combination of left invariant vector fields results in another left invariant vector field. Thus, the set of left invariant vector fields of G forms a vector space, denoted here by $\mathfrak{g}$. Moreover, notice that for any given left invariant vector field $\boldsymbol{v}$, the vector $\boldsymbol{v_g} \in T_gG$ is uniquely determined by the vector $\boldsymbol{v_e} \in T_eG$ as a direct consequence of (3.7):

$$\boldsymbol{v_g} = dL_g(\boldsymbol{v_e}) \quad \text{and} \quad \boldsymbol{v_e} = dL_{g^{-1}}(\boldsymbol{v_g})\,.$$

This one-to-one identification conveys that the vector space of invariant vector fields $\mathfrak{g}$ and the vector space T_eG are isomorphic, $\mathfrak{g} \simeq T_eG$. In particular, the dimension of $\mathfrak{g}$ is equal to the dimension of the group G. In addition to this, from the second of the properties (3.6), for any left invariant vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$dL_g([\boldsymbol{v}, \boldsymbol{w}]) = [dL_g(\boldsymbol{v}), dL_g(\boldsymbol{w})] = [\boldsymbol{v}, \boldsymbol{w}]\,.$$

Therefore, the Lie bracket of left invariant vector fields is again a left invariant vector field. It follows from this result that since $\mathfrak{g}$ is a finite dimensional vector space closed under the Lie bracket of vector fields, then it is a Lie algebra. In fact, a Lie algebra constructed this way is said to be *the* Lie algebra of the group.

Choosing a basis $\{\boldsymbol{X_i}\}$ for the Lie algebra $\mathfrak{g}$ and assuming that $\dim(\mathfrak{g}) = n$,

$$[\boldsymbol{X_i}, \boldsymbol{X_j}] = \sum_{k=1}^{n} C^k{}_{ij} \boldsymbol{X_k}\,,$$

for some constants $C^k{}_{ij}$ called the **structure constants** of $\mathfrak{g}$. It follows from the anti-symmetry and Jacobi identity of Lie brackets that

$$C^k{}_{ji} = -C^k{}_{ij} \quad \text{and} \quad \sum_{l=1}^{n} C^l{}_{[ij} C^m{}_{k]l} = 0\,. \tag{3.8}$$

In fact, these constants fully characterize the Lie algebra $\mathfrak{g}$ and any set of constants satisfying (3.8) are structure constants for the Lie algebra of some Lie group [24].

3.3.1 Constructing the Lie Algebra for the $SU(2)$ Group

In this subsection, some of the most important concepts described above on Lie algebras are exemplified with an actual Lie group, using again the $SU(2)$ group.

Consider the parameterization for the $SU(2)$ group given by (3.3), recalling that $\vec{x}^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. In particular, as we have seen in section 3.2, the group multiplication in these coordinates is given by (3.4). From this, the components of the left translation $L_{g(x)}g(y) = g(x) \cdot g(y)$ can be easily obtained:

$$[L_{g(x)}g(y)]^i = \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon^i{}_{jk} y^j x^k + x^i\sqrt{1 - \vec{y}^2} + y^i\sqrt{1 - \vec{x}^2}\,,$$

where $\vec{y}^2 = (y^1)^2 + (y^2)^2 + (y^3)^2$. Note that, since the identity element e corresponds to the coordinates $y^i = 0$, the expression above gives $[L_{g(x)}g(0)]^i = y^i$, as expected.

Having in hands the components of the left translations $L_g(y)$ in the coordinates (y^i) above, we are now in position to construct the left invariant vector fields of the $SU(2)$ group. To this end, consider the vector $\boldsymbol{v_e} = v^i\partial_i$ defined at T_eG, where the v^i are constants, and then, making use of equation (3.5), we are lead to

$$\boldsymbol{v_g} = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon^i{}_{jk}v^jx^k + v^i\sqrt{1-\vec{x}^2}\right)\frac{\partial}{\partial x^i}\bigg|_x \quad \in \quad T_gG\,,$$

where (x^i) are the coordinates of the arbitrary group element g. Then, by letting these coordinates vary along the domain in which they are defined, the left invariant vector field $\boldsymbol{v} = \boldsymbol{v}(x)$ is then constructed. This expression can be rearranged to provide us the following simpler form:

$$\boldsymbol{v}(x) = \sum_{j=1}^{3} v^j\boldsymbol{X_j}(x)\,, \text{ where } \boldsymbol{X_j}(x) = \sum_{i=1}^{3}\left(\sum_{k=1}^{3}\epsilon^i{}_{jk}x^k + \delta^i_j\sqrt{1-\vec{x}^2}\right)\frac{\partial}{\partial x^i}\,. \tag{3.9}$$

In fact, the left invariant vector fields $\{\boldsymbol{X_j}\}$, for $j = 1, 2, 3$, form a basis for the Lie algebra $\mathfrak{su}(2)$ of the $SU(2)$ group. In particular, we see that $\boldsymbol{X_j} = \partial_j$ at the identity element. Computation of the Lie brackets of these vector fields yields

$$[\boldsymbol{X_i}, \boldsymbol{X_j}] = \sum_{k=1}^{3}(-2)\epsilon^k{}_{ij}\boldsymbol{X_k} \quad \Rightarrow \quad C^i{}_{jk} = -2\epsilon^i{}_{jk}\,. \tag{3.10}$$

The change of basis $\tilde{\boldsymbol{X}}_{\boldsymbol{i}} = -\frac{1}{2}\boldsymbol{X_i}$, changes the structure constants to $\tilde{C}^i{}_{ij} = \epsilon^i{}_{jk}$.

Now, consider the set of 1-form fields $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{i}}\}$ on the $SU(2)$ group defined by

$$\tilde{\boldsymbol{\omega}}^{\boldsymbol{i}} = 2\sum_{j=1}^{3}\left[\sum_{k=1}^{3}\left(-\epsilon^i{}_{jk}x^k + \frac{x^ix^k\delta_{kj}}{\sqrt{1-\vec{x}^2}}\right) + \delta^i_j\right]dx^j\,.$$

It is easy to verify that this is dual to the basis of left invariant vector fields $\tilde{\boldsymbol{X}}_{\boldsymbol{i}}$, since $\tilde{\boldsymbol{\omega}}^{\boldsymbol{i}}(\tilde{\boldsymbol{X}}_{\boldsymbol{j}}) = \delta^i_j$. Then, from the expression above we can calculate the exterior derivatives $d\tilde{\boldsymbol{\omega}}^{\boldsymbol{i}}$ and verify that the Maurer-Cartan equation holds [26]. Namely,

$$d\tilde{\boldsymbol{\omega}}^{\boldsymbol{i}} + \sum_{j=1}^{3}\sum_{k=1}^{3}\tfrac{1}{2}C^i{}_{jk}\tilde{\boldsymbol{\omega}}^{\boldsymbol{j}}\wedge\tilde{\boldsymbol{\omega}}^{\boldsymbol{k}} = 0\,, \quad \text{with} \quad C^i{}_{jk} = \epsilon^i{}_{jk}\,.$$

The existence of this set of 1-forms is not particular to the $SU(2)$ group. Rather, in every Lie group there always exists such a set, which is defined by the notion of *pullback* of 1-form fields, and such that the Maurer-Cartan equation always holds. This is exactly the 1-form counterpart of the left invariant vector fields.

3.3.2 The Exponential Map

In the discussion above, we have seen that every Lie group admits the existence of a Lie algebra of left invariant vector fields. In this section, we present a mechanism called the exponential map used to obtain a Lie group stemming from a Lie algebra.

Before, consider a vector field $\boldsymbol{v}$ defined on a differentiable manifold M with coordinates (x^i), with integral curves parameterized by τ. In other words, given a point x_0 in M, the parameterized curve $x^i = x^i(\tau)$ is such that it starts at $x(0) = x_0$ and walks on M according to the following first order differential equation:

$$\frac{dx^i}{d\tau} = v^i(x)\,.$$

The solution to this equation $x(\tau)$ can be expanded around x_0 at $\tau = 0$, yielding

$$x^i(\tau) = x^i_0 + \tau \frac{dx^i}{d\tau}\bigg|_{\tau=0} + \frac{\tau^2}{2!}\frac{d^2x^i}{d\tau^2}\bigg|_{\tau=0} + \ldots = \left(1 + \tau\boldsymbol{v} + \frac{\tau^2}{2!}\boldsymbol{v}^2 + \ldots\right) x^i\bigg|_{x=x_0}$$

$$\Rightarrow \quad x^i(\tau) \equiv \exp(\tau\boldsymbol{v})x^i\,,$$

where $\boldsymbol{v}^2x^i$ should be understood as the action of $\boldsymbol{v}$ on the function $\boldsymbol{v}(x^i) = v^i(x)$, $\boldsymbol{v}^3x^i$, the action of $\boldsymbol{v}$ on $\boldsymbol{v}(v^i) = \sum_j v^j\partial_j v^i$, and so on.

The map $\exp : \mathbb{R} \times M \to M$ defined above, which takes the pair (t, x_0) to the point $x(t)$, is the well-known **exponential map**, being also called the **flow** of $\boldsymbol{v}$. This map is such that the following properties hold:

1. *composition*: $\exp\left[(\tau_1 + \tau_2)\boldsymbol{v}\right] x = [\exp(\tau_1\boldsymbol{v}) \circ \exp(\tau_2\boldsymbol{v})]x\,;$
2. *identity*: $\exp(0 \cdot \boldsymbol{v})x = x\,;$
3. *inverse*: $\left[\exp(\tau\boldsymbol{v})\right]^{-1} x = \exp(-\tau\boldsymbol{v})x\,.$

In particular, for fixed x and $\boldsymbol{v}$, this map has the structure of a group, being called the **one-parameter group of diffeomorphism**. In this sense, the vector field $\boldsymbol{v}$ is said to be the **infinitesimal generator** of the group.

Specializing to the case where M is a Lie group G and setting both the starting point to be the identity e and $\boldsymbol{v}$ to be a left invariant vector field, the exponential map $\exp(\boldsymbol{v}) \equiv \exp(\tau\boldsymbol{v})e\big|_{\tau=1}$ becomes a map from the Lie algebra $\mathfrak{g}$ to the group G. In fact, this map takes the Lie algebra $\mathfrak{g}$ onto a neighborhood of the identity element in G. In addition to this, although this map might not be defined globally, we can always write any group element g as a finite product of exponential maps,

$$g = \exp\boldsymbol{v_1} \cdot \exp\boldsymbol{v_2} \cdot \ldots \cdot \exp\boldsymbol{v_k}\,,$$

for some $\boldsymbol{v_1}$, ..., $\boldsymbol{v_k}$ in $\mathfrak{g}$ [17]. In this expression, "$\cdot$" is the usual group multiplication. As an illustration, consider again the $SU(2)$ group with the parameterization given by (3.3) and basis for the $\mathfrak{su}(2)$ Lie algebra by (3.9). Then, let us evaluate $\exp(\alpha\boldsymbol{X_1})$, where α is a real constant. In order to perform this calculation, we need to evaluate the action $(\boldsymbol{X_1})^n x^i$, for $i = 1, 2, 3$, at the identity $x^i = 0$, where

$$\boldsymbol{X_1} = (\sqrt{1 - \vec{x}^{\,2}}\,)\partial_1 - x^3\partial_2 + x^2\partial_3\,.$$

It is easy to verify that $(\boldsymbol{X_1})^n x^1|_e = (-1)^{(n-1)/2}$ for n odd and $(\boldsymbol{X_1})^n x^1|_e = 0$ for n even, whereas $(\boldsymbol{X_1})^n x^2|_e = 0$ and $(\boldsymbol{X_1})^n x^3|_e = 0$ in both cases. Hence, we have:

$$x^1(\alpha) = \exp(\alpha\boldsymbol{X_1}) = \sum_{n=1}^{\infty}\frac{\alpha^n}{n!}(\boldsymbol{X_1})^n x^1\bigg|_{x=0} = \sum_{n \text{ odd}}\frac{(-1)^{(n-1)/2}\alpha^n}{n!} = \sin(\alpha)\,,$$

along with $x^2(\alpha) = 0 = x^3(\alpha)$. Thus, the vector fields $\alpha \boldsymbol{X_1}$ of $\mathfrak{su}(2)$ are mapped to the group elements in G with coordinates $x^1(\alpha) = \sin\alpha$, $x^2 = 0$ and $x^3 = 0$. In this case, the matrix representation, given by (3.3), takes the following form:

$$g(\sin(\alpha), 0, 0) = \begin{bmatrix} \cos(\alpha) & i\sin(\alpha) \\ i\sin(\alpha) & \cos(\alpha) \end{bmatrix} . \tag{3.11}$$

For n-parameter matrix Lie groups G, the group elements $g(\alpha)$ can be expanded in Taylor series around the identity matrix $I = g(0)$, leading to

$$g(\alpha) = I + \sum_{i=1}^{n} \alpha^i Z_i + \dots ,$$

where the α^i are n real parameters and the Z_i are matrices defined by

$$Z_i = \frac{\partial}{\partial \alpha^i} [g(\alpha)] \bigg|_{\alpha^i = 0} .$$

These are the infinitesimal generators of the Lie group as they give rise to the same Lie algebra, with respect to the ordinary commutator of matrices, as the one generated by the left invariant vector fields $\boldsymbol{X_i}$. In this case the exponential map of vector fields $\boldsymbol{X_i}$ is equivalent to the matrix exponentiation of the generators Z_i:

$$\exp\left(\sum\nolimits_i \alpha^i \boldsymbol{X_i}\right) \qquad \Longleftrightarrow \qquad \exp\left(\sum\nolimits_i \alpha^i Z_i\right) .$$

Then, similarly, the Lie algebra $\mathfrak{g}$ is spanned by the matrices Z_i, which is mapped onto a neighborhood of the identity I through the matrix exponentiation. For the $SU(2)$ group with elements $g(x)$ parameterized by (3.3), from $Z_i = \partial g(x)/\partial x^i|_{x^i=0}$,

$$Z_1 = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\sigma_1 \,,\ Z_2 = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_2 \,,\ Z_3 = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3 \,, \tag{3.12}$$

where the σ_i are the well-known Pauli matrices. Thus, $Z_i = i\sigma_i$ and we have:

$$[\sigma_i, \sigma_j] = \sum_{k=1}^{3} 2i\epsilon^k{}_{ij}\sigma_k \quad \Rightarrow \quad [Z_i, Z_j] = \sum_{k=1}^{3} (-2)\epsilon^k{}_{ij} Z_k \,.$$

Notice that, this algebra of commutators is the same as that of Lie brackets satisfied by the vector fields $\boldsymbol{X_i}$, (3.10), illustrating thus the equivalence $Z_i \Leftrightarrow \boldsymbol{X_i}$. Besides this, once the exponential of Pauli matrices are known to satisfy the relation:

$$\exp\left[ia(\boldsymbol{\hat{n}} \cdot \boldsymbol{\vec{\sigma}})\right] = \cos(a)I + i\sin(a)(\boldsymbol{\hat{n}} \cdot \boldsymbol{\vec{\sigma}}) \,,$$

where a is a real parameter and $\boldsymbol{\hat{n}}$ is a three-dimensional unit vector, we have that

$$\exp(\alpha Z_1) = \cos(\alpha)I + \sin(\alpha)Z_1 = \begin{bmatrix} \cos(\alpha) & i\sin(\alpha) \\ i\sin(\alpha) & \cos(\alpha) \end{bmatrix} .$$

Comparing this with (3.11), we see that the exponential map of the vector field $\alpha \boldsymbol{X_1}$ and the matrix exponentiation of αZ_1 resulted in the same group element, agreeing, thus, with the statement that the two exponential maps are equivalent.

Before finishing the section, there are some important points regarding the exponential map of Lie algebras that are worthwhile mentioning:

1. *the covering problem*: in general, one single exponential map does not cover the whole group. Instead, it only maps the Lie algebra onto a neighborhood of the identity. For Lie groups G whose Lie algebras $\mathfrak{g}$ contain a subalgebra $\mathfrak{h}$ that generates a compact subgroup H, a theorem due to Cartan states that the product of exponentials, one of the generators in the complement of $\mathfrak{h}$, and the other with the generators in $\mathfrak{h}$, maps $\mathfrak{g}$ onto the entire Lie group G.

2. *isomorphic Lie algebras*: two Lie groups presenting the same Lie algebra are not necessarily the same. Rather, they are locally isomorphic. In fact, for any given Lie algebra there exists a unique Lie group $\bar{G}$ which is simply connected, called the **universal covering group**. In this case, any other group presenting the same Lie algebra is either identical to $\bar{G}$ or else has the form of a quotient $\bar{G}/D$, where D is a discrete invariant subgroup of $\bar{G}$.

3.4 Transformation Groups

In physics, Lie groups are realized as transformations acting on a manifold M of physical relevance for some theory. More precisely, the action of a Lie group G on the manifold M is defined to be a smooth transformation $\boldsymbol{\tau}$ that assigns to every pair of points $(g, x) \in G \times M$ a point $\boldsymbol{\tau_g} \cdot x$ in M such that for any group elements g, g_1 and $g_2 \in G$ and points $x \in M$, the following properties hold:

(i) *composition law*: $\boldsymbol{\tau_{g_1}} \cdot (\boldsymbol{\tau_{g_2}} \cdot x) = (\boldsymbol{\tau_{g_1}} \circ \boldsymbol{\tau_{g_2}}) \cdot x = \boldsymbol{\tau_{(g_1 \cdot g_2)}} \cdot x$;

(ii) *identity transformation:* $\boldsymbol{\tau_e} \cdot x = x$;

(iii) *inverse transformation*: $(\boldsymbol{\tau_g})^{-1} = \boldsymbol{\tau_{g^{-1}}}$, since $\boldsymbol{\tau_{g^{-1}}} \cdot (\boldsymbol{\tau_g} \cdot x) = \boldsymbol{\tau_{(g^{-1} \cdot g)}} \cdot x = x$.

In particular, due to (iii), for any fixed group element g, the map $\boldsymbol{\tau_g} : M \to M$ is a diffeomorphism on M. On the other hand, fixing the point x and letting the group element g vary, the image of the map $\boldsymbol{\tau} : G \to M$ defines a submanifold O_x on M, called the **orbit of** G **through** x. The group G is then said to be **transitive** on its orbits, since for any two points x_1 and x_2 on O_x there exist group elements g_1, g_2 and g_3 in G such that if $\boldsymbol{\tau_{g_1}} \cdot x = x_1$ and $\boldsymbol{\tau_{g_2}} \cdot x = x_2$ then $\boldsymbol{\tau_{g_3}} \cdot x_1 = x_2$, in which case $g_3 = g_2 \cdot g_1^{-1}$. In other words, any two points in an orbit is connected to each other by means of a transformation $\boldsymbol{\tau_g}$, for some g:

$$\boldsymbol{\tau_{g_1}} \cdot x = x_1 \quad \Rightarrow \quad x = \boldsymbol{\tau_{g_1^{-1}}} \cdot x_1 \quad \Rightarrow \quad x_2 = \boldsymbol{\tau_{g_2}} \cdot x = \boldsymbol{\tau_{g_2}} \cdot (\boldsymbol{\tau_{g_1^{-1}}} \cdot x_1) = \boldsymbol{\tau_{g_2 \cdot g_1^{-1}}} \cdot x_1\,.$$

Besides this, the group G is said to be either **transitive** on M, in the case where $O_x = M$, or **intransitive**, if $O_x \neq M$.

A group action is said to be **simply-transitive** if $\boldsymbol{\tau_{g_1}} \cdot x = \boldsymbol{\tau_{g_2}} \cdot x$ implies $g_1 = g_2$, for all $g_1, g_2 \in G$. Conversely, the action is said to be **multiply-transitive** if there exist distinct group elements g_1 and g_2 such that $\boldsymbol{\tau_{g_1}} \cdot x = \boldsymbol{\tau_{g_2}} \cdot x$, a case of particular interest being when $\boldsymbol{\tau_g} \cdot x = \boldsymbol{\tau_e} \cdot x = x$. In this case, the group elements g form a Lie subgroup, called the **stability group** $S(x)$ **of** x, as can easily be proven:

(i) *closure*: given two group elements g_1 and g_2 in $S(x)$, $\boldsymbol{\tau_{g_1}} \cdot x = x$ and $\boldsymbol{\tau_{g_2}} \cdot x = x$, then $g_1 \cdot g_2$ is also in $S(x)$: $\boldsymbol{\tau_{(g_1 \cdot g_2)}} \cdot x = \boldsymbol{\tau_{g_1}} \cdot (\boldsymbol{\tau_{g_2}} \cdot x) = x$;

(ii) *identity*: the identity element e is in $S(x)$, by definition, since $\boldsymbol{\tau_e} x = x$;

(iii) *inverse*: for every g in $S(x)$: $\boldsymbol{\tau_g} \cdot x = x \Rightarrow x = \boldsymbol{\tau_{g^{-1}}} \cdot x$, implying that if g is in $S(x)$, then so is its inverse g^{-1}.

Notice that if the points x_1 and x_2 of the same orbit are connected to each other by means of $\boldsymbol{\tau_g} \cdot x_1 = x_2$ and g_1 is in $S(x_1)$, namely $\boldsymbol{\tau_{g_1}} \cdot x_1 = x_1$, then the element $g \cdot g_1 \cdot g^{-1}$ belongs to the stability group of x_2, $S(x_2)$:

$$\boldsymbol{\tau_g} \cdot x_1 = x_2 \quad \Rightarrow \quad x_1 = \boldsymbol{\tau_{g^{-1}}} \cdot x_2 \quad \Rightarrow \quad x_1 = \boldsymbol{\tau_{g_1}} \cdot x_1 = \boldsymbol{\tau_{g_1}} \cdot (\boldsymbol{\tau_{g^{-1}}} \cdot x_2)\,.$$

Then, since $x_2 = \boldsymbol{\tau_g} \cdot x_1$, it follows that

$$x_1 = \boldsymbol{\tau_{g_1}} \cdot (\boldsymbol{\tau_{g^{-1}}} \cdot x_2) \quad \Rightarrow \quad x_2 = \boldsymbol{\tau_g} \cdot (\boldsymbol{\tau_{g_1}} \cdot (\boldsymbol{\tau_{g^{-1}}} \cdot x_2)) = \boldsymbol{\tau_{(g \cdot g_1 \cdot g^{-1})}} \cdot x_2\,.$$

Therefore, this shows that $S(x_1)$ and $S(x_2)$ are conjugate subgroups of G.

The exponential map when applied to the map $\boldsymbol{\tau} : G \to O_x$ provides a way to map left invariant vector fields of $\mathfrak{g}$ to vector fields tangent to $O_x \subset M$. In fact, by varying the point x, the submanifolds O_x will cover a region of M, and then smooth vector fields will be defined in the same region. Recall that the differential map preserves the Lie brackets of vector fields, meaning that the basis of left invariant vector fields in G will give rise to a set of vector fields in M satisfying the same Lie algebra. Denoting this mapping by $d\boldsymbol{\tau}(\boldsymbol{X_i}) = \boldsymbol{\xi_i}$, we have that:

$$[\boldsymbol{X_i}, \boldsymbol{X_j}] = \sum_k C^k{}_{ij} \boldsymbol{X_k} \quad \Longrightarrow \quad [\boldsymbol{\xi_i}, \boldsymbol{\xi_j}] = \sum_k C^k{}_{ij} \boldsymbol{\xi_k}\,.$$

Conversely, given a finite set of vector fields $\boldsymbol{\xi_i}$ on M satisfying $[\boldsymbol{\xi_i}, \boldsymbol{\xi_j}] = \sum_k C^k{}_{ij} \boldsymbol{\xi_k}$, for some constants $C^k{}_{ij}$, there always exists a Lie group acting on M, whose Lie algebra has as structure constants the same constants $C^k{}_{ij}$ [24]. Besides this, the stability group $S(x)$ of a point x is generated by those $\boldsymbol{v} \in \mathfrak{g}$ such that the mapping $d\boldsymbol{\tau}(\boldsymbol{v})|_x = 0$, namely the Kernel of the linear map $d\boldsymbol{\tau}$ at x. In particular, it follows from this that the dimensions r of the group G, d of the orbit through x and s of the stability group $S(x)$ are related to each other by the relation $r = d + s$.

A very simple example of group action is given by the one-parameter group of diffeomorphism introduced in the previous section. Indeed, provided a vector field $\boldsymbol{\xi}$ defined on a manifold M, the exponential map $\exp : \mathbb{R} \times M \to M$ defined by $\boldsymbol{\tau}_\tau = \exp(\tau \boldsymbol{\xi})$ takes a point $(\tau, x) \in [G = (\mathbb{R}, +)] \times M$ and assigns to it a point $\exp(\tau \boldsymbol{\xi}) x \in M$ along the integral curves of $\boldsymbol{\xi}$, starting from x and walking along the curve for a "time" τ. $G = (\mathbb{R}, +)$ is the Lie group defined by the real line endowed with the sum, similar to the discrete group $(\mathbb{Z}, +)$.

3.5 Isometries

Notice that, so far, nothing has been said about whether the manifolds in which the groups are acting are equipped with structures such as a metric or connection. In the particular case of Riemannian manifolds, it is of great importance to study group transformations that leave the metric invariant, namely diffeomorphisms ϕ such that the pullback of the metric $\boldsymbol{g}$ by ϕ results in $\phi^* \boldsymbol{g} = \boldsymbol{g}$. When this happens, the diffeomorphism ϕ is a **symmetry transformation** called **isometry**.

Assuming (x^μ) to be a coordinate system defined in M and letting ϕ be a diffeomorphism on M which takes points in the coordinates x^μ to the ones in $x'^\mu = x'^\mu(x)$, the pullback of the metric tensor $\boldsymbol{g}$ by the diffeomorphims ϕ reads[2]

$$(\phi^* \boldsymbol{g})_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x')\,. \tag{3.13}$$

Then, the invariance of the metric under the map ϕ is equivalent to the equation

$$(\phi^* \boldsymbol{g})_{\mu\nu}(x) = g_{\mu\nu}(x)\,, \tag{3.14}$$

meaning that isometries are transformations that leave the functional form of the metric invariant. Equations of this form are extremely difficult to be solved directly and, instead, we consider **infinitesimal transformations**:

$$x^\mu \mapsto x'^\mu(x) = x^\mu + \epsilon\, \xi^\mu(x)\,, \tag{3.15}$$

where ϵ is an infinitesimal parameter and $\xi^\mu(x)$ are the components of a vector field $\boldsymbol{\xi}(x)$, called the **infinitesimal generator** of the transformation. Hence, a finite transformation is constructed through successive applications of (3.15), giving rise, ultimately, to the exponential map of the vector field $\boldsymbol{\xi}$ discussed in the previous sections. In order to work out the general form for the infinitesimal generators of isometries, consider the infinitesimal tranformations (3.15). Then, it follows that

$$\frac{\partial x'^\rho}{\partial x^\mu} = \delta^\rho_\mu + \epsilon\, \partial_\mu \xi^\rho \quad \text{and} \quad g_{\mu\nu}(x') = g_{\mu\nu}(x) + \epsilon\, \xi^\lambda \partial_\lambda g_{\mu\nu}(x) + O(\epsilon^2)\,,$$

where the second of these expressions was obtained via Taylor expansion. Plugging these results into equation (3.14), with help of the definition (3.13), we are lead to

$$\begin{aligned} g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') &= \left(\delta^\rho_\mu + \epsilon\, \partial_\mu \xi^\rho\right)\left(\delta^\sigma_\nu + \epsilon\, \partial_\nu \xi^\sigma\right)\left[g_{\rho\sigma}(x) + \epsilon\, \xi^\lambda \partial_\lambda g_{\rho\sigma}(x)\right] \\ &= g_{\mu\nu}(x) + \epsilon\left(\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi^\sigma + g_{\rho\nu}\partial_\mu \xi^\rho\right)\,, \end{aligned}$$

where we have kept only the terms up to first-order in ϵ. Notice that, the content embraced by the brackets in this last expression is the Lie derivative of the metric tensor $\boldsymbol{g}$ along the vector field $\boldsymbol{\xi}$, $\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g}$. Thus, the requirement that the infinitesimal transformation is an isometry translates to the following equation for ξ^μ:

$$(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g})_{\mu\nu} = \left(\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi^\sigma + g_{\rho\nu}\partial_\mu \xi^\rho\right) = 0\,. \tag{3.16}$$

In turn, this equation can be put in the following simpler form:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu}\xi_{\nu)} = 0\,, \tag{3.17}$$

where we have made use of $\partial_\lambda g_{\mu\nu} = \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} + \Gamma^\rho_{\lambda\mu} g_{\rho\nu}$ and $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$, which are direct consequence of the fact that Riemannian manifolds are endowed with a torsion free metric-compatible connection. Equation (3.17) is known as the **Killing equation**, and the infinitesimal generators $\boldsymbol{\xi}$ satisfying it are called **Killing vector fields**.

[2]From now on, it is convenient to make use of the Einstein notation, which says that to any repeated index, there is a summation implied over all the values allowed for that index.

We notice from equation (3.16) that, if in some coordinate system (x^μ) the components of the metric $g_{\mu\nu}$ do not depend on one of the coordinates, say x^1, then the vector field defined by $\boldsymbol{\xi} = \partial_1$ is a Killing vector field. In this case, the isometry is easily seen to be given by $x'^\mu(x) = x^\mu + a\delta_1^\mu$, where a is a constant.

An useful equation that relates the second order derivative of a Killing vector field $\boldsymbol{\xi}$ with components ξ^μ to the Riemann tensor $R_{\mu\nu\rho}{}^\sigma$ can be obtained from the defining equation for the Riemann tensor

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\xi_\rho = R_{\mu\nu\rho}{}^\sigma \xi_\sigma \,.$$

Indeed, calling this equation $(\mu\nu\rho)$, evaluation of the sum $(\mu\nu\rho) - (\nu\rho\mu) - (\rho\mu\nu)$, with further use of the index symmetries of the Riemann tensor and the Bianchi identity $R_{[\mu\nu\rho]}{}^\sigma = 0$, we easily arrive at the following equation:

$$\nabla_\rho \nabla_\nu \xi_\mu = R_{\mu\nu\rho}{}^\sigma \xi_\sigma \,. \tag{3.18}$$

Notice that, equation (3.18) gives the second order partial derivatives of the Killing vector field ξ^μ in terms of its first order derivatives and its components themselves. It is not difficult to see from this that higher order derivatives are also expressed in terms of these same quantities, namely the n components ξ_μ and the $n(n-1)/2$ components $\nabla_\mu \xi_\nu$ (since $\nabla_\mu \xi_\nu = -\nabla_\nu \xi_\mu$), where n is the dimension of the manifold on which the vector field ξ^μ is defined. Thus, Killing vector fields are fully characterized by these $n + n(n-1)/2 = n(n+1)/2$ quantities. In particular, if $\xi_\mu = 0$ and $\nabla_\mu \xi_\nu = 0$ at some point, then $\boldsymbol{\xi} \equiv 0$. Consequently, there cannot be more than $n(n+1)/2$ independent Killing vector fields in a Riemannian manifold.

The set of Killing vector fields in a Riemannian manifold turns out to form a Lie algebra. This fact follows directly from the two following properties of Lie derivatives: for any vector fields $\boldsymbol{v}$, $\boldsymbol{w}$, and constants a, b,

(i) *linearity*: $\mathcal{L}_{a\boldsymbol{v}+b\boldsymbol{w}} = a\mathcal{L}_{\boldsymbol{v}} + b\mathcal{L}_{\boldsymbol{w}}$;

(ii) *commutation*: $\mathcal{L}_{\boldsymbol{v}}\mathcal{L}_{\boldsymbol{w}} - \mathcal{L}_{\boldsymbol{w}}\mathcal{L}_{\boldsymbol{v}} = \mathcal{L}_{[\boldsymbol{v},\boldsymbol{w}]}$.

Indeed, since Killing vector fields $\boldsymbol{\xi}$ are such that $\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} = 0$, given two of such vectors, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the first of these properties provides that the linear combination $a\boldsymbol{\xi} + b\boldsymbol{\eta}$ is also a Killing vector field:

$$\mathcal{L}_{a\boldsymbol{\xi}+b\boldsymbol{\eta}}\boldsymbol{g} = a\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} + b\mathcal{L}_{\boldsymbol{\eta}}\boldsymbol{g} = 0 \,.$$

Thus, since the number of independent Killing vector fields is finite, they form a finite dimensional vector space. In addition to this, from the second of the properties above, this vector space is closed under the Lie bracket of vector fields:

$$\mathcal{L}_{[\boldsymbol{\xi},\boldsymbol{\eta}]}\boldsymbol{g} = (\mathcal{L}_{\boldsymbol{\xi}}\mathcal{L}_{\boldsymbol{\eta}} - \mathcal{L}_{\boldsymbol{\eta}}\mathcal{L}_{\boldsymbol{\xi}})\boldsymbol{g} = 0 \,.$$

From these results, it becomes clear that the set of Killing vector fields on an n-dimensional Riemannian manifold forms, indeed, a Lie algebra. In this case, the dimension of the algebra is no larger than $n(n+1)/2$. Consequently, since to any Lie algebra of vector fields there exists a corresponding Lie group of transformations, the set of Killing vector fields defines a Lie group, called the **isometry group**, the corresponding stability group being known as the **isotropy group**.

Another important class of diffeomorphisms ϕ on Riemannian manifolds is the one that leaves the metric invariant up to a conformal factor $\Omega^2 = \Omega^2(x)$:

$$(\phi^* \boldsymbol{g})_{\mu\nu} = \Omega^2 g_{\mu\nu} \,.$$

Transformations of this type are called **conformal isometries**. By means of the infinitesimal transformation $x^\mu \mapsto x'^\mu(x) = x^\mu + \epsilon\psi^\mu$, and assuming that $\Omega^2(x) \approx 1 + \epsilon f(x)$, by a procedure similar to the above for Killing vector fields, we obtain

$$(\mathcal{L}_{\boldsymbol{\psi}} \boldsymbol{g})_{\mu\nu} = \nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu = f(x) g_{\mu\nu} \,. \tag{3.19}$$

This equation is known as the **conformal Killing equation**, and the generators of conformal isometries $\boldsymbol{\psi}$ are therefore called **conformal Killing vector fields**. Contracting the indices μ and ν in (3.19), and replacing the expression for $f(x)$ into the same equation, we obtain the following equivalent equation:

$$\nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu = \frac{2}{n} \left(\nabla^\rho \psi_\rho \right) g_{\mu\nu} \,. \tag{3.20}$$

Just as for Killing vector fields, the conformal Killing vector fields defined on a Riemannian manifold happen to form a Lie algebra. In fact, this can easily be proven, following the same steps as we did above for the Killing vector fields.

3.5.1 Conserved Quantities Along the Geodesic Motion

An important feature carried by Killing vector fields is that, through them, we are able to build quantities that are conserved along geodesic paths. In fact, these constants of the motion will turn out to be very effective in the attainment of the geodesic motion, as, in general, the geodesic equation cannot be solved analytically without the use of symmetries.

Let M be an n-dimensional manifold with coordinate system (x^μ) defined on it, and assume $u^\mu = dx^\mu/d\tau$ to be the tangent vector to a geodesic path parameterized by the affine parameter τ: $u^\mu \nabla_\mu u^\nu = 0$. Then, the quantity defined by $\xi_\mu u^\mu$ is conserved along the geodesic motion:

$$u^\mu \nabla_\mu (\xi_\nu u^\nu) = u^\mu u^\nu \nabla_\mu \xi_\nu + (u^\mu \nabla_\mu u^\nu) \xi_\nu = 0 \,,$$

where it has been made use of the fact that $u^\mu u^\nu \nabla_\mu \xi_\nu = u^\mu u^\nu \nabla_{(\mu} \xi_{\nu)} = 0$, since ξ^μ is a Killing vector field, and that u^μ is tangent to an affinely parameterized geodesic. If, instead of a Killing vector field we had considered a conformal Killing vector field $\boldsymbol{\psi}$ and had constructed the quantity $\psi_\mu u^\mu$, we would have obtained:

$$u^\mu \nabla_\mu (\psi_\nu u^\nu) = u^\mu u^\nu \nabla_{(\mu} \psi_{\nu)} + (u^\mu \nabla_\mu u^\nu) \psi_\nu = \frac{1}{n} \left(\nabla^\rho \psi_\rho \right) g_{\mu\nu} u^\mu u^\nu \,,$$

where we have made use of $\nabla_{(\mu} \psi_{\nu)} = (\nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu)/2$, along with the defining equation for conformal Killing vector fields (3.20). Notice that the right hand side of the above equation vanishes only for null geodesics, $g_{\mu\nu} u^\mu u^\nu = 0$. Thus, conformal Killing vector fields give rise to conserved quantities along null geodesics.

3.6 Maximally Symmetric Spaces

A **maximally symmetric space** is an n-dimensional Riemannian manifold that admits the action of an isometry group with maximal number of generators, namely $n(n+1)/2$. In this section we list the most important properties of such spaces.

Consider Riemannian manifolds with Riemann tensor given in the form below:

$$R_{\mu\nu\rho\sigma} = \mathcal{R}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,. \tag{3.21}$$

In fact, any two-dimensional Riemannian manifold has a Riemann tensor of this form, where $\mathcal{R}$ represents a measure of curvature called the **sectional curvature**. For manifolds with dimension bigger than two, at each point we may choose a pair of tangent vectors, which will determine a two-dimensional submanifold by the geodesics passing through this point, with initial tangent vector given by the vectors in the tangent subspace spanned by this pair of vectors. In this case, the Riemann tensor acquire the above form when restricted to this two-dimensional submanifold. Consider, then, the special case in which (3.21) holds regardless of the direction we choose. In this case, the sectional curvature $\mathcal{R}$, at any point, does not depend on this direction either. This, in fact, endows this space with a notion of isotropy. Besides this, contracting $R_{\mu\rho\nu}{}^{\rho} = R_{\mu\nu}$, and further $R_{\mu}{}^{\mu} = R$, to obtain Ricci scalar, we arrive at $R = n(n-1)\mathcal{R}$. Bianchi identity $\nabla_{[\lambda}R_{\mu\nu]\rho}{}^{\sigma} = 0$ then yields $(n-2)(n-1)\partial_\mu\mathcal{R} = 0$. In particular, for dimensions bigger than two, $\mathcal{R}$, and hence R, is constant. For this reason, spaces of the form (3.21), and in special of dimension two with constant $\mathcal{R}$, are called **spaces of constant curvature**.

An important feature of spaces of constant curvature is that they admit the action of an isometry group with maximal number of generators. In fact, it is also possible to prove the converse: any Riemannian manifold admitting a maximal isometry group is necessarily a space of constant curvature [24]. Therefore, spaces of constant curvature are maximally symmetric spaces, and vice-versa. Consequently, equation (3.21) with $\mathcal{R}$ constant can be adopted, equivalently, as the definition for maximally symmetric spaces.

The action of the isometry group on maximally symmetric spaces is transitive on the whole manifold. In other words, any point in the manifold can be taken continuously to any other through an isometry transformation. Spaces with this characteristic are said to be **homogeneous**. Thus, maximally symmetric spaces, in addition to being isotropic, as discussed above, are homogeneous. For this reason, this class of spaces are widely used in cosmology to describe the spacelike hypersurfaces of cosmological models [28].

Notice that the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ can be easily evaluated for maximally symmetric spaces of dimension n using equation (3.21):

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \left(\frac{1}{n} - \frac{1}{2}\right) Rg_{\mu\nu}\,.$$

From this, we see that such a class of spaces is necessarily solution for the vacuum Einstein's equation with a cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \qquad \text{where } \Lambda \equiv \left(\frac{1}{2} - \frac{1}{n}\right) R\,. \tag{3.22}$$

In particular, spaces of this type with a Lorentzian signature, $g_{\mu\nu} = \mathrm{diag}(- + \ldots +)$, are called either a **de Sitter space** dS_n, if $R > 0$ ($\Lambda > 0$), or an **anti-de Sitter space** AdS_n, if $R < 0$ ($\Lambda < 0$). The case where $R = 0$ is necessarily the flat space.

It is simple to verify that spaces with Riemann tensor defined by equation (3.21) are such that all the components of the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$$

are identically vanishing. For spaces of dimension equal or larger than four the vanishing of the Weyl tensor is equivalent to the statement that the metric tensor is conformal to the metric of the flat space [29]. In other words, there exists a coordinate system such that

$$g_{\mu\nu} = \Omega^2\eta_{\mu\nu}\,, \tag{3.23}$$

where $\eta_{\mu\nu}$ stands for the components of the metric tensor of the flat space. On the other hand, two-dimensional spaces are always conformal to the flat space, while three-dimensional spaces, which are such that the Weyl tensors is always zero, are conformally flat if, and only if, the Cotton tensor vanishes [30]:

$$C_{\mu\nu\lambda} = \nabla_\lambda R_{\mu\nu} - \nabla_\nu R_{\mu\lambda} + \frac{1}{2(n-1)}(g_{\mu\lambda}\nabla_\nu R - g_{\mu\nu}\nabla_\lambda R) = 0\,.$$

It turns out that, since $R_{\mu\nu} = Rg_{\mu\nu}/n$ and R is constant for maximally symmetric spaces, we have $\nabla_\lambda R_{\mu\nu} = (\nabla_\lambda R)g_{\mu\nu}/n = 0$, which implies that the Cotton tensor vanishes for such spaces, irrespective of the dimension n. Thus, we conclude that maximally symmetric spaces of any dimension are conformally flat. Conversely, starting from a metric tensor with components given in the form (3.23), and then integrating Einstein's equation (3.22), which can be accomplished with the aid of the Cartan vielbein formalism, we obtain:

$$\Omega^2 = \left[1 + \frac{\Lambda(\eta_{\mu\nu}x^\mu x^\nu)}{2(n-1)(n-2)}\right]^{-2} \quad\Rightarrow\quad g_{\mu\nu} = \left[1 + \frac{\Lambda(\eta_{\mu\nu}x^\mu x^\nu)}{2(n-1)(n-2)}\right]^{-2}\eta_{\mu\nu}\,,$$

where $\eta_{\mu\nu} = \mathrm{diag}(+\ldots+-\ldots-)$ are the components of the flat metric in arbitrary dimension and signature. In fact, spaces endowed with the metric above are maximally symmetric, as can be seen from the fact that, by definition, the scalar $\eta_{\mu\nu}x^\mu x^\nu$ is invariant under the $O(k, n-k)$ group, which is of dimension $n(n+1)/2$. Thus, we see that not only maximally symmetric spaces are conformally flat and satisfies the vacuum Einstein's equation with a cosmological constant, but also that conformally flat spaces satisfying such a equation are maximally symmetric.

For homogeneous spaces M with an isometry group G and isotropy at some point x given by H, it can be shown that $M = G/H$. In other words, every group transformation $\boldsymbol{\tau_g}\cdot x$, for group elements that differ only by a multiplication by an element of H on the right, is equivalent to some point in this manifold in a one-to-one manner. In the particular case of n-dimensional de Sitter spaces, the isometry group is given by the generalized orthogonal group $O(1,n)$, while the isotropy group is given by $O(1,n-1)$. Thus, $dS_n = O(1,n)/O(1,n-1)$. Notice that the dimension of the group $O(1,n)$ is $n(n+1)/2$, which leads to $r = n$, the dimension of the orbits. Similarly, we have that $AdS_n = O(2,n-1)/O(1,n-1)$.

Before finishing the section, there are still two other important features shared by this class of spaces that are worthwhile mentioning. The first one concerns the number of conformal isometries these spaces have. In fact, maximally symmetric spaces not only count with the highest number of isometries, but also with the maximum number of conformal isometries. Indeed, it is possible to prove that a space admits the maximal number of independent conformal Killing vector fields if, and only if, it is conformally flat [30]. In this case, the maximal number is $(n+1)(n+2)/2$, where n is the dimension of the manifold. The second feature concerns the uniqueness of maximally symmetric spaces: two maximally symmetric spaces sharing the same signature and sectional curvature $\mathcal{R}$ are locally the same.

3.7 Hidden Symmetries

Besides the conserved quantities linear in the velocity u^μ of a geodesic curve built from Killing vectors, a space may admit the existence of conserved quantities of higher order in these velocities [31, 32]. Unlike the former ones, whose origin lies in the existence of symmetries of the metric tensor, the symmetries associated to higher order conserved quantities, the so-called **hidden symmetries**, are related to the existence of first integrals in the cotangent bundle of Riemannian manifolds.

One of such conserved quantities can be constructed from **conformal Killing tensors**, which are defined to be totally symmetric tensors of rank r satisfying:

$$\nabla_{(\nu} K_{\mu_1\mu_2\ldots\mu_r)} = g_{\nu(\mu_1}\tilde{K}_{\mu_2\ldots\mu_r)} \,, \tag{3.24}$$

with $\tilde{K}_{\mu_2\ldots\mu_r}$ being another totally symmetric tensor of rank $(r-1)$. It is easy to prove that the quantity $K_{\mu_1\mu_2\ldots\mu_r}u^{\mu_1}u^{\mu_2}\cdots u^{\mu_r}$, where u^μ is the tangent vector to an affinely parameterized null geodesic, is conserved along the geodesic. In particular, when $\tilde{K}_{\mu_2\ldots\mu_p} = 0$, equation (3.24) becomes the defining equation for **Killing tensors**. For these latter, the quantity $K_{\mu_1\mu_2\ldots\mu_r}u^{\mu_1}u^{\mu_2}\cdots u^{\mu_r}$ is, in fact, conserved for any kind of geodesic motion, not being restricted only to null geodesics, as can easily be checked. In particular, the metric tensor of Riemannian manifolds provides an example of a rank-2 Killing tensor, since it is, by definition, covariantly constant. Notice, in particular, that this definition of conformal Killing tensors provides a generalization for Killing vector fields, as these can be obtained from equation (3.24) by setting $r = 1$ and $\tilde{\boldsymbol{K}} = 0$.

The generalization towards the antisymmetric tensors also exists and these are called **conformal Killing-Yano tensors**. Such quantities are defined as totally antisymmetric tensors of rank r (differential r-forms) whose components satisfy

$$\nabla_{(\nu} Y_{\mu_1)\mu_2\ldots\mu_r} = g_{\nu\mu_1}\tilde{Y}_{\mu_2\ldots\mu_r} - (r-1)g_{[\mu_2(\nu}\tilde{Y}_{\mu_1)\mu_3\ldots\mu_r]} \,, \tag{3.25}$$

where $\tilde{Y}_{\mu_2\ldots\mu_r}$ is some $(r-1)$-form. Contracting this equation with the inverse metric tensor $g^{\mu_1\nu}$, we easily arrive at the following equation for $\tilde{\boldsymbol{Y}}$:

$$\tilde{Y}_{\mu_2\ldots\mu_r} = \frac{1}{(n-r+1)}\nabla^{\mu_1}Y_{\mu_1\mu_2\cdots\mu_r} \,.$$

As a matter of checking, choosing $r = 1$ in equation (3.25), it reduces to the conformal Killing equation, and to the Killing equation if, in addition to $r = 1$, we set $\tilde{\boldsymbol{Y}} = 0$. For the case in which the tensor $\tilde{\boldsymbol{Y}}$ is vanishing, the rank-r antisymmetric tensor $\boldsymbol{Y}$ is, then, called a **Killing-Yano tensor**. Killing-Yano tensors are such that their "square" is a rank-2 Killing tensor:

$$K_{\mu\nu} = Y_{\mu\mu_2\mu_3\cdots\mu_r}Y_{\nu}{}^{\mu_2\mu_3\cdots\mu_r} \,,$$

as can easily be proved after a little of algebra. Therefore, any space admitting a Killing-Yano tensor of arbitrary rank, necessarily possess a rank-2 Killing tensor. The converse is not always true, though. Besides this, for Einstein space, namely spaces where $R_{\mu\nu} \propto g_{\mu\nu}$, or equivalently (3.22), from a rank-2 Killing-Yano tensor, we can construct a Killing vector field by means of the definition

$$\xi^{\mu} = \frac{1}{n-1} \nabla_{\nu} Y^{\mu\nu} \,. \tag{3.26}$$

Indeed, using the definition of Riemann tensor in terms of covariant derivatives along with the Bianchi identity $R_{[\mu\nu\rho]}{}^{\sigma} = 0$, is it not difficult to show that [33]

$$\nabla_{(\mu}\xi_{\nu)} = -\frac{1}{n-2} R_{\lambda(\mu} Y_{\nu)}{}^{\lambda} \,.$$

Then, for Einstein spaces, the right hand side of this equation vanishes as a consequence of the antisymmetry of the tensor $\boldsymbol{Y}$. Thus, we see that Einstein spaces admitting a rank-2 Killing-Yano tensor possess a Killing vector of the form (3.26).

3.8 Group Action on Spaces with a Separability Structure of Type $\mathcal{S}_{n-2}$

In this section, we shall study spaces admitting a separability structure of type $\mathcal{S}_{n-2}$, as obtained in the previous chapter, in light of the approach of group action described in the present chapter. This will hopefully provide us with insights and definitely with a better understanding of such spaces.

Recall, these Riemannian manifolds are described by the metric given by

$$\boldsymbol{g}^{-1} = \frac{1}{\phi_1+\phi_2}\left[(\eta_1^{\alpha\beta}\psi_1 + \eta_2^{\alpha\beta}\psi_2)\partial_\alpha \otimes \partial_\beta + \psi_1 \partial_x \otimes \partial_x + \psi_2 \partial_y \otimes \partial_y\right] ,$$

where functions with subscript 1 are functions of x, while the ones with subscript 2 are functions of the coordinate y. Besides this, the coordinates with Greek indices, α and β, varies among the coordinates $\{\sigma^3, \sigma^4, \cdots, \sigma^n\}$, where n is dimension of the manifold. In addition to this, remember also that such spaces are endowed with a nontrivial rank-2 Killing tensor given in this same coordinates by

$$\boldsymbol{K} = \frac{1}{\phi_1+\phi_2}\left[(\eta_1^{\alpha\beta}\phi_2\psi_1 - \eta_2^{\alpha\beta}\phi_1\psi_2)\partial_\alpha \otimes \partial_\beta + \phi_2\psi_1 \partial_x \otimes \partial_x - \phi_1\psi_2 \partial_y \otimes \partial_y\right] .$$

We immediately notice from the metric tensor above that, since it does not depend on the coordinates σ^α, the coordinate vectors $\partial_\alpha = \partial_{\sigma^\alpha}$ are Killing vector fields. Hence, once coordinate vectors always commute with each other, we have that the Lie algebra generated by them is abelian, namely all the constants $C^a{}_{bc} = 0$, giving rise, therefore, to an abelian Lie group. Thus, since this Lie algebra is $(n-2)$-dimensional, the corresponding Lie group of isometries is of same dimension.

As we have seen along this chapter, once we have the Lie algebra of Killing vector fields of a Riemannian manifold, the isometry group is then obtained through the exponential map. In the present case, we can easily obtain this group action:

$$\tilde{\sigma}^\alpha = \exp(a^\beta \partial_\beta)\sigma^\alpha = \left(1 + a^\beta\partial_\beta + \frac{1}{2}a^\beta a^\gamma \partial_\beta\partial_\gamma + \cdots\right)\sigma^\alpha = \sigma^\alpha + a^\alpha \,,$$

where the constants a^α are parameters spanning the group elements of the Lie group. Similarly, for the coordinates x and y, we have

$$\tilde{x} = \exp(a^\alpha\partial_\alpha)x = x \quad \text{and} \quad \tilde{y} = \exp(a^\alpha\partial_\alpha)y = y \,.$$

From this transformation, it becomes clear the abelian character of the group. This group is isomorphic to the simply connected Lie group of translations $(\mathbb{R}^{n-2}, +)$ which is, in fact, the universal covering group of the Lie algebra presented.

It is also clear from the action of the isometry group above that their orbits are the submanifolds W_{n-2} defined in the previous chapter as the hypersurfaces of constant x and y. Hence, this group action is transitive on these submanifolds. Moreover, this action is simply-transitive, since

$$\exp(a^\beta \partial_\beta)\sigma^\alpha = \exp(b^\beta \partial_\beta)\sigma^\alpha \quad \Leftrightarrow \quad a^\alpha = b^\alpha \, .$$

In particular, since $\exp(a^\beta \partial_\beta)\sigma^\alpha = \sigma^\alpha \Leftrightarrow a^\beta = 0$, the isotropy group is trivial.

Regarding conserved quantities along the geodesic motion, the isometry group of the class of metrics considered here, generated by the Killing vector fields ∂_α, gives rise to the $(n-2)$ conserved quantities q_α defined by

$$q_\alpha = (\partial_\alpha)_\mu u^\mu = g_{\mu\nu}(\partial_\alpha)^\mu u^\nu = g_{\alpha\nu}\frac{dx^\nu}{d\tau} \, ,$$

where u^μ is the tangent vector to an affinely parameterized curve $x(\tau)$. As stressed in the previous section, these conserved quantities have origin in the symmetries of the Riemannian manifold. On the other hand, the following conserved quantities

$$q_1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad \text{and} \quad q_2 = K_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \, ,$$

where $g_{\mu\nu}$ and $K_{\mu\nu}$ are respectively the components of $\boldsymbol{g}$ and $\boldsymbol{K}$, have origin in the symmetries of the geodesic Hamiltonian defined on the cotangent bundle of our Riemannian manifold. Hence, it follows that, once n independent conserved quantities were obtained through symmetries, the geodesic motion is analytically attainable for such a class of metrics. In fact, this is in total accordance with the results found in chapter 2 and reinforces the usefulness of symmetries.

4 A Class of Integrable Metrics Coupled to Gauge Fields

4.1 Introduction

Since the advent of Einstein's theory of general relativity in 1915 there has been an extensive attempt to obtain exact solutions for the field equations. Since then, general classes of spaces have been investigated, each one taking into account different geometric anzätze. The use of symmetries is one of the most desired features one would like to impose in this search, since the existence of these are known to be connected to the notion of integrability [31, 32]. Particular emphasis is put on the integrability of the Klein-Gordon and Schrödinger equations, which have been attained in some cases thanks to the existence of symmetries [8].

As we have seen in chapter 2, the study of separability of the Hamilton-Jacobi equation for the geodesic Hamiltonian naturally gives rise to a general class of spaces with definite separability structure and symmetries. In particular, the most general n-dimensional metric endowed with $(n-2)$ commuting Killing vectors and a nontrivial rank-2 Killing tensor was found and studied there when we considered separability structures of type $\mathcal{S}_{n-2}$. The importance of investigating such a class resides in the fact that the attainment of the complete integral is always possible in such a case due to the number of symmetries these spaces have. In fact, as shown explicitly in section 3.8, the $(n-2)$ Killing vector fields together with the metric and Killing tensor provide n first integrals, allowing, thus, the complete integrability of the geodesic motion. When restricted to $n=4$, this general line element, given in section 2.9 by equation (2.34), takes the following form:

$$g^{ab}\partial_a\partial_b = \frac{1}{S_1(x)+S_2(y)}\Big[G_1^{ij}(x)\partial_i\partial_j - G_2^{ij}(y)\partial_i\partial_j + \Delta_1(x)\partial_x^2 + \Delta_2(y)\partial_y^2\Big]\,, \tag{4.1}$$

where we have defined $\eta_1^{ij}\psi_1 = G_1^{ij}(x)$, $\eta_2^{ij}\psi_2 = -G_2^{ij}(y)$, $\psi_1 = \Delta_1(x)$, $\psi_2 = \Delta_2(y)$, $\phi_1 = S_1(x)$ and $\phi_2 = S_2(y)$. In this notation, the indices a and b range over the coordinates $\{\tau,\,\sigma,\,x,\,y\}$, while i, j run over $\{\tau,\,\sigma\}$. Moreover, the functions $G_1^{ij}(x)$ and $G_2^{ij}(y)$ are symmetric in their upper indices, i.e., $G_1^{ij} = G_1^{ji}$ and $G_2^{ij} = G_2^{ji}$. Besides this, we have used the notation $\partial_a\partial_b$ for $\partial_a \otimes \partial_b$. In the language of the chapter 2, the coordinates used in this metric are normal separable, τ and σ being first class coordinates, while x and y are of second class. In this case, the two commuting Killing vector fields are the coordinate vectors ∂_τ and ∂_σ, and the nontrivial rank-2 Killing tensor $\boldsymbol{K}$, written in terms of the new functions G_1^{ij}, G_2^{ij}, Δ_1, Δ_2, S_1 and S_2, is given by (see section 2.9)

$$\boldsymbol{K} = \frac{1}{S_1+S_2}\Big[S_1 G_2^{ij}\partial_i\partial_j + S_2 G_1^{ij}\partial_i\partial_j + \Delta_1 S_2 \partial_x^2 - S_1\Delta_2\partial_y^2\Big]\,. \tag{4.2}$$

As stressed earlier in this introduction, this Killing tensor, along with the two Killing vectors ∂_τ and ∂_σ, ensures the complete integrability of the geodesic equation for spaces given by the metric (4.1).

Recently, the authors of Ref. [11], A. Anabalón and C. Batista, attained the complete integration of Einstein's vacuum equation for the general class of metrics of the form (4.1), assuming the following restriction on the functions G_1^{ij} and G_2^{ij}:

$$\det G_1^{ij} \equiv G_1^{\tau\tau} G_1^{\sigma\sigma} - G_1^{\tau\sigma} G_1^{\tau\sigma} = 0 \quad \text{and} \quad \det G_2^{ij} \equiv G_2^{\tau\tau} G_2^{\sigma\sigma} - G_2^{\tau\sigma} G_2^{\tau\sigma} = 0\,. \tag{4.3}$$

These constraints are interesting as they guarantee that both of the terms $G_1^{ij}\partial_i\partial_j$ and $G_2^{ij}\partial_i\partial_j$ appearing in the line element (4.1) can be written as the square of a vector field, so that the spacetime has a naturally defined Lorentz frame and, consequently, a natural null tetrad. In particular, this class of spaces contains the most physically relevant analytical solutions of Einstein's equation, as it contains the Kerr-(A)dS spacetime [5]. In effect, in the particular case of Lorentzian signature, the metric (4.1) represents a stationary and axisymmetric spacetime. As a matter of fact, the constraints (4.3) give rise to two independent geodesic and shear-free null congruences, even without imposing a field equation. Hence, making use of the Goldberg-Sachs theorem, the authors were able to completely integrate Einstein's vacuum equation for the spaces under consideration.

Following these general lines, we aim to broaden their results by allowing the existence of a gauge field on backgrounds described by the line element (4.1) in the so-called Einstein-Yang-Mills theory. More precisely, we intend to analytically integrate the field equations of this theory for an arbitrary gauge group, for the class of metrics (4.1) under (4.3). A drawback that we immediately face is that the Goldberg-Sachs theorem is no longer valid in this case. Nevertheless, we surpass this problem by requiring the gauge fields to be aligned to the principal null directions of the spacetime, a feature shared by almost all known charged black hole solutions.

Einstein-Yang-Mills (EYM) theory is an interacting theory describing the dynamics of a non-abelian gauge field coupled to the gravitational field. Alone, the Yang-Mills (YM) theory plays an essential role in the unification of fundamental interactions in particle physics. In the framework of the EYM theory, due to the complexity and the non-linear character of the field equations, until now almost all known solutions have been found numerically [34, 35, 36, 37, 38], although in a few specific cases, exact solutions could also be obtained [39, 40]. The first of these numerical solutions was presented by R. Bartnik and J. McKinnon in 1988 [41] for the case of a four-dimensional static spherically symmetric spacetime, describing solitonic solutions. Two years later, a black-hole counterpart was found, also numerically, by P. Bizon [36]. Interestingly, this new solution presented hair, contradicting the well-known no-hair conjecture for black hole solutions, as it presented globally vanishing YM-charges characterizing the black hole. In fact, other asymptotically flat *colored* black hole solutions were also presented later on [34, 37]. In all these cases, the $SU(2)$ was the favorite gauge group used to describe the source of matter. Nevertheless, other gauge groups were also investigated. In particular, some special EYM systems with the Lorentz group $SO(n-1,1)$ as the gauge group have revealed to be equivalent to modified theories of gravity [42, 43]. The $SO(n)$, along with the $SU(n)$, has also been used in the context of cosmology to study the evolution of the early Universe in the inflationary epoch [40, 44, 45].

4.2 Geometric Characterization of Spacetimes

In what follows, a geometric characterization of spacetimes is provided by studying the Petrov classification, the optical scalars, the Frobenius theorem, and, finally, the Goldberg-Sachs theorem. These four elements will turn out to be of central importance in both understanding the general geometric properties of the spacetimes (4.1) and in the achievement of nontrivial exact solutions of the EYM theory.

4.2.1 Petrov Classification

The Petrov classification provides a way of characterizing spacetimes based on the the so-called principal null directions of the Weyl tensor. In fact, together with the Goldberg-Sachs theorem, it turns out to be a powerful tool in the process of integration of the Einstein's equation, as we will see in the subsequent sections.

It is well known that in any four-dimensional spacetime, we are always able to find a basis of vector fields $\{\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ such that $\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{ab}$, where $\eta_{ab} = \text{diag}(-1, 1, 1, 1) = \eta^{ab}$. In this case, this orthonormal basis is said to be a **Lorentz frame** and, in terms of it, the inverse metric can be written as

$$g^{ab}\partial_a\partial_b = \eta^{ab}\boldsymbol{e_a} \otimes \boldsymbol{e_b} = -(\boldsymbol{e}_0)^2 + (\boldsymbol{e}_1)^2 + (\boldsymbol{e}_2)^2 + (\boldsymbol{e}_3)^2 . \tag{4.4}$$

Out of this basis, a **null tetrad frame** $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ can be defined by the relations

$$\begin{aligned} \boldsymbol{l} &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_0 + \boldsymbol{e}_3) , \qquad & \boldsymbol{m} &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + i\boldsymbol{e}_2) , \\ \boldsymbol{n} &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_0 - \boldsymbol{e}_3) , \qquad & \bar{\boldsymbol{m}} &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - i\boldsymbol{e}_2) . \end{aligned} \tag{4.5}$$

It follows that the only nonvanishing inner products among the vector fields in this frame are $l^a n_a = -1$ and $m^a \bar{m}_a = 1$. In particular, it is clear that all these four vectors are null, justifying the designation "null tetrad". In terms of this frame, the general metric (4.4) can be written as

$$g^{ab}\partial_a\partial_b = -(\boldsymbol{l} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{l}) + (\boldsymbol{m} \otimes \bar{\boldsymbol{m}} + \bar{\boldsymbol{m}} \otimes \boldsymbol{m}) . \tag{4.6}$$

Notice that a null tetrad frame is not uniquely defined. In fact, once the Lorentz group can be used in a Lorentz frame, giving rise to a continuum of new orthonormal frames, there can be infinitely many distinct null frames defined within the same spacetime. In particular, the action of the Lorentz group in a null tetrad frame is translated into the following set of transformations:

(i) *Lorentz Boost*

$$\boldsymbol{l} \to \lambda \boldsymbol{l} , \quad \boldsymbol{n} \to \lambda^{-1}\boldsymbol{n} , \quad \boldsymbol{m} \to e^{i\theta}\boldsymbol{m} , \quad \bar{\boldsymbol{m}} \to e^{-i\theta}\bar{\boldsymbol{m}} ; \tag{4.7}$$

(ii) *Null Rotation Around* $\boldsymbol{l}$

$$\boldsymbol{l} \to \boldsymbol{l} , \quad \boldsymbol{n} \to \boldsymbol{n} + w\boldsymbol{m} + \bar{w}\bar{\boldsymbol{m}} + |w|^2\boldsymbol{l} , \quad \boldsymbol{m} \to \boldsymbol{m} + \bar{w}\boldsymbol{l} , \quad \bar{\boldsymbol{m}} \to \bar{\boldsymbol{m}} + w\boldsymbol{l} ; \tag{4.8}$$

(iii) *Null Rotation Around* $\boldsymbol{n}$

$$\boldsymbol{l} \to \boldsymbol{l} + \bar{z}\boldsymbol{m} + z\bar{\boldsymbol{m}} + |z|^2\boldsymbol{n} , \quad \boldsymbol{n} \to \boldsymbol{n} , \quad \boldsymbol{m} \to \boldsymbol{m} + z\boldsymbol{n} , \quad \bar{\boldsymbol{m}} \to \bar{\boldsymbol{m}} + \bar{z}\boldsymbol{n} ; \tag{4.9}$$

where λ and θ are real parameters and w and z are complex. Thus, the six real parameters defined from λ, θ, w and z parameterize the Lorentz group $SO(3,1)$. In this case, the most general transformation preserving the metric (4.6) is a composition of the above transformations.

In four-dimensional spaces, the Weyl tensor C_{abcd} possesses just ten independent components. These components, in turn, can be neatly encoded in the five so-called **Weyl scalars** $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, which are complex functions of the spacetime defined by:

$$\begin{aligned}\Psi_0 \equiv C_{abcd} l^a m^b l^c m^d\,, \quad \Psi_1 \equiv C_{abcd} l^a n^b l^c m^d\,, \quad \Psi_2 \equiv C_{abcd} l^a m^b \bar{m}^c n^d\,,\\ \Psi_3 \equiv C_{abcd} n^a l^b n^c \bar{m}^d\,, \quad \Psi_4 \equiv C_{abcd} n^a \bar{m}^b n^c \bar{m}^d\,.\end{aligned} \tag{4.10}$$

Indeed, all the other projections of the Weyl tensor on the null tetrad can be written in terms of these scalars through the symmetries of the Weyl tensor, including the ones stemming from the vanishing traces and from the Bianchi identity $C_{a[bcd]} = 0$. For instance, $C_{abcd} l^a n^b \bar{m}^c m^d = \Psi_2 + \bar{\Psi}_2$ and $C_{abcd} n^a m^b n^c \bar{m}^d = 0$.

Under a null rotation around $\boldsymbol{n}$, the Weyl scalars transform according to

$$\begin{aligned}
\Psi_0 \to \Psi_0'(z) &= \Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4\,,\\
\Psi_1 \to \Psi_1'(z) &= \Psi_1 + 3z\Psi_2 + 3z^2\Psi_3 + z^3\Psi_4 = \tfrac{1}{4}\tfrac{d}{dz}\Psi_0'(z)\,,\\
\Psi_2 \to \Psi_2'(z) &= \Psi_2 + 2z\Psi_3 + z^2\Psi_4 = \tfrac{1}{3}\tfrac{d}{dz}\Psi_1'(z)\,,\\
\Psi_3 \to \Psi_3'(z) &= \Psi_3 + z\Psi_4 = \tfrac{1}{2}\tfrac{d}{dz}\Psi_2'(z)\,,\\
\Psi_4 \to \Psi_4'(z) &= \Psi_4 = \tfrac{d}{dz}\Psi_3'(z)\,,
\end{aligned} \tag{4.11}$$

as can be easily proven using equations (4.9) and (4.10). Then, assuming Ψ_4 to be nonvanishing (which can always be achieved by means of the Lorentz transformations (4.7)-(4.9), as long as the Weyl tensor is nonzero), $\Psi_0'(z)$ is a quartic polynomial in z and, hence, the order of degeneracy of its roots $\{z_1, z_2, z_3, z_4\}$ can be used to define the Petrov types of the spacetime, as it follows:

- *Type O*: the Weyl tensor is vanishing, $C_{abcd} \equiv 0$;
- *Type I*: all roots z_i are different;
- *Type II*: two roots coincide, $z_1 = z_2$, and $z_3 \neq z_4$ are both different from z_1;
- *Type III*: three roots coincide, $z_1 = z_2 = z_3$, and $z_4 \neq z_1$;
- *Type N*: all roots coincide, $z_1 = z_2 = z_3 = z_4$;
- *Type D*: two pairs of roots coincide, $z_1 = z_2$, $z_3 = z_4$, and $z_4 \neq z_1$.

Each of the distinct roots of the equation $\Psi_0'(z) = 0$, namely the elements of $\{z_1, z_2, z_3, z_4\}$, gives rise to a privileged null direction of the spacetime, defined by

$$\boldsymbol{l}_i' = \boldsymbol{l} + \bar{z}_i \boldsymbol{m} + z_i \bar{\boldsymbol{m}} + |z_i|^2 \boldsymbol{n}\,. \tag{4.12}$$

These are the so-called **principal null directions** (PNDs) of the Weyl tensor, and the direction $\boldsymbol{l}_i'$ is said to be a **repeated PND** if the root z_i is degenerated. Thus, for instance, type I spacetimes possess four distinct PNDs, while type D spaces possess just two, both being repeated. In particular, notice that the choice $z = z_i$ in (4.11) will not just annihilate Ψ_0', but also Ψ_1', if the root z_i has degree of degeneracy two; Ψ_1' and Ψ_2', if the root is triple degenerate; and Ψ_1', Ψ_2' and Ψ_3', if $z_1 = z_2 = z_3 = z_4$; as can be easily grasped if we write Ψ_0' as

$$\Psi_0'(z) \propto (z - z_1)(z - z_2)(z - z_3)(z - z_4)\,, \tag{4.13}$$

and use the derivatives on the right-hand side of the expressions (4.11). From this, we also see that the proportionality factor of $\Psi_0'(z)$ is just Ψ_4.

It is worth noting that the exchanges $\boldsymbol{l} \leftrightarrow \boldsymbol{n}$ and $\boldsymbol{m} \leftrightarrow \bar{\boldsymbol{m}}$ will turn a rotation around $\boldsymbol{l}$ into a rotation around $\boldsymbol{n}$, and vice-versa, and, likewise, the Weyl scalars will change as $\Psi_0 \leftrightarrow \Psi_4$, $\Psi_1 \leftrightarrow \Psi_3$ and $\Psi_2 \leftrightarrow \Psi_2$. In particular, with these replacements, we immediately obtain the behavior of the Weyl scalars under a rotation around $\boldsymbol{l}$, starting from equations (4.11). It is easy to see from this that after performing a rotation around $\boldsymbol{n}$ and annihilating some of the Weyl scalars, any arbitrary rotation around the null vector $\boldsymbol{l}$ will keep these scalars unchanged.

Using $\Psi_0'(z)$ as in (4.13) and performing a null rotation around $\boldsymbol{n}$, followed by a rotation around $\boldsymbol{l}$, the Weyl scalar Ψ_4 is taken to a new Ψ_4'' given in the form

$$\Psi_4 \to \Psi_4'' = \Psi_4(1 + w(z - z_1))(1 + w(z - z_2))(1 + w(z - z_3))(1 + w(z - z_4)) \,. \quad (4.14)$$

Besides this, the vector field $\boldsymbol{n}''$ of the new null tetrad frame acquires the form

$$\boldsymbol{n} \to \boldsymbol{n}'' = |w|^2 \left[\boldsymbol{l} + \left| z + \frac{1}{w} \right|^2 \boldsymbol{n} + \left(\bar{z} + \frac{1}{\bar{w}} \right) \boldsymbol{m} + \left(z + \frac{1}{w} \right) \bar{\boldsymbol{m}} \right] \,. \quad (4.15)$$

From (4.14), we see that, after choosing z to be one of the roots of $\Psi_0' = 0$, $z = z_i$, we can annihilate Ψ_4'' by choosing $w = -1/(z_i - z_j)$, for some $z_j \neq z_i$. Notice that this will not be possible only if z_i is a fourth order degenerate root of $\Psi_0' = 0$. In particular, setting $z = z_i$ and plugging $w = -1/(z_i - z_j)$ into (4.15), we obtain:

$$\boldsymbol{n}'' = |z_i - z_j|^{-2} (\boldsymbol{l} + |z_j|^2 \boldsymbol{n} + \bar{z}_j \boldsymbol{m} + z_j \bar{\boldsymbol{m}}) \,,$$

which is a null vector field aligned with the PND corresponding to the root z_j (compare this with equation (4.12)). Thus, we see that the successive annihilation of the Weyl scalars Ψ_0 and Ψ_4 corresponds to the alignment of the null vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ along the principal null directions of the Weyl tensor. In particular, similarly to what happens when we align the null vector field $\boldsymbol{l}$ to one of the PNDs of the spacetime, the alignment of the null vector field $\boldsymbol{n}$ will result in the annihilation just of the Weyl scalar Ψ_4, for nondegenerate roots; Ψ_4 and Ψ_3, for double degenerate roots; Ψ_4, Ψ_3 and Ψ_2, for triple degeneracy; and, finally, Ψ_4, Ψ_3, Ψ_2 and Ψ_1, when all the roots are equal; all of this without affecting the Weyl scalars already annihilated by the alignment of the vector field $\boldsymbol{l}$.

Below we summarize these results, displaying the Weyl scalars that can be annihilated in each of the Petrov types by aligning both $\boldsymbol{l}$ and $\boldsymbol{n}$ with the principal null directions, and choosing $\boldsymbol{l}$ to be the one with the higher degeneracy degree:

Type O - All	Type II - Ψ_0, Ψ_1, Ψ_4	Type N - Ψ_0, Ψ_1, Ψ_2, Ψ_3
Type I - Ψ_0, Ψ_4	Type III - Ψ_0, Ψ_1, Ψ_2, Ψ_4	Type D - Ψ_0, Ψ_1, Ψ_3, Ψ_4

Table 4.1: Weyl scalars that can be made to zero by a judicious choice of the null tetrad frame, for each of the possible types according to the Petrov classification.

4.2.2 Optical Scalars

In this section, we discuss the important concept of optical scalars of a null congruence of geodesics. Understanding these quantities is of fundamental importance in the connection between the Petrov classification and the Goldberg-Sachs theorem.

As previously introduced, assume that $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ is a null tetrad and define vector fields $\boldsymbol{e}_{(1)} = \boldsymbol{e}_1$ and $\boldsymbol{e}_{(2)} = \boldsymbol{e}_2$ from $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ according to (4.5). Namely,

$$\boldsymbol{e}_{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{m} + \bar{\boldsymbol{m}}) \qquad \text{and} \qquad \boldsymbol{e}_{(2)} = \frac{1}{i\sqrt{2}}(\boldsymbol{m} - \bar{\boldsymbol{m}}) . \tag{4.16}$$

Hence, in terms of the frame $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}\}$, the covariant derivative of the null vector field $\boldsymbol{l}$, $\nabla_\nu l_\mu$, can be expressed in the following way:

$$\nabla_\nu l_\mu = K_{00} l_\mu l_\nu + K_{03} l_\mu n_\nu + K_{0i} l_\mu e_{(i)\nu} + K_{i0} e_{(i)\mu} l_\nu + K_{i3} e_{(i)\mu} n_\nu + K_{ij} e_{(i)\mu} e_{(j)\nu} , \tag{4.17}$$

where

$$\begin{array}{lll} K_{00} \equiv n^\mu n^\nu \nabla_\nu l_\mu , & K_{0i} \equiv -n^\mu {e_{(i)}}^\nu \nabla_\nu l_\mu , & K_{i1} \equiv -{e_{(i)}}^\mu l^\nu \nabla_\nu l_\mu , \\ K_{01} \equiv n^\mu l^\nu \nabla_\nu l_\mu , & K_{i0} \equiv -{e_{(i)}}^\mu n^\nu \nabla_\nu l_\mu , & K_{ij} \equiv {e_{(i)}}^\mu {e_{(j)}}^\nu \nabla_\nu l_\mu . \end{array} \tag{4.18}$$

In expression (4.17), the components K_{30}, K_{33} and K_{3i} are not present, which is a consequence of the vanishing of the inner product $l^\mu l_\mu = 0$. For instance,

$$K_{30} \equiv (\nabla_\nu l_\mu) l^\mu n^\nu = \tfrac{1}{2} \nabla_\nu (l^\mu l_\mu) n^\nu = 0 .$$

Notice that the contraction of expression (4.17) with l^μ leads to

$$l^\nu \nabla_\nu l_\mu = -K_{03} l_\mu - K_{i3} e_{(i)\mu} .$$

We see from this that $\boldsymbol{l}$ is tangent to a **congruence of null geodesics** if, and only if, $K_{i3} \equiv 0$. In other words, in the case where $K_{i3} = 0$, there will exist a set of null geodesics filling the whole manifold, in such a way that at each point passes one, and only one, of these curves, each having $\boldsymbol{l}$ as its tangent vector. In this case, by means of a reparameterization by an affine parameter, we can always make $K_{03} = 0$. Hence, assuming $\boldsymbol{l}$ to be tangent to a congruence of null geodesics, expression (4.17) reduces to

$$\nabla_\nu l_\mu = K_{00} l_\mu l_\nu + K_{0i} l_\mu e_{(i)\nu} + K_{ij} e_{(i)\mu} e_{(j)\nu} .$$

Notice that, in this case, performing a null rotation around $\boldsymbol{l}$, in which case the vector fields $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$ transforms as $\boldsymbol{e}'_{(i)} = \boldsymbol{e}_{(i)} + a_i \boldsymbol{l}$, for real parameters a_1 and a_2, the covariant derivative $\nabla_\nu l_\mu$ will transform in the following way:

$$\nabla_\nu l_\mu \to \nabla_\nu l'_\mu = \nabla_\nu l_\mu = \tilde{K}_{00} l_\mu l_\nu + \tilde{K}_{0i} l_\mu e_{(i)\nu} + K_{ij} e_{(i)\mu} e_{(j)\nu} ,$$

for some functions $\tilde{K}_{00}$ and $\tilde{K}_{0i}$. In particular, the components K_{ij} was not changed. The invariance of K_{ij} under null rotations around $\boldsymbol{l}$ can, then, be used to invariantly characterize the geometric properties of the congruence of null geodesics with tangent $\boldsymbol{l}$. We can then decompose the matrix $\boldsymbol{K}$ with components K_{ij} in its traceless symmetric part, its trace, and its antisymmetric parts,

$$\boldsymbol{K} = \boldsymbol{\sigma} + \theta \boldsymbol{I} + \boldsymbol{\omega} , \tag{4.19}$$

with matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ defined in terms of the real parameter ω and complex σ as

$$\boldsymbol{\sigma} = \frac{1}{2} \begin{bmatrix} -(\sigma + \bar{\sigma}) & i(\sigma - \bar{\sigma}) \\ i(\sigma - \bar{\sigma}) & \sigma + \bar{\sigma} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\omega} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \tag{4.20}$$

and $\boldsymbol{I}$ being the identity matrix. The real parameters θ and ω are known respectively as the **expansion** and **twist**, while the complex parameter σ is called the **shear**. These are the **optical scalars** of the null congruence of geodesics, being interpreted as measures of the relative movement of nearby null geodesics.

From (4.19) and (4.20), the optical scalars can be solved in terms of K_{ij}, giving

$$\sigma = \frac{1}{2}[(K_{22} - K_{11}) - i(K_{12} + K_{21})]\,, \quad \theta = \frac{1}{2}(K_{11} + K_{22})\,, \quad \omega = \frac{1}{2}(K_{21} - K_{12})\,.$$

From this and equations (4.16) and (4.18), these scalars can be neatly written as

$$\theta + i\omega = \bar{m}^\mu m^\nu \nabla_\mu l_\nu \quad \text{and} \quad \sigma = -m^\mu m^\nu \nabla_\mu l_\nu\,.$$

Two important classes of spaces, the Kundt and Robinson-Trautman spacetimes, are defined in terms of the optical scalars as those admitting a congruence of null geodesics with all vanishing optical scalars ($\sigma = 0$, $\theta = 0$ and ω), in the first class, and shear-free ($\sigma = 0$) and twist-free ($\omega = 0$), in the latter class. Important examples of spacetimes within the Kundt and Robinson-Trautman spacetimes are respectively the *pp*-wave and Schwarzschild spacetimes.

4.2.3 Frobenius Theorem

Although some elements of the Frobenius theorem have already been used in the second chapter of this dissertation in order to discuss the geometric properties of spaces with a separability structures of type $\mathcal{S}_r$, here we give a precise definition once this theorem has an essential role in the integrability of the Einstein's equation for the problem proposed in the present chapter, as we will see shortly.

To this end, let M be an n-dimensional differentiable manifold and N a submanifold of M. Then, N is said to be an **integral submanifold** if there exists a set of vector fields $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$ that spans all the tangent spaces to N in a smooth fashion. The tangent bundle of an integral submanifold is called a **distribution**, and both the distribution and the system of vector fields $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$ are said to be **integrable** if through every point $q \in M$ there passes one of such integral submanifolds. Based on these definitions, the **Frobenius theorem** states that the set $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$ is integrable if, and only if, it is in involution. Namely, if at every point $p \in M$ there exist smooth functions $h^k{}_{ij} = h^k{}_{ij}(x)$ such that

$$[\boldsymbol{v_i}, \boldsymbol{v_j}] = h^k{}_{ij}\boldsymbol{v_k}\,. \qquad (i, j, k = 1, \ldots, r)$$

In other words, this theorem says that the integral curves of the system $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$ satisfying the above constraint foliates the whole manifold M, each leaf being an integral submanifold generated by the integrable system $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}\}$.

An extension of the Frobenius theorem in terms of differential 1-forms exist and turns out to be of great importance in the discussion that follows. In this formulation, we consider a set of r 1-forms $\{\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^r\}$ with $\boldsymbol{\omega}^1 \wedge \ldots \wedge \boldsymbol{\omega}^r \neq 0$. Then, the statement that these set of 1-form fields satisfy

$$d\boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^1 \wedge \ldots \wedge \boldsymbol{\omega}^r = 0\,, \qquad (i = 1, \ldots, r) \tag{4.21}$$

is equivalent to saying that there exists an $(n - r)$-dimensional integrable distribution, spanned by those vector fields $\boldsymbol{v}$ that annihilates the 1-forms $\boldsymbol{\omega}^i$ everywhere:

$$\boldsymbol{\omega}^1(\boldsymbol{v}) = \ldots = \boldsymbol{\omega}^r(\boldsymbol{v}) = 0\,.$$

4.2.4 The Goldberg-Sachs Theorem

The Goldberg-Sachs theorem is the most important theorem involving the Petrov classification. As originally introduced in Ref. [69], this theorem states that in a Ricci-flat spacetime (spaces where $R_{ab} = 0$ but the Riemann tensor is nonvanishing) the Weyl scalars Ψ_0 and Ψ_1 are vanishing if, and only if, the vector field $\boldsymbol{l}$ is tangent to a shear-free congruence of null geodesics. In other words, in spaces such that $R_{ab} = 0$, it is equivalent to saying either that $\boldsymbol{l}$ is geodesic and shear-free or that it is a repeated principal null direction. Later on, it was shown that the condition $R_{ab} = 0$ could be weakened to include Einstein spaces: $R_{ab} = \Lambda g_{ab}$ [70].

Another important result provided in Ref. [69] states that if the equations

$$R_{ab}k^a k^b = 0\,, \quad R_{ab}k^a \eta^b = 0 \quad \text{and} \quad R_{ab}\eta^a \eta^b = 0 \tag{4.22}$$

hold for a geodesic and shear-free real direction $\boldsymbol{k}$ and some complex null vector $\boldsymbol{\eta}$ orthogonal to $\boldsymbol{k}$, then $\boldsymbol{k}$ is necessarily a repeated PND of the Weyl tensor. Notice that, the cases $R_{ab} = 0$ and $R_{ab} = \Lambda g_{ab}$ above are covered by this stronger result, as does any other spacetime with energy-momentum tensor T_{ab} satisfying equations (4.22) with T_{ab} in place of R_{ab}. Indeed, once the Einstein's equation reads

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab}\,,$$

the conditions (4.22) hold as long as the energy-momentum tensor satisfies

$$T_{ab}k^a k^b = 0\,, \quad T_{ab}k^a \eta^b = 0 \quad \text{and} \quad T_{ab}\eta^a \eta^b = 0\,. \tag{4.23}$$

4.3 The Natural Null Tetrad Frame

Back to our problem, notice that under the assumptions (4.3), the symmetric tensors $G_1^{ij}\partial_i\partial_j$ and $G_2^{ij}\partial_i\partial_j$ acquire simpler forms that will be of great help:

$$G_1^{ij}\partial_i\partial_j = G_1^{\tau\tau}\partial_\tau^2 + G_1^{\sigma\sigma}\partial_\sigma^2 + 2\sqrt{G_1^{\tau\tau}G_1^{\sigma\sigma}}\,\partial_\tau\partial_\sigma = \left[\sqrt{G_1^{\tau\tau}}\,\partial_\tau + \sqrt{G_1^{\sigma\sigma}}\,\partial_\sigma\right]^2\,,$$
$$G_2^{ij}\partial_i\partial_j = G_2^{\tau\tau}\partial_\tau^2 + G_2^{\sigma\sigma}\partial_\sigma^2 + 2\sqrt{G_2^{\tau\tau}G_2^{\sigma\sigma}}\,\partial_\tau\partial_\sigma = \left[\sqrt{G_2^{\tau\tau}}\,\partial_\tau + \sqrt{G_2^{\sigma\sigma}}\,\partial_\sigma\right]^2\,.$$

From this, the metric (4.1) can easily be written as a sum of squares, as in (4.4), if we define the natural Lorentz frame $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ by

$$\boldsymbol{e}_0 = \frac{1}{\sqrt{S_1+S_2}}\left(\sqrt{G_2^{\tau\tau}}\,\partial_\tau + \sqrt{G_2^{\sigma\sigma}}\,\partial_\sigma\right)\,, \qquad \boldsymbol{e}_2 = \frac{1}{\sqrt{S_1+S_2}}\sqrt{\Delta_1}\,\partial_x\,,$$
$$\boldsymbol{e}_1 = \frac{1}{\sqrt{S_1+S_2}}\left(\sqrt{G_1^{\tau\tau}}\,\partial_\tau + \sqrt{G_1^{\sigma\sigma}}\,\partial_\sigma\right)\,, \qquad \boldsymbol{e}_3 = \frac{1}{\sqrt{S_1+S_2}}\sqrt{\Delta_2}\,\partial_y\,.$$

In this case, as learned in the previous section, a null tetrad frame $\{\boldsymbol{l},\boldsymbol{n},\boldsymbol{m},\overline{\boldsymbol{m}}\}$ can be easily defined, using (4.5), resulting in the following natural null frame:

$$\begin{aligned}
\boldsymbol{l} &= \frac{1}{\sqrt{2(S_1+S_2)}}\left[\sqrt{G_2^{\tau\tau}}\partial_\tau + \sqrt{G_2^{\sigma\sigma}}\partial_\sigma + \sqrt{\Delta_2}\partial_y\right] ,\\
\boldsymbol{n} &= \frac{1}{\sqrt{2(S_1+S_2)}}\left[\sqrt{G_2^{\tau\tau}}\partial_\tau + \sqrt{G_2^{\sigma\sigma}}\partial_\sigma - \sqrt{\Delta_2}\partial_y\right] ,\\
\boldsymbol{m} &= \frac{1}{\sqrt{2(S_1+S_2)}}\left[\sqrt{G_1^{\tau\tau}}\partial_\tau + \sqrt{G_1^{\sigma\sigma}}\partial_\sigma + i\sqrt{\Delta_1}\partial_x\right] ,\\
\bar{\boldsymbol{m}} &= \frac{1}{\sqrt{2(S_1+S_2)}}\left[\sqrt{G_1^{\tau\tau}}\partial_\tau + \sqrt{G_1^{\sigma\sigma}}\partial_\sigma - i\sqrt{\Delta_1}\partial_x\right] .
\end{aligned} \tag{4.24}$$

Hence, the metric can be put in the form (4.6) and, in addition, the Killing tensor field (4.2) can, similarly, be written compactly in terms of this null frame as

$$\boldsymbol{K} = S_1(\boldsymbol{l}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{l}) + S_2(\boldsymbol{m}\otimes\bar{\boldsymbol{m}} + \bar{\boldsymbol{m}}\otimes\boldsymbol{m}) .$$

The null tetrad frame constructed this way can easily be checked to satisfy the following nicely displayed algebraic relations:

$$\begin{aligned}
d\boldsymbol{l}\wedge\boldsymbol{l}\wedge\boldsymbol{m} &= 0 \quad\text{and}\quad d\boldsymbol{m}\wedge\boldsymbol{l}\wedge\boldsymbol{m} = 0 ,\\
d\boldsymbol{l}\wedge\boldsymbol{l}\wedge\bar{\boldsymbol{m}} &= 0 \quad\text{and}\quad d\bar{\boldsymbol{m}}\wedge\boldsymbol{l}\wedge\bar{\boldsymbol{m}} = 0 ,\\
d\boldsymbol{n}\wedge\boldsymbol{n}\wedge\boldsymbol{m} &= 0 \quad\text{and}\quad d\boldsymbol{m}\wedge\boldsymbol{n}\wedge\boldsymbol{m} = 0 ,\\
d\boldsymbol{n}\wedge\boldsymbol{n}\wedge\bar{\boldsymbol{m}} &= 0 \quad\text{and}\quad d\bar{\boldsymbol{m}}\wedge\boldsymbol{n}\wedge\bar{\boldsymbol{m}} = 0 .
\end{aligned} \tag{4.25}$$

In these equations, the symbols $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$, and $\bar{\boldsymbol{m}}$ should be understood as the 1-form fields corresponding (through the metric tensor) to the vector fields of the null tetrad frame. Namely, their components are $l_a = g_{ab}l^b$, $n_a = g_{ab}n^b$, $m_a = g_{ab}m^b$, and $\bar{m}_a = g_{ab}\bar{m}^b$. Notice that each of the lines in (4.25) is in the form (4.21), meaning that the vector fields that annihilate the 1-forms in each of these lines generate a bidimensional integrable distribution in accordance with Frobenius theorem. In particular, since each of the pairs of 1-forms present in a line have corresponding vector fields that are their own annihilators, and since they are linearly independent, this foliation is spanned by the corresponding vector fields themselves. Indeed, it is straightforward to see from (4.24) that the Lie brackets $[\boldsymbol{l},\boldsymbol{m}]$, $[\boldsymbol{l},\bar{\boldsymbol{m}}]$, $[\boldsymbol{n},\boldsymbol{m}]$, and $[\boldsymbol{n},\bar{\boldsymbol{m}}]$ are all vanishing. Besides this, once in four dimensional spacetimes the maximum dimension of a null subspace is two, each of these distributions are said to be a **maximally isotropic integrable distribution**. In this case, the distributions corresponding to the pairs in each of the lines in (4.25) are denoted respectively by span$\{\boldsymbol{l},\boldsymbol{m}\}$, span$\{\boldsymbol{l},\bar{\boldsymbol{m}}\}$, span$\{\boldsymbol{n},\boldsymbol{m}\}$, and span$\{\boldsymbol{n},\bar{\boldsymbol{m}}\}$. As a consequence of this, the vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ are geodesic and shear-free [71].

In addition to the interesting algebraic properties (4.25) and their powerful geometric implications, the real vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ of the null tetrad (4.24) are principal null directions of the spaces (4.1), as both Ψ_0 and Ψ_4 are identically zero. Besides this, we have that $\Psi_1 = \Psi_3$, that are generally nonvanishing, so that $\boldsymbol{l}$ is a repeated null direction whenever $\boldsymbol{n}$ is also a repeated null direction. It is interesting to note that, as a consequence of the Goldberg-Sachs theorem, once the vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ are geodesic and shear-free, if we are interested in Einstein spaces $R_{ab} = \Lambda g_{ab}$, we can impose $\Psi_1 = \Psi_3 = 0$ without loss of generality. In fact, this class of type D spaces turn out to be fully integrable and has already been investigated in Ref. [11].

It is important to bear in mind that the addition of non-abelian gauge fields coupled to the gravitational field in this problem precludes, in general, the use of the Goldberg-Sachs theorem. Nevertheless, assuming the gauge fields to be aligned with the principal null directions $\boldsymbol{l}$ and $\boldsymbol{n}$, a concept that will be exploited in the following section, we are led to an energy-momentum tensor T_{ab} that obey (4.23). In this case the real and complex null vector fields $\boldsymbol{k}$ and $\boldsymbol{\eta}$ will be either $\boldsymbol{l}$ and $\boldsymbol{m}$, or $\boldsymbol{n}$ and $\boldsymbol{m}$, ensuring, this way, that the real vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ are both repeated principal null directions of the spacetime. Remember, this is only valid because both $\boldsymbol{l}$ and $\boldsymbol{n}$ are geodesic and shear-free. In fact, as we will see in the following sections, this alignment will enable the integration of Einstein's equation, culminating in the attainment of exact solutions that broaden those of Ref. [11].

4.4 Reparameterizing the Metric

In order to attain the integration of Einstein's equation in the subsequent sections, it is convenient to choose a different parameterization for the unknown functions present in the line element (4.1). Remember, because of the conditions (4.3), the functions $G_1^{\tau\sigma}$ and $G_2^{\tau\sigma}$ are determined in terms of $G_1^{\tau\tau}$ and $G_1^{\sigma\sigma}$, in the first case, and $G_2^{\tau\tau}$ and $G_2^{\sigma\sigma}$, for the latter case. Hence, the freedom in these functions can be transferred to the four new functions N_1, P_1, N_2 and P_2 defined by:

$$N_1(x) = \frac{1}{G_1^{\sigma\sigma}\Delta_1}\,, \quad P_1(x) = -\sqrt{\frac{G_1^{\tau\tau}}{G_1^{\sigma\sigma}}}\,, \quad N_2(y) = \frac{1}{G_2^{\sigma\sigma}\Delta_2}\,, \quad P_2(y) = \sqrt{\frac{G_2^{\tau\tau}}{G_2^{\sigma\sigma}}}\,.$$

Notice that, we can easily invert these relations to express $G_1^{\tau\tau}$ and $G_1^{\sigma\sigma}$ in terms of N_1 and P_1, and $G_2^{\tau\tau}$ and $G_2^{\sigma\sigma}$ in terms of N_2 and P_2. Then, plugging these into the line element (4.1), the spaces under consideration acquire the following simpler form, given not in terms of the inverse metric, but in terms of the metric itself:

$$\begin{aligned} ds^2 = (S_1+S_2)\Bigg[& \frac{-N_2\Delta_2}{(P_1+P_2)^2}\,(d\tau + P_1 d\sigma)^2 \\ & + \frac{N_1\Delta_1}{(P_1+P_2)^2}\,(d\tau - P_2 d\sigma)^2 + \frac{dx^2}{\Delta_1} + \frac{dy^2}{\Delta_2}\Bigg]. \end{aligned} \tag{4.26}$$

4.5 Aligning the Gauge Fields with the PNDs

In this section, we present the so-called Einstein-Yang-Mills theory, and specify precisely the problem we are going to solve. Particularly, we introduce the concept of alignment of the interacting field with the principal null directions $\boldsymbol{l}$ and $\boldsymbol{n}$, in which case it will provide us with a great deal of simplification to our problem.

The Einstein-Yang-Mills theory describes an interacting theory between the gauge field $\boldsymbol{\mathcal{A}}$, a Lie algebra-valued 1-form, with the gravitational field, played, as usual in general relativity, by the metric tensor $\boldsymbol{g}$. This interaction happens through minimal coupling, described by the following action:

$$\mathcal{S} = \int \sqrt{|g|}\left[R - 2\Lambda - \frac{1}{2\lambda}\mathrm{Tr}\left(\mathcal{F}^{ab}\mathcal{F}_{ab}\right)\right] d\tau d\sigma dx dy\,, \tag{4.27}$$

where g stands for the determinant of $\boldsymbol{g}$, R is the Ricci scalar, λ is the coupling constant, and $\boldsymbol{\mathcal{F}}$ is the field strength, a Lie algebra-valued 2-form defined by

$$\boldsymbol{\mathcal{F}} \;=\; d\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}} \wedge \boldsymbol{\mathcal{A}} \,. \tag{4.28}$$

In coordinates, once provided a representation for the Lie algebra of the gauge group describing the chosen theory, this definition reads

$$\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_b] \,. \tag{4.29}$$

The trace in (4.27) is taken over the internal degrees of freedom originating from this Lie algebra. Finally, the equations of motion stemming respectively from the variation of $\mathcal{S}$ with respect to the metric g_{ab} and the gauge field $\mathcal{A}_a$ are:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab} \,, \tag{4.30}$$

$$\mathcal{D} \star \boldsymbol{\mathcal{F}} \equiv d \star \boldsymbol{\mathcal{F}} + [\boldsymbol{\mathcal{A}}, \star \boldsymbol{\mathcal{F}}] = 0 \,. \tag{4.31}$$

In this case, $\mathcal{D}$ stands for the gauge group covariant derivative, $\star$ the Hodge dual operation, and T_{ab} the energy-momentum tensor of the gauge field, defined by

$$T_{ab} = \frac{1}{\lambda} \mathrm{Tr} \left(\mathcal{F}_{ac} \mathcal{F}_b{}^c - \frac{1}{4} g_{ab} \mathcal{F}_{cd} \mathcal{F}^{cd} \right) \,.$$

Notice, in particular, that the abelian case is immediately recovered if we set to zero the last term on the right hand side of equation (4.28) (or equivalently in (4.29)), along with the commutator in equation (4.31).

In what follows, we will assume the field strength $\boldsymbol{\mathcal{F}}$ to be aligned with the principal null directions of the Weyl tensor, $\boldsymbol{l}$ and $\boldsymbol{n}$, meaning, in other words, that we are considering the gauge field to inherit the geometric properties of the spacetime. A generic rank-2 antisymmetric tensor $\boldsymbol{H}$ with components $H_{ab} = H_{[ab]}$ is said to be aligned with the real null direction $\boldsymbol{k}$ if there exists some complex null vector $\boldsymbol{\eta}$ linearly independent and orthogonal to $\boldsymbol{k}$ such that

$$H_{ab} k^a \eta^b \;=\; 0 \,.$$

In the language of Newman-Penrose formalism, this means that $\boldsymbol{k}$ is a principal null direction of $\boldsymbol{H}$ [74], a fact that is neatly understood through the spinorial formalism [75]. Thus, the alignment conditions of our problem are given by

$$\mathcal{F}_{ab} l^a m^b \;=\; 0 \quad \text{and} \quad \mathcal{F}_{ab} n^a m^b \;=\; 0 \,. \tag{4.32}$$

In particular, since $\boldsymbol{\mathcal{F}}$ is a physical field, it follows that its components are real. In this case, equation (4.32) implies that $\mathcal{F}_{ab} l^a \bar{m}^b$ and $\mathcal{F}_{ab} n^a \bar{m}^b$ are also vanishing. A direct algebraic consequence of this this alignment condition is that the following eight components of the energy-momentum tensor vanish:

$$\begin{aligned} T_{ab} l^a l^b = T_{ab} n^a n^b = T_{ab} m^a m^b = T_{ab} \bar{m}^a \bar{m}^b = 0 \,, \\ T_{ab} l^a m^b = T_{ab} l^a \bar{m}^b = T_{ab} n^a m^b \;= T_{ab} n^a \bar{m}^b \;= 0 \,, \end{aligned} \tag{4.33}$$

which can easily be checked by using the pair in equation (4.32), along with its complex conjugate. This justifies the use of the Goldberg-Sachs in the integration of Einstein's equation, and then we can impose $\Psi_1 = \Psi_3 = 0$ as discussed in section 4.3. In addition to the equations (4.33), we also have that

$$T_{ab}\left(l^a n^b - m^a \bar{m}^b\right) = 0\,, \tag{4.34}$$

which stems directly from the fact that the trace of the energy-momentum tensor of the gauge field is zero, $T_{ab}g^{ab} = 0$, and from the fact that the inverse metric can be written as in equation (4.5). These relations provide a great simplification on the integration of the equation of motion (4.30). Indeed, defining the tensor

$$E_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} - T_{ab}\,,$$

the Einstein field equations are given by $E_{ab} = 0$. Remember, this equation comprises a total of ten equations since E_{ab} is symmetric. In terms of the null tetrad frame, these equations can be arranged in the following way:

$$\begin{aligned} E_{ab}l^a l^b = E_{ab}n^a n^b = E_{ab}m^a m^b = E_{ab}\bar{m}^a \bar{m}^b = 0\,, \\ E_{ab}l^a m^b = E_{ab}l^a \bar{m}^b = E_{ab}n^a m^b \ = E_{ab}n^a \bar{m}^b \ = 0\,, \\ E_{ab}(l^a n^b - m^a \bar{m}^b) = 0\,, \end{aligned} \tag{4.35}$$

along with the equation

$$E_{ab}(l^a n^b + m^a \bar{m}^b) = 0\,. \tag{4.36}$$

It is not difficult to see that the conditions (4.33) and (4.34) imply that the nine components of the Einstein field equations in (4.35) do not depend on the gauge field $\boldsymbol{\mathcal{A}}$, since the projection of the energy-momentum tensor also vanishes in these components. In fact, the resulting equations depend just on the functions present in the metric and have already been solved in Ref. [11] for the vacuum case. Thus, we can benefit from this, borrowing the general solutions from this reference. Once we have done this, we just need to solve the last of these components, namely equation (4.36), along with the gauge field equation (4.31).

Besides the alignment conditions, we also assume the gauge fields to be invariant under the Lie dragging along the Killing vectors ∂_τ and ∂_σ, inheriting thereby the symmetries of the spacetime. Formally, this is given by Lie derivatives as

$$\mathcal{L}_{\partial_\tau}\mathcal{A} \;=\; 0 \quad \text{and} \quad \mathcal{L}_{\partial_\sigma}\mathcal{A} \;=\; 0\,.$$

In fact, this invariance means that the components of $\mathcal{A}$ in the coordinate frame do not depend on the coordinates τ and σ, allowing us to write the gauge field as

$$\mathcal{A} = \mathcal{A}_\tau d\tau + \mathcal{A}_\sigma d\sigma + \mathcal{A}_x dx + \mathcal{A}_y dy = \mathcal{A}_a(x, y)dx^a\,, \tag{4.37}$$

where each of the components $\mathcal{A}_a$ is an element of the Lie algebra for the gauge group adopted in the theory, depending just on the coordinates x and y.

It turns out that the Lie algebra-valued 1-form (4.37) can be further simplified due to the alignment conditions, by noticing that the equation (4.32) gives

$$\mathcal{F}_{xy} \propto \mathcal{F}_{ab}(l^a - n^a)(m^b - \bar{m}^b) = 0\,, \tag{4.38}$$

which is equivalent to the equation

$$\tilde{d}\tilde{\boldsymbol{\mathcal{A}}} + \tilde{\boldsymbol{\mathcal{A}}} \wedge \tilde{\boldsymbol{\mathcal{A}}} = 0\,, \tag{4.39}$$

if we define the 1-form field $\tilde{\boldsymbol{\mathcal{A}}}$ on the leaves of constant τ and σ by

$$\tilde{\boldsymbol{\mathcal{A}}} \equiv \mathcal{A}_x dx + \mathcal{A}_y dy \,.$$

In this case, $\tilde{d}$ is the exterior derivative on these submanifolds. In fact, this is only possible thanks to the fact that the components $\mathcal{A}_a$ are functions only of x and y.

Equation (4.39) conveys the vanishing of the corresponding field strength $\tilde{\boldsymbol{\mathcal{F}}}$, meaning that the gauge field $\tilde{\boldsymbol{\mathcal{A}}}$ is related through a gauge transformation to the zero gauge field in each of the leaves that foliates the manifold. Thus, there must have a smooth gauge transformation on the whole manifold that takes the gauge field $\tilde{\boldsymbol{\mathcal{A}}}$ to the zero gauge field. According to this, without losing generality, both of the components $\mathcal{A}_x$ and $\mathcal{A}_y$ can be set to zero in (4.37), yielding

$$\mathcal{A} = \mathcal{A}_\tau d\tau + \mathcal{A}_\sigma d\sigma \,. \tag{4.40}$$

Hence, in the sequence, and throughout the remainder of this chapter, this gauge freedom will be implicitly used to write the gauge field $\mathcal{A}$ in the form (4.40).

So far just one of the alignment conditions was used, which allowed us to write the gauge field in the form (4.40). The two following equations, also stemming from the alignment conditions (4.32),

$$\mathcal{F}_{ab}(l^a - n^a)(m^b + \bar{m}^b) = 0 \,,$$
$$\mathcal{F}_{ab}(l^a + n^a)(m^b - \bar{m}^b) = 0 \,,$$

have general solution given by

$$\mathcal{A}_\tau = \frac{\mathcal{B}_1 + \mathcal{B}_2}{P_1 + P_2} \quad \text{and} \quad \mathcal{A}_\sigma = \frac{P_1\mathcal{B}_2 - P_2\mathcal{B}_1}{P_1 + P_2} \,, \tag{4.41}$$

where $\mathcal{B}_1 = \mathcal{B}_1(x)$ is an arbitrary element of the Lie algebra depending solely on the variable x, and similarly for $\mathcal{B}_2 = \mathcal{B}_2(y)$. Finally, imposing the last of the constraints arising from the alignment conditions, namely,

$$\mathcal{F}_{\tau\sigma} \propto \mathcal{F}_{ab}(l^a + n^a)(m^b + \bar{m}^b) = 0 \,, \tag{4.42}$$

we conclude, using equations (4.29) and (4.40), that

$$\mathcal{A} \wedge \mathcal{A} = 0 \,. \tag{4.43}$$

This equation means that the elements $\mathcal{B}_1$ and $\mathcal{B}_2$ appearing in (4.41) should commute with each other, since this is equivalent, in coordinates, to the equation $[\mathcal{A}_a, \mathcal{A}_b] = 0$. Besides this, it also says that the field strength $\boldsymbol{\mathcal{F}}$ is linear in the gauge field, just as in the abelian case; see either (4.28) or (4.29).

Once we have treated the Einstein's equation, let us now investigate the consequences of the assumptions above on the field equations for the gauge field, equation (4.31), which becomes better manageable if we take the Hodge dual of it:

$$\nabla^a \mathcal{F}_{ab} \,+\, [\mathcal{A}^a, \mathcal{F}_{ab}] = 0 \,. \tag{4.44}$$

Due to the condition (4.38), the components $\nabla^a \mathcal{F}_{ax}$ and $\nabla^a \mathcal{F}_{ay}$ of the divergence of $\boldsymbol{\mathcal{F}}$ vanish identically, as can be easily seen, so that equation (4.44) yields

$$[\mathcal{A}^a, \mathcal{F}_{ax}] = [\mathcal{A}^a, \mathcal{F}_{ay}] = 0 \,.$$

On the other hand, due to the fact that $\mathcal{F}_{\tau\sigma}$ is zero, as shown in equation (4.42), it follows that the contractions $\mathcal{A}^a\mathcal{F}_{a\tau}$, $\mathcal{F}_{a\tau}\mathcal{A}^a$, $\mathcal{A}^a\mathcal{F}_{a\sigma}$, and $\mathcal{F}_{a\sigma}\mathcal{A}^a$ are all vanishing, reducing the components τ and σ of the equation (4.44) to

$$\nabla^a\mathcal{F}_{a\tau} = \nabla^a\mathcal{F}_{a\sigma} = 0\,.$$

A consequence of the above equations, which follows from the alignment conditions, is that the field equation (4.44) can be broken into the following pair of equations:

$$\nabla^a\mathcal{F}_{ab} = 0\,, \tag{4.45}$$

$$[\mathcal{A}^a, \mathcal{F}_{ab}] = 0\,. \tag{4.46}$$

In face of this, we immediately see that our problem has been reduced to first solving the abelian case, once equations (4.43) and (4.45) gives respectively the general form for the field strength and the field equation for the abelian case, and, then, in hands of the most general solution for this problem, the non-abelian case is obtained by simply solving the supplementary algebraic equations (4.43) and (4.46) for the general gauge group. This is, in fact, the outline of the steps we shall follow in the subsequent sections. At this point, it is worthwhile mentioning that the non-abelian solutions contructed from the abelian ones should not be interpreted physically as the same. In fact, as we will see in the next sections, the non-abelian solutions might present a richer algebraic structure with no analogue in the abelian theory. Indeed, even in the simplest case in which the non-abelian gauge field $\mathcal{A}_a$ is proportional to the $U(1)$ electromagnetic field A_a, namely $\mathcal{A}_a = A_a\mathcal{Q}$, where A_a is a solution for the Maxwell equations and $\mathcal{Q}$ is an arbitrary constant element for some choice of Lie algebra, the non-abelian solution is physically distinguishable from the electromagnetic one, as discussed in Ref. [39].

4.6 Solving the Abelian Case

As anticipated in the previous section, the general solution for the Einstein-Yang-Mills theory in a background possessing two commuting Killing vector fields, a nontrivial Killing tensor and a gauge field that is aligned with the principal null directions of the spacetime is obtained by first solving the abelian problem. Since the integration of the equations (4.35) has already been performed in Ref. [11], here we just borrow these results. In particular, the integration of the equation $\Psi_1 = \Psi_3$ leads to three independent cases depending on whether the derivatives $P_1' = dP_1/dx$ and $P_2' = dP_2/dy$ are zero or not. In any of these cases, the integration is attained following the same basic three steps: (i) first, we integrate the nine components of the Einstein's equation (4.35) in order to find the most general form for the metric functions S_1, S_2, N_1, N_2, Δ_1, and Δ_2; (ii) then, using the gauge field in the form given by equations (4.40) and (4.41), and then imposing the field equation for the abelian case, $\nabla^a\mathcal{F}_{ab} = 0$, we obtain the most general form for the Lie algebra functions $\mathcal{B}_1$ and $\mathcal{B}_2$; (iii) finally, we integrate the remaining component of the Einstein's equation, (4.36), which gives rise to a constraint in the integration constants for the functions previously found. After these three steps, our goal of solving the abelian problem is completed.

Due to the similarity among the three possible cases, let us tackle just one of them, when $P1' \neq 0$ and $P2' \neq 0$, which is the most general and interesting case, and discuss the other two. In this case, since the functions $P_1(x)$ and $P_2(y)$ are both nonconstant, we can always make a coordinate choice in such a way to obtain

$$P_1(x) = x^2 \quad \text{and} \quad P_2(y) = y^2\,. \tag{4.47}$$

Indeed, performing the coordinate transformation $(x,y) \to (\hat{x} = \sqrt{P_1}, \hat{y} = \sqrt{P_2})$ in the line element (4.26), along with a redefinition of the functions N_1, Δ_1, N_2, Δ_2, we obtain the desired result after dropping the hats in the new coordinates. Then, following steps completely analogous to the ones taken in Ref. [11] to obtain the general solution for the equations (4.35), we are led to the following results:

$$\begin{aligned} N_1 &= \frac{x^2}{(a_1 + b_1 x^2)(a_2 + b_2 x^2)}\,, \qquad & S_1 &= \frac{a_3 + b_3 x^2}{a_1 + b_1 x^2}\,, \\ N_2 &= \frac{-\,y^2}{(a_1 - b_1 y^2)(a_2 - b_2 y^2)}\,, \qquad & S_2 &= \frac{a_3 - b_3 y^2}{b_1 y^2 - a_1}\,, \end{aligned} \tag{4.48}$$

where the a's and b's are arbitrary integration constants. We also obtain the general form for the functions Δ_1 and Δ_2, which can be put in the compact form

$$\begin{aligned} \Delta_1 &= \frac{1}{x^2}\left[c_1 I_1^{5/2} J_1^{3/2} + c_2 I_1^3 J_1 + c_3 I_1^2 J_1^2 + \frac{\Lambda(a_1 b_3 - a_3 b_1)}{3 b_1^2 (a_2 b_1 - a_1 b_2)} I_1 J_1\right]\,, \\ \Delta_2 &= \frac{1}{y^2}\left[c_4 I_2^{5/2} J_2^{3/2} + c_5 I_2^3 J_2 - c_3 I_2^2 J_2^2 - \frac{\Lambda(a_1 b_3 - a_3 b_1)}{3 b_1^2 (a_2 b_1 - a_1 b_2)} I_2 J_2\right]\,, \end{aligned} \tag{4.49}$$

where the c's are integration constants that are arbitrary for the moment, and the I's and J's stand for the following first order polynomials:

$$I_1 = a_1 + b_1 x^2\,, \quad I_2 = a_1 - b_1 y^2\,, \quad J_1 = a_2 + b_2 x^2\,, \quad J_2 = a_2 - b_2 y^2\,. \tag{4.50}$$

This covers the first of the steps presented at the beginning of this section for the attainment of the general solution for the abelian case. In the second step, we assume the gauge field to be given in the form displayed in the equations (4.40) and (4.41), and integrate the abelian field equations $\nabla^a \mathcal{F}_{ab} = 0$. This can be accomplished without difficulty, yielding as general solutions

$$\mathcal{B}_1 = \mathcal{Q}_1 I_1^{1/2} J_1^{1/2} + \mathcal{Q}_3 x^2 + \mathcal{Q}_4 \quad \text{and} \quad \mathcal{B}_2 = \mathcal{Q}_2 I_2^{1/2} J_2^{1/2} + \mathcal{Q}_3 y^2 - \mathcal{Q}_4\,,$$

where the $\mathcal{Q}$'s are arbitrary constant elements of the Lie algebra. From this, we can easily build the field strength $\boldsymbol{\mathcal{F}}$, in which case we will notice that neither $\mathcal{Q}_3$ or $\mathcal{Q}_4$ makes part of the final result, meaning that both of them are pure gauge and, therefore, can be ignored, without losing generality. Thus, we can write

$$\mathcal{B}_1 = \mathcal{Q}_1 I_1^{1/2} J_1^{1/2} \quad \text{and} \quad \mathcal{B}_2 = \mathcal{Q}_2 I_2^{1/2} J_2^{1/2}\,. \tag{4.51}$$

The constant elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are interpreted as the charges generating the gauge field. In the particular case of the electromagnetic theory, the gauge group is the group $U(1)$, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are respectively the magnetic and electric charges.

Finally, imposing the last component of Einstein's equation to hold, namely (4.36), we obtain that the integration constants c_2 and c_5 appearing in the functions Δ_1 and Δ_2 must be related to each other through the equation

$$c_5 = \frac{(a_1 b_2 - a_2 b_1)}{2\lambda(a_3 b_1 - a_1 b_3)} \mathrm{Tr}\,(\mathcal{Q}_1 \mathcal{Q}_1 - \mathcal{Q}_2 \mathcal{Q}_2) - c_2\,. \tag{4.52}$$

As it will be shown in the section 4.8 below, assuming that the gauge group is $U(1)$, this solution turns out to be the well-known Kerr-Newman-NUT-(A)dS spacetime. Thus, the solution presented in this section is just a straightforward generalization of the charged Kerr-NUT-(A)dS class of spacetimes to the case where there are several electromagnetic fields decoupled from each other.

Following the same basic three steps above, we can solve the case where $P_1(x)$ is nonconstant while $P_2(y)$ is a constant. In this case, we obtain a generalization of a twisting but nonaccelerating spacetime contained in the Plebański-Demiański class, which has as a special case the charged Taub-NUT-(A)dS spacetime. Notice that this is virtually the same as the case in which $P_1' = 0$ and $P_2' \neq 0$, since the latter can be obtained from the first one if we interchange the coordinates x and y in the line element (4.26). The case in which both of the functions P_1 and P_2 are constant splits into two cases: one in which one of the functions S_1 or S_2 are constant while the other is a nonconstant, and the other in which both functions are constant. In the first of these cases, we obtain a generalization of the Reissner-Nordström metric with cosmological constant [76], while the latter provides a generalization of the charged Nariai spacetime [77].

Notice that the integration process performed above did not provide any fundamentally new solutions besides the ones already described in the literature. Indeed, such solutions are contained in the results of Ref. [39]. Nevertheless, this process was of valuable importance, inasmuch as we could exhaust all the abelian solutions to the Einstein-Yang-Mills theory in a background possessing two commuting Killing vector fields and one nontrivial rank-2 Killing tensor when the gauge field is aligned with the principal null directions of the spacetime. Additionally, as we will see in the next section, these abelian solutions will also be important in the construction of nontrivial solutions with non-abelian gauge groups that have never been described in the literature before.

4.7 General Gauge Group

As we have learned in section 4.5, the general solution for an arbitrary gauge group is provided by the abelian solution presented in the previous section, supplemented by the conditions (4.43) and (4.46) constraining the gauge field. We have also seen in the same section that both components $\mathcal{A}_x$ and $\mathcal{A}_y$ of the gauge field can be set to zero without losing generality and that the terms $\mathcal{A}^a\mathcal{F}_{a\tau}$, $\mathcal{F}_{a\tau}\mathcal{A}^a$, $\mathcal{A}^a\mathcal{F}_{a\sigma}$, $\mathcal{F}_{a\sigma}\mathcal{A}^a$ are all vanishing as a consequence of the alignment conditions. On account of such properties, (4.43) and (4.46) become equivalent to the three following equations:

$$[\mathcal{A}_\tau, \mathcal{A}_\sigma] = 0\,, \quad [\mathcal{A}^a, \mathcal{F}_{ax}] = 0 \quad \text{and} \quad [\mathcal{A}^a, \mathcal{F}_{ay}] = 0\,.$$

As it turns out, for all the solutions to the problem considered in this chapter, including those cases not treated explicitly here, namely the cases where $P_1' \neq 0$ and $P_2' = 0$, and $P_1' = 0$ and $P_2' = 0$, these conditions surprisingly boil down to

$$[\mathcal{Q}_1, \mathcal{Q}_2] = 0\,. \tag{4.53}$$

Thus, the two conditions (4.43) and (4.46) can be replaced by the one simple constraint above. In particular, since in the abelian case this requirement represents no constraint at all, in the case of an arbitrary k-dimensional abelian Lie algebra, our solution will have $2k$ arbitrary constant parameters stemming from the gauge group, once the elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are completely arbitrary in this case.

Although in the general case we need first to specify the gauge group in order to work out the constraint (4.53), we immediately see that a trivial solution for this constraint is given by assuming that the Lie algebra elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are proportional to each other, so that they commute trivially:

$$\mathcal{Q}_2 = e\mathcal{Q}_1 \,, \tag{4.54}$$

where e is some arbitrary real constant. In this case, we easily see that the total number of independent parameters is $k+1$, k being the components of $\mathcal{Q}_1$, plus the parameter e. In fact, this particular type of solutions have already been discussed in Ref. [39] and can be understood as simple generalization of $U(1)$ solutions, like the ones presented in the previous section. Notice, however, that this does not represent the most general solution, as there might have more intricate solutions depending on the particular structure of the gauge field. Indeed, we will see in the following examples that nontrivial solutions for this constraint do exist.

In summary, the general solution for the Einstein-Yang-Mills theory in spacetimes possessing two commuting Killing vector fields, one nontrivial rank-2 Killing tensor, assuming the constraint (4.3) to hold, and such that the gauge fields are aligned with the principal null directions of the Weyl tensor and subjected to the symmetries of the spacetime is provided by the line element (4.26), with functions P_1, P_2, N_1, N_2, S_1, S_2, Δ_1 and Δ_2 defined by equations (4.47)-(4.50), along with (4.52). The gauge field of this solution is defined by equations (4.41) and (4.51), where the constant Lie algebra elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are general elements of the gauge group Lie algebra, satisfying (4.53). Both the functions of the metric and the functions defining the gauge field will be different for the other two cases not tackled explicitly here. However, the structure of such solutions are similar to the case presented here. In particular, in both of them there will be constant elements of the Lie algebra $\mathcal{Q}_1$ and $\mathcal{Q}_2$ that must commute according to (4.53), see Ref. [9].

4.7.1 Group $SU(2)$

In this first example, we consider the $SU(2)$ group and denote a basis for its Lie algebra by $\{\mathcal{L}_\alpha\}$. As introduced in the previous chapter, a representation for these elements is provided by the Pauli matrices σ_α (equation (3.12)), assuming the form

$$\mathcal{L}_\alpha = \frac{-i}{2}\sigma_\alpha \,.$$

The corresponding algebra of commutators can be summarized in the relation

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] = \varepsilon_{\alpha\beta}{}^{\gamma}\mathcal{L}_\gamma \,,$$

where $\varepsilon_{\alpha\beta\gamma}$ is the three-dimensional Levi-Civita symbol. Then, a metric in the Lie algebra vector space can be defined as

$$\langle \mathcal{L}_\alpha, \mathcal{L}_\beta \rangle = \mathrm{Tr}\left(\mathcal{L}_\alpha \mathcal{L}_\beta\right) = -\frac{1}{2}\delta_{\alpha\beta} \,.$$

Assuming that the charges $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are independent elements of the Lie algebra, they can be generically written as

$$\mathcal{Q}_1 = {q_1}^{\alpha}\mathcal{L}_\alpha = {q_1}^{1}\mathcal{L}_1 + {q_1}^{2}\mathcal{L}_2 + {q_1}^{3}\mathcal{L}_3 \,,$$
$$\mathcal{Q}_2 = {q_2}^{\alpha}\mathcal{L}_\alpha = {q_2}^{1}\mathcal{L}_1 + {q_2}^{2}\mathcal{L}_2 + {q_2}^{3}\mathcal{L}_3 \,.$$

From this, it follows that the constraint (4.53) implies that

$$q_1{}^\alpha\, q_2{}^\beta\, \varepsilon_{\alpha\beta\gamma} = 0\,,$$

whose general solution gives that $q_1{}^\alpha$ and $q_2{}^\alpha$ must be proportional to each other:

$$q_2{}^\alpha = e q_1{}^\alpha\,, \tag{4.55}$$

where e is some arbitrary parameter. As a matter of fact, this solution has already been described in the literature, inasmuch as it is among those solutions considered in Ref. [39]; compare equations (4.55) and (4.54). Finally, notice that the traces appearing in the solution of section 4.6, equation (4.52), are written as

$$\mathrm{Tr}\,(\mathcal{Q}_1\mathcal{Q}_1) = -\frac{1}{2}\sum_{i=1}^{3}(q_1{}^i)^2 \quad \text{and} \quad \mathrm{Tr}\,(\mathcal{Q}_2\mathcal{Q}_2) = -\frac{e^2}{2}\sum_{i=1}^{3}(q_1{}^i)^2\,.$$

Although the $SU(2)$ group revealed to be one of the simple cases with $\mathcal{Q}_1$ proportional to $\mathcal{Q}_2$, in the following subsection we present an explicit example in which this is not the case, showing that, depending on the gauge group chosen, we can have solutions that are not just trivial generalizations of the $U(1)$ case.

4.7.2 The Lorentz Group $SO(3,1)$

Now, let us assume that the gauge group is the Lorentz group $SO(3,1)$, whose Lie algebra can be generated by the following six 4×4 matrices:

$$\mathcal{J}_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{bmatrix}, \quad \mathcal{J}_3 = \begin{bmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix},$$

$$\mathcal{K}_1 = \begin{bmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \mathcal{K}_2 = \begin{bmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \mathcal{K}_3 = \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}.$$

In this case, the algebra satisfied by these generators reads

$$[\mathcal{J}_\alpha, \mathcal{J}_\beta] = \varepsilon_{\alpha\beta}{}^\gamma \mathcal{J}_\gamma\,, \qquad [\mathcal{J}_\alpha, \mathcal{K}_\beta] = \varepsilon_{\alpha\beta}{}^\gamma \mathcal{K}_\gamma\,, \qquad [\mathcal{K}_\alpha, \mathcal{K}_\beta] = -\varepsilon_{\alpha\beta}{}^\gamma \mathcal{J}_\gamma\,.$$

In terms of this basis, a symmetric bilinear form in the Lie algebra vector space is then provided by

$$\begin{aligned} &\langle \mathcal{J}_\alpha, \mathcal{J}_\beta\rangle = \mathrm{Tr}\,(\mathcal{J}_\alpha\mathcal{J}_\beta) = -2\delta_{\alpha\beta}\,, \quad \langle \mathcal{K}_\alpha, \mathcal{K}_\beta\rangle = \mathrm{Tr}\,(\mathcal{K}_\alpha\mathcal{K}_\beta) = 2\delta_{\alpha\beta}\,,\\ &\langle \mathcal{J}_\alpha, \mathcal{K}_\beta\rangle = \mathrm{Tr}\,(\mathcal{J}_\alpha\mathcal{K}_\beta) = 0\,. \end{aligned}$$

In this case, the arbitrary elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ can be written as

$$\mathcal{Q}_1 = q_1{}^\alpha \mathcal{J}_\alpha + \tilde{q}_1{}^\alpha \mathcal{K}_\alpha \quad \text{and} \quad \mathcal{Q}_2 = q_2{}^\alpha \mathcal{J}_\alpha + \tilde{q}_2{}^\alpha \mathcal{K}_\alpha\,,$$

where the twelve charges $q_1{}^\alpha$, $\tilde{q}_1{}^\alpha$, $q_2{}^\alpha$, and $\tilde{q}_2{}^\alpha$ are, for the moment, arbitrary. In order to find the most general form allowed for these elements under the constraint (4.53), it is convenient to define a new basis

$$\mathcal{N}_\alpha \equiv \frac{1}{2}(\mathcal{J}_\alpha - i\mathcal{K}_\alpha) \qquad \text{and} \qquad \mathcal{N}_\alpha^\dagger \equiv \frac{1}{2}(\mathcal{J}_\alpha + i\mathcal{K}_\alpha)\,,$$

which is such that

$$[\mathcal{N}_\alpha, \mathcal{N}_\beta] = \varepsilon_{\alpha\beta}{}^\gamma \mathcal{N}_\gamma\,, \qquad [\mathcal{N}_\alpha^\dagger, \mathcal{N}_\beta^\dagger] = \varepsilon_{\alpha\beta}{}^\gamma \mathcal{N}_\gamma^\dagger\,, \qquad [\mathcal{N}_\alpha, \mathcal{N}_\beta^\dagger] = 0\,,$$

producing two independent copies of the $\mathfrak{su}(2)$ Lie algebra. In this basis, the elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ acquire the form

$$\begin{aligned}\mathcal{Q}_1 &= ({q_1}^\alpha + i{\tilde{q}_1}{}^\alpha)\mathcal{N}_\alpha + ({q_1}^\alpha - i{\tilde{q}_1}{}^\alpha)\mathcal{N}_\alpha^\dagger\,,\\ \mathcal{Q}_2 &= ({q_2}^\alpha + i{\tilde{q}_2}{}^\alpha)\mathcal{N}_\alpha + ({q_2}^\alpha - i{\tilde{q}_2}{}^\alpha)\mathcal{N}_\alpha^\dagger\,.\end{aligned}$$

Then, the requirement that these two charges must commute translates into the following equation:

$$({q_1}^\alpha + i{\tilde{q}_1}{}^\alpha)({q_2}^\beta + i{\tilde{q}_2}{}^\beta)\varepsilon_{\alpha\beta}{}^\gamma\mathcal{N}_\gamma + ({q_1}^\alpha - i{\tilde{q}_1}{}^\alpha)({q_2}^\beta - i{\tilde{q}_2}{}^\beta)\varepsilon_{\alpha\beta}{}^\gamma\mathcal{N}_\gamma^\dagger = 0\,,$$

with general solution given by

$$({q_2}^\alpha + i{\tilde{q}_2}{}^\alpha) = \epsilon({q_1}^\alpha + i{\tilde{q}_1}{}^\alpha) \qquad \text{and} \qquad ({q_2}^\alpha - i{\tilde{q}_2}{}^\alpha) = \tilde{\epsilon}({q_1}^\alpha - i{\tilde{q}_1}{}^\alpha)\,.$$

From this, we can easily solve for ${q_2}^\alpha$ and ${\tilde{q}_2}{}^\alpha$ in terms of the ${q_1}^\alpha$ and ${\tilde{q}_1}{}^\alpha$, in which case we will see that in order to keep these solutions real, we need to assume that $\bar{\epsilon} = \tilde{\epsilon}$. Thus, defining $\epsilon = e + i\tilde{e}$ and $\tilde{\epsilon} = e - i\tilde{e}$, with e and $\tilde{e}$ real, we obtain

$${q_2}^\alpha = e{q_1}^\alpha - \tilde{e}{\tilde{q}_1}{}^\alpha \qquad \text{and} \qquad {\tilde{q}_2}{}^\alpha = \tilde{e}{q_1}^\alpha + e{\tilde{q}_1}{}^\alpha\,.$$

With this, the final form for the elements $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is given by

$$\mathcal{Q}_1 = {q_1}^\alpha\mathcal{J}_\alpha + {\tilde{q}_1}{}^\alpha\mathcal{K}_\alpha \qquad \text{and} \qquad \mathcal{Q}_2 = (e{q_1}^\alpha - \tilde{e}{\tilde{q}_1}{}^\alpha)\mathcal{J}_\alpha + (\tilde{e}{q_1}^\alpha + e{\tilde{q}_1}{}^\alpha)\mathcal{K}_\alpha\,. \tag{4.56}$$

In particular, the traces appearing in the solution of section 4.6 can be written as

$$\begin{aligned}\mathrm{Tr}\,(\mathcal{Q}_1\mathcal{Q}_1) &= -2\sum_{\alpha=1}^{3}\left[({q_1}^\alpha)^2 - ({\tilde{q}_1}{}^\alpha)^2\right]\,,\\ \mathrm{Tr}\,(\mathcal{Q}_2\mathcal{Q}_2) &= -2\sum_{\alpha=1}^{3}\left\{(e^2 - \tilde{e}^2)\left[({q_1}^\alpha)^2 - ({\tilde{q}_1}{}^\alpha)^2\right] - 4e\tilde{e}{q_1}^\alpha{\tilde{q}_1}{}^\alpha\right\}\,.\end{aligned}$$

Since in general $\tilde{e} \neq 0$, it follows that $\mathcal{Q}_1$ is not proportional to $\mathcal{Q}_2$, so that the general solution (4.56) is not of the simple type presented in equation (4.54) and, therefore, it is not contained in the class of solutions presented in Ref. [39]. In this case, the most general solution for the group $SO(3,1)$ has eight charge parameters, six stemming from the components of $\mathcal{Q}_1$ plus the two parameters e and $\tilde{e}$. For a similar example using the $SO(4)$ Lie group, see Ref. [9]

4.8 Coordinate Transformations

As it was pointed out in the previous sections, for the case where the gauge group is $U(1)$, corresponding to the electromagnetic interaction, the solutions obtained above have already been described in the literature and does not present any novelty. In particular, as it was argued in section 4.6, the case tackled explicitly there leads to the well-known Kerr-Newman-NUT-(A)dS spacetime. To show that this is indeed the case, in the present section we present the explicit coordinate transformations that take the line element found at that section to its known form.

The general solution presented in section 4.6 for the gauge group $U(1)$ is given by the line element (4.26), with the functions P_1, P_2, N_1, N_2, S_1, S_2, Δ_1 and Δ_2 given by equations (4.47)-(4.50), along with the constraint (4.52). In addition, the gauge field of this solution is given by (4.41) and (4.51). Then, instead of using the constant parameters c_1, c_2, c_3, c_4, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, which should be emphasized that, in this case, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are just numbers, we use the constants $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, $\tilde{c}_4$, e_1 and e_2 defined by the following relations:

$$\begin{aligned} \tilde{c}_1 &= \frac{b_1^{3/2}(a_2b_1-a_1b_2)^{3/2}}{h_2^3(a_1b_3-a_3b_1)}c_1\,, & \tilde{c}_3 &= -\frac{b_1(a_1b_2-a_2b_1)^2}{h_2^2(a_1b_3-a_3b_1)}c_3-\frac{2b_2\Lambda}{3h_2^2(a_1b_2-a_2b_1)}\,,\\ \tilde{c}_4 &= \frac{b_1^{3/2}(a_1b_2-a_2b_1)^{3/2}}{h_2^3(a_1b_3-a_3b_1)}c_4\,, & \tilde{c}_2 &= \frac{b_1^2(a_1b_2-a_2b_1)}{a_1b_3-a_3b_1}c_2-\frac{b_2^2\Lambda}{3(a_1b_2-a_2b_1)^2}\,,\\ e_1 &= \frac{h_2^2b_1(a_1b_2-a_2b_1)}{\sqrt{2\lambda}(a_1b_3-a_3b_1)}\mathcal{Q}_1\,, & e_2 &= \frac{ih_2^2b_1(a_1b_2-a_2b_1)}{\sqrt{2\lambda}(a_1b_3-a_3b_1)}\mathcal{Q}_2\,. \end{aligned}$$

In addition, let us perform the coordinate transformation $(\tau,\sigma,x,y)\to(t,\phi,p,q)$ defined by

$$\begin{aligned} x^2 &= b_1^{-1}\left(h_2^2p^2+\frac{b_2}{a_1b_2-a_2b_1}\right)^{-1}-b_1^{-1}a_1\,,\\ y^2 &= b_1^{-1}\left(h_2^2q^2-\frac{b_2}{a_1b_2-a_2b_1}\right)^{-1}+b_1^{-1}a_1\,,\\ \tau &= \frac{\sqrt{b_1^3(a_2b_1-a_1b_2)}}{h_1(a_1b_3-a_3b_1)}\left(\frac{a_1}{h_2b_1}t+\frac{a_2}{h_2^3(a_1b_2-a_2b_1)}\phi\right)\,,\\ \sigma &= \frac{\sqrt{b_1^3(a_2b_1-a_1b_2)}}{h_1(a_1b_3-a_3b_1)}\left(\frac{1}{h_2}t+\frac{b_2}{h_2^3(a_1b_2-a_2b_1)}\phi\right)\,. \end{aligned}$$

Notice that we have introduced two new constant parameters h_1 and h_2 that will turn out to be very convenient in the discussion below. In fact, as can be easily seen, these parameters can be trivially absorbed into the new constants and coordinates, and therefore do not present physical content. They are just mathematical assets.

The result of performing the coordinate transformations and redefinition of integration constants above is given neatly by the line element

$$ds^2=-\frac{Q(q)}{h_1^2\rho^2}(dt-p^2d\phi)^2+\frac{\rho^2}{Q(q)}dq^2+\frac{P(p)}{\rho^2}(dt+q^2d\phi)^2+\frac{\rho^2}{h_1^2P(p)}dp^2\,,$$

with $\rho^2=p^2+q^2$ and functions $Q(q)$ and $P(p)$ defined by the following polynomials:

$$P(p)=h_1^{-2}\left(\frac{\tilde{c}_2}{h_2^4}+\tilde{c}_1p+\tilde{c}_3p^2-\frac{\Lambda}{3}p^4\right)\,, \tag{4.57}$$

$$Q(q)=\frac{\tilde{c}_2}{h_2^4}+e_1^2+e_2^2+\tilde{c}_4q-\tilde{c}_3q^2-\frac{\Lambda}{3}q^4\,. \tag{4.58}$$

This result is impressive due to its simplicity. Notice in particular that all the integration constants a's and b's could be eliminated from the functions defining the metric, conveying that they are devoid of physical meaning. Thus, we are left with just the six constants $\tilde{c}_1$,

$\tilde{c}_2$, $\tilde{c}_3$, $\tilde{c}_4$, e_1 and e_2. However, not all of them present physical meaning inasmuch as the constant $\tilde{c}_2$ can be eliminated by a convenient choice in the arbitrary parameter h_2, as can be grasped from equations (4.57) and (4.58). Thus, just five parameters are necessary to characterize the spacetime.

Notwithstanding, the line element presented above is still not in its most recognized form. In order to bring this about, we redefine the three physical constants $\tilde{c}_1$, $\tilde{c}_3$ and $\tilde{c}_4$ in terms of the new constants a, l and m as

$$\tilde{c}_1 = 2l + \frac{2}{3}a^2 l\Lambda - \frac{8}{3}l^3\Lambda\,, \quad \tilde{c}_3 = \Lambda\left(\frac{1}{3}a^2 + 2l^2\right) - 1\,, \quad \tilde{c}_4 = -2m\,,$$

and use the freedom in the choice of the parameters h_1 and h_2 to set

$$h_1 = a \quad \text{and} \quad h_2 = \tilde{c}_2^{1/4}(a^2 - l^2)^{-1/4}(1 - \Lambda l^2)^{-1/4}\,.$$

Following this with a further coordinate change $(t, \phi, p, q) \to (\tilde{t}, \tilde{\phi}, \theta, r)$ defined by

$$t = (a + l)^2\tilde{\phi} - a\tilde{t}\,, \quad \phi = \tilde{\phi}\,, \quad p = l + a\cos\theta\,, \quad q = r\,,$$

we arrive at the desired form for our line element:

$$\begin{aligned} ds^2 = {} & \frac{\rho^2}{Q}dr^2 - \frac{Q}{\rho^2}\left[d\tilde{t} - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\tilde{\phi}\right]^2 \\ & + \frac{P}{\rho^2}\left[ad\tilde{t} - (r^2 + (a+l)^2)d\tilde{\phi}\right]^2 + \frac{\rho^2}{P}\sin^2\theta d\theta^2\,, \end{aligned} \tag{4.59}$$

where

$$\rho^2 = r^2 + (l + a\cos\theta)^2, \qquad P = \sin^2\theta\left(1 + \frac{4}{3}\Lambda a l\cos\theta + \frac{1}{3}\Lambda a^2\cos^2\theta\right),$$

$$Q = a^2 - l^2 + e_1^2 + e_2^2 - 2mr + r^2 - \Lambda\left[(a^2 - l^2)l^2 + \left(\frac{1}{3}a^2 + 2l^2\right)r^2 + \frac{1}{3}r^4\right].$$

Indeed, this is precisely the Kerr-Newman-NUT-(A)dS solution as written in Ref. [78]. In this case, the constants m, a and l are interpreted respectively as the mass, angular momentum per mass and NUT parameter of the black hole, while e_2 and e_1 are its electric and magnetic charges. For a comparison, see equation (17) of Ref. [78] and the choice of parameters adopted in section 4.2 of this reference.

Finally, in terms of these coordinates and parameters, the gauge field becomes

$$\begin{aligned} \mathcal{A} = {} & \frac{1}{\rho^2}\left[e_1(l + a\cos\theta) + e_2 r\right]d\tilde{t} \\ & - \frac{1}{a\rho^2}\left\{e_1\left[r^2 + (l+a)^2\right](l + a\cos\theta) + e_2 r\left[(l+a)^2 - (l + a\cos\theta)^2\right]\right\}d\tilde{\phi}\,. \end{aligned}$$

This form for the gauge field is obtained once we have set $\lambda = 1/2$ in the action (4.27), which is the coupling constant usually adopted for the electromagnetic field.

It is not difficult to see that, for a general k-dimensional gauge group, the coordinate transformations and redefinitions of parameters given above can still be used and it will result in the same metric, with the only difference that, now, Q will not only have two squared charge parameters, but rather there will be k sums of squared charge parameters.

To achieve this, we just need to define k constants $e_1, \ldots, e_k$ in terms of the k parameters of the Lie algebra, in such a way that

$$\mathrm{Tr}(\mathcal{Q}_1\mathcal{Q}_1 - \mathcal{Q}_2\mathcal{Q}_2) = \frac{2\lambda(a_1b_3 - a_3b_1)^2}{h_2^4 b_1^2 (a_1b_2 - a_2b_1)^2} \sum_{i=1}^{k} e_i^2 \,,$$

just as we did in the $U(1)$ case. Besides this, notice that the coordinate transformations taking the separable coordinate system $\{x, y, \tau, \sigma\}$ to the system $\{r, \theta, \tilde{\phi}, \tilde{t}\}$ is of the type presented in section 2.6 (property (iv)). Hence, we see that the coordinate system $\{r, \theta, \tilde{\phi}, \tilde{t}\}$ is also separable and is equivalent to the first one. In particular, since the second class coordinates x and y are taken respectively to θ and r by means of separated transformations, and the first class coordinates τ and σ transform to $\tilde{\phi}$ and $\tilde{t}$ according to a linear combination, the coordinates in $\{r, \theta, \tilde{\phi}, \tilde{t}\}$ are also normal. Indeed, the inverse of the metric (4.59) can be computed, in which case we will see that it is of the form given by equation (2.30) if we define the inverse Stäckel matrix $[\phi_{(a)}{}^b]$ using (2.33) for functions defined by

$$\psi_1(r) = Q\,, \quad \psi_2(\theta) = \frac{P}{\sin^2\theta}\,, \quad \phi_1(r) = r^2 \quad \text{and} \quad \phi_2(\theta) = (l + a\cos\theta)^2\,,$$

along with

$$\eta_1^{\tilde{\phi}\tilde{\phi}} = -\frac{a^2}{Q^2}\,, \quad \eta_1^{\tilde{t}\tilde{t}} = -\frac{1}{Q^2}[r^2 + (a+l)^2]^2\,, \quad \eta_2^{\tilde{t}\tilde{t}} = \frac{\sin^2\theta}{P^2}\left(4l\sin^2\frac{\theta}{2} + a\sin^2\theta\right)^2\,,$$
$$\eta_2^{\tilde{\phi}\tilde{\phi}} = \frac{\sin^2\theta}{P^2}\,, \quad \eta_1^{\tilde{\phi}\tilde{t}} = -\frac{a}{Q^2}[r^2 + (a+l)^2]\,, \quad \eta_2^{\tilde{\phi}\tilde{t}} = \frac{\sin^2\theta}{P^2}\left(4l\sin^2\frac{\theta}{2} + a\sin^2\theta\right)\,.$$

From the functions defined above, we can immediately construct the nontrivial rank-2 Killing tensor of the solution using the results of section (2.9). In this case it will take the following form:

$$\begin{aligned}\boldsymbol{K} = Q(l + a\cos\theta)^2\,\partial_r^2 - \frac{r^2}{P}\left[\partial_{\tilde{\phi}} + \left(4l\sin^2\frac{\theta}{2} + a\sin^2\theta\right)\partial_{\tilde{t}}\right]^2 \\ - \frac{r^2P}{\sin^2\theta}\,\partial_\theta^2 - \frac{1}{Q}(l + a\cos\theta)^2\left[a\,\partial_{\tilde{\phi}} + (r^2 + (a+l)^2)\,\partial_{\tilde{t}}\right]^2\,.\end{aligned}$$

This is the irreducible rank-2 Killing tensor of the Kerr-Newman-NUT-(A)dS (and generalizations thereof) that generates Carter's quadratic conserved quantity when the nut charge l, the cosmological constant Λ, and the charges that generate the gauge field are zero.

5 Conclusion

In the chapter 2 of this dissertation, we presented a study on the separability of the Hamilton-Jacobi equation for the geodesic Hamiltonian. The close relationship between the notion of separability structures and the existence of sufficiently many symmetries that allow us to completely integrate the geodesic equation for spaces admitting such an structure was established. Furthermore, the most general form for these spaces was presented, being dubbed the normal form. The particular case where the separability structure is of type $\mathcal{S}_{n-2}$ was worked out in section 2.9, in which case we learned that these spaces are naturally equipped with a set of $(n-2)$ commuting Killing vector fields along with a nontrivial rank-2 Killing tensor. Moreover, we learned that they possess two orthogonal foliations: $\{Z_2\}$, which presents an orthogonal separable structure of type $\mathcal{S}_2$, and $\{W_{n-2}\}$, which is a flat foliation. The canonical form for both the metric and the Killing tensor in this case was also given in section 2.9.

In chapter 3, a presentation of the definition and general properties of groups, Lie groups and Lie algebras was given, exemplifying wherever possible. The action of a Lie group on differentiable manifolds was then studied, followed by a specialization to the case where the action maintained the metric invariant, given rising to the spacetime symmetries. The symmetries of spaces with a separability structure of type $\mathcal{S}_{n-2}$ were studied in light of this group action. Hence, we learned that the orbits of the abelian isometry group generated by the $(n-2)$ commuting Killing vectors are precisely the flat submanifolds W_{n-2} that foliates the manifold, thus providing a better understanding on the underlying geometry shared by these spaces. The symmetries existing in this class of metrics was then used explicitly in section 3.8 to build the corresponding first integrals.

In face of all the knowledge obtained throughout these chapters on spaces with a separability structure of type $\mathcal{S}_{n-2}$, in chapter 4, restricting the dimension to $n = 4$ and conditioning these spaces to the constraints (4.3), we aimed at obtaining exact solutions for the Einstein-Yang-Mills theory. As we could see, the restrictions (4.3) gave rise to a natural null frame, where the two real null vector fields were aligned with the principal null directions of the spacetime. In fact, these two vector fields were shown to generate two independent shearfree null congruences of geodesics even without imposing a field equation. Hence, coupling gauge fields to these spaces, and requiring them to be aligned with these principal null directions, the Goldberg-Sachs theorem could be used to facilitate the integration process, showing that the solutions had to be of Petrov type D. We further assumed these gauge fields to inherit the symmetries of the spacetime, namely that they are not dependent on the cyclic coordinates of the metric in the canonical form, and hence a whole class of exact solutions could successfully be obtained in this framework.

Unlike all the solutions of the Einstein-Yang-Mills theory presented so far in the literature, the exact solutions found in chapter 4 did not require initial specification of the particular gauge group used in the investigation. Rather, we could obtain solutions for any gauge group chosen. In this same chapter, explicit construction of solutions using the

$SU(2)$ and the Lorentz group $SO(3,1)$ as the gauge groups were performed, the latter providing nontrivial solutions never presented in the literature before. The chapter is then finished by explicit presentation of the known form for the general solutions, which turned out to represent generalizations of the Kerr-NUT-(A)dS spacetime coupled to an arbitrary gauge field. In particular, assuming the gauge group to be $U(1)$, corresponding to the the electromagnetic interaction, we could exhaust all the possible solutions within the class of metrics considered, in which case, no new solution was found.

In conclusion, the full characterization of symmetries and geometric features studied from different approaches along the chapters 2, 3 and 4 for the class of metrics with a separability of type $\mathcal{S}_{n-2}$, and particularly for $n = 4$, equipped us with a deeper and more complete understanding of the origin of the symmetries of this important class of spaces that contains, as its most important members, the Kerr-NUT-(A)dS metric and generalizations thereof. Besides this, its worth stressing that the wide class of nontrivial solutions found in chapter 4 were never presented in the literature before.

Possible future directions in the same line of the research presented in this dissertation would be pursing a different choice on the restriction over the functions $G_1^{ij}(x)$ and $G_2^{ij}(y)$, present in the line element (4.1), replacing the constraint (4.3). In addition, different choices regarding the kind of matter filling the spacetime can be made. In fact, setting one of these functions to zero was the path chosen by the author of the present work, together with his advisor, in the investigation described in Ref. [10], for the four dimensional case, and Ref. [12], for arbitrary dimensions. In the first of these papers, we searched for exact solutions of the Einstein's vacuum equation, in which case an interesting new solution was found. In the second case, a whole class of higher dimensional exact solutions in the presence of a perfect fluid was obtained, the diagonal solutions being generalizations of the Kasner metrics, while the nondiagonal case yielded solutions revealing a curious algebraic structure.

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