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WELLINGTON MARTINS FILHO

FERMION-SOLITON INTERACTION IN 1+1 DIMENSIONS

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"Genius is one percent inspiration, ninety-nine percent perspiration" [1]
EDISON, 1932

Abstract

This thesis deals with the concept of fermion-soliton interactions. In the very beginning, we provide a historical background on how solitons were discovered and how they led to other solutions with similar features in many branches of physics. Then we study the main solitonic models in 1+1 dimensions and discuss some important results concerning these types of models, including Derrick theorem stating that there is no stable soliton solution for Lagrangians with only scalar fields in spatial dimensions above 2. Besides that, we present some solitonic models in higher dimensions, e.g. magnetic monopoles and vortices. We also study instantons which are soliton solutions on a Euclidean spacetime and compare the formalism of instantons for tunneling processes in quantum mechanics with the corresponding one in Yang-Mills theory. Moreover, we present the formalism of fermions interacting with solitons as background fields, as well as some essential mathematical tools including Stationary Phase Approximation, Grassman numbers, Path Integral formalism. We also investigate one of the most important consequences of the interaction, the so-called Casimir Energy induced in systems containing non-trivial background fields such as solitons, using the phase shift method. Finally, we study a nonlinear interaction of a fermion field with a solitonic solution called compaction and compare the results with the known limiting cases in literature.

Keywords: Casimir energy. Solitons. Quantum Field Theory. Fermion soliton interaction. Dirac equation

Resumo

Nessa dissertação nós lidamos com o conceito de interações entre férmions e solitons. Inicialmente apresentamos um ponto de vista histórico de como solitons foram descobertos e como eles levaram a outras soluções com características semelhantes em diferentes áreas da física. Em seguida estudamos os principais modelos de sólitons em $1+1$ dimensões e discutimos alguns resultados importantes relacionados a esses modelos, incluindo o teorema de Derrick que garante que não existem soluções tipo sóliton estáveis em Lagrangeanas de apenas campos escalares em dimensão maior que 2. Além disso, apresentamos alguns modelos de sólitons em dimensões maiores, e.g. monopolos magnéticos e vórtices. Também estudamos instantons, que são soluções tipo sóliton em espaço-tempo Euclidiano e comparamos o formalismo de instantons para processos de tunelamento em mecânica quântica com esse mesmo formalismo para o caso de teoria de Yang-Mills. Além do que foi citado, apresentamos o formalismo de interação de fermions com campos de sóliton como background assim como com algumas ferramentas matemáticas essenciais includindo Aproximação de Fase Estacionária, números de Grassman e formalismo de integral de caminho. Também investigamos uma das consequencias mais importantes desse tipo de interação, a chamada Energia de Casimir, induzida em sistemas contendo campos de fundo com topologia não trivial como sólitons, para isso usamos o método da diferença de fase. Finalmente, estudamos a interação não linear de um campo fermiônico com o campo de um sóliton chamado compacton e comparamos nossos resultados com alguns casos limitantes presentes na literatura.

Palavras-chave: Energia de Casimir. Sólitons. Teoria Quântica de Campos. Interação férmion-sóliton. Equação de Dirac

Contents

1 Introduction: Solitons and how they manifest in nature

In this thesis we present a contained discussion regarding solitons in quantum field theory (QFT). But first we should introduce the concept of soliton, how it is related with QFT, when it first appeared and what makes it so special. The first person who noticed and tried to study a soliton was John Scott Russel in August 1834 [2]. An engineer that became amazed when he first observed what emerged from the motion of a boat that had been drawn through a channel, a solitary well-behaved, round and stable wave [3]. It captivated the attention of J.S.Russel. Later he conducted experiments investigating the shape of the wave and some interesting properties such as the very small damping observed.

After J.S.Russel's discovery of such phenomenon some mathematical approach started to be developed [4, 5, 6]. But the most important contribution and still studied today is due to Korteweg and de Vries (1895) [6] who first formulated a non-linear partial differential equation whose solution possesses the same features as observed by J. S. Russel. It is worthhwhile to outline how Korteweg and de Vries (KdV) first formulated this expression. First they considered an incompressible and irrotational fluid together with the inviscid Euler equation for fluid dynamics and appropriate boundary conditions. Second they have expanded the components of the velocity vector and considered only first approximation of the solution. That is why the KdV equation gives soliton solutions in shallow water, where the wavelength is much larger than the depth of the water [6].

Their calculations yield to the following equation:

$$\frac{\partial \eta}{\partial t} - \frac{3}{2}\sqrt{\frac{g}{H}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right) = 0 \tag{1.1}$$

knowing η is the elevation above the equilibrium level, $\sigma = \frac{1}{3}H^3 - \frac{TH}{\rho g}$ where H is the height of the water (relative distance from the surface to the bottom of the reservoir), T is the surface tension term, ρ is the density of the water, g is the gravitational acceleration, α is a small parameter when compared to H, coming from the shallow water approximation of the solutions.

We can try to find a solution in a reference frame that travels with velocity v, for doing so we suppose a solution such that $\eta(x - vt)$ with the following boundary conditions $\eta(x \to \pm\infty) = 0$. These steps are part of the process to find the famous solution due to Korteweg and de Vries, for the KdV soliton:

$$\eta\left(x\right) = a\,\mathrm{sech}^2\left(\sqrt{\frac{a}{4\sigma}}x\right) \tag{1.2}$$

with σ positive and a the amplitude of the wave
Korteweg and de Vries also derived the velocity of the propagating steady wave [2]:

$$c = \sqrt{gh}\left(1 + \frac{\alpha}{2H}\right) \tag{1.3}$$

Those were the main contributions of Korteweg and de Vries, such contributions that changed the way physicists look at non-linearity in nature.

More recent investigations about KdV equation envolve mKdV (modified Korteweg de Vries) equation, supersymmetric KdV equation [7, 8] and more. Actually further investigation in the KdV equation after its discovery provided insight in several methods for understanding integrability of many physical systems, among these techniques are inverse scattering transform and LAX pairs [9]. There is actually other solutions for the KdV equation besides the solitonic one, the interested reader can see [10].

Nowdays we can see that Korteweg, de Vries, Russel, and others (Rayleigh, Boussinesq..), started creating a huge field of study, the study of soliton theory. This type of structure appears in many unrelated fields in physics, in many different dimensions and different contexts. We cite three more examples to convince the reader.

In 1955 in Los Alamos, Enrico Fermi, John Pasta and Stan Ulam were looking for an interesting problem, suitable for investigation in one of the first computers MANIAC 1 [11]. And there was an open problem at that time. The question was, why solids have finite heat conductivity? Some years before, Debye in 1914 proposed a conjecture, he thought that in a model of a solid that consists of many masses attached to each other by springs, the anhamornicity would have an important role in the finiteness of the heat conductivity [12]. He argued that if the Hook's law is the force acting on each spring and if the initial condition of the problem is all energy localized in the lowest mode then the evolution in time of the configuration leads to the energy being carried freely along the independent fundamental normal modes of the configuration. Moreover, he thought, if there is a weak non-linearity then the normal modes of the previous configuration will be mixed and the energy supposed to be transmitted between these fundamental modes and eventually the system will reach thermal equilibrium [13]. The relaxation time of this initial configuration will be intrinsically connected with the diffusion coefficient. Motivated by this conjecture, Fermi, Pasta, Ulan and Tsingou (FPUT) started studying the one dimensional analog problem. They supposed a nonlinear force as the following:

$$F = k\left(x + \alpha x^2\right) \tag{1.4}$$

Where k is the Hook's constant for the linear case and α is a parameter to control the strength of the nonlinearity, x is the displacement from the equilibrium length of the spring.

FPU supposed as initial condition a smooth state in which all energy was in the lowest mode of the corresponding linear problem. They expected to confirm Debye hypothesis but they found a different result. The energy were distributed across the whole system, but for a sufficiently large number of time steps the system went back to the initial condition. This was an astonishing result and showed them that the equipartition of energy was not guaranteed when non-linearity is present. After that people started realizing that many interesting solutions could arise from non-linear systems. But this is what we should expect, nature is essentially non-linear!

Another interesting example is the huge vortex in Jupiter's atmosphere [14], called Jupiter's Great Red Spot (JGRS for short), a single stable vortex that has already been observed without significant changes for about 300 years [15]. Jupiter is the most massive planet in our solar system, being composed mainly of helium and hidrogen. Small quantities of methane and amonia are also present. But Jupiter's Red Spot was something that always drawn attention and puzzled astronomers for many years. This great tornado lives in the Jupiter's atmosphere, almost fixed in its latitude but sometimes varying in its longitude. Nowadays the best explanation for the generation of the JGRS relies mainly on three facts, the rapid rotation velocity of Jupiter's atmosphere, the nearly dissipationless atmosphere and the strong shearing east-west winds [14]. The different wind velocities are responsable for making this structure stable compensating viscosity and momentum losses. Despite the argument presented above, some authors [16] stress that it does not rule out the possibility that the solid core of Jupiter can play a role in the formation of the JGRS. The vortex can be modeled by what is called Rossby soliton [17]. This type of soliton is also known as planetary waves and is formed at the atmosphere or oceans of a rotating planet. If the depth of the atmosphere or ocean does not depend on the geographical coordinates then the Rossby waves are generated mainly by the spatial inhomogeneities of the Coriollis force [18].

Another interesting example of a soliton model that today still plays an important role in high energy and condensed matter physics is the Skyrme [19] model. The skyrme model arose in the context of quantum chromodynamics where people were trying to study low energy hadrons. Hadrons are composite particles formed by two or more quarks. Skyrmions are solutions from a proposed Lagrangian that effectively describe the system of interest. In their case the theory was simplified in a way that the proposed Lagrangian refers to meson fields (quark anti-quark pair) and baryons (made of three quarks) would arise as soliton solutions of the equations of motion [20]

$$\mathcal{L} = \frac{F_\pi^2}{16} Tr\left[\partial^\mu U \partial_\mu U^\dagger\right] + \frac{1}{32e^2} Tr\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\right]^2 \tag{1.5}$$

Here F_π is the pion decay constant and U is a unitary 2×2 matrix.

Initial considerations were made concerning only one baryon system and after that two baryons and further collections with many baryons were investigated.

There is a very famous ansatz to solve this equation. The ansatz is called "hedgehog" solution, and plugging it in equations of motion gives a non-linear equation for F(r). These equations cannot be solved analytically, but we can use numerical techniques to solve it. The ansatz has the following form

$$U = exp(iF(r)\vec{\tau} \cdot \hat{r}) \tag{1.6}$$

with $\vec{\tau}$ Isospin Pauli matrices and $F(r)$ a function of the radial variable only.
This ansatz is a good trial in order to solve the equations of motion but unfortunately breaks Lorentz invariance. Further investigation was made in order to fix this and other issues. Eventualy the Skyrme model became a good effective model in certain cases and even today further investigations of QCD have been made based on the Skyrme's initial approach [21]. Besides that, a similar model not in $3+1$ (3 dimensions in space and 1 in time) but in $2+1$ dimension, regarding the first Skyrme model is currently investigated, it is called baby Skyrme model (the name comes from the fact that it is a planar analogue of the first model) and is of great interest in condensed matter physics [20**?**]. One final, but not less important example on how non-trivial topology creates interesting physics is

cosmic string. The theory of cosmic strings was first introduced by Kibble [22]. These objects were supposed to be formed in the early Universe, when the Universe passed through a series of phase transitions. People usually refer to these topological defects as evidence for the occurrence of the big bang, not only cosmic strings but other topological defects arise due to non-trivial topological properties of the system in the same way that happens to solitons.

In the 80's, cosmic strings were really good candidates to explain the formation of galaxies in our Universe [23]. But this idea was put aside since no evidence was found for the predicted contributions of cosmic strings in the CMB radiation. After some time people realized that these entities could also arise in the context of string theory. Besides the similar name, these two things are completely different stories. String theory is a theoretical model that suppose all elementary particles are generated as excitations of different vibrating modes of tiny little strings, such strings are the size of the Planck scale [24]. What happens is that for certain cases in string theory these tiny strings can turn into what is called cosmic superstrings, those are the historical cosmic strings.

Then, detection of cosmic strings could also confirm important signatures of string theory. The thing is, until now no experimental evidence of this structures was found [25]. [1]

Those were some examples of how solitons manifest in nature. In the following chapters throughout this thesis, we provide a detailed discussion about others types of solitons and how they interact with fermions. In chapter 2 we talk about instantons, monopoles, kinks and SG-solitons. In chapter 3 we discuss interaction of solitons with fermions and as a particular example we study the interactions in 1+1 dimensions. In chapter 4 we discuss Casimir energy and Levinson theorem and in the final chapter we present new results regarding interaction of solitons with fermions in 1+1 dimensions where our soliton possesses a compact profile. We discuss and compare our results with previous works in literature.

[1] Particularly in "$http : //www.damtp.cam.ac.uk/research/gr/public/cs_home.html$" one can find small movies of numerical simulations showing the evolution of the density of cosmic strings in the universe together with a pedagogic material on the subject

2 Solitons

2.1 Solitons in scalar field theories

We start our discussion with a very simple Lagrangian, and we are mainly concerned with the static localised solutions arising from the potential involved rather than other solutions of the equation of motion. It means that the potentials we are considering must possess, at least two minima in order to admit a non-trivial solution connecting these two minima. As discussed briefly in the introduction this solution is a non-dissipative (static) solution with finite energy. First, we consider the general form for the Lagrangian of a real scalar field in $1+1$ dimensions as:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.1}$$

where the $\phi(x,t)$ is a scalar field and $\mu = 0,1$ labels spatial and time coordinates.

If we want to look for solutions with the properties described above we need to consider fields that do not diverge as $x \to \infty$. We expect $\phi(x \to \pm\infty) \to \pm\phi_0$ where ϕ_0 is some constant. We find the following equation of motion after applying the E-L equation:

$$\begin{aligned} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) &= \frac{\partial\mathcal{L}}{\partial\phi} \\ \partial_\mu\partial^\mu\phi &= -\frac{\partial V}{\partial\phi} \end{aligned} \tag{2.2}$$

We are using the metric signature $diag(1,-1)$ which leads to

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x,t) = -\frac{\partial V}{\partial\phi} \tag{2.3}$$

and assuming also $\frac{\partial\phi}{\partial t} = 0$.

We can perform a smart trick to solve this equation, multiply both sides by $\frac{d\phi}{dx}$, as

$$\frac{d^2\phi}{dx^2}\frac{d\phi}{dx} = \frac{dV(\phi)}{d\phi}\frac{d\phi}{dx} \tag{2.4}$$

The left hand side of this equality can be rewritten as a total derivative, and the right hand side as a derivative of the potential but with respect to the x variable. Then we obtain

$$\frac{d}{dx}\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \frac{dV}{dx} \tag{2.5}$$

Integrating both sides, gives:

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi) + C \tag{2.6}$$

We should assume here $C = 0$. This is due to the fact that the solutions should possess finite energy. Note that if we want $E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}(\frac{d\phi}{dx})^2 + V(\phi)\right)$ to be finite, any constant of integration different from zero in this step would yield an infinite kinectic term contribution to the energy.

We can integrate again to obtain:

$$\begin{aligned} \frac{d\phi}{dx} &= \pm\sqrt{2V(\phi)} \\ \int_{\phi(x_0)}^{\phi(x)} d\phi' \frac{1}{\sqrt{2V(\phi')}} &= \int_{x_0}^{x} dx \\ x - x_0 &= \pm \int_{\phi(x_0)}^{\phi(x)} d\phi' \frac{1}{\sqrt{2V(\phi')}} \end{aligned} \tag{2.7}$$

After computing the integral we only invert the function to get our solution for $\phi(x)$.

2.1.1 BPS (Bogomol'nyi-Prasad-Sommerfield) condition

Now, we are able to compute the total energy. After finding an expression for ϕ one can plug $\phi(x)$ in the expression for the energy density and integrate it to find the total energy of the static solution as

$$E = \int \left(\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right) dx \tag{2.8}$$

But even without any information about ϕ, one can look at (2.8) and perform a smart manipulation of terms that will lead us to the same result we have found before. Knowing

$$\left(\frac{1}{\sqrt{2}}\frac{d\phi}{dx} \mp \sqrt{V(\phi)}\right)^2 = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 \mp \sqrt{2}\frac{d\phi}{dx}\sqrt{V(\phi)} + V(\phi) \tag{2.9}$$

One can rewrite the energy in the form

$$\begin{aligned} E &= \int \left(\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right) dx \\ &= \int \left(\left(\frac{1}{\sqrt{2}}\frac{d\phi}{dx} \pm \sqrt{V(\phi)}\right)^2 \pm \frac{d\phi}{dx}\sqrt{2V(\phi)}\right) dx \end{aligned} \tag{2.10}$$

Note that this integral is always larger than $E = \int_{-\infty}^{\infty} \left(\frac{d\phi}{dx}\sqrt{2V(\phi)}\right) dx$, due to the fact that the first term is always non-negative. That is

$$E \geq \int \left(\frac{d\phi}{dx} \sqrt{2V(\phi)} \right) dx \tag{2.11}$$

The quantity in the right hand side of the inequality is called BPS energy. If we want to have a solution that extremizes the energy functional, that is a minimum energy solution, we should agree with the condition that the first term in parenthesis should vanish, this leading to:

$$\frac{1}{\sqrt{2}} \frac{d\phi}{dx} = \sqrt{V} \tag{2.12}$$

which is the same result we obtained before. If we find a solution satisfying this equation, our solution has an energy given by:

$$E_{BPS} = 2 \int_{-\infty}^{\infty} V(\phi) dx \tag{2.13}$$

We have obtained the same equation of motion from two distinct approaches but as we might see in the future BPS approach is very powerful. Other systems do not allow us to rewrite the equations as we did in (2.4). The BPS approach is more general and can reduce, in many cases e.g monopoles in Yang-Mills theory, our second order equation of motion in first order equations of motion. For some systems this means solving analytycally a problem that for a second order differential equation it was impossible. Besides that, the BPS energy only depends on the assymptotic behavior of the fields and is related to the topological charge of it, not depending on peculiarities of spatial dependence [26]. Actually sometimes relevant information about the system can only be extracted using BPS inequality approach [27].

2.1.2 $\lambda\phi^4$ potential and kink solution

It is instructive if we start discussing a specific model to see how useful the tools that we have developed so far can be.

Let us study the well-known $\lambda\phi^4$ potential, which has the specific profile:

$$V(\phi) = \frac{\lambda}{4} \left(\phi^2 - \frac{m^2}{\lambda} \right)^2 \tag{2.14}$$

where, λ is the strength of the self-interaction of the scalar field and m is proportional to the "mass" of our field[1]. We can see the potential depicted in Figure 2.1.
Using (2.14) expression in (2.7) we have:

$$x - x_0 = \pm \sqrt{\frac{4}{\lambda}} \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi'}{\left(\phi'^2 - \frac{m^2}{\lambda} \right)} \tag{2.15}$$

[1]Just as a reminder, since the beginning we are working with natural units $\hbar = c = 1$.

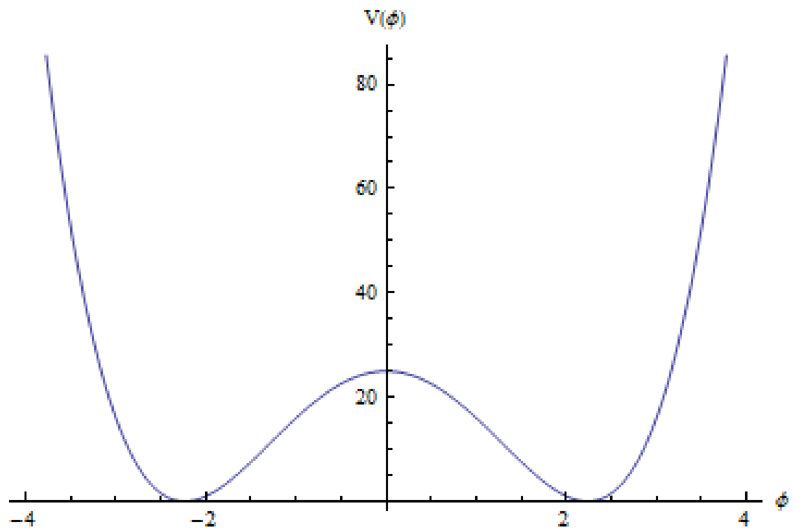

Figure 2.1. Plot of our potential for $\frac{m^2}{\lambda} = 5$

Solving the integral and inverting the function we obtain :

$$\phi(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh(\frac{m}{\sqrt{2}}(x - x_0)) \tag{2.16}$$

The solution with the + sign is the kink and the solution with the − sign is the anti-kink. The x_0 term is a constant of integration that dictates the center of the soliton. We can choose $x_0 = 0$ to obtain what is in Fig. 2.2.

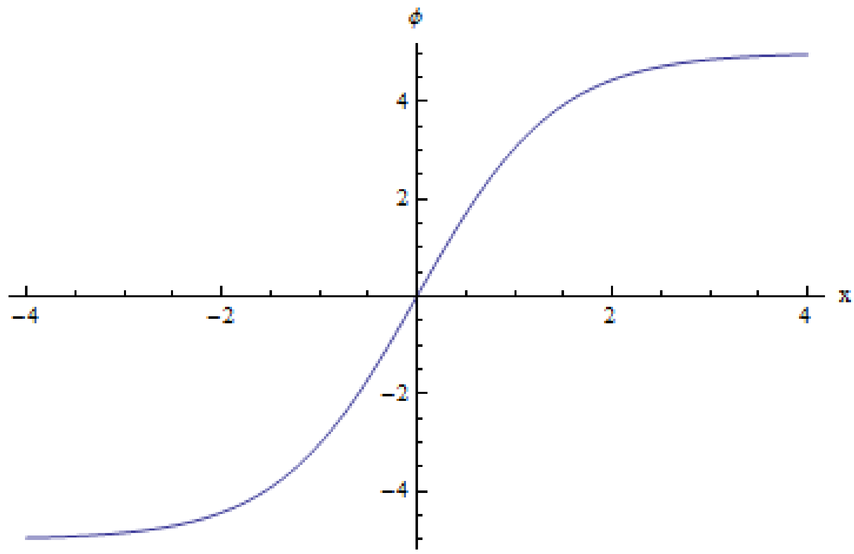

Figure 2.2. Plot of soliton solution for $\frac{m^2}{\lambda} = 5$ and $m = 1$

2.1.3 Stability equation

We can study the dynamical stability of the soliton if we suppose a small pertubation on the soliton field. We suppose $\phi(x,t) = \phi_s(x) + \eta(x,t)$, where $\phi_s(x)$ is the static solution of the equation of motion, which means we can expand $V(\phi)$ as:

$$V(\phi_s + \eta) = V\Big|_{\phi_s} + \frac{\partial V}{\partial \phi}\Big|_{\phi_s} \eta + \dots \tag{2.17}$$

We are supposing that a small pertubation acts on our solution. Therefore, we consider up to a linear term in η in the above expansion.
We can consider a periodic pertubation in the form

$$\eta = \sum_n \eta_n(x) cos(\omega_n t) \tag{2.18}$$

resulting in:

$$\begin{aligned} \partial_\mu \partial^\mu \phi(x,t) + \frac{\partial V}{\partial \phi} &= \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial V(\phi)}{\partial \phi} \\ &= -\sum_n \eta_n \cos(\omega_n t) \omega_n^2 - \sum_n \left(\frac{d^2 \eta_n}{dx^2} \right) \cos(\omega_n t) + \frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi_s} \eta_n \\ &= 0 \end{aligned} \tag{2.19}$$

We obtain an eigenvalue equation for each η_n

$$\left(-\frac{d^2}{dx^2} + \frac{\partial^2 V(\phi)}{\partial \phi^2}\Big|_{\phi_s} \right) \eta_n = \omega_n^2 \eta_n \tag{2.20}$$

For the $\lambda\phi^4$ potential we obtain:

$$\left(-\frac{d^2}{dx^2} + m^2 \left(-4 + 3\,\mathrm{sech}^2 \left(\frac{mx}{\sqrt{2}} \right) \right) \right) \eta_n = \omega_n^2 \eta_n \tag{2.21}$$

If we perform a change of variables as $z = \frac{mx}{\sqrt{2}}$, we find the following equation:

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} - \left(3\tanh^2 z - 1 \right) \right) \eta_n = \frac{1}{m} \omega_n^2 \eta_n \tag{2.22}$$

This is similar to a Schrödinger equation where the potential envolved belongs to a family of Pösch-Teller potentials. The bound state solutions are:

$$\eta_0 = \frac{1}{\cosh^2(z)}, \quad \eta_1 = \frac{\sinh z}{\cosh^2(z)} \tag{2.23}$$

For $n \geq 2$ we have continuum states[2] [28]. These η_0 and η_1 are bound modes of the kink solution with respective eigenvalues $\omega_0 = 0$ and $\omega_2 = \frac{3}{2} m^2$. The mode with η_1 can be excited whenever the kink is perturbed, an example is the collision of two kinks,

[2] This particular case solved analytically can be found in chapter 12 of the citted reference

where kinetic energy can be interchanged and these modes excited. The η_0 mode is a consequence of the translational symmetry of our solution, which means that our solution can be centered at any point in space without changing its internal energy.

2.1.4 Some topological properties

Now, since solitons usually have a non-trivial topology, we can imagine some topological concepts will be useful in characterizing these structures. Topological charge plays an important role here. The main reason to define this quantity is to classify our solutions. We will see shortly that different topological charges label topologically different maps. These maps are grouped in homotopy sectors (classes). If two different maps can be continuously deformed in each other they belong to the same homotopy class. Usually these mappings connect the internal space of fields we are working with and the real physical space of space-time coordinates. Soon these concepts will be clear. For now is sufficient for us to understand the following example. In one dimension it is readily intuitive and tell us the solutions that belong to each homotopy sector. These sectors are responsible for separating different topological solutions, which means that one topological solution from a sector labeled by topological charge $Q = 1$ cannot be deformed into a solution belonging to a topological sector labeled by $Q = 0$ or $Q \neq 1$. We will discuss more examples regarding this subject in the future.
In one dimension we define the topological charge as:

$$\begin{aligned} Q &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon^{\mu 0}\partial_\mu \phi(x) \\ &= \frac{1}{2\pi}\left(\phi(\infty) - \phi(-\infty)\right) \end{aligned} \tag{2.24}$$

The fields with non-trivial topology have $Q \neq 0$.
One could associate the topological charge with the topological current given by:

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu \phi(x) \tag{2.25}$$

through $Q = \int_{-\infty}^{\infty} j^0$ and $\partial_\mu j^{\mu\nu} = 0$.

2.1.5 Sine-Gordon soliton

This model was first introduced in [29] when studying surfaces with constant negative curvature in 1862. It was rediscovered independently by Frenkel and Kontorova in 1939 when studying dislocations in crystals [30]. This potential is a more interesting case since it has analogy with a very interesting topic in physics today, duality.
This model has the following potential:

$$V(\phi) = \frac{m^4}{\lambda}\left(1 - \cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right) \tag{2.26}$$

From the beginning it is easy to see how this system is different from the first one. Note that here we have an infinite set of minima and in the first one we had only two minima.
Again, replacing the potential in equation (2.7), we can find our finite energy static solution:

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} d\phi' \frac{1}{\sqrt{\frac{2m^4}{\lambda}\left(1 - \cos\left(\frac{\sqrt{\lambda}}{m}\phi'\right)\right)}} \tag{2.27}$$

Where we have chosen $\phi(x \to \infty) \to \frac{2\pi m}{\sqrt{\lambda}}$ and $\phi(x \to -\infty) \to 0$ for $\phi(x)$ to be single valued.

Solving the integral and inverting the function we find:

$$\phi(x) = \pm 4\frac{\sqrt{\lambda}}{m} \arctan(e^{m(x-x_0)}) \tag{2.28}$$

with + sign describing a SG-kink solution and − sign an SG-anti-kink[3] solution. The propagation of an SG-soliton can be pictured as in Fig.2.3.
It can also be obtained from the following equation of motion, that gives the name to the model:

$$\partial_\mu \partial^\mu \phi + \frac{m^3}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{m}\phi\right) = 0 \tag{2.29}$$

with the following boundary conditions $\phi(x \to \infty) = \phi(x \to -\infty) + \frac{2\pi m}{\sqrt{\lambda}}$ to be consisted with our previous choice and $\frac{\partial \phi}{\partial t} = 0$.

Not only isolated kinks and anti-kinks are solutions to this model, there is also the so-called "breather" solution, which is a composition of kink and anti-kink solutions. If we perform the following change of variables $\bar{x} = mx$, $\bar{t} = mt$ and $\bar{\phi} = \frac{\sqrt{\lambda}}{m}\phi$ in our equation of motion we can check that:

$$\bar{\phi}_v(\bar{x}, \bar{t}) = 4 \arctan\left(\frac{\sinh(v\bar{t}/\sqrt{1+v^2})}{v \cosh(\bar{x}/\sqrt{1+v^2})}\right) \tag{2.30}$$

Is one of the solutions of (2.29), where $v = iu$. One of the first works studying the complete solutions of this equation is [31]. This is a moving soliton, the appearance of the u term is due to a Lorentz-boost, that is, replacing the x dependence in the solutions from (2.29) with $(x - ut)/\sqrt{1 - u^2}$, where u is the velocity of the moving soliton.

This model possesses also other interesting features, namely its duality with the massive Thirring model, discovered by S. Coleman in [32]. Duality in physics happens when we have the same mathematical model describing two distinct physical phenomena. The Sine-Gordon Lagrangian is a suitable model to describe non-linear interaction between fermions:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{g}{2} j_\mu j^\mu \tag{2.31}$$

[3] Usually in literature people use the word "kink" to name any soliton in $1 + 1$ dimensions. In this thesis we choose to call the solutions of the SG equations of motion "SG-kink" and "SG-anti-kink" and the solution of the $\lambda\phi^4$ model simply "kink".

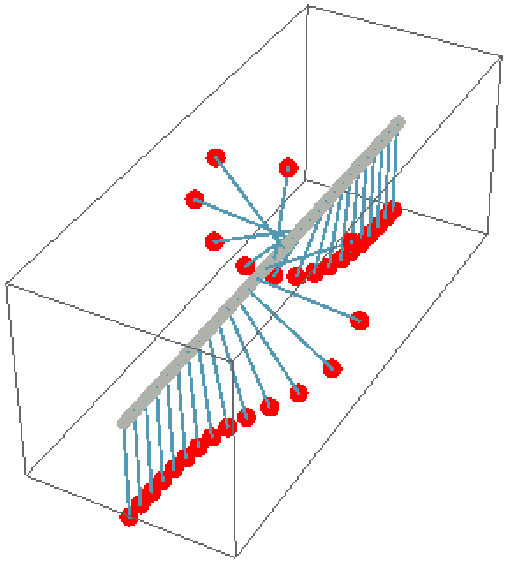

Figure 2.3. Set of pendulums supporting propagation of a sine-gordon soliton. This row of pendulums hang from a rod and are coupled by torsion springs

where $j^\mu = \bar{\psi}\gamma^\mu\psi$ and γ^μ are the gamma matrices[4].
The following correspondence is what make the two models equivalent:

- kink $\leftrightarrow$ ψ, fermion
- anti-kink $\leftrightarrow$ $\bar{\psi}$, anti-fermion
- boson ϕ, $\leftrightarrow$ ψ-$\bar{\psi}$, fermion anti-fermion pair
- topological charge $\leftrightarrow$ fermion number

This model describing fermions coupled non-linearly is known as the Thirring model [33]. The duality between these two systems is an interesting feature. One can find how the coupling constants of the two systems are related. It turns out that the weak coupling regime of one of the theories is the strong coupling of the other (the coupling constants are inversely proportional to each other). Studying this one dimensional system can give us useful insight concerning strong coupling regime of non-linear interacting fermions in (1+1)-dimensions. Actually some authors investigated experimentally this model in the context of optical lattices [34].

2.1.6 Derrick's Theorem

Dealing with fields in 1+1 dimensions sometimes reveals really interesting physics but more realistic systems would require models in higher dimensions. The Derrick's theorem

[4]More information about these models can be found in the reference citted and therein or in the notes of Hugo Laurell "A summary on Solitons in quantum field theory".

introduce severe constraints as we try to complete this task, it states that theories with only scalar fields cannot admit soliton solutions in $(D+1)$ systems if $D \geq 3$.
Let us suppose we have the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - V(\vec{\phi}) \tag{2.32}$$

where $\vec{\phi}$ is a scalar field in D dimensions, that is: $\vec{\phi}(x,t) = (\phi_1(x,t)\ldots,\phi_D(x,t))$. If we look at the equations of motion in the static case we reach:

$$\nabla^2 \vec{\phi} = \frac{\partial V(\vec{\phi})}{\partial \phi} \tag{2.33}$$

Naturally we can also calculate the Hamiltonian of our system:

$$H = \frac{1}{2}\int d^D x \left((\nabla\vec{\phi})^2 + V(\vec{\phi})\right) = V_1 + V_2 \tag{2.34}$$

where

$$V_1 = \frac{1}{2}\int d^D x (\nabla\vec{\phi})^2, \quad V_2 = \frac{1}{2}\int d^D x (V(\vec{\phi})) \tag{2.35}$$

The solution given by the equation of motion above is of course an extremum of the energy functional H, $\delta H{=}0$.

We can analogously define a $\vec{\phi}_\lambda$ field such as $\vec{\phi}_\lambda = \vec{\phi}(\lambda x)$ and it will lead to:

$$H_\lambda = \int d^D x (\lambda^{2-D} V_1 + \lambda^{-D} V_2) \tag{2.36}$$

Note that $\vec{\phi}_{\lambda=1} = \vec{\phi}(x)$. It is easy to be convinced that if $\vec{\phi}$ (solution) of the equation of motion is an extremum of the hamiltonian H it is also an extremum of H_λ. Then we calculate the functional derivative of H_λ and evaluate $\frac{\delta H_\lambda}{\delta \lambda}|_{\lambda=1} = 0$, resulting in:

$$\left.\frac{\delta H_\lambda}{\delta \lambda}\right|_{\lambda=1} = (2-D)V_1 - DV_2 = 0 \tag{2.37}$$

We clearly see that for D=2 we must have $V(\vec{\phi}_{\lambda=1}) = 0$. That is, if our potential has a discrete number of minima our solution for $\vec{\phi}_{\lambda=1}$ is trivial. But if the potential has a continuum minima[5], then $\vec{\phi}_{\lambda=1}$ can have an x dependence but it's dynamics will be constrained to the minima of V.
For the case where $D > 2$ we must have $V_1 = V_2 = 0$ and of course this is a trivial solution. That is why higher dimensional systems with only scalar fields involved do not support solitons. Aware of this people started to look for soliton solutions in models of scalar fields coupled to other fields, for instance gauge fields. The following topic is an example of an interesting structure found after considering this approach. But first we introduce another approach following historical derivation of this entity.

[5]such as the vortex potential depicted in Fig.2.9

2.2 Monopoles

During last centuries people were amazed by magnetism. After the discovery of positive and negative electric charges researchers started wondering about the existence of magnetic charges as fundamental constituents of the magnetic force. This assertion was proved wrong after the discovery of the electron that is the fundamental "magnet" generating the magnetic field of all materials. But in 1931 Dirac made his effort to not let this ideia on magnetic charges (magnetic monopoles) die [35]. He came up with a discussion for the existence of magnetic monopoles that actually justified the discrete character of the electric charge. It brought back the interest of physicists on studying this structures. Despite the development of this field over the years, not a single magnetic charge (or magnetic monopole) was detected. But why study magnetic monopoles if they were never detected? Well, nowadays its importance relies on the fact that monopoles are expected to exist in some GUT'S (Grand Unified Theories)[6] and from the data we collect about the abscence of this monopoles we can put constraints on GUT models.

The Dirac investigation of magnetic monopoles starts with a very simple reasoning. First we recall the Maxwell's equation for a point-like charge localized in the origin of a coordinate system. Maxwell's equation defined in the $\mathcal{R}^3 - \{0,0,0\}$ domain has the following form:

$$\nabla \cdot \vec{E} = 0 \tag{2.38}$$
$$\nabla \times \vec{E} = 0 \tag{2.39}$$
$$\nabla \cdot \vec{B} = 0 \tag{2.40}$$
$$\nabla \times \vec{B} = 0 \tag{2.41}$$

With our fields being $\vec{E} = q\hat{r}/r^2$ since we are suposing the charge at rest, gives $\vec{B} = 0$. This set of equations probably made Dirac think that Maxwell's equation also support solutions as $\vec{B} = g\hat{r}/r^2$ and $\vec{E} = 0$, where g is a 'magnetic charge'. It turns out that these equations for $\vec{B}$ and $\vec{E}$ actually satisfy Maxwell's equation.
We should also try to find the corresponding potential field for this new solution of the Maxwell's equation.

$$\nabla \times \vec{B} = 0 \tag{2.42}$$
$$\nabla \cdot \vec{B} = 0 \tag{2.43}$$

Since the $\mathcal{R}^3 - \{0,0,0\}$ is simply connected and we have (2.42), we can say that there is a potential V such that $\nabla V = \vec{B}$. We can agree that this potential is $V = -g/r$. But we will find a richest physics if we try to find a vector potential for the field $\vec{B}$ such as $\nabla \times \vec{A} = \vec{B}$ as we usually consider for the magnetic field. In the near future this gauge vector field will play a key role in describing the quantum version of this discussion.
Another set of interesting details includes the second equation for the magnetic field (2.43) and our domain. It is temptating to say that since we have (2.43) and the fact that the domain is simply connected, we have sufficient conditions to guarantee the existence of a vector potential such that $\vec{\nabla} \times \vec{A} = \vec{B}$, due to the divergence theorem. We will see that

[6]Those are theories in particle physics that at sufficiently high energies, describe weak force, strong force and electromagnetism, with a single model.

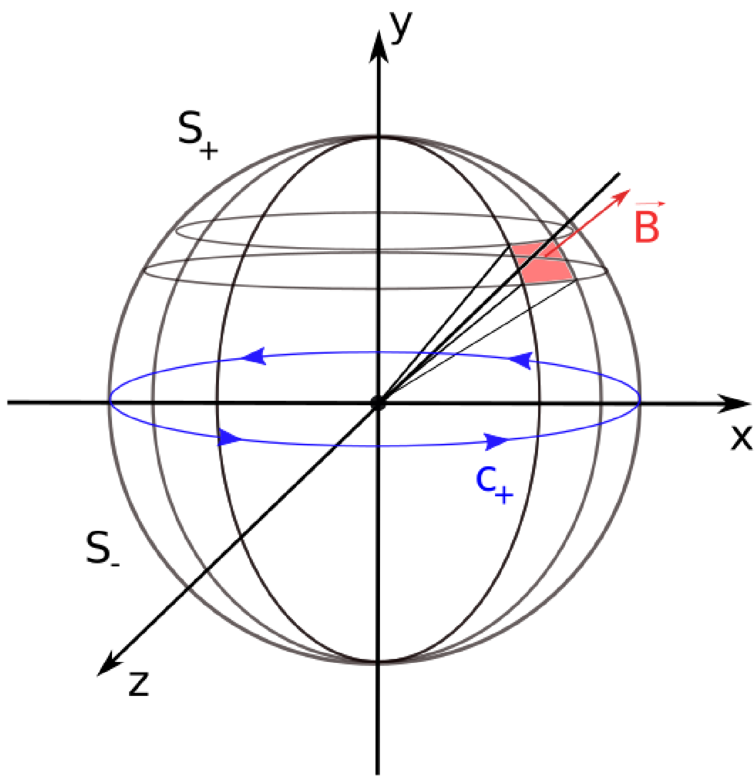

Figure 2.4. Magnetic charge placed in the center of a coordinate system

it is not true. To convince the reader, let us suppose that $\vec{A}$, $(\nabla \times \vec{A} = \frac{g}{r^2}\hat{r})$ does exist and let us try to calculate the flux of a point magnetic charge localized at the origin of a coordinate system. We suppose a spherical shell S, with upper hemisphere S^+ and lower hemisphere S^- and surface vector oriented in the direction of the radial coordinate as depicted in Fig. 2.4, then:

$$\begin{aligned} \int\int \nabla \times \vec{A} \cdot d\vec{S} &= \int\int g/r^2 \cdot r^2 \sin(\theta) dr d\theta \\ &= g\int\int \sin\theta dr d\theta \\ &= 4\pi g \end{aligned} \tag{2.44}$$

We can perform the same calculation using Stoke's theorem:

$$\begin{aligned} \int\int \nabla \times \vec{A} \cdot d\vec{S} &= \int\int_{S^+} \nabla \times \vec{A} \cdot d\vec{S} + \int\int_{S^-} \nabla \times \vec{A} \cdot d\vec{S} \\ &= \int_{C^+} \vec{A} \cdot d\vec{l} + \int_{C^-} \vec{A} \cdot d\vec{l} \\ &= \int_{C^+} \vec{A} \cdot d\vec{l} - \int_{C^+} \vec{A} \cdot d\vec{l} \\ &= 0 \end{aligned} \tag{2.45}$$

where the C curve is the equator of this spherical shell oriented counter clockwise. As we can see, those are not consistent results and then, makes no sense agree that the supposed vector potential $\vec{A}$ exists. Actually the conditions for a vector potential to exist in the domain we are working with are required by a theorem stating that:

1. $\vec{B}$ must have $\nabla \cdot \vec{B} = 0$

2. The subset, U, in which $\vec{B}$ is defined must possess a trivial second homotopy group $\pi_2(U) = 0$

But what does $\pi_2(U) = 0$ mean?

It means we can define a 2-dimensional sphere inside an open subset U and all points enclosed by this sphere are elements of U. It is the 3 dimensional version of what we say is a simply connected subset ($\pi_1(U) = 0$). But as we can see, the set we have been working with does not possess $\pi_2(U) = 0$. It means that we cannot define a vector potential as we desired. But there is actually a way we can contour this problem. Let us define a subset U_1 such that $U_1 = \mathcal{R}^3 - \{z|z \geq 0\}$ and let us define $U_2 = \mathcal{R}^3 - \{z|z \leq 0\}$. U_1 and U_2 are useful because we can see that $U = U_1 \cap U_2$ and we have $\pi_2(U_1) = \pi_2(U_2) = 0$ for each region separately. With these we are allowed to define a vector potential in each subset separately, then we can check that the following potentials below [7] satisfy our requirements. Note that the value of the potential for each independent region is different, as it should be, otherwise we would be able to define $\vec{A}$ in $\mathcal{R} - \{0, 0, 0\}$, which contradicts what was concluded before. Then the vector potential can be defined in the following form, first computed by T.T Wu and C.N Yang in [37]

$$\vec{A^+} = \frac{g}{r \sin\theta} (1 - \cos\theta)\, \hat{\theta} \tag{2.46}$$

$$\vec{A^-} = -\frac{g}{r \sin\theta} (1 + \cos\theta)\, \hat{\theta} \tag{2.47}$$

We can see that we satisfy all the conditions, and of course $\nabla \times \vec{A} = \vec{B}$. But why is this result interesting? Despite that for classical electromagnetism where $\vec{A}$ and V fields are just auxiliary fields, in quantum mechanics this fields play an important role and we see that they cannot be thought anymore as just "mathematical tools". The Aharanov-Bohm effect [38] is a perfect example of this. Hence our non-relativistic quantum theory of electromagnetic interaction should include $\vec{A}$ in the Hamiltonian, consequently our wave function will depend on $\vec{A}$. Besides that, since $\vec{A}$ is a vector field such that $\nabla \times \vec{A} = \vec{B}$, a change in $\vec{A}$ as $\vec{A} \to \vec{A} + \nabla\Omega$ should not change the wave function ψ more than adding a phase to it. That is for $\vec{A} \to \vec{A} + \nabla\Omega$ we should have $\psi \to e^{iq\Omega}\psi$. Now let us suppose again that we are dealing with a magnetic charge at the origin of a coordinate system, at rest, and this charge is interacting with an electric charge q passing nearby as depicted in Fig. 2.5. The wave function will depend on $\vec{A}$ and since physically there is no difference between the electric charge being below $z = 0$ plane or above $z = 0$ plane (what separate our U_1 and U_2 domains) it means that the wave functions ψ_+ consequent of $\vec{A^+}$ and ψ_- consequent of $\vec{A^-}$ should be the same in the intersection $U_1 \cap U_2$. Actually one can see that $A^+ - A^- = \nabla(2g\theta)$ and the wave functions should be related as $\psi_+ = \psi_- e^{iq2g\theta}$ in this region. Note also that the $(r, \phi, \theta) = (0, \frac{\pi}{2}, \theta)$ circle is included in $U_1 \cap U_2$ and as well as the rest of the domain the wave functions ψ_+ and ψ_- should be single valued. This means that a change in the angle θ by $\theta \to \theta + 2\pi$ must reproduce $\psi_+(\theta) = \psi_+(\theta + 2\pi)$. Then, our conclusion is:

[7] One can find a complete derivation of these expressions in [36].

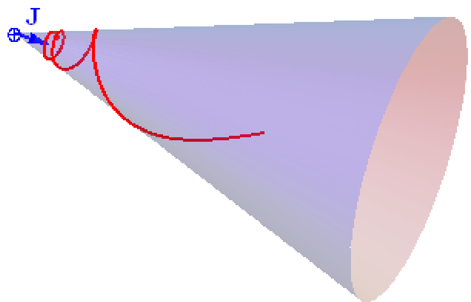

Figure 2.5. Diagram of a classical electron charge interacting with the field of a magnetic monopole. $\vec{J}$ is the total momentum of the configuration, accounting mechanical and electromagnetic contributions. This problem was first solved, long ago by Poincaré in 1896 [39].

leading to

$$\psi_+ = e^{(iq2g\theta)}\psi_- = e^{iq2g(\theta+2\pi)}\psi_- \tag{2.48}$$

$$\begin{aligned} 4qg\pi &= 2\pi n \\ qg &= \frac{n}{2} \end{aligned} \tag{2.49}$$

If monopoles exist, this result expresses the first explanation to the well-known fact that the electric charge is discrete. That is, if Dirac's monopoles exist then its charge can be computed using the above expression.

For a mathematically detailed investigation of the topic check [40].

2.2.1 't Hooft Polyakov Monopole

Another system from where monopole solutions also arises is the Georgi-Glashow Model. A system where the Yang-Mills Lagrangian is coupled to a triplet scalar field. This scalar field is called Higgs field, because it follows the Higgs mechanism of symmetry breaking. Gerard 't Hooft and A. Polyakov have found, independently, monopole solution in this model [41, 42, 43]. The Georgi-Glashow Lagrangian is gauge invariant and possesses $SU(2)$ symmetry. It means that it is invariant under the following transformations:

$$(L^a A^a_\mu)_{bc} \to U_{bd}\left[L^a A^a_\mu + \frac{i}{g} I\partial_\mu\right]\left(U^{-1}\right)_{ec}, \quad \phi^a(x,t) \to [U(x,t)]_{ab}\,\phi^b(x,t) \tag{2.50}$$

where

$$U(x,t) = exp(-iL^a\theta^a(x,t)) \tag{2.51}$$

The parameters $\theta^a(x,t)$ characterize the $SU(2)$ group transformation and L^a are the generators of the group symmetry, $(L^a)_{bc} = i\epsilon_{abc}$.
The proposed Lagrangian by Georgi and Glashow is given by:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu(\phi^a)D_\mu(\phi^a) - V(\phi) \tag{2.52}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu, \quad D_\mu(\phi^a) = \partial_\mu\phi^a + g\epsilon^{abc}A^b_\mu\phi^c \tag{2.53}$$

with $a, b, c = 1, 2, 3$.
The potential is the typical Higgs potential:

$$V(\phi) = \lambda(\phi^a\phi^a - \nu^2)^2/4 \tag{2.54}$$

Calculating the equations of motion associated with the A^a_ν and ϕ^a fields, one finds

$$D_\nu F^{a\mu\nu} = e\epsilon^{abc}\phi^b D^\nu\phi^c \tag{2.55}$$
$$(D^\mu D_\mu)\phi^a = -\lambda\phi^a(\phi^b\phi^b - \nu^2) \tag{2.56}$$

The energy of the system, for the static case, is the following:

$$E = \int d^3x \left(\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_i\phi^a)(D_i\phi^a) + \frac{1}{4}\lambda(\phi^a\phi^a - \nu^2)^2\right) \tag{2.57}$$

with

$$E^{ai} = -F^{a0i}, \quad B^{ai} = -\frac{1}{2}\epsilon^{ijk}F^a_{jk} \tag{2.58}$$

where E^{ai} and B^{ai} are the electric and magnetic fields, respectively. It is clear that $E \geq 0$. In order to have equality we should have $D^\mu(\phi^a) = V(\phi^a) = A^a_i = 0$ and $\phi^a\phi^a = \nu^2$.

If we want to look for finite energy configurations the fields have to satisfy the following constraints: $r^{\frac{3}{2}}D_i\phi \to 0$, $|\phi|^2 \to \nu^2$ as $r \to \infty$. The first constraint is just related to the fact that we expect our derivative of the field to fall faster than $D_i\phi^a \propto \frac{1}{r^{3/2}}$ otherwise this term in the energy functional diverges. The last constraint is straightforward but we still stress that the condition is only upon the absolute value of the triplet. The ϕ^a field is free to point any direction in the internal space.
We can define the vacuum of the theory as:

$$M_H = \left\{\vec{\phi} : V(\vec{\phi}) = 0\right\} \tag{2.59}$$

In this case the space is defined by the two-sphere $\sum_a \phi^a\phi^a = \nu^2$.

Finite energy configuration does not need to lie in the Higgs field for any value of $\phi^a(x)$. Actually only for $\vec{\phi}(r \to \infty)$ at infinity our triplet field should be degenerate with all possible configurations of $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ lying on a sphere and with absolute value $|\phi| = \nu$. Then we can say that there is a map between the vacuum $M_H = S^2_\infty$ and S^2_{phy}, meaning:

$$\vec{\phi} : S^2_\infty \to S^2_{phy} \tag{2.60}$$

S^2_{phys} is the physical space, the space defined by the coordinates $x^\mu = (ct, x^1, x^2, x^3)$. This mapping is characterized by an integer that counts the winding of one S^2_∞ around the S^2_{phys} and it is exactly the topological charge[8]. This integers label mathematical objects called homotopy sectors, as cited before. For our studied model the relevant homotopy group is $\pi_2(S^2) = \mathbb{Z}$ (since we are dealing with maps of the sphere on itself). This result $\pi_2(S^2) = \mathbb{Z}$ means that we have as many homotopy sectors as integers when mapping $S^2 \to S^2$.

If one tries to deform one map from the sector $Q = 1$ into a map from the sector $Q = 2$ will realize that it is impossible.

The topological charge is defined as the integral of the zero-th component of the following topological current [44]

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_{abc} \partial^\nu \hat{\phi}^a \partial^\rho \hat{\phi}^b \partial^\sigma \hat{\phi}^c \tag{2.61}$$

with

$$\hat{\phi}^a = \frac{\phi^a}{|\phi|}, \quad \text{where} \quad |\phi| = \sqrt{\sum_a \phi^a \phi^a} \tag{2.62}$$

Thus, the topological charge should be calculated as follows

$$Q = \int k_0 d^3x \tag{2.63}$$

But how is this related to monopoles? Let us take a look at the Maxwell's equations written in terms of the field strengh tensor

$$\partial_\nu F^{\mu\nu} = 4\pi j^\nu, \quad \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0 \tag{2.64}$$

With the assumptions made in (2.58) we can see that the last equation results in:

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.65}$$

[8] A mapping with $Q = 1$ can be pictured as a piece of paper wrapping a sphere once. A $Q = 2$ mapping can be pictured as a bag wrapping a sphere once, but when it closes in the top one can twist this bag and turn it inside out in order to redo the same procedure. In the end the bag had wrapped the same sphere twice.

As happens with the equation $\nabla \cdot \vec{E} = \rho/\epsilon_0$ when integrated it results in the total electric charge of the system. The same does not happen with $\nabla \cdot \vec{B} = 0$, when integrated it results in 0. That is the reason we say there is no magnetic charge in the universe. Based on the Dirac's reasoning presented in the very first discussion of this topic we assume that if there were magnetic charges (let us call it m) one should expect that $\int \nabla \cdot \vec{B} d^3x = m$, now with $\nabla \cdot \vec{B} \neq 0$ being equivalent to some sort of density of magnetic charges.

We will show that conventional electromagnetism is embedded in the model we are working with and we will derive a similar equation to (2.65) regarding the magnetic field. In order to achieve this 't Hooft proposed a gauge invariant version of the electromagnetic field tensor in this theory, given by

$$F_{\mu\nu} = \hat{\phi}^a F^a_{\mu\nu} - \frac{1}{g}\epsilon^{abc}\hat{\phi}^a (D_\mu \hat{\phi}^b)(D_\nu \hat{\phi}^c) \tag{2.66}$$

where in the case of $\phi^a = (0, 0, 1)$ we get:

$$F^{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu \tag{2.67}$$

This way we regain our conventional electromagnetic theory. The reason why 't Hooft have proposed such field strenght is the fact that, calculating the derivative of its dual field $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ we obtain exactly the topological charge defined above. That happens in the case where we chose the field $\phi^a = (0, 0, 1)$ recovering the Maxwell's equations, now supporting magnetic monopoles!
We start with

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2g}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu \hat{\phi}^a \partial^\rho \hat{\phi}^b \partial^\sigma \hat{\phi}^c \tag{2.68}$$

Using (2.58) we obtain:

$$\frac{1}{2}\epsilon_{0\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2g}\epsilon_{0\nu\rho\sigma}\epsilon_{abc}\partial^0 \hat{\phi}^a \partial^\rho \hat{\phi}^b \partial^\sigma \hat{\phi}^c \tag{2.69}$$

$$\frac{1}{2}(2\partial^1 F^{23} + 2\partial^2 F^{31} + 2\partial^3 F^{12}) = \frac{4\pi}{g}k_0 \tag{2.70}$$

$$\nabla \cdot \vec{B} = \frac{4\pi}{g}k_0 \tag{2.71}$$

Integrating both sides, gives

$$\int \nabla \cdot \vec{B} d^3x = \int \frac{4\pi}{g}k_0 d^3x = Q/g = m \tag{2.72}$$

where Q is the topological charge.
't Hooft and Polyakov have proposed an ansatz with $Q = 1$ in order to obtain an explicit solution for the equations of motion

$$\phi^a = \delta_{ia}(x^i/r)F(r), \quad A^a_i(x) = \epsilon_{aij}(x^j/r)W(r), \quad A^a_0 = 0 \tag{2.73}$$

we will be looking for static solutions, where $r \equiv |x|$. 't Hooft and Polyakov ansatz is subjected to the following boundary conditions:

$$F(r) \to \nu, \quad W(r) \to \frac{1}{gr} \tag{2.74}$$

as $r \to \infty$, yielding to the following equations of motion

$$r^2 \frac{d^2K(r)}{dr^2} = K(r)(K^2(r) - 1) + H^2(r)K(r), \quad r^2 \frac{d^2H(r)}{dr^2} = 2H(r)K^2(r) \tag{2.75}$$

where

$$H(r) = grF(r), \quad K(r) = 1 - grW(r) \tag{2.76}$$

The numerical solution of these equations are discussed in [36].

2.2.2 Bogomoln'yi bound, Prasad and Sommerfield solutions

From the ansatz we presented above we find differential equations for $W(r)$ and $F(r)$ that are not analytically solvable. However Bogomoln'yi, Prasad and Sommerfield (with the requirement that $\lambda \to 0$, with ν and g fixed) have found very simple solutions
They have found:

$$K(r) = \frac{rg\nu}{\sinh(g\nu r)}, \quad H(r) = \frac{rg\nu}{\tanh(g\nu r) - 1} \tag{2.77}$$

Besides $\lambda \to 0$ condition, it is important to keep in mind that the authors have used the same boundary conditions considered by 't Hooft and Polyakov. Moreover, Bogomoln'yi also found other interesting properties in this model in the case where there is no Higgs potential ($\lambda \to 0$). The discussions proposed by Bogomoln'yi includes solutions of the equation of motion with any topological charge.
Let us take a look at the energy functional of the model:

$$E = \int d^3x \left(\frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2}(D_i\phi^a)(D_i\phi^a) \right) \tag{2.78}$$

Remember that in the beginning in the one dimensional case we have found a lower bound for the energy (2.11) and obtained a condition for the energy to be finite in an approach that did not depend on (2.7). Here a very similar strategy is considered:
Let us rewrite the energy functional as:

$$E = \int d^3x \left(\frac{1}{4}(F^a_{ij} - \epsilon_{ijk} D_k\phi^a)^2 + \frac{1}{2}\epsilon_{ijk}(D_k\phi^a)F^a_{ij} \right) \tag{2.79}$$

with

$$E \geq \int d^3x \left(\frac{1}{2}\epsilon_{ijk}(D_k\phi^a)F^a_{ij} \right) = E_{BPS} \quad (2.80)$$

To find the lowest energy solution, E_{BPS}, we have to assume that the first term in (2.79) must vanish resulting in the following equation[9]

$$F^a_{ij} = \epsilon_{ijk} D_k \phi^a \quad (2.81)$$

But for this to work, the second term should be positive and always present no matter which ϕ^a we are considering. We can calculate this term as

$$\int d^3x \frac{1}{2}\epsilon_{ijk}F^a_{ij}D_k\phi^a = \int d^3x \partial_k(\frac{1}{2}\epsilon_{ijk}F^a_{ij}\phi^a) = \oint_{S_2} d\sigma_k(\frac{1}{2}\epsilon_{kij}F^a_{ij}\phi^a) \quad (2.82)$$

Using

$$D_\mu \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma} = 0 \quad (2.83)$$

A small comment on how to obtain the first equality is that, choosing $\nu = 0$ in (2.83), we will get $\frac{1}{2}D_k\epsilon^{ikj}F^a_{ij} = 0$ to realize that:

$$\int d^3x \frac{1}{2}\epsilon_{ijk}F^a_{ij}D_k\phi^a = \int d^3x \frac{1}{2}\epsilon_{ijk}D_k(F^a_{ij}\phi^a) \quad (2.84)$$

and comparing

$$\epsilon^{ikj}D_k(F^a_{ij}\phi^a) = \epsilon^{ikj}\partial_k(F^a_{ij}\phi^a) + \epsilon^{ikj}\epsilon^{abc}A^b_kF^c_{ij}\phi^c \quad (2.85)$$
$$\epsilon^{ikj}F^a_{ij}D_k(\phi^a) = \epsilon^{ikj}F^a_{ij}\partial_k(\phi^a) + F^a_{ij}\epsilon^{ikj}\epsilon^{abc}A^b_k\phi^c \quad (2.86)$$

We note that, since F^a_{ij} and ϕ^a are contracting in $\epsilon^{ikj}F^a_{ij}\partial_k(\phi^a)$, we can choose the dummy indices as $a \to c$, that is, we conclude that the last term in each equation is the same. With this we prove the first equality in (2.81).
If we also consider the the gauge invariant electromagnetic field tensor defined earlier, with the boundary conditions:

$$B_k = \frac{1}{2}\epsilon_{kij}F_{ij} \to \left(\frac{1}{2\nu}\right)\epsilon_{kij}F^a_{ij}\phi^a, \quad \nu\oint_{S_2} d\sigma_k B_k = 4\pi m\nu = 4\pi\frac{Q}{g}\nu \quad (2.87)$$

then $4\pi\frac{Q}{g}\nu$ is the mass of our monopole. Then we can rewrite the total energy of our

[9] Understanding the approach we have used to derive (2.11) and counting the indices of the terms in the first parenthesis one can agree with (2.79).

system as:

$$E = \frac{4\pi Q\nu}{g} + \int d^3x \sum_{i,j,a} \frac{1}{4} \left(F^a_{i,j} - \epsilon_{ijk} D_k \phi^a \right)^2 \tag{2.88}$$

Where of course we can see that this energy is minimised when we have (2.81).

2.2.3 The connection between the Dirac and 't Hooft Polyakov Monopoles

The first monopole solution was found by Dirac in the Maxwell's equations themselves. For a long time its quantum version was an interesting explanation for the quantization of the electric charge. Besides Dirac approach we saw that monopole solutions also arise in Yang-Mills theory, particularly in the Georgi-Glashow model. The interesting fact is that, even though these two solutions were found in different scenarios they share some similarities. First it starts with the fact that the classical Dirac monopole has a point-like structure, while in the 't Hooft-Polyakov case it has an internal structure. By studying the numerical solutions found by Prasad and Sommerfeld, one can see that the monopole field reach its assymptotic value rather fast, which means that outside a characteristic distance the Higgs field reaches its vacua. From this solutions one can actually estimate the size of monopole's core R_c. In this sense it is possible to show that at large distances ($r >> R_c$) the classical Dirac monopole is a good approximation to the 't Hooft Polyakov monopole because in this regime the SU(2) symmetry is broken into U(1) what is conventional electromagnetism [36]. In the cited reference and therein one can find also discussions regarding topological similarities between the two models and how the homotopy sectors defined by the mappings $S^2_\infty \to S^2_{phys}$ are related to the gauge field invariance on the the Dirac's monopole case.

2.3 Instantons

Historically instanton solutions were first obtained by Polyakov *et.al* [45] when investigating Yang-Mills theory looking for non-trivial topological solutions motivated by the discovery of the 't Hooft Polyakov monopole. As solitons, instantons are topological solutions connecting two distinct vacua. Based on this we introduce the subject in a less conventional way and we also study for the case of (1+1) dimensions. We start revisiting some potentials in quantum mechanics, now from a different point of view, and try to understand how instantons arise when tunneling is a possibility in these potentials. After that we relate periodic potentials in quantum mechanics to the QCD vacua.
Instantons are solitons that arise in the Euclidian version of theories we studied before. That is, $t \to it$

$$d^2s = c^2d^2t - d^2x \to c^2dt^2 + d^2x \tag{2.89}$$

we will adopt $c = 1$.
We start our approach with the following Hamiltonian system

$$H = \frac{p^2}{2} + V(x) \tag{2.90}$$

Let us suppose $V(x)$ is a symmetric potential ($V(x) = V(-x)$).

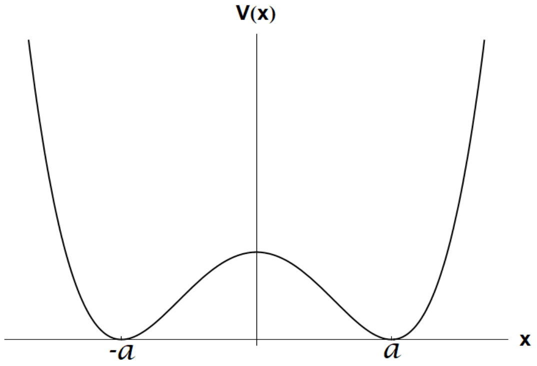

Figure 2.6. Symmetric potential

In our treatment $x(t)$ is a field depending only on time. Since we will focus on finite energy solutions, it means $\varepsilon \to 0$ as $x \to \infty$, ε is the energy density, it is easy to obtain[10]

$$\frac{dx}{dt} = \sqrt{2V} \tag{2.91}$$

we can note that the action of the proposed Hamiltonian is

$$\begin{aligned} S &= \int_{t_0}^{t} \left(\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + V(x) \right) dt \\ &= \int_{-a}^{a} dx \sqrt{2V} \end{aligned} \tag{2.92}$$

Another thing to note is that in a case where we are very close to $x = a$ (large t) we can expand $V(x)$ around a and use (2.91) to obtain

$$\frac{dx}{dt} \approx \omega(a - x) \tag{2.93}$$

where $\omega^2 = \frac{\partial^2 V}{\partial x^2}|_{x=a}$. Here we have used that $V(a) = \frac{\partial V}{\partial x}|_{x=a} = 0$. If we solve for x we see that $x \propto e^{-\omega T}$ which tell us that the characteristic size of the instanton is $\frac{1}{\omega}$, then our conclusion is that our instanton solution is localized. Note that in our consideration we have chose the minus sign in $\pm\sqrt{2V}$, otherwise we would get a wrong approximation.
So far, what was presented concerns the classical version of this system. An interesting discussion can be made if we quantize this model around a and $-a$.
We will be interested in using an Euclidean version [46] of the path integral formalism to calculate approximately the energy spectrum of the system, we will rely on the following relation:

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N \int \mathcal{D}[x] \, e^{-S/\hbar} \tag{2.94}$$

[10]Note that since we are working with $t \to it$ a minus sign contribution arises from the time derivatives in the Hamilton-Jacobi equations of motion.

after inserting the completness relation for the Hamiltonian eigenstates one obtain

$$\langle x_f|e^{-HT/\hbar}|x_i\rangle = \sum_n e^{-E_nT/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle \tag{2.95}$$

We can see that, for large T, an approximation to the first order will give us the energy of the lowest lying wave function. In the right hand side this is the same as calculating the Stationary Phase Approximation (SPA) [11] of our functional integral. For doing so we should expand the action around the classical solution of the equation of motion, in this case our instanton.
We expand the action as

$$S = S(x_{cl})+\int \frac{\delta S}{\delta x(t_1)}\delta x(t_1)dt_1+\frac{1}{2}\int dt_1\int dt_2\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\delta x(t_1)\delta x(t_2)dt_1dt_2+\ldots \tag{2.96}$$

to obtain

$$\mathcal{N}\int \mathcal{D}[\delta x]e^{-S} = \mathcal{N}e^{-S_0}\left(Det\left[-\frac{1}{2}\frac{\delta^2 S}{\delta x^2}\right]\right)^{-\frac{1}{2}} \tag{2.97}$$

note that in our expansion the term with first variation in the action vanishes since it is our equation of motion, but the term of zero-th order is present $S_0 = S(x_{cl})$. After quantizing the system around each vacuum we can calculate the following quantities.

$$\langle -a|e^{-HT/\hbar}|a\rangle\,,\quad \langle a|e^{-HT/\hbar}|a\rangle \tag{2.98}$$

where $e^{-HT/\hbar}$ is the time evolution operator. These terms are exactly the tunneling amplitudes. But rather than calculating this directly with (2.97) we will present a more simple approach. For the harmonic oscillator (single well) problem the result is known to be [46]

$$\langle a|e^{-HT/\hbar}|a\rangle_{HO} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}e^{-\omega T/2} \tag{2.99}$$

for the double well, just for convenience we will assume that the result is such that

$$\langle -a|e^{-HT/\hbar}|a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}e^{-\omega T/2}K \tag{2.100}$$

At the end of the section we will present considerations on how to calculate this tricky K term. This is only true if we are considering T smaller than the lifetime of the tunneling process, for T larger than that we should take into account multi-instanton configurations connecting the vacua. That is due to sucessive tunneling processes.

We should also pay attention to another delicate feature of our system. We know that in the stationary phase approximation we compute a gaussian integral that is related to the determinant of the operator which is the second functional derivative of the action.

[11] The formalism of path integral will be briefly introduced in the next chapter

If we have a zero mode among the eigenvalues of this operator it means we are in trouble since the determinant appears in the denominator and is equal to the product of all eigenvalues. This result leads to a divergence[12] that we should contour. Generally zero modes arise in systems that possess translational symmetry, that is, the solution of our equation of motion is free to be centered at any point in space. This is exactly the kind of problem we have here and with which we should be careful. If one calculate the eigenfunctions of the operator in (2.97) one should write a general solution to this differential operator as

$$x = \sum_n c_n x_n \tag{2.101}$$

where x_0 is the zero mode with eigenvalue $\epsilon_0 = 0$ and amplitude c_0.
Since this eigenfunction has translational symmetry if we perform an infinitesimal shift on its center $t_0 \to t_0 + dt_0$ this must be proportional to an infinitesimal change in the amplitude c_0, that is $dc_0 \propto dt_0$.
And the contribution to the functional integral of this zero mode should be computed as

$$\int_{-T/2}^{T/2} \frac{e^{-\frac{dc_0 \epsilon_0 c_0^2}{2}}}{\sqrt{2\pi}} = \int_{-T/2}^{T/2} \frac{dc_0}{\sqrt{2\pi}} \tag{2.102}$$

This is exactly the integral present in 2.94, but note that since $\epsilon_0 = 0$ the c_0 term does not contribute as a gaussian integral, and this is the divergent term. We should then factor out this integral in t_0 variable in order to have K well defined. We will see shortly how to deal with this integral

$$\int D[\delta x] e^{-S} = e^{-S_0} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} K \int_{-T/2}^{T/2} dt_0 \tag{2.103}$$

So our K term is now, such that we have factored out the contribution from the zero mode from the path integral. The proportionality constant relating dc_0 and dt_0 is included in K.

Then, as we said, for large T we would expect that not only one instanton would interpolate the two vacua, but a string of instanton and anti-instanton consecutive pairs where the distance of instantons-anti-instanton pair is way larger than the characteristic size of the single solutions $\frac{1}{\omega}$ that is, $\frac{1}{\omega} << |t_i - t_j|$. This scenario can be pictured as:

We can think in this as a string of instantons interpolating a and $-a$ and we can calculate the transition amplitude of a tunneling process between these two vacua. Based on what we presented before we can agree that this result should be [13] [46]

$$\langle \pm a | e^{-HT/\hbar} | a \rangle = e^{\frac{-S_0 n}{\hbar}} \sqrt{\frac{\omega}{\pi\hbar}} e^{-\frac{\omega T}{2}} K^n \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{-\frac{T}{2}}^{t_1} dt_2 \dots \int_{-\frac{T}{2}}^{t_{n-1}} dt_n \tag{2.104}$$

Note that in this approximation the action of the multi-instanton solution can be approximated as $S \approx S_0 n$ since they are widely separated. Considering the instantons to be distributed each centered at a discrete $t_n - t_{n-1}$ interval, the zero mode contributions of

[12] In our case we integrate with respect to the interval $[-\frac{T}{2}, \frac{T}{2}]$, but as large as our T the better is our approximation and in the limit $T \to \infty$ this divergence can't be ignored

[13] Chapter 7 of "Aspects of Symmetry. "Check also the lecture notes of Prof. Kazama "Instantons in quantum mechanics"

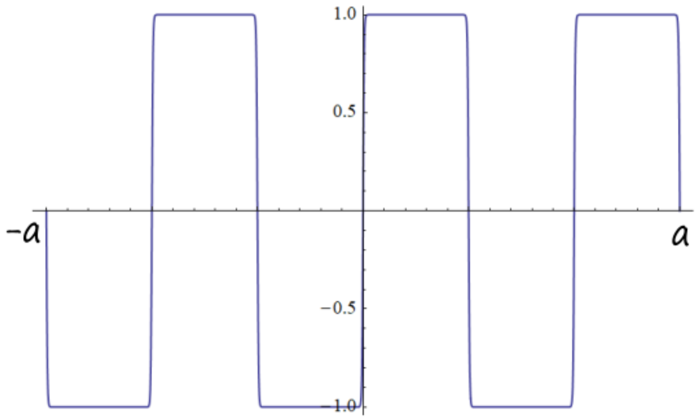

Figure 2.7. Multi-instanton solution connecting the two vacua. One first instanton with a vacuum at $x(-\frac{T}{2})$ is connected to an anti-instanton which is connected to another instanton and this process is repeated untill the last solution is connected to $x(\frac{T}{2})$. This is a numerical representation of the Multi-instanton solution with a large separation between instantons, this means that the characteristic instanton "size" is too small compared to the interval T.

each instanton to the path integral results in

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{-\frac{T}{2}}^{t_1} dt_2 \ldots \int_{-\frac{T}{2}}^{t_{n-1}} dt_n = \frac{T^n}{n!} \tag{2.105}$$

where $-\frac{T}{2} < t_1 \ldots < t_{n-1} < \frac{T}{2}$. Fig. 2.8 is a simple example in order to understand the result in (2.105).

If we sum the contribution of every possible n instantons to our potential we obtain:

$$\langle -a| e^{-HT/\hbar} |a\rangle = \sqrt{\frac{\omega}{\hbar\pi}} e^{-\frac{\omega T}{2}} \sum_{odd \quad n} \frac{(Ke^{-\frac{S_0}{\hbar}}T)^n}{n!}[1 + O(\hbar)] \tag{2.106}$$

This sum includes all contributions of strings of odd number of instantons connecting a and $-a$. It supports only odd number of instantons since we must connect different minima. These solutions corresponds to back and forth tunneling processes between minima of the potential [46]. If we want to compute contributions of multi-instanton solutions connecting the same minimum, for example $-a$ and $-a$, we must have an even number of instantons.

$$\langle -a| e^{-HT/\hbar} |-a\rangle = \sqrt{\frac{\omega}{\hbar\pi}} e^{-\frac{\omega T}{2}} \sum_{even \quad n} \frac{(Ke^{-\frac{S_0}{\hbar}}T)^n}{n!}[1 + O(\hbar)] \tag{2.107}$$

Comparing this with the previous similar equation we see that our result for the lowest energy level approximation is:

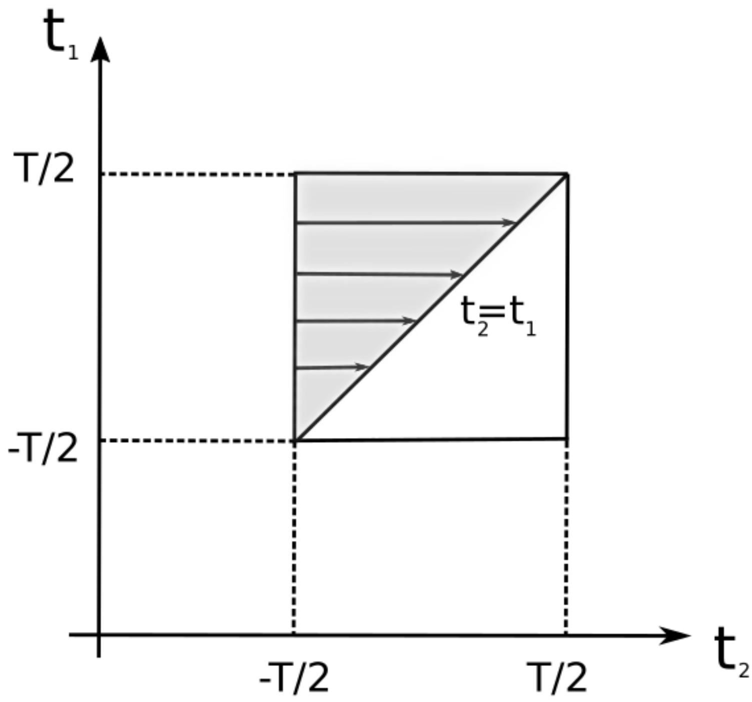

Figure 2.8. We can imagine a $n = 2$ example in order to understand how this result is possible. We integrate $\int_{-T/2}^{T/2}\int_{-T/2}^{t_1} dt_2 dt_1$, by shadowing the region of integration is easy to see that $\frac{T^2}{2}$. The result stated above is just a generalization of this example.

$$E_{\pm} = \frac{\hbar\omega}{2} \pm \hbar K e^{-\frac{S_0}{\hbar}} \tag{2.108}$$

We can make the same investigation regarding periodic potentials. In that case our instantons can be summed in any way we want it does not need to be like the previous case where we alternated between instantons and anti instantons, in this context

$$\langle l|e^{-HT}|m\rangle = \sum_{N_+}\sum_{N_-} \frac{\delta_{N_+-N_--(m-l)}(KTe^{-S_0})^{N_++N_-}}{N_+!N_-!} \tag{2.109}$$

we can use the trick of rewriting a Kronecker delta in the following form:

$$\delta_{ab} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(a-b)} \tag{2.110}$$

and also conveniently insert unity ($1 = e^{i\theta}e^{-i\theta}$) in the expression

$$\begin{aligned} \langle l|e^{-HT}|m\rangle &= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(m-l)} \left(\sum_{N_+=0}^{\infty} \frac{\left(e^{-S_0}KTe^{-i\theta}\right)^{N_+}}{N_+!} \right) \left(\sum_{N_-=0}^{\infty} \frac{\left(e^{-S_0}KTe^{i\theta}\right)^{N_-}}{N_-!} \right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(m-l)} \exp[2KT\cos(\theta)\exp(-S_0)] \end{aligned} \tag{2.111}$$

resulting in the following formula for the spectrum

$$E(\theta) = \frac{1}{2}\hbar\omega - 2KT\cos\theta e^{-S_0/\hbar} \tag{2.112}$$

The system possess negative ground state energy, as it should be, tunneling is the favorable choice for the system rather than staying in the $E = \frac{\hbar\omega}{2}$ state. The interesting fact is that this same derivation holds in the context of QCD vacua, although it is not widely accepted. The tunneling processes happens between asymptotic values of the gluonic field (such field describes particles in QCD called gluons), leading to instantons solutions. In this context, the θ parameter can be measured and nowadays the most strong constraint in this term is $\theta < 10^{-9}$ [47], that is, pragmatically $\theta = 0$. This implies that CP symmetry is broken [48]. This CP symmetry is just the combination of charge conjugation symmetry and parity symmetry, it changes θ to $-\theta$. So far no one knows any physical process that justify why this value of θ is so small, this is called the Strong CP problem.
As we promised we should discuss briefly how to compute K. It arise from

$$\mathcal{N}\left(Det\left[-\frac{1}{2}\frac{\delta^2 S}{\delta x^2}\right]\right)^{-\frac{1}{2}} \tag{2.113}$$

Computing this term will lead to

$$\mathcal{N}\left(Det\left[-\frac{1}{2}\frac{\delta^2 S}{\delta x^2}\right]\right)^{-\frac{1}{2}} = \left(Det\left[-\partial_t^2 + V''(x_{cl})\right]\right)^{-\frac{1}{2}} \tag{2.114}$$

A common procedure to calculate this determinant is to rewrite it as

$$\begin{aligned}\left(Det\left[-\partial_t^2 + V''(x_{cl})\right]\right)^{-\frac{1}{2}} &= (S_0/2\pi\hbar)^{\frac{1}{2}}\left[Det(-\partial_t^2+\omega^2)\right]^{-\frac{1}{2}} \\ &\times \left|\frac{Det(-\partial_t^2+\omega^2)}{Det'(-\partial_t^2+V''(x_{cl})}\right|^{\frac{1}{2}}\end{aligned} \tag{2.115}$$

where the $(S_0/2\pi\hbar)^{\frac{1}{2}}$ factor appearing in the expression is the proportionality term between dc_0 and dt_0 discussed when regarding the zero mode present in (2.113). And $Det^{'}$ means that we are not considering the zero mode when computing the determinant. It happens that

$$\mathcal{N}\left[Det(-\partial_t^2+\omega^2)\right] = \left(\frac{\omega}{\pi\hbar}\right)e^{-\omega T/2} \tag{2.116}$$

then we conclude that

$$K = (S_0/2\pi\hbar)^{\frac{1}{2}}\left|\frac{Det(-\partial_t^2+\omega^2)}{Det'(-\partial_t^2+V''(x_{cl}))}\right|^{\frac{1}{2}} \tag{2.117}$$

This procedure is important because the determinant in the denominator possess a continuum spectrum of eigenvalues that need to be normalized. In the end the resulting ration can be computed and it is finite. In the references cited in this section one can find detailed derivation of this ratio.

2.4 Vortices

They were first discovered in superconductivity with the work of Abrikosov studying type two superconductors, [49]. After that theirs relativistic counter part was introduced by [50] in the context of high energy physics. They were first motivated by the work of Veneziano [51] that showed that string like solutions are also likely to model hadronic matter. They tried to generalize the Abrikosov vortex found in superconductivity to the relativistic case, coupling the gauge field of the theory to a complex scalar field, via $U(1)$ group. They showed that there is a link between the approximated solutions of the proposed Lagrangian and the Nambu dual string. This model is known as Maxwell-Higgs model[14].

2.4.1 Maxwell-Higgs Model

The Lagrangian in $3+1$ dimensions we will be working with is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi(D_\mu\phi)^* - U(\phi,\phi^*) \tag{2.118}$$

where $F_{\mu\nu}$ is the faraday tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.119}$$

And D_μ is the covariant derivative defined in the following way:

$$D_\mu\phi = \partial_\mu\phi + iA_\mu\phi \tag{2.120}$$

The potential term in the Lagrangian is of Higgs type given by the following expression:

$$U(\phi,\phi^*) = \frac{\lambda}{4}(|\phi|^2 - \nu^2)^2 \tag{2.121}$$

We can use the Euler Lagrange equation for the fields ϕ, ϕ^* and A_μ to get respectively the following equations of motion:

$$\begin{aligned} D_\mu D^\mu\phi + \frac{\partial U}{\partial\phi^*} &= 0 \\ D_\mu D^\mu\phi^* + \frac{\partial U}{\partial\phi} &= 0 \\ \partial_\mu F^{\mu\nu} &= J^\nu \end{aligned} \tag{2.122}$$

The conserved current interacting with the $U(1)$ gauge field is given by

$$J^\nu = ie\left[\phi^*(D^\nu\phi) - \phi(D^\nu\phi)^*\right] \tag{2.123}$$

[14]Other models and new perspectives on vortex research can be found in the up to date notes of B.Malomed, "Vortices: Old results new perspectives"

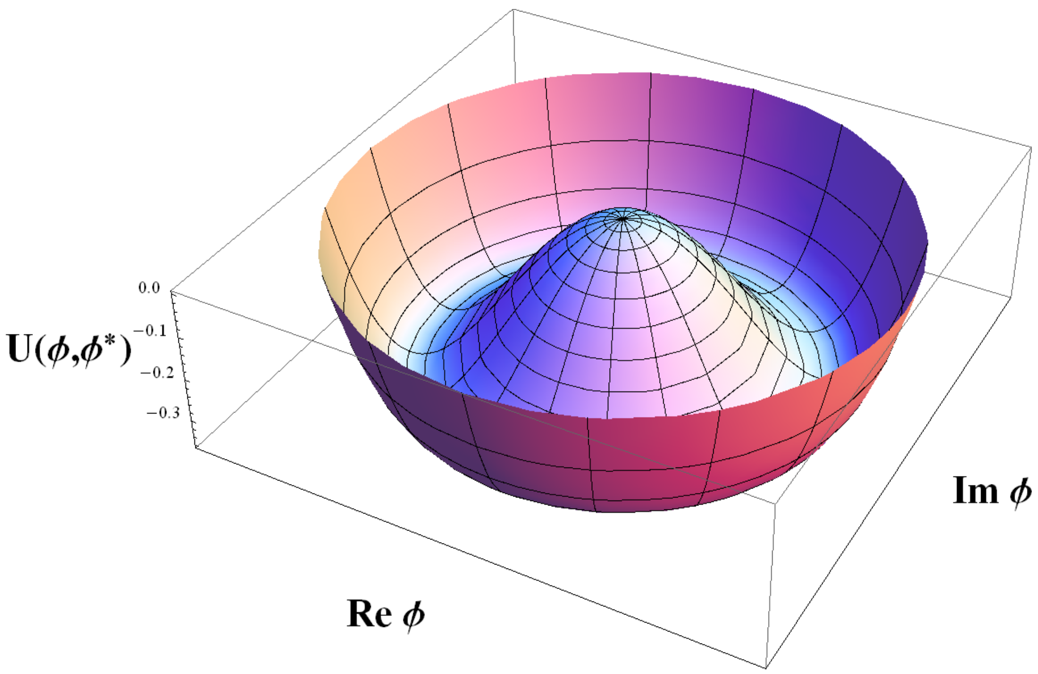

Figure 2.9. $U(\phi, \phi^*)$ potential

« Fulfilling the conservation condition $\partial_\nu J^\nu = 0$.

The Faraday tensor components are $-F^{i0} = E_i$ and $-F^{ij} = \frac{1}{2}\epsilon^{ijk}B_k$. Since vortices are usually static solutions in $2+1$ dimensions we choose only $-F_{12} = B_z \equiv B$ component to survive. We will look for solutions with cylindrical symmetry, similar to the Abrikosov vortex solution, meaning that there is a non-vanishing magnetic field only in the z-direction, B_z.

We plan to search for static solutions, that is $\partial_0 A_\mu = \partial_0 \phi = 0$. For convenience we also redefine our fields as:

$$\begin{aligned} A_\mu &= (A_0, -\vec{A}) = (\varphi, -\vec{A}) \\ \partial_\mu F^{\mu 0} &= \partial_\mu \left(\partial^\mu A^0 - \partial^0 A^\mu\right) \\ &= \nabla^2 \varphi \\ &= -J^0 = ie\left[\phi^*(\partial^0\phi - iA^0\phi) + \phi(\partial^0\phi + iA^0\phi)\right] \end{aligned} \tag{2.124}$$

We get to the following equation for the $A^0 = \varphi$ field :

$$\nabla^2\varphi = -2e^2|\phi|^2\varphi \tag{2.125}$$

We take advantage of the symmetry of the problem and suppose an *Ansatz* for solving the equation in the form

$$\begin{aligned} \vec{A} &= -\frac{1}{er}(a(r) - n)\hat{\theta} = A(r)\hat{\theta} \\ \phi &= \nu g(r) e^{in\theta} \end{aligned} \tag{2.126}$$

with the boundary conditions as: $a(r \to 0) \to n$, $g(r \to 0) \to 0$, $a(r \to \infty) \to 0$, $g(r \to \infty) \to 1$, here of course ν is such that $|\phi(x \to \infty)| \to \nu$. These boundary

conditions guarantee that the vortex fields go to zero at infinity and the solutions carry finite energy.
Consider the equation of motion appeared in (2.122)

$$D_\mu D^\mu \phi + \frac{\partial U}{\partial \phi^*} = 0 \tag{2.127}$$

Now plugging our ansatz also into the equation for the ϕ field, (2.125) we get:

$$-g^{''} - \frac{g^{'}}{r} + \frac{a^2}{r^2} g + \frac{\lambda^2 \nu^2}{2}(g^2 - 1) = 0 \tag{2.128}$$

And also from $\partial_\mu F^{\mu\nu} = +J^\nu$ we obtain

$$\frac{d^2 a}{dr^2} - \frac{1}{r}\frac{da}{dr} = 2e^2\nu^2 g^2 a \tag{2.129}$$

Solving these two equations above we determine the vortex solutions. Usually these equations are not analytically solvable, or at least, any analytical solutions were found. Some approximations based on the aforementioned boundary conditions can be performed to have a guess in the behavior of the solutions, as carefully done by [50], for example.

2.4.2 Chern-Simons Vortices

Another interesting theory regarding vortices, is called Chern-Simons theory. The interesting fact about this theory is that from the very beginning, at the level of the Lagrangian, there is a topological term present, called chern-simons term. The Lagrangian of this theory is written as follows:

$$\mathcal{L}_{CS} = D_\mu \phi (D^\mu \phi)^* + \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - V(|\phi|) \tag{2.130}$$

where κ is called Chern-Simons term, it will contribute to a non-trivial topology of the system. The potential here can be any that supports a symmetry breaking process. In particular it can be $V(|\phi|) = \frac{1}{\kappa^2}|\phi|^2(|\phi|^2 - \nu^2)^2$.
This theory possesses many applications ranging from coupling of dynamical matter fields to gravity models in $2+1$ dimensions. A well known success of this theory and its developments came when people realized it could be used to model and explain the quantum hall effect. This effect happens in devices where we can have a two dimensional gas of electrons subjected to a strong magnetic field. The previous classical hall condutance is now quantized in the form:

$$\sigma_{xy} = \frac{e^2}{h}\nu \tag{2.131}$$

where ν is an integer or fractional number, characterizing the integer quantum hall effect or fractionl quantum hall effect. It is a huge subject to dive into and some related interesting subjects can be found in [52] for example[15].

[15] Also in the well written lectures of Steven M. Girvin on quantum hall effect and Gerard Dune on Chern-Simons theory both lectures prepared to the Les Houches Summer School. Chern-Simons Dynamics and the quantum hall effect A.P Balachandran

3 Solitons interacting with fermions

3.1 Path integral formalism

After this chapter we will have learnt how solitons interact with fermions. In order to get there we need to introduce some tools, the first tool is path integral formalism, with this formalism we will obtain the energy spectrum of the interaction. Not only this, our fermions interacts with quantum solitons and this quantization procedure will be discussed as well. The last but not less important tool we will present is the formalism of Grassman numbers, essential when it comes to the path integral description of such unique entities as fermions.
We start with path integrals method, this subject is generally introduced in the context of quantum mechanics and after that generalized to the context of quantum field theory. It is a formulation first introduced by Dirac [53] and after developed by Feynman [54] to include the probability amplitude of all possible contributions of all possible paths a particle can undergo between two points in space and time.

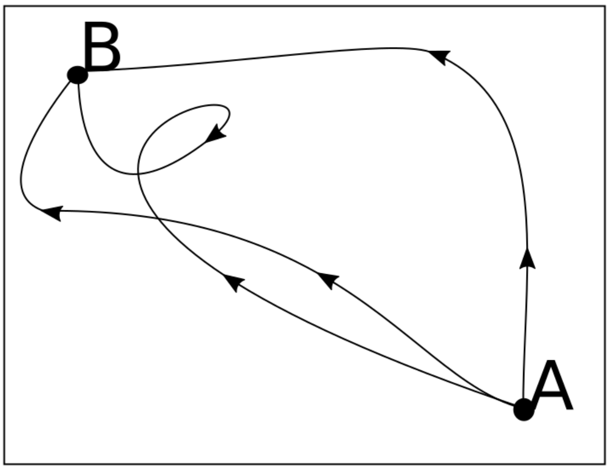

Figure 3.1. Different paths a quantum particle can undergo between two points

We introduce it as:

$$K(x_i, x_f, y_i, y_f) = \int \prod_{N'=1}^{\infty} dx_{N'} \Psi_{N'}(x, y) \tag{3.1}$$

where $K(x_f, y_f, x_i, y_i)$ is the transition probability amplitude of a particle leaving position (x_i, y_i) and reaching position (x_f, y_f).
This expression is obtained from the following thought experiment. One can first imagine the double-slit experiment where an electron in order to leave point A and reach point B

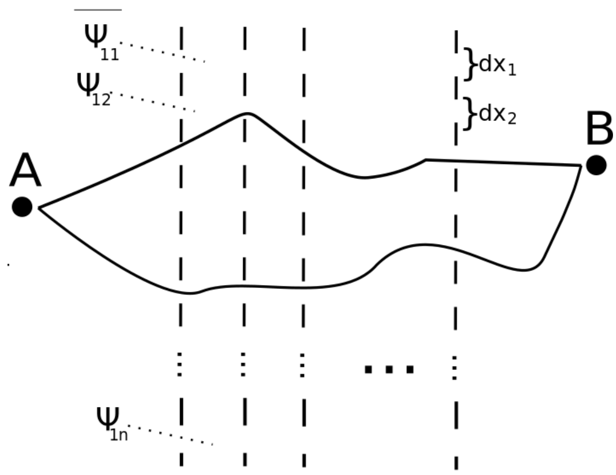

Figure 3.2. This is a diagram of what would happen if we took the first double slit experiment and generalize it to an infinite number of slits and screens.

must pass either through one slit or the other of a given screen, realizing that we can take any number of slits for a single screen and also any number of screens (as it is pictured in Fig. (3.2)) we can make the numbers of screen and slits arbitrarily large. Ψ_{11}, Ψ_{12} and Ψ_{1n} are the probability transition amplitude density through first (second or n-th) slit of the first screen. We take the the limit of screens and slits to infinity and we compute the expression for the final amplitude (above) following probability theory rules. For each screen we should sum all slit amplitude densities since each slit constitutes a possible choice as a path the particle can undertake. For each screen we should take the product of the resulting amplitude explained before since go through each screen is an independent event.
That is

$$\Psi = \lim_{n\to\infty} (\Psi_{11} + \dots \Psi_{1n}) \times \dots (\Psi_{n1} + \dots \Psi_{nn}) \tag{3.2}$$

But this approach will only be valuable if find an specific form of the resulting wave function. We need to be based on some properties the resulting transition amplitude should follow, namely linear property of adding paths and we should be able to recover the classical results when we go to the limit of $\hbar \to 0$
For this we suppose that Ψ should be dependent on the action of the model $S(x, y)$ that is something which is uniquely determined for each path and linear when adding paths. For this reason we generally suppose $\Psi = \sum_\gamma e^{iS(\gamma)/\hbar}$, where γ is a given curve or path "chosen" by the particle. The $\hbar$ factor is introduced to enable us to use the so-called SPA (stationary phase approximation) method when taking the classical limit $\hbar \to 0$. As we considered before we need to sum over all possible paths, which means that for the continuous limit:

$$K = \int \mathcal{D}[q] e^{iS(q,\dot{q})/\hbar} \tag{3.3}$$

actually

$$\langle q_b| exp(-iHT/\hbar)|q_a\rangle = \int \mathcal{D}\,[q]\, exp(\frac{i}{\hbar}S(q,\dot{q})) \tag{3.4}$$

where $|q_a\rangle$ and $|q_b\rangle$ are initial and final states from a quantum system respectively and $e^{-iHT/\hbar}$ is the time evolution operator that dictates the time dynamics of the wave packet. In order to prove this relation one might suppose that, rather than going from $|q_a\rangle$ state directly to $|q_b\rangle$ state, the particle departures from $|q_a\rangle$ and reaches $|q_1\rangle$ state after time $T/N = \epsilon$. Then, it leaves $|q_1\rangle$ towards $|q_2\rangle$ after time $2T/N$. We successively do that in small steps until we reach $|q_b\rangle$, N is the number of steps we suppose to divide our time interval. At the end, we take the limit $N \to \infty$ to obtain what is in the right hand side. With this approach we reach that

$$\int \mathcal{D}[q] = \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} \frac{dx_i}{A(N)} \tag{3.5}$$

which is called the measure of our path integral.
From this result we can proceed as following:

$$\sum_n \langle q_b|\phi_n\rangle \langle\phi_n|q_a\rangle\, exp(-iE_nT/\hbar) = \int \mathcal{D}\,[q]\, exp(\frac{i}{\hbar}S(q,\dot{q})) \tag{3.6}$$

that is, we have inserted the completness relation for the eigenstates of the hamiltonian in the expression in the left hand side. After that, we can take the following clever step:

$$\begin{aligned} \int_{-\infty}^{\infty} dq_o \int \mathcal{D}\,[q]\, exp(\frac{i}{\hbar}S(q,\dot{q})) &= \int_{-\infty}^{\infty} \sum_n \langle q_0|\phi_n\rangle \langle\phi_n|q_0\rangle\, exp(-iE_nT/\hbar) dq_o \\ &= \sum_n exp\,(-iE_nT/\hbar) \\ &= Tr\left(e^{-iHT/\hbar}\right) \end{aligned} \tag{3.7}$$

we consider $|q_a\rangle = |q_b\rangle = |q_0\rangle$ and integrate q_0. We redefine the measure to include the q_0 integration in the symbol $\int \mathcal{D}[q_0]$.
From the regime of quantum theory we will jump to the regime of quantum field theory (we do not derive it here but the result also holds for QFT). Now our Hamiltonian and action both depend on fields and no longer on canonical coordinates:

$$\int \mathcal{D}_{q_0}\,[\phi(q)]\, exp(\frac{i}{\hbar}S[\phi,\partial_\mu\phi]) = Tr\left(e^{-iHT/\hbar}\right) \tag{3.8}$$

This result will be of great value for us in finding the spectrum of systems we will study, in the next sections. We calculate the path integral and try to write the left hand side as similar as possible to the right hand side, in order to match terms and find an expression for the energy.

For this thesis we have chosen a short and concise motivation concerning path integral formalism. Besides that there is a myriad of material about this subject applying this approach in several physical examples and with deep discussions regarding this approach.

3.2 Quantizing soliton fields

3.2.1 Some main concepts

This section is of great importance to understand fundamental features about the topics that will be discussed further. For a pedagogical purpose we start proposing a system and we compare its classical and quantum features in order to understand the quantum behaviour of soliton solutions.
To start let us propose the following equation of motion:

$$\mathrm{d}^2 x/\mathrm{d}t^2 = -\mathrm{d}V/\mathrm{d}x \tag{3.9}$$

where we are choosing $m = 1$. Our potential has the following profile.

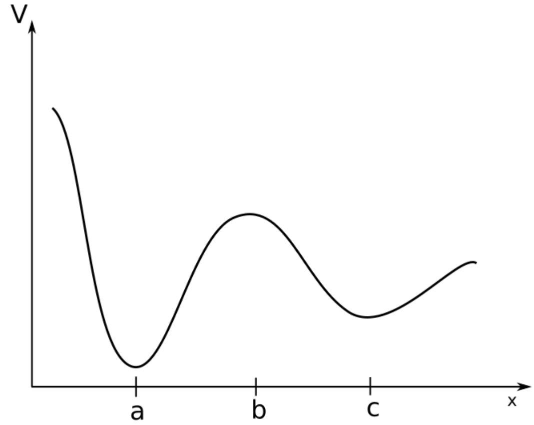

Figure 3.3. Asymmetric potential example

We can see that the static solutions of our system are $x = a$, $x = b$, $x = c$, where $x = a$ and $x = c$ are stable solutions ($\mathrm{d}^2V/\mathrm{d}x^2 > 0$) and $x = b$ is an unstable solution $\mathrm{d}^2V/\mathrm{d}x^2 < 0$. In order to try an approximate solution for $x\,(t)$ we can expand $V\,(x)$ around these extrema. Let us first try to expand it around $x = a$.

$$V(x) = V(a) + \frac{1}{2}\omega^2(x-a)^2 + \frac{1}{3!}\xi_3(x-a)^3 + \frac{1}{4!}\xi_4(x-a)^4 + \dots \tag{3.10}$$

note that $V\,(a)$ is a minimum and $\frac{dV(a)}{dx}$ should naturally vanish. Here we have also defined $\frac{d^2V(x)}{dx^2}|_{x=a} = \omega^2$ and $\frac{d^iV(x)}{dx^i}|_{x=a} = \xi_i$
Our first assumption is that these higher derivatives of the potential are small enough for us to consider the following equation of motion a good approximation to our problem

$$\mathrm{d}^2 x/\mathrm{d}t^2 \approx -\omega^2\,(x-a) \tag{3.11}$$

We know that this is our old good friend, the harmonic oscillator, with the additional detail that it gives oscillating around $x = a$. As one knows, the energy spectrum and the solution to the above equation are given by

$$E_{cl}^{a} = V\left(a\right) + \omega^2 A^2 \tag{3.12}$$

$$x\left(t\right) = A\cos(\omega t + \Delta) \tag{3.13}$$

where A and Δ are to be determined depending on the boundary conditions.
The quantum version of this problem is dictated by the following Hamiltonian:

$$\hat{H}\psi\left(x\right) = E\psi\left(x\right) \tag{3.14}$$

where the eigenfunction $\psi\left(x\right)$ is the representation in the $\left|x\right\rangle$ basis of the hamiltonian eigenstate $\left|n\right\rangle$, that is $\left\langle x|n\right\rangle = \psi\left(x\right)$.

$$\left(\frac{\hat{P}^2}{2m} + V(\hat{X})\right)\left|n\right\rangle = E_n\left|n\right\rangle \tag{3.15}$$

Again we can expand our potential around one of its minima and try to find an expression for the energy spectrum. We are allowed to perform the following Taylor expansion of the operator $V(\hat{X})$ around $x = a$.

$$V(\hat{X}) = \sum_i \frac{f^i}{i!}(x-a)^i \tag{3.16}$$

where $f^i = \frac{d^i V(\hat{X}x)}{dx^i}|_{x=a}$ and $\left\langle n\right| f^i \left|n\right\rangle = \lambda_i$.
We are interested in those satisfying the condition $\lambda_{i>2}\left\langle(x-a)^i\right\rangle << \omega^2\left\langle(x-a)^2\right\rangle$.
$\omega^2 = \left\langle n\right| \frac{d^2V(\hat{X}x)}{dx^2}|_{x=a}\left|n\right\rangle$. This assumption is called "weak coupling approximation" [55]. With this assumption, very similar to the classical case we can already agree that this problem will also resembles the harmonic oscillator but in the quantum regime. And we know already how to find its energy spectrum what leads us to the conclusion that the total energy of our quantum system is:

$$E_n^a = V\left(a\right) + \hbar\omega(n + \frac{1}{2}) + O\left(\lambda_i\right) \tag{3.17}$$

Note: that despite the fact the we have founded a solution around the minima of the potential, $\left\langle x|n\right\rangle = \psi(x)$ is not a static solution, because we cannot tell exactly its momentum and position due to the Heisenberg's uncertanty principle [1]. But we know for sure, that the expectation value of the position operator with respect to the ground state[2] is:

$$\langle 0|\hat{X}|0\rangle = \int \psi_0^*\left(x\right)x\psi_0\left(x\right) \approx a + O\left(\lambda_i\right) \tag{3.18}$$

We can see that the energy of the ground state is given by

$$E_0^a = V\left(a\right) + \frac{\hbar\omega}{2} \tag{3.19}$$

[1] $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$

[2] We can find an expression for the ground state solving the differential equation arising from the condition $\left\langle x\right| a\left|0\right\rangle = 0$.

The same reasoning can be adapted to the discussion regarding the other minimun in $x = c$. And based on the arguments presented above, we can conclude that:

$$E_n^c = V(c) + \hbar\omega\left(n + \frac{1}{2}\right) + O(\lambda_i) \tag{3.20}$$

with its ground state energy given by:

$$E_0^c = V(c) + \frac{\hbar\omega}{2} \tag{3.21}$$

note that although the two cases $x = a$ and $x = c$ are minima of the system, $x = c$ is only a local minimum with an energy E_0^c bigger than E_0^a for this reason we say that E_0^a is the "true vacuum" of the theory.

In this subsection we have discussed some main concepts regarding how we can construct an analogy between a classical and quantum versions of the same system. Now, we should proceed to the delicate task of quantizing static soliton solutions. In the following discussions we will deal with fields in quantum field theory, that naturally possesses infinite degrees of freedom. We cannot interpret the solutions of our equations of motion as quantum particles as we usually do with non-relativistic quantum mechanics. So our interpretations between this quantum-classical analogy changes a little, but we will present additional mathematical tools that will help us to achieve our goal in obtaining a quantum version of a soliton.

3.2.2 An example: Quantizing the kink static solution

Now, let us start with the following Lagrangian for the scalar field $\phi(x,t)$:

$$L = \int \mathrm{d}x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right] \tag{3.22}$$

here λ is the self-coupling constant in the $\lambda\phi^4$ potential, $U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$, and m is proportional to the mass of the soliton present in this model.

Just for convenience we define:

$$V(\phi) \equiv \int \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right] dx \tag{3.23}$$

We are dealing with static solutions, $\frac{\partial\phi}{\partial t} = 0$. Naturally, we use the Euler-Lagrange equation to obtain the equation of motion of the system,

$$\frac{\delta V(\phi)}{\delta\phi(x)} = -\frac{\partial^2\phi}{\partial x^2} - m^2\phi + \lambda\phi^3 = 0 \tag{3.24}$$

This equation has four solutions, two are trivial ones $\phi(x) = \frac{m}{\sqrt{\lambda}} = \phi_1$, $\phi(x) = -\frac{m}{\sqrt{\lambda}} = \phi_2$ and the third and forth ones are non-trivial solutions $\phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right) = \phi_{\pm k}$. The soliton ϕ_k is called kink solution and ϕ_{-k} is called anti-kink solution.

Our potential is a quartic symmetric potential, which means that the two trivial vacua possess the same potential energy $V(\phi_1) = V(\phi_2) = 0$. So, as we did earlier, we pick up one of the vacua and perform our analysis. Due to the symmetry of the potential if instead of choosing ϕ_1 we choose ϕ_2 we would obtain the same result.

Hence, let us choose ϕ_1 and in the same fashion as before, expand $V(\phi)$ around this vacuum. Just as a reminder, $V(\phi)$ in this discussion is now a functional, and we perform a functional Taylor expansion:

$$V(\tilde{\phi}) = V(\phi_1) + \int \left.\frac{\delta V}{\delta\phi(x_1)}\right|_{\phi_1} \tilde{\phi}(x_1)dx_1 + \int\int \frac{1}{2} \left.\frac{\delta^2 V}{\delta\phi(x_1)\phi(x_2)}\right|_{\phi_1} \tilde{\phi}(x_1)\tilde{\phi}(x_2)dx_1dx_2 + \dots \tag{3.25}$$

where $\tilde{\phi}$ is the variation in the field $\tilde{\phi} = \phi(x,t) - \phi_1$, and ϕ_1 is the trivial solution present in the equations of motion. Our terms in the expansion will be:

$$\int \left.\frac{\delta V}{\delta\phi(x_1)}\right|_{\phi_1} \tilde{\phi}(x_1)dx_1 = \int \left[\left.\frac{\partial V}{\partial\phi(x_1)}\right|_{\phi_1} \tilde{\phi}(x_1) + \left.\frac{\partial V}{\partial\partial_{x_1}\phi(x_1)}\right|_{\phi_1} \partial_{x_1}\tilde{\phi}(x_1) \right] dx \tag{3.26}$$

we can use integration by parts and since the surface term vanishes we obtain our well known Euler-Lagrange equation and after evaluating it in the classical solution we obtain zero, as expected expected.
To calculate $\frac{1}{2}\frac{\delta^2 V}{\delta\phi^2}$ we need to know that:

$$\begin{aligned}\frac{1}{2}\int\int \left.\frac{\delta^2 V}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi_1} \tilde{\phi}(x_1)\tilde{\phi}(x_2)dx_1dx_2 = \int\int dx_1dx_2 \left[\left.\frac{\partial^2 V}{\partial\phi(x_1)\partial\phi(x_2)}\right|_{\phi_1} \tilde{\phi}(x_1)\tilde{\phi}(x_2) + \right. \\ \left. 2\left.\frac{\partial^2 V}{\partial\partial_{x_1}\phi(x_1)\partial\phi(x_2)}\right|_{\phi_1} \partial_{x_1}\tilde{\phi}(x_1)\tilde{\phi}(x_2) + \left.\frac{\partial^2 V}{\partial\partial_{x_1}\phi(x_1)\partial\partial_{x_2}\phi(x_2)}\right|_{\phi_1} \partial_{x_1}\tilde{\phi}(x_1)\partial_{x_2}\tilde{\phi}(x_2) \right.\end{aligned} \tag{3.27}$$

performing the calculations and remembering $\frac{\partial}{\partial\phi(x_2)}\phi(x_1) = \delta(x_1 - x_2)$, we can conclude that only first and second terms contribute. As a result we have

$$\frac{1}{2}\int\int \left.\frac{\delta^2 V}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi_1} \phi(\tilde{x}_1)\phi(\tilde{x}_2)dx_1dx_2 = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_1^2\right) \tag{3.28}$$

We know that $V(\phi_1) = 0$, and we know also that the first derivative in the expansion is our equation of motion, what leads to $\frac{\delta V}{\delta\phi}|_{\phi_1}\tilde{\phi} = 0$. With this we can conclude that the contribution for our expansion starts with second order in the derivative. Analogously we can calculate the higher contributions to the expansion. If we remember how we defined (3.23) we can finally obtain:

$$V\left(\tilde{\phi}\right) = \int \left[-\frac{1}{2}\tilde{\phi}\frac{\partial^2\tilde{\phi}}{\partial x} - \frac{1}{2}m^2\tilde{\phi}^2 + \frac{3}{2}\lambda\phi_1^2\tilde{\phi}^2 \right] dx + \lambda\int \left[\phi_1\tilde{\phi}^3 + \frac{1}{4}\tilde{\phi}^4 \right] dx \tag{3.29}$$

Substituing $\phi_1 = \frac{m}{\sqrt{\lambda}}$, we get:

$$V = \int \mathrm{d}x\frac{\tilde{\phi}}{2}\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\tilde{\phi} + m\sqrt{\lambda}\int \tilde{\phi}^3\mathrm{d}x + \frac{\lambda}{4}\int \tilde{\phi}^4\mathrm{d}x \tag{3.30}$$

again, if we consider that our coupling constant is small (weak coupling approximation) the approximated expression is given by the first term.

Now let us put together everything we have. We have expanded V around a classical solution. As a result, the action now takes the following form:

$$S = \int Ldt = \int\int \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\tilde{\phi}(x)\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\tilde{\phi}(x)\right] dtdx - \int V(\phi_1)dt \quad (3.31)$$

Since our expansion of V is around the static solution, that is $\tilde{\phi}(x) = \phi - \phi_1$, we have $\partial_t\tilde{\phi} = \partial_t\phi(x)$. By using integration by parts, we obtain

$$S = \int Ldt = \frac{1}{2}\int\int \left[-\tilde{\phi}(x)\partial_t^2\tilde{\phi}(x) - \tilde{\phi}(x)\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\tilde{\phi}(x)\right] dtdx - \int V(\phi_1)dt \quad (3.32)$$

which can be written in the following form

$$S = \int Ldt = \frac{1}{2}\int\int \tilde{\phi}(x)\left[-\frac{\partial^2}{\partial t^2} - \left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\right]\tilde{\phi}(x)dtdx - \int V(\phi_1)dt \quad (3.33)$$

separating the classical part contribution of the integral, we have

$$Tr\left(e^{-iHT/\hbar}\right) = e^{\frac{i}{\hbar}V(\phi_1)} \int \mathcal{D}[\tilde{\phi}]exp\left[-\frac{i}{2\hbar}\delta\phi(x_1)\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\delta\phi(x_1)\right] \quad (3.34)$$

And the result of the last integral is:

$$Tr\left(e^{-iHT/\hbar}\right) = const \times e^{\frac{i}{\hbar}V(\phi_1)}\left[Det\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - 2m^2\right)\right]^{-\frac{1}{2}} \quad (3.35)$$

In order to calculate this determinant we assume our variation has the form, $\tilde{\phi}(x,t) = \eta_n(t)e^{ik_n x}$, which gives

$$\tilde{\phi(x)}\left[-\frac{\partial^2}{\partial t^2} - \left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\right]\tilde{\phi(x)} = \eta_n(t)\left[-\frac{\partial^2}{\partial t^2} - \omega_n^2\right]\eta_n(t) \quad (3.36)$$

where ω_n are the eigenvalues of the following operator:

$$\left(-\frac{\partial^2}{\partial x^2} + 2m^2\right)\xi_n(x) = \omega_n^2\xi_n(x) \quad (3.37)$$

with eigenfunctions $\xi_n(x) = \frac{1}{L^{1/2}}e^{ik_n x}$. The L parameter originates from the box normalization procedure.

What we have done here is nothing but finding the eigenmodes of the pertubation in $\phi(x)$. Our problem now is broken into a set of independent harmonic oscillators. We simply need to compute:

$$Det\left[-\frac{\partial^2}{\partial t^2} - \omega_n^2\right] \quad (3.38)$$

which is the single quantum harmonic oscillator problem. [3]

After computing this and plugging it in the action we can rearrange terms to obtain the following expression for the spectrum:

$$E_{\{N_n\}} = \hbar \sum_n \left(N_n + \frac{1}{2} \right) \left(k_n^2 + 2m^2\right)^{1/2} + \mathrm{O}(\lambda) \tag{3.39}$$

Here, N_n represents the number of quanta present in each n mode, each of them with momentum given by $\hbar k_n$. The momentum $\hbar k_n$ arises from the boundary conditions of box normalization performed to solve the equations of motion for $\xi_n(x)$.

Our lowest energy value is obtained when there is no quanta present:

$$E_0 = \frac{1}{2}\hbar \sum_n \left(k_n^2 + 2m^2\right)^{1/2} + \mathrm{O}(\lambda) \tag{3.40}$$

We say that the quantum states are constructed around ϕ_1, the "vacuum sector" and postulate that there is a state with energy E_0, $|0\rangle$.

As we discussed above in the non-relativistic quantum example we can calculate the expectation value of the quantized scalar field ϕ with respect to this ground state given by:

$$\langle 0|\phi(x,t)|0\rangle = \phi_1 + \mathrm{O}(\lambda) \tag{3.41}$$

It is easy to convince the reader that the same approach where we expand V around ϕ_2 leads to the same spectrum. The only difference is the expectation value $\langle 0|\, \phi_2 \,|0\rangle = \phi_2 + O\left(\lambda\right)$.

After second quantization the field now describes particles. The particles arising from the quantisation of this trivial solutions are sometimes called mesons.

Acquainted with all of this let us start now by quantizing the soliton solution of our system:

$$\phi_K(x) = (m/\sqrt{\lambda}) \tanh[mx/\sqrt{2}] \tag{3.42}$$

Our expansion of the functional potential remains the same. The only difference is the fact that now, $\tilde{\phi} = \phi(x,t) - \phi_K = \phi(x,t) - (m/\sqrt{\lambda}) \tanh[mx/\sqrt{2}]$.

The resulting potential then is:

$$\begin{aligned} V[\phi] =& V\left[\phi_K\right] + \int \mathrm{d}x \frac{1}{2}\tilde{\phi}(x) \left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_K^2 \right) \tilde{\phi}(x) \\ &+ \lambda \int \mathrm{d}x \left(\phi_K \tilde{\phi}(x)^3 + \frac{1}{4}\tilde{\phi}(x)^4 \right) \end{aligned} \tag{3.43}$$

Again, considering the potential up to zeroth order in λ we get the following equation of motion

$$\left[-\frac{\partial^2}{\partial x^2} - m^2 + 3m^2 \tanh^2\left(\frac{mx}{\sqrt{2}}\right) \right] \tilde{\phi}_n(x) = \omega_n^2 \tilde{\phi}_n(x) \tag{3.44}$$

[3] To calculate this result one should take care of divergencies and sometimes depending, on the approach, this calculation can be tricky. We reference the interested reader chapter 6 of [55] for a complete derivation of this result and also the well-written notes of Ricardo Rattazzi on "Path integral approach in Quantum Mechanics". Further in this section we will follow a very similar approach to derive another result.

the result of this equation was found analytically in [28] after performing a change of variable, $z = \frac{mx}{\sqrt{2}}$. More recently an independent derivation was proposed by [56].
Then, the result is:

$$\begin{array}{ll} \omega_0^2 = 0 & \text{with } \tilde{\phi}_0(z) = 1/\cosh^2 z \\ \omega_1^2 = \frac{3}{2}m^2 & \text{with } \tilde{\phi}_1(z) = \sinh z/\cosh^2 z \end{array} \tag{3.45}$$

those are followed by continuous levels that for convenience we represent here as:

$$\omega_q^2 = m^2 \left(\frac{1}{2} q^2 + 2 \right) \tag{3.46}$$

with q being any real number.
Hence, the energy is given by

$$E_c = V\left(\phi_c\right) + \hbar \sum_{n=0}^{\infty} \left(N_n + \frac{1}{2} \right) \omega_n \tag{3.47}$$

including the continuum spectrum. Actually the energy do not carry any contribution from the $n = 0$ mode, since it is associated with the eigenvalue $\omega_0^2 = 0$ and as we know it does not tell anything about the vibrational normal modes of this low order approximation. Since with $\omega_0 = 0$ there is no vibration at all, it is like the "spring constant" of the harmonic approximation vanishes. This is why it does not make sense to include this mode in our results. Further attempt to include higher order contributions to the energy expression regarding the $n = 0$ mode leads to divergences, confirming our point expressed here. Here we will not dive into this particular case. Then our final result, inlcuding the contributions from the discrete and continuous parte of the potential, at zero-th order in λ is

$$E = \frac{2\sqrt{2}\mathrm{m}^3}{3\lambda} + \left(N_1 + \frac{1}{2} \right) \hbar \sqrt{\frac{3}{2}} m + m\hbar \sum_{q_\mathrm{n}} \left(N_{q_\mathrm{n}} + \frac{1}{2} \right) \left(\frac{1}{2} q_\mathrm{n}^2 + 2 \right)^{1/2} \tag{3.48}$$

There are, some considerations that need to be discussed.

- The lowest energy of this spectrum, $\{N_n\} = 0$, is interpreted as the ground state energy of the quantum kink. It is the ground state energy of the "kink sector". Just as a reminder, it is not the vacuum of this theory. The vacuum of this theory was built around the $\phi = \phi_1$ solution.

- The first excited state with only $N_1 \neq 0$ tells us the energy of the excited state of the quantum kink.

- The cases in which $N_q \neq 0$, are interpreted in a different manner though. These states are actually thought as the scattering states of the mesons, such scattering induced by the presence of the kink. We can be convinced about it with a more carefull discussion of this case. We have written ω_q as $\omega_q = m\sqrt{\left(\frac{1}{2}q^2 + 2\right)}$ then we can agree that the scattered meson carries an energy of $\hbar\omega_q = m\hbar\sqrt{\left(\frac{1}{2}q^2 + 2\right)}$ we can think of the term $\frac{m\hbar q}{\sqrt{2}}$ as the kinetic energy of the meson and $m\sqrt{2}$ as its rest mass. Furthermore, the eigenfunctions of the operator above are given by [28]

$$\tilde{\phi}_q(z) = \mathrm{e}^{iqz}\left(3\tanh^2 z - 1 - q^2 - 3iq\tanh z\right) \tag{3.49}$$

Taking the limit when $z \to \infty$ we realize that $\tanh z \to 1$ and we can write it as:

$$e^{iqz}\left(2 - q^2 - 3iq\right) = e^{i(qz \pm \frac{1}{2}\delta(q))} \tag{3.50}$$

Taking logarithm of both sides of the above equation we find that the phase is equal to:

$$\delta(q) = \mp 2\tan^{-1}\left(3q/2 - q^2\right) \tag{3.51}$$

meaning

$$\tilde{\phi}_q(z) \underset{z\to\pm\infty}{\longrightarrow} \exp\left[i\left(qz \pm \frac{1}{2}\delta(q)\right)\right] \tag{3.52}$$

Then, now it seems we have clear concepts and many tools to understand a bit more about quantum solitons. In the following section we digress a little from the physics to introduce a more sophisticated mathematical machinery that will enable us include fermions in the path integral formalism.

Besides this approach, some authors also investigate the quantization of supersymmetric solitons. This investigation can give remarkable insights on how to understand quantum chromodynamics and some gauge theories. The whole point is that for these supersymmetric theories a BPS sector may exist and important information can be exctracted even considering the strong coupling regime of this theories. The following reference investigate this subject regarding kinks, vortices and monopoles[27].

3.3 Grassman algebra

Up to this point the discussion of the semiclassical methods in this book was made only regarding bosonic fields. These fields follow some fundamental commutation relations that are of major importance in the discussion of their quantization. For Fermi fields one needs to work with anticommutaion relations instead. This change brings more than only a negative sign, it enables us to start calling these fermion fields, *Grassman fields*, following Grassman algebra.

Now imagine one have an operator a_0, such that $\{a_0, a_0\} = 0$ which implies that $a_0^2 = 0$ and of course $a_0^3 = a_0^2 a_0 = a_0^4 = a_0^3 a_0 ... = 0$. Hence, there are only two elements of this set that are independent functions $(a_0)^0 = 1$ and a_0. In the same fashion we can define a set of N operators as $\{a_1, ..., a_N\}$ with the following requirement:

$$\{a_i, a_j\} = 0 \tag{3.53}$$

for any $i, j = 1, \dots, N$.

This gives linearly independent functions:

$$\begin{array}{r} 1 \\ a_1, a_2, ..., a_N \\ a_1a_2, a_1a_3, ..., a_1a_N \\ \vdots \\ a_Na_1, ..., a_Na_{N-1} \\ ... \\ ... \\ a_1a_2...a_N. \end{array}$$

together, these 2^N independent functions span a 2^N dimensional space which we call G_N. It defines a Grassman algebra, where any element of this algebra can be expressed as finite sum:

$$f(a_1, ..., a_N) = f^0 + \sum_i f_i^1 a_i + \sum_{i,j} f_{i,j}^2 a_i a_j + ... \sum_{i,...,z} f_{i,...,z}^N a_1...a_z \tag{3.54}$$

where $f^0 ... f_{i,...,z}^N$ are ordinary commuting c-numbers. Note that under the antisymmetry of the operator $a_i a_j$ this decomposition is not unique.

A calculus, both differential and integral, can be developed in this algebra but not with the standard interpretation from the calculus of c-numbers. The integral defined in this context is not the area under some curve and the derivative is not the slope of that curve. Since we are dealing with operators, they do not vary continuously, but we can take advantage of the same symbology to define derivatives and integrals of c-numbers, together with some extra component which we explain below.
Let us start with definition of derivative in this algebra. We define

$$\frac{\partial a_i}{\partial a_j} = \delta ij \tag{3.55}$$

For example we can try to compute

$$\frac{\partial}{\partial a_2} a_1 a_2 a_3 \tag{3.56}$$

Since the operators anti commute $\{a_i, a_j\} = 0$, in this case we just need to anticommute the a_2 operator untill it reaches to the far left and use the definition of the derivative:

$$\frac{\partial}{\partial a_2} a_1 a_2 a_3 = -a_1 a_3 \tag{3.57}$$

Similar to derivatives we can define integrals of these operators in the Grassman algebra. We define this integrals as just functionals that associate a Grassman number to a c-number. We just want this integrals to possesses two properties, linearity and translational symmetry

$$\begin{aligned} \int f(a)da &= \int f(a+b)da \\ \int \left[\alpha f(a) + \beta h(a)\right] da &= \alpha \int f(a)da + \beta \int h(a)da \end{aligned} \tag{3.58}$$

We start postulating two integrals that match both requirements

$$\int a da = 1, \qquad \int 1 da = 0 \tag{3.59}$$

Now, if we consider two operators a_1, a_2 that generating G_2 we can define a double integral. This algebra is generated by the following elements, $1, a_1, a_2, a_1 a_2$. To be consistent with the one dimensional integrals we calculated, we should agree with the following results

$$\begin{aligned} &\int\int a_2 da_2 da_1 = \int 1 da_1 = 0 \\ &\int\int a_1 da_2 da_1 = -\int\int (a_1 da_1)\, da_2 = -\int 1 da_2 = 0 \\ &\int\int 1 da_1 da_2 = 0 \\ &\int\int a_1 a_2 da_1 da_2 = -\int a_1 \int (a_2 da_2)\, da_1 = -1 \end{aligned}$$

which means that when performing an integral of Grassman numbers only the following terms contribute:

$$\int da_1 \dots da_n a_1 \dots a_n = (-1)^P \tag{3.60}$$

where P is the number of permutations needed. If there is a function $f(a_1 \dots a_n)$ that can be expanded in terms of the generators of the algebra we can say that

$$\begin{aligned} &\int f(a_1 \dots a_n) da_1 \dots da_n = \\ &\int \left(f^0 + \sum_i f^1_i a_i + \sum_{i,j} f^2_{i,j} a_i a_j + ... \sum_{i,...,z} f^n_{i,...,z} a_1 ... a_z \right) da_1 \dots da_n \end{aligned} \tag{3.61}$$

and this function contributes to the integral with only the last term:

$$\int \left(\sum_{i,...,z} f^n_{i,...,z} a_1 ... a_z \right) da_1 \dots da_n = -\sum_{i,...,z} \epsilon_{i,...,z} f^n_{i,...,z} \tag{3.62}$$

It is useful also if we study how a linear change of variables affect the integral

Suppose first a set of generators $da_1 \dots da_n$ where we do the following transformation of variables $b_i = B_{ij}a_j$ where every element of B_{ij} is a c-number.
If the b operators are equally good basis to span G_N, then of course, $\{a_i, a_j\} = \{b_i, b_j\} = 0$
We get to the conclusion that:

$$\int da_1 \dots da_n(a_1 \dots a_n) = \int db_1 \dots db_n(b_1 \dots b_n) = 1 \tag{3.63}$$

We can see that $b_i = B_{ij}a_j$ it meas $(B^{-1})_{ji}b_i = a_j$ where there is an implicit sum $\sum_i (B^{-1})_{ji}b_i = a_j$. Consequently $a_1 \dots a_n = [\sum_i (B^{-1})_{1i}b_i] [\sum_i (B^{-1})_{2i}b_i] \dots [\sum_i (B^{-1})_{ni}b_i]$ but since only terms that match $da_1 \dots da_n$ that contributes, we then obtain

$$\begin{aligned} a_1 \dots a_n = \sum_{i,j,k} \epsilon_{i,j,k}(B^{-1})_{1i}(B^{-1})_{2j}(B^{-1})_{3k}b_1 \dots b_n \\ a_1 \dots a_n = (Det(B))^{-1}b_1 \dots b_n \end{aligned} \tag{3.64}$$

where $Det(B)$ is the determinant of the transformation. Then we should have:

$$da_1 \dots da_n = (Det(B))db_1 \dots db_n \tag{3.65}$$

for the integration in b variable to hold.
An instructive example of what we have learnt untill here is the following integral:

$$\int da_1 da_2 e^{-\lambda a_1 a_2} \tag{3.66}$$

If we expand it in terms of the operators of the basis, remembering that the only terms contributing are $a_1 a_2$ terms:

$$\int da_1 da_2 (1 - \lambda a_1 a_2) \tag{3.67}$$

We finally obtain that:

$$\int da_1 da_2 e^{-\lambda a_1 a_2} = \lambda \tag{3.68}$$

A more complete material on the subject can be found in the references [57, 58]

3.4 Path integral for the Dirac field

Here we use also the stationary phase approximation but regarding fermionic fields. We take advantage of the Grassman algebra formalism developed before in this task. It turns out that we obtain an exact result rather than just an approximation.
After we introduced Grassmam numbers we can evaluate the generalized path integral for Fermionic fields.

$$Tr(e^{-iHT}) = N \int \int \mathcal{D}(\Psi^\dagger)\mathcal{D}(\Psi)e^{iS(\Psi^\dagger,\Psi)} \tag{3.69}$$

We choose to work with natural units, $\hbar = 1$.
As we did for the bosonic case, is instructive for us evaluate the following integral that will appear many times in our considerations:

$$Tr(e^{-iHT}) = N \int\int \mathcal{D}(\Psi^\dagger)\mathcal{D}(\Psi) e^{i\int dxdt\Psi^\dagger \hat{A}\Psi} \tag{3.70}$$

where $\hat{A}$ is also an operator as we considered in the last section.

Thanks to the the Grassman number formalism we discussed previously one can expand Ψ in the following way:

$$\Psi = \sum_i^\infty \varphi_i(x,t)a_i, \quad \Psi^\dagger = \sum_j^\infty \varphi_j^*(x,t)\bar{a}_j \tag{3.71}$$

where the integration measure is

$$\int\int \mathcal{D}[\Psi^\dagger]\mathcal{D}[\Psi] = \int \prod_i id\bar{a}_i da_i \tag{3.72}$$

In this expansion we assume there is infinity number of of operators $(a1, \cdots, a_n, \cdots, \bar{a}_1, \cdots, \bar{a}_n, \cdots)$. It is important to stress that the coefficients multiplying a and $\bar{a}$ are c-numbers and not spinors, besides that in the expression for the measure we have $i = \sqrt{-1}$.Because of the nature of the coefficients in 3.71, these Ψ and $\Psi^\dagger$ are called classic Dirac fields.
These coefficients obey orthogonality condition:

$$\int\int \varphi_j^*\varphi_i dxdt = \delta_{ij} \tag{3.73}$$

We are considering $\hat{A}$ a differential operator, we can say it acts on the expansion coefficients as $\hat{A}\varphi_i = A_{ij}\varphi_j$.
Then our integral is:

$$\begin{aligned} N\int\int \mathcal{D}[\Psi^\dagger]\mathcal{D}[\Psi]e^{i\Psi^\dagger A\Psi} &= N\int \prod_i (id\bar{a}_i da_i) e^{\int\int dxdt i\Psi^\dagger A\Psi} \\ &= N\int \prod_i (id\bar{a}_i da_i) e^{i\bar{a}_i A_{ij} a_j} \\ &= N Det(\hat{A}) \end{aligned} \tag{3.74}$$

Note how different this result is compared to the Bosonic case. We here have a factor of $Det(\hat{A})$ in the numerator while in the previous case it was $\sqrt{Det(\hat{O})}$ in the denominator. It will play an essential role when we study the interaction of kinks and fermions in the following sections.
The expansion chosen in Ψ is only with the intention of justifying the result above. We could have expanded in the canonical way where rather than c-numbers there would be

spinors. In this case the calculations would be much more difficult than what was presented, but should recover the same result.
Let us now take the tools we developed up to this point to calculate the energy of the free fermion solution of the Dirac equation.
We start with the Lagrangian that dictates the equation of motion:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{3.75}$$

Taking the variation with respect to $\bar{\Psi}$, we obtain

$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi = 0 \tag{3.76}$$

where here $\gamma_0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\alpha_i = \beta\gamma_i$ remembering that we are in $3+1$ dimensions.
And, we can write the fermionic path integral as:

$$Tr(e^{-iHT}) = N\int\int \mathcal{D}\,[\Psi^{\dagger}]\mathcal{D}\,[\Psi]\,e^{i\int\bar{\Psi}(\partial\!\!\!/ - m)\Psi} \tag{3.77}$$

here N is a normalization constant, we are also denoting $\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$. Our differential operator now is:

$$\hat{A} = (+i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - m\beta) \tag{3.78}$$

It means that we need to find the eigenvalues of the differential operator above if we plan to evaluate

$$Tr(e^{-iHT}) = NDet(\hat{A}) \tag{3.79}$$

We call the reader's attention to a minor point that should be emphasized in this calculation. Note that this result is for the case where the Ψ and $\Psi^{\dagger}$ fields are operators. But the operator $\hat{A}$ acts on the space of the Dirac spinors where $\Psi \to \psi$ and ψ is $\psi(x,t)$ is a four component spinor where each component is a c-number.
Going further, we need to solve the following eigenvalue problem:

$$\left(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - m\beta\right)\psi(x,t) = \lambda_i\psi(x,t) \tag{3.80}$$

The solutions are well-known plane waves [59], one more time these eigenfunctions were box normalized :

$$\psi(x,t) = \frac{1}{L^{3/2}}e^{i(\vec{k}\vec{x} - wt)}u(w,k) \tag{3.81}$$

Inserting this in the eigenvalue equation above we reach the eigenvalue equation:

$$Det(w - i\vec{\alpha}\vec{\nabla} - m\beta - \lambda) = 0 \tag{3.82}$$

With the results of Bjorken and Drell [59] the possible values of λ are :

$$\lambda_k = w \pm \sqrt{k^2+m^2} \tag{3.83}$$

Note that there are four values of energy, each 2 fold degenerate. Then our expression is:

$$Tr(e^{-iHT}) = N\prod_{w,k}(w+\epsilon_k)^2(w-\epsilon_k)^2 \tag{3.84}$$

where $\epsilon_k = \sqrt{k^2+m^2}$.

To solve the differential equation we need to set some boundary conditions. The allowed values of k follows $kL = 2\pi n$ given $k_n = 2\pi n/L$. Then after box normalization we take $L \to \infty$. To find the allowed values of w we need a less intuitive condition. Usually we choose the function to be periodic, when we are dealing with bosons $\phi(x,0) = \phi(x,T)$. However, for dealing with fermions we should impose anti-periodicity $\psi(x,0) = -\psi(x,T)$ conditions, which gives $w_nT = (2n+1)\pi$ (for more details check [55])
Looking carefully at each term, we realize that:

$$\prod_{n=-\infty}^{\infty}(\epsilon_k - w_n)^2 = \prod_{n=-\infty}^{\infty}(\epsilon_k + w_n)^2 \tag{3.85}$$

meaning

$$\begin{aligned} Tr(e^{-iHT}) &= N\prod_k\left(\prod_{-\infty}^{\infty}(\epsilon_k + w_n)^4\right) \\ &= N\prod_k\left(\prod_{-\infty}^{\infty}(w_n)^4(1+\epsilon_k/w_n)^4\right) \\ &= N\prod_k\prod_{-\infty}^{\infty}\left(\frac{(2n+1)\pi}{T}\right)^4\left(1+\frac{\epsilon_kT}{(2n+1)\pi}\right)^4 \end{aligned} \tag{3.86}$$

We can conveniently choose $N\prod_k\prod_w\left(\frac{(2n+1)\pi}{T}\right)^4$ to be $N\prod_k\prod_w\left(\frac{(2n+1)\pi}{T}\right)^4 = \prod_k(2)^4$, since N is arbitrary. The rest can be simplified to

$$\prod_{n=-\infty}^{\infty}\left(1+\frac{\epsilon_kT}{(2n+1)\pi}\right) = \prod_{n=0}^{\infty}\left(1-\frac{\epsilon_k^2T^2}{(2n+1)^2\pi^2}\right) = \cos\left(\frac{\epsilon_kT}{2}\right) \tag{3.87}$$

where we have used the result in [60] , eq. (1.413.3). Putting the results together we have

$$\prod_k\left(2\cos(\frac{\epsilon_kT}{2})\right)^4 = \prod_k\left(e^{i\frac{\epsilon_kT}{2}} + e^{-i\frac{\epsilon_kT}{2}}\right)^4 = \prod_k\left(e^{i\frac{\epsilon_kT}{2}}\right)^4\left(1+e^{-i\epsilon_kT}\right)^4 \tag{3.88}$$

which leads to

$$
\begin{aligned}
Tr(e^{-iHT}) &= \prod_k (e^{i\frac{\epsilon_k T}{2}})^4 \left(1+e^{-i\epsilon_k T}\right)^4 \\
&= \prod_k (e^{i2\epsilon_k T}) \prod_k \left(\sum_{\{n_k\}} \binom{4}{n_k} e^{-i\epsilon_k T n_k} \right) \\
&= (e^{i2\sum_k \epsilon_k T}) \left(\sum_{\{n_k\}} \left[\prod_k \binom{4}{n_k} \right] e^{-i\sum_k \epsilon_k T n_k} \right)
\end{aligned}
\tag{3.89}
$$

where in the final expression the product is limited only to the terms in square brackets , finally we end up with:

$$
Tr(e^{-iHT}) = \sum_{\{n_k\}} D(\{n_k\}) \exp\left[-iT\sum_k (-2\epsilon_k + n_k\epsilon_k)\right] \tag{3.90}
$$

where $D(\{n_k\}) = \prod_k \binom{4}{n_k}$. To get to this we should manipulate the exponentials in such a way to factorize the term $-iT$, whats is left of it is our total energy. This energy spectrum of the free Dirac field is

$$
E_{\{n_k\}} = \sum_k (-2\epsilon_k + n_k\epsilon_k) \tag{3.91}
$$

Of course as one can see this spectrum is divergent. It counts for the Dirac sea, so it should naturally be. The vacuum is where there is no states excited, i.e. $n_k = 0$, giving

$$
E_0 = -\sum_k 2\epsilon_k \tag{3.92}
$$

This is interpreted as the "fully filled negative energy sea" . This E_0 can be removed by normal ordering or by adding it to the Lagrangian.

3.5 Fermion soliton interaction and the possibility of soliton charge $\frac{1}{2}$

3.5.1 General theory of solitons interacting with fermions

In this subsection we take advantage of the mathematical formalism developed for bosonic and fermionic fields to understand what happens when solitons interact with fermions.

First, based on the quantization of the kink we performed before we can start developing some ideas. Our approach is based on the discussion of Jackiw and Rebbi [61], regarding solitons states.

Again we have the same Lagrangian as before and we assume $\hbar = 1$:

$$
L = \int \mathrm{d}x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 - \frac{m^4}{4\lambda} \right] \tag{3.93}
$$

As discussed previously our quantum theory of the bosonic Lagrangian possesses not only the vacuum sector and a multi meson sector as mentioned, but also possesses a kink sector where in our Hilbert space is described by the following states, that spans the whole kink sector:

- The states $|P\rangle$, where only the first mode of the kink energy is excited, and we interpreted as our quantum soliton particle, carrying an energy: $E = \sqrt{M^2 + P^2}$ with momentum P and mass M.
- The kink excited state $|P*\rangle$ carrying energy: $E = \sqrt{M^{*2} + P}$ with momentum P and mass M^*.
- The scattering states $|P, k_1 \ldots k_n\rangle$, that are the states created by excitations of the continuum part of the energy spectrum and $k_1 \ldots k_n$ are the assimptotic momenta of the mesons scattered.
- The scattering states of the excited soliton state $|P*, k_1 \ldots k_n\rangle$ where again $k_1 \ldots k_n$ are the assymptotic momenta of the mesons scattered.
- The mass of the kink is considered constant in this weak coupling approximation up to $O(\lambda)$ order.

Let us start our discussion about fermion soliton interaction supposing the following Lagrangean, with a Yukawa type interaction:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - U(\phi) + \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + g\bar{\Psi}\Psi\phi \tag{3.94}$$

where $U(\phi) = \frac{\lambda}{4}(\phi^2 - \frac{m^2}{\lambda})^2$. From the results we developed before we know we can write:

$$Tr\left[e^{-iHt}\right] = N \int \mathcal{D}[\phi] \int \mathcal{D}[\Psi^\dagger]\mathcal{D}[\Psi]\, e^{i(S_{int}+S_\phi+S_\Psi)} \tag{3.95}$$

We know already how to integrate the contribution to the action that comes from the fermion field. We just need to note this time that there is a background field that differs from the spectrum found before:

$$\hat{A}_\phi = i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - \beta\,(m - g\phi(x,t)) \tag{3.96}$$

We need to find the eigenvalues of this operator. It means that we need to solve the following eigenvalue equation:

$$\left(i\frac{\partial}{\partial t} - i\vec{\alpha}\cdot\nabla + \beta(m - g\phi(x))\right) f(x) = \epsilon_r f(x) \tag{3.97}$$

that is similar to what was solved before. Then we need only to identify the corresponding terms to find the result. We have

$$NDetA_{\phi_c} = \sum_{\{n_r\}} \left(\tilde{C}e^{-iT\left[\sum_r(-\epsilon_r + n_r\epsilon_r)\right]}\right) \tag{3.98}$$

where $\tilde{C} = \prod_k \begin{pmatrix} 2 \\ n_k \end{pmatrix}$. Hence, we obtain:

$$Tr[e^{-\frac{iHt}{\hbar}}] = \int D\left[\phi\right] e^{iS_\phi + \ln NDet(A_\phi)} \tag{3.99}$$

Just for convenience we define $S_{eff} = S_\phi - i\ln(NDet(A_\phi))$. At this step we could use the stationary phase approximation but it is very difficult to calculate the result.[4] We can use an already known calculation to evaluate the trace of the hamiltonian. Since we consider the weak coupling approximation we can approximate the extremum of S_{eff} by the extremum of S_ϕ that is ϕ_{cl}, and this is an already known result.
Therefore, our final expression is nothing more than

$$Tr\left[e^{-iHt}\right] = e^{iS[\phi_{cl}]} NDet(A_{\phi_{cl}})\Delta_0 \tag{3.100}$$

where Δ_0

$$\Delta_0 = \int\int dtdx \exp\left(\frac{i}{2} y \left(\frac{\partial^2 S}{\partial \phi^2}\right)\bigg|_{\phi_{cl}} y\right) \tag{3.101}$$

with $y = \phi - \phi_{cl}$, knowing ϕ_{cl} is the soliton solution arising from the potential $U(\phi)$ in the absence of the fermionic field.

Following the same steps we discussed before, we obtain

$$\Delta_0 = \sum_{\{N_p\}} exp\left\{-iT\left[\sum_{p=0}^{\infty}\left(N_p + \frac{1}{2}\right)\omega_p\right]\right\} \tag{3.102}$$

where ω_p^2 are the eigenvalues of the operator $\left(-\nabla^2 + \frac{d^2U}{dx^2}\big|_{\phi_{cl}}\right)$
Collecting all these terms we end up with:

$$E_{\{N_p,n_r\}} = E_{cl}(\phi_{cl}) + \hbar\left(\sum_r^{\infty}(-\epsilon_r + n_r\epsilon_r)\right) + \hbar\left(\sum_{p=0}^{\infty}(N_p + \frac{1}{2})\omega_p\right) \tag{3.103}$$

the above result is divergent and counter terms need to be added to the expression to have a finite energy.

In spite of the fact that the soliton energy has a divergent term there is a contribution to the energy (due to the interaction with the fermion field) that comes exactly from the zero point energy of the filled sea.
Also, compare the middle term with the term for the spectrum of the free particle. Here n_r can only assume values $n_r = 1, 2$. Back there in the calculation of (3.98) our operator eigenvalues are ϵ_r and not ϵ_k as in the previous case. In the former case we do not have the degeneracy present in the second case we have.
Apart from the path integral formalism we could also look at this system from the point

[4] A more complete discussion about it can be found in Rajaraman [55], chapter 9

of view of the canonical operator formalism
Our fields now are promoted to operators and if there is no zero mode our fields can be expanded as:

$$\Phi(x,t) = \phi_{cl}(x) + \sum_r \left(a_r \frac{e^{-it\omega_r}}{\sqrt{2\omega_r}} \eta_r(x) + a_r^\dagger \frac{e^{it\omega_r}}{\sqrt{2\omega_r}} \eta_r^\star(x) \right) \tag{3.104}$$

We do the same with fermion fields, and expand it in the presence of the soliton solution, where $f_r^{(\pm)}$ are solutions of the single particle Dirac equation for positive and negative energy respectively:

$$\Psi(x,t) = \sum_r \left(b_r \mathrm{e}^{-\mathrm{i}\varepsilon_r t} f_r^{(+)}(x) + d_r^\dagger \mathrm{e}^{\mathrm{i}\varepsilon_r t} f_r^{(-)}(x) \right) \tag{3.105}$$

We can define a normal ordered current operator as

$$j^\mu = \frac{1}{2} \left[\Psi^\dagger, \gamma^0 \gamma^\mu \Psi \right] \tag{3.106}$$

This current is defined this way because the canonical expression of the current leads to well-known divergencies due to the vacuum of the Dirac field, mentioned in the previous section.
the mean value of the charge with respect to the quantum soliton ground state is

$$\langle P| Q |P\rangle = \langle P| \frac{1}{2} \int (\Psi^\dagger \Psi - \Psi \Psi^\dagger) |P\rangle \, dx \tag{3.107}$$

$$= \langle P| (\sum_r (b_r^\dagger b_r - d_r^\dagger d_r)) |P\rangle = 0 \tag{3.108}$$

since $b_r |P\rangle = d_r |P\rangle = 0$.

These states $|P\rangle$ are the same as defined and explained previously, the vacuum state of the soliton sector

3.5.2 Solitons with fermionic charge $\frac{1}{2}$

In this chapter we have quantized solitons and have introduced the path integral formalism for fermions together with the main features of fermion-soliton interaction as tools to understand the following considerations. Starting with the non-trivial result obtained by Jackiw and Rebbi in their proeminent work [61]. They have shown how solitons can acquire fermion number by interacting with fermions (valuable discussions can also be found in [62])). It starts with the following Lagrangian:

$$\mathcal{L}(x,t) = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \left(\phi^2 - 1 \right)^2 + \overline{\Psi} (\mathrm{i} \gamma^\mu \partial_\mu) \Psi + g \overline{\Psi} \Psi \phi \tag{3.109}$$

the same as (3.94) but choosing $m = 0$ and $\lambda = 1$ in where $\Psi(x,t)$ is a (1+1) fermionic

field that is a spinor with two components $f_r = \begin{pmatrix} u_r(x,t) \\ v_r(x,t) \end{pmatrix}$, $u(x,t)$ and $v(x,t)$ c-numbers. The γ^μ are gamma matrices which are poportional to Pauli matrices in 1+1 dimensions. We use the representation $\gamma^1 = \sigma_1, \gamma^0 = i\sigma_3$ here.
We can find the equation of motion for the ϕ field, free of the interaction with the fermion field and in the static case:

$$\frac{d^2\phi}{dx^2} - \frac{\partial U(\phi)}{\partial \phi} = 0 \tag{3.110}$$

From this equation we find the classical static soliton solution:

$$\phi_{\pm k} = \pm \tanh(x) \tag{3.111}$$

And for the two components spinor field we have the following set of coupled equations

$$\begin{pmatrix} -\partial_x + g\phi_k & \epsilon_r \\ \epsilon_r & \partial_x + g\phi_k \end{pmatrix} \begin{pmatrix} u_r \\ v_r \end{pmatrix} = 0 \tag{3.112}$$

An interesting feature of the system arise when we study the bound states of this model. The fermionic bound energy spectrum [5], is as follows where $\epsilon_r < g$

$$\epsilon_r^2 = 2gr - r^2 \tag{3.113}$$

This result is not easy to be obtained. Once decoupled, this set of equations can be solved and its eigenfunctions written in terms of hypergeometric functions. Imposing the right boundary conditions we obtain (3.113), r can be any integer up to $r < g$. With this we see directly that there is a bound state with zero energy, such bound state called zero mode. If $\epsilon_r \neq 0$ then we have always a pair of energy ϵ_r and $-\epsilon_r$ for each r. Energy always appear in pairs, if f_r with energy ϵ_r is a solution to the equations of motion, it means that Cf_r^* is also a solution and possesses energy $-\epsilon_r$. In particular, note that the zero state is not degenerate. Then we conclude that, $Cf_0^* = f_0$ that is, the state is it self conjugate! Now, if we want to expand the Dirac spinor as before, again in the presence of the soliton, we should separate the zero mode

$$\Psi(x,t) = b_0 f_0(x) + \sum_{r \geqslant 1} \left(b_r \mathrm{e}^{-\mathrm{i}\varepsilon_r t} f_r^{(+)}(x) + d_r^{+} \mathrm{e}^{\mathrm{i}\varepsilon_r t} f_r^{(-)}(x) \right) \tag{3.114}$$

through the same procedure we can define a normal ordered current and calculate the charge induced in the soliton when interacting with the fermion field.
The creation and anihilation operators in (3.114) follows

$$\{b_0, b_0^\dagger\} = 1, \qquad \{b_0, b_r\} = \{b_0, d_r\} = 0 \tag{3.115}$$

[5]The equations arising for this model are the same as the Schrödinger equation for the potential $g^2 \tanh^2(x) \pm g\,\mathrm{sech}^2(x)$. The discussion on how to find the bound states for this and other potentials can be found in [28].

We can define the normal ordered current operator as in [59], to avoid divergences:

$$j^{\mu} = \frac{1}{2}\left[\Psi^{\dagger}, \gamma^{0}\gamma^{\mu}\Psi\right] \tag{3.116}$$

resulting in

$$\begin{aligned} Q &= \frac{1}{2}\int \left(\Psi^{\dagger}\Psi - \Psi\Psi^{\dagger}\right) dx \\ &= b_0^{\dagger}b_0 - \frac{1}{2} + \sum_r \left(b_r^{\dagger}b_r - d_r^{\dagger}d_r\right) \end{aligned} \tag{3.117}$$

And since $b_r \left|P,-\right\rangle = d_r \left|P,-\right\rangle = b_0 \left|P,-\right\rangle = 0$ it implies that

$$\langle P,-|Q|P,-\rangle = -\frac{1}{2} \tag{3.118}$$

The $\left|P,-\right\rangle$ is the state where no zero mode is excited. The states $\left|P,-\right\rangle$ and $\left|P,+\right\rangle$ can be related by:

$$\left|P,+\right\rangle = b_0^{\dagger}\left|P,-\right\rangle \tag{3.119}$$

We also calculate the mean value of the fermion number operator with respect to the $\left|P,+\right\rangle$ state to obtain:

$$\begin{aligned} \left\langle P,+\right| Q \left|P,+\right\rangle &= \frac{1}{2}\int \left(\Psi^{\dagger}\Psi - \Psi\Psi^{\dagger}\right) \\ \langle Q\rangle &= +\frac{1}{2} \end{aligned} \tag{3.120}$$

It is possible to show that these states are self conjugate of each other.

There is also investigation regarding the interaction of solitons and fermions in higher dimensions, e.g monopoles, vortices, instantons among others [63, 64, 65]. A particular example is the investigation of coupling between the so-called skyrmion and fermions. A skyrmion is a soliton introduced first by Skyrme in (3+1) dimensions that can effectively describe baryons in the limit of a large color number. The 2+1 analogue of this structure is known as the baby skyrmion model and arises in many contexts in condensed matter physics. The following paper examines how fermions couple to this structures as well as the consequences of this interaction [66].

3.6 Adiabatic and non-adiabatic methods

In this section we present and explain more recent results regarding fermion soliton interaction. First we start with presenting the adiabatic method developed by Goldstone and Wilczek [67]. This method consists of calculating background fields evolving adiabatically

from the trivial topological regime to the non-trivial topological regime and its contribution to the fermionic vacuum polarization. The integral of the zero-th component of the induced current due to the presence of soliton is exactly the fermionic charge acquired by the soliton. Their work also showed that Jackiw and Rebbi's result was only a particular case among all possible values that the fermion number of a soliton can assume. Actually, it can be any real number. The method can be summarized as follows:
Let us first study the following Lagrangian with two bosonic fields

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - g\left(\phi_1 + i\gamma^5\phi_2\right)\right)\psi \tag{3.121}$$

This Lagrangian is invariant under chiral transformations

$$\begin{aligned} \phi_1 &\to \phi_1\cos\theta + \phi_2\sin\theta \\ \phi_2 &\to -\phi_1\sin\theta + \phi_2\cos\theta \\ \psi &\to e^{i\gamma^5\theta/2}\psi \end{aligned} \tag{3.122}$$

to make this invariance more explicit we write

$$\begin{aligned} \phi_1 + i\gamma^5\phi_2 &= \rho e^{i\gamma^5\theta} \\ \psi &= e^{-i\gamma^5\theta/2}\chi \end{aligned} \tag{3.123}$$

where $\rho = \sqrt{\phi_1^2(x) + \phi_2^2(x)}$, $V_\mu = -\frac{1}{2}\partial_\mu\theta$, $\theta = \tan^{-1}\left(\frac{\phi_2}{\phi_1}\right)$ and χ is the spinorial component of the transformed ψ.
Including an additional source term $\mathcal{L} = -\bar{\psi}\gamma^\mu\psi J_\mu$, this yields to:

$$\mathcal{L} = \bar{\chi}(i\partial\!\!\!/ - g\rho - V\!\!\!/\gamma^5 - J\!\!\!/)\chi \tag{3.124}$$

This procedure of adding a source term is commonly used in the steps to calculate the induced current [68]. Then if we calculate the transition amplitude of this Lagrangian using path integral formalism and integrate over the Grassman fields (as we did previously) we obtain:

$$G = \mathcal{N}Det[i\partial\!\!\!/ - g\rho - V\!\!\!/\gamma^5 - J\!\!\!/] \tag{3.125}$$

we can use an important identity valid for any matrix M as

$$Det(M) = exp[tr(ln(M))] \tag{3.126}$$

Then we get

$$Det(i\partial\!\!\!/ - g\rho - V\!\!\!/\gamma^5 - J\!\!\!/) = exp[tr(ln(i\partial\!\!\!/ - g\rho - V\!\!\!/\gamma^5 - J\!\!\!/))] \tag{3.127}$$

And consider ρ to be constant $\rho \approx \rho_0$, we can call $\bar{\rho} = g\rho_0$ so we can perform the following

$$ln(i\partial\!\!\!/ - \bar{\rho} - V\!\!\!/\gamma^5 - J\!\!\!/) = ln\left[(i\partial\!\!\!/ - \bar{\rho})\,(1 - S_f(V\!\!\!/\gamma^5 + J\!\!\!/))\right] \quad (3.128)$$
$$= ln\left[(i\partial\!\!\!/ - \bar{\rho})\right] + ln[(1 - S_f(V\!\!\!/\gamma^5 + J\!\!\!/))] \quad (3.129)$$

where S_f is the fermion propagator. This condition over $\rho(x)$ is necessary since we want to expand the logarithmic function in terms of the x variable, otherwise we would have to consider derivatives of $\rho(x)$ making even harder our computation of this determinant [68].

The first term of (3.129) can be absorbed in the normalization constant $\mathcal{N}$, we will be interested in the second contribution of (3.129) that leads to

$$G = \mathcal{N} exp\,[\Gamma_1]\,, \quad \Gamma_1(J) = -iTr[ln(1 - S_F(V\!\!\!/\gamma^5 + J\!\!\!/)] \quad (3.130)$$

where

$$Tr = \int d^2x \int \frac{d^2k}{(2\pi)^2} \quad (3.131)$$

And $S_f = \frac{1}{k\!\!\!/ - \bar{\rho}}$. This Γ_1 term is the one-loop effective action which can be expanded as

$$\Gamma_1 = i\sum_{n=1}^{\infty}\frac{1}{n}Tr(S_f(V\!\!\!/\gamma^5 + J\!\!\!/))^n \quad (3.132)$$

Then we can look again to the expression in (3.130) now having the following form

$$G = \mathcal{N} e^{\int d^2x \int \frac{d^2k}{(2\pi)^2} ln(1 - S_F(V\!\!\!/\gamma^5 + J\!\!\!/))} \quad (3.133)$$

and we can interpet the integrand of $\int d^2x$ as an effective local Lagrangian, which means

$$\Gamma_1 = \int d^2x \mathcal{L}_{eff} = S_{eff} \quad (3.134)$$

This is the contribution to the fermionic induced current at one loop level. We will also consider only first order contribution to the induced current, linear in J_μ so we can use the following result to compute the first contribution to the induced current

$$\langle j^\mu_{eff} \rangle = -\frac{\delta S_{eff}}{\delta J_\mu} \quad (3.135)$$

where S_{eff} is the effective action. The fermion propagator in (3.132) is given by

$$S_F = \frac{1}{k\!\!\!/ - \bar{\rho}} = \frac{k\!\!\!/ + \bar{\rho}}{k^2 - \bar{\rho}^2 + i\eta} \quad (3.136)$$

If we look at (3.132) up to the second term in the expansion we see that

$$\Gamma_1 = \frac{i}{2}\left(Tr[S_f(V\!\!\!/\gamma^5)S_f J\!\!\!/] + Tr[S_f(J\!\!\!/)S_f(V\!\!\!/\gamma^5)]\right) \quad (3.137)$$

These will be the first terms to contribute to our current, others are either high order in J_μ or possess null trace.
Using cyclic property of the trace one obtains:

$$\Gamma_1 = iTr\frac{\not{k}+\bar{\rho}}{k^2-\bar{\rho}^2+i\eta}\not{V}\gamma^5\frac{\not{k}+\bar{\rho}}{k^2-\bar{\rho}^2+i\eta}\not{J} \tag{3.138}$$

where one can calculate each trace separately and use the following identities for the gamma matrices in 1+1-dimension

$$Tr\gamma^\nu\gamma^\lambda\gamma^5 = -2\epsilon^{\nu\lambda}, \quad Tr\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^5 = 2\left(-g^{\mu\nu}\epsilon^{\sigma\lambda} + g^{\mu\sigma}\epsilon^{\nu\lambda} + g^{\nu\sigma}\epsilon^{\lambda\mu}\right) \tag{3.139}$$

which gives

$$\begin{aligned}\mathcal{L}_1 = 2i\epsilon^{\mu\nu}\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2-\bar{\rho}^2+i\eta)^2}\big(\bar{\rho}^2V_\mu J_\nu \\ -k\cdot Vk_\mu J_\nu + k^2V_\mu J_\nu + k\cdot VJ_\mu k_\nu\big)\end{aligned} \tag{3.140}$$

Note that the last three terms have at least k^2 in the integrand and if we change coordinates to polar coordinates we obtain $\approx \int_0^\infty \frac{drr^3}{r^4+\bar{\rho}^2} = ln(r^4+\bar{\rho}^2)|_0^\infty$. This characterizes a UV divergence which needs to be properly regularized. We choose the dimensional regularization procedure, where we suppose those integrals to be performed in $d = 2-\epsilon$ dimensions where ϵ is an infinitesimal parameter. After computing the integrals we should take the limit $\epsilon \to 0$.
We use two common integral results

$$\int\frac{d^dk}{(2\pi)^d}\frac{\bar{\rho}^2}{(k^2-\bar{\rho}^2+i\eta)^n} = i\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\frac{\bar{\rho}^2}{(\bar{\rho}^2-i\eta)^{n-d/2}} \tag{3.141}$$

$$\begin{aligned}\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu g_{\nu\mu}}{(k^2-\bar{\rho}^2+i\eta)^n} &= \frac{i}{2}\frac{(-1)^{n-1}}{(4\pi)^{d/2}}\frac{\Gamma(n-1-d/2)}{\Gamma(n)}\frac{g^{\mu\nu}g_{\nu\mu}}{(\bar{\rho}^2-i\eta)^{n-1-d/2}} \\ &= \frac{i}{2}\frac{(-1)^{n-1}}{(4\pi)^{d/2}}\frac{\Gamma(n-1-d/2)}{\Gamma(n)}\frac{2}{(\bar{\rho}^2-i\eta)^{n-1-d/2}}\end{aligned} \tag{3.142}$$

after taking the limit $\epsilon \to 0$ we have

$$\mathcal{L}_1 = \frac{1}{\pi}\epsilon^{\mu\nu}J_\mu V_\nu \tag{3.143}$$

As we said before, our main goal is to compute the induced current as in (3.135). Using the result above we get

$$\langle j^\nu_{eff}\rangle = -\frac{1}{\pi}\epsilon^{\mu\nu}V_\nu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\theta = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\tan^{-1}\left(\frac{\phi_2}{\phi_1}\right) \tag{3.144}$$

and integrating the zero-th component of the induced current gives us the associated charge in the following

$$\langle Q\rangle = \int_{-\infty}^{\infty} \frac{1}{2\pi}\partial_x \tan^{-1}\left(\frac{\phi_2}{\phi_1}\right) dx \tag{3.145}$$

If we consider the massive case of this theory, we set $\phi_1 = \frac{m}{g}$ and assume that the soliton reaches its vacua as $x \to \pm\infty$ and $\phi_2 \to \pm\nu$. We simple get to:

$$\langle Q\rangle = \frac{1}{\pi}\tan^{-1}\left(\frac{\nu g}{m}\right) \tag{3.146}$$

Taking the limit $m \to 0$ we obtain

$$\langle Q\rangle = \pm\frac{1}{2} \tag{3.147}$$

We obtain different signs for the charge depending on how we take the limit $m \to 0^{\pm}$. All of this discussion is regarding to a linear coupling between soliton field and fermions, one can study a non-linear coupling, for example in the form

$$\mathcal{L} = g\bar{\psi}e^{i\gamma_5\theta}\psi \tag{3.148}$$

Following the same steps discussed above we can conclude that the expectation value of the probability current operator is:

$$\langle j^{\mu}\rangle = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\theta \tag{3.149}$$

which yields to

$$\langle Q\rangle = \frac{1}{2\pi}\Delta\theta \tag{3.150}$$

Note that the soliton with $0 \leq \theta \leq \pi$ has charge $\langle Q\rangle = +\frac{1}{2}$ and the soliton in the interval $0 \leq \theta \leq \frac{2\pi}{3}$ has $\langle Q\rangle = +\frac{1}{3}$. In this fashion one can create a soliton with any real fermion number in this non-linear model!
This model was first introduced in [69]. The non-linear model is also present in different investigations regarding condensed matter physics, including investigations in topological superconductivity [70].
This is the known Adiabatic contribution to the soliton fermion number. It is called adiabatic because it does not depend on the profile of the scalar fields but only its vacua. The only requirement is that the fields shown slowly vary in space.

Another approach first introduced by Mackenzie and Wilczek [71] study how background fields that do not satisfy the conditions imposed by Goldstone and Wilczek can also contribute to the soliton fermionic number. We start with the following Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m_1 - m_2 e^{i\gamma_5\phi(x)})\psi \tag{3.151}$$

We consider the case of what is called infinitely thin soliton. Let us suppose $m_1 = 0$,

$m_2 = m$, and the background soliton field in the form

$$\phi(x) = \alpha \frac{x}{|x|} \tag{3.152}$$

It possesses one discontinuity at $x = 0$. Our attempt is to expand the soliton field in terms of the soliton Hamiltonian eigenstates. Just as always the coeficients of this expansion is the creation and anihilation operators of particles in the solitonic eigenstates of energy. Expanding the fermion number in terms of these creation and anihilation operators can show from where the contributions to the fractional fermion number are coming.
Now we outline the steps cited above for writing the number operator in terms of creation and anihilation of fermionic operators. For the free fermion fields we can write:

$$\begin{aligned} \psi(x) &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left[b_p u_p(x) + d_p^\dagger v_p(x) \right] \\ &= e\chi_b(x) + \int_0^{\infty} \frac{dp}{2\pi} \sum_{j=+,-} \left[a_b^j \mu_b^j(x) + c_b^{j\dagger} \nu_b^j(x) \right] \end{aligned} \tag{3.153}$$

We can use the anticommutation relations and the orthonormality conditions to write some of the operators in term of the others. Because of knowing how the number operator is defined $N = b_p^\dagger b_p - d_p^\dagger d_p$ we write b_p and d_p as functions of χ_b, a_p^j, c_p^j operators which gives

$$N = e^\dagger e + a_p^{j\dagger} a_p^j - c_p^{j\dagger} c_p^j + \langle \nu_p^{j\dagger} | \nu_p^j \rangle - \langle v_k^{j\dagger} | v_k^j \rangle \tag{3.154}$$

where $\int \frac{dp}{2\pi}$ and $\sum_{j=+,-}$ are implicit.

Note that the term $\langle \nu_p^{j\dagger} | \nu_p^j \rangle - \langle v_k^{j\dagger} | v_k^j \rangle$ in the fermion number is equal to $N = -\frac{\Delta\theta}{2\pi}$ which is the adiabatic contribution, after the soliton has been already built up. Thus, we have

$$N = e^\dagger e + a_p^{j\dagger} a_p^j - c_p^{j\dagger} c_p^j - \frac{\Delta\theta}{2\pi} \tag{3.155}$$

leading to

$$\langle N \rangle = -\frac{\Delta\theta}{2\pi} \tag{3.156}$$

We need to discuss a little the consequences of this result. Imagine a soliton being created infinitely slowly from the ground state. Consider the following regimes

When $0 < \alpha < \frac{\pi}{2}$ the state constructed adiabatically is the ground state (E=0) and is natural to accept that since the soliton is constructed adiabatically, then fluctuations in the induced current must be very small. This garantees that the Dirac sky and bound states remains empty and we obtain only the adiabatic contribution to the charge. Taking the expectation value of the number operator, gives $\langle N \rangle = 0 + 0 + 0 - \frac{\Delta\Theta}{2\pi}$

When the parameter $\alpha = \frac{\pi}{2}$, the adiabatic state is the ground state. Besides that, the bound state and the ground state possess the same energy $E = 0$! In fact, now there are two ground states, one for each value of $\langle N \rangle = \pm\frac{1}{2}$ (this is exactly what we calculated above in the formalism of path integral). The ground state charge with the empty bound state possesses $\langle N \rangle = -\frac{1}{2}$ and with filled bound state possesses charge $\langle N \rangle = \frac{1}{2}$. The adiabatic state is the ground state with charge $\langle N \rangle = -\frac{1}{2}$

When $\frac{\pi}{2} < \alpha < \pi$ the state constructed adiabatically is no longer the ground state, since the bound state has negative energy. For this reason the adiabatic state (3.156) and the ground state ($\langle N \rangle = 1 + 0 - 0 - \frac{\Delta\Theta}{2\pi}$) charge differ by 1.

When $\alpha = \pi$ something really interesting happens. In this case the energy spectrum must be identical to the case $\alpha = 0$ and it really is since the bound state goes into the sea and our adiabatic state has charge given by eq. (3.156). Since after $\alpha = \frac{\pi}{2}$ our ground state and adiabatic state differ in charge by 1 this time the same happens and the charge of our ground state should be $\langle N \rangle = 0$, as there is no soliton. It means that for a topological charge of $\frac{\Delta\Theta}{2\pi} = 1$ the soliton is transparent to the fermions.

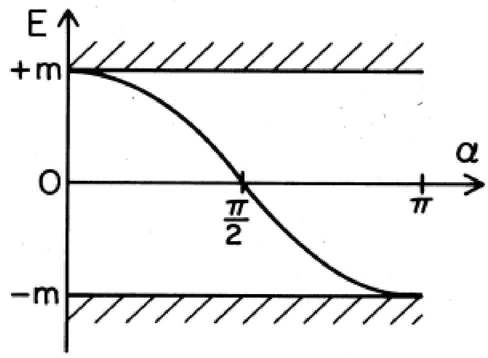

Figure 3.4. The bound state energies as a function of the parameter α of the soliton studied in [28]

4 Casimir energy and Levinson's theorem

4.1 Casimir energy

4.1.1 Historical introduction

After London formulated his theory of force between atoms in colloids [72], finding that it is proportional to $F \propto \frac{1}{r^6}$, with r being the distance between atoms, experiments showed that is was only valid for small r. As r increases the force seems to fall faster than London's result. The first guess researchers had at that moment relied on the retarded effect of the electromagnetic field, based on the transition frequency between atoms that had the order of magnitude of r/c, where such effects start to become relevant. Casimir and Polder worked on the suposition and performed all lenghty QED calculations. They have found that for large distances, $F \propto \frac{1}{r^7}$ [73]. After this, Casimir came to discuss with Bohr the results he had obtained. Bohr gave the hint on try to look at the zero point energy of the configuration in order to simplify the computations. Casimir realized that considering changes in the zero point energy lead to the same result he had obtained with Polder. In the following year Casimir have published his seminal paper [74] where he compute the change in the vacuum energy due to the presence of two neutral conducting plates and presented what became kwnon as Casimir Force and Casimir Effect.

The approach of Casimir was to calculate the difference in the spectrum of electromagnetic field as the following:

$$H = \sum_{\alpha=1}^{2} \sum_{k} \left(a^{\dagger}_{\alpha k} a_{\alpha k} + \frac{1}{2} \right) \omega_k \hbar \tag{4.1}$$

where α and k are respectively the polarization and the momentum of the electromagnetic field. For a zero number of photons emitted we have the vacuum energy of the electromagnetic field[1]:

$$H_0 = \sum_{\alpha} \sum_{\alpha k} \frac{\omega_k \hbar}{2} \tag{4.2}$$

[1]Clearly this result diverges and need a proper renormalization. A very good discussion on this details can be found in [75]

When this free electromagnetic field is constrained to the boundary conditions imposed by the two parallel plates, we have

$$\hat{n} \times \vec{E} = 0, \qquad \hat{n} \cdot \vec{B} = 0 \tag{4.3}$$

These conditions naturally change the allowed values of frequency for the field, which yields to the Casimir energy

$$E_{Casimir} = \left[\sum_{\alpha} \sum_{k} \frac{\omega_k^{'} \hbar}{2} - \sum_{\alpha} \sum_{k} \frac{\omega_k \hbar}{2} \right] \tag{4.4}$$

where in the subraction the summations are well regularized. $\omega_k^{'}$ denotes the new allowed values for the zero point frequency in accordance with the boundary conditions in (4.3).

4.1.2 Casimir energy for fermion-soliton system

The Casimir energy is non-zero whenever some physical processes change the free energy spectrum of the fields envolved. In the case of Hendrik Casimir it was the parallel plates but it could have been a background field such as a soliton field [76].
In the case of a non-trivial background field, like a soliton, the change in the energy spectrum of our system is exactly:

$$E_{Casimir} = \langle \Omega | H | \Omega \rangle - \langle 0 | H_{free} | 0 \rangle \tag{4.5}$$

where the hamiltonian is:

$$H = \int_{-\infty}^{\infty} (\psi^{\dagger} \mathcal{H} \psi) dx \tag{4.6}$$

and $\mathcal{H}$ the our one-particle hamiltonian.
Gousheh and Mobilia [77] have demonstrated that the fermion solution in the presence of a background field is complete, which means that we can expand our fermion field as the following.

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left[b_k u_k(x,t) + d_k^{\dagger} v_k(x,t) \right] \\
&= \int_{0}^{+\infty} \frac{dp}{2\pi} \left\{ \sum_{j=\pm} \left[a_p^j \mu_p^j(x,t) + c_p^{j\dagger} \nu_p^j(x,t) \right] \right. \\
&+ \left. \sum_i \left[e_i \chi_{1b_i}(x,t) + f_i^{\dagger} \chi_{2b_i}(x,t) \right] \right\}
\end{aligned}$$

where $a_p^j(c_p^j)$ are the creation (annihilation) operators associated with $\mu_p^j(\nu_p^j)$, positive (negative) eigenfunctions of the electron field in the presence of the background. The second term counts for the creation(annihilation) operators of bound states, $f_i^{\dagger}(e_i)$. χ_{1b_i} and χ_{2b_i} are respectively the positive and negative energy bound states. We will see shortly how the separation of bound states in this form is useful.

We obtain:

$$
\begin{aligned}
H = & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \left\{ \frac{dp}{2\pi} \sum_{j=\pm} \left[a_p^j \mu_p^j(x,t) + c_p^{j\dagger} \nu_p^j(x,t) \right] + \sum_i \left[e_i \chi_{1b_i}(x,t) + f_i^\dagger \chi_{2b_i}(x,t) \right] \right\} \\
& \times \int_0^\infty \left\{ \frac{dq}{2\pi} \sum_{n=\pm} \left[\sqrt{q^2+m^2} a_q^n \mu_q^n(x,t) - \sqrt{q^2+m^2} c_q^{n\dagger} \nu_q^n(x,t) \right] \right. \\
& \left. + \sum_l \left[E_{bound}^{l+} e_l \chi_{1b_l}(x,t) + E_{bound}^{l-} f_l^\dagger \chi_{2b_l}(x,t) \right] \right\}
\end{aligned} \tag{4.7}
$$

When we calculate the expectation value of the Hamiltonian with respect to the vacuum in the presence of the background field, only terms with annihilation operator on the left or creation operator on the right survive. This leads us to:

$$
\begin{aligned}
\langle \Omega | H | \Omega \rangle = & \int dx \int \frac{dp}{2\pi} \sum_{j=\pm} (-\sqrt{p^2+M^2}) \nu_p^{j\dagger} \nu_p^j \\
& + \int dx \sum_i E_{bound}^{i-} \chi_{2b_i}^\dagger(x,t) \chi_{2b_i}(x,t)
\end{aligned} \tag{4.8}
$$

where we have used

$$
\begin{aligned}
c_p^j c_q^{m\dagger} + c_q^{m\dagger} c_p^j = \delta^{mj} \delta_{pq} \\
f_i f_l^\dagger + f_l^\dagger f_i = \delta_{il}
\end{aligned} \tag{4.9}
$$

And the casimir energy can be obtained in the following form

$$
\begin{aligned}
E_{Casimir} & = \langle \Omega | H | \Omega \rangle - \langle 0 | H_{free} | 0 \rangle \\
& = \int dx \int \frac{dp}{2\pi} \sum_{j=\pm} (-\sqrt{p^2+M^2}) \nu_p^{j\dagger} \nu_p^j + \int dx \sum_i E_{bound}^{-i} \chi_{2b_i}^\dagger(x,t) \chi_{2b_i}(x,t) \\
& - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} (-\sqrt{k^2+M^2}) v_k^\dagger v_k
\end{aligned} \tag{4.10}
$$

what is important to note is that our system possesses charge conjugation symmetry, which means that we could have obtained the Casimir energy by summing symmetrically over all modes, positive and negative frequency ones, and at the end multiply by $\frac{1}{2}$. The two terms that are not related to the bound states can be written as the difference between the continuum states densities of the sea in the presence and the absence of the background field.

Now we can explicitly compute the Casimir energy, using the phase shift method [62]:

$$E_{Casimir} = \sum_i E^{i,\text{sea}}_{\text{bound}} - \int_0^{+\infty} \mathrm{d}k\sqrt{k^2+M^2}\left(\rho^{\text{sea}}(k) - \rho_0^{\text{sea}}(k)\right) + \frac{M}{2} \tag{4.11}$$

where

$$\rho^{sea}(k) = \int_{-\infty}^{\infty} dx \nu_p^{\dagger j}\nu_p^j, \quad \rho_0^{sea}(k) = \int_{-\infty}^{\infty} dx v_p^\dagger v_p \tag{4.12}$$

The first term in this expression is what is left after integrating the normalized bound states eigenfunctions. The second term comes from the difference between densities of states in the continuum with and without the presence of soliton. The last term came from the contribution from the half bound state already present in the sea in the absence of the soliton [78, 79]. With this expression we can use a well known result from quantum mechanics to relate the difference in densities of continuum states to the phase shift of the scattering[2].

$$\rho(k) - \rho_0(k) = \frac{1}{\pi}\frac{d\delta(k)}{dk} \tag{4.13}$$

Substituing this result in our expression we obtain:

$$\int_0^{+\infty} \mathrm{d}k\sqrt{k^2+M^2}\left(\rho^{\text{sea}}(k) - \rho_0^{\text{sea}}(k)\right) = \int_0^{\infty} \frac{dk}{\pi}\sqrt{k^2+M^2}\left(\frac{d\delta(k)}{dk}\right) \tag{4.14}$$

We can add a zero term to avoid the divergence obtained from the surface term as

$$\int \frac{dk}{\pi}\sqrt{k^2+M^2}\left(\frac{d\delta(k)}{dk}\right) = \int \frac{dk}{\pi}\sqrt{k^2+M^2}\left(\frac{d}{dk}(\delta(k) - \delta(\infty))\right) \tag{4.15}$$

Our final expression for this term is:

$$\begin{aligned} &\int_0^{+\infty} \mathrm{d}k\sqrt{k^2+M^2}\left(\rho^{\text{sea}}(k) - \rho_0^{\text{sea}}(k)\right) \\ = -&\int_0^{+\infty} \frac{\mathrm{d}k}{\pi}\frac{k}{\sqrt{k^2+M^2}}\left(\delta^{\text{sea}}(k) - \delta^{\text{sea}}(\infty)\right) - \frac{1}{\pi}M\left(\delta^{\text{sea}}(0) - \delta^{\text{sea}}(\infty)\right). \end{aligned} \tag{4.16}$$

Finally the Casimir energy is given by:

$$\begin{aligned} E_{Casimir} = +\sum_i E^{i,\text{sea}}_{\text{bound}} + \int_0^{+\infty} \frac{\mathrm{d}k}{\pi}\frac{k}{\sqrt{k^2+M^2}}\left(\delta^{\text{sea}}(k) - \delta^{\text{sea}}(\infty)\right) + \\ \frac{1}{\pi}M\left(\delta^{\text{sea}}(0) - \delta^{\text{sea}}(\infty)\right) + \frac{M}{2} \end{aligned} \tag{4.17}$$

[2]We will provide a simple proof of this expression, shortly

4.2 Levinson's theorem

4.2.1 Levinson's theorem, simple presentation

The following presentation is an attempt to derive Levinson's theorem in the most intuitive way relying on a one-dimensional quantum mechanics approach[3]. We start our discussion with the one dimensional general potential $V(x)$.

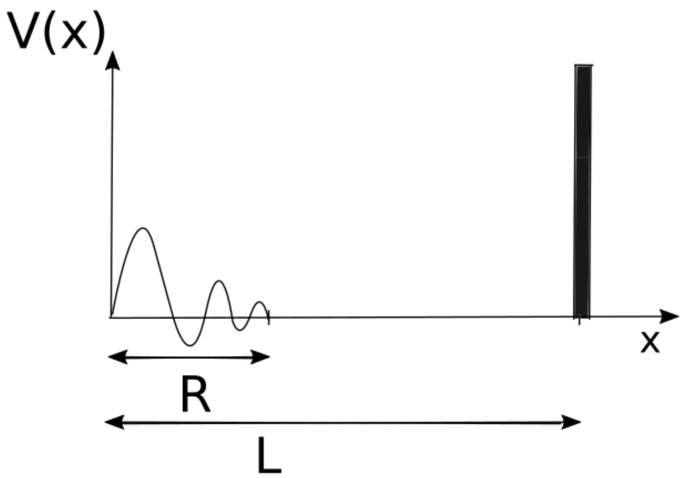

Figure 4.1. General potential of range R.[4]

Let us consider first, a particle that is traveling towards a localized potential with range R as the one depicted in the Fig. 4.1. We also suppose a wall in $x = 0$ position. In this example may induce bound states and a positive denumerable energy spectrum. For this to happen we need to introduce also a second wall at distance L from the origin. This last wall acts as a regulator, to make our positive energy eigenstates to be countable[5]. After finding the wave function we remove this wall.
Solving the static Schrödinger equation in the region where $V = 0$, we have:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x) = E\psi \tag{4.18}$$

$$\tag{4.19}$$

which has the general solution

$$\psi(x) = A\sin kx + B\cos kx \tag{4.20}$$

[3]One should also check MIT opencourses "Levinson's theorem, part II"
[5]The procedure we are employing is called box normalization. For L being finite what we have is a discrete set of states, that is a countabele set. When we take $L \to \infty$ we expect the states to be countable as well.

With the appropriate boundary conditions

$$\psi(0) = 0, \quad \left.\frac{d\psi(x)}{dx}\right|_{x=0} = A \tag{4.21}$$

we obtain :

$$\psi(x) = A \sin kx \tag{4.22}$$

We also impose that $\psi(L) = A \sin kL = 0$ then we can set the possible values of k, that is $k = \frac{n\pi}{L}$
If we now look at an infinitesimal variation in k we conclude that $dn = L/\pi dk$ when $V(x) \neq 0$, after our solution being scattered by the potential in the large x limit, we obtain

$$\psi(x) \approx A \sin (kx + \delta) \tag{4.23}$$

where δ is a phase acquired by the initial wave function after the scattering process
The condition $\psi(L) = 0$ implies $kL + \delta = n\pi$ and an infinitesimal change in k yields:

$$dkL + \frac{d\delta(k)}{dk}dk = dn^{'}\pi \tag{4.24}$$

now, one can calculate the difference between these two cases:

$$\frac{1}{\pi}\frac{d\delta(k)}{dk}dk = (dn^{'} - dn) \tag{4.25}$$

integrating both sides gives

$$\frac{1}{\pi}(\delta(0) - \delta(\infty)) = N_b \tag{4.26}$$

where N_b is the number of bound states.
This simple derivation of the Levinson's theorem is just a pedagogical strategy to help us understand better the result. A more complete and rigorous derivation of this result in one dimension can be found in [62]. It was first introduced by [80] regarding the Schrödinger equation with a spherically symmetric potential and the zero angular momentum bound states. Later, some authors also studied this result in different cases, to cite few of them [81, 82], including cases with spin $\frac{1}{2}$ fermions scattering.

We also present a simple derivation of what we have used in (4.13) with this same example. We start with the same wave function after being scattered by the soliton as in (4.23) with same boundary conditions (4.21). Imposing the boundary condition on $\psi_k(L)$ we obtain:

$$k_n L + \delta(k_n) = n\pi \tag{4.27}$$

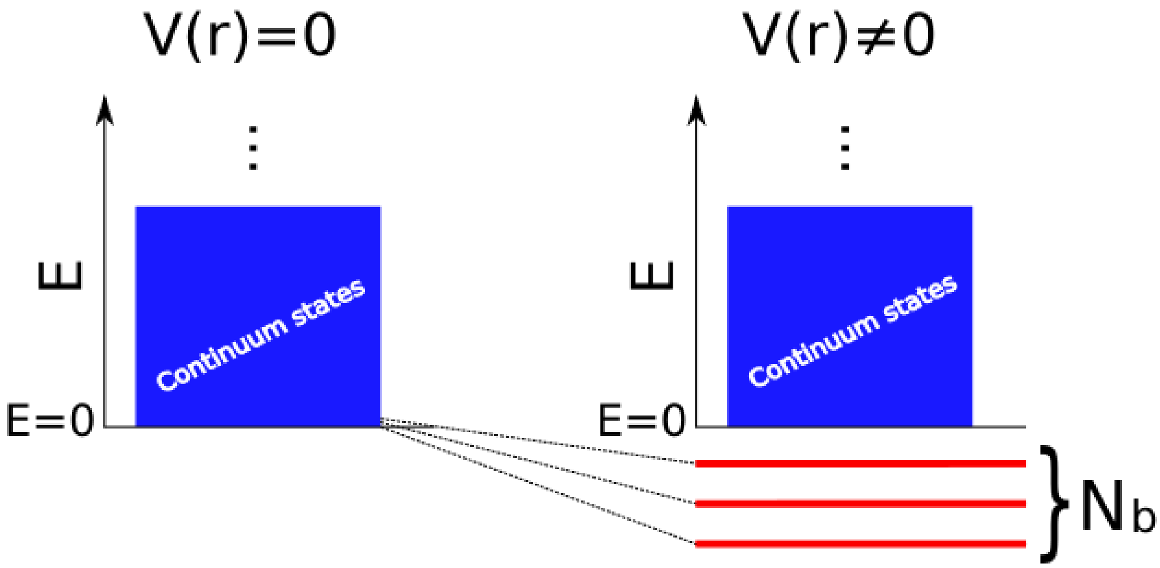

Figure 4.2. States from the continuum "peeling off" into the discrete part of the spectrum.

We can write this condition for consecutive values of n and calculate the difference between expressions, as

$$k_n L + \delta(k_n) = n\pi, \quad k_{n+1} L + \delta(k_{n+1}) = (n+1)\pi \tag{4.28}$$

subtracting the two expression gives

$$(k_{n+1} - k_n)\frac{L}{\pi} + (\delta(k_{n+1}) - \delta(k_n))\,\frac{1}{\pi} = 1 \tag{4.29}$$

One can realize that $(k_{n+1} - k_n)$ starts to get smaller as $L \to \infty$. The limit $L \to \infty$ means we are reaching continuum and if we divide our expression by $(k_{n+1} - k_n)$ we obtain:

$$\frac{L}{\pi} + \left(\frac{d\delta(k)}{dk}\right)\frac{1}{\pi} = \frac{1}{(k_{n+1} - k_n)} \tag{4.30}$$

The term in the right hand side is exactly the density of continuum states after being scattered by the soliton. As $L \to \infty$ we see that this density is becoming infinity. If we actually compare it with the free case, the difference between the results is finite and given by[6]:

$$\rho(k) - \rho_0(k) = \frac{1}{\pi}\frac{d\delta(k)}{dk} \tag{4.31}$$

[6] A more rigorous derivation of this result can be found in [62]

where ρ and ρ_0 are respectively the density of states in the presence and abscence of the potential. As we showed in our discussion about Casimir energy, this expression is a key result if one wants to compute this energy using the phase shift procedure.

4.2.2 Levinson theorem and fermion-soliton interaction

We start with presenting following the lagrangian

$$\mathcal{L} = \bar{\psi}(x,t)\left(i\partial\!\!\!/ - me^{i\gamma^5\phi(x)}\right)\psi(x,t) \tag{4.32}$$

In this model we are considering a fermion field coupled to a soliton where the soliton is too massive to "feel" the presence of the fermion field. Therefore, we only take into account the effect on of the fermion due to the presence of the soliton. We are using a representation of the Dirac matrices such that $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_3, \gamma^5 = \sigma_2$, where $\sigma_{1,2,3}$ are the Pauli matrices.
The field ϕ is a static field depending only on the position and constructed in the convenient way:

$$\phi(x) = \begin{cases} -\theta_0 & \text{for} \quad x \leq -l, \\ \mu x & \text{for} \quad -l \leq x \leq l, \\ +\theta_0 & \text{for} \quad l \leq x \end{cases} \tag{4.33}$$

This is the so-called SESM (Simple Exactly Solvable Model) [77]. Here $\mu = \frac{d\phi(x)}{dx}|_{x=0}$ and $\phi(x \to \pm\infty) = \pm\theta_0$ .This picewise field was first proposed by [77]. This proposed field has the advantage of enabling the system to be solved exactly. Furthermore, it showed to be a very didactic way of understanding also the Levinson theorem for a Dirac field coupled to a soliton in 1 spatial dimension .
If we suppose the following form for the Dirac field

$$\psi(x,t) = e^{-iEt}\begin{pmatrix}\psi_1 + i\psi_2 \\ \psi_1 - i\psi_2\end{pmatrix} = e^{-iEt}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}, \tag{4.34}$$

we obtain the equation of motion in term of the components of the field

$$(\partial_x^2 \mp i\phi'\partial_x + E^2 - m^2 + \phi' E)\xi_{1,2} = 0 \tag{4.35}$$

The equations of motion for the case of the scattering states are

$$\xi_0(x) = \begin{cases} a_0\begin{pmatrix} ime^{-i\frac{\theta_0}{2}} \\ (E+k)e^{i\frac{\theta_0}{2}}\end{pmatrix}e^{ik(x+l)} + b_0\begin{pmatrix} ime^{-i\frac{\theta_0}{2}} \\ (E-k)e^{\frac{i\theta_0}{2}}\end{pmatrix}e^{-ik(x+l)} & \text{for} \quad x \leq -l, \\ e_0\begin{pmatrix}(k_1+E)^{i\frac{x\mu}{2}} \\ -ime^{-i\frac{x\mu}{2}}\end{pmatrix}e^{\frac{-ix\zeta}{2}} + f_0\begin{pmatrix} ime^{i\frac{x\mu}{2}} \\ (k_1+E)e^{-\frac{ix\mu}{2}}\end{pmatrix}e^{i\frac{x\zeta}{2}} & \text{for} \quad -l \leq x \leq l, \\ c_0\begin{pmatrix}(E-k)e^{\frac{i\theta_0}{2}} \\ -ime^{\frac{-i\theta_0}{2}}\end{pmatrix}e^{ik(x-l)} & \text{for} \quad l \leq x, \end{cases} \tag{4.36}$$

Our approach is to calculate the S-matrix components. We realize that there are distinct S-matrix coefficients for each component of the spinor field. It means that there are two phase shifts and we need to introduce a procedure to associate these two phases to a unique phase which is in agreement with Levinson's theorem. The S-matrix can be calculated as [83]

$$S = -i\frac{m}{E+k}\frac{c_0}{a_0}e^{-2ikl}\begin{pmatrix} e^{i\phi l} \\ e^{-i\phi l} \end{pmatrix} \propto e^{i\delta(k)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.37}$$

where the proportionality factor is the transmition coefficient that is not of our interest in this considerations. In [83] the procedure to find the proper $\delta(k)$ in agreement with the Levinson's theorem is proposed:

$$\delta(k) = \frac{1}{2}(\delta^{up}(k) + \delta^{down}(k)) \tag{4.38}$$

where δ^{up} and δ^{down} are respectively the up and down components of the S-matrix coefficient. The phase shift of these scattering states possess the following limits, as reported by [83]

$$\delta^{sky}_{sea}(k) = \begin{cases} \frac{(2n+1)\pi}{2} & \text{for } \; k \to \infty \quad \text{(considering no threshold states)} \\ n\pi & \text{for } \; k \to \infty, \quad \text{(considering threshold states)} \\ \pm\theta_0 & \text{for } \; k \to 0 \quad \text{(always)} \end{cases} \tag{4.39}$$

To obtain this we should separate the real and imaginary part of (4.37) in order to compute its complex argument (if z is complex number $arg(z) = atan^{-1}(\frac{Imz}{Rez})$). We do it for each spinor component, and proceed as in (4.38) to take the limits of interest . We can also write the difference in total phase shift as:

$$\Delta\delta = \Delta\delta^{sea} + \Delta\delta^{sky} = [\delta^{sea}(0) - \delta^{sea}(\infty)] + \left[\delta^{sky}(0) - \delta^{sky}(\infty)\right] \tag{4.40}$$

resulting in

$$\Delta\delta = \left(N + \frac{N_t}{2}\right)\pi \tag{4.41}$$

where N_t is the number of threshold bound states in the presence of the background field. This threshold bound state can be understood as the following. As the strength of the potential increases (increasing θ_0) there is a threshold value of parameter that separates bound and continuum states. For this value of parameter the wave function is not normalizable, that is, it cannot be normalizable as a bound state neither as a free state [83]. A particular feature of this system is the presence of such states even in the absence of the background field. Then to write (4.41) properly we should subtract these states from the $\Delta\delta$ expression.

$$\Delta\delta = \left(N + \frac{N_t}{2} - \frac{N_0}{2}\right)\pi \tag{4.42}$$

Hence N_0 is the number of half-bound states in the absence of the soliton and N is the number of half-bound states in the presence of the soliton. In [83], the authors discusses about the Levinson's theorem including this threshold bound state in the computation of $\Delta\delta$. They show that the proposed expression in (4.38) is in agreement with the above equation. To get to this conclusion investigations were made concerning the energy spectrum of the bound states, for different values of θ_0 and μ parameters. To get full details on this discussion check the cited reference.

Up to now we have only discussed about the weak form of Levinson's theorem. It relates the value of phase shift for $k \to 0$ and $k \to \infty$ with the number of bound states. But there is a strong form of this theorem that relates the value of the entering bound states and exiting bound states in the spectrum as the strength of the potential increases, for each boundary separately [84].
We can write it as:

$$\delta(0) = (N_{exit} - N_{enter})\pi, \quad \delta(\infty) = (N_{enter} - N_{exit})\pi \tag{4.43}$$

This result also holds for each of the continua separately.
Finally we can see that the difference in the phase shift giving rise to Casimir energy can be rewritten in the following form with the help of the Levinson theorem:

$$\begin{aligned} E_{Casimir} = &+ \sum_i E^{i,\mathrm{sea}}_{\mathrm{bound}} + \int_0^{+\infty} \frac{\mathrm{d}k}{\pi} \frac{k}{\sqrt{k^2+M^2}} \left(\delta^{\mathrm{sea}}(k) - \delta^{\mathrm{sea}}(\infty)\right) + \\ &M(N + \frac{N_t}{2} - \frac{N_t^0}{2}) + \frac{M}{2} \end{aligned} \tag{4.44}$$

It is instructive to discuss a practical example and how Levinson's theorem agrees with the bound spectrum of the model.

We take for example the model we have been discussing, given by (4.32) and (4.33)

The result shown in Fig.4.3 is the spectrum of bound states formed by the interaction of the fermion field with the soliton field. This result is similar to what was obtained in [71] considering an infinitely thin soliton.

The spectrum where $\mu = 10$ is actually very interesting since we have, in the interval $0 < \theta_0 < 2\pi$, a bound state that "peels off" from the Dirac sky into the discrete part of the spectrum and after that "sinks" into the Dirac sea.

We study the bound spectrum for the case where $\mu = 10$ and $\theta_0 = \pi$. We obtain the number of induced bound states by directly counting them from the spectrum but also indirectly from information extracted from the phase shift. We have discussed already the following expression:

$$\Delta\delta \equiv [\delta_{\mathrm{sky}}(0) - \delta_{\mathrm{sky}}(\infty)] + [\delta_{\mathrm{sea}}(0) - \delta_{\mathrm{sea}}(\infty)] = \left(N + \frac{N_{\mathrm{t}}}{2} - \frac{N_{\mathrm{t}}^0}{2}\right)\pi, \tag{4.45}$$

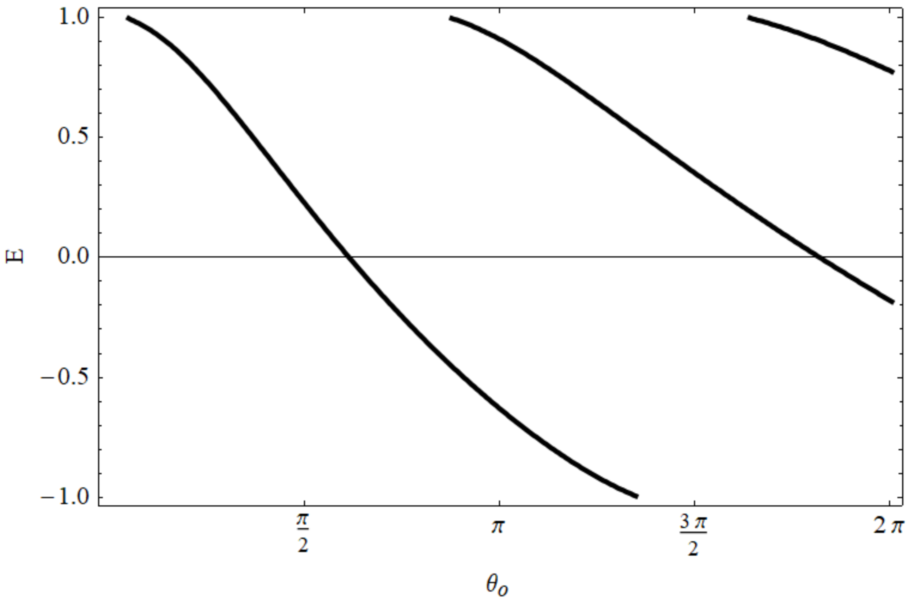

Figure 4.3. Bound state spectrum for $\mu = 10$ and $\theta_0 = \pi$.

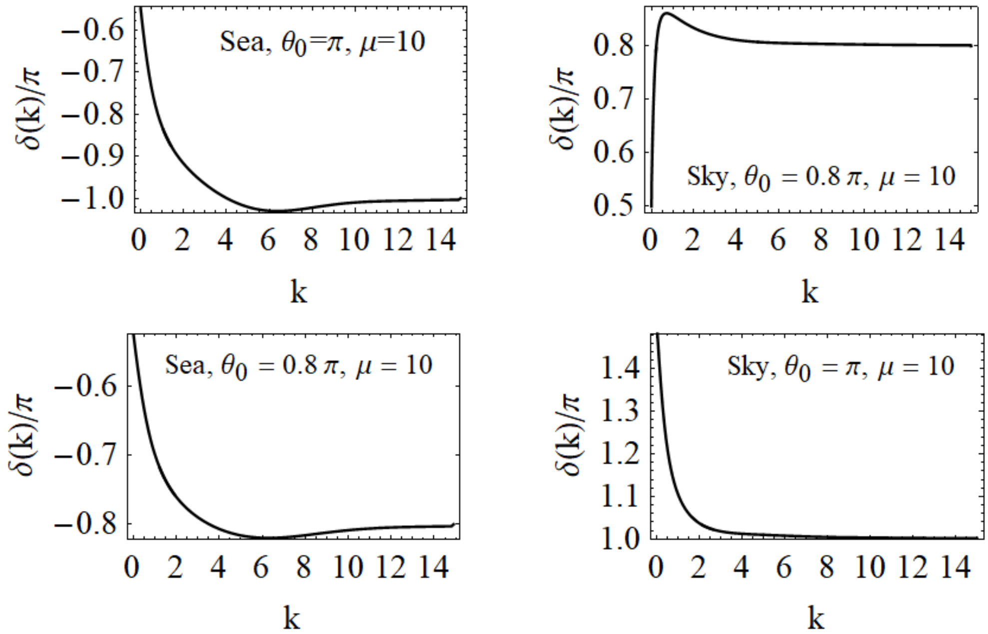

Figure 4.4. Phase shifts for the SESM model.

If we take a look at Fig.4.3 we see that for $\theta_0 = \pi$ there are two bound states. We also have two initial threshold bound states or half-bound states[7] ($N_0^t = 2$, these are always present) in the absence of the background field, but zero threshold state induced by the soliton ($N^t = 0$)[8]. Then the right hand side of (5.40) results in $\Delta\delta = \pi$. In order to compare this with the left hand side we should calculate the phase shifts induced in the fermion field after being scattered by the soliton, with the same parameters $\mu = 10, \theta_0 = \pi$. We have calculated this in Fig.4.4. With the phase shift one can compute the left hand side

[7] This is present particularly in unidimensional and bidimensional systems [85]

[8] To have an induced half-bound state it would be necessary to have a bound state being formed exactly at $\theta_0 = \pi$ which is not our case, but is what happens for $\theta_0 = 0.88\pi$

of (5.40). Which results in

$$\Delta\delta \equiv [\delta_{\rm sky}(0) - \delta_{\rm sky}(\infty)] + [\delta_{\rm sea}(0) - \delta_{\rm sea}(\infty)] \tag{4.46}$$
$$= [(1.5 - 1) + (-0.5 - (-1))]\,\pi = \pi \tag{4.47}$$

Of course the results from both sides must match. Another example for a different value of parameters can be calculated. Let us choose $\theta_0 = 0.8\pi$ and $\mu = 10$. We also check that our results are consistent.

$$[0.5 - 0.8 - 0.5 + 0.8]\,\pi = (1 + 0 - 1)\,\pi = 0 \tag{4.48}$$

There is also the strong form of Levinson's theorem [78]. In this case the number of bound states entering or leaving is what is associated with the phase shift, in the limits $k \to \infty$ and $k \to 0$.

$$\delta(0) = (N_{\rm exit} - N_{\rm enter})\,\pi, \quad \delta(\infty) = (N_{\rm enter} - N_{\rm exit})\,\pi. \tag{4.49}$$

This form of the result also works for each continua separately and computes the number of entering and exiting states from the continua. Let us also elaborate an example

Consider again $\theta_0 = \pi, \mu = 10$. This leads to $\delta(0) = (2 - \frac{1}{2})$ which is in agreement with Fig.4.3[9]. Where, for the positive energy spectrum we have $\delta^{sky}(0) = \left(N^{sky}_{\rm exit} - N^{sky}_{\rm enter}\right)\pi = (1 - \frac{1}{2})\pi = \frac{1}{2}\pi$[10]

Unfortunately this counting procedure only works for $\delta(0)$, it cannot be done for $\delta(\infty)$ case, check [84] and references therein.

4.2.3 Concluding remarks

The Casimir Energy, as we have demonstrated, can be non-zero when there is a topologically non-trivial background field in the configuration of the system. Casimir energy computations are present in many branches of soliton physics. For example, the Casimir energy contributes to the lowest order quantum correction to the mass of the soliton [55] and also make possible the study of the stability of the configuration for the coupled fields, [78]. Many authors also investigate the contribution of the Casimir Energy in the mass of the soliton in models concerning supersymmetric solitons [86, 87].Besdies that, Casimir effects are also present in models in quantum field theory that resembles QCD and QED in lower dimensions [88].

[9] The number of bound states exiting the in continuum is computed counting the number of continuum states that have "peeled off" from the Dirac sky into the region $-1 \leq \frac{E}{m} \leq 1$. In the case we have considered between $0 < \theta_0 < \pi$ there are two continuum states entering into the discrete part of the spectrum.

[10] The half terms appearing are due to the half bound states

5 New results

5.1 The Model

The model we have studied is based on a coupling dictated by the following Lagrangian:

$$\mathcal{L}_f = \bar{\psi}(x,t)\left(i\gamma^\mu\partial_\mu - M\mathrm{e}^{i\phi(x)\gamma^5}\right)\psi(x,t)\,. \tag{5.1}$$

This type of interaction is given by the well known chiral non-linear sigma model, the $\phi(x)$ field is a pseudoscalar, which means that this background field changes its sign under spatial coordinate inversion. But, together with the γ^5 Dirac matrix we guarantee that our system possesses parity symmetry. This is an interesting symmetry present in the model. The presence of this symmetry in our Lagrangian means that the Hamiltonian commutes with the parity operator and the eigenstates of the hamiltonian are also eigenstates of the parity operator.
If we consider separately the eigenvector with positive and negative parity we can write the time-independent Hamiltonian equation as:

$$H\psi_\pm = E_\pm\psi_\pm \tag{5.2}$$

Therefore, $P\psi_\pm(x) = \pm\psi_\pm(x)$. Meaning that

$$\begin{aligned}
[P,H]\psi_+(x) &= PH\psi_+(x) - HP\psi_+(x) \\
&= PE_+\psi_+(x) - H\psi_+(x) \\
&= E_+\psi_+(x) - E_+\psi_+(x) \\
&= 0
\end{aligned} \tag{5.3}$$

As mentioned before, this implies that eigenvectors of the parity operator are also eigenvectors of the Hamiltonian and other than working in the x interval $[-\infty,+\infty]$ we can start working with $[0,+\infty]$. But in order to take advantage of this symmetry we must find our parity operator, we proceed as following
We take the Dirac equation

$$(i\gamma^\mu\partial_\mu - Me^{i\gamma^5\phi})\psi = 0 \tag{5.4}$$

and factor out the time dependent part of the wave function as

$$\psi = \psi e^{-iEt} \tag{5.5}$$

We rewrite (5.4) and multiply it by Pγ^0

$$P\gamma^0\left(\gamma^0 E + i\gamma^i\partial_i - Me^{i\gamma^5\phi}\right)\psi = 0 \tag{5.6}$$

We obtain

$$P(i\gamma^0\gamma^i\partial_i - M\gamma^0 e^{i\gamma^5\phi})\psi = -EP\psi \tag{5.7}$$

The system have parity symmetry, it means that $P\psi$ is also a solution of the Dirac equation. If we write (5.4) for $P\psi$ we obtain

$$(-i\gamma^0\gamma^i\partial_i - M\gamma^0 e^{-i\gamma^5\phi(-x)})P\psi = -EP\psi \tag{5.8}$$

Then if we compare (5.8) and (5.27) we conclude

$$P\gamma^0\gamma^i = -\gamma^0\gamma^i P, \quad P\gamma^0 e^{i\phi\gamma^5} = \gamma^0 e^{-i\phi\gamma^5} P \tag{5.9}$$

We can look to the last condition and rewrite $e^{i\gamma^5\phi}$ as

$$P\gamma^0(\cos\phi + i\gamma^5\sin\phi) = \gamma^0(\cos\phi - i\gamma^5\sin\phi)P \tag{5.10}$$

in order to obtain

$$P\gamma^0 = \gamma^0 P, \quad P\gamma^0\gamma^5 = -\gamma^0\gamma^5 P \tag{5.11}$$

to find that, P must be

$$P = \gamma^0 \tag{5.12}$$

In the next section we will use the parity operator to set our boundary conditions for the parity states.
Besides that we assume that the background field is a solution of the equation of motion arising from the given Lagrangian:

$$\mathcal{L}_b = \frac{1}{2}\left(\partial_\mu\phi\left(x\right)\right)^2 - \frac{\mu^2}{2\theta_0^{4n}}\left(\phi^{2n}\left(x\right) - \theta_0^{2n}\right)^2 \tag{5.13}$$

where $\mu = \frac{d\phi(x)}{dx}\Big|_{x=0}$, $\phi\left(\pm\infty\right) = \pm\theta_0$ and n belongs to the set of natural numbers. The solutions of the equations of motion dictated by this potential are called compactons[1]. We plot some numerical solutions for different values of n in Fig. 5.1

[1]We use this name here, because as n increases we see our soliton getting more "compact" that is, the soliton acquires finite span.

Hence the complete Lagrangian of our model has the following form:

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f \tag{5.14}$$

The type of interacting present in $\mathcal{L}_f$ is called chiral non-linear σ model (ChNLM). It was first proposed by Feza Gürsey [69] when studying the symmetries of strong and weak interactions.

There are other classes of non-linear sigma models (NLσM) and those has been widely studied. With investigations in cosmology [89], particle physics [69], string theory [90], condensed matter physics [55] and other fields. Particularly, in particle physics, ChNLM models are present in the theory of Skyrmions [91] and chiral bag models which are important theories to describe low energy QCD [92].

There are also, investigations applying ChNLM models to describe the one dimensional version of topological superconductivity [70]. Based on the rich physics this model has provided and also following the investigations in the literature on fermion soliton interactions [78, 93, 77] we consider the model described by the Lagrangian in (5.1). The profile of the prescribed soliton is also of great interest. The model we are studying here, with a free n parameter, is also an interesting one compared to previous works [93, 78, 77]. It actually interpolates the SESM profile for the background field and the kink one as we change n. This is interesting because it enables us to investigate the properties of both models when increasing n. Particularly, we are interested in the behavior of the Casimir energy for each parity channel. Investigations reported an oscillating behavior of the Casimir energy for the SESM model, see for example Fig. 5 of [93].
Our goal is also to study how this behavior changes as we start with the kink model ($n = 1$) and increase n toward the SESM model ($n \to \infty$). This will help us understand this unique oscillatory behavior of $E_{casimir}$, for each parity channel.

In summary this $1 + 1$ dimensional model presents the more fundamental physical properties arising from the coupling of solitons with fermions. Such properties can be of great importance in enlightening future investigations of such kind of couplings in higher dimensional systems. Throughout this work we compare our results with previous results cited earlier [78, 93], for the limiting cases $n = 1, n \to \infty$.

5.2 Equations of motion

The precise way of solving the equations of motion for this coupled system is to write the complete Lagrangian of the problem and solve the equations dynamically, which is too complicated. In fact, we perform some simplifications. In the same fashion as the authors of [77], we are interested only in the effect on the fermion field due to the exsitence of the soliton. This means that we solve first the equations of motion arising from $\mathcal{L}_b$ and assume ϕ to be a background field in $\mathcal{L}_f$. This approach is chosen since the mass of the soliton is supposed to be much bigger than the mass of the fermion field (weak coupling regime) which means that the pertubation induced in the soliton due to the presence of the fermion is almost negligible.

Since our system is not analytically solvable we need to use a numerical method to find the solutions of the following equation of motion for the fermion field. Following the literature we choose the representation to be $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_3, \gamma^5 = \gamma^0\gamma^5 = \sigma_2$.

$$i\sigma_1\partial_t\psi - \sigma_3\partial_x\psi - M\left[\cos\phi(x) + i\sigma_2\sin\phi(x)\right]\psi = 0 \tag{5.15}$$

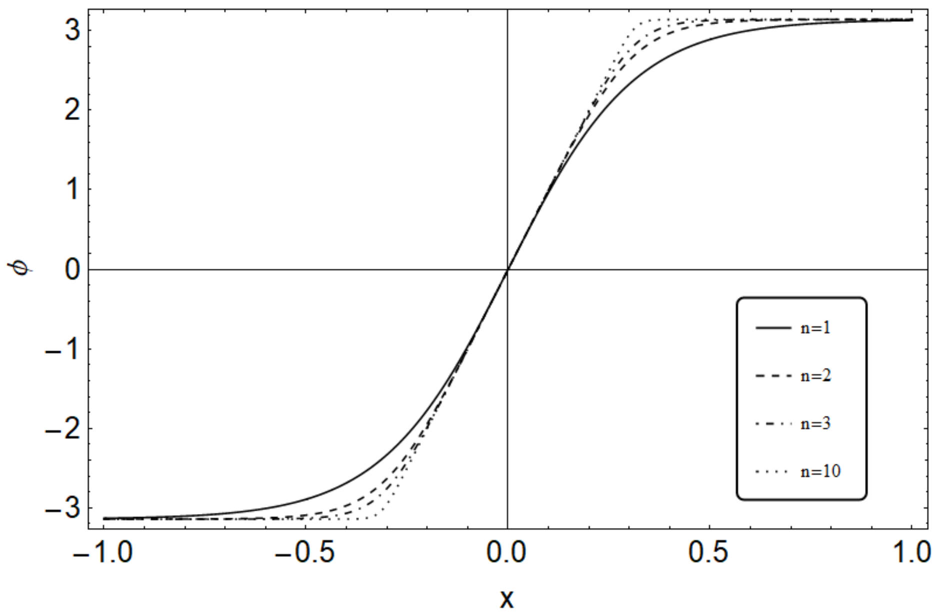

Figure 5.1. Different profiles of solitons arising from (5.13) for several n. We have used $\mu = 10$ and $\theta_0 = \pi$

where ψ is a two component spinor in the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{5.16}$$

Following literature we define new variables:

$$\psi(x,t) = e^{-iEt}\psi(x) = e^{-iEt}\begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_1 - i\psi_2 \end{pmatrix} = e^{-iEt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi(x,t) \tag{5.17}$$

which leads us to the following equations of motion

$$\begin{pmatrix} i\partial_x - E & iMe^{i\phi(x)} \\ -iMe^{-i\phi(x)} & -i\partial_x - E \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.18}$$

For computational purposes we write explicitly the real and imaginary parts of each component of the spinor field as:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} y_1 + iy_2 \\ y_3 + iy_4 \end{pmatrix} \tag{5.19}$$

Besides that we work with the dimensionless parameters in the equation of motion using the mass of the fermion as the reference scale considering $x^\mu \to Mx^\mu$, $\psi \to \frac{\psi}{\sqrt{M}}$, $\phi \to \phi$, $E \to \frac{E}{\sqrt{M}}$, which gives:

$$y_1^{'} + \cos\phi\,(x)\,y_3 - Ey_2 - \sin\phi\,(x)\,y_4 = 0 \quad (5.20)$$
$$y_2^{'} + \cos\phi\,(x)\,y_4 + Ey_1 + \sin\phi\,(x)\,y_3 = 0 \quad (5.21)$$
$$y_3^{'} + \cos\phi\,(x)\,y_1 + Ey_4 + \sin\phi\,(x)\,y_2 = 0 \quad (5.22)$$
$$y_4^{'} + \cos\phi\,(x)\,y_2 - Ey_3 - \sin\phi\,(x)\,y_1 = 0 \quad (5.23)$$

At this point we take advantage of the Runge Kutta Fehlberg method of order 5 to find the bound states of the system. We proceed as following: After rescaling the energy with the fermion mass, it can assume any real value in the interval $[-1, 1]$. The method of solving these equations is based on choosing the lowest bound value of the energy ($E = -1$) to solve the coupled equations of motion and with small steps raise the energy value until it reaches $E = 1$. For each value of energy we assert to solve the equations of motion we obtain y_1, y_2, y_3, y_4 and for each step of the numerical algorithm we calculate $sum = y_1^2 + y_2^2 + y_3^2 + y_4^2$ which is our numerical way of determining which are the normalizable solutions. To clearly see the values of energy for which we obtain normalizable solutions we plot $\ln(sum)$ versus E, check Fig. (5.4). The deep valleys present in the graphic profile are the values for which our sum variable "converge". Those are the eigenvalues of energy. This method was first adapted by A.Mohammadi and A.Amado in a previous work.

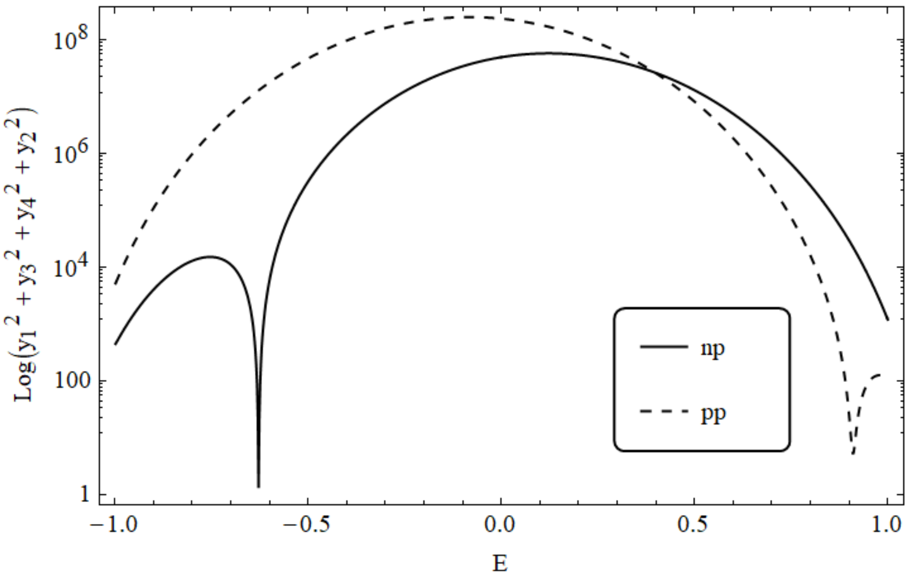

Figure 5.2. Example of the algorithm used for finding the bound state energies. For this example we choose $\mu = 10$, $\theta_0 = \pi$, $n = 1$ and the valleys are centered at $E = -0.628000$ and $E = 0.910000$. We have calculated these eigenvalues for each parity, positive (pp) and negative (np), of the eigenstates.

We solve the spectrum of energy as a function of μ and θ_0 parameters. We repeat the procedure explained above for each value of these parameters to obtain the bound energy spectrum shown in Fig. 5.4 and Fig. 5.5

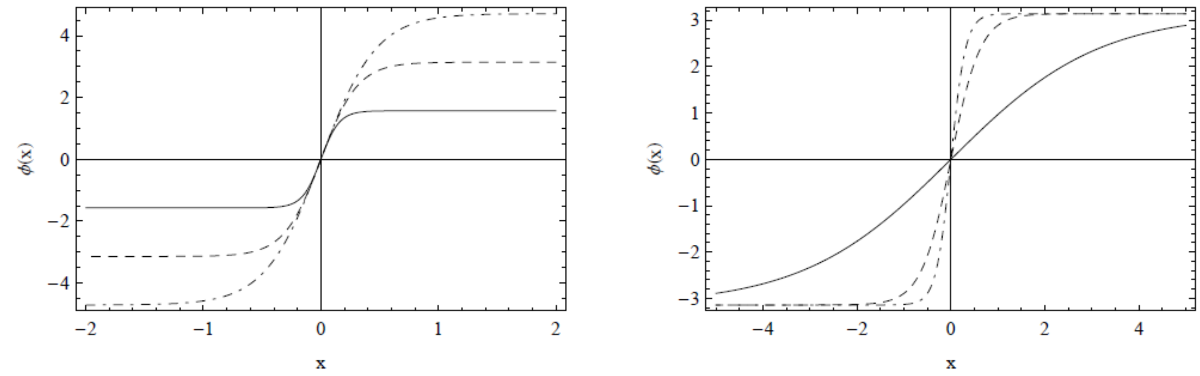

Figure 5.3. In the left graph we have the profile of the soliton for $n = 1$ fixed $\mu = 10$ and $\theta_0 = 0.5\pi, \pi, 1.5\pi$ in solid, dashed and dotdashed lines respectively. In the right graph we have the profile of the soliton $n = 1$ for fixed $\theta_0 = \pi$ and $\mu = 1, 5, 10$ in solid, dashed and dotdashed line respectively.

In order to solve these equations of motion we have used boundary conditions determined by the parity symmetry of our system. After the transformation (5.17) we obtain the following representation for the Dirac matrices, $\gamma^0 = -\sigma_2, \gamma^1 = i\sigma_1, \gamma^5 = \gamma^0\gamma^1 = -\sigma_3$ Where we can say that

$$P\xi_{\pm}(x,t) = \pm\xi_{\pm}(x,t) \tag{5.24}$$

If we have $\xi_+ = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$, this should satisfy

$$-\sigma_2 \begin{pmatrix} f(-x) \\ g(-x) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{5.25}$$

and if we evaluate $f(x)$ and $g(x)$ at $x = 0$

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} \tag{5.26}$$

we can conclude that

$$\xi_+(0) \propto \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{5.27}$$

With the same reasoning

$$\xi_-(0) \propto \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{5.28}$$

It is important to mention that we have faced a numerical difficulty when solving the bound state energies for small μ. In this case our soliton is very broad and converges very slowly to the limiting value θ_0. As a result it takes more time to find the bound energies and numerical errors starts to stack. Even though Fig 5.5 matches the results in [78]

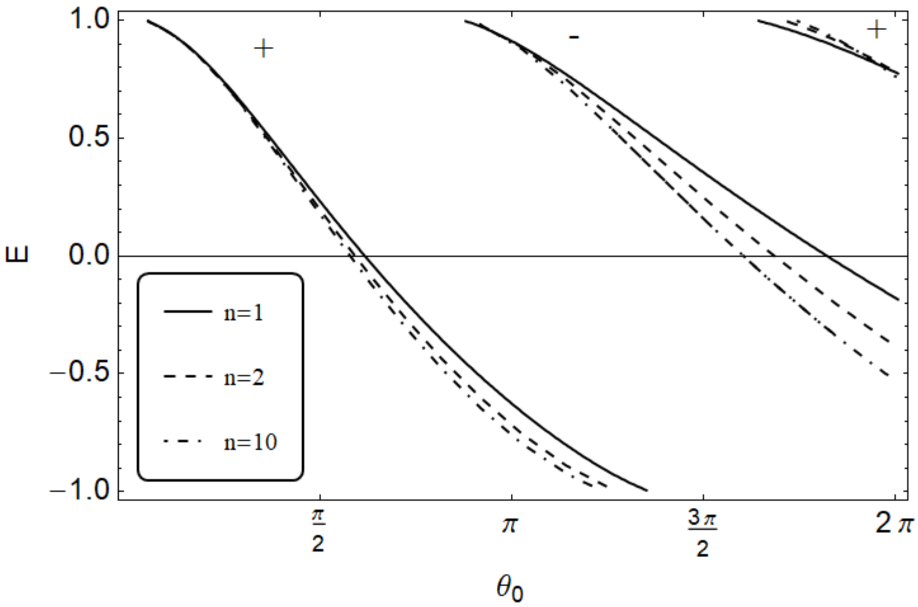

Figure 5.4. Bound state energies as a function of the parameter θ_0, for three values of n. The μ parameter is set at $\mu = 10$. The $+$ and $-$ signs labels the bound states found depending on the boundary condition we have used, (5.27) or (5.8)

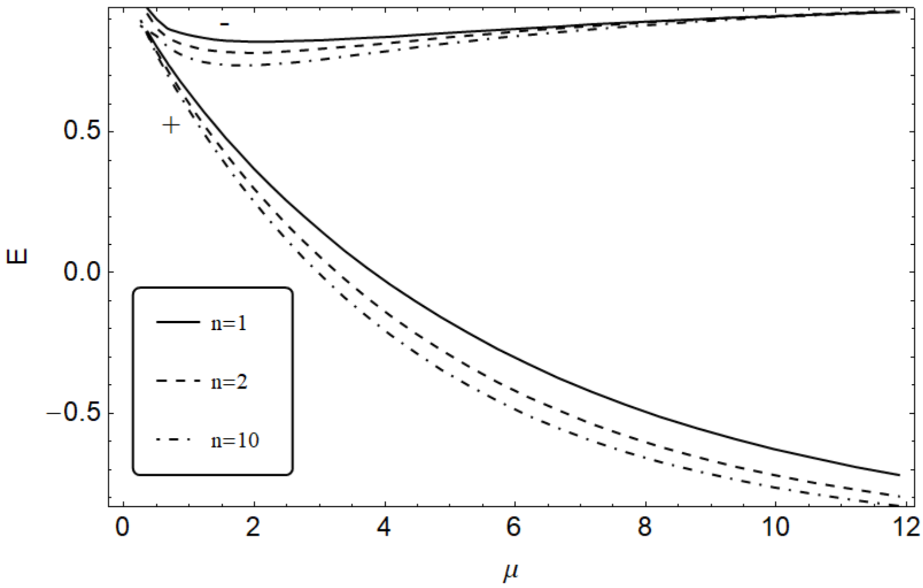

Figure 5.5. Bound state energies as a function of the slope of the field μ, for three values of n. The θ_0 parameter is set as $\theta_0 = \pi$. The $+$ and $-$ signs labels the bound states found depending on the boundary condition we have used, (5.27) or (5.8)

5.3 Scattering states

We have also calculated the fermion wave function scattered by the soliton. We have considered an incident plane wave from the left, this wave is partially transmitted but also partially reflected, and have the following form

$$\xi_k(x) = \begin{pmatrix} y_1(x) + iy_2(x) \\ y_3(x) + iy_4(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix} \mathrm{e}^{-ikx} + \begin{pmatrix} b_1 + ib_2 \\ b_3 + ib_4 \end{pmatrix} \mathrm{e}^{ikx} & \text{for} \quad x \to -\infty \\ \begin{pmatrix} z_1(x) + iz_2(x) \\ z_3(x) + iz_4(x) \end{pmatrix} & \text{for} \quad \text{finite } x \\ \begin{pmatrix} c_1 + ic_2 \\ c_3 + ic_4 \end{pmatrix} \mathrm{e}^{ikx} & \text{for} \quad x \to +\infty \end{cases} \tag{5.29}$$

For computational purposes we can factorize the oscillatory factor present in $\xi_k(x)$.

$$\xi_k(x) = \mathrm{e}^{ikx}\eta_k(x) = \mathrm{e}^{ikx} \begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ \eta_3(x) + i\eta_4(x) \end{pmatrix} \tag{5.30}$$

This will allow us to easily extract the relevant coefficients in (5.29). Then, inserting (5.30) in our equations of motion leads to

$$\eta_1' + \cos\phi\,(x)\,\eta_3 - (E+k)\eta_2 - \sin\phi\,(x)\,\eta_4 = 0 \tag{5.31}$$
$$\eta_2' + \cos\phi\,(x)\,\eta_4 + (E+k)\eta_1 + \sin\phi\,(x)\,\eta_3 = 0 \tag{5.32}$$
$$\eta_3' + \cos\phi\,(x)\,\eta_1 + (E-k)\eta_4 + \sin\phi\,(x)\,\eta_2 = 0 \tag{5.33}$$
$$\eta_4' + \cos\phi\,(x)\,\eta_2 - (E-k)\eta_3 - \sin\phi\,(x)\,\eta_1 = 0 \tag{5.34}$$

where we choose the boundaries conditions to be dictated by the same problem analytically solved in (4.36). At large x the scattered states will have the values

$$\begin{aligned} \eta_1(\infty) &= c_0(E-k)\cos\left(\frac{\theta_0}{2} - k\frac{\theta_0}{\mu}\right), & \eta_2(\infty) &= c_0(E-k)\sin\left(\frac{\theta_0}{2} - k\frac{\theta_0}{\mu}\right) \\ \eta_3(\infty) &= c_0\sin\left(-\frac{\theta_0}{2} - k\frac{\theta_0}{\mu}\right), & \eta_4(\infty) &= -c_0\cos\left(-\frac{\theta_0}{2} - k\frac{\theta_0}{\mu}\right) \end{aligned} \tag{5.35}$$

with $c_0 = ((k_1+E)^2-1)$, $k_1 = \frac{1}{2}(\mu+\zeta)$ and $\zeta = \sqrt{(\mu^2 - 4(1-E^2-\mu E))}$, all parameters defined in the SESM model. Knowing this we numerically integrate backwards to obtain Fig. 5.6

One can see that increasing n we obtain an oscillation amplitude more proeminent than the two other cases. It means that the fermion wave function is more reflected in the $n=10$ than in the other $n=1,2$ cases, since the oscillatory behaviour in (5.30) comes from the term of the reflected wave (this can be noticed more clearly in 5.37). Here we are adopting that we reach the SESM model for $n=10$, since in this case our result is barely distinguishable from the SESM ($n\to\infty$). Particularly, the kink ($n=1$) case is the one that less reflects the wave function

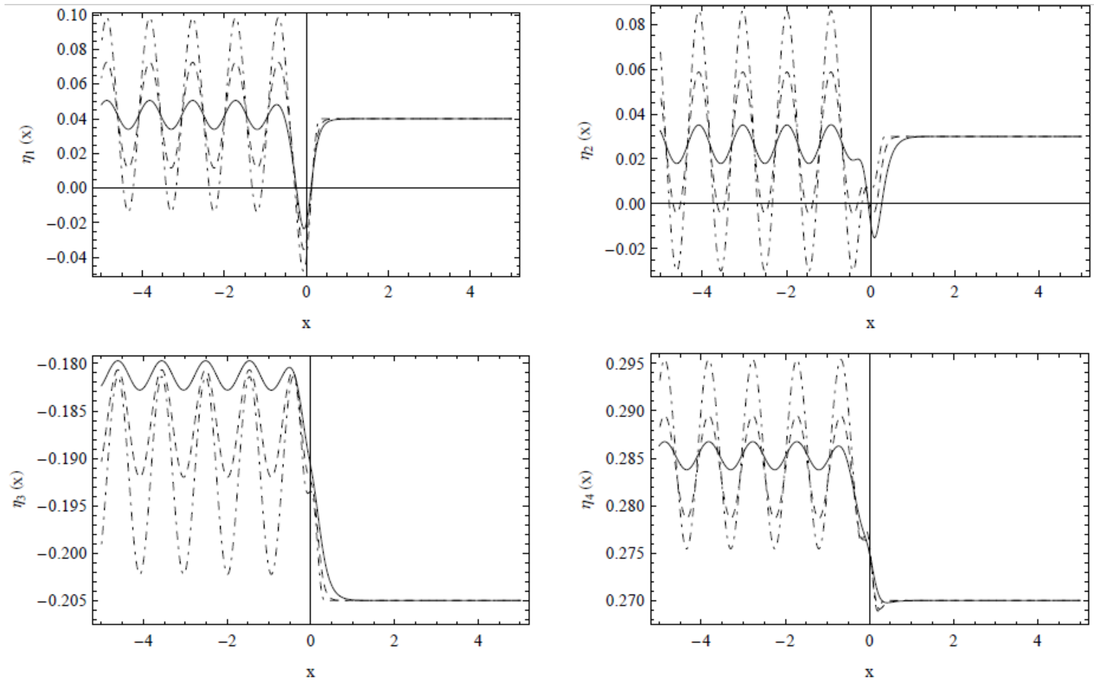

Figure 5.6. These are the scattered wave functions of the fermion induced by the soliton for the following parameters' values $k = 3.0$, $\theta_0 = \pi$, $\mu = 10$ and $E = \sqrt{k^2 + M^2}$. The cases $n = 1$, $n = 2$ and $n = 10$ are given respectively by solid, dashed and dotdashed lines.

5.4 Phase shift

It is also instructive to study the phase shift of the scattering process. For doing so we follow the procedure indicated by S.Gousheh in [83] which is in agreement with Levinson's theorem.
The phase shift is given by :

$$\delta(k) = \frac{1}{2}\left(\tan^{-1}\left|\frac{c_1 + ic_2}{b_1 + ib_2}\right| + \tan^{-1}\left|\frac{c_3 + ic_4}{b_3 + ib_4}\right|\right) \tag{5.36}$$

where the both terms comes from the S-matrix element, here we are averaging up and down component of the S-matrix element spinor as explained in (4.38). We have determined already the coefficients c_1, c_2, c_3, c_4, those are $\eta_1(\infty), \eta_2(\infty), \eta_3(\infty), \eta_4(\infty)$, respectively. In order to obtain the coefficients left we need to proceed as following.

We insert η_k in the equations of motion

$$\eta_k = \begin{pmatrix} \eta_1 + i\eta_2 \\ \eta_3 + i\eta_4 \end{pmatrix} = \begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix} e^{-i2kx} + \begin{pmatrix} b_1 + b_2 \\ b_3 + ib_4 \end{pmatrix} \tag{5.37}$$

and realize

$$\begin{aligned} \eta_1^{'} &= 2k\left(\eta_2 - b_2\right), \quad & \eta_2^{'} &= -2k\left(\eta_1 - b_1\right) \\ \eta_3^{'} &= 2k\left(\eta_4 - b_4\right), \quad & \eta_4^{'} &= -2k\left(\eta_3 - b_3\right) \end{aligned} \tag{5.38}$$

where $\eta_i^{'}(x) = \frac{d\eta_i(x)}{dx}$. Using (5.38) we obtain 4 equations and 4 unknowns where we can solve for b_1, b_2, b_3, b_4.
When calculating the scattering states of the sea, for some values of k we realize that our variables contributing to the real and imaginary parts gets mixed. The source of this issue relies in the c_0 term which depends on ζ and of course the values we choose for μ. We have found that for values of momentum between k_1^* and k_2^*

$$k_1^* = +\sqrt{\left(\frac{\mu}{2} - 1\right)^2 - 1}, \quad k_2^* = +\sqrt{\left(\frac{\mu}{2} + 1\right)^2 - 1} \tag{5.39}$$

We need to separate again real and imaginary contributions. This comes from studying the sign of the following equation $\mu^2 - 4(1 - E^2 - \mu E) = 0$ which is the same as $\zeta = 0$. After that one can calculate the phase shift for the Dirac sea and sky.

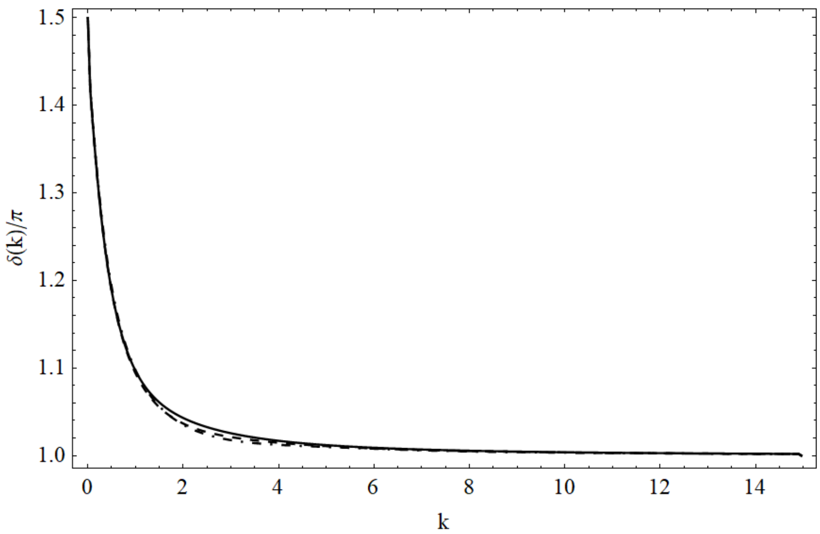

Figure 5.7. Phase shift for the Dirac sky, $E = \sqrt{k^2 + M^2}$, for $\mu = 10$ and $\theta_0 = \pi$ parameter values. We have $n = 1, 2, 10$ in the solid, dashed and dotdashed lines respectively. Note that $\delta(0)/\pi = \frac{3}{2}$ and $\delta(\infty)/\pi = 1$

Note that despite that the phase shift for the Dirac sky Fig.5.7 does not show us significant differences for the models we are considering $n = 1, 2, 10$ the phase shift for the sea does, check Fig. 5.8. This differences will for sure be reflected in the Casimir energy (4.44).

We can use Levinson's theorem to check the consistency of our results. Let us take a look at Fig.5.8 and Fig.5.7. We use (5.40) to conclude that for the case of $\mu = 10$ and $\theta_0 = \pi$:

$$\Delta\delta \equiv \left[\frac{3}{2} - 1\right] + \left[-\frac{1}{2} - (-1)\right] = \left(2 + 0 - \frac{2}{2}\right)\pi = \pi, \tag{5.40}$$

The results for the phase shifts are in complete agreement with the Levinson's theorem. In the continuation of this work we plan to find the Casimir energy for two different cases: fixing the slope of the background field, μ, and changing the asymptotic value of the field, θ_0, and vice versa. Besides that we plan to study the Casimir energy for each parity.

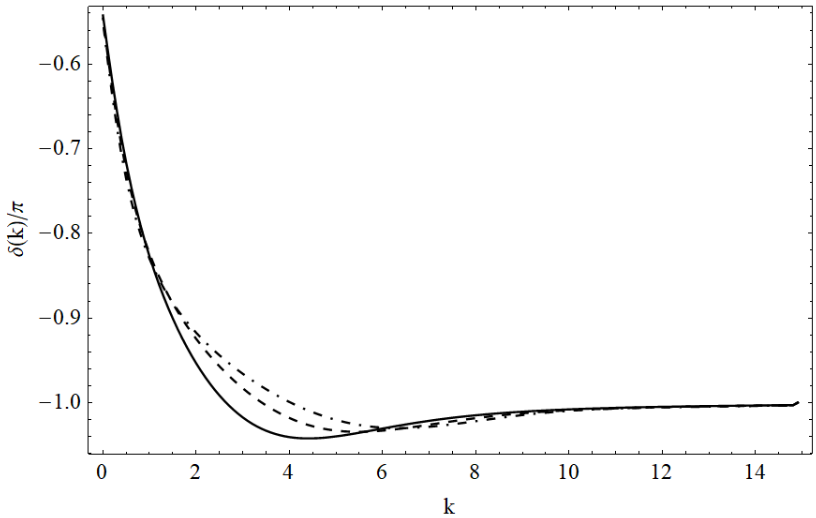

Figure 5.8. Phase shift for the Dirac sea, $E = -\sqrt{k^2 + M^2}$, for $\mu = 10$ and $\theta_0 = \pi$ are the parameter values. We have $n = 1, 2, 10$ in the solid, dashed and dotdashed lines respectively. Note that $\delta(0)/\pi = -\frac{1}{2}$ and $\delta(\infty)/\pi = -1$

6 Conclusion, remarks and perspectives

In this thesis we have studied the concept of soliton. A well behaved, finite energy solution that arises in many contexts in physics due to the interplay of nonlinearity and dissipation that stabilizes this structure. In the introduction we have started from a historical point of view highlighting the first considerations that yielded to understand the properties of this new structure or "wave" compared to what was known at the time of J. S. Russel[2]. Later we briefly discussed how nonlinearity also brought unexpected results besides solitons in hidrodynamics modeled by KdV equation. The FPUT problem that amazed Fermi Pasta, Ulam and Tsingou showing that the equipartition of energy do not hold when considering nonlinear oscillators. Also we have cited other soliton-like structures such as skyrmions and cosmic strings that play important role in nowadays physics research.
In chapter II we have discussed and derived thoroughly general properties of soliton in $(1+1)$-dimensions, deriving an important theorem known as the Derrick's theorem. We have also introduced soliton solutions in higher dimensions. We presented the concept of monopoles, first derived by Dirac searching for symmetric solutions of the Maxwell's equations at the level of E, B fields. We have also considered monopoles in the Yang-Mills theory of the Georgi-Glashow model and discussed briefly the relation between these two structures. We discussed vortices and instantons. Particularly, we have shown how instantons can be introduced from a discussion of periodic potentials in non-relativistic quantum mechanics and how they describe QCD vacuum.
In the chapter III we presented the main focus of this study, the interaction of fermions with solitons. We have shown an approach to this problem using the formalism of path integral and the tools we developed up to this point. We rederived the prominent result of Jackiw and Rebbi of a fractional fermion number induced in the soliton field due to the existence of a zero mode in the spectrum of the coupled system. We also discussed the works that introduced the adiabatic and non-adiabatic contributions to the fermion charge. For the adiabatic contribution we have derived the well known Goldstone-Wilczek current and discussed the main steps of this derivation. In the chiral limit $(m \to 0)$ we have also recovered the result of Jackiw and Rebbi. Moreover we investigated an specific example of non-adiabatic contribution induced by an infinitly thin soliton and have written the fermion number operator as a function of the quantum field operators. We provided a detailed discussion on how the fermion number changes as a function of the vacua of the soliton. In chapter IV we discussed Casimir energy and Levinson's theorem and related them to the phase shift of the fermion field scattered by the soliton.
In chapter IV we have provided a historical introduction to the Casimir energy and we derived this result using the phase shift approach for the case of fermion-soliton interaction. We discussed the weak and strong form of Levinson's theorem and performed some examples.
In chapter V we presented the system we have been working with and report our recent

results. We have stressed that an important symmetry of our system is the parity symmetry and we have used this as a way to choose our boundary conditions for the equations of motion. We have discussed the bound states with numerical details on how to calculate such bound energies and also discussed limitations of our method for certain values of parameters. We have discussed the scattering states and phase shifts induced by the presence of the soliton and have raised some expectations on the profile for the Casimir energy. Our future perspectives relies on calculating the Casimir Energy as depending on θ_0 and μ separately. Adopting different values of n parameter and providing discussion regarding the same values of n we have been considering. In order to achieve this we need to choose a numerical integration method that can compute the integral in (4.44) of the phase shift. Besides that we plan to calculate the parity states of our system. To obtain that we take advantage again of the parity symmetry and remember that the eigenfunctions of the parity operator are also eigenfunctions of the Hamiltonian operator. Previous works [83] have found that these eigenfunctions are related as $\xi_{\pm}^{scatt}(x) = \xi_{k}^{scatt}(x) + \xi_{-k}^{scatt}(x)$ with $P\xi_{k}^{scatt}(x) = \xi_{-k}^{scatt}(x)$ and where ξ^{scatt} and $\xi_{\pm}^{scatt}$ are the scattering and parity eigenfunctions respectively. After computing these entities we plan to compute the phase shift and Casimir energy for each parity channel. Investigating the Casimir energy for each parity channel is one of the main goals of this work, we will calculate this result for the case $n = 2$ and expect it to be an enlightening result since it is a model between the known kink profile and the SESM.

References

[1] EDISON, Thomas A. Genius is one percent inspiration and ninety-nine percent perspiration. *Quoted in Harper's Monthly Magazine*, 1932.

[2] JE Allen. The early history of solitons (solitary waves). *Physica Scripta*, 57(3):436, 1998.

[3] Peter J Olver. *Applications of Lie groups to differential equations*, volume 107. Springer Science & Business Media, 2000.

[4] J. Boussinesq. "théorie de l'intumescence liquide, applelée onde solitaire ou de translation, se propageant dans un canal rectangulaire". *Comptes Rendus de l'Académie des Sciences.*, 72:755–759.

[5] Xxxii. on waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1(4):257–279, 1876.

[6] Diederik Johannes Korteweg and Gustav De Vries. Xli. on the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 39(240):422–443, 1895.

[7] Shunichi Tanaka. Some remarks on the modified korteweg-de vries equations. *Publications of the Research Institute for Mathematical Sciences*, 8(3):429–437, 1972.

[8] Tuncay Aktosun. Inverse scattering transform, kdv, and solitons. In *Current trends in operator theory and its applications*, pages 1–22. Springer, 2004.

[9] Alessandro Torrielli. Classical integrability. *Journal of Physics A: Mathematical and Theoretical*, 49(32):323001, 2016.

[10] E Infeld, A Karczewska, G Rowlands, and P Rozmej. Exact cnoidal solutions of the extended kdv equation. *Acta Physica Polonica, A.*, 133(5), 2018.

[11] Mason A Porter, Norman J Zabusky, Bambi Hu, and David K Campbell. Fermi, pasta, ulam and the birth of experimental mathematics: A numerical experiment that enrico fermi, john pasta, and stanislaw ulam reported 54 years ago continues to inspire discovery. *American Scientist*, 97(3):214–221, 2009.

[12] Giovanni Gallavotti. *The Fermi-Pasta-Ulam problem: a status report*, volume 728. Springer, 2007.

[13] Joseph Ford. The fermi-pasta-ulam problem: Paradox turns discovery. *Physics Reports*, 213(5):271 – 310, 1992.

[14] Eli Galanti, Yohai Kaspi, Frederik J Simons, Daniele Durante, Marzia Parisi, and Scott J Bolton. Determining the depth of jupiter's great red spot with juno: a slepian approach. *The Astrophysical Journal Letters*, 874(2):L24, 2019.

[15] SV Antipov, MV Nezlin, EN Snezhkin, and AS Trubnikov. Rossby autosoliton and stationary model of the jovian great red spot. *Nature*, 323(6085):238, 1986.

[16] SV Antipov, MV Nezlin, EN Snezhkin, and AS Trubnikov. Models of the great red spot. *Nature*, 343(6258):517–517, 1990.

[17] Carl-Gustaf Rossby. Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. mar. Res.*, 2:38–55, 1939.

[18] SV Antipov, MV Nezlin, EN Snezhkin, and AS Trubnikov. Rossby solitons in the laboratory. 1982.

[19] T.H.R. Skyrme. A unified field theory of mesons and baryons. *Nuclear Physics*, 31:556 – 569, 1962.

[20] Gerald Edward Brown and Mannque Rho. *The multifaceted skyrmion*. World Scientific, 2010.

[21] Olaf Lechtenfeld and Alexander D. Popov. Skyrme and faddeev models in the low-energy limit of 4d yang–mills–higgs theories. *Nuclear Physics B*, 945:114675, 2019.

[22] T W B Kibble. Topology of cosmic domains and strings. *Journal of Physics A: Mathematical and General*, 9(8):1387–1398, aug 1976.

[23] Alexander Vilenkin and E Paul S Shellard. *Cosmic strings and other topological defects*. Cambridge University Press, 2000.

[24] Barton Zwiebach. *A first course in string theory*. Cambridge university press, 2004.

[25] Pierre Auclair, Jose J. Blanco-Pillado, Daniel G. Figueroa, Alexander C. Jenkins, Marek Lewicki, Mairi Sakellariadou, Sotiris Sanidas, Lara Sousa, Daniele A. Steer, Jeremy M. Wachter, and Sachiko Kuroyanagi. Probing the gravitational wave background from cosmic strings with lisa, 2019.

[26] Luiz Agostinho Agostinho Ferreira, Pawel Klimas, Andrzej Wereszczynski, and Wojtek J Zakrzewski. Some comments on bps systems. *Journal of Physics A: Mathematical and Theoretical*, 2019.

[27] M. Shifman and A. Yung. Supersymmetric solitons. *Rev. Mod. Phys.*, 79:1139–1196, Oct 2007.

[28] Mark Kac. Review: P. m. morse and h. feshbach, methods of theoretical physics. *Bull. Amer. Math. Soc.*, 62(1):52–54, 01 1956.

[29] Edmond Bour. *Theorie de la deformation des surfaces.* [Paris], 1861.

[30] J Frenkel and T Kontorova. On the theory of plastic deformation and twinning. *Izv. Akad. Nauk, Ser. Fiz.*, 1:137–149, 1939.

[31] Leon Armenovich Takhtadzhyan and Lyudvig Dmitrievich Faddeev. Essentially nonlinear one-dimensional model of classical field theory. *Teoreticheskaya i Matematicheskaya Fizika*, 21(2):160–174, 1974.

[32] Sidney Coleman. Quantum sine-gordon equation as the massive thirring model. *Phys. Rev. D*, 11:2088–2097, Apr 1975.

[33] Walter E Thirring. A soluble relativistic field theory. *Annals of Physics*, 3(1):91 – 112, 1958.

[34] J. Ignacio Cirac, Paolo Maraner, and Jiannis K. Pachos. Cold atom simulation of interacting relativistic quantum field theories. *Physical Review Letters*, 105(19), Nov 2010.

[35] Paul Adrien Maurice Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 133(821):60–72, 1931.

[36] Yakov M Shnir. *Magnetic monopoles*. Springer Science & Business Media, 2006.

[37] Tai Tsun Wu and Chen Ning Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Physical Review D*, 12(12):3845, 1975.

[38] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Phys. Rev.*, 115:485–491, Aug 1959.

[39] Henri Poincaré. Remarques sur une expérience de m. birkeland. *Comptes Rendus Acad. Sci*, 123:530, 1896.

[40] G.L. Naber. *Topology, Geometry and Gauge fields: Foundations*. Texts in Applied Mathematics. Springer New York, 2010.

[41] G.'t Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Physics B*, 79(2):276 – 284, 1974.

[42] Alexander M Polyakov. Particle spectrum in quantum field theory. In *30 Years Of The Landau Institute—Selected Papers*, pages 540–541. World Scientific, 1996.

[43] Alexander M. Polyakov. Particle Spectrum in the Quantum Field Theory. *JETP Lett.*, 20:194–195, 1974. [Pisma Zh. Eksp. Teor. Fiz.20,430(1974); ,300(1974); ,300(1974)].

[44] J. Arafune, P. G. O. Freund, and C. J. Goebel. Topology of higgs fields. *Journal of Mathematical Physics*, 16(2):433–437, 1975.

[45] A. A. Belavin, Alexander M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin. Pseudoparticle Solutions of the Yang-Mills Equations. *Phys. Lett.*, 59B:85–87, 1975. [,350(1975)].

[46] Sidney Coleman. *Aspects of Symmetry: Selected Erice Lectures*. Cambridge University Press, 1985.

[47] R.J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten. Chiral estimate of the electric dipole moment of the neutron in quantum chromodynamics. *Physics Letters B*, 88(1):123 – 127, 1979.

[48] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov. Can confinement ensure natural cp invariance of strong interactions? *Nuclear Physics B*, 166(3):493 – 506, 1980.

[49] A. A. Abrikosov. On the Magnetic properties of superconductors of the second group. *Sov. Phys. JETP*, 5:1174–1182, 1957. [Zh. Eksp. Teor. Fiz.32,1442(1957)].

[50] Holger Bech Nielsen and P. Olesen. Vortex Line Models for Dual Strings. *Nucl. Phys.*, B61:45–61, 1973. [,302(1973)].

[51] DA Magalhaes. Sobre a importância do modelo de veneziano para a teoria de cordas. *Revista Brasileira de Ensino de Fısica*, 35(4):4303, 2013.

[52] J. Fröhlich and T. Kerler. Universality in quantum hall systems. *Nuclear Physics B*, 354(2):369 – 417, 1991.

[53] Paul AM Dirac. The lagrangian in quantum mechanics. pages 111–119, 2005.

[54] Richard P Feynman, Albert R Hibbs, and Daniel F Styer. Quantum mechanics and path integrals, emended ed. *Mineola, NY: Dover*, 2010.

[55] R. Rajaraman. *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*. North-Holland personal library. North-Holland Publishing Company, 1982.

[56] Joás Venâncio and Carlos Batista. Quasinormal modes in generalized nariai spacetimes. *Phys. Rev. D*, 97:105025, May 2018.

[57] Felix Aleksandrovich Berezin. Automorphisms of a grassman algebra. *Mathematical Notes*, 1(3):180–184, 1967.

[58] Alice Rogers. Fermionic path integration and grassmann brownian motion. *Communications in Mathematical Physics*, 113(3):353–368, 1987.

[59] J.D. Bjorken and S.D. Drell. *Relativistic quantum fields*. International series in pure and applied physics. McGraw-Hill, 1965.

[60] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.

[61] R. Jackiw and C. Rebbi. Solitons with fermion number . *Phys. Rev. D*, 13:3398–3409, Jun 1976.

[62] Noah Graham, Markus Quandt, and Herbert Weigel. *Spectral methods in quantum field theory*, volume 777. Springer, 2009.

[63] Wang Rong-tai and Ni Guang-jiong. The fractional fermion number and fractional charge induced by magnetic monopole. *Chinese Physics Letters*, 2:565, 04 2009.

[64] R. Jackiw and P. Rossi. Zero modes of the vortex-fermion system. *Nuclear Physics B*, 190(4):681 – 691, 1981.

[65] A.J. Niemi and G.W. Semenoff. Fermion number fractionization in quantum field theory. *Physics Reports*, 135(3):99 – 193, 1986.

[66] Ilya Perapechka, Nobuyuki Sawado, and Ya Shnir. Soliton solutions of the fermion-skyrmion system in $(2+1)$ dimensions. *Journal of High Energy Physics*, 2018(10):81, 2018.

[67] Jeffrey Goldstone and Frank Wilczek. Fractional quantum numbers on solitons. *Phys. Rev. Lett.*, 47:986–989, Oct 1981.

[68] Ian Johnston Rhind Aitchison and CM Fraser. Derivative expansions of fermion determinants: Anomaly-induced vertices, goldstone-wilczek currents, and skyrme terms. *Physical Review D*, 31(10):2605, 1985.

[69] Feza Gürsey. On the symmetries of strong and weak interactions. *Il Nuovo Cimento (1955-1965)*, 16(2):230–240, 1960.

[70] AG Abanov and Paul B Wiegmann. Chiral nonlinear σ models as models for topological superconductivity. *Physical review letters*, 86(7):1319, 2001.

[71] R. MacKenzie and Frank Wilczek. Illustrations of vacuum polarization by solitons. *Phys. Rev. D*, 30:2194–2200, Nov 1984.

[72] Fritz London. Zur theorie und systematik der molekularkräfte. *Zeitschrift für Physik*, 63(3-4):245–279, 1930.

[73] H. B. G. Casimir and D. Polder. The influence of retardation on the london-van der waals forces. *Phys. Rev.*, 73:360–372, Feb 1948.

[74] Hendrick BG Casimir. On the attraction between two perfectly conducting plates. In *Proc. Kon. Ned. Akad. Wet.*, volume 51, page 793, 1948.

[75] Carlos Farina. The casimir effect: some aspects. *Brazilian journal of physics*, 36(4A):1137–1149, 2006.

[76] Leila Shahkarami, Azadeh Mohammadi, and Siamak Sadat Gousheh. Casimir energy for a coupled fermion-soliton system. *Journal of High Energy Physics*, 2011(11):140, 2011.

[77] Siamak S Gousheh and Rafael Lopez-Mobilia. Vacuum polarization by solitons in $(1+1)$ dimensions. *Nuclear Physics B*, 428(1-2):189–208, 1994.

[78] Siamk Sadat Gousheh, Azadeh Mohammadi, and Leila Shahkarami. Casimir energy for a coupled fermion-kink system and its stability. *Physical Review D*, 87(4):045017, 2013.

[79] R MacKenzie and Frank Wilczek. Illustrations of vacuum polarization by solitons. *Physical Review D*, 30(10):2194, 1984.

[80] Norman Levinson. On the uniqueness of the potential in a schrodinger equation for a given asymptotic phase. *Kgl. Danske Videnskab Selskab. Mat. Fys. Medd.*, 25, 1949.

[81] Richard Blankenbecler and Daniel Boyanovsky. Fractional charge and spectral asymmetry in one dimension: A closer look. *Phys. Rev. D*, 31:2089–2099, Apr 1985.

[82] Zhong-Qi Ma and Guang-Jiong Ni. Levinson theorem for dirac particles. *Physical Review D*, 31(6):1482, 1985.

[83] Siamak Gousheh. Levinson theorem for the equation in the presence of solitons in (1+1) dimensions. *Phys. Rev. A*, 65, 02 2002.

[84] Siamak S Gousheh, Maryam Taheri-Nejad, and Mohammad R Fathollahi. The strong form of the levinson theorem for a distorted kp potential. *arXiv preprint arXiv:0911.3582*, 2009.

[85] WA Berger, HG Miller, and D Waxman. Threshold bound states. *The European Physical Journal A*, 37(3):357–360, 2008.

[86] Roger F. Dashen, Brosl Hasslacher, and André Neveu. Nonperturbative methods and extended-hadron models in field theory. ii. two-dimensional models and extended hadrons. *Phys. Rev. D*, 10:4130–4138, Dec 1974.

[87] Ljudvig D Faddeev and Vladimir E Korepin. Quantum theory of solitons. *Physics Reports*, 42(1):1–87, 1978.

[88] MN Chernodub, VA Goy, and AV Molochkov. Nonperturbative casimir effects in field theories: aspects of confinement, dynamical mass generation and chiral symmetry breaking. *arXiv preprint arXiv:1901.04754*, 2019.

[89] SV Chervon. On the chiral model of cosmological inflation. *Russian Physics Journal*, 38(5):539–543, 1995.

[90] H Osborn. String theory effective actions from bosonic σ models. *Nuclear Physics B*, 308(2-3):629–661, 1988.

[91] Royle Skyrme. Skyrmion makeover. *Nature*, 465:17, 2010.

[92] G. E. Brown and Mannque Rho. The Chiral Bag. *Comments Nucl. Part. Phys.*, 18(1):1–29, 1988.

[93] Siamak S Gousheh, Azadeh Mohammadi, and Leila Shahkarami. An investigation of the casimir energy for a fermion coupled to the sine-gordon soliton with parity decomposition. *The European Physical Journal C*, 74(8):3020, 2014.