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CONTROL AND GEOMETRIC INVERSE PROBLEMS FOR SOME LINEAR AND NONLINEAR PARTIAL DIFFERENTIAL SYSTEMS

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ABSTRACT

In this Thesis we present results for control and geometric inverse problems associated with certain linear and non-linear PDEs. First, in Chapter 1 we perform a detailed analysis of the geometric inverse problem that consists to identify, from boundary measurements, an unknown obstacle to passage of a fluid governed by a system of linear elliptic equations. Then, by using the so-called local Carleman estimates, we get a uniqueness result, that is, we show that if two obstacles leading to the same boundary measurements are, necessarily, equals. Moreover, by applying some techniques of differentiation with respect to domains, we can obtain a stability result and then apply a reconstruction algorithm. In Chapter 2, we analyze the controllability properties of the so-called inviscid and viscous Burgers-α equations. More specifically, in the first part of the chapter we can get, by applying the so-called return method, time-reversibility and scale change arguments, a global exact controllability result for the inviscid Burgers-α system. Then, in the second part, we prove that the viscous Burgers-α equation is globally exactly controllable to constant trajectories following three steps: (1) We apply a smoothing effect result for parabolic PDEs; (2) We use a controllability result for the inviscid Burgers-α system to deduce an approximate controllability result for the viscous system; (3) We prove a local exact controllability result for regular time-dependent trajectories. In Chapter 3, we deal with a two-phase free-boundary problem associated with the heat equation. Then, by using a classical technique that reduces controllability to minimization of an appropriated functional, parabolic regularity and the Schauder Fixed-Point Theorem, we prove that it is possible to drive both temperatures and the interface to desired targets in an arbitrary small time, as long as the initial data are small enough in a suitable norm.

Keywords: Geometric inverse problem. Burgers-α system. Global exact controllability. Free-boundary problem.

RESUMO

Nesta tese são apresentados resultados para problemas inversos geométricos e de controle associados à certas EDPs lineares e não-lineares. De fato, fizemos no capítulo 1 uma análise detalhada do problema inverso geométrico que consiste em identicar, via medições na fronteira, um obstáculo desconhecido à passagem de um fluido regido por um sistema linear de equações elípticas. Então, aplicando estimativas de Carleman locais, obtemos um resultado de unicidade, ou seja, mostramos que dois obstáculos provocando as mesmas medições de fronteira devem ser iguais. Além disso, aplicando técnicas de diferenciação com relação a domínios, é possível obter um resultado de estabilidade e esboçar um algoritmo de reconstrução. No capítulo 2, analisamos a controlabilidade das equações Burgers-α invíscida e viscosa. Mais precisamente, numa primeira parte, aplicamos o método do retorno e argumentos de reversibilidade temporal e mudança de escala para obter um resultado global de controlabilidade exata para a equação Burgers-α invíscida. Numa segunda parte, seguimos três etapas para provar que a equação Burgers-α viscosa é globalmente exatamente controlável à trajetórias constantes: 1. aplicamos o efeito regularizante para EDPs parabólicas; 2. usamos o resultado obtido para a equação de Burgers-α invíscida para provar um resultado de controle aproximado; 3. provamos um resultado local de controlabilidade exata à trajetórias regulares que dependem somente do tempo. No capítulo 3, trabalhamos com um problema de fronteira-livre bifásico associado à equação do calor. Então, usando uma técnica clássica que consiste em demonstrar controlabilidade via minimização de um funcional apropriado, resultados de regularidade local para EDPs parabólicas e o Teorema do Ponto-Fixo de Schauder, demonstramos que é possível conduzir tanto as temperaturas quanto a interface à objetivos desejados, desde que os dados iniciais sejam suficientemente pequenos numa norma apropriada.

Palavras-chaves: Problema inverso geométrico. Sistema Burgers-α. Controlabilidade global exata. Problema de fronteira-livre.

CONTENTS

1 INTRODUCTION

The main aims of this Thesis are the analysis of control and inverse problems of some boundary/initial value problems for partial differential systems. Throughout this introduction we will present a description of such systems, we will provide a short historical overview of the related control and inverse problems theory and we will describe precisely the problems under study.

1.1 MATHEMATICAL MODELING VIA DIFFERENTIAL EQUATIONS

One of the greatest achievements in mathematics is the invention of Differential Calculus by Isaac Newton and Gottfried W. Leibniz at the 17th Century. Ever since we are able to elaborate mathematical models to describing a plethora of phenomena in the most diverse fields, such as biology, chemistry, physics, engineering, etc. We can see a mathematical model as an attempt to write, in mathematical language, certain aspects observed at the real world. In order to illustrate better this concept, we briefly analyse two well-known models.

$a)$ A simple spring-particle system is a system formed by a negligible mass spring with a particle of mass m connected to one of its extremes and the other extreme stuck to a wall. We suppose that the particle can move, without friction, along a straight line endowed with a coordinate system where the rest position is $x = 0$. When the particle undergoes a small displacement x, then the spring exerts a restoration force F_1 to bring it back to the rest position and we can see empirically that F_1 has a intensity proportional to the displacement, with a proportionality constant given by the elasticity constant k of the spring and acting in the opposite direction to the particle's movement:

$$F_1 = -kx. \tag{1.1}$$

Even supposing that the particle is moving without friction, the viscosity of the surrounding media (air, water, oil, etc.) can create a force F_2 which acts in the opposite direction of the displacement and it is proportional to particle's velocity; here, the friction constant η of the environment is the proportionality constant, that is:

$$F_2 = -\eta x'. \tag{1.2}$$

We could also consider the presence of external forces (for example, the wall's vibrations in which the spring is stuck) acting on the spring-particle system but, for simplicity, we will neglect them. Then, using (1.1), (1.2) and Newton's Second Law, we see that trajectory of the particle satisfies the following differential equation:

$$mx''(t) = -kx(t) - \eta x'(t). \tag{1.3}$$

The constants m, k and η are the model's parameters of the simple spring-particle system.

The equation (1.3) is an example of ODE (Ordinary Differential Equation), since the unknown function x only depends of a unique independent variable, which is the time variable. Moreover, if the friction force F_2 is negligible, that is, $\eta = 0$ in (1.3) and we suppose that x_0 and y_0 represent, respectively, the initial position and initial velocity of the particle, then the trajectory is given by:

$$x(t) = x_0 \cos \omega t + \frac{y_0}{\omega} \sin \omega t,$$

where $\omega = \sqrt{k/m}$.

b) Let us consider a thin and straight bar of length L located on the x-axis with its ends positioned at $x = 0$ and $x = L$. Suppose that the bar is formed by a homogeneous material with uniform cross-section and its sides are perfectly isolated, so that a heat flow can pass only through the ends. Then, it is usually accepted that the temperature $u = u(t, x)$ evolves according to the following equation, known as the classical heat equation:

$$u_t = \nu u_{xx}, \tag{1.4}$$

where $\nu > 0$ is the so-called diffusion coefficient.

The equation (1.4) is an example of what we call PDE (Partial Differential Equation), since the solution u depends on more than one variable, which are the time and the position. We must observe that there is one important difference between ODEs and PDEs: whereas we need to know only the initial condition to analyze an ODE (the numbers $x(0) = x_0$ and $x'(0) = y_0$ in equation (1.3)) the PDEs on bounded spatial domains demand us to know the initial and boundary conditions to analyze them appropriately (we can suppose, for instance, in the equation (1.4) that $u(0, x) = u_0(x)$, for certain function $u_0 : [0, L] \mapsto R$, and $u(t, 0) = u(t, L) = 0$, for all $t \in [0, \infty)$).

1.2 INVERSE PROBLEMS IN MATHEMATICAL MODELING

Mathematical models using differential equations contain certain parameters related to the physical phenomena under analysis (for example, the elasticity constant of a spring, the specific heat of a surface, the kinematic viscosity of a fluid, etc.). Once all the parameters are known, a classical approach leads to existence and uniqueness results and stability estimates in the standard functional spaces. Next, one can try to find (at least approximately) the solution of the model via techniques from numerical analysis and with this one can measure some effects generated by the model parameters. The approach described in this paragraph is known as the solution to a direct problem associated to a mathematical model and it is summarized below:

- Measure all the parameters involved in the model and find the associated solution;
- Make measurements of the effects generated by the parameters.

To exemplify this approach in a more concrete way, we consider the following elliptic equation associated with a Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset R^N$ is a bounded open domain of class C^∞, $\gamma \in C^2(\Omega)$ is a strictly positive function and $f \in \ H^{1/2}(\partial\Omega)$. The mathematical model (1.5) describes the behavior of an electrical current throughout a region Ω where the parameters γ and f stand for the electrical conductivity of Ω and the voltage applied on the boundary $\partial\Omega$, respectively; the solution u is the voltage in Ω. It is a classical result that, for each positive function $\gamma \in C^0(\overline{\Omega})$ and every $f \in H^{1/2}(\Omega)$, (1.5) has a unique weak solution $u \in H^1(\Omega)$.

An interested physical property that we can compute from u (induced by γ and f) is the current flowing through the boundary, which is mathematically given by $\gamma\, \partial u/\partial n|_{\partial\Omega}$, where n stands the exterior unit normal vector on $\partial\Omega$. Fixing a γ and using the uniqueness solution result for (1.5), we can introduce the called Steklov-Poincaré map (also known as Dirichlet to Neumann map) $\Lambda_\gamma : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$, which is given by:

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}, \quad \forall f \in H^{1/2}(\partial\Omega).$$

We can easily see that the direct problem related to observing the current flowing through the boundary consists, basically, in evaluating the map Λ_γ.

We can analyze a mathematical model following an "inverse"pathway, that is, to identify the unknown model parameters from the knowledge of their effects. This approach is known as the solution of an inverse problem associated to the model and it has a lot of applications, such as the following: recovering the density of the Earth from measurements of the graviational field, identification of tumors from exterior electrical, ultrasonic or magnetic measurements and locating of underground mineral deposits from electrical measurements on the Earth's surface. For more details and other examples, see (ISAKOV, 2006). An important example of inverse problem is the so called Calderón problem (or problem of inverse conductivity), which it was introduced by A. P. Calderón in (CALDERÓN, 1980) and it was studied in a lot of works, see (ALESSANDRINI, 1998; BROWN R.; UHLMANN, 1997) and the references therein. The problem consists of determining the electrical conductivity of a medium from measurements of the voltage and the current on the boundary. In other words, in the Calderón problem we apply a voltage f on the boundary of Ω and we are able to measure the electrical current $\Lambda_\gamma(f)$ flowing through of $\partial\Omega$ and our aim is to identify the conductivity γ that generates this current.

One of the goals of this Thesis is the solution of geometric inverse problems. This consists, in general, in determining the shape and localization of an unknown geometric structure, such as a cavity or a crack in a medium, from measurements on its boundary. To illustrate better the concept, let us consider a conductor represented by a simply connected and bounded set $\Omega \subset R^2$ that contains a cavity represented by a simply connected subset $D \subset\subset \Omega$. We assume that the conductivity tensor is given by a square matrix $A \in L^\infty(\Omega)^{2\times 2}$ and the behavior of the electrostatic potential u can be described by the following elliptic problem:

$$\begin{cases} \nabla \cdot (A(x)\nabla u) = 0 & \text{in} \quad \Omega\backslash\overline{D} \\ A(x)\nabla u \cdot \nu = 0 & \text{on} \quad \partial D, \\ A(x)\nabla u \cdot \nu = \psi & \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

where $\psi \in L^2(\partial\Omega)$ is a current density applied on $\partial\Omega$ and satisfies the compatibility condition:

$$\int_{\partial\Omega} \psi = 0.$$

The main geometric inverse problem associated with (1.6) consists in determining the unknown cavity D when Ω, A and ψ are given and we can measure the potential u on an open portion $\Gamma \subset \partial\Omega$. That is, determining (at least approximately) the shape and localization of D from the knowledge of the Steklov-Poincaré mapping $\Lambda_D : L^2(\partial\Omega) \mapsto L^\infty(\Gamma)$, given by

$$\Lambda_D(\psi) = u|_\Gamma \quad \forall\ \psi \in L^2(\partial\Omega). \tag{1.7}$$

The three main aspects to be analyze in a geometric inverse problems are : uniqueness, stability and reconstruction. In the context of the problem $(1.6) - (1.7)$, this can be presented as follows:

- Uniqueness: If two admissible cavities D_0 and D_1 generate the same potential, i.e. $\Lambda_{D_0}(\psi) = \Lambda_{D_1}(\psi)$ for all $\psi \in L^2(\partial\Omega)$, then $D_0 = D_1$.
- Stability: Find an estimate of the "distance" $\mu_d(D_0, D_1)$ from D_0 to D_1 in terms of a "distance" $\mu_0(\Lambda_{D_0}, \Lambda_{D_1})$ from Λ_{D_0} to Λ_{D_1} of the form

$$\mu_d(D_0, D_1) \leq \Phi(\mu_0(\Lambda_{D_0}, \Lambda_{D_1})),$$

where the function $\Phi : R^+ \mapsto R^+$ satisfies $\Phi(s) \to 0$ as $s \to 0$, at least when Λ_{D_0} and Λ_{D_1} are "close" to a fixed $\Lambda_{\widetilde{D}}$.
- Reconstruction: Find an iterative algorithm to compute the unknown domain D from the values of the Steklov-Poincaré mapping Λ_D.

The proof of uniqueness results for geometric inverse problems relies, fundamentally, on unique continuation properties of differential operators in simply connected domains. Some concepts from complex analysis, such as stream functions and theorems of representation by holomorphic functions can be used to prove the unique continuation property for elliptic operators in R^2 (see (ALESSANDRINI G.; RONDI, 2001; ALESSANDRINI G.; RONDI, 1998)). This is the case of (1.6)-(1.7) and Hormander's Theorem can be applied in any euclidean space R^N, but only for differential operators with constant coefficients (see (DOUBOVA A.; FERNÁNDEZ-CARA, 2015; HORMANDER, 1969)). Contrarily, the tool used in this Thesis is local Carleman inequality, which allows to get unique continuation property for elliptic (and even parabolic) operators with non-smooth coefficients and spatial dimension (for more details, see also (DOUBOVA, 2006; DOUBOVA, 2007; FABRE, 1995)).

In general, our measurements of some effect (like, for instance, the electrical potential on a portion of a medium's boundary) provoked by the presence of an unknown cavity or obstacle D is not totally precise and a stability analysis permits to quantify how much those measure errors may influence the identification process of D. That is, after introducing a suitable distance between two observations $\Lambda_{D_0}(\psi)$ and $\Lambda_{D_1}(\psi)$ (for instance, the norm $\|\Lambda_{D_0}(\psi) - \Lambda_{D_1}(\psi)\|_{L^\infty(\Gamma)}$) and between the corresponding domains D_0 and D_1 (for instance, the Hausdorff distance $d_{\mathcal{H}}(D_0, D_1)$), we want to prove a result like

$$\|\Lambda_{D_0}(\psi) - \Lambda_{D_1}(\psi)\|_{L^\infty(\Gamma)} \leq \varepsilon \Longrightarrow d_{\mathcal{H}}(D_0, D_1) \leq \Phi(\varepsilon), \tag{1.8}$$

for all $\varepsilon > 0$ sufficiently small, where $\Phi : R^+ \mapsto R^+$ (error function) satisfyies $\Phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. It is important to note that the inequality (1.8) means that, for "acceptable"measurement errors, we are “close”to find the unknown domain D that provokes the observed effect on the boundary.

Likewise for uniqueness, the study of stability properties of (1.6)-(1.7) can be achieved using tools from complex analysis. In that case, the standard approach consists in finding the stream function v (a kind of generalized harmonic conjugate) associated with the unique weak solution u to (1.6) and assume that ∂D has regularity properties that allow to find a quasi-conformal mapping $\chi : \Omega \backslash \overline{D} \mapsto B$, where B is a circular domain. Then, by using a Representetion Theorem by L. Bers and L. Nirenberg one proves that the complex function $f = u + iv$ can be identified to $f = F \circ \chi$, where $F : B \mapsto C$ is a holomorphic function and, also, one can get appropriate estimates for f that lead to stability results similar to (1.8). For more details, we refer (ALESSANDRINI G.; RONDI, 2001; ALESSANDRINI G.; RONDI, 1998; RONDI, 2000) .

Another important approach to study the stability of geometric inverse problems associated with elliptic (and even parabolic) operators relies on analyticity arguments. To sum up, assume a priori that the effects measured from the boundary are obtained for domains D that are “suitable”deformations of a fixed domain D_0, that is, they can be written as $D = (I + \sigma\mu)(D_0)$, where $I : R^N \mapsto R^N$ is the identity, $\sigma \in (-1, 1)$ and μ belongs to an appropriated subset of $W^{1,\infty}(R^N; R^N)$. Then, the deformations lead to an analytic mapping from $(-1, 1)$ to the space where the observed effects live (for instance, $H^{-1/2}(\Gamma)$). After writing the mapping as a power series expansion, we find a sufficiently small $\sigma_0 \in (-1, 1)$ such that an inequality like (2.6) in Theorem 2.2 of Chapter 2 holds. While the method based on complex analysis allows to get stability estimates for elliptic operators whose coefficients are only $L^\infty(\Omega)$, the latter method, based on analyticity, as far as we know, only works for constant coefficients.

There are several papers dealing with reconstruction algorithms to solve geometric inverse problems (see, for instance, (ABDA, 2009; ALVAREZ, 2005; ALVAREZ, 2008; DOUBOVA A.; FERNÁNDEZ-CARA, 2015; DOUBOVA A.; FERNÁNDEZ-CARA, 2018)) and, in most of them, the main idea consists in reducing to finite dimension and reformulating the search of the unknown D as a constrained (maybe numerically ill-conditioned) extremal problem and, after this, using gradient, quasi-Newton or even Newton methods to compute approximate solutions. In this Thesis we outlined a reconstruction method (see Section 2.4.4 of the Chapter 2) that relies on the application of techniques of differentiation with respect to the domains that can be found in (HENROT A.; PIERRE, 2018) and (SIMON, 1987).

1.3 MATHEMATICAL MODELING IN FLUID MECHANICS

One of the most notable mathematical models using Differential Calculus comes from fluid mechanics and is represented by the so-called Navier-Stokes Equations (NSEs) that, in turn, are applied to modeling a lot of phenomena (weather, ocean currents, water flows in oceans, lakes and rivers, star motions inside and outside of a galaxy, smoke spread in fires and industrial chimneys, etc.).

A fluid is a large number of molecules in motion that, contrarily to a solid, has not a precise shape at rest. A physical property that characterizes all fluids is the so-called mean free path, that is, the average distance travelled by a fluid particle between collisions that, in turn, change the direction of its movement, energy, etc. The mean free path determines the choice of the approach kinetic theory or continuum mechanics. The kinetic theory is used when the number of fluid particles contained in a region is very small (which means a mean free path relatively large); the effects of the particular molecules are then important and, consequently, the movement of the fluid must be treated individually for each molecule. However, if the mean free path is very small compared to the characteristic lengths of the problem, then the fluid can be considered as a continuum medium and the movement of its particles must be viewed as a whole. Through this approach, the fluid can be described by macroscopic quantities, such as: density, velocity, etc.

From now on, we will consider a fluid as a continuum medium and we will suppose that a fluid fills a region in space represented by a bounded and simply connected open set $\Omega \subset R^N$ during a time interval $(0, T)$. Also, we denote by y and p, respectively, the velocity field and the pressure of the fluid and we assume that the fluid is incompressible (conservation of volume), homogeneous (the density of the fluid is a constant ρ_0) and Newtonian, that is, the so-called stress tensor σ obeys the relation:

$$\sigma = -pId + \mu(\nabla y + \nabla y^t),$$

where $\mu > 0$ is the so-called dynamical viscosity coefficient. Then, we get that the velocity field and pressure of the fluid obey the so-called Navier-Stokes equations (for more details see, for instance, (BOYER F.; FABRIE, 2002; CHORIN A. J.; MARSDEN, 1990; FERNÁNDEZ-CARA, 2012)):

$$\begin{cases} y_t + (y \cdot \nabla)y = -\dfrac{1}{\rho_0}\nabla p + \nu\Delta y + f & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot y = 0 & \text{in} \quad (0,T) \times \Omega, \end{cases} \tag{1.9}$$

where $\nu = \mu/\rho_0$ is the so-called kinematic viscosity.

Remark 1 *When we let $\nu \to 0$ in (1.9), we get in the limit the incompressible Euler equations:*

$$\begin{cases} y_t + (y \cdot \nabla)y = -\dfrac{1}{\rho_0}\nabla p + f & \textit{in} \quad (0,T) \times \Omega, \\ \nabla \cdot y = 0 & \textit{in} \quad (0,T) \times \Omega. \end{cases} \tag{1.10}$$

When a Newtonian, homogeneous and incompressible fluid flows in a spatial region $\Omega \subset R^3$ during a time interval $(0,T)$, we can establish a characteristic length L_* (usually, the diameter of Ω) and a characteristic velocity Y_* that can be related, for example, to the forces applied on the boundary $\partial\Omega$ or to a pressure gradient. Once L_* and Y_* are defined, we can rewrite the NSEs in (1.9) in a nondimensional form by setting:

$$x_* = \frac{x}{L_*},\ t_* = \frac{t}{T_*},\ y_* = \frac{y}{Y_*},\ p_* = \frac{p}{\rho_0 P_*} \text{ and } f_* = \frac{T_*^2}{L_*}f, \tag{1.11}$$

where $T_* = L_*/Y_*$ and $P_* = Y_*^2$. We can indeed check that the new independent variables x_*, t_* and the physical quantities y_*, p_*, f_* are nondimensional and, moreover,

$$\nabla_* = L_*\nabla \text{ and } \Delta_* = L_*^2\Delta. \tag{1.12}$$

Then, by applying (1.11) - (1.12) in (1.9), we obtain the following nondimensional NSEs:

$$y_{*,t} - \frac{1}{Re}\Delta_* y_* + (y_* \cdot \nabla)y_* + \nabla_* p_* = f_*$$

where the $Re := \dfrac{L_* Y_*}{\nu}$ is the so-called Reynolds number.

We see that previous process has a lot of interest because the parameter Re, that naturally appears, helps to predict the behaviour pattern of a fluid in different situations. Indeed, in experiments, it is observed that at values of Re below the so-called critical Reynolds number, the fluid moves smoothly with a minimal amount of mixing between adjacent layers. In this regime, we say that the fluid is laminar. However, for higher values, the fluid exhibits a chaotic behaviour, with rapid variations in its velocity field and pressure and swirling regions appear, called eddies. In this regime, we say that the fluid is turbulent. By observing a turbulent flow modeled by the NSEs (1.9), a process can be observed where large eddies (large-scale components of the fluid) break up into smaller eddies (fine-scale components of the fluid). That process is known as energy cascade and it is provoked by the presence of the inertial term $(y \cdot \nabla)y$ in the NSEs, responsible of the transport of kinetic energy from the larger structures to the smaller ones.

If a mesh sufficiently fine is used, a numerical resolution of the NSEs allows to visualize all the fluid eddies, even the smallest ones close to a length-scale l_d, where the molecular dissipation begins to dominate the inertial term. This approach is called Direct Numerical Simulation (DNS) and for large Re, it demands a really high computational cost what makes it impractical. On the other hand, a numerical method proposal to overcome this difficult is the so-called Large Eddy Simulation (LES) that consists in directly resolve the turbulent structures larger than a certain prescribed length l_* and inhibit the creation of structures smaller than l_* by filtering the inertial term $(y \cdot \nabla)y$ with a spatial smoothing kernel Φ. This means that this term is replaced by $(z \cdot \nabla)y$, with $z = \Phi * y$. However, the effects of the unresolved smallest eddies cannot be ignored and, therefore, they must be modeled. To sum up, the LES methods directly account for the large eddies (what does not require a very fine mesh and, therefore, reduce the computational cost) and model the effects of the smallest eddies. It is worth mentioning that the representation of those effects by subgrid scale modeling is an active subject of research in the theory of turbulent fluids.

In the last years, two mathematical models has raised with a great potential to become a subgrid scale LES model for $3D$ turbulence phenomena. The first one is well-described in (FOIAS, 2001) and it is called Lagrangian Averaged Navier-Stokes-α model (LANS-α model) (also called Navier-Stokes-α or viscous Camassa-Holm equations):

$$\begin{cases} y_t - \nu \Delta y + (z \cdot \nabla) y + \sum_{j=1}^{3} y_j \nabla z_j + \nabla p = f & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot y = \nabla \cdot z = 0 & \text{in} \quad (0,T) \times \Omega, \\ z - \alpha^2 \Delta z = y & \text{in} \quad (0,T) \times \Omega, \end{cases} \tag{1.13}$$

where y is periodic in the periodic box $\Omega = [0, 2\pi L]^3$. From the empirical point of view in (CHEN, 1998; CHEN, 1999b; CHEN, 1999a), the authors showed that the steady LANS-α numerical solutions compare successfully with empirical data from turbulent flows and pipes, for a wide range of Reynolds numbers. Inspired by the LANS-α model (1.13), the authors in (CHESKIDOV, 2005) introduced and described the second model mentioned above, the so-called Leray-α model to $3D$ turbulence:

$$\begin{cases} y_t - \nu \Delta y + (z \cdot \nabla) y + \nabla p = f & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot y = \nabla \cdot z = 0 & \text{in} \quad (0,T) \times \Omega, \\ z - \alpha^2 \Delta z = y & \text{in} \quad (0,T) \times \Omega. \end{cases} \tag{1.14}$$

Here, we assume that y is periodic in the box $\Omega = [0, 2\pi L]^3$. Notice that, as in the LANS-α model the low-pass filter used is the Green's function associated with the Helmholtz operator $I - \alpha^2\Delta$.

The Euler equations for compressible fluids have two problems at small length-scales that demand attention: the first one is the turbulence (like in incompressible fluids) and the second one is the formation of shocks. Then, inspired by the good results obtained for LANS-α and Leray-α models to treat with turbulent flows and with the aim of regularize the shock discontinuities that appear in the compressible Euler equations' solutions, the authors in (BHAT, 2005) applied the Lagrangian averaging approach to deduce averaged models for barotropic and compressible Euler equations. The Lagrangian averaging approach results in a filtered convective velocity in the nonlinear term $(y \cdot \nabla)y$. Then, taking this fact into account and due to the great difficulties involved in the numerical treatment of the averaged models obtained in (BHAT, 2005), the authors used in (NORGARD G.; MOHSENI, 2008) a filtered velocity in the nonlinear term yy_x of the Burgers equation, with the intention of discovering if this technique is reasonable to capture shock formation. Precisely, the starting $1D$ fluid model is the inviscid Burgers equation. It is given by:

$$y_t + yy_x = 0 \tag{1.15}$$

while the viscous Burgers equation is given by:

$$y_t - \nu y_{xx} + yy_x = 0, \tag{1.16}$$

where $\nu > 0$ represents the viscosity.

There are a lot of works dealing with the inviscid and viscous Burgers equations (see for example (BURGERS, 1948; COLE, 1951; GOTOH T.; KRAICHNAN, 1993; LAX, 1973; LIGHTHILL, 1956; OBERAI A. A.; WANDERER, 2006; TADMOR, 2004; WHITHAM, 1974)) and the reasons why they have been a useful testing ground for dynamic fluids rely, mainly, on two aspects:

- Simplicity: Since the equations (1.15) - (1.16) are one-dimensional and they have not a pressure term, then they are simpler than the Euler and Navier-Stokes equations.
- Similarity: Like the Euler equations (1.10) the inviscid Burgers equation (1.15) can be expressed as a conservation law and just like the term $(y \cdot \nabla)y$ in system (1.10), the presence of the one-dimensional convective term yy_x is responsible by shock formation for the solutions to (1.15). On the other hand, the viscous Burgers equation (1.16) can be regarded as a simplified version of theNavier-Stokes equations (1.9).

Taking into account the aspects mentioned above, in (NORGARD G.; MOHSENI, 2008) the authors made a analytical and numerical study of the so-called inviscid Burgers-α equation:

$$\begin{cases} y_t + zy_x = 0, \\ z - \alpha^2 z_{xx} = y, \end{cases} \tag{1.17}$$

where, like in the LANS-α and Leray-α models, there is a filtered convective velocity $z = \Phi^\alpha * y$; here, Φ^α is the Green's function associated with the one-dimensional Helmholtz operator $Id - \alpha^2 \partial^2_{xx}$.

In (NORGARD G.; MOHSENI, 2008) the authors showed numerical evidence that (1.17) is in fact a shock regularization for (1.15) and the behaviour of its solutions is similar to the one for viscous Burgers equation's solutions, which can also be interpreted as a regularization of (1.15). Moreover, it is important to highlight that (1.17) appears when we take $b \to 0$ in the so-called $b-$family:

$$\begin{cases} y_t + zy_x + bz_x y = 0, \\ z - \alpha^2 z_{xx} = y. \end{cases} \tag{1.18}$$

A physical motivation of the equations (1.18) is found in (DULLIN, 2003; DULLIN, 2004), where the authors showed that it is an asymptotically equivalent approximation of the shallow water equations.

It is worth mentioning that, beyond their utility as test platforms for the Euler and the Navier-Stokes equations, the equations (1.15) - (1.16) have an interest by themselves as models for many areas, such as: acoustic waves (GARDNER C. S.; HSING SU, 1969), road traffic modeling (HIGUCHI H.; MUSHA, 1978; NAGATANI, 2000), runoff in soils (SU, 2004), shock formation in inelastic gases (BEN-NAIM, 1999), cosmology turbulence dynamics as in (BEC J; KHANIN, 2007; BOUCHAUD J.-P.; MÉZARD, 1996), etc.

1.4 FREE-BOUNDARY PROBLEMS IN MATHEMATICAL MODEL

In all mathematical models presented before, either the boundary of the spatial domain where the PDE is posed was given or there is a part of the spatial boundary that we have to determine. However, there are some physical phenomena modeled by differential equations defined in spatial domains such that a part of the boundary (called free boundary or moving boundary) is unknown and evolves in time. One of them is the so-called $1D$ two-phase melting phenomenon that we briefly describe below.

Let $L > 0$ a material length, $T > 0$ a positive time and suppose that at the initial time $t = 0$, the spatial region $0 \leq x \leq L$ is filled with ice and the initial distribution of temperature is given by a non-positive function $u_0 : [0, L] \to R$. Furthermore, consider that a time-dependent heat source $q(t) > 0$ acts at the left boundary $x = 0$ starting, therefore, a melting process of the ice. Therefore, for any $0 < t \leq T$, there exists a material length $\ell(t)$ (also called liquid/solid interface) such that the region $0 \leq x \leq L$ consists of two parts: $[0, \ell(t))$ (filled with water) and $(\ell(t), L]$ (filled with ice).

If we denote by u_l and u_s, respectively, the temperatures in the water and ice regions, then the $1D$ two-phase melting problem (also called the two-phase Stefan problem) is to find a triplet (u_l, u_s, ℓ) satisfying the following:

Water region:

$$\begin{cases} c_l \rho_l u_{l,t} = \kappa_l u_{l,xx}, & 0 < t < T,\ 0 \leq x < \ell(t), \\ u_l(0,t) = q(t), & 0 < t < T. \end{cases} \tag{1.19}$$

Ice region:

$$\begin{cases} c_s \rho_s u_{s,t} = \kappa_s u_{s,xx}, & 0 < t < T,\ \ell(t) < x \leq L, \\ u_s(L,t) = 0, & 0 < t < T, \\ u_s(x,0) = u_0(x), & 0 < x \leq L \end{cases} \tag{1.20}$$

Additional conditions:

$$\begin{cases} \ell(0) = 0, \\ u_l(\ell(t),t) = u_s(\ell(t),t) = 0, & 0 < t < T, \\ \kappa_s u_{s,x}(\ell(t),t) - \kappa_l u_{l,x}(\ell(t),t) = \sigma \rho \ell'(t), & 0 < t < T. \end{cases} \tag{1.21}$$

The positive constants c_i, ρ_i, κ_i, for $i \in \{l, s\}$ and σ stand for, respectively, the specific heat, density, thermal conductivity and the latent heat required to melt the ice. Moreover, notice that in (1.21) we have two boundary conditions on the moving boundary $\ell(t)$. The first one $(1.21)_2$ allows to complete the definition of solution for the system of differential equations (1.19)-(1.20) and the second one $(1.21)_3$, called the Stefan condition, helps to determine the unknown interface.

Besides the mathematical role described above, the Stefan condition $(1.21)_3$ can be deduced physically (see (CRANK, 1984)) as a consequence of the behaviour of the heat flux from the water to the ice region.

An interesting control question for Stefan problems like (1.19)-(1.21) that we consider in this Thesis (see Chapter 4), is to prove that, in a finite time $T > 0$, one can drive both the temperatures u_l, u_s and the interface ℓ to desired targets.

CONTROLLABILITY PROBLEMS IN MATHEMATICAL MODELING

To fix ideas, let us consider the non-homogeneous version of the heat equation (1.4):

$$\begin{cases} u_t - u_{xx} = f1_\omega & \text{in} \quad (0,T)\times(0,L), \\ u(\cdot,0) = u(\cdot,L) = 0 & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,L), \end{cases} \tag{1.22}$$

where $L > 0$ is the bar length, $T > 0$ is the time in which the phenomenon is observed, $\omega \subset\subset (0,L)$ is an open subset, 1_ω represents the characteristic function on ω and the source term f and initial condition u_0 are taken in appropriated spaces.

System (1.22) describes the time evolution of the temperature distribution in a thin bar whose extremes have null temperature and the initial distribution of temperature is given by a function $u_0 : (0,L) \mapsto R$. On the other hand, the function $f1_\omega : (0,T)\times\omega \mapsto R$ represents a heat source acting on a portion ω of the bar. One can deal with the following question: given a time $T > 0$, an open subset $\omega \subset (0,L)$ and a state $u_T : (0,L) \mapsto R$ is it possible to find a heat source f such that the solution u to (1.22) satisfies $u(T,\cdot) \equiv u_T$ in $(0,L)$? In other words, is it possible, by applying a suitable heat source on a portion of the bar, to steer the temperature distribution from a given initial u_0 to a desired u_T at time $t = T$?

In the above question, the source f plays the role of a control. As a consequence of the regularizing effect for the heat equation, we have a negative answer for a general u_T. However, we can drive the temperature close enough (in some topology) to any final state. The issue formulated in the previous paragraph is what we call controllability problem for a PDE, that is, the study of the trajectories of a system governed by PDEs connecting two states in a finite time T.

The types of controllability problems can vary according to three factors: time, target and initial state:

Regarding time, we deal with long-time controllability when the time of control T depends on the states to be connected and with small-time controllability when two states can be linked at any time $T > 0$.

With respect to the target, we have the following types of controllability problems: $i)$ null controllability, when our aim is to lead any initial state to the null-state; $ii)$ controllability problem to trajectories, when we desire to drive an initial state to a trajectory (a particular solution of the system); $iii)$ approximate controllability, when starting from an arbitrary initial state, we want to steer the solution of the system arbitrarily close to any target; $iv)$ exact controllability, when any initial state can be driven exactly to every target.

Concerning the initial state, we have controllability questions of two kinds: global and local controllability. The first one holds when any initial state can be driven (exactly or not) to a prescribed target and the second one holds when we only can reach a target if we start sufficiently close to it. A widely used method in Control Theory to solve controllability problems for nonlinear partial differential equations consists in linearizing the system around a well chosen trajectory and then, by applying a fixed-point argument or an inverse function theorem, get a local controllability result (we have applied this technique in this Thesis, see Chapter 3).

1.5 CONTEXT AND AIMS OF THIS THESIS

Geometric inverse problems associated with elliptic and parabolic operators and controllability of fluid mechanics systems and of free-boundary problems are focus of intense research in mathematics, physics, engineering, etc. The main goal of this Thesis is to present new contributions to all of them.

The first chapter of this Thesis relies on the paper (ARAUJO, 2020), which consists in a detailed study of geometric inverse problems associated with systems of elliptic and parabolic equations. The second chapter relies on the submitted manuscript, dealing with the global uniform controllability properties of a family of inviscid and viscous regularizations of the inviscid and viscous Burqers equations. Finally, the third chapter relies on the submitted manuscript, concerning a local controllability result for a two-phase Stefan problem, where we control both temperatures and also the interface.

1.6 PREVIOUS RESULTS

There are a lot of works dealing with geometric inverse problems for elliptic equations. Let us first mention (FRIEDMAN A.; VOGELIUS, 1989), where the authors obtained a uniqueness result for the inverse problem of determining the shape and localization of a singular crack σ (a simple arc) from two boundary measurements. Then, this result was extended for the case of multiple cracks $\sigma_1, \ldots, \sigma_n$ by Alessandrini and Valenzuela in (ALESSANDRINI G.; VALENZUELA, 1996) and stability results were obtained by Alessandrini and Rondi in (ALESSANDRINI G.; RONDI, 1998). It is worth mentioning the paper (ALESSANDRINI G.; RONDI, 2001) of Alessandrini and Rondi, where they found optimal stability estimates for the problem of determining the locations of several cavities $D_1, \ldots, D_n$ from only one boundary measurement.

As mentioned above, in this Thesis we use local Carleman inequalities to get a uniqueness result for geometric inverse problems. This method was previously applied, for example, in (DOUBOVA, 2006; DOUBOVA, 2007; FABRE, 1995) for inverse problems associated with the systems of Boussinesq, Navier-Stokes and Stokes, respectively. We also prove stability results by applying techniques used by C. Alvarez et al in (ALVAREZ, 2005) for the Navier-Stokes equations and we present a reconstruction algorithm based in arguments used by Doubova et al in (DOUBOVA, 2006; DOUBOVA, 2007).

On the other hand, the more notable controllability results in the field of fluid mechanics are related to Navier-Stokes, Euler and Burgers equations. Concerning the Navier-Stokes equations, Fursikov and Imanuvilov in (FURSIKOV A. V.; IMANUVILOV, 1999) proved, using Carleman inequality and an Inverse Function Theorem, local controllability results to trajectories of class C^∞ and Fernández-Cara et al in (FERNÁNDEZ-CARA, 2004) improved these results, by considering bounded trajectories. Later, Fernández-Cara et al proved in (FERNÁNDEZ-CARA, 2006), under some specific geometric conditions, a local exact controllability result to trajectories of the $N-$dimensional Navier-Stokes and Boussinesq systems with a reduced number of scalar distributed controls. More results in this direction has been obtained later; see (CORON J.-M; LISSY, 2014) and the references therein.

Regarding the Euler equations, Coron in (CORON, 1993; CORON, 1996) proved, by applying the return method, a global controllability result for the $2D$ incompressible Euler equations and in (GLASS, 1997; GLASS, 2000) Glass proved a global controllability result for the $3D$ incompressible Euler equations. Also, it is worth mentioning that Coron in (CORON, 1995), using the results in (CORON, 1993; CORON, 1996), proved a global controllability result for the $2D$ incompressible Navier-Stokes equations with slip boundary conditions.

Some remarkable works were carried out in the framework of the inviscid Burgers equations. In (ANCONA F.; MARSON, 1998), Ancona and Marson described the attainable states for the general conservation law of the type $y_t + [f(y)]_x$, with f strictly convex and of class C^2, in the positive half-space. Later, Horsin, in (HORSIN, 1998), described the attainable states for the inviscid Burgers equation in a line segment using two boundary controls. Furthermore, similar to the result of Ancona and Marson, Perrolaz in (PERROLLAZ, 2012), studied the controllability properties of the system $y_t + [f(y)]_x$ in but in the context of the entropic solutions.

In the case of the viscous Burgers equation, let us highlight the following positive controllability results: in (FURSIKOV A. V.; IMANUVILOV, 1996), Fursikov and Imanuvilov proved a small time local controllability result using only one boundary control; long time controllability results towards steady states were obtained by Fursikov and Imanuvilov in (FURSIKOV A.; IMANUVILOV, 1995); furthermore, Chapouly proved in (CHAPOULY, 2009) that the viscous Burgers equation is small time globally controllable with three scalar controls and Marbach improved in (MARBACH, 2014) the Chapouly's result considering only two controls.

On the other hand, there are negative controllability results in the framework of the viscous Burgers equation. For instance, Fernández-Cara and Guerrero, in (FERNÁNDEZ-CARA E.; GUERRERO, 2007), presented optimal estimates for the minimal time of null controllability $T(r)$ of the initial data of L^∞ norm $\leq r$ and Marbach proved in (MARBACH, 2018) the viscous Burgers equation is not small time locally null controllable with one space-independent distributed control and no boundary controls.

On the other hand, there are some controllability results related the "$\alpha -$ modifications" of the Burgers equations. Indeed, in (ARARUNA, 2013), Araruna et al proved a local null controllability result for the viscous Burgers-α equations. This result was extended in (FERNÁNDEZ-CARA E.; SOUSA, 2019) to the viscous $b-$family and in higher dimensional cases, a null controllability result for the Leray-α equations was obtained in (ARARUNA, 2014).

Finally, the study of the controllability properties of free boundary problems related to parabolic PDEs is a topic few explored in the literature. Nevertheless, important results arised in the last years, specially for one-phase Stefan problems and their variants: in (FERNÁNDEZ-CARA, 2016), using classical results of parabolic regularity theory, standard energy estimates and Schauder fixed-point Theorem, the authors managed to steer to zero the temperature in an $1D$ one-phase Stefan Problem, when the initial condition is small enough. By similar arguments, in (FERNÁNDEZ-CARA E.; SOUSA, 2017a) and (DEMARQUE R.; FERNÁNDEZ-CARA, 2018), the local null controllability for the temperature was extended to cover $1D$ semilinear one-phase Stefan problems and $2D$ Stefan problems in star-shaped domains, respectively. Moreover, in (FERNÁNDEZ-CARA, 2018), the authors treat the null controllability problem for $1D$ one-phase Stefan problem by using a local inversion argument (more precisely, Liusternik's

Inverse Function Theorem in Banach spaces).

Related to the two-phase Stefan problems, the best results up to our knowledge concern stabilization. Precisely, in (KOGA S.; KRSTIC, 2020), a melting/solidification temperature $\mathcal{T}_m$, is introduced; the medium is assumed to fill a region $(0, L)$ and a regular function $\ell : [0, +\infty) \mapsto [0, +\infty)$ describing the evolution in time of the interface liquid/solid, is introduced. Then, using the Backstepping Transformation Method, the authors design a feedback boundary control acting at $x = 0$ that stabilizes the temperatures and drive them to $\mathcal{T}_m$, while the interface $\ell(t)$ is driven towards a desired ℓ_T. In other words, they prove the existence of a control such that:

$$\lim_{t\to\infty} \|u_l(\cdot,t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = \lim_{t\to\infty} \|u_s(\cdot,t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = 0 \quad \text{and} \quad \lim_{t\to\infty} \ell(t) = \ell_T.$$

1.7 DESCRIPTION OF THE RESULTS: INVERSE PROBLEM FOR AN ELLIPTIC SYSTEM

Let $\Omega \subset R^N$ be a simply connected bounded domain whose boundary $\partial\Omega$ is of class $W^{2,\infty}$, D^* a fixed nonempty open set with $D^* \subset\subset \Omega$ and $\gamma \subset \partial\Omega$ a nonempty open set. Consider the following family of subsets of D^*:

$$\mathcal{D} = \left\{ D \subset \Omega : D \neq \emptyset \text{ is a simply connected domain, } \overline{D} \subset D^* \text{ and } \partial D \text{ is of class } W^{2,\infty} \right\}$$

and let us denote by $\mathcal{A}$ the set of all (a, b, A, B) such that $a,\ b,\ A,\ B \in L^\infty(\Omega)$ and

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}^t \begin{bmatrix} a(x) & b(x) \\ A(x) & B(x) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \geq -\lambda(|\xi_1|^2 + |\xi_2|^2) \ \ \forall (\xi_1, \xi_2) \in R^2, \text{ a.e. in } \Omega,$$

for some λ with $0 < \lambda < \mu_1(\Omega)^{-1}$, where $\mu_1(\Omega)$ is the smallest positive constant such that

$$\|u\|_{L^2}^2 \leq \mu_1(\Omega) \|\nabla u\|_{L^2}^2 \quad \forall u \in H_0^1(\Omega).$$

We will always assume that $(\varphi, \psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $(a, b, A, B) \in \mathcal{A}$. Under these conditions, it is well known that, for any $D \in \mathcal{D}$, there exists a unique solution $(y, z) \in H^1(\Omega\backslash\overline{D}) \times H^1(\Omega\backslash\overline{D})$ to the elliptic system

$$\begin{cases} -\Delta y + ay + bz = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ -\Delta z + Ay + Bz = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ y = \varphi,\ z = \psi & \text{on} \quad \partial\Omega, \\ y = z = 0 & \text{on} \quad \partial D. \end{cases} \tag{1.23}$$

Furthermore, there exists a constant $C(\Omega, D^*) > 0$ such that

$$\|(y,z)\|_{H^1(\Omega\setminus\overline{D})} \leq C(\Omega, D^*)\|(\varphi,\psi)\|_{H^{1/2}(\partial\Omega)}.$$

In Chapter 2, we will deal with the following geometric inverse problem:

Given $(\varphi,\psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $(\alpha,\beta) \in H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, find a set $D \in \mathcal{D}$ such that the solution (y,z) to the linear system (1.23) satisfies the additional conditions:

$$\left.\frac{\partial y}{\partial n}\right|_\gamma = \alpha \ \text{ and } \ \left.\frac{\partial z}{\partial n}\right|_\gamma = \beta. \tag{1.24}$$

A motivation of problems of this kind can be found, for instance, when one tries to compute the stationary temperature of a chemically reacting plate whose shape is unknown. More precisely, (1.23)-(1.24) has the following interpretation: assume that a chemical product, sensible to temperature effects, fills an unknown domain $\Omega\setminus\overline{D}$; its concentration $y = y(x)$ and its temperature $z = z(x)$ are imposed on the whole outer boundary $\partial\Omega$, the associated normal fluxes are measured on $\gamma \subset \partial\Omega$ and both y and z vanish on the boundary of the non-reacting unknown set D; what we pretend to do is to determine D from these data and measurements.

The two main theorems proved in Chapter 2 involve uniqueness and stability results of the inverse problem (1.23)-(1.24). Specifically, our first main result is the following:

Theorem 1 *Assume that* $(\varphi,\psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ *is nonzero. For* $i = 0, 1$, *let* (y^i, z^i) *be the unique weak solution to* (1.23) *with* D *replaced by* D^i *and let* α^i *and* β^i *be given by the corresponding equalities* (1.24). *Then one has the following:*

$$(\alpha^0,\beta^0) = (\alpha^1,\beta^1) \quad \Longrightarrow \quad D^0 = D^1.$$

The proof relies on some ideas from (FABRE, 1995). More precisely, by applying a well-known local Carleman inequality for elliptic operators, we obtain a unique continuation property for the elliptic system $(1.23)_{1,2}$ defined in a ball and, after that, we use a compactness argument to extend this property for general domains.

Next let us introduce the sets $D := D^0 \cup D^1$, $O^0 := \Omega\setminus\overline{D}$ and let O be the unique connected component of O^0 such that $\partial\Omega \subset \partial O$. Consider the functions $y = y^0 - y^1$ and $z = z^0 - z^1$. Then, using the hypothesis that $(\alpha^0,\beta^0) = (\alpha^1,\beta^1)$, we can define a appropriated extensions O' of O and $(\widetilde{y},\widetilde{z})$ of (y,z) such that

$$\begin{cases} -\Delta\widetilde{y}+a\widetilde{y}+b\widetilde{z}=0 & \text{in} \quad O', \\ -\Delta\widetilde{z}+A\widetilde{y}+B\widetilde{z}=0 & \text{in} \quad O', \\ \widetilde{y}=\widetilde{z}=0 & \text{in} \quad O'\backslash\overline{O}. \end{cases}$$

As a consequence of the unique continuation property obtained in a first step, we get $(\widetilde{y},\widetilde{z})=(0,0)$ in O', what implies $(y,z)=(0,0)$ in O.

Finally, we must prove that $D^0\backslash\overline{D}^1$ and $D^1\backslash\overline{D}^0$ are empty sets. This can be done using a contradiction argument and the well-posedness of our elliptic system. Indeed, suppose by contradiction that $D^1\backslash\overline{D}^0\neq\emptyset$. Then, introducing the set $D^2:=D^1\cup[(\Omega\backslash\overline{D}^0)\cap(\Omega\backslash\overline{O})]$, we have $D^2\backslash\overline{D}^0$ is a non-empty set. Furthermore, we note that $\partial(D^2\backslash\overline{D}^0)=\Gamma_0\cup\Gamma_1$ where $\Gamma_0:=\partial(D^2\backslash\overline{D}^0)\cap\partial D^0$ and $\Gamma_1:=\partial(D^2\backslash\overline{D}^0)\cap\partial D^1$. Therefore, since $(y^0,z^0)=(y^1,z^1)$ in O, the pair (y^0,z^0) is the solution to

$$\begin{cases} -\Delta y^0+ay^0+bz^0=0 & \text{in} \quad D^2\backslash\overline{D}^0, \\ -\Delta z^0+Ay^0+Bz^0=0 & \text{in} \quad D^2\backslash\overline{D}^0, \\ y^0=z^0=0 & \text{on} \quad \Gamma_0\cup\Gamma_1. \end{cases}$$

The above system has a unique solution in $H_0^1(D^2\backslash\overline{D}^0)^2$, what yields $(y^0,z^0)=(0,0)$ in $D^2\backslash\overline{D}^0$. Then, applying unique continuation, we conclude that $(y^0,z^0)=(0,0)$ in $\Omega\backslash\overline{D}^0$, which contradicts the fact that $(\varphi,\psi)\not\equiv(0,0)$. Analogously, we can prove that $D^0\backslash\overline{D}^1=\emptyset$.

Remark 2 (Parabolic Case) *Let us consider the set $Q:=\Omega\times(0,T)$ and let us take functions $a,b,A,B\in L^\infty(Q)$. Then, we can formulate the following geometric inverse problem associated with a parabolic system:*

> *Given (φ,ψ) and (α,β) in appropriate spaces and a nonempty open set $\gamma\subset\partial\Omega$, find an open set $D\in\mathcal{D}$ such that the solution (y,z) to the linear evolution system:*

$$\begin{cases} y_t-\Delta y+ay+bz=0 & \textit{in} \quad \Omega\backslash\overline{D}\times(0,T), \\ z_t-\Delta z+Ay+Bz=0 & \textit{in} \quad \Omega\backslash\overline{D}\times(0,T), \\ y=\varphi,\ z=\psi & \textit{on} \quad \partial\Omega\times(0,T), \\ y=0,\ z=0 & \textit{on} \quad \partial D\times(0,T), \\ y(\cdot\,,0)=0,\ z(\cdot\,,0)=0 & \textit{in} \quad \Omega\backslash\overline{D}, \end{cases} \tag{1.25}$$

satisfies the additional conditions:

$$\left.\frac{\partial y}{\partial n}\right|_{\gamma\times(0,T)} = \alpha \quad \textit{and} \quad \left.\frac{\partial z}{\partial n}\right|_{\gamma\times(0,T)} = \beta. \tag{1.26}$$

If one assume that $(\varphi,\psi) \not\equiv (0,0)$ *then, using arguments similar to those in the proof of Theorem 1, we can also establish an uniqueness result for the inverse problem* (1.25)-(1.26)*.*

In order to present our second main result, concerning stability, let us introduce some notations. Let $D^0 \in \mathcal{D}$ be a fixed subdomain, let $\mu \in W^{1,\infty}(R^N;R^N)$ satisfying

$$\|\mu\|_{W^{1,\infty}} < \epsilon < 1, \quad \mu = 0 \;\text{ in }\; \Omega\backslash\overline{D}^* \tag{1.27}$$

and, for any $\sigma \in (-1,1)$, let us denote by m_σ, D^σ and (y_σ, z_σ), respectively, the mapping $m_\sigma := I + \sigma\mu$ (where $I : R^N \mapsto R^N$ stands the identity), the open set $D^\sigma := m_\sigma(D^0)$ and the solution to (1.23) with D replaced by D^σ. Moreover, it will be assumed that the coefficients a, b, A, B are constant. The following holds:

Theorem 2 *There exists* $\sigma_0 > 0$ *with the following properties:*

1. *The mapping*
$$\sigma \mapsto \left.\left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n}\right)\right|_\gamma \tag{1.28}$$
is well defined and analytic in $(-\sigma_0, \sigma_0)$*, with values in* $H^{-1/2}(\gamma)^2$*.*
2. *Either* $m_\sigma(D^0) = D^0$ *for all* $\sigma \in (-\sigma_0,\sigma_0)$ *(and then the mapping in* (1.28) *is constant), or there exist* $\sigma_* \in (0,\sigma_0)$*,* $C > 0$ *and* $k \geq 1$ *(an integer) such that*
$$\left\|\left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n}\right) - \left(\frac{\partial y_0}{\partial n}, \frac{\partial z_0}{\partial n}\right)\right\|_{H^{-1/2}(\gamma)^2} \geq C|\sigma|^k \quad \forall \sigma \in (-\sigma_*, \sigma_*).$$

To prove the Theorem 2, we assume that $\mu \in W^{1,\infty}(R^N;R^N)$ satisfies (1.27), we define the function $m := I + \mu \in W^{1,\infty}(R^N;R^N)$ and we introduce

$$m' := \left(\frac{\partial m^i}{\partial x_j}\right)_{i,j=1}^N, \;\; \text{Jac}(m) := |\text{det}(m')| \;\text{ and }\; M := ((m')^*)^{-1}.$$

We can check in (HENROT A.; PIERRE, 2018, p. 193) that m is bijective for $\epsilon > 0$ small enough and $m^{-1} \in W^{1,\infty}(R^N;R^N)$. Thus, $D^1 := m(D^0)$ is also a domain in $\mathcal{D}$ and we can consider the unique solution $(y^1, z^1) \in H^1(\Omega\backslash\overline{D}^1)^2$ to (1.23), with D replaced by D^1. On the other hand, due the regularity of the boundary $\partial(\Omega\backslash\overline{D}^*)$ and the couple (φ, ψ), we have the existence of $(\varphi_1, \psi_1) \in H^1(\Omega)$ satisfying:

$$\begin{cases} (\varphi_1, \psi_1) = (0, 0) & \text{in} \quad \overline{D}^*, \\ (\varphi_1, \psi_1) = (\varphi, \psi) & \text{on} \quad \partial\Omega. \end{cases}$$

Next, we can write $(y_1, z_1) = (u_1 + \varphi_1, v_1 + \psi_1)$, where $(u_1, v_1) \in H_0^1(\Omega\backslash\overline{D}^1)^2$ is the unique weak solution to the elliptic problem:

$$\begin{cases} -\Delta u_1 + au_1 + bv_1 = F_1 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ -\Delta v_1 + Au_1 + Bv_1 = G_1 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ u_1 = 0, \;\; v_1 = 0 & \text{on} \quad \partial\Omega \cap \partial D^1, \end{cases} \tag{1.29}$$

where $(F_1, G_1) \in H^{-1}(\Omega\backslash\overline{D}^1)^2$ is given by

$$F_1 = \Delta\varphi_1 - a\varphi_1 - b\psi_1, \quad G_1 = \Delta\psi_1 - A\varphi_1 - B\psi_1.$$

Let us now introduce the functions:

$$u_0 := \widetilde{m}(u_1), \;\; v_0 := \widetilde{m}(v_1), \;\; \varphi_0 := \widetilde{m}(\varphi_1) \text{ and } \psi_0 := \widetilde{m}(\psi_1),$$

with $\widetilde{m} : H_0^1(\Omega\backslash\overline{D}^1) \mapsto H_0^1(\Omega\backslash\overline{D}^0)$ being the isomorphism induced by m, that is,

$$\widetilde{m}(f) := f \circ m, \;\; \forall f \in H_0^1(\Omega\backslash\overline{D}^1).$$

Then, by using the definition of weak solution, we get easily that (u_1, v_1) is the unique solution to (1.29) if and only if (u_0, v_0) is the unique solution to the system:

$$\begin{cases} -\nabla \cdot (\mathsf{Jac}(m) M^* M \nabla u_0) + (au_0 + bv_0)\, \mathsf{Jac}(m) = F_0 & \text{in} \quad \Omega\backslash\overline{D}^0, \\ -\nabla \cdot (\mathsf{Jac}(m) M^* M \nabla v_0) + (Au_0 + Bv_0)\, \mathsf{Jac}(m) = G_0 & \text{in} \quad \Omega\backslash\overline{D}^0, \\ u_0 = 0, \;\; v_0 = 0 & \text{in} \quad \partial\Omega \cup \partial D^0, \end{cases} \tag{1.30}$$

where $(F_0, G_0) \in H^{-1}(\Omega\backslash\overline{D}^0)^2$ is given by

$$F_0 = \nabla \cdot (\mathsf{Jac}(m) M^* M \nabla\varphi_0) - (a\varphi_0 + b\psi_0)\, \mathsf{Jac}(m),$$
$$G_0 = \nabla \cdot (\mathsf{Jac}(m) M^* M \nabla\psi_0) - (A\varphi_0 + B\psi_0)\, \mathsf{Jac}(m).$$

Notice that, for each $\mu \in W^{1,\infty}(R^N; R^N)$ satisfying (1.27), one can simplify the formulation of system (1.30) by introducing the operator $T(\mu) \in \mathcal{L}((H_0^1(\Omega\backslash\overline{D}^0)^2; H^{-1}(\Omega\backslash\overline{D}^0)^2)$, with:

$$T(\mu)(u, v) := (-\nabla \cdot (\mathsf{Jac}(m) M^* M \nabla u) + (au + bv)\, \mathsf{Jac}(m),$$
$$-\nabla \cdot (\mathsf{Jac}(m) M^* M \nabla v) + (Au + Bv)\, \mathsf{Jac}(m)),$$

for all $(u,v) \in H_0^1(\Omega\backslash\overline{D}^0)^2$.

It can be proved that mapping $\mu \mapsto T(\mu)$ is analytic in a neighbourhood of $\mu = 0$ and, therefore, using (1.29) and the fact that $T \mapsto T^{-1}$ is also analytic, we get that $\mu \mapsto (u_0, v_0) = T^{-1}(\mu)(F_0, G_0)$ is again analytic in a neighbourhood of $\mu = 0$. This proves the first statement of the Theorem 2. The second statement is obtained by writing the difference $\left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n}\right) - \left(\frac{\partial y_0}{\partial n}, \frac{\partial z_0}{\partial n}\right)$ as a power series in $H^{-1/2}(\gamma)^2$ and making suitable computations.

Finally, in Section 2.4.4 of Chapter 2, we outline a reconstruction algorithm using the techniques of differentiation with respect to the domains mentioned before.

The results of this chapter can be found in (ARAUJO, 2020), written in collaboration with Enrique Fernández-Cara and Diego A. Souza.

1.8 DESCRIPTION OF THE RESULTS: GLOBAL CONTROLLABILITY OF THE BURGERS-ALPHA SYSTEM

Let $L > 0$ and $T > 0$ be given. In Chapter 3, we study the following two families of controlled systems:

$$\begin{cases} y_t + z y_x = p(t) & \text{in} \quad [0,T] \times [0,L], \\ z - \alpha^2 z_{xx} = y & \text{in} \quad [0,T] \times [0,L], \\ z(\cdot,0) = v_l, \quad z(\cdot,L) = v_r & \text{on} \quad [0,T], \\ y(\cdot,0) = v_l & \text{on} \quad I_l, \\ y(\cdot,L) = v_r & \text{on} \quad I_r, \\ y(0,\cdot) = y_0 & \text{in} \quad [0,L], \end{cases} \tag{1.31}$$

where $I_l = \{t \in [0,T] : \ v_l(t) > 0\}$ and $I_r = \{t \in [0,T] : \ v_r(t) < 0\}$ and

$$\begin{cases} y_t - \gamma y_{xx} + z y_x = p(t) & \text{in} \quad (0,T) \times (0,L), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,T) \times (0,L), \\ z(\cdot,0) = y(\cdot,0) = v_l & \text{in} \quad (0,T), \\ z(\cdot,L) = y(\cdot,L) = v_r & \text{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{1.32}$$

They are, respectively, the inviscid and viscous Burgers-α systems. The couple (y, z) and the triplets (p, v_l, v_r) respectively stand for the corresponding states and controls. For the sake of simplicity, we always assume $\gamma = 1$, since all the results shown below can be extended without difficulty to the case where γ is an arbitrary positive number.

Our two main results deal with uniform global exact controllability (uniform with respect to α) for (1.31) and (1.32). Regarding the inviscid Burgers-α system, we have the following result:

Theorem 3 *Let $\alpha > 0$ and $T > 0$ be given. The inviscid Burgers-α system* (1.31) *is globally exactly controllable in C^1. That is, for any given $y_0, y_T \in C^1([0,L])$, there exist a time-dependent control $p^\alpha \in C^0([0,T])$, a couple of boundary controls $(v_l^\alpha, v_r^\alpha) \in C^1([0,T]; R^2)$ and an associated state $(y^\alpha, z^\alpha) \in C^1([0,T] \times [0,L]; R^2)$ satisfying* (1.31) *and*

$$y^\alpha(T,\cdot) = y_T \quad \textit{in} \quad (0,L).$$

Moreover, there exists a positive constant $C > 0$ (depending on y_0 and y_T, but independent of α) such that

$$\|(z^\alpha, y^\alpha)\|_{C^1([0,T]\times[0,L];R^2)} + \|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0,T];R^2)} \leq C.$$

As far as we know, this is the first work analyzing the uniform controllability properties of the inviscid Burguers-α system and the main tool used to get Theorem 3 is Coron's return method, which was introduced in (CORON, 1992).

Let us present some ideas of the proof. To do this, consider $L, T > 0$ and a non-negative $k \in Z$ and let us introduce the set

$$\Lambda_{L,T,k} := \{\lambda \in C^k([0,T]; [0,\infty)) : \ \|\lambda\|_{L^1(0,T)} > L\}.$$

It is not difficult to see that, for each $\lambda \in \Lambda_{L,T,k}$ with $k \geq 1$, the pair of functions $(\widehat{y}(t,x), \widehat{z}(t,x)) := (\lambda(t), \lambda(t))$ is a trajectory of (1.31), that is, a particular solution to (1.31) associated with $(\widehat{p}(t), \widehat{v}_l(t), \widehat{v}_r(t)) := (\lambda'(t), \lambda(t), \lambda(t))$. We can see easily that, since the function $\lambda \in \Lambda_{L,T,k}$ then there exists $\eta \in (0, L/2)$ such that

$$\int_0^T \lambda(s)\, ds > L + 2\eta. \tag{1.33}$$

Let us consider the flux $\Phi : [0,T] \times [0,T] \times R \mapsto R$ associated with λ, that is, the function $\Phi = \Phi(s;t,x)$ defined by:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(s;t,x) = \lambda(t), \\ \Phi(s;s,x) = x. \end{cases}$$

We note that, given $y_0 \in C^1([0,L])$, it is possible to define an extension $y_0^* \in C_0^1(R)$ such that supp $y_0^* \subset (-\eta,\eta)$, where $\eta > 0$ is introduced in (1.33). Then, we get from (1.33) that $\Phi(T;0,x) < -2\eta$, for all $x \in [0,L]$, which means that, after the time T, the "particles"inside $[0,L]$ are driven out of the region supp y_0^* by Φ. This is the main argument used in the proof of the following:

Proposition 1 *Let T, $L > 0$ be given and assume that $\lambda \in \Lambda_{L,T,0}$. Then, for any $\alpha > 0$ and any $y_0 \in C^1([0,L])$, there exists $(y,z) \in C^1([0,T]\times[0,L];R^2)$ such that*

$$\begin{cases} y_t + \lambda(t) y_x = 0 & \textit{in} \quad (0,T)\times(0,L), \\ z - \alpha^2 z_{xx} = y & \textit{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = y(\cdot,0), \quad z(\cdot,L) = y(\cdot,L) & \textit{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \textit{in} \quad (0,L), \\ y(T,\cdot) = 0 & \textit{in} \quad (0,L). \end{cases} \tag{1.34}$$

Th result ensures that the linearization of (1.31) around the trajectory $(\widehat{y}(t,x), \widehat{z}(t,x)) := (\lambda(t),\lambda(t))$ is null-controllable. Then, it can be expected that "small perturbations"of (1.34) furnishes a flux function $\Phi^* : [0,T]\times[0,T]\times R \mapsto R$ sufficiently close of Φ (for example, in the space $C^0([0,T]\times[0,T]\times R))$ such that, after T, the "particles"in $[0,L]$ are also driven off supp y_0^* by Φ^*. This intuitive idea, together with a fixed-point argument, are the main ingredients of the proof of the following local controllability result:

Proposition 2 *Let T, $L > 0$ be given and assume that $\lambda \in \Lambda_{L,T,0}$. Then, there exist $\delta > 0$ and $C > 0$ (both independent of α) such that, for any $y_0 \in C^1([0,L])$ with $\|y_0\|_{C^1([0,L])} \leq \delta$ and any $\alpha > 0$, there exist boundary controls $(v_l, v_r) \in C^1([0,T];R^2)$ and an associated state $(y,z) \in C^1([0,T]\times[0,L];R^2)$ satisfying*

$$\begin{cases} y_t + (\lambda(t) + z) y_x = 0 & \textit{in} \quad (0,T)\times(0,L), \\ z - \alpha^2 z_{xx} = y & \textit{in} \quad (0,T)\times(0,L), \\ y(\cdot,0) = z(\cdot,0) = v_l & \textit{in} \quad (0,T), \\ y(\cdot,L) = z(\cdot,L) = v_r & \textit{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \textit{in} \quad (0,L), \\ y(T,\cdot) = 0 & \textit{in} \quad (0,L) \end{cases} \tag{1.35}$$

and

$$\|y\|_{C^1([0,T]\times[0,L])} \leq C\|y_0\|_{C^1([0,L])} \quad \forall\alpha > 0.$$

Then, if we take $\lambda \in \Lambda_{L,T,1}$, with supp $\lambda \subset (0,T)$, we can obtain, as an immediate consequence of Proposition 2, the following:

Proposition 3 *Let $T, L, \alpha > 0$ be given. Then, there exist $\delta > 0$ and $C > 0$ (both independent of α) such that the following property holds: for each $y_0 \in C^1([0,L])$ with $\|y_0\|_{C^1([0,L])} \leq \delta$, there exist $p^\alpha \in C^0([0,T])$ with $p^\alpha(T) = 0$, $v_l^\alpha, v_r^\alpha \in C^1([0,T])$ and associated states $(y^\alpha, z^\alpha) \in C^1([0,T]\times[0,L]; R^2)$ satisfying* (1.31),

$$y^\alpha(T,\cdot) = 0 \quad \textit{in} \quad (0,L)$$

and

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0,T];R^2)} \leq C \quad \forall\alpha > 0.$$

Thus, if we take $\lambda \in \Lambda_{L,T,1}$ with supp $\lambda \subset (0,T)$, we obtain Theorem 3 by using a scaling argument (in order to apply Proposition 3) and the time-reversibility of the inviscid Burgers-α equation.

On the other hand, regarding the viscous Burgers-α system, we have the following result:

Theorem 4 *Let $\alpha > 0$ and $T > 0$ be given. The viscous Burgers-α system* (1.32) *is globally exactly controllable in L^∞ to constant trajectories. That is, for any given $y_0 \in L^\infty(0,L)$ and $M \in R$, there exist controls $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T;R^2)$ and associated states $(y^\alpha, z^\alpha) \in L^2(0,T;H^1(0,L;R^2)) \cap L^\infty(0,T;L^\infty(0,L;R^2))$ satisfying* (1.32), *the controllability condition below*

$$y^\alpha(T,\cdot) = M \quad \textit{in} \quad (0,L)$$

and the following estimates

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \leq C,$$

where C is a positive constant independent of α, but depending on y_0 and M. Moreover, if $y_0 \in H_0^1(0,L)$, the same conclusion holds with $(y^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$.

The proof of Theorem 4 is divided in three steps: smoothing effect, approximate controllability and local exact controllability to the trajectories.

One of the main features of the parabolic equations is the so-called smoothing effect. This property asserts, roughly speaking, that the solutions become regular after an arbitrary positive time. Then, by combining the smoothing effects and standard energy estimates, we obtain the following result:

Proposition 4 *Let $y_0 \in L^\infty(0,L)$ be given and let (y^α, z^α) be the solution to* (1.32)*, with $p = v_l = v_r = 0$. Then, there exist $T^* \in (0, T/2)$ and $C > 0$ (independent of α) such that y^α belongs to $C^0([T^*, T]; C^2([0, L]))$ and satisfies*

$$\|y^\alpha\|_{C^0([T^*,T];C^2([0,L]))} \leq C\Lambda(\|y_0\|_\infty),$$

where $\Lambda : R_+ \to R_+$ is a continuous function satisfying $\Lambda(s) \to 0$ as $s \to 0^+$.

The approximate controllability step is contained in the following result:

Proposition 5 *Let $y_0, y_f \in C^2([0, L])$ be given. There exist positive constants τ_* and K, independent of α, such that, for any $\tau \in (0, \tau_*]$, there exist controls $p^\alpha \in C^0([0, \tau])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0, \tau; R^2)$ and associated states (y^α, z^α) satisfying* (1.32) *(with T replaced by τ), the condition*

$$\|y^\alpha(\tau, .) - y_f\|_{H^1(0,L)} \leq K\sqrt{\tau} \tag{1.36}$$

and, moreover,

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \leq C,$$

for some positive constant C independent of α.

A heuristic argument for the proof of the Proposition 5 consists in splitting the state (y^α, z^α) in the form:

$$(y^\alpha, z^\alpha) := (u^{\alpha,\tau} + r^{\alpha,\tau} + \lambda^\tau, w^{\alpha,\tau} + q^{\alpha,\tau} + \lambda^\tau), \tag{1.37}$$

where $\lambda^\tau \in \Lambda_{L,\tau,1}$, $(u^{\alpha,\tau}, w^{\alpha,\tau})$ is a controlled solution of an inviscid system like (1.35) and $(r^{\alpha,\tau}, q^{\alpha,\tau})$ is the "remainder"(solution of an appropriated parabolic-elliptic system). Therefore, the task consists in proving the existence of suitable λ^τ, $(u^{\alpha,\tau}, w^{\alpha,\tau})$ and $(r^{\alpha,\tau}, q^{\alpha,\tau})$ such that the solution in (1.37) satisfies the result in Proposition 5.

Then, arguing as in the proof of Theorem 3 we get that, given $y_0, y_f \in C^2([0, L])$, there exists a pair of boundary controls $(v_l^{\alpha,\tau}, v_r^{\alpha,\tau}) \in C^2([0, \tau]; R^2)$ and an associated state

$(u^{\alpha,\tau}, w^{\alpha,\tau}) \in C^2([0,\tau]\times[0,L];R^2)$ that solves an inviscid system like (1.35), with λ replaced by λ^τ and T replaced by τ. We get yet $u^{\alpha,\tau}(0,\cdot) = y_0$ and $u^{\alpha,\tau}(\tau,\cdot) = y_f$ in $[0,L]$.

The construction of suitable λ^τ and $(u^{\alpha,\tau}, w^{\alpha,\tau})$ is not difficult. The main aim is to get good estimates for the "remainder"$(r^{\alpha,\tau}, q^{\alpha,\tau})$. Indeed, together with $(u^{\alpha,\tau}, w^{\alpha,\tau})$, one can define a suitable boundary-initial problem associated to a parabolic-elliptic system, whose solution $(r^{\alpha,\tau}, q^{\alpha,\tau})$ is such that the decomposition in (1.37) solves (1.32), replacing T by τ. Moreover, from standard energy estimates for parabolic equations, we get a constant $K>0$ (independent of α) such that

$$\|r^{\alpha,\tau}\|_{C^0([0,\tau];H^1(0,L))} \leq K\sqrt{\tau},$$

which implies that y^α, given in (1.37), satisfies (1.36).

Finally, inspired on some ideas of Araruna, Fernández-Cara and Souza in (ARARUNA, 2013), we are able to prove the following local controllability result:

Proposition 6 *Let $T, L, \alpha > 0$ and $\widehat{m} \in C^1([0,T])$ be given. There exists $\delta > 0$ (independent of α) such that, for any initial data $y_0 \in H^1(0,L)$ satisfying $\|y_0 - \widehat{m}(0)\|_{H^1} \leq \delta$ there exist $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T;R^2)$ and associated states $(y^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$ satisfying (3.2) and*

$$y^\alpha(T,\cdot) \equiv z^\alpha(T,\cdot) \equiv \widehat{m}(T).$$

Moreover, $p^\alpha = \widehat{m}'$ and the following estimates hold:

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \leq C,$$

where $C > 0$ is a positive constant independent of α.

To sum up, using the three steps above we can construct a triplet of controls $(p^\alpha, v_l^\alpha, v_r^\alpha)$ and a pair of associated states (y^α, z^α) satisfying Theorem 4.

1.9 DESCRIPTION OF THE RESULTS: LOCAL CONTROLLABILITY OF A TWO-PHASE STEFAN PROBLEM

Let $L > 0$ be a material length, let $T > 0$ be a positive time and let $\ell_l, \ell_0, \ell_r \in (0,L)$ be three positive real numbers such that $\ell_l < \ell_0 < \ell_r$. Furthermore, let us consider functions $u_0 \in W_0^{1,4}(0,\ell_0)$, with $u_0 \geq 0$, and $v_0 \in W_0^{1,4}(\ell_0,L)$, with $v_0 \leq 0$ and two open sets

$\omega_l \subset\subset (0, \ell_l)$ and $\omega_r \subset\subset (\ell_r, L)$. The material domain is separated in two parts: $x \in [0, \ell(t))$ (liquid phase) and $x \in (\ell(t), L]$ (solid phase). Here, $\ell = \ell(t)$ is the position of the interface between liquid and solid phases and satisfies $\ell(0) = \ell_0$ and $\ell(t) \in (\ell_l, \ell_r)$, for all t. The main aim of Chapter 4 is to study the controllability properties of the following two-phase Stefan problem:

$$\begin{cases} u_t - d_l u_{xx} = h_l 1_{\omega_l} & \text{in} \quad Q_l, \\ v_t - d_r v_{xx} = h_r 1_{\omega_r} & \text{in} \quad Q_r, \\ u(0,t) = v(L,t) = 0 & \text{on} \quad (0,T), \\ u(\cdot, 0) = u_0 & \text{in} \quad (0, \ell_0), \\ v(\cdot, 0) = v_0 & \text{in} \quad (\ell_0, L), \\ u(\ell(t), t) = v(\ell(t), t) = 0 & \text{on} \quad (0,T), \\ -\ell'(t) = d_l u_x(\ell(t), t) - d_r v_x(\ell(t), t) & \text{on} \quad (0,T). \end{cases} \tag{1.38}$$

Here and in the sequel, d_l and d_r are positive diffusion coefficients and we use the notation

$$\begin{cases} Q := (0, L) \times (0, T), \\ Q_l := \{(x,t) \in Q;\ t \in (0,T) \text{ and } x \in (0, \ell(t))\}, \\ Q_r := \{(x,t) \in Q;\ t \in (0,T) \text{ and } x \in (\ell(t), L)\}, \\ \mathcal{O}_l = \omega_l \times (0,T) \text{ and } \mathcal{O}_r = \omega_r \times (0,T). \end{cases}$$

Notice that, as in (1.19) - (1.21), the time evolution of the temperature distribution in the solid-liquid phases is described by two parabolic equations and the Stefan condition, given by the ODE $(1.38)_7$, meaning that the time evolution of the interface between the solid-liquid phases is influenced by the heat flux induced by the process of melting-solidification.

Then, the main result obtained in Chapter 4 is the following:

Theorem 5 *Let $\ell_T \in (\ell_l, \ell_r)$. Then, there exists $\delta > 0$ such that for any $u_0 \in W_0^{1,4}(0, \ell_0)$ with $u_0 \geq 0$, any $v_0 \in W_0^{1,4}(\ell_0, L)$ with $v_0 \leq 0$, and $\ell_0 \in (\ell_l, \ell_r)$ satisfying*

$$\|u_0\|_{W_0^{1,4}} + \|v_0\|_{W_0^{1,4}} + |\ell_0 - \ell_T| < \delta,$$

there exist controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$*and associated states* (u, v, ℓ) *with*

$$\begin{cases} \ell \in C^1([0,T]) \text{ and } \ell(t) \in (\ell_l, \ell_r)\, \forall\, t \in [0,T], \\ u,\, u_x,\, u_t,\, u_{xx} \in L^2(Q_l) \text{ and } v,\, v_x,\, v_t,\, v_{xx} \in L^2(Q_r) \end{cases} \tag{1.39}$$

and

$$\ell(T) = \ell_T, \quad u(\cdot,T) = 0 \;\; \text{in} \;\; (0,\ell_T) \text{ and } v(\cdot,T) = 0 \text{ in } (\ell_T, L). \tag{1.40}$$

The proof of Theorem 5 relies on an extension of the ideas in (DOUBOVA A.; FERNÁNDEZ-CARA, 2005; FERNÁNDEZ-CARA, 2016). More precisely, the proof is divided in five steps that we briefly describe below:

1. We define a suitable diffeomorphism $\Phi : Q \mapsto Q$ that transforms the non-cylindrical problem (1.38) into a system of parabolic PDEs, whose coefficients depend on ℓ, defined in a cylindrical domain. Therefore, after we define the new coordinates $(\xi, t) := \Phi(x,t)$, we consider the new initial data $p_0(\xi) := u_0(\Phi^{-1}(\xi,0))$ and $q_0(\xi) := v_0(\Phi^{-1}(\xi,0))$ and we denote (p,q) the unique strong solution to the new system with initial data (p_0, q_0).
2. It is not difficult to see that the controllability result in Theorem 5 is equivalent to prove the same result for the new cylindrical problem obtained in the first step. Then, the task is reduced to obtain, via global Carleman estimates and classical arguments of compactness-uniqueness, an improved observability inequality for the adjoint system associated to the new parabolic cylindrical domain.
3. Since our goal is to control both the temperatures (p,q) and the interface ℓ, we must to solve an approximate controllability problem with a constraint. More precisely, for fixed ℓ satisfying (1.39) and $\varepsilon > 0$, we search for controls $(h^\ell_{l,\varepsilon}, h^\ell_{r,\varepsilon}) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ such that the associated state (p,q) satisfies the approximate controllability condition:

$$\|(p(\cdot,T), q(\cdot,T))\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \leq \varepsilon \tag{1.41}$$

   and the linear constraint

$$\iint_{\mathcal{O}_l} h^\ell_{l,\varepsilon}\psi_\ell \, d\xi\, dt + \iint_{\mathcal{O}_r} h^\ell_{r,\varepsilon}\zeta_\ell \, d\xi\, dt = \ell_T - \ell_0 - \int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\, d\xi - \int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\, d\xi. \tag{1.42}$$

   Here, the couple (ψ_ℓ, ζ_ℓ) is the weak solution of a suitable augmented adjoint system.

   To get controls such that (1.41)-(1.42) hold, we apply the improved observability inequality obtained in the second step to minimize an appropriate functional $J_{\ell,\varepsilon}$ and, after this, we build the controls $(h^\ell_{l,\varepsilon}, h^\ell_{r,\varepsilon})$.

4. Let $R > 0$ and $\ell_l < \tilde{\ell}_l < \tilde{\ell}_r < \ell_r$ be given and let us define the set

$$\mathcal{A}_R := \{\ell \in C^1([0,T]) : \tilde{\ell}_l \leq \ell(t) \leq \tilde{\ell}_r,\ \ell(0) = \ell_0,\ \|\ell'\|_{C^0([0,T])} \leq R\}.$$

For each fixed $\varepsilon > 0$, let us introduce the mapping $\Lambda_\varepsilon : \mathcal{A}_R \mapsto C^1([0,T])$ given by

$$\Lambda_\varepsilon(\ell) = \mathcal{L}, \quad \text{with} \quad \mathcal{L}(t) = \ell_0 - \int_0^t [d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)]\, d\tau,$$

where (p,q) is the controlled state associated with the controls $(h^\ell_{l,\varepsilon}, h^\ell_{r,\varepsilon})$, obtained in the third step and, therefore, $\mathcal{L}(T) = \ell_T$. Then, by using standard local regularity results for parabolic PDEs, we can prove that, for any couple of initial data (p_0,q_0) small enough in $\left[W^{1,4}_0(0,\ell_0) \times W^{1,4}_0(\ell_0,L)\right]$, the mapping Λ_ε satisfies all the conditions of the Schauder Fixed-Point Theorem. Therefore, there exists $\ell_\varepsilon \in \mathcal{A}_R$ such that $\Lambda_\varepsilon(\ell_\varepsilon) = \ell_\varepsilon$. In other words, given $\varepsilon > 0$, there exist controls $(h^\varepsilon_l, h^\varepsilon_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ whose associated states $(p_\varepsilon, q_\varepsilon, \ell_\varepsilon)$ satisfy (1.41) and $\ell_\varepsilon(T) = \ell_T$.

5. In this last step, we prove that the family of controls $\{(h^\varepsilon_l, h^\varepsilon_r)\}_{\varepsilon>0}$ and associate states $\{(p_\varepsilon, q_\varepsilon, \ell_\varepsilon)\}_{\varepsilon>0}$ are uniformly bounded in appropriate spaces which, in turn, leads to the existence of controls (h_l,h_r) and associated states (p,q,ℓ) satisfying

$$\ell(T) = \ell_T,\ p(\cdot,T) = 0 \text{ in } (0,\ell_0) \text{ and } q(\cdot,T) = 0 \text{ in } (\ell_0,L).$$

Then, it is not difficult to see the controls (h_l,h_r) and the interface ℓ, together with the temperatures defined by $(u,v) := (p,q)\circ\Phi$, satisfy (1.38), (1.39) and (1.40).

Remark 3 *Using a classical extension domain argument, one can prove a local boundary controllability result for* (1.38)*, by using two boundary controls.*

After proving Theorem 5, a natural question is whether it is possible to get a positive controllability result for (1.38) by using only one control. Then, using the maximum principle for parabolic equations, one finds an obstruction and one has the following negative result:

Theorem 6 *Assume that $u_0 \in W^{1,4}_0(0,\ell_0)$ with $u_0 \geq 0$, $v_0 \in W^{1,4}_0(0,\ell_0)$ with $v_0 \leq 0$ and $v_0 \not\equiv 0$. Then, if $(h_l,h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$, $h_r \equiv 0$ and the associated strong solution to* (1.38) *satisfies $\ell(t) < L$ for all $t \in [0,T]$, we necessarily have*

$$v(\cdot,T) \not\equiv 0 \textit{ in } (\ell(T),L).$$

2 ON SOME GEOMETRIC INVERSE PROBLEMS FOR NONSCALAR ELLIPTIC SYSTEMS

In this chapter, we consider several geometric inverse problems for linear elliptic systems. We prove uniqueness and stability results. In particular, we show the way that the observation depends on the perturbations of the domain. In some particular situations, this provides a strategy that could be used to compute approximations to the solution of the inverse problem. In the proofs, we use techniques related to (local) Carleman estimates and differentiation with respect to the domain.

2.1 INTRODUCTION

Let $\Omega \subset R^N$ be a simply connected bounded domain whose boundary $\partial\Omega$ is of class $W^{2,\infty}$, let D^* be a fixed nonempty open set with $D^* \subset\subset \Omega$ and let $\gamma \subset \partial\Omega$ be a nonempty open set. In what follows, the symbols $C, C_1, C_2, \ldots$ will be used to denote generic positive constants and sometimes, we will indicate the data on which they depend by writting, for example, $C(\Omega, D^*)$.

Let us consider the following family of subsets of D^*:

$$\mathcal{D} = \left\{ D \subset \Omega : D \neq \emptyset \text{ is a simply connected domain, } \overline{D} \subset D^* \text{ and } \partial D \text{ is of class } W^{2,\infty} \right\}$$

and let us denote by $\mathcal{A}$ the set of all (a, b, A, B) such that $a,\ b,\ A,\ B \in L^\infty(\Omega)$ and

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}^t \begin{bmatrix} a(x) & b(x) \\ A(x) & B(x) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \geq -\lambda(|\xi_1|^2 + |\xi_2|^2) \ \ \forall (\xi_1, \xi_2) \in R^2, \text{ a.e. in } \Omega, \tag{2.1}$$

for some λ with $0 < \lambda < \mu_1(\Omega)^{-1}$, where $\mu_1(\Omega)$ is the smallest positive constant such that

$$\|u\|_{L^2}^2 \leq \mu_1(\Omega) \|\nabla u\|_{L^2}^2 \quad \forall u \in H_0^1(\Omega).$$

In this chapter, we will always assume that $(\varphi, \psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $(a, b, A, B) \in \mathcal{A}$. Under these circumstances it is well known that, for any $D \in \mathcal{D}$, there exists a unique solution $(y, z) \in H^1(\Omega \backslash \overline{D}) \times H^1(\Omega \backslash \overline{D})$ to the system

$$\begin{cases} -\Delta y + ay + bz = 0 & \text{in} \quad \Omega \backslash \overline{D}, \\ -\Delta z + Ay + Bz = 0 & \text{in} \quad \Omega \backslash \overline{D}, \\ y = \varphi,\ z = \psi & \text{on} \quad \partial\Omega, \\ y = z = 0 & \text{on} \quad \partial D, \end{cases} \tag{2.2}$$

furthermore satisfying

$$\|(y,z)\|_{H^1(\Omega\setminus\overline{D})} \leq C(\Omega, D^*)\|(\varphi,\psi)\|_{H^{1/2}(\partial\Omega)}.$$

In many physical phenomena, in order to predict the result of a measurement, we need a model of the system under investigation (typically a PDE system) and an explanation or interpretation of the observed quantities. If we are able to compute the solution to the model and quantify relevant observations, we say that we have solved the forward or direct problem. Contrarily, the inverse problem consists of using the observations to recover unknown data that characterize the model. For details about the main questions concerning inverse problems for PDEs from the theoretical and numerical viewpoints, see for instance the book (ISAKOV, 2006).

In this chapter, we will deal with the following geometric inverse problem:

> Given $(\varphi,\psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $(\alpha,\beta) \in H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, find a set $D \in \mathcal{D}$ such that the solution (y,z) to the linear system (2.2) satisfies the additional conditions:
>
> $$\left.\frac{\partial y}{\partial n}\right|_{\gamma} = \alpha \text{ and } \left.\frac{\partial z}{\partial n}\right|_{\gamma} = \beta. \tag{2.3}$$

A physical motivation of problems of this kind can be found, for instance, when one tries to compute the stationary temperature of a chemically reacting plate whose shape is unknown. More precisely, (2.2)-(2.3) has the following interpretation: assume that a chemical product, sensible to temperature effects, fills an unknown domain $\Omega\setminus\overline{D}$; its concentration $y = y(x)$ and its temperature $z = z(x)$ are imposed on the whole outer boundary $\partial\Omega$, the associated normal fluxes are measured on $\gamma \subset \partial\Omega$ and both y and z vanish on the boundary of the non-reacting unknown set D; what we pretend to do is to determine D from these data and measurements.

In the context of the inverse problem (2.2)-(2.3), three main questions appear. They are the following:

- Uniqueness: Let (α^0,β^0) and (α^1,β^1) be two observations and let (y^0,z^0) and (y^1,z^1) be solutions to (2.2) satisfying the identities (2.3) associated to the sets D^0 and D^1, respectively. The question is: do we have $D^0 = D^1$ whenever $(\alpha^0,\beta^0) = (\alpha^1,\beta^1)$?
- Stability: Find an estimate of the "distance" $\mu_d(D^0,D^1)$ from D^0 to D^1 in terms of the "distance" $\mu_0((\alpha^0,\beta^0),(\alpha^1,\beta^1))$ from (α^0,β^0) to (α^1,β^1) of the form

$$\mu_d(D^0,D^1) \leq \Phi(\mu_0((\alpha^0,\beta^0),(\alpha^1,\beta^1))),$$

where the function $\Phi : R^+ \mapsto R^+$ satisfies $\Phi(s) \to 0$ as $s \to 0$, valid at least whenever (α^0, β^0) and (α^1, β^1) are "close" to a fixed $(\bar{\alpha}, \bar{\beta})$.

- Reconstruction: Find an iterative algorithm to compute the unknown domain D from the observation (α, β).

In the sequel, we will on the uniqueness and the stability of the inverse problem (2.2)-(2.3). Specifically, our first main result is the following:

Theorem 2.1 *Assume that* $(\varphi, \psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ *is nonzero. For* $i = 0, 1,$ *let* (y^i, z^i) *be the unique weak solution to* (2.2) *with* D *replaced by* D^i *and let* α^i *and* β^i *be given by the corresponding equalities* (2.3). *Then one has the following:*

$$(\alpha^0, \beta^0) = (\alpha^1, \beta^1) \quad \Longrightarrow \quad D^0 = D^1.$$

The proof is given in Section 2.2. It relies on some ideas from (FABRE, 1995); more precisely, we use two well known properties of (2.2): unique continuation and well-posedness in the Sobolev space H^1.

Remark 2.1 *Note that, if* $(\varphi, \psi) = (0, 0)$, *then the associated solution to* (2.2) *is zero, for any* $D \in \mathcal{D}$. *Therefore, the uniqueness problem has no sense when* $(\varphi, \psi) = (0, 0)$.

Remark 2.2 *In the one-dimensional case, if one consider* (2.2) *with only one boundary observation, uniqueness may not hold. Indeed, suppose that* $\Omega = (0, 1)$, $L \in (0, 1)$ *and consider the system*

$$\begin{cases} -y_{xx} + \eta^2 y + bz = 0 & \textit{in} \quad (0, L), \\ -z_{xx} + Ay + \zeta^2 z = 0 & \textit{in} \quad (0, L), \\ y(0) = z(0) = 0, \\ y_x(0) = 0, \end{cases} \tag{2.4}$$

where $A, b, \eta, \zeta \in R$ *(all them different from zero) and* $|A| + |b| < 2|\eta||\zeta|$. *Then, the numbers* $(\eta^2, b, A, \zeta^2) \in \mathcal{A}$ *and, using the parameter variation method, we get that a solution* (y, z) *to* (2.4) *is given by*

$$z(x) = \frac{K}{\zeta}\sinh(\zeta x) + \frac{A}{\zeta}\int_0^x y(s)\sinh[\zeta(x-s)]\,ds, \quad y(x) = \frac{b}{\eta}\int_0^x z(s)\sinh[\eta(x-s)]ds$$

for each $K \in R$. *Therefore, if* $K \neq 0$ *we have* $z_x(0) \neq 0$ *and this implies non-uniqueness.*

For $N \geq 2$, *the uniqueness in the N-dimensional case with only one information on* γ *is, to our knowledge, an open question.*

In order to state our main stability result, let us introduce some notation. Thus, let $D^0 \in \mathcal{D}$ be a fixed subdomain and let's assume $\mu \in W^{1,\infty}(R^N; R^N)$ satisfy

$$\|\mu\|_{W^{1,\infty}} \leq \epsilon < 1, \quad \mu = 0 \text{ in } \Omega \backslash \overline{D}^*$$

and, for any $\sigma \in (-1, 1)$, we denote by m_σ, D^σ and (y_σ, z_σ), respectively, the mapping $m_\sigma := I + \sigma\mu$, the open set $D^\sigma := m_\sigma(D^0)$ and the solution to (2.2) with D replaced by D^σ. It will be assumed still that the coefficients a, b, A, B are constant. The following holds:

Theorem 2.2 *There exists $\sigma_0 > 0$ with the following properties:*

1. *The mapping*
$$\sigma \mapsto \left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n}\right)\Big|_\gamma \tag{2.5}$$
is well defined and analytic in $(-\sigma_0, \sigma_0)$, with values in $H^{-1/2}(\gamma)^2$.

2. *Either $m_\sigma(D^0) = D^0$ for all $\sigma \in (-\sigma_0, \sigma_0)$ (and then the mapping in* (2.5) *is constant), or there exist $\sigma_* \in (0, \sigma_0)$, $C > 0$ and $k \geq 1$ (an integer) such that*
$$\left\| \left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n}\right) - \left(\frac{\partial y_0}{\partial n}, \frac{\partial z_0}{\partial n}\right) \right\|_{H^{-1/2}(\gamma)^2} \geq C|\sigma|^k \quad \forall \sigma \in (-\sigma_*, \sigma_*). \tag{2.6}$$

In (BUCKGEIM, 1999) and (KAVIAN, 2003), a similar geometric inverse problem for one scalar elliptic equation is studied. For geometric inverse problems for nonlinear models, like Navier-Stokes and Boussinesq systems, the uniqueness has been analyzed in (DOUBOVA, 2006) and (DOUBOVA, 2007), respectively. Reconstruction algorithms have been considered and applied in (ABDA, 2009) and (ALVAREZ, 2008) for the stationary Stokes system and in (DOUBOVA, 2006) and (DOUBOVA, 2007) for the Navier-Stokes and Boussinesq systems.

Note that, in the applications to fluid mechanics, the goal is to identify the shape of a body around which a fluid flows from measurements performed far from the body. In other contexts, the domain D can represent a rigid body immersed in an elastic medium. Thus, related inverse problems with relevant applications in Elastography have been analyzed in (DOUBOVA A.; FERNÁNDEZ-CARA, 2015) for the wave equation and (DOUBOVA A.; FERNÁNDEZ-CARA, 2018) for the Lamé system.

This chapter is organized as follows. In Section 2.2, we prove a unique continuation property for the solutions to (2.2) and, then, we prove the uniqueness result (Theorem 2.1). Section 2.3 is devoted to proof of the stability result (Theorem 2.2). Finally, in Section 2.4, we present some additional comments and open questions.

2.2 UNIQUE CONTINUATION AND UNIQUENESS

In this section, we analyze a unique continuation property for (2.2). More precisely, we have the following result:

Theorem 2.3 *Let $G \subset R^N$ be a bounded domain whose boundary ∂G is of class $W^{1,\infty}$, let $\omega \subset G$ be a nonempty open set and assume that $a, b, A, B \in L^\infty(G)$. Then, any solution $(y, z) \in H^1(G) \times H^1(G)$ to the linear system*

$$\begin{cases} -\Delta y + ay + bz = 0 & \text{in} \quad G, \\ -\Delta z + Ay + Bz = 0 & \text{in} \quad G, \end{cases} \tag{2.7}$$

satisfying

$$y = z = 0 \text{ in } \omega, \tag{2.8}$$

is zero everywhere.

As already mentioned, the proof relies on some ideas from (FABRE, 1995). In fact, we will divide the proof in two parts: (i) the proof of Theorem 2.3 when ω and G are open balls and (ii) the proof for general domains ω and G, using a compactness argument.

2.2.1 A unique continuation property for balls

In this Section, we prove a very particular result concerning the unique continuation of the solutions to (2.7):

Lemma 2.1 *Assume that $R > 0$, $x_0 \in R^N$ and $a, b, A, B \in L^\infty(B_{2R}(x_0))$, where $B_{2R}(x_0)$ denotes the open ball of radius $2R$ centered at x_0. For any solution $(y, z) \in H^1(B_{2R}(x_0)) \times H^1(B_{2R}(x_0))$ to the linear system (2.7) in $B_{2R}(x_0)$, the following property holds:*

$$(y, z) = (0, 0) \text{ in } B_R(x_0) \quad \Rightarrow \quad (y, z) = (0, 0) \text{ in } B_{2R}(x_0).$$

Before proving this lemma, let us introduce $\varphi \in C_0^\infty(R^N)$ and let us define

$$a_0(x, \xi) := |\xi|^2 - |\nabla\varphi(x)|^2 \quad \text{and} \quad b_0(x, \xi) := 2\xi \cdot \nabla\varphi(x).$$

Let us also recall that the Poisson bracket of a_0 and b_0 is given by

$$[a_0, b_0] := \nabla_\xi a_0 \cdot \nabla_x b_0 - \nabla_x a_0 \cdot \nabla_\xi b_0.$$

A crucial result in the proof of Lemma 2.1 is the following:

Theorem 2.4 ((FABRE, 1995, Proposition 2.3**))** *Let* $U \subset R^N$ *be a nonempty bounded open set,* $K \subset U$ *a nonempty compact set and assume that* $\varphi \in C_0^\infty(R^N)$*. Suppose that* φ *is bi-convex in* U *with respect to the characteristics of* a_0 *and* b_0*, i.e.* φ *satisfies the following:*

$$\begin{cases} \nabla\varphi(x) \neq 0 \quad \forall x \in U, \\ \exists\, C_0 > 0 \text{ so that } [a_0, b_0](x,\xi) \geq C_0 \text{ when } (x,\xi) \in U \times R^N \text{ and } a_0(x,\xi) = b_0(x,\xi) = 0. \end{cases} \tag{2.9}$$

Then, there exist $C_1 > 0$ *and* $h_1 > 0$ *such that, for all* $0 < h < h_1$ *and any function* $u \in H_0^2(K)$*, one has:*

$$I_0(u) := \int_K e^{2\varphi/h}|u|^2\,dx + h^2\int_K e^{2\varphi/h}|\nabla u|^2\,dx \leq C_1 h^3 \int_K e^{2\varphi/h}|\Delta u|^2\,dx.$$

[Proof of Lemma 2.1] Without loss of generality we may assume that $x_0 = 0$. Let $(y,z) \in H^1(B_{2R}) \times H^1(B_{2R})$ be a solution to (2.7) such that $y = 0$ and $z = 0$ in B_R.

We will try to apply Theorem 2.4 to the functions y and z. To do this, let us fix $\varepsilon > 0$ and let us introduce the sets

$$K := \left\{x \in R^N : \frac{3}{4}R \leq |x| \leq 2R - \varepsilon\right\} \quad \text{and} \quad U := \left\{x \in R^N : \frac{1}{2}R < |x| < 2R\right\}$$

and a function $\varphi \in C_0^\infty(R^N)$, with

$$\varphi(x) := e^{-\delta|x|^2}\ \forall x \in \overline{B}_{2R}, \quad \delta > 4/R^2. \tag{2.10}$$

It is not difficult to see that

$$\partial_{x_j}\varphi(x) = -2\delta x_j\varphi(x) \quad \text{and} \quad \partial_{x_j}\partial_{x_k}\varphi(x) = -2\delta\varphi(x)\delta_{jk} + 4\delta^2 x_j x_k\varphi(x), \tag{2.11}$$

where the δ_{jk} are the Kronecker symbols. From (2.10) and (2.11), we have that

$$\begin{aligned}[a_0, b_0](x,\xi) &= 64\,\delta^3\varphi(x)^3|x|^2\left[\delta|x|^2 - 1\right] \\ &\geq 16\,\delta^3 R^2 e^{-12R^2\delta}\left(\frac{\delta R^2}{4} - 1\right)\end{aligned}$$

for any $(x,\xi) \in U \times R^N$ such that $a_0(x,\xi) = b_0(x,\xi) = 0$. Therefore, we see that (2.9) is satisfied by the function φ in U.

Let us introduce a cut-off function $\zeta \in C_0^\infty(\mathring{K})$ satisfying $\zeta \equiv 1$ for $R - \varepsilon \leq |x| \leq 2R - 2\varepsilon$ and let us set $\tilde{y} := \zeta y$ and $\tilde{z} := \zeta z$. It is then clear that $(\tilde{y}, \tilde{z}) \in H_0^2(K) \times H_0^2(K)$. After some computations, we obtain that:

$$\begin{cases} \Delta\tilde{y} = a\tilde{y} + b\tilde{z} + H_1, \\ \Delta\tilde{z} = A\tilde{y} + B\tilde{z} + H_2, \end{cases}$$

where

$$H_1 := 2\nabla\zeta \cdot \nabla y + y\Delta\zeta \quad \text{and} \quad H_2 := 2\nabla\zeta \cdot \nabla z + z\Delta\zeta. \tag{2.12}$$

Consequently, we can apply Theorem 2.4 to $\tilde{y}$ and deduce that there exist $C_2 > 0$ and $h_2 > 0$ such that

$$I_0(\tilde{y}) \leq C_2 h^3 \left(\int_K e^{2\varphi/h} |\tilde{z}|^2 \, dx + \int_K e^{2\varphi/h} |H_1|^2 \, dx \right) \tag{2.13}$$

for all $h \in (0, h_2)$. Here, we have absorbed the lower order term for $\tilde{y}$ from the right hand side by taking h_2 small enough. Analogously, there exist positive constants $C_3 > 0$ and $h_3 > 0$ such that,

$$I_0(\tilde{z}) \leq C_3 h^3 \left(\int_K e^{2\varphi/h} |\tilde{y}|^2 \, dx + \int_K e^{2\varphi/h} |H_2|^2 \, dx \right) \tag{2.14}$$

for all $h \in (0, h_3)$.

Next, adding (2.13) and (2.14), taking h_4 sufficiently small and C_4 sufficiently large and absorbing again the lower order terms for $\tilde{y}$ and $\tilde{z}$ from the right hand side, we have

$$I_0(\tilde{y}) + I_0(\tilde{z}) \leq C_4 h^3 \int_K e^{2\varphi/h} \left(|H_1|^2 + |H_2|^2 \right) \, dx, \tag{2.15}$$

for all $h \in (0, h_4)$.

To conclude the proof, we note that (2.12), the fact that $y = z = 0$ in B_R and $\nabla\zeta = \Delta\zeta = 0$ for $R - \varepsilon \leq |x| \leq 2R - 2\varepsilon$ imply that H_1 and H_2 vanish in $\overline{B}_{2R-2\varepsilon}$. Now, we have from (2.10) that φ is positive and radially decreasing in U. Thus, one has

$$\int_K e^{2\varphi/h} (|H_1|^2 + |H_2|^2) \, dx \leq e^{2\varphi(2R-2\varepsilon)/h} \int_K (|H_1|^2 + |H_2|^2) \, dx.$$

On the other hand,

$$\begin{aligned} \int_K e^{2\varphi/h} (|\tilde{y}|^2 + |\tilde{z}|^2) \, dx &\geq \int_{R \leq |x| \leq 2R-3\varepsilon} e^{2\varphi/h} (|y|^2 + |z|^2) \, dx \\ &\geq e^{2\varphi(2R-3\varepsilon)/h} \int_{R \leq |x| \leq 2R-3\varepsilon} (|y|^2 + |z|^2) \, dx. \end{aligned} \tag{2.16}$$

It follows from (2.15)–(2.16) that

$$\int_{R \leq |x| \leq 2R-3\varepsilon} (|y|^2 + |z|^2) \, dx \leq C_4 h^3 e^{2[\varphi(2R-2\varepsilon) - \varphi(2R-3\varepsilon)]/h} \int_K (|H_1|^2 + |H_2|^2) \, dx.$$

Since H_1 and H_2 are independent of h and $\varphi(2R - 2\varepsilon) < \varphi(2R - 3\varepsilon)$, we can let $h \to 0$ and get that

$$y = z = 0 \quad \text{in} \quad R \leq |x| \leq 2R - 3\varepsilon$$

and, consequently, y and z vanish in $B_{2R-3\varepsilon}$. Since $\varepsilon > 0$ is arbitrarily small, we conclude that y and z vanish identically in B_{2R}.

2.2.2 Unique continuation for general domains

The goal of this section is to prove Theorem 2.3 in the general case.

Let $(y,z) \in H^1(G) \times H^1(G)$ be a solution to (2.7) satisfying (2.8) and let us assume that $\overline{B_{\rho_0}(x_0)} \subset \omega$. Let x_1 be a point of G and let us see that $y = 0$ and $z = 0$ in a neighborhood of x_1.

Since G is connected, there exists a curve $\eta \in C^\infty([0,1];G)$ such that $\eta(0) = x_0$ and $\eta(1) = x_1$.

Notice that for any $t \in [0,1]$ there exists $r_t > 0$ such that $\overline{B_{2r_t}(\eta(t))} \subset G$. Since $\Gamma := \eta\left([0,1]\right)$ is a compact set, there exist $m \geq 1$ and $0 \leq t_1 < \ldots < t_m \leq 1$ satisfying

$$\Gamma \subset \bigcup_{j=1}^{m} B_{r_j}\left(\eta(t_j)\right), \text{ where we have set } r_j := r_{t_j}.$$

By construction, setting $\rho_1 := \min\{r_1, \ldots, r_m, \rho_0\}$, we have that $\overline{B_{\rho_1}(x)} \subset G$, for all $x \in \Gamma$.

Finally, we set $r_0 := \rho_1/2$ and fix $0 < r < r_0$. It is clear that (y,z) vanishes in $B_r(x_0)$ whence, by Lemma 2.1, (y,z) also vanishes in $B_{2r}(x_0)$. Let $\xi_1 := \eta(\tau_1) \in \partial B_r(x_0) \cap \Gamma$. Then, we have that $(y,z) = (0,0)$ in $B_r(\xi_1)$ and, in view of Lemma 2.1, $(y,z) = (0,0)$ in $B_{2r}(\xi_1)$. Applying the same idea a finite number of times, we obtain $y = 0$ and $z = 0$ in $B_r(x_1)$. This ends the proof.

2.2.3 Proof of Theorem 1: uniqueness

Let us introduce the open sets $D := D^0 \cup D^1$ and $O^0 := \Omega \backslash \overline{D}$ and let O be the unique connected component of O^0 such that $\partial\Omega \subset \partial O$. Also, let us set $y := y^0 - y^1$ and $z := z^0 - z^1$ in O. Since $\alpha^0 = \alpha^1$ and $\beta^0 = \beta^1$, the couple $(y,z) \in H^1(O) \times H^1(O)$ satisfies:

$$\begin{cases} -\Delta y + ay + bz = 0 & \text{in} \quad O, \\ -\Delta z + Ay + Bz = 0 & \text{in} \quad O, \\ y = z = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} = 0 & \text{on} \quad \gamma. \end{cases} \tag{2.17}$$

Now, we fix $x_0 \in \gamma$ and we choose $r > 0$ such that $\overline{B}_r(x_0) \cap \partial\Omega \subset \gamma$. Let us set $O' := O \cup B_r(x_0)$ and consider the extension by zero $(\tilde{y}, \tilde{z})$ of (y,z) to the whole set O'.

From (2.17), it follows that

$$\begin{cases} -\Delta\tilde{y} + a\tilde{y} + b\tilde{z} = 0 & \text{in} \quad O', \\ -\Delta\tilde{z} + A\tilde{y} + B\tilde{z} = 0 & \text{in} \quad O'. \end{cases}$$

Also, since O' is connected and $(\tilde{y}, \tilde{z}) = (0, 0)$ in $O' \backslash \overline{O}$, Theorem 2.3 implies $(\tilde{y}, \tilde{z}) = (0, 0)$ in O'. In particular, $(y, z) = (0, 0)$ in O.

To conclude, let us prove that $D^1 \backslash \overline{D}^0$ and $D^0 \backslash \overline{D}^1$ must be empty. Thus, let us suppose that $D^1 \backslash \overline{D}^0 \neq \emptyset$ and let us introduce the set $D^2 := D^1 \cup [(\Omega \backslash \overline{D}^0) \cap (\Omega \backslash \overline{O})]$. By hypothesis, $D^2 \backslash \overline{D}^0$ is nonempty.On the other hand, note that $\partial(D^2 \backslash \overline{D}^0) := \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 = \partial(D^2 \backslash \overline{D}^0) \cap \partial D^0$ and $\Gamma_1 = \partial(D^2 \backslash \overline{D}^0) \cap \partial D^1$.

Therefore, since $(y^0, z^0) = (y^1, z^1)$ in O, the pair (y^0, z^0) verifies

$$\begin{cases} -\Delta y^0 + a y^0 + b z^0 = 0 & \text{in} \quad D^2 \backslash \overline{D}^0, \\ -\Delta z^0 + A y^0 + B z^0 = 0 & \text{in} \quad D^2 \backslash \overline{D}^0, \\ y^0 = 0, \; z^0 = 0 & \text{on} \quad \Gamma_0, \\ y^0 = 0, \; z^0 = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{2.18}$$

Since the linear system (2.18) possesses exactly one solution, we necessarily have $(y^0, z^0) = (0, 0)$ in $D^2 \backslash \overline{D}^0$. Consequently, in view of Theorem 2.3, $(y^0, z^0) = (0, 0)$ in $\Omega \backslash \overline{D}^0$. This contradicts the fact that (φ, ψ) is not identically zero on $\partial\Omega$. Hence, $D^1 \backslash \overline{D}^0$ is the empty set.

Analogously, one can prove that $D^0 \backslash \overline{D}^1$ is empty and, finally, one has $D^0 = D^1$.

Remark 2.3 *Consider a multiple domain $\Sigma = D^0 \cup D^1 \cup \ldots \cup D^k$, with $D^i \cap D^j = \emptyset$, if $i \neq j$, and $D^i \in \mathcal{D}$, for all $i = 1, \ldots, k$. Then, by applying similar techniques as these above, one can see that Theorem 2.1 can be extended to a geometric inverse problem which consists to find an unknown multiple domain Σ such that the corresponding solution to* (2.2) *(with D replaced by Σ) generates the observation* (2.3). *That is, if two multiple domains Σ^0 and Σ^1 generate the same observation (α, β) in* (2.3), *then $\Sigma^0 = \Sigma^1$.*

2.3 STABILITY

2.3.1 Preliminary results

Let us introduce some basic notation. Let $m = (m^1, \ldots, m^N) \in W^{1,\infty}(R^N; R^N)$ be given and let us set

$$m' := \left(\frac{\partial m^i}{\partial x_j}\right)_{i,j=1}^N, \quad \text{Jac}(m) := |\det(m')|, \quad M := ((m')^*)^{-1}.$$

In the sequel, we will consider the set

$$\mathcal{W}_\epsilon := \{\mu \in W^{1,\infty}(R^N; R^N) : \|\mu\|_{W^{1,\infty}} < \epsilon, \quad \mu = 0 \ \text{ in } \ \Omega\backslash\overline{D}^*\},$$

where $0 < \epsilon < 1$. We will work with mappings of the form $m := I + \mu$, where $I : R^N \mapsto R^N$ is the identity and $\mu \in \mathcal{W}_\epsilon$. For any $\mu \in \mathcal{W}_\epsilon$, $I + \mu$ is obviously bijective, $(I+\mu)^{-1} \in W^{1,\infty}(R^N; R^N)$ (see (HENROT A.; PIERRE, 2018), p. 193) and

$$(I + \mu)(D) \in \mathcal{D}, \quad \forall D \in \mathcal{D}.$$

Also, the corresponding functions Jac(m), M and M^{-1} satisfy

$$\text{Jac}(m) \geq C(\epsilon) > 0, \quad \|M\|_{L^\infty} + \|M^{-1}\|_{L^\infty} \leq C(\epsilon). \tag{2.19}$$

Let $D^0 \in \mathcal{D}$ and $\mu \in \mathcal{W}_\epsilon$ be given, let us set again $m := I + \mu$ and $D^1 = m(D^0)$ and let us consider the solution $(y_1, z_1) \in H^1(\Omega\backslash\overline{D}^1)^2$ to

$$\begin{cases} -\Delta y_1 + a y_1 + b z_1 = 0 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ -\Delta z_1 + A y_1 + B z_1 = 0 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ y_1 = \varphi, \ z_1 = \psi & \text{on} \quad \partial\Omega, \\ y_1 = 0, \ z_1 = 0 & \text{on} \quad \partial D^1. \end{cases}$$

Since $(\varphi, \psi) \in H^{1/2}(\partial\Omega)^2$, there exists $(\varphi_1, \psi_1) \in H^1(\Omega)^2$ such that

$$(\varphi_1, \psi_1) = 0 \text{ in } \overline{D^*} \quad \text{and} \quad (\varphi_1, \psi_1) = (\varphi, \psi) \text{ on } \partial\Omega.$$

Thus, we can write $(y_1, z_1) = (u_1 + \varphi_1, v_1 + \psi_1)$, where (u_1, v_1) is the solution to

$$\begin{cases} -\Delta u_1 + a u_1 + b v_1 = F_1 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ -\Delta v_1 + A u_1 + B v_1 = G_1 & \text{in} \quad \Omega\backslash\overline{D}^1, \\ u_1 = 0, \ v_1 = 0 & \text{on} \quad \partial\Omega \cup \partial D^1 \end{cases} \tag{2.20}$$

and

$$F_1 = \Delta\varphi_1 - a\varphi_1 - b\psi_1, \quad G_1 = \Delta\psi_1 - A\varphi_1 - B\psi_1.$$

We see from (2.20) that for any $(w,p) \in H_0^1(\Omega\backslash\overline{D}^1)^2$ the following holds:

$$\begin{aligned}&\int_{\Omega\backslash\overline{D}^1}(\nabla u_1\cdot\nabla w+\nabla v_1\cdot\nabla p)\;dy+\int_{\Omega\backslash\overline{D}^1}(au_1w+bv_1w+Au_1p+Bv_1p)\;dy\\&=-\int_{\Omega\backslash\overline{D}^1}(\nabla\varphi_1\cdot\nabla w+\nabla\psi_1\cdot\nabla p)\;dy-\int_{\Omega\backslash\overline{D}^1}(a\varphi_1w+b\psi_1w+A\varphi_1p+B\psi_1p)\;dy.\end{aligned} \tag{2.21}$$

Let us introduce the functions:

$$u_0 := \tilde{m}(u_1), \quad v_0 := \tilde{m}(v_1), \quad \varphi_0 := \tilde{m}(\varphi_1), \quad \psi_0 := \tilde{m}(\psi_1),$$

where $\tilde{m}$ is the isomorphism from $H_0^1(\Omega\backslash\overline{D}^1)^2$ onto $H_0^1(\Omega\backslash\overline{D}^0)^2$ induced by m, that is,

$$\tilde{m}(f) := f\circ m, \quad \forall f \in H_0^1(\Omega\backslash\overline{D}^1)^2. \tag{2.22}$$

We get easily from (2.19) that there exists a positive constant $C = C(\epsilon)$ such that

$$\|\widetilde{m}(f)\|_{H_0^1(\Omega\backslash\overline{D}^0)^2} \leq C(\epsilon)\|f\|_{H_0^1(\Omega\backslash\overline{D}^1)^2}, \;\forall\; f\in H_0^1(\Omega\backslash\overline{D}^1)^2.$$

Moreover, it's easy to see that $\widetilde{m}^{-1}$ is the linear mapping induced by the inverse m^{-1} of the m, that is, $\widetilde{m}^{-1}(g) = g\circ m^{-1}$, for all $g \in H_0^1(\Omega\backslash\overline{D}^0)^2$.

Observe that, since $(\varphi_1,\psi_1) = 0$ in D^* and $m = I$ in $\Omega\backslash\overline{D}^*$, we have $(\varphi_0,\psi_0) = (\varphi_1,\psi_1)$ in Ω. In other words, (φ_1,ψ_1) is invariant under the isomorphism $\tilde{m}$ associated to m.

It can be easily shown that solving the variational problem (2.21) is equivalent to find $(u_0,v_0) \in H_0^1(\Omega\backslash\overline{D}^0)^2$ such that

$$\begin{aligned}&\int_{\Omega\backslash\overline{D}^0}(M\nabla u_0\cdot M\nabla z+M\nabla v_0\cdot M\nabla q)\,\mathsf{Jac}(m)\,dx\\&+\int_{\Omega\backslash\overline{D}^0}(au_0z+bv_0z+Au_0q+Bv_0q)\,\mathsf{Jac}(m)\,dx\\&\quad=-\int_{\Omega\backslash\overline{D}^0}(M\nabla\varphi_0\cdot M\nabla z+M\nabla\psi_0\cdot M\nabla q)\,\mathsf{Jac}(m)\,dx\\&\quad-\int_{\Omega\backslash\overline{D}^0}(a\varphi_0z+b\psi_0z+A\varphi_0q+B\psi_0q)\,\mathsf{Jac}(m)\,dx,\end{aligned}$$

for all $(z,q) \in H_0^1(\Omega\backslash\overline{D}^0)^2$ that is, a solution to the system:

$$\begin{cases}-\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla u_0)+(au_0+bv_0)\,\mathsf{Jac}(m)=F_0 & \text{in}\quad \Omega\backslash\overline{D}^0,\\ -\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla v_0)+(Au_0+Bv_0)\,\mathsf{Jac}(m)=G_0 & \text{in}\quad \Omega\backslash\overline{D}^0,\\ u_0=0,\;\; v_0=0 & \text{on}\quad \partial\Omega\cup\partial D^0,\end{cases} \tag{2.23}$$

where $(F_0, G_0) \in H^{-1}(\Omega\backslash\overline{D}^0)^2$ is given by

$$F_0 = \nabla \cdot (\mathsf{Jac}(m) M^* M \nabla \varphi_0) - (a\varphi_0 + b\psi_0)\,\mathsf{Jac}(m),$$

$$G_0 = \nabla \cdot (\mathsf{Jac}(m) M^* M \nabla \psi_0) - (A\varphi_0 + B\psi_0)\,\mathsf{Jac}(m).$$

For convenience, we will rewrite (2.23) in the abridged form

$$\begin{cases} T(u_0, v_0) = (F_0, G_0) & \text{in} \quad \Omega\backslash\overline{D}^0, \\ u_0 = 0,\ v_0 = 0 & \text{on} \quad \partial\Omega \cup \partial D^0, \end{cases} \tag{2.24}$$

where the notation is self-explanatory.

Lemma 2.2 *The linear operator $T : H_0^1(\Omega\backslash\overline{D}^0)^2 \mapsto H^{-1}(\Omega\backslash\overline{D}^0)^2$ is an isomorphism. Furthermore, if $\|\cdot\|_{\mathcal{L}_0}$ and $\|\cdot\|_{\mathcal{L}_0'}$ denotes the usual norm in $\mathcal{L}(H_0^1(\Omega\backslash\overline{D}^0)^2; H^{-1}(\Omega\backslash\overline{D}^0)^2)$ and $\mathcal{L}(H^{-1}(\Omega\backslash\overline{D}^0)^2; H_0^1(\Omega\backslash\overline{D}^0)^2)$, respectively, one has*

$$\|T\|_{\mathcal{L}_0} + \|T^{-1}\|_{\mathcal{L}_0'} \leq C(\epsilon).$$

It is easy to see that $T \in \mathcal{L}(H_0^1(\Omega\backslash\overline{D}^0)^2; H^{-1}(\Omega\backslash\overline{D}^0)^2)$ and $\|T\|_{\mathcal{L}_0} \leq C(\epsilon)$. On the other hand, the continuous and bilinear form $\tau(\cdot\,,\cdot)$, given by

$$\tau(\,(u,v),(z,q)\,) := \langle T(u,v),(z,q)\rangle_{H^{-1},H_0^1} \quad \forall (u,v),(z,q) \in H_0^1(\Omega\backslash\overline{D}^0)^2,$$

is coercive. Indeed, given $(u,v) \in H_0^1(\Omega\backslash\overline{D}^0)^2$ we have from the fact that mapping $\widetilde{m}$ defined in (2.22) is an isomorphism that there exists a unique pair $(u_1, v_1) \in H_0^1(\Omega\backslash\overline{D}^1)^2$ such that $(u,v) = \widetilde{m}(u_1, v_1)$. Thus,

$$\begin{aligned} \tau((u,v),(u,v)) &= \int_{\Omega\backslash\overline{D}^0} (|M\nabla u|^2 + |M\nabla v|^2)\,\mathsf{Jac}(m)\,dx \\ &+ \int_{\Omega\backslash\overline{D}^0} (a|u|^2 + B|v|^2 + (b+A)uv)\mathsf{Jac}(m)\,dx \\ &\geq \int_{\Omega\backslash\overline{D}^0} (|M\nabla u|^2 + |M\nabla v|^2)\,\mathsf{Jac}(m)\,dx \\ &-\lambda \int_{\Omega\backslash\overline{D}^0} (|u|^2 + |v|^2)\,\mathsf{Jac}(m)\,dx \\ &\geq (1 - \lambda\mu_1(\Omega))\|(u_1,v_1)\|^2_{H_0^1(\Omega\backslash\overline{D}^1)^2}, \end{aligned}$$

where in the last inequality we used the fact that (2.1) is satisfied by the coefficients a, b, A and B. Then, it follows from this and from fact $\widetilde{m}$ is a isomorphism that

$$\begin{aligned} \tau((u,v),(u,v)) &\geq (1 - \lambda\mu_1(\Omega))\|(u_1,v_1)\|^2_{H_0^1(\Omega\backslash\overline{D}^1)^2} \\ &= (1 - \lambda\mu_1(\Omega))\|\widetilde{m}^{-1}(u,v)\|^2_{H_0^1(\Omega\backslash\overline{D}^1)^2} \\ &\geq C(\epsilon)(1 - \lambda\mu_1(\Omega))\|(u,v)\|^2_{H_0^1(\Omega\backslash\overline{D}^0)^2}. \end{aligned}$$

Therefore, from Lax-Milgram's Lemma, we get the result.

Theorem 2.5 *The mapping $\mu \mapsto (u_0, v_0)$ is analytic in a neighbourhood of the origin in $\mathcal{W}_\epsilon$.*

We note first that (F_0, G_0) does not depend of μ, because $(\varphi_0, \psi_0) = 0$ where $\mu \neq 0$. Then, since the mapping T is a isomorphism we have by (2.24) that

$$(u_0, v_0) = T^{-1}(F_0, G_0).$$

We prove in the Appendix A the mapping $\mu \mapsto T$ is analytic in a neighbourhood of 0. Consequently, this is also the case for $\mu \mapsto (u_0, v_0)$ and the proof is done.

2.3.2 Proof of Theorem 2: stability

Let $D^0 \in \mathcal{D}$ and $\mu \in W_\epsilon$ be given, with $\mu \not\equiv 0$ in D^0. Recall that, in Theorem 2, for any $\sigma \in (-1, 1)$, we have set $m_\sigma := I + \sigma\mu$ and $D^\sigma := m_\sigma(D^0)$ and $(y_\sigma, z_\sigma) \in H^1(\Omega \backslash \overline{D}_\sigma)^2$ is the solution to

$$\begin{cases} -\Delta y_\sigma + a y_\sigma + b z_\sigma = 0 & \text{in} \quad \Omega \backslash \overline{D}_\sigma, \\ -\Delta z_\sigma + A y_\sigma + B z_\sigma = 0 & \text{in} \quad \Omega \backslash \overline{D}_\sigma, \\ y_\sigma = \varphi, \ z_\sigma = \psi & \text{on} \quad \partial\Omega, \\ y_\sigma = 0, \ z_\sigma = 0 & \text{on} \quad \partial D^\sigma. \end{cases} \tag{2.25}$$

We will argue as in (ALVAREZ, 2005):

1. First, it follows from Theorem 2.5 and the fact that $\mu \equiv 0$ in $\Omega \backslash \overline{D}^*$ that there exists $\sigma_0 > 0$ such that the mapping in (2.5) is well defined and analytic in $(-\sigma_0, \sigma_0)$. Hence, there exist $F_1, F_2, \ldots$ in $H^{-1/2}(\gamma)^2$ such that
$$\left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n} \right) - \left(\frac{\partial y_0}{\partial n}, \frac{\partial z_0}{\partial n} \right) = \sum_{j=1}^{\infty} \sigma^j F_j \quad \forall \sigma \in (-\sigma_0, \sigma_0), \tag{2.26}$$
where the series converges in $H^{-1/2}(\gamma)^2$.

2. Now, let us assume that $m_\sigma(D^0) \neq D^0$ for some $\sigma \in (-\sigma_0, \sigma_0)$. In view of Theorem 1, not all the F_j can be zero. Let k_0 be the smallest j such that $F_j \neq 0$. It is then clear that there exists $\sigma_* \in (0, \sigma_0)$ such that
$$\left\| \sum_{j=k_0+1}^{\infty} \sigma^j F_j \right\|_{H^{-1/2}} \leq \frac{1}{2} |\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}}, \quad \forall \sigma \in (-\sigma_*, \sigma_*). \tag{2.27}$$
Indeed, using the definition of series' convergence, we can see easily that fixed $\sigma \in (-\sigma_0, \sigma_0)$ and taking $\varepsilon = 1/4 |\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}} > 0$, there exists $p_0 \in N$ such that if

$p \geq p_0$ then

$$\left\| \sum_{j=p+1}^{\infty} \sigma^j F_j \right\|_{H^{-1/2}} \leq \varepsilon.$$

If $p_0 \leq k_0$ then the inequality in (2.27) is easily obtained. Now, suppose that $p_0 > k_0$, that is, $p_0 = k_0 + k_1$, for some positive integer $k_1 \geq 1$. Thus,

$$\left\| \sum_{j=k_0+k_1+1}^{\infty} \sigma^j F_j \right\|_{H^{-1/2}} \leq \frac{1}{4}|\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}}$$

and, consequently,

$$\left\| \sum_{j=k_0+1}^{\infty} \sigma^j F_j \right\|_{H^{-1/2}} \leq \frac{1}{4}|\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}} + \left\| \sum_{j=k_0+1}^{k_1+1} \sigma^j F_j \right\|_{H^{-1/2}}.$$

Then, by taking $\sigma_* \in (0, \sigma_0)$ so that

$$\sum_{k_0+1}^{k_0+k_1} |\sigma|^j \|F_j\|_{H^{-1/2}} \leq \frac{1}{4}|\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}},$$

for all $\sigma \in (-\sigma_*, \sigma_*)$, we get the inequality (2.27).

Then, for these $\sigma \in (-\sigma_*, \sigma_*)$, one must also have

$$|\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}} \leq \left\| \left(\frac{\partial y_\sigma}{\partial n}, \frac{\partial z_\sigma}{\partial n} \right) - \left(\frac{\partial y_0}{\partial n}, \frac{\partial z_0}{\partial n} \right) \right\|_{H^{-1/2}} + \frac{1}{2}|\sigma|^{k_0} \|F_{k_0}\|_{H^{-1/2}},$$

which allows to achieve the proof.

Remark 2.4 *Similar to Remark 2.3 above, one can consider a fixed multiple domain $\Sigma_0 = D_0^1 \cup \ldots \cup D_0^k$, with $D_0^i \cap D_0^j = \emptyset$ if $i \neq j$, and $D_0^i \in \mathcal{D}$, for all $i = 1, \ldots, k$. Then, assuming that the set Σ_0 is subjected to perturbations, as described in this section, it is possible to prove a version of the Theorem 2.2 for multiple domains.*

2.4 ADDITIONAL COMMENTS AND QUESTIONS

2.4.1 A similar inverse problem with internal observation

Let $\omega \subset\subset \Omega \backslash \overline{D}^*$ be a nonempty open set. Consider the following geometric inverse problem, where the observation is performed on ω:

Given $(\varphi, \psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $\alpha \in H^1(\omega)$, find a set $D \in \mathcal{D}$ such that the solution (y, z) to the linear system (2.2) satisfies the following additional condition:

$$y\Big|_{\omega} = \alpha. \tag{2.28}$$

We have the following uniqueness result:

Theorem 2.6 *Assume that* $(\varphi, \psi) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ *is nonzero and suppose that there exists a nonempty open set* $\omega_0 \subset \omega$ *such that* $b \neq 0$ *a.e. in* ω_0. *Let* (y^i, z^i) *be the unique weak solution to* (2.2) *with* D *replaced by* D^i *for* $i = 0, 1$ *and let* α^i *be given by the corresponding equality* (2.28). *Then, one has:*

$$\alpha^0 = \alpha^1 \quad \Longrightarrow \quad D^0 = D^1.$$

The proof is very similar to the proof of Theorem 2.1.

As before, we can consider the open sets $D := D^0 \cup D^1$, $O^0 := \Omega \backslash \overline{D}$ and the unique connected component O of O^0 such that $\partial\Omega \subset \partial O$. Again, let us set $y := y^0 - y^1$ and $z := z^0 - z^1$ in O. Then, using the facts that $\alpha^0 = \alpha^1$ and $b \neq 0$ a.e. in ω_0, we have that $(y, z) \in H^1(O) \times H^1(O)$ and satisfies

$$\begin{cases} -\Delta y + ay + bz = 0 & \text{in} \quad O, \\ -\Delta z + Ay + Bz = 0 & \text{in} \quad O \end{cases}$$

and

$$y = z = 0 \ \text{ in } \ \omega_0.$$

Consequently, Theorem 2.3 guarantees that $(y, z) = (0, 0)$ in O. Arguing as in the proof of Theorem 2.1, we deduce that $D^0 \backslash \overline{D}^1$ and $D^1 \backslash \overline{D}^0$ are empty sets and, consequently, $D^0 = D^1$.

We also have a stability result similar to Theorem 2.2. Thus, let us fix $D^0 \in \mathcal{D}$ and $\mu \in W_\epsilon$ with $\mu \neq 0$ in D^0, let us take $m_\sigma = I + \sigma\mu$ and $D^\sigma = m_\sigma(D^0)$ and let (y_σ, z_σ) be the solution to (2.25). The following holds:

Theorem 2.7 *Under the assumptions in Theorem 2.6 on* (φ, ψ) *and* b, *there exists* $\sigma_0 > 0$ *with the following properties:*

1. *The mapping*

$$\sigma \mapsto y_\sigma\Big|_\omega \tag{2.29}$$

   is well defined and analytic in $(-\sigma_0, \sigma_0)$, *with values in* $L^2(\omega)$.

2. *Either* $m_\sigma(D^0) = D^0$ *for all* $\sigma \in (-\sigma_0, \sigma_0)$ *(and then the mapping in* (2.29) *is constant), or there exist* $\sigma_* \in (0, \sigma_0)$, $C > 0$ *and* $k \geq 1$ *(an integer) such that*

$$\left\| (y_\sigma - y_0)\Big|_\omega \right\|_{L^2} \geq C|\sigma|^k \quad \forall \sigma \in (-\sigma_*, \sigma_*).$$

Again, the proof is very similar to the proof of stability in the boundary observation case (Theorem 2.2). In fact, the unique difference appears in the last part of the argument, when we write

$$(y_\sigma - y_0)\Big|_\omega = \sum_{j=1}^{\infty} \sigma^j F_j \quad \forall \sigma \in (-\sigma_0, \sigma_0),$$

instead of (2.26).

For brevity, we omit the details.

Remark 2.5 *Recall that, in the case of problem* (2.2)–(2.3)*, we need two boundary observations, the normal derivatives of y and z on γ, to deduce uniqueness and stability. The last two results show that, with internal observations, this holds with the information supplied by just one variable.*

2.4.2 A geometric inverse problem for a parabolic system

Let us present some ideas that allow to extend Theorems 2.1 and 2.6 to time-dependent parabolic systems. For brevity, we will only consider the boundary observation case. Thus, let $T > 0$ be given and let us consider the following inverse problem:

> We consider $(\varphi, \psi) \in C^1([0,T]; H^{3/2}(\partial\Omega)^2)$, with $\varphi(\cdot, 0) = \psi(\cdot, 0) = 0$ on $\partial\Omega$ and $(\alpha, \beta) \in L^2(0,T; H^{-1/2}(\gamma)^2)$, where $\gamma \subset \partial\Omega$ is a nonempty open set. Then, our goal is find an open set $D \in \mathcal{D}$ such that the unique solution (y, z) to the linear evolution system:
>
> $$\begin{cases} y_t - \Delta y + ay + bz = 0 & \text{in} \quad \Omega\backslash\overline{D} \times (0,T), \\ z_t - \Delta z + Ay + Bz = 0 & \text{in} \quad \Omega\backslash\overline{D} \times (0,T), \\ y = \varphi, \ \ z = \psi & \text{on} \quad \partial\Omega \times (0,T), \\ y = 0, \ \ z = 0 & \text{on} \quad \partial D \times (0,T), \\ y(\cdot\,,0) = 0, \ \ z(\cdot\,,0) = 0 & \text{in} \quad \Omega\backslash\overline{D}, \end{cases} \tag{2.30}$$
>
> satisfies the additional conditions:
>
> $$\frac{\partial y}{\partial n}\Big|_{\gamma\times(0,T)} = \alpha \quad \text{and} \quad \frac{\partial z}{\partial n}\Big|_{\gamma\times(0,T)} = \beta. \tag{2.31}$$

If one assume that $(\varphi, \psi) \not\equiv (0,0)$, then arguments similar to those in the proof of Theorem 2.1 can be used to deduce uniqueness for (2.30)–(2.31).

Indeed, the first step is to deduce a unique continuation property:

Proposition 2.1 *Let $G \subset R^N$ be a bounded domain whose boundary is of class $W^{2,\infty}$ and let us set $Q := G \times (0,T)$. Suppose that $a, b, A, B \in L^\infty(Q)$ and let O be a nonempty open subset of Q. Then, any solution $(y,z) \in L^2(0,T; H^2(G) \times H^2(G))$ to*

$$\begin{cases} y_t - \Delta y + ay + bz = 0 & \text{in} \ \ Q, \\ z_t - \Delta z + Ay + Bz = 0 & \text{in} \ \ Q. \end{cases} \tag{2.32}$$

satisfies the following property:

$$(y,z) = (0,0) \ \ \text{in} \ \ O \implies (y,z) = (0,0) \ \ \text{in} \ \ C(O),$$

where $C(O)$ is the horizontal component of O, defined by

$$C(O) := \{(x,t) \in Q : \exists\, x_0 \text{ such that } (x_0,t) \in O\}.$$

The proof of this result is similar to the proof of Theorem 1.4 in (FABRE, 1995).

Remark 2.6 *Let $G \subset R^N$ be a bounded domain whose boundary is of class $W^{2,\infty}$ and let Γ_0 be an open nonempty subset of $\partial G \times (0,T)$. Then, any solution (y,z) in $L^2(0,T; H^2(G) \times H^2(G))$ to (2.32) satisfies the following property:*

$$(y,z) = (0,0) \ \ \text{and} \ \ \left(\frac{\partial y}{\partial n}, \frac{\partial z}{\partial n}\right) = (0,0) \ \ \text{on} \ \ \Gamma_0 \implies (y,z) = (0,0) \ \ \text{in} \ \ C(\Gamma_0).$$

Indeed, take $(x_0,t_0) \in \Gamma_0$. Since Γ_0 is open in $\partial G \times (0,T)$, there exist constants $r, \delta > 0$ such that $(B_r(x_0) \cap \partial G) \times (t_0 - \delta, t_0 + \delta) \subset \Gamma_0$. Let us denote by $(\tilde{y}, \tilde{z})$ the extension by zero of (y,z) to $O := (B(x_0; r) \cap G^c) \times (t_0 - \delta, t_0 + \delta)$. Then, writing $\widetilde{G} := G \cup [B(x_0; r) \cap G^c]$, we have that $(\tilde{y}, \tilde{z}) \in L^2(0,T; H^2(\widetilde{G}) \times H^2(\widetilde{G}))$ is a solution of (2.32) in $\widetilde{G} \times (t_0 - \delta, t_0 + \delta)$ and $(\tilde{y}, \tilde{z}) = (0,0)$ in O. Consequently, by Proposition 3.2, $(\tilde{y}, \tilde{z})$ vanishes in $C(O)$. Since (x_0,t_0) is arbitrary in Γ_0, we get the result.

Let us now achieve the proof of uniqueness for the geometric inverse problem (2.30)-(2.31). To this end, let D^0 and D^1 be two open sets in $\mathcal{D}$ and let (y^i, z^i) be the solution to (2.30) with $D = D^i$. Let us also assume that

$$\left(\frac{\partial y^0}{\partial n}, \frac{\partial z^0}{\partial n}\right) = \left(\frac{\partial y^1}{\partial n}, \frac{\partial z^1}{\partial n}\right) \ \text{ on } \gamma \times (0,T).$$

As before, by introducing the open sets $D := D^0 \cup D^1$, $O^0 := \Omega \backslash \overline{D}$ and O, with $y := y^0 - y^1$ and $z := z^0 - z^1$, we have

$$\begin{cases} y_t - \Delta y + ay + bz = 0 & \text{in} \quad O \times (0,T), \\ z_t - \Delta z + Ay + Bz = 0 & \text{in} \quad O \times (0,T), \\ y = 0, \ z = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ \dfrac{\partial y}{\partial n} = 0, \ \dfrac{\partial z}{\partial n} = 0 & \text{on} \quad \gamma \times (0,T). \end{cases}$$

From Remark 2.6, we find that $(y,z) = (0,0)$ in $O \times (0,T)$.

We can prove that $D^1 \backslash \overline{D}^0$ is the empty set. Indeed, suppose the contrary, i.e. that $D^1 \backslash \overline{D}^0$ is nonempty. Let us introduce the open set $D^2 = D^1 \cup ((\Omega \backslash \overline{D^0} \cap (\Omega \backslash \overline{O}))$. As before, using the fact that $(y^0, z^0) = (y^1, z^1)$ in $O \times (0,T)$, we see that

$$\begin{cases} y_t^0 - \Delta y^0 + ay^0 + bz^0 = 0 & \text{in} \quad (D^2 \backslash \overline{D}^0) \times (0,T), \\ z_t^0 - \Delta z^0 + Ay^0 + Bz^0 = 0 & \text{in} \quad (D^2 \backslash \overline{D}^0) \times (0,T), \\ y^0 = z^0 = 0 & \text{on} \quad \partial(D^2 \backslash \overline{D}^0) \times (0,T), \\ y^0(\cdot\,,0) = z^0(\cdot\,,0) = 0 & \text{in} \quad D^2 \backslash \overline{D}^0. \end{cases} \tag{2.33}$$

Consequently, thanks to the uniqueness of solution to (2.33) and Proposition 3.2, we must have $(y^0, z^0) = (0,0)$ in $(\Omega \backslash \overline{D}^0) \times (0,T)$, which implies $(\varphi, \psi) \equiv (0,0)$, an absurd. This proves that $D^1 \subset D^0$.

Similarly, we can also prove that $D^0 \subset D^1$ and, therefore, $D^0 = D^1$.

Remark 2.7 *If, in* (2.30)*, we impose nonzero initial conditions on y and/or z, the situation is much more complex. In particular, the previous argument does not work.*

Remark 2.8 *Notice that, in this time-dependent case, no assumption of the kind* (2.1) *is needed.*

Stability results like Theorems 2.2 and 2.7 can also be established in this framework. We will not give the details for brevity, since the arguments are not very different and can easily be completed by the reader.

2.4.3 Additional comments on stability

In the context of the stability problem, we can adopt another (more geometrical) viewpoint. To clarify the situation, let us consider the scalar systems

$$\begin{cases} -\Delta y^i = 0 & \text{in} \quad \Omega\backslash\overline{D^i}, \\ y = \varphi^i & \text{on} \quad \partial\Omega, \\ y = 0 & \text{on} \quad \partial D^i, \end{cases}$$

where Ω, is as before, D^0 and D^1 are convex and have nonempty intersection and the following regularity properties hold:

$$\varphi^i \in C^2(\partial\Omega), \quad \tilde{\alpha}^i := \frac{\partial y^i}{\partial n} \in C^1(\overline{\gamma}), \quad y^i \in C^2(\overline{\Omega} \setminus D^i).$$

Let us assume that

$$\begin{cases} \|\varphi^i\|_{C^0(\partial\Omega)} \geq m > 0, \quad \|\varphi^0 - \varphi^1\|_{C^2(\partial\Omega)} \leq \epsilon \\ \|\tilde{\alpha}^0 - \tilde{\alpha}^1\|_{C^1(\overline{\gamma})} \leq \epsilon, \quad \|y^i\|_{C^2(\overline{\Omega}\backslash D^i)} \leq M. \end{cases}$$

Then, it can be proved that the Haussdorf distance $d_H(D^0, D^1)$ satisfies the estimate

$$d_H(D^0, D^1) \leq \frac{C}{\left(\log(\log \frac{1}{\epsilon})\right)^2},$$

where C only depends on Ω, M and m; see (BUCKGEIM, 1999).

The proof is based on the following well-posedness results, where C is as above:

- Let us set again $G = \Omega \setminus \overline{D^0 \cup D^1}$ and let us assume that

  $$|y^0(\hat{x}) - y^1(\hat{x})| = \max_{x \in \overline{G}} |y^0(x) - y^1(x)|.$$

  Then, under the previous hypotheses, one has $|y^0(\hat{x}) - y^1(\hat{x})| \leq C \left(\log \frac{1}{\epsilon}\right)^{-1}$.
- Assume that $\|y^0 - y^1\|_{C^0(\overline{G})} \leq \delta$. Then $d_H(D^0, D^1) \leq C \left(\log \frac{1}{\delta}\right)^{-2}$.

Under additional properties for the φ^i, $\tilde{\alpha}^i$ and y^i, the previous estimates can be improved; see (BUCKGEIM, 1999) for more details.

It would be interesting to extend this approach to the inverse problems (2.2), (2.3) and (2.2), (2.28). At present, to our knowledge, whether or not this is possible is an open question.

2.4.4 Reconstruction

As already said, reconstruction algorithms for the solution of problems of the kind (2.2)–(2.3) have been considered in several papers. In all them, the main idea is to reduce to finite dimension and reformulate the search of the unknown D as a constrained (maybe numerically ill-conditionned) extremal problem. Then, usual gradient, quasi-Newton or even Newton methods can be used to compute approximate solutions; see for instance (ABDA, 2009; ALVAREZ, 2005; ALVAREZ, 2008).

Let us present in this section a different approach that relies on the domain variation techniques introduced in (SIMON, 1987).

The main idea is to describe how the observation depends on small perturbations of D as explicitly as possible. For such, let us introduce the Banach space

$$\mathcal{W}_*^{2,\infty} := \{\mu \in W^{2,\infty}(R^N; R^N); \mu \equiv 0 \text{ in } \Omega\backslash\overline{D}^*\}$$

and the open set

$$\mathcal{W}_\epsilon := \{\mu \in \mathcal{W}_*^{2,\infty};\ \|\mu\|_{2,\infty} < \epsilon \text{ and } \mu \equiv 0 \text{ in } \Omega\backslash\overline{D}^*\}.$$

Moreover, for a fixed domain $D \in \mathcal{D}$ and for each $\mu \in \mathcal{W}_\epsilon$, let us set

$$D + \mu := \Big\{z \in R^N, z : x + \mu(x), \quad x \in D\Big\}$$

and let us recall that, whenever $D \in \mathcal{D}$, we also have $D + \mu \in \mathcal{D}$.

Now, let us assume that a, b, A, $B \in C^1(\overline{\Omega})$ satisfy (2.1), $(\varphi, \psi) \in H^{3/2}(\partial\Omega)^2$ and we consider the unique solution $(y_\mu, z_\mu) \in H^2(\Omega\backslash(\overline{D+\mu})^2$ of the perturbed system

$$\begin{cases} -\Delta y_\mu + a y_\mu + b z_\mu = 0 & \text{in} \quad \Omega\backslash(\overline{D+\mu}), \\ -\Delta z_\mu + A y_\mu + B z_\mu = 0 & \text{in} \quad \Omega\backslash(\overline{D+\mu}), \\ y_\mu = \varphi,\ z_\mu = \psi & \text{on} \quad \partial\Omega, \\ y_\mu = 0,\ z_\mu = 0 & \text{on} \quad \partial(D+\mu). \end{cases}$$

Then, we have the following lemma whose proof is in Appendix B:

Lemma 2.3 *Assume that $a, b, A, B \in C^1(\overline{\Omega})$ satisfy (2.1). Then,*

1. *The mapping $\mu \mapsto (y_\mu, z_\mu) \circ (I + \mu)$ is differentiable at $\mu = 0$ from $\mathcal{W}_\epsilon$ into $H^2(\Omega\backslash\overline{D})^2$. That is, there exists a linear mapping $\mu \in \mathcal{W}_*^{2,\infty} \mapsto (\dot{y}_\mu, \dot{z}_\mu) \in H^2(\Omega\backslash\overline{D})^2$ such that*

$$(y_\mu, z_\mu) \circ (I + \mu) = (y, z) + (\dot{y}_\mu, \dot{z}_\mu) + o(\mu) \tag{2.34}$$

where $(y,z) \in H^2(\Omega\backslash\overline{D})^2$ *is the unique solution of* (2.2) *and*

$$\frac{o(\mu)}{\|\mu\|_{2,\infty}} \to 0 \ \text{ when } \ \|\mu\|_{2,\infty} \to 0. \tag{2.35}$$

2. *For each domain* $\mathcal{O} \subset\subset \Omega\backslash\overline{D}^*$*, the mapping* $\mu \mapsto (y_\mu, z_\mu)|_{\mathcal{O}}$*, which is defined in* $\mathcal{W}_\epsilon$ *and takes values in* $H^2(\mathcal{O})^2$ *is differentiable at* $\mu = 0$ *from* $\mathcal{W}_\epsilon$ *into* $H^1(\mathcal{O})$ *and we stand by* (y'_μ, z'_μ) *the local derivative in the direction of the vector* μ.

3. *Furthermore,* (y'_μ, z'_μ) *is the unique solution of the linear system*

$$\begin{cases} -\Delta y'_\mu + a y'_\mu + b z'_\mu = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ -\Delta z'_\mu + A y'_\mu + B z'_\mu = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ y'_\mu = 0, \; z'_\mu = 0 & \text{on} \quad \partial\Omega, \\ y'_\mu = -(\mu\cdot n)\dfrac{\partial y}{\partial n}, \; z'_\mu = -(\mu\cdot n)\dfrac{\partial z}{\partial n} & \text{on} \quad \partial D, \end{cases}$$

and the following holds:

$$(y'_\mu, z'_\mu) = (\dot{y}_\mu, \dot{z}_\mu) + \mu\cdot\nabla(y,z). \tag{2.36}$$

Then, using (2.34), (2.36) and the fact each $\mu \in \mathcal{W}_\epsilon$ is null in a neighbourhood of the boundary $\partial\Omega$, we get easily

$$\left(\frac{\partial y_\mu}{\partial n}, \frac{\partial z_\mu}{\partial n}\right) - \left(\frac{\partial y}{\partial n}, \frac{\partial z}{\partial n}\right) = \left(\frac{\partial y'_\mu}{\partial n}, \frac{\partial z'_\mu}{\partial n}\right) + o(\mu) \quad \text{on} \quad \gamma,$$

where $o(\mu)$ satisfies (2.35).

Moreover, for any $(\overline{\eta}, \overline{\theta}) \in C^2(\overline{\gamma})$, one has

$$\int_\gamma \left[\left(\frac{\partial y_\mu}{\partial n} - \frac{\partial y}{\partial n}\right)\overline{\eta} + \left(\frac{\partial z_\mu}{\partial n} - \frac{\partial z}{\partial n}\right)\overline{\theta}\right] d\Gamma = -\int_{\partial D} (\mu\cdot n)\left(\frac{\partial y}{\partial n}\frac{\partial \eta}{\partial n} + \frac{\partial z}{\partial n}\frac{\partial \theta}{\partial n}\right) d\Gamma + o(\mu), \tag{2.37}$$

where (η, θ) is the solution to the adjoint system

$$\begin{cases} -\Delta\eta + a\eta + A\theta = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ -\Delta\theta + b\eta + B\theta = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ \eta = \overline{\eta} 1_\gamma, \; \theta = \overline{\theta} 1_\gamma & \text{on} \quad \partial\Omega, \\ \eta = 0, \; \theta = 0 & \text{on} \quad \partial D. \end{cases} \tag{2.38}$$

Let us see how, starting from an already computed candidate $\tilde{D}$ to the solution of the geometric inverse problem (2.2)–(2.3), we can compute a better candidate of the form $\tilde{D} + \mu$.

Let $\mathcal{M}$ be a finite dimensional subspace of $L^\infty(\partial\tilde{D})$ and let $\{f_1, \ldots, f_p\}$ be a basis of $\mathcal{M}$. We will take μ such that $\mu \cdot n|_{\partial\tilde{D}} \in \mathcal{M}$. Then, we can write

$$\mu \cdot n|_{\partial\tilde{D}} = \sum_{i=1}^{p} \lambda_i f_i$$

for some $\lambda_i \in R$ to be determined.

Now, let us introduce p linearly independent functions $(\overline{\eta}^i, \overline{\theta}^i) \in C^2(\overline{\gamma})$. Using (2.37), we obtain

$$\int_\gamma \left[\left(\frac{\partial y_\mu}{\partial n} - \tilde{\alpha} \right) \overline{\eta}^j 1_\gamma + \left(\frac{\partial z_\mu}{\partial n} - \tilde{\beta} \right) \overline{\theta}^j 1_\gamma \right] d\Gamma = - \sum_{i=1}^{p} K_{ij} \lambda_i + o(\mu),$$

where

$$K_{ij} := \int_{\partial\tilde{D}} f_i \left(\frac{\partial \tilde{y}}{\partial n} \frac{\partial \eta^j}{\partial n} + \frac{\partial \tilde{z}}{\partial n} \frac{\partial \theta^j}{\partial n} \right) d\Gamma,$$

we have denoted by (η^j, θ^j) is the solution to (2.38) corresponding to $(\overline{\eta}^j, \overline{\theta}^j)$ and $(\tilde{\alpha}, \tilde{\beta})$ is the observation corresponding to $(\tilde{y}, \tilde{z})$. We thus see that an appropriate strategy to compute the coefficients λ_i is to solve, if possible, the finite-dimensional algebraic system

$$\sum_{i=1}^{p} K_{ij} \lambda_i = - \int_\gamma \left[(\alpha - \tilde{\alpha}) \overline{\eta}^j + (\beta - \tilde{\beta}) \overline{\theta}^j \right] d\Gamma, \quad 1 \leq j \leq p.$$

Remark 2.9 *Let us devote some words to other reconstruction issues. A natural way to compute a sequence of open sets D^k such that, in some sense, D^k converges to a solution to the inverse problem is the following:*

1. *Reformulate* (2.2)-(2.3) *as an extremal (direct) problem*

$$\begin{cases} \text{Minimize } \dfrac{1}{2} \left\| \left(\dfrac{\partial y}{\partial n}, \dfrac{\partial z}{\partial n} \right) - (\alpha, \beta) \right\|^2_{H^{-1/2}(\gamma)^2} \\ \text{Subject to } y = y_D, \ z = z_D, \ D \in \mathcal{D}_{ad}. \end{cases} \tag{2.39}$$

   Here, $\mathcal{D}_{ad}$ is an appropriated family of admissible domains and, for each $D \in \mathcal{D}_{ad}$, (y_D, z_D) is the unique solution to (2.2)*. For practical purposes, it is of course interesting to take, for instance, classes $\mathcal{D}_{ad}$ similar to those in Theorem 2.2.*

2. *Try to solve* (2.39) *by applying an iterative algorithm. For example, if the domains in $\mathcal{D}_{ad}$ are parametrized (as in Theorem 2.2), it makes sense to apply constrained descent techniques. This approach has been chosen in a lot of references up to date in connection with many different problems.*

3 ON THE UNIFORM CONTROLLABILITY OF THE INVISCID AND VISCOUS BURGERS-ALPHA SYSTEMS

In this chapter we study the global controllability of families of non-viscous and viscous convectively filtered Burgers equations (also known as inviscid and viscous Burgers-α systems) in a finite interval by using boundary and space independent distributed controls. In these equations, the usual convective velocity of the Burgers equation is replaced by a regularized velocity, induced by a Helmholtz filter of characteristic wavelength α. First, we prove a global (uniform with respect to α) exact controllability result for the family of non-viscous Burgers-α equations, using the return method and a fixed-point argument. Then, we establish the global uniform exact controllability to constant states for the similar family of viscous equations. To this purpose, we first prove a local exact controllability property and a global approximate controllability property for smooth initial and target states.

3.1 INTRODUCTION

Let $L > 0$ and $T > 0$ be given. Let us present the notations used along this chapter. The symbols C, $\widehat{C}$ and C_i, $i = 0, 1, \dots$ stand for positive constants (usually depending on L and T). For any $r \in [1, +\infty]$ and any given Banach space X, $\|\cdot\|_{L^r(X)}$ will denote the usual norm in Lebesgue-Bochner space $L^r(0,T;X)$. In particular, the norms in $L^r(0,L)$, $L^r(0,T)$ and $L^r((0,T)\times(0,L))$ will be denoted by $\|\cdot\|_r$. In this chapter, we will consider the following two families of controlled systems:

$$\begin{cases} y_t + zy_x = p(t) & \text{in} \quad [0,T]\times[0,L], \\ z - \alpha^2 z_{xx} = y & \text{in} \quad [0,T]\times[0,L], \\ z(\cdot,0) = v_l, \quad z(\cdot,L) = v_r & \text{in} \quad [0,T], \\ y(\cdot,0) = v_l & \text{in} \quad I_l, \\ y(\cdot,L) = v_r & \text{in} \quad I_r, \\ y(0,\cdot) = y_0 & \text{in} \quad [0,L], \end{cases} \tag{3.1}$$

where $I_l = \{t \in [0,T] :\ v_l(t) > 0\}$ and $I_r = \{t \in [0,T] :\ v_r(t) < 0\}$, and

$$\begin{cases} y_t - \gamma y_{xx} + z y_x = p(t) & \text{in} \quad (0,T)\times(0,L), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = y(\cdot,0) = v_l & \text{in} \quad (0,T), \\ z(\cdot,L) = y(\cdot,L) = v_r & \text{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{3.2}$$

These are respectively the so called non-viscous and viscous convectively filtered Burgers equations (also known in the literature as the Burgers-α or the Leray-Burgers equations). The pairs (y, z) and the triplets (p, v_l, v_r) stand for the corresponding states and controls. The parameter $\gamma > 0$ is the fluid viscosity and α is the characteristic wavelength of the Helmholtz filter. For simplicity, throughout this chapter we will take $\gamma = 1$. All the results can be extended without difficulty to the case where γ is an arbitrary positive number.

Obviously, (3.1) and (3.2) can be regarded as nonlinear regularizations of the inviscid and viscous Burgers equations. These systems and some related variants have already been studied. More precisely, (3.2) is the $b = 0$ case of the b-family:

$$\begin{cases} y_t - \gamma\, y_{xx} + z y_x + b\, z_x y = p(t) \\ z - \alpha^2 z_{xx} = y \end{cases}$$

or, equivalently

$$z_t - \alpha^2 z_{xxt} - \gamma\, z_{xx} + \gamma\, \alpha^2 z_{xxxx} + (b+1) z z_x - \alpha^2 z z_{xxx} - b\alpha^2 z_x z_{xx} = p(t).$$

This has been studied in (HOLM D.; STALEY, 2003) as a model for the 1D nonlinear wave dynamics in fluids which includes the effects of convection and stretching. The dimensionless parameter b measures the relative strength of these effects. The variable z can be viewed as the fluid velocity in the x direction (or equivalently the height of the free surface of the fluid above a flat bottom).

When $b = 2$, this equation is the so-called 1D viscous Camassa-Holm equation; it describes the unidirectional surface waves at a free surface of shallow water under the influence of gravity, see (CAMASSA R.; HOLM, 1993). When $b = 3$, we are dealing with the viscous Degasperis-Procesi equation which plays a similar role in water wave theory.

It is interesting to highlight that this regularization idea was first employed by Leray in (LERAY, 1934) to prove the existence of a solution to the Navier-Stokes equations. This Leray-type regularization has been used to capture shocks in the Burgers equation in (BHAT H. S.; FETECAU, 2006; BHAT H. S.; FETECAU, 2008; BHAT H. S.; FETECAU, 2009b; NORGARD G.; MOHSENI, 2008; NORGARD G.; MOHSENI, 2009). It has also been employed in other contexts, such as the analysis of compressible Euler equations, scalar conservations laws and aggregation equations, see (BERTOZZI, 2012; BHAT, 2005; BHAT H. S.; FETECAU, 2009a; CRAIG K.; BERTOZZI, 2016; SHEN, 2014). Finally, systems like (3.1) and (3.2) can also be viewed as simplified 1D versions of the so called Leray-α system introduced some time ago to describe turbulent flows as an alternative to the classical averaged Reynolds models, see (CHESKIDOV, 2005; FOIAS, 2004; FOIAS, 2002).

Our two main results deal with the global uniform exact controllability (with respect to α) for systems (3.1) and (3.2). More precisely, one has:

Theorem 3.1 *Let $\alpha > 0$ and $T > 0$ be given. The inviscid Burgers-α system* (3.1) *is globally exactly controllable in C^1. That is, for any given $y_0, y_T \in C^1([0,L])$, there exist a time-dependent control $p^\alpha \in C^0([0,T])$, a couple of boundary controls $(v_l^\alpha, v_r^\alpha) \in C^1([0,T]; R^2)$ and an associated state $(y^\alpha, z^\alpha) \in C^1([0,T] \times [0,L]; R^2)$ satisfying* (3.1) *and*

$$y^\alpha(T, \cdot) = y_T \quad \textit{in} \quad (0, L). \tag{3.3}$$

Moreover, there exists a positive constant $C > 0$ (depending on y_0 and y_T but independent of α) such that

$$\|(z^\alpha, y^\alpha)\|_{C^1([0,T]\times[0,L];R^2)} + \|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0,T];R^2)} \le C.$$

Theorem 3.2 *Let $\alpha > 0$ and $T > 0$ be given. The viscous Burgers-α system* (3.2) *is globally exactly controllable in L^∞ to constant trajectories. That is, for any given $y_0 \in L^\infty(0,L)$ and $N \in R$, there exist controls $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T; R^2)$ and associated states $(y^\alpha, z^\alpha) \in L^2(0,T; H^1(0,L; R^2)) \cap L^\infty(0,T; L^\infty(0,L; R^2))$ satisfying* (3.2)*,*

$$y^\alpha(T, \cdot) = N \quad \textit{in} \quad (0, L)$$

and the following estimates

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \le C,$$

where C is a positive constant (depending on y_0 and N but independent of α). Moreover, if $y_0 \in H_0^1(0,L)$ then the same conclusion holds with $(y^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$.

We will see in the proofs of the above results that the distributed control p^α is independent of α, it only depends on T, L, the initial condition and the target state. In this chapter, we are going to deal with situations that lead to new difficulties compared to previous works on nonlinear parabolic equations. Let us discuss these differences:

- Nonlocal nonlinearities. In the (3.1) and (3.2), the usual convective term is replaced and a filtered (averaged) velocity appears. As a consequence, the arguments in (CHAPOULY, 2009) must be modified, as shown below.
- Uniform controllability. Performing careful estimates of the controls, global uniform controllability results are obtained. This way, we are able to generalize some previous control results to the context of nonlinear parabolic equations with nonlocal nonlinearities, see (CHAPOULY, 2009; MARBACH, 2014).

For completeness, let us mention some previous works on the well-posedness and the control of our main systems and other similar models. Concerning well-posedness, global well-posedness for the inviscid Bugers-α system is established in (ESCHER J.; YIN, 2009) in the case of homogeneous boundary conditions and local well-posedness can be found in (COCLITE, 2009; PERROLLAZ, 2010) for non-homogenous boundary conditions. Regarding the viscous Burgers-α system we prove a global well-posedness below in the Section 3.3. On the other hand, there are many important works dealing with the controllability properties of parabolic equations and systems, see (FERNÁNDEZ-CARA E.; ZUAZUA, 2000b; FURSIKOV A. V.; IMANUVILOV, 1996) and inviscid and viscous Burgers equations, see (CHAPOULY, 2009; FERNÁNDEZ-CARA E.; GUERRERO, 2007; FURSIKOV A. V.; IMANUVILOV, 1996; GLASS O; GUERRERO, 2007; GUERRERO S.; IMANUVILOV, 2007; HORSIN, 1998; MARBACH, 2014; PERROLLAZ, 2012).

For Burgers-α sytems, the uniform local null controllability for the viscous system (3.2) with distributed and boundary controls was studied in (ARARUNA, 2013); later, the results have been extended to any equation of the b-family in (FERNÁNDEZ-CARA E.; SOUSA, 2019). In higher dimensions, uniform local null control results for the Leray-α system have been obtained in (ARARUNA, 2014).

The rest of this chapter is organized as follows. In Section 3.2, we prove some results concerning the existence, uniqueness and regularity of the solution to the viscous and inviscid Burgers-α systems. Sections 3.4 and 3.5 deal with the proofs of Theorems 3.1 and 3.2, respectively. Finally, in Section 4.5, we present some additional comments and questions.

3.2 PRELIMINARIES

3.2.1 Notations and Classical Results

Let us denote by $C_b^0(R)$ the Banach space of bounded continuous functions on R, let $C_b^{0,1}(R)$ be the Banach space of bounded Lipschitz-continuous functions on R and let $C_x^{0,1}([0,T]\times R)$ be the space of functions $f:[0,T]\times R\mapsto R$ that are continuous in x and t and globally Lipschitz-continuous in space, with Lipschitz constant independent of t.

In the sequel, for any function $f\in C^0([0,T]\times R)$, the associated flux function $\Phi=\Phi(s;t,x)$ is defined as follows:

$$\begin{cases} \dfrac{\partial\Phi}{\partial t}(s;t,x)=f(t,\Phi(s;t,x)), \\ \Phi(s;s,x)=x. \end{cases} \tag{3.4}$$

The mapping Φ contains all the information on the trajectories of the particles transported by the velocity f. Furthermore, we have the following existence, uniqueness and regularity result:

Proposition 3.1 (Theorem 10.19**, (DOERING C.I.; LOPES, 2014))** *Assume that* $f\in C_x^{0,1}([0,T]\times R)$ *and* $\frac{\partial f}{\partial x}$ *belongs to* $C^0([0,T]\times R)$*. Then, there exists a unique flux associated to* f*, that is, a unique function* $\Phi:[0,T]\times[0,T]\times R\mapsto R$ *satisfying* (3.4) *for all* $(s,x)\in[0,T]\times R$*. Moreover,* $\Phi\in C^1([0,T]\times[0,T]\times R)$*.*

For results like the Proposition 3.1 above, the reader can consult also the references (HALE, 1980; HARTMAN, 1964). Under the assumptions of Proposition 3.1, it is well known that, for each s,t in $[0,T]$, the mapping $\Phi(s;t,\cdot):R\mapsto R$ is a diffeomorphism, with

$$\Phi(s;t,\cdot)^{-1}=\Phi(t;s,\cdot).$$

Let us now recall a classical result related the solution of a transport equation. To this purpose, let us first note that, for any given Banach space X with norm $\|\cdot\|_X$ and any function $u\in C^1([0,T];X)$, the following inequality holds:

$$\frac{d}{dt^+}\|u(t)\|_X\leq\|u_t(t)\|_X \quad \text{in} \quad (0,T), \tag{3.5}$$

where d/dt^+ represents the right derivative.

Proposition 3.2 *Let us consider a velocity* $v \in C^0([0,T]; C_b^{0,1}(R)) \cap C_x^{0,1}([0,T] \times R)$ *and a source* $g \in C^0([0,T]; C_b^0(R))$. *Then, any solution* $y \in C^0([0,T]; C_b^1(R)) \cap C^1([0,T]; C_b^0(R))$ *to the equation*

$$y_t + v y_x = g \quad \text{in } (0,T) \times R \tag{3.6}$$

satisfies the following inequality

$$\frac{d}{dt^+} \|y(t,\cdot)\|_{C_b^0(R)} \leq \|g(t,\cdot)\|_{C_b^0(R)} \quad \text{in } (0,T).$$

Let Φ be the flow associated to v. For any $(s,t,x) \in [0,T] \times [0,T] \times R$, we have by (3.6) that

$$\frac{d}{dt} y(t, \Phi(s;t,x)) = g(t, \Phi(s;t,x)).$$

Using this identity and the fact that $\Phi(s;t,\cdot)$ is a diffeomorphism, we get

$$\left\| \frac{d}{dt} y(t,\cdot) \right\|_{C_b^0(R)} \leq \|g(t,\cdot)\|_{C_b^0(R)}.$$

Now, the result follows easily from this and from (3.5).

The last result of this section is an immediate consequence of the Banach Fixed-Point Theorem:

Theorem 3.3 *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be Banach spaces with* F *continuously embedded in* E. *Let* B *be a subset of* F *and let* $G : B \mapsto B$ *be a mapping such that*

$$\|G(u) - G(v)\|_E \leq \gamma \|u - v\|_E \quad \forall\, u, v \in B, \text{ for some } \gamma \in (0,1).$$

Let us denote by $\widetilde{B}$ *the closure of* B *for the norm* $\|\cdot\|_E$. *Then,* G *can be uniquely extended to a continuous mapping* $\widetilde{G} : \widetilde{B} \mapsto \widetilde{B}$ *that possesses a unique fixed-point in* $\widetilde{B}$.

3.3 WELL-POSEDNESS OF THE VISCOUS BURGERS-ALPHA SYSTEM

Let us introduce the Hilbert space $E := H^{3/4}(0,T) \times H^{3/4}(0,T)$. It is not difficult to check that the trace operator $\Gamma : L^2(0,T; H^2(0,L)) \cap H^1(0,T; L^2(0,L)) \mapsto E$, defined by $\Gamma(\xi) := (\xi(\cdot,0), \xi(\cdot,L))$ is surjective, see (LIONS J.-L.; MAGENES, 1972, p. 18). Furthermore, there exists a linear continuous mapping $S : E \mapsto L^2(0,T; H^2(0,L)) \cap H^1(0,T; L^2(0,L))$ such that $\Gamma \circ S = I_E$. Thus, for each $(v_l, v_r) \in E$ we can get $\xi \in L^2(0,T; H^2(0,L)) \cap H^1(0,T; L^2(0,L))$ such that

$$\|\xi\|_{L^2(H^2) \cap H^1(L^2)} \leq C(\|v_l\|_{H^{3/4}} + \|v_r\|_{H^{3/4}}),$$

for some $C > 0$.

The following result concerns global existence and uniqueness for viscous Burgers-α systems:

Proposition 3.3 *Let $\alpha > 0$, $f \in L^\infty((0,T)\times(0,L))$, $y_0 \in H^1(0,L)$ and $v_l, v_r \in H^{3/4}(0,T)$ be given. Assume that the following compatibility relations hold:*

$$v_l(0) = y_0(0) \quad \textit{and} \quad v_r(0) = y_0(L).$$

Then, there exists a unique solution (y^α, z^α) to the Burgers-α system:

$$\begin{cases} y^\alpha_t - y^\alpha_{xx} + z^\alpha y^\alpha_x = f & \textit{in} \quad (0,T)\times(0,L), \\ z^\alpha - \alpha^2 z^\alpha_{xx} = y^\alpha & \textit{in} \quad (0,T)\times(0,L), \\ z^\alpha(\cdot,0) = y^\alpha(\cdot,0) = v_l & \textit{in} \quad (0,T), \\ z^\alpha(\cdot,L) = y^\alpha(\cdot,L) = v_r & \textit{in} \quad (0,T), \\ y^\alpha(0,\cdot) = y_0 & \textit{in} \quad (0,L). \end{cases} \tag{3.7}$$

with

$$\begin{cases} y^\alpha \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1(0,L)), \\ z^\alpha \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^4(0,L)) \cap C^0([0,T];H^3(0,L)). \end{cases} \tag{3.8}$$

Let us set $M_T := \|y^0\|_\infty + \|v_l\|_\infty + \|v_r\|_\infty + T\|f\|_\infty$. Then, the following estimates holds:

$$\begin{gathered} \|y^\alpha\|_\infty \le M_T, \qquad \|z^\alpha\|_\infty \le M_T, \\ \|y^\alpha\|_{H^1(L^2)\cap L^2(H^2)} + \|y^\alpha\|_{L^\infty(H^1)} \le Ce^{CM_T^2}\left(\|f\|_2 + \|y_0\|_{H^1} + \|v_l\|_{H^{3/4}} + \|v_r\|_{H^{3/4}}\right) \\ \|z^\alpha\|_2 + \alpha\|z^\alpha_x\|_2 + \alpha^2\|z^\alpha_{xx}\|_2 \le Ce^{CM_T^2}\left[\|f\|_2 + \|y_0\|_{H^1} + (1+\alpha^2)\|(v_l,v_r)\|_{H^{3/4}\times H^{3/4}}\right]. \end{gathered} \tag{3.9}$$

The proof of existence can be reduced to find a fixed-point of an appropriate mapping Λ_α. Thus, note first that there exists $\xi \in L^2(0,T;H^2(0,L)) \cap H^1(0,T;L^2(0,L))$ with

$$\xi(\cdot,0) = v_l \ \text{ and } \ \xi(\cdot,L) = v_r \quad \text{in} \quad (0,T).$$

Accordingly, for each $\bar{y} \in L^\infty(0,T;L^\infty(0,L))$ there exists exactly one $z \in L^\infty(0,T;H^2(0,L))$ with

$$\begin{cases} z - \alpha^2 z_{xx} = \bar{y} & \text{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = v_l, \ z(\cdot,L) = v_r & \text{in} \quad (0,T), \end{cases}$$

satisfying:

$$\|z\|_2^2 + 2\alpha^2\|z_x\|_2^2 + \alpha^4\|z_{xx}\|_2^2 \le C\left(\|\bar{y}\|_2^2 + \|\xi\|_2^2 + \alpha^2\|\xi_x\|_2^2 + \alpha^4\|\xi_{xx}\|_2^2\right),$$

$$\|z\|_{L^\infty(L^\infty)} \le \|\bar{y}\|_{L^\infty(L^\infty)} + \|v_l\|_\infty + \|v_r\|_\infty.$$

With this z, by applying (for instance) the Faedo-Galerkin method, we can easily prove the existence of a y to the linear parabolic equation

$$\begin{cases} y_t - y_{xx} + zy_x = f & \text{in} \quad (0,T)\times(0,L), \\ y(\cdot,0) = v_l,\ y(\cdot,L) = v_r & \text{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \text{in} \quad (0,L) \end{cases} \tag{3.10}$$

that satisfies

$$y \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1(0,L))$$

and

$$\|y_t\|_{L^2(L^2)} + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H^1)} \le C\left(\|y_0\|_{H^1} + \|f\|_{L^2(L^2)} + \|v_l\|_{H^{3/4}} + \|v_r\|_{H^{3/4}}\right)e^{C\|z\|_\infty^2}. \tag{3.11}$$

Arguing as in the proof of (ARARUNA, 2013, Lemma 1), we can deduce that the solution to (3.10) belongs to the space $C^0([0,T];H^1(0,L))$ and, in particular, $y_{L^\infty(L^\infty)} \le M_T$. Accordingly, we can introduce the bounded closed convex set

$$K := \{\bar{y} \in L^\infty(0,T;L^\infty(0,L)) \,:\, \|\bar{y}\|_{L^\infty(L^\infty)} \le M_T\}$$

and the mapping $\Lambda_\alpha : K \mapsto K$, with $\Lambda_\alpha(\bar{y}) = y$. Obviously, Λ_α is well-defined and continuous and, moreover, we can see from the estimates in (3.11) that $G := \Lambda_\alpha(K)$ is bounded in $L^\infty(0,T;H^1(0,L))$ and $G_t := \{u_t; u \in G\}$ is bounded in $L^2(0,T;L^2(0,L))$. From classical results of the Aubin-Lions kind (see (SIMON, 1980)), we deduce that G is relatively compact in $L^\infty(0,T;L^\infty(0,L))$. Therefore, by Schauder's fixed point Theorem, Λ_α has a fixed point in K, which obviously implies the existence of a solution to (3.7).

We prove now that the solution is unique. Let (y^α, z^α) and $(\widehat{y}^\alpha, \widehat{z}^\alpha)$ be two solutions to (3.7) and let us introduce $u := y^\alpha - \widehat{y}^\alpha$ and $v := z^\alpha - \widehat{z}^\alpha$. Then,

$$\begin{cases} u_t - u_{xx} + z^\alpha u_x = -v\widehat{y}^\alpha_x & \text{in} \quad (0,T)\times(0,L), \\ v - \alpha^2 v_{xx} = u & \text{in} \quad (0,T)\times(0,L), \\ u(\cdot,0) = u(\cdot,L) = v(\cdot,0) = v(\cdot,L) = 0 & \text{in} \quad (0,T), \\ u(0,\cdot) = 0 & \text{in} \quad (0,L). \end{cases} \tag{3.12}$$

Using the fact that $\widehat{y}^\alpha \in L^2(0,T;H^2(0,L)) \hookrightarrow L^2(0,T;C^1[0,L])$ and multiplying the first equation of the system above by u, we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|u_x\|_2^2 &\leq \|z^\alpha\|_\infty \|u_x\|_2\|u\|_2 + \|\widehat{y}_x^\alpha\|_\infty \|v\|_2\|u\|_2 \\ &\leq \frac{1}{2}\|u_x\|_2^2 + \frac{\|z^\alpha\|_\infty^2}{2}\|u\|_2^2 + \|\widehat{y}_x^\alpha\|_\infty\|u\|_2^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u\|_2^2 + \|u_x\|_2^2 \leq \left(\|z^\alpha\|_\infty^2 + 2\|\widehat{y}_x^\alpha\|_\infty\right)\|u\|_2^2.$$

Since $u(0,\cdot) = 0$, Gronwall's Lemma implies $u \equiv 0$ and, consequently, $v \equiv 0$.

Finally, let us check that z^α satisfies the regularity properties in (3.8). To get this, let us introduce the function given by

$$h(t,x) := \frac{v_l(t)(L-x) + x\,v_r(t)}{L}.$$

Then, we obtain from (3.7) that $z^\alpha = w^\alpha + h$, where w^α solves

$$\begin{cases} w^\alpha - \alpha^2 w_{xx}^\alpha = y^\alpha - h & \text{in} \quad (0,T)\times(0,L), \\ w^\alpha(\cdot,0) = w^\alpha(\cdot,0) = 0 & \text{in} \quad (0,T). \end{cases}$$

Consequently, $w^\alpha \in L^2(0,T;H^4(0,L)\cap H_0^1(0,L)) \cap C^0([0,T];H^3(0,L)\cap H_0^1(0,L))$ and the estimates are uniform, with respect to α, in the space $L^2(0,T;H^2(0,L)\cap H_0^1(0,L)) \cap C^0([0,T];H_0^1(0,L))$.

Now, let us present a result concerning global existence and uniqueness of a (weak) solution with initial conditions in $L^\infty(0,L)$:

Proposition 3.4 *Let $\alpha > 0$, $f \in L^\infty((0,T)\times(0,L))$, $y_0 \in L^\infty(0,L)$ and $v_l, v_r \in H^{3/4}(0,T)$ be given. Then, there exists a unique solution (y^α, z^α) to the Burgers-α system:*

$$\begin{cases} y_t^\alpha - y_{xx}^\alpha + z^\alpha y_x^\alpha = f & \textit{in} \quad (0,T)\times(0,L), \\ z^\alpha - \alpha^2 z_{xx}^\alpha = y^\alpha & \textit{in} \quad (0,T)\times(0,L), \\ z^\alpha(\cdot,0) = y^\alpha(\cdot,0) = v_l & \textit{in} \quad (0,T), \\ z^\alpha(\cdot,L) = y^\alpha(\cdot,L) = v_r & \textit{in} \quad (0,T), \\ y^\alpha(0,\cdot) = y_0 & \textit{in} \quad (0,L) \end{cases} \tag{3.13}$$

with

$$\begin{cases} y^\alpha \in H^1(0,T;H^{-1}(0,L)) \cap L^2(0,T;H_0^1(0,L)) \cap C^0([0,T];L^2(0,L)) \cap L^\infty(0,T;L^\infty(0,L)), \\ z^\alpha \in H^1(0,T;H^{-1}(0,L)) \cap L^2(0,T;H^3(0,L)) \cap C^0([0,T];H^2(0,L)). \end{cases} \tag{3.14}$$

Let us set $M_T := \|y^0\|_\infty + \|v_l\|_\infty + \|v_r\|_\infty + T\|f\|_\infty$*. Then, the following estimates holds:*

$$\|y^\alpha\|_\infty \leq M_T, \quad \|z^\alpha\|_\infty \leq M_T,$$

$$\|y^\alpha\|_{H^1(H^{-1})\cap L^2(H^1)} + \|y^\alpha\|_{L^\infty(L^2)} \leq Ce^{CM_T^2}\left(\|f\|_2 + \|y_0\|_2 + \|v_l\|_{H^{3/4}} + \|v_r\|_{H^{3/4}}\right),$$

$$\|z^\alpha\|_2 + \alpha\|z_x^\alpha\|_2 + \alpha^2\|z_{xx}^\alpha\|_2 \leq Ce^{CM_T^2}\Big[\|f\|_2 + \|y_0\|_2 + (1+\alpha^2)(\|(v_l,v_r)\|_{H^{3/4}\times H^{3/4}})\Big]. \tag{3.15}$$

For any $\bar{y} \in L^2(0,T;L^\infty(0,L))$, there exists a unique solution to the elliptic problem

$$\begin{cases} z - \alpha^2 z_{xx} = \bar{y} & \text{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = v_l, \ z(\cdot,L) = v_r & \text{in} \quad (0,T), \end{cases}$$

furthermore satisfying

$$\|z\|_2^2 + 2\alpha^2\|z_x\|_2^2 + \alpha^4\|z_{xx}\|_2^2 \leq C\left(\|\bar{y}\|_2^2 + \|\xi\|_2^2 + \alpha^2\|\xi_x\|_2^2 + \alpha^4\|\xi_{xx}\|_2^2\right),$$

$$\|z\|_{L^2(L^\infty)} \leq \|\bar{y}\|_{L^2(L^\infty)} + \|v_l\|_2 + \|v_r\|_2.$$

With this z, we solve the linear problem

$$\begin{cases} y_t - y_{xx} + z y_x = f & \text{in} \quad (0,T)\times(0,L), \\ y(\cdot,0) = v_l, \ y(\cdot,L) = v_r & \text{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \text{in} \quad (0,L) \end{cases} \tag{3.16}$$

and we find a solution y that satisfies

$$\|y_t\|_{L^2(H^{-1})} + \|y\|_{L^2(H_0^1)} + \|y\|_{L^\infty(L^2)} \leq C\left(\|y_0\|_2 + \|f\|_2 + \|v_l\|_{H^{3/4}} + \|v_r\|_{H^{3/4}}\right) e^{C\|z\|^2_{L^2(L^\infty)}}.$$

Again, as in the proof of (ARARUNA, 2013, Lemma 1), we can deduce that the solution to (3.16) satisfies

$$\|y\|_{L^2(L^\infty)} \leq T^{1/2} M_T.$$

Let us introduce the set

$$K := \{\bar{y} \in L^2(0,T;L^\infty(0,L)) \, : \, \|\bar{y}\|_{L^2(L^\infty)} \leq T^{1/2} M_T\}$$

and the mapping $\Lambda_\alpha : K \mapsto K$ with $\Lambda_\alpha(\bar{y}) = y$. Then, arguing as in the proof of Proposition 3.3, it is not difficult to prove that Λ_α possesses a fixed-point in K.

Finally, in order to prove uniqueness, we consider to solutions $u := y^\alpha - \widehat{y}^\alpha$ and $v := z^\alpha - \widehat{z}^\alpha$ and we get (3.12). Then, multiplying the first equation of (3.12) by u, we easily get the differential inequality

$$\frac{d}{dt}\|u(t,\cdot)\|_2^2 + \|u_x(t,\cdot)\|_2^2 \leq \left(\|z^\alpha(t,\cdot)\|_\infty^2 + \frac{2C\|\widehat{y}_x^\alpha(t,\cdot)\|_2}{\alpha} \right) \|u(t,\cdot)\|_2^2.$$

Since $u(0,\cdot) \equiv 0$, Gronwall's Lemma implies $u \equiv 0$ and, consequently, $v \equiv 0$.

Let (y^α, z^α) be the solution to (3.13). From (3.13) and the fact that $y \in L^\infty(0,T;L^\infty(0,L))$, the maximum principle implies that

$$\|z^\alpha\|_{L^\infty(L^\infty)} \leq \|y^\alpha\|_{L^\infty(L^\infty)} \leq M_T.$$

This ends the proof.

3.4 CONTROLLABILITY OF THE INVISCID BURGERS-ALPHA SYSTEM

In this section we present a proof of the global exact controllability property of the inviscid Burgers-α system. We split the proof in two parts: (i) a local null controllability result; (ii) an argument based on a time scale-invariance and reversibility in time that leads to the desired global result.

3.4.1 Local null controllability

We have the following result:

Proposition 3.5 *Let $T, L, \alpha > 0$ be given. Then, there exist $\delta > 0$ and $C > 0$ (both independent of α) such that the following property holds: for each $y_0 \in C^1([0,L])$ with $\|y_0\|_{C^1([0,L])} \leq \delta$, there exist $p^\alpha \in C^0([0,T])$ with $p^\alpha(T) = 0$, $v_l^\alpha, v_r^\alpha \in C^1([0,T])$ and associated states $(y^\alpha, z^\alpha) \in C^1([0,T] \times [0,L]; R^2)$ satisfying* (3.1),

$$y^\alpha(T,\cdot) = z^\alpha(T,\cdot) = 0 \quad \textit{in} \quad (0,L)$$

and

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0,T];R^2)} \leq C \quad \forall \alpha > 0.$$

The proof is obtained applying the return method, see (CHAPOULY, 2009; CORON, 1992; CORON, 1996; GLASS, 2000). It relies on a linearization process in combination with a fixed-point argument: (i) first, we need to find a “good"trajectory (a particular solution for the nonlinear system) steering 0 to 0 such that the linearization around it is controllable; (ii) then, we must recover (for instance by a fixed-point argument) the exact controllability result, at least locally, for the nonlinear system.

In our case, it is not difficult to verify that the linearization around zero is not controllable. Accordingly, we build an appropriate nontrivial trajectory connecting $(0,0)$ to $(0,0)$.

To this purpose, let us introduce the set

$$\Lambda_{L,T,k} := \left\{ \lambda \in C^k([0,T];[0,\infty)) \; : \; \|\lambda\|_{L^1(0,T)} > L \right\}.$$

Let us consider the couple $(\widehat{y}(x,t), \widehat{z}(x,t)) := (\lambda(t), \lambda(t))$ and the triplet $(\widehat{p}(t), \widehat{v}_l(t), \widehat{v}_r(t)) := (\lambda'(t), \lambda(t), \lambda(t))$, with $\lambda \in \Lambda_{L,T,1}$ and supp $\lambda \subset (0,T)$. Note that $(\widehat{y}, \widehat{z})$ is a particular solution to (3.1), associated to the control $(\widehat{p}, \widehat{v}_l, \widehat{v}_r)$. We have the following general controllability result:

Proposition 3.6 *Let T, $L > 0$ be given and assume that $\lambda \in \Lambda_{L,T,0}$. Then, for any $\alpha > 0$ and any $y_0 \in C^1([0,L])$, there exists $(y,z) \in C^1([0,T] \times [0,L]; R^2)$ such that*

$$\begin{cases} y_t + \lambda(t) y_x = 0 & \text{in} \quad (0,T) \times (0,L), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,T) \times (0,L), \\ z(\cdot,0) = y(\cdot,0), \quad z(\cdot,L) = y(\cdot,L) & \text{in} \quad (0,T), \\ y(0,\cdot) = y_0 & \text{in} \quad (0,L), \\ y(T,\cdot) = 0 & \text{in} \quad (0,L). \end{cases} \tag{3.17}$$

For the proof, it suffices to use (CHAPOULY, 2009, Proposition 8) to find $y \in C^1([0,T] \times [0,L])$ satisfying $(3.17)_1$,$(3.17)_4$ and $(3.17)_5$ and then solve the elliptic problem $(3.17)_2$-$(3.17)_3$ to construct $z \in C^1([0,T] \times [0,L])$.

Thanks to Proposition 3.6, one may expect that the null controllability for the nonlinear system (3.1) holds. Indeed, we have the following result from which Proposition 3.5 is an immediate consequence:

Proposition 3.7 *Let T, $L > 0$ be given and assume that $\lambda \in \Lambda_{L,T,0}$. Then, there exist $\delta > 0$ and $C > 0$ (both independent of α) such that, for any $y_0 \in C^1([0,L])$ with $\|y_0\|_{C^1([0,L])} \leq$*

δ and any $\alpha > 0$, there exist $(v_l, v_r) \in C^1([0,T];R^2)$ and an associated state $(y,z) \in C^1([0,T]\times[0,L];R^2)$ satisfying

$$\begin{cases} y_t + (\lambda(t)+z)y_x = 0 & \textit{in} \quad (0,T)\times(0,L), \\ z - \alpha^2 z_{xx} = y & \textit{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = v_l, \quad z(\cdot,L) = v_r & \textit{in} \quad [0,T], \\ y(\cdot,0) = v_l & \textit{in} \quad I_l, \\ y(\cdot,L) = v_r & \textit{in} \quad I_r, \\ y(0,\cdot) = y_0 & \textit{in} \quad (0,L), \\ y(T,\cdot) = 0 & \textit{in} \quad (0,L) \end{cases} \tag{3.18}$$

and

$$\|y\|_{C^1([0,T]\times[0,L])} \le C\|y_0\|_{C^1([0,L])} \quad \forall \alpha > 0.$$

We will reformulate the null controllability problem as a fixed-point equation. To do this, we will first introduce some auxiliar functions and establish some helpful results. Thus, to any given $h \in C^0([0,T];C^0([0,L])) \cap L^\infty(0,T;C^{0,1}([0,L]))$ we can associated the unique solution to the time-dependent problem

$$\begin{cases} z - \alpha^2 z_{xx} = h & \text{in} \quad (0,T)\times(0,L), \\ z(\cdot,0) = h(\cdot,0) & \text{in} \quad (0,T), \\ z(\cdot,L) = h(\cdot,L) & \text{in} \quad (0,T). \end{cases} \tag{3.19}$$

From the maximum principle for elliptic equations, we get

$$\|z\|_{C^0([0,T];C^1([0,L]))} \le \|h\|_{C^0([0,T];C^0([0,L]))} + \|h\|_{L^\infty(0,T;C^{0,1}([0,L]))}. \tag{3.20}$$

Since $\lambda \in \Lambda_{L,T,0}$, we can find $\eta \in (0, L/2)$ such that

$$\int_0^T \lambda(s)\,ds > L + 2\eta. \tag{3.21}$$

Now, we consider the following extension of z to the closed interval $[-\eta, L+\eta]$:

$$z^\eta(t,x) := \begin{cases} 5z(t,-x) - 20z\left(t,-\frac{x}{2}\right) + 16z\left(t,-\frac{x}{4}\right), & (t,x) \in [0,T]\times[-\eta,0], \\ z(t,x) & (t,x), \in [0,T]\times[0,L], \\ 5z(t,2L-x) - 20z\left(t,\frac{3L-x}{2}\right) + 16z\left(t,\frac{5L-x}{4}\right), & (t,x) \in [0,T]\times[L,L+\eta]. \end{cases}$$

It is not difficult to check that $z^\eta \in C^0([0,T];C^2([-\eta,L+\eta]))$ and there exists $C_1 > 0$ (independent of α) such that

$$\|z^\eta\|_{C^0([0,T];C^1([-\eta,L+\eta]))} \leq C_1 \|z\|_{C^0([0,T];C^1([0,L]))}. \tag{3.22}$$

Then, let χ be given, with $\chi \in C_0^\infty(-\eta/2, L+\eta/2)$, $\chi = 1$ in $[0,L]$ and $0 \leq \chi \leq 1$.

This way, we can introduce $z^* \in C^0([0,T];C^2(R))$, with

$$z^*(t,x) = \begin{cases} \chi(x) z^\eta(t,x), & (t,x) \in [0,T] \times [-\eta, L+\eta], \\ 0, & (t,x) \in [0,T] \times (R \setminus [-\eta, L+\eta]). \end{cases} \tag{3.23}$$

and, using (3.22) and (3.23), we see that

$$\|z^*\|_{C^0([0,T];C_b^1(R))} \leq C_2 \|z\|_{C^0([0,T];C^1([0,L]))}, \tag{3.24}$$

for some $C_2 > 0$, again independent of α.

Let us set $R := \frac{\eta}{C_2 T}$ and let us assume from now on that

$$\|h\|_{C^0([0,T];C^0([0,L])) \cap L^\infty(0,T;C^{0,1}([0,L]))} \leq R. \tag{3.25}$$

Then, it follows from (3.20), (3.24) and (3.25) that

$$\|z^*\|_{C^0([0,T];C_b^1(R))} \leq \frac{\eta}{T}. \tag{3.26}$$

Let ϕ^* be the flow associated with the ordinary differential equation $\xi' = \lambda(t) + z^*(t,\xi)$, that is, the solution to

$$\begin{cases} \dfrac{\partial \phi^*}{\partial t}(s;t,x) = \lambda(t) + z^*(t, \phi^*(s;t,x)), \\ \phi^*(s;s,x) = x. \end{cases} \tag{3.27}$$

Claim 3.1 *The function $\phi^* = \phi^*(s;t,x)$ is well-defined for any $(t,x) \in [0,T] \times R$ and any $s \in [0,T]$.*

Let $\phi : [0,T] \times [0,T] \times R \mapsto R$ be the flow associated to the ODE $\xi' = \lambda(t)$. Then, for every (s,t,x) we get from (3.27) that

$$\begin{aligned} |\phi^*(s;t,x) - \phi(s;t,x)| &= \left| \int_s^t \left(\frac{\partial \phi^*}{\partial \tau}(s;\tau,x) - \frac{\partial \phi}{\partial \tau}(s;\tau,x) \right) d\tau \right| \\ &\leq \int_s^t \left| \frac{\partial \phi^*}{\partial \tau}(s;\tau,x) - \frac{\partial \phi}{\partial \tau}(s;\tau,x) \right| d\tau \\ &= \int_s^t |z^*(\tau, \phi^*(s;\tau,x))|\, d\tau \\ &\leq T \|z^*\|_{C^0([0,T];C^0(R))}. \end{aligned}$$

Hence, for any (s,t,x) such that $\phi^*(s;t,x)$ is well-defined, one has

$$|\phi^*(s;t,x) - \phi(s;t,x)| \leq \eta. \tag{3.28}$$

This and the fact that $\phi(s;t,x)$ is well-defined for all $(s,t,x) \in [0,T] \times [0,L] \times R$ lead to the desired conclusion.

Let $y_0 \in C^1([0,L])$ be given and let us introduce $y_0^\eta \in C^1([-\eta, L+\eta])$ with

$$y_0^\eta(x) = \begin{cases} -y^0(-x) + 2y^0(0), & x \in [-\eta, 0], \\ y^0(x), & x \in [0, L], \\ -y^0(2L-x) + 2y^0(L), & x \in [L, L+\eta] \end{cases}$$

and

$$y_0^*(x) = \begin{cases} \chi(x) y_0^\eta(x), & x \in [-\eta, L+\eta], \\ 0, & x \in R \setminus [-\eta, L+\eta]. \end{cases}$$

Then, it is easy to see that y_0^* is an extension of y_0 to the whole real line and

$$\|y_0^*\|_{C_b^1(R)} \leq C_3 \|y_0\|_{C^1([0,L])}. \tag{3.29}$$

for some $C_3 > 0$.

Let us set $y \in C^1([0,T] \times R)$, with

$$y(t, \bar{x}) := y_0^*(\phi^*(t; 0, \bar{x})) \quad \forall (t, \bar{x}) \in [0,T] \times R. \tag{3.30}$$

Then, we have the following :

Claim 3.2 *The function y satisfies:*

$$\begin{cases} y_t + (\lambda(t) + z^*(t,x)) y_x = 0 & \textit{in} \quad (0,T) \times R, \\ y(0, \cdot) = y_0^* & \textit{in} \quad R, \\ y(T, \cdot) = 0 & \textit{in} \quad [0, L]. \end{cases} \tag{3.31}$$

For any $t \in [0,T]$, $\phi^*(0;t,\cdot) : R \to R$ is a diffeomorphism and (3.30) is equivalent to $y(t, \phi^*(0,t,x))) \equiv y_0^*(x)$. Then, for each $x \in R$, we deduce that

$$y_t(t, \phi^*(0;t,x)) + [\lambda(t) + z^*(t, \phi^*(0;t,x))]\, y_x(t, \phi^*(0;t,x)) = 0.$$

Using (3.30) and $(3.27)_2$, we get

$$y(0,x) = y_0^*(x) \quad \forall\, x \in R.$$

Moreover, it is not difficult to see that, for any $0 < \eta < L/2$ such that (3.21) holds, the flow associated to the ODE $\xi' = \lambda(t)$ satisfies $\phi(T;0,L) < -2\eta$ and we obtain from (3.28) that $\phi^*(T;0,L) < -\eta$. Since $\phi^*(s;t,\cdot)$ is increasing for any $s,t \in [0,T]$, we see that

$$\phi^*(T;0,x) < -\eta \quad \forall\, x \in (-\infty, L].$$

This inequality, together with the fact that

$$\text{supp } y_0^* \subset [-\eta, L+\eta]\,,$$

implies $y(T,\cdot) = 0$ in $[0,L]$.

An immediate consequence of (3.26), C^1 estimates for (3.27), (3.29) and (3.30) is that

$$\|y\|_{C^1([0,T]\times[0,L])} \leq C_4 \|y_0\|_{C^1([0,L])},$$

for a positive constant C_4 depending on R but independent of α. Taking $y_0 \in C^1([0,L])$ such that

$$\|y_0\|_{C^1([0,L])} \leq R/3C_4,$$

we have that $\|y\|_{C^0([0,T];C^0([0,L]))} + \|y\|_{L^\infty(0,T;C^{0,1}([0,L]))} \leq R$ and we can therefore introduce the mapping $\mathcal{F} : B_R \mapsto B_R$, where B_R is the closed ball of radius R in the Banach space $C^0([0,T];C^0([0,L])) \cap L^\infty(0,T;C^{0,1}([0,L]))$ and, for each $h \in B_R$, $y = \mathcal{F}(h)$ is given by (3.30). Thanks to (3.30), we have that $\mathcal{F}(B_R) \subset C^1([0,T]\times[0,L])$. Moreover, the following holds:

Claim 3.3 *There exists a positive constant C that depends on $\|y_0\|_{C^1([0,L])}, L, R$ and T, such that, for any $m \geq 1$ and any $h^1, h^2 \in B_R$, one has*

$$\|(\mathcal{F}^m(h^1) - \mathcal{F}^m(h^2))(t,\cdot)\|_{C^0([0,L])} \leq \frac{(Ct)^m}{m!} \|h^1 - h^2\|_{C^0([0,T];C^0([0,L]))} \text{ in } [0,T],$$

where $\mathcal{F}^m$ represents the m-th iteration of $\mathcal{F}$.

The proof relies on an induction argument. Let $h^i \in B_R$ be given for $i = 1,2$. Then, let us consider the functions $z^{i,*}$ and y^i, respectively given by (3.23) and (3.30) and set $y := y^1 - y^2$ and $z^* := z^{1,*} - z^{2,*}$. By Claim 3.2, we have

$$y_t + (\lambda + z^{1,*}) y_x = -z^* y_x^2 \quad \text{in} \quad (0,T) \times R,$$

whence, from Proposition 3.2,

$$\frac{d}{dt^+}\|y(t,\cdot)\|_{C_b^0(R)} \leq \|z^*(t,\cdot)y_x^2(t,\cdot)\|_{C_b^0(R)}.$$

Therefore, integrating from 0 to t and using that $y_x^2 \in C^0([0,T];C_b^0(R))$ and the maximum principle for elliptic PDE's, we find a positive constant C depending on $\|y_0\|_{C^1([0,L])}, L, R$ and T, such that

$$\begin{aligned}\|y(t,\cdot)\|_{C_b^0(R)} &\leq C\int_0^t \|z^*(\tau,\cdot)\|_{C_b^0(R)}\,d\tau\\ &\leq C\int_0^t \|z^1(\tau,\cdot)-z^2(\tau,\cdot)\|_{C^0([0,L])}\,d\tau\\ &\leq C\int_0^t \|h^1(\tau,\cdot)-h^2(\tau,\cdot)\|_{C^0([0,L])}\,d\tau.\end{aligned}$$

It follows that

$$\|(\mathcal{F}(h^1)-\mathcal{F}(h^2))(t,\cdot)\|_{C^0([0,L])} \leq Ct\|h^1-h^2\|_{C^0([0,T];C^0([0,L]))} \tag{3.32}$$

and the result is true for $m=1$.

Now, assume that the claim is true for a fixed m and let us prove that it holds also for $m+1$. Performing computations similar to those above, we get

$$\|(\mathcal{F}^{m+1}(h^1)-\mathcal{F}^{m+1}(h^2))(t,\cdot)\|_{C^0([0,L])} \leq C\int_0^t \|(\mathcal{F}^m(h^1)-\mathcal{F}^m(h^2))(\tau,\cdot)\|_{C^0([0,L])}\,d\tau,$$

where C is the same positive constant in (3.32).

Using the induction hypothesis, we deduce that

$$\begin{aligned}\|(\mathcal{F}^{m+1}(h^1)-\mathcal{F}^{m+1}(h^2))(t,\cdot)\|_{C^0([0,L])} &\leq C\|h^1-h^2\|_{C^0([0,T];C^0([0,L]))}\int_0^t \frac{(C\tau)^m}{m!}\,d\tau\\ &= \frac{(Ct)^{m+1}}{(m+1)!}\|h^1-h^2\|_{C^0([0,T];C^0([0,L]))}.\end{aligned}$$

Therefore, the result is also true for $m+1$ and the proof is done.

Let $\widetilde{B}_R$ be the closure of B_R with the norm of $C^0([0,T];C^0([0,L]))$ and let $\widetilde{\mathcal{F}}$ be the unique continuous extension of $\mathcal{F}$ to $\widetilde{B}_R$. Let us now present additional properties for the extension $\widetilde{\mathcal{F}}$:

Claim 3.4 *The continuous extension $\widetilde{\mathcal{F}}$ satisfies the following properties:*

a) For any h in $\widetilde{B}_R$, the function $\widetilde{\mathcal{F}}(h)$ belongs to $C^1([0,T]\times[0,L])$ and satisfies equation (3.31);

b) $\widetilde{\mathcal{F}}(\widetilde{B}_R) \subset B_R$.

Let us begin by proving the item a). For a given $h \in \widetilde{B}_R$, let us consider a sequence $(h_n)_{n\in N}$ in B_R such that $h_n \to h$ in $C^0([0,T];C^0([0,L]))$. Therefore, the corresponding elliptic solutions to (3.19) and the associated flows (given in (3.27)) satisfy the convergences

$$z_n \to z \;\text{ in }\; C^0([0,T];C^2([0,L])) \quad\text{and}\quad \Phi_n^* \to \Phi^* \;\text{ in }\; C^0([0,T]\times[0,T]\times R).$$

Moreover, since the $\Phi_n^*, \Phi^* \in C^1([0,T]\times[0,T]\times R)$, the corresponding functions, defined in (3.30), belong to $C^1([0,T]\times[0,L])$, verify the transport equation (3.31) and, furthermore,

$$y_n \to y \;\text{ in }\; C^0([0,T];C^0([0,L])).$$

Thus, it follows easily from this and from the definition of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that $\widetilde{\mathcal{F}}(h) = y$.

Let us now prove the item b). In fact, notice that, by definition of $\mathcal{F}$,for any $h \in \widetilde{B}_R$ there exists a sequence $(h_n)_{n=1}^{\infty}$ in B_R such that $h_n \to h$ in $C^0([0,T];C^0([0,L]))$ and

$$\widetilde{\mathcal{F}}(h) = \lim_{n\to\infty} \mathcal{F}(h_n) \;\text{ in }\; C^0([0,T];C^0([0,L])).$$

On the other hand, it is not difficult to prove that

$$\|\widetilde{\mathcal{F}}(h)(t,\cdot)\|_{C^0([0,L])} + \|\widetilde{\mathcal{F}}(h)(t,\cdot)\|_{C^{0,1}([0,L])} \le \|\mathcal{F}(h_n)(t,\cdot)\|_{C^0([0,L])} + \|\mathcal{F}(h_n)(t,\cdot)\|_{C^{0,1}([0,L])}$$
$$4\|\mathcal{F}(h_n)(t,\cdot) - \widetilde{\mathcal{F}}(h)(t,\cdot)\|_{C^0([0,L])}.$$

Therefore, using the fact that $\widetilde{\mathcal{F}}(h) \in C^1([0,T]\times[0,L])$, we certainly have that $\widetilde{\mathcal{F}}(h) \in B_R$.

It follows from Claim 3.3 that $\mathcal{F}^m$ is a contraction for m large enough. Then, from Banach Fixed-Point Theorem 3.3, $\widetilde{\mathcal{F}}$ possesses a unique fixed-point $y \in \widetilde{B}_R$. Finally, taking into account Claim 3.4, the proof of Proposition 3.7 is achieved.

Remark 3.1 *Let T, $L > 0$, assume that $\lambda \in \Lambda_{L,T,1}$ and consider the Banach space*

$$\mathcal{X} = C^0([0,T];C^1([0,L])) \cap C^1([0,T];C^0([0,L])) \cap L^\infty(0,T;C^{1,1}([0,L]))$$

If $y_0 \in C^2([0,L])$ is small enough, then the fixed-point mapping $\mathcal{F}$ can be defined in a closed ball of $\mathcal{X}$ centered at zero of radius $R > 0$. Then, one applies Banach Fixed-Point Theorem in the closure of this ball with the norm of $C^0([0,T];C^1([0,L])) \cap C^1([0,T];C^0([0,L]))$. Performing similar computations of Proposition 3.7, one can deduce that there exists $\delta > 0$

(independent of α) such that, for any $y_0 \in C^2([0,L])$ with $\|y_0\|_{C^2([0,L])} \le \delta$, there exists a solution $y \in C^2([0,T]\times[0,L])$ to (3.18), satisfying

$$\|y\|_{C^2([0,T]\times[0,L]))} \le C\|y_0\|_{C^2([0,L])} \quad \forall\alpha > 0, \tag{3.33}$$

for a constant $C > 0$ that is independent of α.

Remark 3.2 *From the proof of the previous result, one sees that $z(\cdot,0) = y(\cdot,0)$ and $z(\cdot,L) = y(\cdot,L)$ in $(0,T)$. This is important to guarantee that in the limit, as α goes to 0, z and y converge to the same limit.*

3.4.2 Global exact controllability

In order to prove Theorem 3.1, we have to use scaling arguments and the time-reversibility of the inviscid Burgers-α system. Thus, let $T, L > 0$ be given, let us consider initial and final states $y_0, y_T \in C^1([0,L])$, let $\delta > 0$ be given by Proposition 3.5 and let $\gamma_0, \gamma_T \in (0,1)$ be such that $\gamma_0 < \gamma_T$,

$$\|\gamma_0 y_0\|_{C^1([0,L])} \le \delta \text{ and } \|(1-\gamma_T)y_T\|_{C^1([0,L])} \le \delta.$$

Then, by Proposition 3.5, there exist distributed controls $\widetilde{p}$, $\widehat{p}$ in $C^0_c((0,T))$, boundary controls $(\widetilde{v}_l, \widetilde{v}_r)$, $(\widehat{v}_l, \widehat{v}_r)$ in $C^1([0,T])$ and associated states $(\widetilde{y}, \widetilde{z})$, $(\widehat{y}, \widehat{z})$ in $C^1([0,T]\times[0,L])$ such that

$$\begin{cases} \widetilde{y}_t + \widetilde{z}\,\widetilde{y}_x = \widetilde{p}(t) & \text{in } (0,T)\times(0,L), \\ \widetilde{z} - \alpha^2\widetilde{z}_{xx} = \widetilde{y} & \text{in } (0,T)\times(0,L), \\ \widetilde{z}(\cdot,0) = \widetilde{y}(\cdot,0) = \widetilde{v}_l & \text{in } (0,T), \\ \widetilde{z}(\cdot,L) = \widetilde{y}(\cdot,L) = \widetilde{v}_r & \text{in } (0,T), \\ \widetilde{y}(0,\cdot) = \gamma_0 y_0(x) & \text{in } (0,L), \\ \widetilde{y}(T,\cdot) = 0 & \text{in } (0,L) \end{cases} \tag{3.34}$$

and

$$\begin{cases} \widehat{y}_t + \widehat{z}\widehat{y}_x = \widehat{p}(t) & \text{in} \quad (0,T)\times(0,L), \\ \widehat{z} - \alpha^2 \widehat{z}_{xx} = \widehat{y} & \text{in} \quad [0,T]\times[0,L], \\ \widehat{z}(\cdot,0) = \widehat{y}(\cdot,0) = \widehat{v}_l & \text{in} \quad (0,T), \\ \widehat{z}(\cdot,L) = \widehat{y}(\cdot,L) = \widehat{v}_r & \text{in} \quad (0,T), \\ \widehat{y}(0,\cdot) = (1-\gamma_T)y_T & \text{in} \quad (0,L), \\ \widehat{y}(T,\cdot) = 0 & \text{in} \quad (0,L). \end{cases} \tag{3.35}$$

Using (3.34), (3.35) and the facts that $\widetilde{p}(T) = \widehat{p}(T) = 0$ and $\gamma_0 0$ and $\gamma_T 1$, we can introduce the functions $Y, Z : [0,T]\times[0,L] \mapsto R$ and $P, V_l, V_r : [0,T] \mapsto R$, given by

$$Y(t,x) := \begin{cases} \gamma_0^{-1}\,\widetilde{y}\left(t\,\gamma_0^{-1}, x\right) & (t,x)\in[0,\gamma_0 T]\times[0,L], \\ 0 & (t,x)\in[\gamma_0 T,\gamma_T T]\times[0,L], \\ \dfrac{1}{1-\gamma_T}\widehat{y}\left(\dfrac{T-t}{1-\gamma_T}, L-x\right) & (t,x)\in[\gamma_T T, T]\times[0,L], \end{cases}$$

$$Z(t,x) := \begin{cases} \gamma_0^{-1}\,\widetilde{z}\left(t\,\gamma_0^{-1}, x\right) & (t,x)\in[0,\gamma_0 T]\times[0,L], \\ 0 & (t,x)\in[\gamma_0 T,\gamma_T T]\times[0,L], \\ \dfrac{1}{1-\gamma_T}\widehat{z}\left(\dfrac{T-t}{1-\gamma_T}, L-x\right) & (t,x)\in[\gamma_T T, T]\times[0,L], \end{cases}$$

$$P(t) := \begin{cases} \gamma_0^{-2}\,\widetilde{p}\left(t\,\gamma_0^{-1}\right) & t\in[0,\gamma_0 T], \\ 0 & t\in[\gamma_0 T,\gamma_T T], \\ -\dfrac{1}{(1-\gamma_T)^2}\widehat{p}\left(\dfrac{T-t}{1-\gamma_T}\right) & t\in[\gamma_T T, T], \end{cases}$$

$$V_l(t) := \begin{cases} \gamma_0^{-1}\,\widetilde{v}_l\left(t\,\gamma_0^{-1}, x\right) & t\in[0,\gamma_0 T], \\ 0 & t\in[\gamma_0 T,\gamma_T T], \\ \dfrac{1}{1-\gamma_T}\widehat{v}_r\left(\dfrac{T-t}{1-\gamma_T}\right) & t\in[\gamma_T T, T] \end{cases}$$

and

$$V_r(t) := \begin{cases} \gamma_0^{-1}\,\widetilde{v}_r\left(t\,\gamma_0^{-1}, x\right) & t\in[0,\gamma_0 T], \\ 0 & t\in[\gamma_0 T,\gamma_T T], \\ \dfrac{1}{1-\gamma_T}\widehat{v}_l\left(\dfrac{T-t}{1-\gamma_T}\right) & t\in[\gamma_T T, T]. \end{cases}$$

It is now straightforward to check that $(Y,Z) \in C^1([0,T]\times[0,L];R^2)$, $P \in C^0([0,T])$, $V_l, V_r \in C^1([0,T])$ and (3.1) and (3.3) are satisfied.

3.5 GLOBAL CONTROLLABILITY OF THE VISCOUS BURGERS-ALPHA SYSTEM

3.5.1 Smoothing effect

The goal of this section is to prove that, starting from a H_0^1 initial data, there exists a small time where the solution begins to be smooth. More precisely, we have the following result:

Proposition 3.8 *Let $y_0 \in H_0^1(0,L)$ be given and let (y^α, z^α) be the solution to*

$$\begin{cases} y_t^\alpha - y_{xx}^\alpha + z^\alpha y_x^\alpha = 0 & \textit{in} \quad (0,T)\times(0,L), \\ z^\alpha - \alpha^2 z_{xx}^\alpha = y^\alpha & \textit{in} \quad (0,T)\times(0,L), \\ y^\alpha(\cdot,0) = y^\alpha(\cdot,L) = z^\alpha(\cdot,0) = z^\alpha(\cdot,L) = 0 & \textit{in} \quad (0,T), \\ y^\alpha(0,\cdot) = y_0 & \textit{in} \quad (0,L). \end{cases} \tag{3.36}$$

Then, there exist $T^* \in (0,T/2)$ and $C > 0$ (independent of α) such that the solution y^α belongs to $C^0([T^*,T];C^2([0,L]))$ and satisfies

$$\|y^\alpha\|_{C^0([T^*,T];C^2([0,L]))} \le \Lambda(\|y_0\|_{H_0^1}),$$

where $\Lambda : R_+ \to R_+$ is a continuous function satisfying $\Lambda(s) \to 0$ as $s \to 0^+$.

We will divide the proof in several steps. Throughout the proof, all the constants are independent of α.

Step 1: Strong estimates in $(0,T/2)$. Since $y_0 \in H_0^1(0,L)$, $f \equiv 0$ and $v_l \equiv v_r \equiv 0$, Proposition 3.3 implies the existence and uniqueness of a solution (y^α, z^α) to (3.36) satisfying (3.8) and (3.9). In particular, we have from (3.9) that

$$\|y^\alpha\|_{L^2(H^2\cap H_0^1)} \le C_1\|y_0\|_{H_0^1} e^{C\|y_0\|_{H_0^1}^2}.$$

Consequently, there exists $t_1 \in (0,T/2)$ such that

$$\|y^\alpha(t_1,\cdot)\|_{H^2\cap H_0^1} \le \sqrt{\frac{2}{T}}C_1\|y_0\|_{H_0^1} e^{C\|y_0\|_{H_0^1}^2}.$$

Step 2: Estimates in $(t_1, T/2)$. Let us set $y_1 := y^\alpha(t_1,\cdot)$, $g := z^\alpha y^\alpha_x$. Then, we can easily check that y^α is the unique solution to the heat equation:

$$\begin{cases} y^\alpha_t - y^\alpha_{xx} = g & \text{in} \quad (t_1,T)\times(0,L), \\ y^\alpha(\cdot,0) = y^\alpha(\cdot,L) = 0 & \text{in} \quad (t_1,T), \\ y^\alpha(t_1,\cdot) = y_1 & \text{in} \quad (0,L). \end{cases} \tag{3.37}$$

From the regularity of y^α and z^α, we have $g \in L^2(0,T;H^1_0(0,L)) \cap H^1(0,T;H^{-1}(0,L))$ and

$$\begin{aligned} \|g\|_{L^2(H^1_0)} + \|g_t\|_{L^2(H^{-1})} &\le C\|y^\alpha\|_{L^\infty(H^1_0)}(\|y^\alpha\|_{L^2(H^2)} + \|y^\alpha_t\|_{L^2(L^2)}) \\ &\le Ce^{C\|y_0\|^2_{H^1_0}}\|y_0\|^2_{H^1_0}. \end{aligned}$$

Using this estimate, the fact that $g \in C^0(0,T;L^2(0,L))$ (see (EVANS, 2010, Ch. 5, Thm. 3)), (3.37) and the parabolic regularity result (EVANS, 2010, Ch. 3, Thm. 5), we find that

$$y^\alpha \in L^\infty(t_1,T;H^2(0,L)),\ y^\alpha_t \in L^\infty(t_1,T;L^2(0,L)) \cap L^2(t_1,T;H^1_0(0,L)) \cap H^1(t_1,T;H^{-1}(0,L))$$

and, in the time interval (t_1,T),

$$\begin{aligned} \|y^\alpha\|_{L^\infty(H^2)} + \|y^\alpha_t\|_{L^\infty(L^2)\cap L^2_1(H^1_0)} + \|y^\alpha_{tt}\|_{L^2(H^{-1})} &\le C\left(\|g\|_{L^2(H^1_0)\cap H^1(H^{-1})} + \|y_1\|_{H^2}\right) \\ &\le \frac{1}{2}\Lambda_1(\|y_0\|_{H^1_0}) \end{aligned} \tag{3.38}$$

where

$$\Lambda_1(\|y_0\|_{H^1_0}) = 2Ce^{C\|y_0\|^2_{H^1_0}}\|y_0\|_{H^1_0}(1 + \|y_0\|_{H^1_0}).$$

From (3.37), we have that

$$\begin{cases} -y^\alpha_{xx}(t,\cdot) = g(t,\cdot) - y^\alpha_t(t,\cdot) \\ y^\alpha(t,0) = y^\alpha(t,L) = 0 \end{cases}$$

for t a.e in (t_1,T). Thus, using (3.38) and elliptic regularity results, (see (EVANS, 2010, Ch. 6, Thm. 5)), we deduce that $y^\alpha \in L^2(t_1,T;H^3(0,L))$ and

$$\|y^\alpha\|_{L^2(t_1,T;H^3(0,L))} \le \frac{1}{2}\Lambda_1(\|y_0\|_{H^1_0}).$$

We also deduce that, for some $t_2 \in (t_1, T/2)$, one has

$$\|y^\alpha_t(t_2,\cdot)\|_{H^1_0} + \|y^\alpha(t_2,\cdot)\|_{H^3\cap H^1_0} \le \sqrt{\frac{2}{T-2t_1}}\Lambda_1(\|y_0\|_{H^1_0}).$$

Step 3: Estimates in $(t_2, T/2)$. Let us set $y_2 := y^\alpha(t_2,\cdot)$. Note that

$$\begin{aligned}\|g\|_{L^2(t_1,T;H^2(0,L))\cap H^1(t_1,T;L^2(0,L))} &\leq C\|y^\alpha\|_{L^\infty(t_1,T;H_0^1(0,L))}\|y^\alpha\|_{L^2(t_1,T;H^3(0,L))\cap H^1(t_1,T;H_0^1(0,L))}\\ &\leq C\|y_0\|_{H_0^1}\Lambda_1(\|y_0\|_{H_0^1})\, e^{C\|y_0\|^2_{H_0^1}}\end{aligned}$$

and the needed compatibility conditions for regularity results holds:

$$g(t_2,\cdot) + (y_2)_{xx}(t,\cdot) = y_t^\alpha(t_2,\cdot) \in H_0^1(0,L).$$

Using (EVANS, 2010, Ch. 7, Thm. 6), we get that

$$y^\alpha \in L^2(t_2,T;H^4(0,L)) \cap H^1(t_2,T;H^2(0,L)) \cap H^2(t_2,T;L^2(0,L))$$

and, moreover, in the time interval (t_2, T)

$$\begin{aligned}\|y^\alpha\|_{L^2(H^4)\cap H^1(H^2)\cap H^2(L^2)} &\leq C\left(\|g\|_{L^2(H^2)\cap H^1(L^2)} + \|y_2\|_{H^3}\right)\\ &\leq \Lambda_2(\|y_0\|_{H_0^1}),\end{aligned} \tag{3.39}$$

where

$$\Lambda_2(\|y_0\|_{H_0^1}) := C\left(1 + \|y_0\|_{H_0^1} e^{C\|y_0\|^2_{H_0^1}}\right)\Lambda_1(\|y_0\|_{H_0^1}).$$

Step 4: Conclusion. Finally, the result in (EVANS, 2010, Ch. 5, Thm. 4) applied to (3.39) leads to the regularity $C^0([t_2,T];H^3(0,L))$ for y^α. Therefore, the conclusion follows from Sobolev's embedding, taking $T^* = t_2$ and $\Lambda(\|y_0\|_{H_0^1}) = \Lambda_2(\|y_0\|_{H_0^1})$.

Proposition 3.8 is also true when $y_0 \in L^\infty(0,L)$. Indeed, we can start using Proposition 3.3 that guarantees the existence and uniqueness of a solution (y^α, z^α) to (3.36) satisfying (3.14) and (3.15). In particular, we have from (3.15) that

$$\|y^\alpha\|_{L^2(H_0^1)} \leq C_1\|y_0\|_\infty e^{C\|y_0\|^2_\infty}.$$

Therefore, there exists $t_1 \in (0, T/2)$ such that

$$\|y^\alpha(t_1,\cdot)\|_{H_0^1} \leq \sqrt{\frac{2}{T}}C_1\|y_0\|_\infty e^{C\|y_0\|^2_\infty}.$$

Then, we can achieve arguing as in the proof of Proposition 3.8.

3.5.2 Uniform approximate controllability

In this section, the goal is to prove the following approximate controllability result starting from sufficiently smooth initial data:

Proposition 3.9 *Let $y_0, y_f \in C^2([0,L])$ be given. There exist positive constants τ_* and $K > 0$, independent of α, such that, for any $\tau \in (0,\tau_*]$, there exist $p^\alpha \in C^0([0,\tau])$, $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,\tau;R^2)$ and associated states (y^α, z^α) with the following regularity*

$$\begin{cases} y^\alpha \in L^2(0,\tau;H^2(0,L)) \cap H^1(0,\tau;L^2(0,L)) \cap C^0([0,\tau];H^1(0,L)) \\ z^\alpha \in L^2(0,T;H^4(0,L)) \cap H^1(0,\tau;L^2(0,L)) \cap C^0([0,\tau];H^3(0,L)), \end{cases} \tag{3.40}$$

satisfying

$$\begin{cases} y_t^\alpha - y_{xx}^\alpha + z^\alpha y_x^\alpha = p^\alpha(t) & \textit{in} \quad (0,\tau)\times(0,L), \\ z^\alpha - \alpha^2 z_{xx}^\alpha = y^\alpha & \textit{in} \quad (0,\tau)\times(0,L), \\ z^\alpha(\cdot,0) = y^\alpha(\cdot,0) = v_l^\alpha & \textit{on} \quad (0,\tau), \\ z^\alpha(\cdot,L) = y^\alpha(\cdot,L) = v_r^\alpha & \textit{on} \quad (0,\tau), \\ y^\alpha(0,\cdot) = y_0 & \textit{in} \quad (0,L) \end{cases} \tag{3.41}$$

and, moreover,

$$\|y^\alpha(\tau,.) - y_f\|_{H^1(0,L)} \le K\sqrt{\tau} \tag{3.42}$$

and

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \le C \quad \forall\alpha > 0.$$

In order to prove this result, let us introduce $\lambda \in C_0^1(0,1)$ with $\|\lambda\|_{L^1(0,1/2)} > L$ and $\lambda(t) = \lambda(1-t)$ for all $t \in [0,1]$. Let us set $\lambda^\tau(t) := \frac{1}{\tau}\lambda\left(\frac{t}{\tau}\right)$ for all $t \in [0,\tau]$.

The following two results hold:

Lemma 3.1 *Let $M > 0$ be a positive constant. Then, if $u_0, u_f \in C^2([0,L])$ and*

$$\max\{\|u_0\|_{C^2([0,L])}, \|u_f\|_{C^2([0,L])}\} \le M, \tag{3.43}$$

there exists $\tau_0 \in (0,1)$ such that for every $\tau \in (0,\tau_0]$ we can find controls $v_l^{\alpha,\tau}, v_r^{\alpha,\tau}$ in $C^2([0,\tau])$ and associated states $u^{\alpha,\tau}, w^{\alpha,\tau}$ in $C^2([0,\tau]\times([0,L]))$, satisfying

$$\begin{cases} u_t^{\alpha,\tau} + (\lambda^\tau(t) + w^{\alpha,\tau})u_x^{\alpha,\tau} = 0 & \textit{in} \quad (0,\tau)\times(0,L), \\ w^{\alpha,\tau} - \alpha^2 w_{xx}^{\alpha,\tau} = u^{\alpha,\tau} & \textit{in} \quad (0,\tau)\times(0,L), \\ u^{\alpha,\tau}(\cdot,0) = w^{\alpha,\tau}(\cdot,0) = v_l^{\alpha,\tau} & \textit{in} \quad (0,\tau), \\ u^{\alpha,\tau}(\cdot,L) = w^{\alpha,\tau}(\cdot,L) = v_r^{\alpha,\tau} & \textit{in} \quad (0,\tau), \\ u^{\alpha,\tau}(0,\cdot) = u_0 & \textit{in} \quad (0,L), \\ u^{\alpha,\tau}(\tau,\cdot) = u_f & \textit{in} \quad (0,L). \end{cases} \tag{3.44}$$

Furthermore, there exists $C>0$, independent of α and τ, such that

$$\|u^{\alpha,\tau}\|_{C^0([0,\tau];C^2([0,L]))} \le CM. \tag{3.45}$$

First, thanks to the fact that $\|\lambda\|_{L^1(0,1/2)} > L$ and Remark 3.1, we know that there exists $\delta > 0$ (independent of α) such that, for any initial datum in a ball of $C^2([0,L])$ centered at origin and radius δ, there exists a solution to (3.18) belonging to the space $C^2([0,1/2]\times[0,L])$ satisfying (3.33).

Let us now take $\tau_0 \in (0,1)$ such that $\tau_0 M \le \delta$. Then, according to the previous construction, for each $\tau \in (0,\tau_0]$ there exist functions $(\widetilde{y}^\alpha, \widetilde{z}^\alpha)$, $(\widehat{y}^\alpha, \widehat{z}^\alpha)$ in the space $C^2([0,1/2]\times[0,L];R^2)$, solutions to (3.18), satisfying the $\widetilde{y}^\alpha(0,x) = \tau u_0(x)$ and $\widehat{y}^\alpha(0,x) = \tau u_f(L-x)$, for all $x \in [0,L]$, and satisfying the inequality (3.33).

Then, one defines the states

$$u^{\alpha,\tau}(t,x) := \begin{cases} \tau^{-1}\widetilde{y}^\alpha(\tau^{-1}t, x) & \text{in} \quad [0,\tau/2]\times[0,L], \\ \tau^{-1}\widehat{y}^\alpha(\tau^{-1}(\tau-t), L-x) & \text{in} \quad [\tau/2,\tau]\times[0,L] \end{cases}$$

and

$$w^{\alpha,\tau}(t,x) := \begin{cases} \tau^{-1}\widetilde{z}^\alpha(\tau^{-1}t, x) & \text{in} \quad [0,\tau/2]\times[0,L], \\ \tau^{-1}\widehat{z}^\alpha(\tau^{-1}(\tau-t), L-x) & \text{in} \quad [\tau/2,\tau]\times[0,L], \end{cases}$$

that satisfy $(u^{\alpha,\tau}, w^{\alpha,\tau}) \in C^2([0,\tau]\times[0,L];R^2)$ and the associated boundary controls

$$v_l^{\alpha,\tau}(t) := u^{\alpha,\tau}(t,0) \quad \text{and} \quad v_r^{\alpha,\tau}(t) := u^{\alpha,\tau}(t,L).$$

Since $\lambda(\tau^{-1}t) \equiv \lambda(\tau^{-1}(\tau-t))$ and Remark 3.2, the couple $(u^{\alpha,\tau}, w^{\alpha,\tau})$ satisfies (3.44) and (3.45).

Lemma 3.2 *Assume that $M > 0$, $u_0, u_f \in C^2([0,L])$ satisfy* (3.43) *and τ_0 is furnished by Lemma 3.1. There exists $\tau_* \in (0, \tau_0]$ such that, for any $\tau \in (0, \tau_*]$ and any $(u^{\alpha,\tau}, w^{\alpha,\tau}) \in C^2([0,\tau] \times [0,L]; R^2)$ satisfying* (3.44) *and* (3.45)*, there exists a unique solution to*

$$\begin{cases} r_t^{\alpha,\tau} + (q^{\alpha,\tau} + w^{\alpha,\tau} + \lambda^\tau) r_x^{\alpha,\tau} - r_{xx}^{\alpha,\tau} + q^{\alpha,\tau} u_x^{\alpha,\tau} - u_{xx}^{\alpha,\tau} = 0 & \text{in} \quad (0,\tau) \times (0,L), \\ q^{\alpha,\tau} - \alpha^2 q_{xx}^{\alpha,\tau} = r^{\alpha,\tau} & \text{in} \quad (0,\tau) \times (0,L), \\ r^{\alpha,\tau}(\cdot, 0) = 0, \quad r_x^{\alpha,\tau}(\cdot, L) = 0, & \text{on} \quad (0,\tau), \\ q^{\alpha,\tau}(\cdot, 0) = 0, \quad q^{\alpha,\tau}(\cdot, L) = r^{\alpha,\tau}(\cdot, L), & \text{on} \quad (0,\tau), \\ r^{\alpha,\tau}(0, \cdot) = 0 & \text{in} \quad (0,L), \end{cases}$$

satisfying

$$\begin{cases} r^{\alpha,\tau} \in L^2(0,\tau; H^2(0,L)) \cap H^1(0,\tau; L^2(0,L)) \cap C^0([0,\tau]; H^1(0,L)), \\ q^{\alpha,\tau} \in L^2(0,\tau; H^4(0,L)) \cap H^1(0,\tau; L^2(0,L)) \cap C^0([0,\tau]; H^3(0,L)) \end{cases}$$

and

$$\|r^{\alpha,\tau}\|_{L^2(0,\tau;H^2(0,L)) \cap H^1(0,\tau;L^2(0,L))} \leq C.$$

Here, C is a positive constant that depends on L, T, M and τ, but it is independent of α. Moreover, there exists a constant K that depends on L, T and M (independent of α and τ), such that

$$\|r^{\alpha,\tau}\|_{C^0([0,\tau];H^1(0,L))} \leq K\sqrt{\tau}. \tag{3.46}$$

The proof is standard. It can be easily obtained, for instance, via a Faedo-Galerkin technique in combination with well known energy estimates.

We can now achieve the proof of Proposition 3.9. Indeed, given $\tau \in (0, \tau_*]$, it is not difficult to see that (y^α, z^α) given by

$$(y^\alpha, z^\alpha) := (u^{\alpha,\tau} + r^{\alpha,\tau} + \lambda^\tau, w^{\alpha,\tau} + q^{\alpha,\tau} + \lambda^\tau)$$

satisfies (3.40) and (3.41) with $p^\alpha(t) = \lambda_t^\tau$ and boundary controls $v_l^\alpha(t) = u^{\alpha,\tau}(t,0) + r^{\alpha,\tau}(t,0) + \lambda^\tau(t)$ and $v_r^\alpha(t) = u^{\alpha,\tau}(t,L) + r^{\alpha,\tau}(t,L) + \lambda^\tau(t)$. Moreover, using $(3.44)_6$, (3.46) and the fact that λ^τ vanishes in the neighbourhood of τ, we obtain (3.42).

3.5.3 Uniform local exact controllability to the trajectories

The goal of this section is to prove the local exact controllability to space-independent trajectories for the Burgers-α system, with controls and associated states uniformly bounded with respect to α in appropriate spaces. Thus, let $\widehat{m} \in C^1([0,T])$ be given and note that $(\widehat{y}^\alpha, \widehat{z}^\alpha) = (\widehat{m}, \widehat{m})$ is a trajectory of viscous Burgers-α system with $(\widehat{p}^\alpha(t), \widehat{v}_l^\alpha(t), \widehat{v}_r^\alpha(t)) = (\widehat{m}'(t), \widehat{m}(t), \widehat{m}(t))$. We have the following result:

Theorem 3.4 *Let $T, L, \alpha > 0$ and $\widehat{m} \in C^1([0,T])$ be given. There exists $\delta > 0$ (independent of α) such that, for any initial data $y_0 \in H^1(0,L)$ satisfying $\|y_0 - \widehat{m}(0)\|_{H^1} \leq \delta$ there exist $p^\alpha \in C^0([0,T])$ and $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0,T;R^2)$ and associated states $(y^\alpha, z^\alpha) \in L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$ satisfying* (3.2) *and*

$$y^\alpha(T,\cdot) \equiv \widehat{m}(T). \tag{3.47}$$

Moreover, $p^\alpha = \widehat{m}'$ and the following estimates hold:

$$\|p^\alpha\|_{C^0([0,T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \leq C \quad \forall \alpha > 0, \tag{3.48}$$

where $C > 0$ is a positive constant independent of α.

Let us set $(y^\alpha, z^\alpha) = (u^\alpha + \widehat{m}, w^\alpha + \widehat{m})$ and $p^\alpha = \widehat{m}'$. Then, (u^α, w^α) must satisfy

$$\begin{cases} u_t^\alpha - u_{xx}^\alpha + (w^\alpha + \widehat{m})u_x^\alpha = 0 & \text{in} \quad (0,T)\times(0,L), \\ w^\alpha - \alpha^2 w_{xx}^\alpha = u^\alpha & \text{in} \quad (0,T)\times(0,L), \\ u^\alpha(\cdot,0) = w^\alpha(\cdot,0) = h_l^\alpha & \text{in} \quad (0,T), \\ u^\alpha(\cdot,0) = w^\alpha(\cdot,L) = h_r^\alpha & \text{in} \quad (0,T), \\ u^\alpha(0,\cdot) = u_0 & \text{in} \quad (0,L), \end{cases} \tag{3.49}$$

where $u_0 := y_0 - \widehat{m}(0)$ and $(h_l^\alpha, h_r^\alpha) := (v_l^\alpha - \widehat{m}, v_r^\alpha - \widehat{m})$. Therefore, Theorem 3.4 is equivalent to the local null-controllability to (3.49).

Proposition 3.10 *Let the conditions of Theorem 3.4 be satisfied. There exists $\delta > 0$ (independent of α) such that, for any initial data $u_0 \in H^1(0,L)$ satisfying $\|u_0\|_{H^1} \leq \delta$, there exist $(h_l^\alpha, h_r^\alpha) \in H^{3/4}(0,T;R^2)$ and $(u^\alpha, w^\alpha) \in L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$ satisfying* (3.49) *and*

$$u^\alpha(T,\cdot) \equiv 0. \tag{3.50}$$

Moreover, there exists a positive constant $C > 0$ (independent of α) such that

$$\|(h_l^\alpha, h_r^\alpha)\|_{H^{3/4}([0,T];R^2)} \leq C \quad \forall \alpha > 0. \tag{3.51}$$

The proof of this result relies on a fixed-point argument. Thus, given $u_0 \in H^1(0,L)$ and $\eta > 0$, one can get by reflection method an extension $u_0^* \in H_0^1(-\eta, L+\eta)$, with

$$\|u_0^*\|_{H_0^1(-\eta,L+\eta)} \leq C\|u_0\|_{H^1(0,L)}.$$

Let $R > 0$ be given and consider the set

$$B_R^\eta := \{\bar{u} \in L^\infty(0,T;C^0([-\eta, L+\eta]) : \|\bar{u}\|_{L^\infty(0,T;C^0([-\eta,L+\eta]))} \leq R\}.$$

For any $\bar{u} \in B_R^\eta$, we can easily deduce that there exists a unique solution to

$$\begin{cases} w - \alpha^2 w_{xx} = \bar{u}1_{(0,L)} & \text{in} \quad (0,T) \times (0,L), \\ w(\cdot,0) = \bar{u}(\cdot,0), \;\; w(\cdot,L) = \bar{u}(\cdot,L) & \text{in} \quad (0,T). \end{cases} \tag{3.52}$$

Moreover, using the maximum principle, we obtain that

$$\|w\|_{L^\infty(0,T;C^0([0,L]))} \leq C\|\bar{u}\|_{L^\infty(0,T;C^0([-\eta,L+\eta]))} \leq CR.$$

Then, again reflection method, we get an extension $w^* \in L^\infty(0,T;C^2([-\eta, L+\eta]))$ with

$$\|w^*\|_{L^\infty(0,T;C^0([-\eta,L+\eta]))} \leq C\|w\|_{L^\infty(0,T;C^0([0,L]))} \leq CR.$$

We assume that $L < a < b < L+\eta$. Then, arguing as in the proof of (ARARUNA, 2013, Theorem 1), we find $v \in L^\infty((0,T)\times(a,b))$ and $u \in L^2(0,T;H^2(-\eta,L+\eta)) \cap L^\infty(0,T;H_0^1(-\eta,L+\eta))$ such that

$$\begin{cases} u_t - u_{xx} + (w^* + \widehat{m})u_x = v1_{(a,b)} & \text{in} \quad (0,T)\times(-\eta,L+\eta), \\ u(\cdot,-\eta) = u(\cdot,L+\eta) = 0 & \text{in} \quad (0,T), \\ u(0,\cdot) = u_0^* & \text{in} \quad (-\eta,L+\eta), \\ u(T,\cdot) = 0 & \text{in} \quad (-\eta,L+\eta), \end{cases} \tag{3.53}$$

and

$$\|v\|_{L^\infty(0,T;L^\infty(a,b))} \leq C\|u_0\|_{H^1(0,L)},$$

for some $C > 0$ of the form

$$C := e^{C_0[1+1/T+(1+T)(\|w^*\|^2_{L^\infty(L^\infty)}+\|\widehat{m}^2_\infty)]}.$$

where $C_0 > 0$ depends on a, b, L and η. Therefore, it is not difficult to deduce that the norm of u in $H^1(0,T;L^2(-\eta,L+\eta))$, $L^2(0,T;H^2(-\eta,L+\eta))$ and $L^\infty(0,T;H_0^1(-\eta,L+\eta))$ are bounded by $C\|u_0\|_{H^1}$, where C is independent of α.

Consequently, there exists $\delta > 0$ (independent of α) such that, if $\|u_0\|_{H^1} \leq \delta$, one has $\|u\|_{L^\infty(0,T;C^0([-\eta,L+\eta]))} \leq R$ and the mapping $\Lambda_\alpha : B_R^\eta \mapsto B_R^\eta$, $\Lambda_\alpha(\bar{u}) := u$ is well defined. Note that

1. Λ_α is well defined and continuous. Indeed, this follows from the uniqueness of solution of (3.52) and (3.53); the continuity is obtained by using standard parabolic estimates and the fact that, if $\bar{u}_n \to \bar{u}$ in $L^\infty(0,T;C^0([-\eta,L+\eta]))$, then $w_n^* \to w^*$ in $L^\infty(0,T;C^0([-\eta,L+\eta]))$ and, therefore, $u_n \to u$ in $L^\infty(0,T;C^0([-\eta,L+\eta]))$.
2. $F^\eta := \Lambda_\alpha(B_R^\eta)$ is relatively compact in $L^\infty(0,T;C^0([-\eta,L+\eta]))$. Indeed, one easily obtains that F^η is bounded in $L^\infty(0,T;H^1_{\eta,0}(-\eta,L+\eta))$ and F_t^η is bounded in $L^2(0,T;L^2(-\eta,L+\eta))$. Hence, applying again (SIMON, 1980, Corollary 4), we get the desired compactness.

Finally, by applying Schauder's Fixed-Point Theorem, we see that there exists $u \in B_R^\eta$ such that $\Lambda_\alpha(u) = u$. Then, the couple (u^α, v^α), where u^α is the restriction to $(0,T)\times(0,L)$ of u and w is the solution to (3.52), belongs to $L^2(0,T;H^2(0,L;R^2)) \cap H^1(0,T;L^2(0,L;R^2))$ and satisfies (3.49), (3.50) and (3.51) with controls $h_l^\alpha := u(\cdot,0)$ and $h_r^\alpha := u(\cdot,L)$.

3.5.4 Global exact controllability

In this section we prove Theorem 3.2 by combining the results obtained in Sections 3.5.1, 3.5.2 and 3.5.3. First recall that given $y_0 \in L^\infty(0,L)$ and the unique associated solution (y_1^α, z_1^α) to (3.36), Proposition 3.8 provides a time $T^* \in (0,T/2)$ and a constant $M^* > 0$ (both independent of α) such that $y_1^\alpha \in C^0([T^*,T];C^2([0,L]))$ and, moreover,

$$\|y_1^\alpha\|_{C^0([T^*,T];C^2([0,L]))} \leq M^*. \tag{3.54}$$

Now, let us fix $N \in R$, let us set $M := \max\{M^*, |N|\}$ and assume that the constant $\tau^* > 0$, furnished by Proposition 3.9 is small enough, such that $T^* < T/2 - \tau^*$. Then, $y_{2,0}^\alpha := y_1^\alpha(T/2-\tau,\cdot)$ belongs to $C^2([0,L])$ and, from (3.54) and Proposition 3.9, there exist $p_2^\alpha \in C^0([0,\tau])$, $(v_{l,2}^\alpha, v_{r,2}^\alpha) \in H^{3/4}(0,\tau;R^2)$ and associated states $(y_2^\alpha, z_2^\alpha) \in L^2(0,\tau;H^2(0,L;R^2)) \cap H^1(0,\tau;L^2(0,L;R^2))$ satisfying (3.40), (3.41) and (3.42), with initial datum $y_{2,0}^\alpha$ and target $y_f = N$.

Finally, decreasing τ if necessary and setting $y_{3,0}^\alpha := y_2^\alpha(\tau,\cdot)$, we deduce, thanks to (3.42), that $\|y_{3,0}^\alpha - N\|_H^1 \leq \delta$, where $\delta > 0$ is the constant given in Theorem 3.4 for a control time $T/2$. Hence, this theorem (applied with $\widehat{m} \equiv N$), guarantees the existence of controls $(v_{l,3}^\alpha, v_{r,3}^\alpha) \in H^{3/4}(0,T/2;R^2)$ such that the associated states (y_3^α, z_3^α) satisfying (3.2), (3.47) and (3.48), with $p^\alpha \equiv 0$ and initial datum $y_{3,0}^\alpha$.

To conclude, using (y_1^α, z_1^α), (y_2^α, z_2^α) and (y_3^α, z_3^α), and the associated controls, we can build the required solution, as stated in Theorem 3.2.

3.6 ADDITIONAL COMMENTS AND QUESTIONS

3.6.1 Controllability for Lipschitz-continuous data

Proposition 3.7 and Theorem 3.1 also hold for y_0 in $C^{0,1}([0,L])$. Indeed, arguing as in the proof, one can guarantee that, as soon as the initial condition is small enough in the Lipschitz-continuous class, there exist $(v_l, v_r) \in C^{0,1}([0,T];R^2)$ and an associated state $(y,z) \in C^{0,1}([0,T] \times [0,L];R^2)$ satisfying (3.18) almost everywhere; furthermore, it is not difficult to check that all the estimates are uniform with respect to α. Note that this result improves (CHAPOULY, 2009, Theorem 1).

3.6.2 Passage to the limit

Theorem 3.1 establishes the existence of uniformly bounded controls for the inviscid Burgers-α system; the family of associated solutions is uniformly bounded in the space $C^1([0,T]\times[0,L])$. What happens as α goes to 0? For uncontrolled nonlocal conservation law, a similar question related to singular limit was studied in (COLOMBO, 2019).

Thanks to Theorem 3.2, assuming that $y_0 \in H_0^1(0,L)$, the family of controls $\{(p^\alpha, v_l^\alpha, v_r^\alpha)\}_{\alpha>0}$ of the viscous Burgers-α systems is uniformly bounded in $C^0([0,T]) \times H^{3/4}(0,T;R^2)$ and the associated family of states $\{y^\alpha\}_{\alpha>0}$ is uniformly bounded in $L^2(0,T;H^2(0,L)) \cap H^1(0,T;L^2(0,L))$. It is not difficult to verify that $\{y^\alpha\}_{\alpha>0}$ converges, as α goes to 0, to a controlled solution to the viscous Burgers equation with same initial datum y_0.

An additional interesting question is to determine the order of convergence of y^α, in the convergence space.

3.6.3 Null controllability with 2 controls

In Theorems 3.1 and 3.2, we have used 3 scalar controls. It remains open to see whether, using arguments similar to those in (MARBACH, 2014), it is also possible to prove global uniform null controllability with only 2 scalar controls.

3.6.4 Global exact controllability to the trajectories

At least two additional questions remain open here: (i) to obtain uniform global exact controllability to trajectories for the viscous Burgers-α system with trajectories in the space $W^{1,\infty}(0,T;W^{1,\infty}(0,L;R^2))$; (ii) to reduce the number of scalar controls.

3.6.5 Less regular initial conditions

In (MARBACH, 2014), the author proved a null controllability result for the viscous Burgers equation with initial datum in $L^2(0,L)$. Is it also possible to control uniformly L^2 initial conditions in the case of the viscous Burgers-α system?

4 REMARKS ON THE CONTROL OF FREE BOUNDARY PROBLEMS

This chapter analyzes the null-exact controllability of the two-phase Stefan problem with distributed controls. The two-phase Stefan problem is a class of free boundary problems modelling solidification or melting processes where each phase satisfies a parabolic equation which are separated by phase-change interface. We prove that the temperatures and the interface can be steered to zero and to a prescribed location, respectively, since the initial data and interface position are sufficiently close to the targets using two localized sources of heating/-cooling (one in each of the phases). The proofs rely on compactness-uniqueness argument to deduce observability estimates adapted to constraints and fixed point arguments to deduce the result for the non-linear system. Moreover, a negative controllability result is obtained in the case where there are no controls acting in the phases simultaneously and the interface does not collapse to the boundary.

4.1 INTRODUCTION

The two-phase Stefan problem is a mathematical model (a coupled system composed of two PDEs and one ODE) used to describe liquid-solid phase transition processes. These processes appear frequently in science and engineering like, for example, in the cancer treatment by cryosurgeries (RABIN Y.; SHITZER, 1998), crystal growth (CONRAD, 1990), lithium-ion batteries, among others. It is also important to highlight that, besides the interest in thermodynamics processes, similar systems represent models for phenomena of other kinds: analysis and computation of the flux in free-surfaces (HERMANS, 2011; STOKER, 1957), fluid-solid interaction (DOUBOVA A.; FERNÁNDEZ-CARA, 2005; TAKAHASHI L. Y.; TUCSNAK, 2013; VÁZQUEZ J.L.; ZUAZUA, 2003), gases flow through porous medium (VÁZQUEZ, 2007), growth of tumors and others mathematical modelling in biology (FRIEDMAN, 2012).

Let us present the mathematical formulation of the two-phase Stefan problem: let $L > 0$, $T > 0$ and $\ell_l, \ell_0, \ell_r \in (0, L)$ be given with $\ell_l < \ell_0 < \ell_r$. Moreover, let us also consider two functions $u_0 \in W_0^{1,4}(0, \ell_0)$ with $u_0 \geq 0$ and $v_0 \in W_0^{1,4}(\ell_0, L)$ with $v_0 \leq 0$ and two open sets $\omega_l \subset\subset (0, \ell_l)$ and $\omega_r \subset\subset (\ell_r, L)$. At each t, the material domain is separated in two parts: $x \in [0, \ell(t))$ (liquid phase) and $x \in (\ell(t), L]$ (solid phase). Here, $\ell = \ell(t)$ is the position of the interface between liquid and solid phases; it satisfies $\ell(0) = \ell_0$ and $\ell(t) \in (\ell_l, \ell_r)$ for all t.

The aim of this chapter is to study the controllability properties of the two-phase Stefan problem:

$$
\begin{cases}
u_t - d_l u_{xx} = h_l 1_{\omega_l} & \text{in} \quad Q_l, \\
v_t - d_r v_{xx} = h_r 1_{\omega_r} & \text{in} \quad Q_r, \\
u(0,t) = 0 & \text{on} \quad (0,T), \\
v(L,t) = 0 & \text{on} \quad (0,T), \\
u(\cdot,0) = u_0 & \text{in} \quad (0,\ell_0), \\
v(\cdot,0) = v_0 & \text{in} \quad (\ell_0,L), \\
u(\ell(t),t) = v(\ell(t),t) = 0 & \text{on} \quad (0,T), \\
-\ell'(t) = d_l u_x(\ell(t),t) - d_r v_x(\ell(t),t) & \text{in} \quad (0,T).
\end{cases}
\tag{4.1}
$$

Here and in the sequel, d_l and d_r must be viewed as diffusion coefficients and we use the notation

$$
\begin{cases}
Q := (0,L) \times (0,T), \\
Q_l := \{(x,t) \in Q : \; t \in (0,T), \; x \in (0,\ell(t))\}, \\
Q_r := \{(x,t) \in Q : \; t \in (0,T), \; x \in (\ell(t),L)\}, \\
\mathcal{O}_l = \omega_l \times (0,T) \text{ and } \mathcal{O}_r = \omega_r \times (0,T).
\end{cases}
$$

The main result in this chapter is the following:

Theorem 4.1 *Let $\ell_T \in (\ell_l, \ell_r)$. Then there exists $\delta > 0$ such that, for any $u_0 \in W_0^{1,4}(0,\ell_0)$ with $u_0 \geq 0$, any $v_0 \in W_0^{1,4}(\ell_0,L)$ with $v_0 \leq 0$ and any $\ell_0 \in (\ell_l,\ell_r)$ satisfying*

$$\|u_0\|_{W_0^{1,4}} + \|v_0\|_{W_0^{1,4}} + |\ell_0 - \ell_T| \leq \delta,$$

there exist controls $(h_l,h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated states (u,v,ℓ) with

$$
\begin{cases}
\ell \in C^1([0,T]), \; \ell(t) \in (\ell_l,\ell_r) \;\; \forall \; t \in [0,T], \\
u, \, u_x, \, u_t, \, u_{xx} \in L^2(Q_l) \text{ and } v, \, v_x, \, v_t, \, v_{xx} \in L^2(Q_r),
\end{cases}
$$

such that

$$\ell(T) = \ell_T, \;\; u(\cdot,T) = 0 \;\; \text{in} \;\; (0,\ell_T) \;\; \text{and} \;\; v(\cdot,T) = 0 \;\; \text{in} \;\; (\ell_T,L).$$

Remark 4.1 *We will see in Section 4.5.1 that the maximum principle for parabolic equations implies that the null controllability for* (4.1) *does not hold if one of the controls (for instance* h_r*) vanishes and the interface satisfies* $0 < \ell(T) < L$*. However, the possibility of getting a null control result with only one control when one of the phases collapses to the boundary, that is,* $\ell(T) = L$ *or* $\ell(T) = 0$*, is open.*

For completeness, let us mention some previous works on the control of our main system and other similar models.

The analysis of the controllability properties for linear and non-linear parabolic PDEs defined in cylindrical domains is a classical problem in control theory and the some of the main contributions are in the references (FATTORINI H.O.; RUSSELL, 1971; FERNÁNDEZ-CARA E.; ZUAZUA, 2000a; FURSIKOV A. V.; IMANUVILOV, 1996; LEBEAU G.; ROBBIANO, 1995). On the other hand, the study of the controllability properties of free-boundary problems for PDEs has not been much explored, although some important results have been obtained in the last years, specially for one-phase Stefan problems and variants; see (FERNÁNDEZ-CARA, 2016; FERNÁNDEZ-CARA E.; SOUSA, 2017a; DEMARQUE R.; FERNÁNDEZ-CARA, 2018; FERNÁNDEZ-CARA, 2018).

In what respects the two-phase Stefan problem, the best result to our knowledge concerns stabilization. More precisely, it is proved in (KOGA S.; KRSTIC, 2020) that, under some assumptions, there exist Neumann boundary controls and associated states (u, v, ℓ) defined for all $t > 0$ such that

$$\lim_{t\to\infty} \|u(\cdot, t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = \lim_{t\to\infty} \|v(\cdot, t) - \mathcal{T}_m\|_{L^2(0,\ell(t))} = 0 \quad \text{and} \quad \lim_{t\to\infty} \ell(t) = \ell_T,$$

where $\mathcal{T}_m$ is a melting/solidification temperature.

A natural question is whether or not it is possible to drive both the temperature and the interface to prescribed targets at a finite time. In this chapter we give a positive partial answer to this question. Recall that, in (DOUBOVA A.; FERNÁNDEZ-CARA, 2005; FERNÁNDEZ-CARA E.; SOUSA, 2017b; TAKAHASHI L. Y.; TUCSNAK, 2013), a similar problem was considered for a 1D fluid-structure problem, with the following equations on the interface:

$$u(\ell(t), t) = v(\ell(t), t) = \ell'(t), \quad v_x(\ell(t), t) - u_x(\ell(t), t) = m\ell''(t) \quad \text{for} \quad t \in (0, T).$$

In this chapter, we deal with situations leading to new difficulties compared to previous works on free-boundary controllability. Let us discuss some of these differences:

- Control of the interface. Here, we are also going to control the temperature and also the interface between liquid and solid regions. This will bring an extra difficulty. They main strategy will rely on linearization, then reformulation as an observability problem with a linear constraint and then resolution of a fixed-point equation.

- Existence of two phases. Obviously, this complicates a lot the structure and properties of the state and requires an appropriate analysis.

The rest of this chapter is organized as follows. In Section 4.2, we will reformulaqte the free-boundary problem as a nonlinear parabolic system in a cylindrical domain. In Section 4.3, we will present an improved observability inequality which leads to the null controllability for a related linearized system subject to a linear constraint. In Section 4.4, we will give a proof of Theorem 4.1. To this purpose, we will apply a fixed-point argument. Finally, in Section 4.5, we will present some additional comments and questions.

4.2 REFORMULATION OF THE FREE-BOUNDARY PROBLEM

As a first step, let us find a suitable diffeomorphism Φ that transforms the free-boundary problem for the parabolic system (4.1) into an equivalent problem for a nonlinear parabolic system in a cylindrical domain.

To do that, let us fix a function $\ell \in C^1([0,T])$ such that $\ell(t) \in (\ell_l, \ell_r)$ for all $t \in [0,T]$ and let us take $\sigma > 0$ sufficiently small such that

$$\ell_l + \sigma < \ell(t) - \sigma \ \text{ and } \ \ell(t) + \sigma < \ell_r - \sigma \ \text{ in } \ [0,T].$$

Then, for any $\ell_l + 2\sigma < y < \ell_r - 2\sigma$, we build a function $m(\cdot, y) : R \to R$ by linear interpolation of the points $(\ell_l - \sigma, \ell_l - \sigma)$, $(\ell_l + \sigma, \ell_l + \sigma)$, $(y - \sigma, \ell_0 - \sigma)$, $(y + \sigma, \ell_0 + \sigma)$, $(\ell_r - \sigma, \ell_r - \sigma)$ and $(\ell_r + \sigma, \ell_r + \sigma)$ and then extend the extreme segments toward infinity. Specifically, we have the following definition for $m(\cdot, y)$:

$$m(x,y) := \begin{cases} x, & \text{if } \ x \le \ell_l + \sigma, \\ \ell_l + \sigma + \dfrac{(\ell_l - \ell_0 + 2\sigma)(x - \ell_l - \sigma)}{\ell_l + 2\sigma - y}, & \text{if } \ \ell_l + \sigma < x < y - \sigma, \\ x - y + \ell_0, & \text{if } \ y - \sigma < x < y + \sigma, \\ \ell_0 + \sigma + \dfrac{(\ell_r - \ell_0 - 2\sigma)(x - y - \sigma)}{\ell_r - y - 2\sigma}, & \text{if } \ y + \sigma < x < \ell_r - \sigma, \\ x, & \text{if } \ x \ge \ell_r - \sigma. \end{cases}$$

Let us now consider a function $\eta \in C^\infty(R)$ such that

$$\text{supp}\ \eta \subset (-\sigma, \sigma) \quad \int_{-\sigma}^{\sigma} \eta(x)\, dx = 1 \ \text{ and } \ \eta(x) = \ \eta(-x) \ \ \forall x \in R.$$

Then, we can define a smooth function $G : R \times (\ell_l + 2\sigma, \ell_r - 2\sigma) \mapsto R$ as follows:

$$G(x, y) := [\eta * m(\cdot, y)](x).$$

A simple computation leads to the equalities

$$G(y, y) = \ell_0, \quad \partial_x G(y, y) = 1 \tag{4.2}$$

and

$$\nabla G(x, y) = \left([\eta' * m(\cdot, y)](x), [\eta * \partial_y m(\cdot, y)](x) \right),$$

where

$$\partial_y m(x, y) := \begin{cases} 0, & \text{if} \quad x \leq \ell_l + \sigma, \\ \dfrac{(\ell_l - \ell_0 + 2\sigma)(x - \ell_l - \sigma)}{(\ell_l + 2\sigma - y)^2}, & \text{if} \quad \ell_l + \sigma < x < y - \sigma, \\ -1, & \text{if} \quad y - \sigma < x < y + \sigma, \\ \dfrac{(\ell_r - \ell_0 - 2\sigma)(x - \ell_r + \sigma)}{(\ell_r - y - 2\sigma)^2}, & \text{if} \quad y + \sigma < x < \ell_r - \sigma, \\ 0, & \text{if} \quad x \geq \ell_r - \sigma. \end{cases}$$

Let us introduce the mapping

$$\Phi : Q \mapsto Q, \quad \text{with} \quad \Phi(x, t) := (G(x, \ell(t)), t).$$

It can be seen that Φ is a diffeomorphism in Q, coincides with the identity in the regions $(0, \ell_l + \sigma) \times (0, T)$ and $(\ell_r - \sigma, L) \times (0, T)$ and, moreover, $\Phi(\ell(t), t) = (L_0, t)$ for all $t \in [0, T]$. Let us introduce the sets $Q_{0,l} := (0, \ell_0) \times (0, T)$ and $Q_{0,r} := (\ell_0, L) \times (0, T)$ and let us define $p : Q_{0,l} \to R$ and $q : Q_{0,r} \to R$, with

$$p(\xi, t) := u(x, t) = u(\Phi^{-1}(\xi, t)) \text{ and } q(\xi, t) := v(x, t) = v(\Phi^{-1}(\xi, t)),$$

where $(\xi, t) := \Phi(x, t)$. Then, we have that the couple (p, q) satisfies:

$$\begin{cases} p_t - d_l^\ell p_{\xi\xi} + b_l^\ell p_\xi = h_l 1_{\omega_l} & \text{in} \quad Q_{0,l}, \\ q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi = h_r 1_{\omega_r} & \text{in} \quad Q_{0,r}, \\ p(0, \cdot) = q(L, \cdot) = 0 & \text{on} \quad (0, T), \\ p(\cdot, 0) = p_0 & \text{in} \quad (0, \ell_0), \\ q(\cdot, 0) = q_0 & \text{in} \quad (\ell_0, L), \\ p(\ell_0, \cdot) = q(\ell_0, \cdot) = 0 & \text{on} \quad (0, T), \\ d_l p_\xi(\ell_0, t) - d_r q_\xi(\ell_0, t) = -\ell'(t) & \text{on} \quad (0, T). \end{cases} \tag{4.3}$$

where $p_0 = u_0 \circ [G(\cdot,\ell_0)]^{-1} \in W_0^{1,4}(0,\ell_0)$, $q_0 = v_0 \circ [G(\cdot,\ell_0)]^{-1} \in W_0^{1,4}(\ell_0,L)$ and

$$\begin{aligned}
d_l^\ell(\cdot,t) &:= d_l \left(G_x\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right)\right)^2,\\
d_r^\ell(\cdot,t) &:= d_r \left(G_x\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right)\right)^2,\\
b_l^\ell(\cdot,t) &:= G_y\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right)\ell'(t) + d_l G_{xx}\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right),\\
b_r^\ell(\cdot,t) &:= G_y\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right)\ell'(t) + d_r G_{xx}\left([G(\cdot,\ell(t))]^{-1},\ell(t)\right).
\end{aligned}$$

Remark 4.2 *Using the fact that G and G^{-1} are smooth and $\ell \in C^1([0,T])$, we can prove that d_l^ℓ and b_l^ℓ belong, respectively, to $C^1(\overline{Q}_{0,l})$ and $C^0(\overline{Q}_{0,l})$. Furthermore, the second spatial derivative of d_l^ℓ and the first spatial derivative of b_l^ℓ are functions in $C^0(\overline{Q}_{0,l})$. The same can be obtained for the coefficients d_r^ℓ and b_r^ℓ.*

This way, we have that Theorem 4.1 is equivalent to prove a local controllability result for (4.3). Actually, we will prove the following:

Theorem 4.2 *Let $\ell_T \in (\ell_l,\ell_r)$. Then there exists $\delta > 0$ such that, for any $p_0 \in W_0^{1,4}(0,\ell_0)$ with $p_0 \geq 0$, any $q_0 \in W_0^{1,4}(\ell_0,L)$ with $q_0 \leq 0$ and any $\ell_0 \in (\ell_l,\ell_r)$ satisfying*

$$\|p_0\|_{W_0^{1,4}} + \|q_0\|_{W_0^{1,4}} + |\ell_0 - \ell_T| < \delta,$$

there exist controls $(h_l,h_r) \in L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r)$ and associated solutions (p,q,ℓ) to (4.3) with

$$\begin{cases}
\ell \in C^1([0,T]), \quad \ell(t) \in (\ell_l,\ell_r) \quad \forall t \in [0,T],\\[2mm]
p, p_\xi, p_t, p_{\xi\xi} \in L^2(Q_{0,l}) \quad \textit{and} \quad q, q_\xi, q_t, q_{\xi\xi} \in L^2(Q_{0,r}),
\end{cases}$$

such that

$$\ell(T) = \ell_T, \quad p(\cdot,T) = 0 \quad \textit{in} \quad (0,\ell_0) \quad \textit{and} \quad q(\cdot,T) = 0 \quad \textit{in} \quad (\ell_0,L).$$

4.3 APPROXIMATE CONTROLLABILITY OF A LINEARIZED SYSTEM

In this section, we are going to complete a first step in the proof of Theorem 4.2. More precisely, we are going to prove a controllability result for a suitable (natural) linearization of (4.3).

To do this, let us fix $\ell \in C^1([0,T])$ with $\ell(0) = \ell_0$ and $\ell([0,T]) \subset (\ell_l, \ell_r)$ and let us consider the system:

$$\begin{cases} M_l^\ell(p) = h_l 1_{\omega_l} & \text{in} \quad Q_{0,l}, \\ M_r^\ell(q) = h_r 1_{\omega_r} & \text{in} \quad Q_{0,r}, \\ p(0,\cdot) = p(\ell_0,\cdot) = q(\ell_0,\cdot) = q(L,\cdot) = 0 & \text{on} \quad (0,T), \\ p(\cdot,0) = p_0 & \text{in} \quad (0,\ell_0), \\ q(\cdot,0) = q_0 & \text{in} \quad (\ell_0,L), \end{cases} \tag{4.4}$$

where the operators M_l^ℓ and M_r^ℓ are respectively defined by:

$$M_l^\ell(p) := p_t - d_l^\ell p_{\xi\xi} + b_l^\ell p_\xi \quad \text{and} \quad M_r^\ell(q) := q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi.$$

Also, let us introduce the function $\mathcal{L} : [0,T] \mapsto R$ given by

$$\mathcal{L}(t) := \ell_0 - \int_0^t [d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)]\, d\tau.$$

The main goal of this section is to obtain a (robust) approximate controllability result for (4.4) subject to the linear constraint

$$\mathcal{L}(T) = \ell_T. \tag{4.5}$$

In other words, we want to find controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ such that the associated solutions to (4.4) satisfy (4.5).

Let us first reformulate (4.5). Thus, consider the augmented adjoint system

$$\begin{cases} (M_l^\ell)^*(\psi) = 0 & \text{in} \quad Q_{0,l}, \\ (M_r^\ell)^*(\zeta) = 0 & \text{in} \quad Q_{0,r}, \\ \psi(0,\cdot) = 0, \quad \psi(\ell_0,\cdot) = 1 & \text{on} \quad (0,T), \\ \zeta(\ell_0,\cdot) = 1, \quad \zeta(L,\cdot) = 0, & \text{on} \quad (0,T), \\ \psi(\cdot,T) = 0 & \text{in} \quad (0,\ell_0), \\ \zeta(\cdot,T) = 0 & \text{in} \quad (\ell_0,L). \end{cases} \tag{4.6}$$

It is not difficult to check that (4.6) possesses a unique weak solution (ψ_ℓ, ζ_ℓ), with

$$\psi_\ell \in L^2(0,T; H_0^1(0,\ell_0)) \cap H^1(0,T; H^{-1}(0,\ell_0)),$$

$$\zeta_\ell \in L^2(0,T; H_0^1(\ell_0,L)) \cap H^1(0,T; H^{-1}(\ell_0,L)).$$

A crucial property of (ψ_ℓ, ζ_ℓ) is the following:

Proposition 4.1 *Given $R > 0$, let us consider the set $\mathcal{B}_R := \{\ell \in C^1([0,T]);\ \ \|\ell'\|_{C^0([0,T])} \leq R\}$. Then, there exists a positive constant C_0, only depending on ℓ_0, ℓ_l, ℓ_r, ω_l, ω_r, T and R such that, for any $\ell \in \mathcal{B}_R$, one has:*

$$\|\psi_\ell\|_{L^2(\mathcal{O}_l)} + \|\zeta_\ell\|_{L^2(\mathcal{O}_r)} \geq C_0.$$

We argue by contradiction. Thus, if the assertion were false, then there would exist $\ell_1, \ell_2, \ldots$ and associated pairs $(\psi^1, \zeta^1), (\psi^2, \zeta^2), \ldots$ (weak solutions to (4.6)), such that

$$\|\ell_n'\|_\infty \leq R \ \text{ and } \ \|\psi^n\|_{L^2(\mathcal{O}_l)} + \|\zeta^n\|_{L^2(\mathcal{O}_r)} \leq \frac{1}{n} \ \ \forall\, n \geq 1. \tag{4.7}$$

Due to the smoothing effect of parabolic operators and the fact that the $(d_l^{\ell_n}, b_l^{\ell_n})$ are uniformly bounded in $C^1(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$, there would exist $\sigma > 0$ such that

$$\|\psi^n\|_{L^2(0,T-\sigma;H^2(0,\ell_0))} + \|\psi_t^n\|_{L^2(0,T-\sigma;L^2(0,\ell_0))} \leq C \ \ \forall\, n \geq 1,$$

with $C > 0$ depending only on ℓ_0, ℓ_l, ℓ_r, T and R. Consequently, after extraction of a subsequence, we would have

$$\begin{cases} \ell_n \to \ell & \text{strongly in } \ C^0([0,T-\sigma]), \\ \ell_n \to \ell & \text{weakly in } \ H^1(0,T-\sigma), \\ \psi^n \to \psi & \text{weakly in } \ L^2(0,T-\sigma;H^2(0,\ell_0)) \cap H^1(0,T-\sigma;L^2(0,\ell_0)) \end{cases}$$

and we would be able to pass to the limit in the equation and the boundary condition satisfied by ψ^n to deduce that

$$\begin{cases} (M_l^\ell)^*(\psi) = 0 & \text{in} \quad (0,\ell_0) \times (0,T-\sigma), \\ \psi(0,\cdot) = 0,\ \psi(\ell_0,\cdot) = 1 & \text{on} \quad (0,T-\sigma). \end{cases} \tag{4.8}$$

But we would also have, by (4.7), that $\psi \equiv 0$ in $\omega_l \times (0,T-\sigma)$, which is impossible, in view of the unique continuation property and $(4.8)_2$. This ends the proof.

Let us multiply $(4.4)_1$ by ψ_ℓ and let us integrate in $Q_{0,l}$ to obtain

$$\iint_{\mathcal{O}_l} h_l \psi_\ell \, d\xi \, dt = -\int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\, d\xi - \int_0^T d_l p_\xi(\ell_0,\tau)\, d\tau. \tag{4.9}$$

Analogously, multiplying $(4.4)_2$ by ζ_ℓ and integrating in $Q_{0,r}$ we get

$$\iint_{\mathcal{O}_r} h_r \zeta_\ell \, d\xi \, dt = -\int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\, d\xi + \int_0^T d_r q_\xi(\ell_0,\tau)\, d\tau. \tag{4.10}$$

It follows from (4.9)–(4.10) that a couple of controls $(h_l, h_r) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ is such that $\mathcal{L}(T) = \ell_T$ if and only if

$$\iint_{\mathcal{O}_l} h_l \psi_\ell \, d\xi \, dt + \iint_{\mathcal{O}_r} h_r \zeta_\ell \, d\xi \, dt = \ell_T - \ell_0 - \int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi, 0)\, d\xi - \int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi, 0)\, d\xi. \tag{4.11}$$

In Section 4.3.2, we will establish the approximate controllability of (4.4) subject to the linear constraint (4.11). Before, we will need an adequate (improved) observability inequality.

4.3.1 An improved observability inequality

To do this, let us first consider open sets $\omega_{0,l} \subset\subset \omega_l$, $\omega_{0,r} \subset\subset \omega_l$ and let us introduce the weight functions $\eta_{0,l} \in C^2([0, \ell_0])$ and $\eta_{0,r} \in C^2([\ell_0, L])$ satisfying

$$\begin{cases} \eta_{0,l} > 0 \text{ in } (0, \ell_0), \ \eta_{0,l}(0) = \eta_{0,l}(\ell_0) = 0 \text{ and } |\eta_{0,l}'| > 0 \text{ in } [0, \ell_0] \backslash \omega_{0,l}, \\ \eta_{0,r} > 0 \text{ in } (\ell_0, L), \ \eta_{0,r}(\ell_0) = \eta_{0,r}(L) = 0 \text{ and } |\eta_{0,r}'| > 0 \text{ in } [\ell_0, L] \backslash \omega_{0,r}. \end{cases}$$

Also, for any $\lambda > 0$, let us set

$$\begin{cases} \mu_l(\xi, t) := \dfrac{e^{\lambda \eta_{0,l}(\xi)}}{t(T-t)}, \ \ \alpha_l(\xi, t) := \dfrac{e^{2\lambda \|\eta_{0,l}\|_\infty} - e^{\lambda \eta_{0,l}(\xi)}}{t(T-t)}, \\ \\ \mu_r(\xi, t) := \dfrac{e^{\lambda \eta_{0,r}(\xi)}}{t(T-t)}, \ \ \alpha_r(\xi, t) := \dfrac{e^{2\lambda \|\eta_{0,r}\|_\infty} - e^{\lambda \eta_{0,r}(\xi)}}{t(T-t)}. \end{cases}$$

Then, by using the regularity of the coefficients of the adjoint operators $(M_l^\ell)^*$ and $(M_r^\ell)^*$ (see Remark 4.2) and following the ideas in (FERNÁNDEZ-CARA E.; GUERRERO, 2006; FURSIKOV A. V.; IMANUVILOV, 1996), we get the following global Carleman estimates:

Proposition 4.2 *Let $R > 0$ and assume that $\ell \in C^1([0, T])$ satisfies $\ell(0) = \ell_0$, $\ell([0, T]) \subset (\ell_l, \ell_r)$ and $\|\ell'\|_{C^0([0,T])} \leq R$. Then, there exist positive constants λ_0, s_0 and C (depending on $\ell_l, \ell_r, R, \omega_l, \omega_r$ and T) such that, for any $s \geq s_0$ and $\lambda \geq \lambda_0$, we have:*

$$\begin{aligned} &\iint_{Q_{0,l}} e^{-2s\alpha_l} \left[(s\mu_l)^{-1} \left(|\varphi_t|^2 + |\varphi_{\xi\xi}|^2 \right) + \lambda^2 (s\mu_l) |\varphi_\xi|^2 + \lambda^4 (s\mu_l)^3 |\varphi|^2 \right] d\xi \, dt \\ &\quad \leq C \left[\iint_{Q_{0,l}} e^{-2s\alpha_l} \left| (M_l^\ell)^*(\varphi) \right|^2 d\xi \, dt + \iint_{\mathcal{O}_l} e^{-2s\alpha_l} \lambda^4 (s\mu_l)^3 |\varphi|^2 \, d\xi \, dt \right] \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\iint_{Q_{0,r}} e^{-2s\alpha_r} \left[(s\mu_r)^{-1} \left(|\phi_t|^2 + |\phi_{\xi\xi}|^2 \right) + \lambda^2 (s\mu_r) |\phi_\xi|^2 + \lambda^4 (s\mu_r)^3 |\phi|^2 \right] d\xi \, dt \\ &\quad \leq C \left[\iint_{Q_{0,r}} e^{-2s\alpha_r} |(M_r^\ell)^*(\phi)|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} e^{-2s\alpha_r} \lambda^4 (s\mu_r)^3 |\phi|^2 \, d\xi \, dt \right], \end{aligned} \tag{4.13}$$

for any pair (φ, ϕ) *in the Bochner-Sobolev space*

$$[L^2(0,T;H_0^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0))] \times [L^2(0,T;H_0^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L))]$$

such that $\Big((M_l^\ell)^*(\varphi), (M_r^\ell)^*(\phi)\Big)$ *belongs to* $L^2(Q_{0,l}) \times L^2(Q_{0,r})$.

A straightforward argument, based on the estimates (4.12)–(4.13), leads the following observability inequality:

Proposition 4.3 *Let* $R > 0$ *and assume that* $\ell \in C^1([0,T])$ *satisfies* $\ell(0) = \ell_0$, $\ell([0,T]) \subset (\ell_l, \ell_r)$ *and* $\|\ell'\|_{C^0([0,T])} \leq R$. *There exist positive constants* λ_0, s_0 *and* C, *depending on* ℓ_l, ℓ_r, R, ω_l, ω_r *and* T, *such that, for any* $s \geq s_0$ *and any* $\lambda \geq \lambda_0$, *we have:*

$$\|(\varphi(\cdot,0), \phi(\cdot,0))\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \leq C\|(\varphi,\phi)\|_{L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r)}, \tag{4.14}$$

for any pair (φ, ϕ) *in the Bochner-Sobolev space*

$$[L^2(0,T;H_0^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0))] \times [L^2(0,T;H_0^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L))]$$

such that $\Big((M_l^\ell)^*(\varphi), (M_r^\ell)^*(\phi)\Big) = (0,0)$.

In order to present an improved observability inequality, let us introduce the linear projectors $P_l^\ell : L^2(Q_{0,l}) \mapsto L^2(Q_{0,l})$ and $P_r^\ell : L^2(Q_{0,r}) \mapsto L^2(Q_{0,r})$, respectively given by

$$P_l^\ell \varphi := \beta_l^\ell(\varphi)\psi_\ell \quad \text{and} \quad P_r^\ell \phi := \beta_r^\ell(\phi)\zeta_\ell,$$

where we have set

$$\beta_l^\ell(\varphi) := \frac{\displaystyle\iint_{\mathcal{O}_l} \psi_\ell \varphi \, d\xi \, dt}{\displaystyle\iint_{\mathcal{O}_l} |\psi_\ell|^2 \, d\xi \, dt} \quad \text{and} \quad \beta_r^\ell(\phi) := \frac{\displaystyle\iint_{\mathcal{O}_r} \zeta_\ell \phi \, d\xi \, dt}{\displaystyle\iint_{\mathcal{O}_r} |\zeta_\ell|^2 \, d\xi \, dt}$$

and (ψ_ℓ, ζ_ℓ) is the unique weak solution to (4.6).

Remark 4.3 *Note that the ranges of* P_l^ℓ *and* P_r^ℓ *are 1D vector spaces. Therefore, these operators are compact.*

For any $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0, L)$, there exists a unique couple (φ, ϕ) satisfying

$$\begin{cases} \varphi \in L^2(0,T;H_0^1(0,\ell_0)) \cap H^1(0,T;H^{-1}(0,\ell_0)), \\ \phi \in L^2(0,T;H_0^1(\ell_0,L)) \cap H^1(0,T;H^{-1}(\ell_0,L)), \end{cases} \tag{4.15}$$

that solves in the weak sense the linear system

$$\begin{cases} (M_l^\ell)^*(\varphi) = 0 & \text{in} \quad Q_{0,l}, \\ (M_r^\ell)^*(\phi) = 0 & \text{in} \quad Q_{0,r}, \\ \varphi(0,\cdot) = \varphi(\ell_0,\cdot) = 0 & \text{on} \quad (0,T), \\ \phi(0,\cdot) = \phi(\ell_0,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} \quad (0,\ell_0), \\ \phi(\cdot,T) = \phi_T & \text{in} \quad (\ell_0,L). \end{cases} \tag{4.16}$$

Accordingly, we can introduce the following functional in $L^2(0,\ell_0) \times L^2(\ell_0,L)$:

$$\begin{aligned} I(\varphi_T,\phi_T) := & \iint_{\mathcal{O}_l} |\varphi|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi(\xi,0)|^2 \; d\xi + |\beta_l^\ell(\varphi)|^2 \\ & + \iint_{\mathcal{O}_r} |\phi|^2 \, d\xi \, dt + \int_{\ell_0}^{L} |\phi(\xi,0)|^2 \, d\xi + |\beta_r^\ell(\phi)|^2, \end{aligned}$$

where (φ,ϕ) satisfies (4.15)–(4.16).

We can prove the following result:

Proposition 4.4 *Let $R > 0$ and let us assume that $\ell \in C^1([0,T])$ satisfies $\ell(0) = \ell_0$, $\ell([0,T]) \subset (\ell_l,\ell_r)$ and $\|\ell'\|_{C^0([0,T])} \leq R$. Then, there exists a positive constant C, depending on ℓ_0, ℓ_l, ℓ_r, R, ω_l, ω_r and T, such that, for any $(\varphi_T,\phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L)$, the following holds:*

$$I(\varphi_T,\phi_T) \leq C \left[\iint_{\mathcal{O}_l} |\varphi - P_l^\ell \varphi|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi - P_r^\ell \phi|^2 \, d\xi \, dt \right]. \tag{4.17}$$

The prove will be by contradiction. It is inspired by the results in (NAKOULIMA, 2007). Let us first prove that there exists a constant $C_1 > 0$ (depending on $\ell_0, \ell_l, \ell_r, R, \omega_l, \omega_r$ and T) such that, for any couple of functions $(\varphi_T,\phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L)$, we get:

$$\begin{aligned} & \iint_{\mathcal{O}_l} |\varphi|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi(\xi,0)|^2 \; d\xi + \iint_{\mathcal{O}_r} |\phi|^2 \, d\xi \, dt + \int_{\ell_0}^{L} |\phi(\xi,0)|^2 \, d\xi \\ & \leq C_1 \left[\iint_{\mathcal{O}_l} |\varphi - P_l^\ell \varphi|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi - P_r^\ell \phi|^2 \, d\xi \, dt \right]. \end{aligned} \tag{4.18}$$

To prove this, we argue by contradiction. Thus, suppose that (4.18) does not hold. Then, there exists a sequence $\{(\varphi_{T,n},\phi_{T,n})\}_{n=1}^{\infty}$ in $L^2(0,\ell_0) \times L^2(\ell_0,L)$ such that

$$\begin{cases} 1 = \displaystyle\iint_{\mathcal{O}_l} |\varphi_n|^2 \, d\xi \, dt + \int_0^{\ell_0} |\varphi_n(\xi,0)|^2 \; d\xi + \iint_{\mathcal{O}_r} |\phi_n|^2 \, d\xi \, dt + \int_{\ell_0}^{L} |\phi_n(\xi,0)|^2 \, d\xi, \\ \dfrac{1}{n} \geq \displaystyle\iint_{\mathcal{O}_l} |\varphi_n - P_l^\ell \varphi_n|^2 \, d\xi \, dt + \iint_{\mathcal{O}_r} |\phi_n - P_r^\ell \phi_n|^2 \, d\xi \, dt. \end{cases} \tag{4.19}$$

Now, notice that

$$\begin{aligned} & \frac{1}{2}\iint_{\mathcal{O}_l} |P_l^\ell \varphi_n|^2 \, d\xi \, dt + \frac{1}{2}\iint_{\mathcal{O}_r} |P_r^\ell \phi_n|^2 \, d\xi \, dt \\ \leq \quad & \iint_{\mathcal{O}_l} \left[|\varphi_n|^2 + |\varphi_n - P_l^\ell \varphi_n|^2 \right] \, d\xi \, dt + \iint_{\mathcal{O}_r} \left[|\phi_n|^2 + |\phi_n - P_l^\ell \phi_n|^2 \right] \, d\xi \, dt. \end{aligned}$$

Therefore, we get easily from (4.19) that the right hand side of the inequality above is uniformly bounded and, in particular, the sequence $\{(\beta_l^\ell(\varphi_n), \beta_r^\ell(\phi_n))\}_{n=1}^\infty$ is uniformly bounded in R^2. Consequently, there exists a subsequence, still indexed by n, and $(\beta_l^*, \beta_r^*) \in R^2$ such that

$$(\beta_l^\ell(\varphi_n), \beta_r^\ell(\phi_n)) \to (\beta_l^*, \beta_r^*) \quad \text{in } R^2. \tag{4.20}$$

Now, we consider a sequence of positive numbers $\{k_j\}_{j=1}^\infty$ with $k_j = k_1 + (j-1)$, so that $T - 1/k_1 > 0$. Moreover, given j, let us introduce the spaces:

$$\begin{aligned} \mathcal{U}_j^l &:= L^2(1/k_j, T - 1/k_j; H^2(0, \ell_0) \cap H_0^1(0, \ell_0)) \cap H^1(1/k_j, T - 1/k_j; L^2(0, \ell_0)), \\ \mathcal{U}_j^r &:= L^2(1/k_j, T - 1/k_j; H^2(\ell_0, L) \cap H_0^1(\ell_0, L)) \cap H^1(1/k_j, T - 1/k_j; L^2(\ell_0, L)). \end{aligned}$$

Then, using the inequalities (4.12), (4.13) and $(4.19)_1$, we can see the sequence $\{(\varphi_n, \phi_n)\}_{n=1}^\infty$ is uniformly bounded in $\mathcal{U}_1^l \times \mathcal{U}_1^r$ and, consequently, there exist a subsequence $\{(\varphi_n^1, \phi_n^1)\}_{n=1}^\infty$ and functions $(\varphi_1, \phi_1) \in \mathcal{U}_1^l \times \mathcal{U}_1^r$ such that

$$\begin{cases} \varphi_n^1 \to \varphi_1 & \text{weakly in} \quad L^2(1/k_1, T - 1/k_1; H^2(0, \ell_0) \cap H_0^1(0, \ell_0)), \\ \varphi_{n,t}^1 \to \varphi_{1,t} & \text{weakly in} \quad L^2(1/k_1, T - 1/k_1; L^2(0, \ell_0)), \\ \phi_n^1 \to \phi_1 & \text{weakly in} \quad L^2(1/k_1, T - 1/k_1; H^2(\ell_0, L) \cap H_0^1(\ell_0, L)), \\ \phi_{n,t}^1 \to \phi_{1,t} & \text{weakly in} \quad L^2(1/k_1, T - 1/k_1; L^2(\ell_0, L)). \end{cases} \tag{4.21}$$

Analogously, the sequence $\{(\varphi_n^1, \phi_n^1)\}_{n=1}^\infty$ is uniformly bounded in $\mathcal{U}_2^l \times \mathcal{U}_2^r$, where we can extract a new subsequence $\{(\varphi_n^2, \phi_n^2)\}_{n=1}^\infty$ and functions $(\varphi_2, \phi_2) \in \mathcal{U}_2^l \times \mathcal{U}_2^r$ satisfying the weak limits in (4.21), for k_1 replaced by k_2. In particular,

$$(\varphi_2, \phi_2)|_{(1/k_1, T - 1/k_1)} = (\varphi_1, \phi_1).$$

Then, by an induction procedure we can get, for each j, sequences $\{(\varphi_n^j, \phi_n^j)\}_{n=1}^\infty \subset \{(\varphi_n^{j-1}, \phi_n^{j-1})\}_{n=1}^\infty \subset \ldots \subset \{(\varphi_n^1, \phi_n^1)\}_{n=1}^\infty \subset \{(\varphi_n, \phi_n)\}_{n=1}^\infty$ and functions $(\varphi_j, \phi_j) \in \mathcal{U}_j^l \times \mathcal{U}_j^r$ satisfying the weak limits in (4.21), for k_1 replaced by k_j and, furthermore

$$(\varphi_j, \phi_j)|_{(1/k_{j-1}, T - 1/k_{j-1})} = (\varphi_{j-1}, \phi_{j-1}), \quad \text{for } j = 2, 3, \ldots.$$

Now, let us denote again by $\{(\varphi_n,\phi_n)\}_{n=1}^{\infty}$ the sequence $\{(\varphi_n^n,\phi_n^n)\}_{n=1}^{\infty}$ obtained by Cantor diagonalization. Then, it's easy to see that

$$\begin{cases} \varphi_n \to \varphi_j & \text{weakly in} \quad L^2(1/k_j, T-1/k_j; H^2(0,\ell_0)\cap H_0^1(0,\ell_0)), \\ \varphi_{n,t} \to \varphi_{j,t} & \text{weakly in} \quad L^2(1/k_j, T-1/k_j; L^2(0,\ell_0)), \\ \phi_n \to \phi_j & \text{weakly in} \quad L^2(1/k_j, T-1/k_j; H^2(\ell_0,L)\cap H_0^1(\ell_0,L)), \\ \phi_{n,t} \to \phi_{j,t} & \text{weakly in} \quad L^2(1/k_j, T-1/k_j; L^2(\ell_0,L)), \end{cases} \tag{4.22}$$

for all j.

Therefore, the functions $\varphi : (0,\ell_0)\times(0,T)\mapsto R$ and $\phi : (\ell_0,L)\times(0,T)\mapsto R$, given by

$$(\varphi,\phi)|_{(1/k_j,T-1/k_j)} = (\varphi_{j-1},\phi_{j-1}), \quad \forall\, j\in N$$

are well defined and it follows easily from (4.22) that

$$\begin{cases} (M_l^\ell)^*(\varphi) = 0 & \text{in} \quad Q_{0,l}, \\ (M_r^\ell)^*(\phi) = 0 & \text{in} \quad Q_{0,r}, \\ \varphi(\cdot,0) = \varphi(\cdot,\ell_0) = 0 & \text{in} \quad (0,T), \\ \varphi(\cdot,\ell_0) = \varphi(\cdot,L) = 0 & \text{in} \quad (0,T). \end{cases} \tag{4.23}$$

Moreover, since $(\varphi_n,\phi_n) = (\varphi_n - P_l^\ell\varphi_n, \phi_n - P_r^\ell\phi_n) + (P_l^\ell\varphi_n, P_r^\ell\phi_n)$ in $\mathcal{O}_l\times\mathcal{O}_r$, using (4.19)$_2$ and (4.20), it is not difficult to see that

$$(\varphi_n,\phi_n) \to (P_l^*\varphi, P_r^*\phi) \quad \text{strongly in} \;\; L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r),$$

where $(P_l^*\varphi, P_r^*\phi) = (\beta_l^*\psi_\ell, \beta_r^*\zeta_\ell)$.

We have from (4.6) and (4.23) that $((M_l^\ell)^*(\varphi - P_l^*\varphi), (M_r^\ell)^*(\phi - P_r^*\phi)) = (0,0)$ in $Q_{0,l}\times Q_{0,r}$ and also, in view of (4.19) and (4.20), $(\varphi - P_l^*\varphi, \phi - P_l^*\phi) = (0,0)$ in $\mathcal{O}_l\times\mathcal{O}_r$. Then, by applying a classical unique continuation argument, we conclude that $(\varphi,\phi) = (P_l^*\varphi, P_r^*\phi)$ in $Q_{0,l}\times Q_{0,r}$. However, this implies $(\varphi,\phi) = (0,0)$ in $Q_{0,l}\times Q_{0,r}$, since

$$(0,0) = (\varphi(\ell_0,\cdot),\phi(\ell_0,\cdot)) = (\beta_l^*\psi_\ell(\ell_0,\cdot), \beta_r^*\zeta_\ell(\ell_0,\cdot)) = (\beta_l^*,\beta_r^*).$$

In other words,

$$(\varphi_n,\phi_n) \to (0,0) \quad \text{in } L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r).$$

Then, taking into account (4.14) and (4.19), we see that

$$\iint_{\mathcal{O}_l} |\varphi_n|^2\,d\xi\,dt + \int_0^{\ell_0} |\varphi_n(\xi,0)|^2\,d\xi + \iint_{\mathcal{O}_r} |\phi_n|^2\,d\xi\,dt + \int_{\ell_0}^{L} |\phi_n(\xi,0)|^2\,d\xi \to 0,$$

which is obviously absurd.

This proves (4.18). The remaining terms in $I(\varphi_T, \phi_T)$ can also be bounded by the right hand side of (4.17), as an immediate consequence of Proposition 4.1.

4.3.2 Approximate controllability problem with linear constraint

In this section, we prove the approximate controllability of (4.4) subject to the linear constraint (4.11). More precisely, the following holds:

Proposition 4.5 *Assume that $R > 0$, $\ell_0 \in (\ell_l, \ell_r)$ and $\ell \in C^1([0,T])$ satisfies $\ell_l < \ell(t) < \ell_r$ for all $t \in [0,T]$, $\ell(0) = \ell_0$ and $\|\ell'\|_{C^0([0,T])} \leq R$. Then, for any $\varepsilon > 0$, any data $p_0 \in H_0^1(0,\ell_0)$ and $q_0 \in H_0^1(\ell_0, L)$ and any $\ell_T \in (\ell_l, \ell_r)$, there exist controls $(h_{l,\varepsilon}^{\ell}, h_{r,\varepsilon}^{\ell}) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated solutions to* (4.4)*, with*

$$\begin{cases} p \in L^2(0,T; H^2(0,\ell_0)) \cap H^1(0,T; L^2(0,\ell_0)), \\ q \in L^2(0,T; H^2(\ell_0,L)) \cap H^1(0,T; L^2(\ell_0,L)), \end{cases}$$

satisfying the approximate controllability condition

$$\|(p(\cdot,T), q(\cdot,T)\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \leq \varepsilon \tag{4.24}$$

and the linear constraint (4.11)*. Furthermore, the controls can be chosen satisfying*

$$\|(h_{l,\varepsilon}^{\ell} 1_{\omega_l}), (h_{r,\varepsilon}^{\ell} 1_{\omega_l}))\|_{L^2(Q_{0,l})\times L^2(Q_{0,r})} \leq C\left(\|(p_0,q_0)\|_{L^2\times L^2} + |\ell_0 - \ell_T|\right), \tag{4.25}$$

where the constant $C > 0$ depends only on $\ell_l, \ell_r, \omega_l, \omega_r, T$ and R.

Let us first introduce the notation

$$M_\ell := \ell_T - \ell_0 - \int_0^{\ell_0} p_0(\xi)\psi_\ell(\xi,0)\,d\xi - \int_{\ell_0}^{L} q_0(\xi)\zeta_\ell(\xi,0)\,d\xi,$$

where the couple (ψ_ℓ, ζ_ℓ) is the unique solution to (4.6).

Now, for any given $\varepsilon > 0$, let us introduce the functional $J_{\ell,\varepsilon} : L^2(0,\ell_0) \times L^2(\ell_0, L) \mapsto R$, defined as follows: given $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0,L)$, we have

$$\begin{aligned} J_{\ell,\varepsilon}(\varphi_T,\phi_T) := &\iint_{\mathcal{O}_l} |\varphi - P_l^\ell \varphi|^2\,d\xi\,dt + \iint_{\mathcal{O}_r} |\phi - P_r^\ell \phi|^2\,d\xi\,dt + \frac{\varepsilon}{2}\|(\varphi_T,\phi_T)\|_{L^2\times L^2} \\ &- \int_0^{\ell_0} p_0(\xi)\varphi(\xi,0)\,d\xi - \int_{\ell_0}^{L} q_0(\xi)\phi(\xi,0)\,d\xi - \left[\beta_l^\ell(\varphi) + \beta_r^\ell(\phi)\right]\frac{M_\ell}{2}, \end{aligned} \tag{4.26}$$

where the couple (φ, ϕ) satisfies (4.15)–(4.16).

Using Hölder and Young inequalities, it is not difficult to check that $J_{\ell,\varepsilon}$ is a continuous, coercive and strictly convex functional. Therefore, $J_{\ell,\varepsilon}$ possesses a unique minimizer $(\varphi_T^\varepsilon, \phi_T^\varepsilon) \in L^2(0,\ell_0) \times L^2(\ell_0, L)$. The corresponding solution to (4.16) will be denoted by $(\varphi_\varepsilon, \phi_\varepsilon)$. Then

$$J'_{\ell,\varepsilon}(\varphi_T^\varepsilon, \phi_T^\varepsilon)(\varphi_T, \phi_T) = 0 \quad \forall (\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0, L), \tag{4.27}$$

where

$$\begin{aligned} J'_{\ell,\varepsilon}(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})(\varphi_T, \phi_T) &= \iint_{\mathcal{O}_l} [\varphi_\varepsilon - P_l^\ell(\varphi_\varepsilon)]\varphi \, d\xi \, dt + \iint_{\mathcal{O}_r} [\phi_\varepsilon - P_r^\ell(\phi_\varepsilon)]\phi \, d\xi \, dt \\ &+ \frac{\varepsilon}{2\|\varphi_T^\varepsilon\|_{L^2}} \int_0^{\ell_0} \varphi_T^\varepsilon(\xi)\varphi_T(\xi) \, d\xi + \frac{\varepsilon}{2\|\phi_T^\varepsilon\|_{L^2}} \int_{\ell_0}^L \phi_T^\varepsilon(\xi)\phi_T(\xi) \, d\xi \\ &- \int_0^{\ell_0} p_0(\xi)\varphi(\xi,0) \, d\xi - \int_{\ell_0}^L q_0(\xi)\phi(\xi,0) \, d\xi - \left[\beta_l^\ell(\varphi) + \beta_r^\ell(\phi)\right] \frac{M_\ell}{2}, \end{aligned}$$

Here, we have used the fact that $\langle \varphi_\varepsilon - P_l^\ell(\varphi_\varepsilon), P_l^\ell(\varphi)\rangle_{L^2(\mathcal{O}_l)} = 0$ and $\langle \phi_\varepsilon - P_r^\ell(\phi_\varepsilon), P_r^\ell(\phi)\rangle_{L^2(\mathcal{O}_r)} = 0$.

Let us introduce

$$h_{l,\varepsilon}^\ell := \left[P_l^\ell(\varphi_\varepsilon) - \varphi_\varepsilon\right] + \frac{M_\ell}{2} \frac{\psi_\ell}{\|\psi_\ell\|^2_{L^2(\mathcal{O}_l)}} \quad \text{and} \quad h_{r,\varepsilon}^\ell := \left[P_r^\ell(\phi_\varepsilon) - \phi_\varepsilon\right] + \frac{M_\ell}{2} \frac{\zeta_\ell}{\|\zeta_\ell\|^2_{L^2(\mathcal{O}_r)}}. \tag{4.28}$$

Let $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0, L)$ be given and let (p, q) be the solution to (4.4) associated to the control pair $(h_{l,\varepsilon}^\ell, h_{r,\varepsilon}^\ell)$. Then, multiplying (4.4) by the solution (φ, ϕ) to (4.16) and integrating in $Q_{0,l}$ and $Q_{0,r}$, we obtain

$$\begin{aligned} \iint_{\mathcal{O}_l} h_{l,\varepsilon}^\ell \varphi \, d\xi \, dt + \iint_{\mathcal{O}_r} h_{r,\varepsilon}^\ell \phi \, d\xi \, dt = &\int_0^{\ell_0} [p(\xi,T)\varphi(\xi,T) - p_0(\xi)\varphi(\xi,0)] \, d\xi \\ &+ \int_{\ell_0}^L [q(\xi,T)\phi(\xi,T) - q_0(\xi)\phi(\xi,0)] \, d\xi. \end{aligned} \tag{4.29}$$

Taking into account (4.28) and comparing (4.27) with (4.29), we get

$$\begin{aligned} \int_0^{\ell_0} p(\xi,T)\varphi_T(\xi) \, d\xi + \int_{\ell_0}^L q(\xi,T)\phi_T(\xi) \, d\xi = \frac{\varepsilon}{2} \Bigg(&\int_0^{\ell_0} \frac{\varphi_T^\varepsilon(\xi)}{\|\varphi_T^\varepsilon\|_{L^2}} \varphi_T(\xi) \, d\xi \\ &+ \int_{\ell_0}^L \frac{\phi_T^\varepsilon(\xi)}{\|\phi_T^\varepsilon\|_{L^2}} \phi_T(\xi) \, d\xi \Bigg), \end{aligned}$$

for all $(\varphi_T, \phi_T) \in L^2(0,\ell_0) \times L^2(\ell_0, L)$. Therefore, the approximate controllability condition (4.24) follows. Since we also have

$$\begin{aligned} \iint_{\mathcal{O}_l} h_{l,\varepsilon}^\ell \psi_\ell \, d\xi \, dt + \iint_{\mathcal{O}_r} h_{r,\varepsilon}^\ell \zeta_\ell \, d\xi \, dt = &\iint_{\mathcal{O}_l} \left[P_l^\ell(\varphi_\varepsilon) - \varphi_\varepsilon\right] \psi_\ell \, d\xi \, dt + \frac{M_\ell}{2} \\ &+ \iint_{\mathcal{O}_r} \left[P_r^\ell(\phi_\varepsilon) - \phi_\varepsilon\right] \zeta_\ell \, d\xi \, dt + \frac{M_\ell}{2} \\ &= M_\ell, \end{aligned}$$

the pair $(h^\ell_{l,\varepsilon}, h^\ell_{r,\varepsilon})$ satisfies (4.11) and, consequently, $\mathcal{L}(T) = \ell_T$.

Finally, due to the fact that $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ is the minimum of $J_{\ell,\varepsilon}$, we have the inequality $J_{\ell,\varepsilon}(\varphi^\varepsilon_T, \phi^\varepsilon_T) \leq J_{\ell,\varepsilon}(0,0) = 0$. Using this fact and the definition of M_ℓ and (4.17), we deduce that there exist positive constants C (depending on $\ell_l, \ell_r, R, \omega_l, \omega_r$ and T) such that

$$\|(\varphi_\varepsilon - P^\ell_l(\varphi_\varepsilon))\|_{L^2(\mathcal{O}_l)} + \|(\phi_\varepsilon - P^\ell_r(\phi_\varepsilon))\|_{L^2(\mathcal{O}_r)} \leq \; C\left(\|p_0\|_{L^2(0,\ell_0)} + \|q_0\|_{L^2(\ell_0,L)} + |\ell_0 - \ell_T|\right)$$

and

$$\begin{aligned}\|h^\ell_{l,\varepsilon}\|_{L^2(\mathcal{O}_l)} + \|h^\ell_{l,\varepsilon}\|_{L^2(\mathcal{O}_r)} &\leq C\left(\|(\varphi_\varepsilon - P^\ell_l(\varphi_\varepsilon))\|_{L^2(\mathcal{O}_l)} + \|(\phi_\varepsilon - P^\ell_r(\phi_\varepsilon))\|_{L^2(\mathcal{O}_r)} + |M_\ell|\right)\\ &\leq C\left(\|p_0\|_{L^2(0,\ell_0)} + \|q_0\|_{L^2(\ell_0,L)} + |\ell_0 - \ell_T|\right).\end{aligned}$$

This ends the proof.

4.4 CONTROLLABILITY OF THE TWO-PHASE STEFAN PROBLEM

In this Section we prove Theorem 4.2. The proof relies on a fixed-point argument. It will be convenient to first recall some regularity properties for linear parabolic systems.

4.4.1 A regularity property

Assume that $(p_0, q_0) \in \left[W^{1,4}_0(0,\ell_0)\right] \times \left[W^{1,4}_0(\ell_0, L)\right]$. For any open interval $I \subset R$, let us introduce the Banach space

$$X^4(0,T;I) := L^4(0,T;W^{2,4}(I)) \cap W^{1,4}(0,T;L^4(I)).$$

On the other hand, let us consider the cylinder $G_l := (\ell_l, \ell_0) \times (0,T)$, the Hölder semi-norms

$$\langle u\rangle^\kappa_{\xi,G_l} := \sup_{\substack{(\xi,t),(\xi',t)\in\overline{G}_l\\ \xi\neq\xi'}} \frac{|u(\xi,t) - u(\xi',t)|}{|\xi-\xi'|^\kappa}$$

and

$$\langle u\rangle^\kappa_{t,G_l} := \sup_{\substack{(\xi,t),(\xi,t')\in\overline{G}_l\\ t\neq t'}} \frac{|u(\xi,t) - u(\xi,t')|}{|t-t'|^\kappa}$$

where $0 < \kappa < 1$ and the space $C^{\kappa,\kappa/2}(\overline{G}_l)$ formed by the functions $u \in C^0(\overline{G}_l)$ whose corresponding $\langle u\rangle^\kappa_{\xi,G_l}$ and $\langle u\rangle^{\kappa/2}_{t,G_l}$ are finite. It is known that $C^{\kappa,\kappa/2}(\overline{G}_l)$ is a Banach space (see (LADYZHENSKAYA, 1968)) with the following norm:

$$\|u\|_{\kappa,\kappa/2;\overline{G}_l} := \|u\|_{C^0(\overline{G}_l)} + \langle u\rangle^\kappa_{\xi,G_l} + \langle u\rangle^{\kappa/2}_{t,G_l}.$$

Finally, let us introduce the Banach space

$$C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) := \{u \in C^0(\overline{G}_l) : u_\xi \in C^{\kappa,\kappa/2}(\overline{G}_l), \quad \langle u\rangle_{t,G_l}^{(1+\kappa)/2} < +\infty\}.$$

Obviously, we can introduce similar quantities and spaces for functions defined in $G_r := (\ell_0, \ell_r) \times (0,T)$. The following result holds:

Lemma 4.1 *Let us assume that $\ell_0, \ell_T \in (\ell_l, \ell_r)$ and the couple of initial data $(p_0, q_0) \in \left[W_0^{1,4}(0,\ell_0)\right] \times \left[W_0^{1,4}(\ell_0, L)\right]$. Then, the states (p,q), furnished by Proposition 4.5 satisfy*

$$(p,q) \in C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) \times C^{1+\kappa,(1+\kappa)/2}(\overline{G}_r) \quad \textit{for } \kappa = \frac{1}{4}.$$

Furthermore, there exists $C>0$, depending on ℓ_l, ℓ_r, ω_l, ω_r, T and R, such that

$$\|p\|_{1+\kappa,(1+\kappa)/2;\overline{G}_l} + \|q\|_{1+\kappa,(1+\kappa)/2;\overline{G}_r} \leq C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right). \tag{4.30}$$

Clearly, due the regularity of p_0, there exists a function $f \in X^4(0,T;(0,\ell_0))$ such that $f(0,t) = f(\ell_0,t) = 0$, for $t \in (0,T)$, and $f(\xi,0) = p_0(\xi)$, for $\xi \in (0,\ell_0)$. Consequently, the state p, provided by Proposition 4.5, can be written in the form $p = y + f$, where $y \in L^2(0,T;H^2(0,\ell_0)) \cap H^1(0,T;L^2(0,\ell_0))$ is the unique strong solution of the following problem:

$$\begin{cases} y_t - d_l^\ell y_{\xi\xi} + b_l^\ell y_\xi = F & \text{in} \quad Q_{0,l} \\ y(0,\cdot) = y(\ell_0,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = 0 & \text{in} \quad (0,\ell_0), \end{cases} \tag{4.31}$$

where $F = h_{l,\varepsilon}^\ell 1_{\omega_l} - f_t + d_l^\ell f_{\xi\xi} - b_l^\ell f_\xi$.

Now, let $\sigma > 0$ be such that $\omega_l \subset\subset (0, \ell_l - \sigma)$ and, moreover, $G_l^\sigma := (\ell_l - \sigma, \ell_l + \sigma) \times (0,T) \subset Q_{0,l}$. We can easily check that $F \in L^4(0,T;L^4(\ell_l - \sigma, \ell_l + \sigma))$. Therefore, from local parabolic regularity results (see Proposition C.1 of Appendix C), we obtain that $y \in X^4(0,T;(\ell_l - \sigma/2, \ell_l + \sigma/2))$ and

$$\|y\|_{X^4(0,T;(\ell_l-\sigma/2,\ell_l+\sigma/2))} \leq C\left(\|F\|_{L^4(0,T;L^4(\ell_l-\sigma,\ell_l+\sigma))} + \|y\|_{L^2(0,T;H^2(0,\ell_0))\cap H^1(0,T;L^2(0,\ell_0))}\right),$$

where C only depends on $\|d_l^\ell\|_\infty$, $\|b_l^\ell\|_\infty$, ℓ_l, ℓ_0 and σ.

Next, using standard parabolic energy estimates and (4.25), we get

$$\|y\|_{X^4(0,T;(\ell_l-\sigma/2,\ell_l+\sigma/2))} \leq C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right)$$

for some $C > 0$ as above. Here, we have used that $\|d_l^\ell\|_\infty$ and $\|b_l^\ell\|_\infty$ are bounded in terms of R. Finally, using this inequality, the regularity of the trace $y(\ell_l, \cdot)$, the fact that y is a strong solution to (4.31) and (WU, 2006, Propositions 9.2.3 and 9.2.5), we conclude that $y \in X^4(0,T;(\ell_l,\ell_0))$ and, moreover,

$$\|y\|_{X^4(0,T;(\ell_l,\ell_0))} \leq C \left(\|(p_0, q_0)\|_{W_0^{1,4} \times W_0^{1,4}} + |\ell_0 - \ell_T| \right) \tag{4.32}$$

for a new $C > 0$.

In a similar way, we can write $q = z + g$, where $g \in X^4(0,T;(\ell_0,L))$ is a shift function for the initial data q_0 and $z \in X^4(0,T;(\ell_0,\ell_r))$ satisfies

$$\|z\|_{X^4(0,T;(\ell_0,\ell_r))} \leq C \left(\|(p_0, q_0)\|_{W_0^{1,4} \times W_0^{1,4}} + |\ell_0 - \ell_T| \right). \tag{4.33}$$

Then, the estimate in (4.30) is a immediate consequence of (4.32)-(4.33) and the following embedding from (BODART, 2004, Lemma 2.2)

$$X^4(0,T;(\ell_l,\ell_0)) \times X^4(0,T;(\ell_0,\ell_r)) \hookrightarrow C^{1+\kappa,(1+\kappa)/2}(\overline{G}_l) \times C^{1+\kappa,(1+\kappa)/2}(\overline{G}_r),$$

where $\kappa = 1/4$.

Let us introduce the function $\theta : [0,T] \mapsto R$, given by

$$\theta(t) = d_r q_\xi(\ell_0, t) - d_l p_\xi(\ell_0, t). \tag{4.34}$$

Then, as an immediate consequence of (4.30), we get that $\theta \in C^{1/8}([0,T])$ and, moreover, there exists a positive constant C (depending on $\ell_l, \ell_r, \omega_l, \omega_r, T$ and R) such that

$$\|\theta\|_{C^{1/8}([0,T])} \leq C \left(\|(p_0, q_0)\|_{W_0^{1,4} \times W_0^{1,4}} + |\ell_0 - \ell_T| \right). \tag{4.35}$$

4.4.2 A fixed-point argument

In this Section we will achieve the proof of Theorem 4.2. It will be a consequence of the following uniform approximate controllability result:

Theorem 4.3 *Assume that $R > 0$ is given. Then, there exists $\delta > 0$ such that, for any $p_0 \in W_0^{1,4}(0,\ell_0)$ with $p_0 \geq 0$, any $q_0 \in W_0^{1,4}(\ell_0, L)$ with $q_0 \leq 0$ and any $\ell_0, \ell_T \in (\ell_l, \ell_r)$ satisfying*

$$\|p_0\|_{W_0^{1,4}(0,\ell_0)} + \|q_0\|_{W_0^{1,4}(\ell_0,L)} + |\ell_0 - \ell_T| \leq \delta$$

and any $\varepsilon > 0$, there exist controls $(h_l^\varepsilon, h_r^\varepsilon) \in L^2(\mathcal{O}_l) \times L^2(\mathcal{O}_r)$ and associated solutions to (4.3), *with*

$$\begin{cases} \ell_\varepsilon \in C^1([0,T]) \text{ and } \ell_\varepsilon(t) \in (\ell_l, \ell_r)\, \forall\, t \in [0,T],\ \|\ell_\varepsilon'\|_{C^0([0,T])} \le R, \\ p_\varepsilon \in L^2(0,T; H^2(0,\ell_0)) \cap H^1(0,T; L^2(0,\ell_0)), \\ q_\varepsilon \in L^2(0,T; H^2(\ell_0, L)) \cap H^1(0,T; L^2(\ell_0, L)), \end{cases}$$

satisfying the exact-approximate controllability condition

$$\ell_\varepsilon(t) = \ell_T \quad \text{and} \quad \|(p_\varepsilon(\cdot,T), q_\varepsilon(\cdot,T))\|_{L^2(0,\ell_0)\times L^2(\ell_0,L)} \le \varepsilon. \tag{4.36}$$

Moreover, the controls can be found satisfying the following uniform estimate with respect to ε:

$$\|(h_l^\varepsilon 1_{\omega_l}), (h_r^\varepsilon 1_{\omega_r}))\|_{L^2(Q_{0,l})\times L^2(Q_{0,r})} \le C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}} + |\ell_0 - \ell_T|\right)$$

for some positive C (depending on $\ell_l, \ell_r, \omega_l, \omega_r, T$ and R).

Given $\ell_l < \tilde{\ell}_l < \tilde{\ell}_r < \ell_r$ and $R > 0$, we define the set:

$$\mathcal{A}_R := \{\ell \in C^1([0,T]) : \tilde{\ell}_l \le \ell(t) \le \tilde{\ell}_r,\ \forall t \in [0,T],\quad \ell(0) = \ell_0,\ \|\ell'\|_{C^0([0,T])} \le R\}.$$

Obviously, $\mathcal{A}_R$ is a non-empty, closed and convex subset of $C^1([0,T])$. Let us also introduce the mapping $\Lambda_\varepsilon : \mathcal{A}_R \mapsto C^1([0,T])$, given by

$$\Lambda_\varepsilon(\ell) = \mathcal{L}, \quad \text{with} \quad \mathcal{L}(t) := \ell_0 - \int_0^t \left[d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)\right] d\tau,$$

where (p,q) is the state associated to the control pair $(h_{l,\varepsilon}^\ell, h_{r,\varepsilon}^\ell)$ constructed as in the proof of Proposition 4.5 (recall Lemma 4.1) and, therefore, $\mathcal{L}(T) = \ell_T$. Thanks to (4.34) and (4.35), we have that $\mathcal{L} \in C^1([0,T])$.

Let us check that Λ_ε satisfies the conditions of Schauder's Fixed-Point Theorem.

- Λ_ε is continuous: Indeed, let the $\ell_n (n \ge 1)$ and ℓ belong to $\mathcal{A}_R$ and assume that $\ell_n \to \ell$ in $C^1([0,T])$. We must prove that $\Lambda_\varepsilon(\ell_n) \to \Lambda_\varepsilon(\ell)$ in $C^1([0,T])$. To that end, we will first prove that the corresponding solutions to (4.6) satisfy

$$(\psi_{\ell_n}, \zeta_{\ell_n}) \to (\psi_\ell, \zeta_\ell) \quad \text{strongly in} \quad L^2(Q_{0,l}) \times L^2(Q_{0,r}). \tag{4.37}$$

Let $f \in L^2(0,T; H^2(0,\ell_0)) \times H^1(0,T; L^2(0,\ell_0))$ be so that $f(0,\cdot) = 0$ and $f(\ell_0,\cdot) = 1$ on $(0,T)$ and let us put $\psi_{\ell_n} = \Psi_{\ell_n} + f$ and $\psi_\ell = \Psi_\ell + f$. It is then clear that $y_{\ell_n} :=$

$\Psi_{\ell_n} - \Psi_\ell$ is the unique weak solution to:

$$\begin{cases} (M_l^{\ell_n})^*(y_{\ell_n}) = F_{\ell_n} & \text{in} \quad Q_{0,l}, \\ y_{\ell_n}(0,\cdot) = y_{\ell_n}(\ell_0,\cdot) = 0 & \text{on} \quad (0,T), \\ y_{\ell_n}(\cdot,0) = 0 & \text{in} \quad (0,\ell_0), \end{cases}$$

where $F_{\ell_n} \in L^2(0,T;H^{-1}(0,\ell_0))$ is given by

$$\begin{aligned} F_{\ell_n} :=& (d^\ell_{l,\xi\xi} - d^{\ell_n}_{l,\xi\xi})f + 2(d^\ell_{l,\xi} - d^{\ell_n}_{l,\xi})f_\xi + (d^\ell_l - d^{\ell_n}_l)f_{\xi\xi} + (b^\ell_{l,\xi} - b^{\ell_n}_{l,\xi})f \\ &+ (b^\ell_l - b^{\ell_n}_l)f_\xi + ((d^\ell_l - d^{\ell_n}_l)\Psi_\ell)_{\xi\xi} + ((b^\ell_l - b^{\ell_n}_l)\Psi_\ell)_\xi. \end{aligned}$$

Then, using the fact that $(d^{\ell_n}_l, b^{\ell_n}_l)$ are uniformly bounded in the space $C^1(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$ and the $(d^{\ell_n}_{l,\xi\xi}, b^{\ell_n}_{l,\xi})$ are uniformly bounded in $C^0(\overline{Q}_{0,l}) \times C^0(\overline{Q}_{0,l})$, the standard parabolic energy estimates and the regularity of the function G (as well as the regularity of its inverse G^{-1}), we get that $y_{\ell_n} \to 0$ strongly in $L^2(Q_{0,l})$ which, in turn, implies

$$\psi_{\ell_n} \to \psi_\ell \ \text{ strongly in } \ L^2(Q_{0,l}).$$

Analogously, we can prove that $\zeta_{\ell_n} \to \zeta_\ell$ strongly in $L^2(Q_{0,r})$.

Now, we recall that, for each $\varepsilon > 0$, there exists a unique $(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon})$ in $L^2(0,\ell_0) \times L^2(\ell_0,L)$ that minimizes the functional $J_{\ell_n,\varepsilon}$, defined in (4.26). Due to the facts that $J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}) \le 0$ and the constant appearing in the right side of (4.17) does not depend on n, we get that the minimizers are uniformly bounded with respect to n in the space $L^2(0,\ell_0) \times L^2(\ell_0,L)$ and the corresponding $(\varphi^n_\varepsilon, \phi^n_\varepsilon)$, solutions to (4.16), are uniformly bounded spaces given in (4.15). Therefore, there exist $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ in $L^2(0,\ell_0) \times L^2(\ell_0,L)$ and $(\varphi_\varepsilon, \phi_\varepsilon)$ in $L^2(Q_{0,l}) \times L^2(Q_{0,r})$ such that, at least for a subsequence, one has

$$\begin{cases} (\varphi^n_{T,\varepsilon}, \phi^n_{T,\varepsilon}) \to (\varphi_{T,\varepsilon}, \phi_{T,\varepsilon}) \ \text{ weakly in } \ L^2(0,\ell_0) \times L^2(\ell_0,L), \\ \\ (\varphi^n_\varepsilon(\cdot,0), \phi^n_\varepsilon(\cdot,0)) \to (\varphi_\varepsilon(\cdot,0), \phi_\varepsilon(\cdot,0)) \ \text{ weakly in } \ L^2(0,\ell_0) \times L^2(\ell_0,L) \ \text{ and} \\ \\ (\varphi^n_\varepsilon, \phi^n_\varepsilon) \to (\varphi_\varepsilon, \phi_\varepsilon) \ \text{ strongly in } \ L^2(Q_{0,l}) \times L^2(Q_{0,r}). \end{cases} \tag{4.38}$$

We will show now that $(\varphi_{T,\varepsilon}, \phi_{T,\varepsilon})$ is the unique minimizer of the functional $J_{\ell,\varepsilon}$. Indeed, we first note from the convergences in (4.37) and $(4.38)_3$ that

$$(P^{\ell_n}_l(\varphi^n_\varepsilon), P^{\ell_n}_r(\phi^n_\varepsilon)) \to (P^\ell_l(\varphi_\varepsilon), P^\ell_r(\phi_\varepsilon)) \quad \text{strongly in} \quad L^2(Q_{0,l}) \times L^2(Q_{0,r}).$$

Then, using this fact and the weak convergences $(4.38)_{1,2}$, we easily get

$$J_{\ell,\varepsilon}(\varphi_{T,\varepsilon},\phi_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon},\phi^n_{T,\varepsilon}). \tag{4.39}$$

Now, let (φ_T,ϕ_T) be given in $L^2(0,\ell_0)\times L^2(\ell_0,L)$ and let the (φ^n,ϕ^n) be the solutions to the system (4.16), with ℓ replaced by ℓ_n, for $n=1,2,\ldots$. Then, using the same ideas that led to prove of (4.37), we can ensure that the (φ^n,ϕ^n) converge strongly in $L^2(Q_{0,l})\times L^2(Q_{0,r})$, to the solution (φ,ϕ) to (4.16) and the $(\varphi^n(\cdot,0),\phi^n(\cdot,0))$ converge weakly in $L^2(0,\ell_0)\times L^2(\ell_0,L)$ to $(\varphi(\cdot,0),\phi(\cdot,0))$. Therefore, from (4.37) and (4.39) we deduce that

$$J_{\ell,\varepsilon}(\varphi_{T,\varepsilon},\phi_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi^n_{T,\varepsilon},\phi^n_{T,\varepsilon}) \le \liminf_n J_{\ell_n,\varepsilon}(\varphi_T,\phi_T) = J_{\ell,\varepsilon}(\varphi_T,\phi_T). \tag{4.40}$$

Since $(\varphi_T,\phi_T)\in L^2(0,\ell_0)\times L^2(\ell_0,L)$ is arbitrary, we conclude that $(\varphi_{T,\varepsilon},\phi_{T,\varepsilon})$ minimizes $J_{\ell,\varepsilon}$.

Now, let us consider, for each n, the pair $(h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r})$ associated by Proposition 4.5 to ℓ_n. It follows easily from (4.37), $(4.38)_3$ and (4.40) that

$$(h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r}) \to (h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r}) \ \text{ strongly in } \ L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r), \tag{4.41}$$

where $(h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r})$ is the control corresponding to ℓ. Let us denote by $(p^n_\varepsilon,q^n_\varepsilon)$ and $(p_\varepsilon,q_\varepsilon)$ the solutions to (4.4) associated, respectively, to $(h^{\ell_n}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r})$ and $(h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell}_{r,\varepsilon}1_{\omega_r})$. Then, if we set $(y^n,z^n):=(p^n_\varepsilon-p_\varepsilon, q^n_\varepsilon-q_\varepsilon)$ and $(w^n_l1_{\omega_l}, w^n_r1_{\omega_r}) := (h^{\ell_n}_{l,\varepsilon}1_{\omega_l}-h^{\ell}_{l,\varepsilon}1_{\omega_l}, h^{\ell_n}_{r,\varepsilon}1_{\omega_r}-h^{\ell}_{r,\varepsilon}1_{\omega_r})$, we find that:

$$\begin{cases} y^n_t - d^{\ell_n}_l y^n_{\xi\xi} + b^{\ell_n}_l y^n_\xi = w^n_l 1_{\omega_l} + F^n_l & \text{in } \ Q_{0,l},\\ z^n_t - d^{\ell_n}_r z^n_{\xi\xi} + b^{\ell_n}_r z^n_\xi = w^n_r 1_{\omega_r} + F^n_r & \text{in } \ Q_{0,r},\\ y^n(0,\cdot)=y^n(\ell_0,\cdot)=z^n(\ell_0,\cdot)=z^n(L,\cdot)=0 & \text{in } \ (0,T),\\ y^n(\cdot,0)=0 & \text{in } \ (0,\ell_0),\\ z^n(\cdot,0)=0 & \text{in } \ (\ell_0,L), \end{cases} \tag{4.42}$$

where

$$\begin{cases} F^n_l := (d^{\ell_n}_l - d^{\ell}_l)p_{\varepsilon,\xi\xi} + (b^{\ell_n}_l - b^{\ell}_l)p_{\varepsilon,\xi},\\ F^n_r := (d^{\ell_n}_r - d^{\ell}_r)q_{\varepsilon,\xi\xi} + (b^{\ell_n}_r - b^{\ell}_r)q_{\varepsilon,\xi}. \end{cases}$$

Recall that $(p_0,q_0)\in W^{1,4}_0(0,\ell_0)(0,\ell_0)\times W^{1,4}_0(0,\ell_0)(\ell_0,L)$. Therefore, arguing as in Section 4.4.1 and Lemma 4.1, we first deduce that $(F^n_l,F^n_r)\in L^s((\ell_l,\ell_0)\times(0,T))\times$

$L^4((\ell_0,\ell_r)\times(0,T))$ and $(y_\xi^n(\ell_0,\cdot),z_\xi^n(\ell_0,\cdot))\in C^{1/8}([0,T])\times C^{1/8}([0,T])$ and also, that

$$\|y_\xi^n(\ell_0,\cdot)\|_{C^{1/8}}+\|z_\xi^n(\ell_0,\cdot)\|_{C^{1/8}}\leq C\left(\|(F_l^n,F_r^n)\|_{L^s(L^s)\times L^s(L^s)}+\|(y^n,z^n)\|_{L^2(H^2)\times L^2(H^2)}\right)$$

for some $C>0$, independent of n.

It is not difficult to check that, in this inequality, the first term go to 0 when $n\to\infty$. From standard parabolic estimates applied to (4.42) and (4.41), we also have that convergence to zero of the second term. Therefore, we deduce that $(p^n_{\varepsilon,\xi}(\ell_0,\cdot),q^n_{\varepsilon,\xi}(\ell_0,\cdot))\to(p_{\varepsilon,\xi}(\ell_0,\cdot),q_{\varepsilon,\xi}(\ell_0,\cdot))$ in $C^{1/8}([0,T])$, which implies the continuity of Λ_ε.

- Λ_ε is compact. Note that $\Lambda_\varepsilon(\ell)'(t)=\theta(t)$ for all $\ell\in\mathcal{A}_R$ and all $t\in[0,T]$, where θ is the function defined in (4.34). Thus, we conclude easily from (4.35) that $\Lambda_\varepsilon(\mathcal{A}_R)$ is a bounded subset of $C^{1+1/8}([0,T])$, which is a compact subset of $C^1([0,T])$.
- There exists $\delta>0$, such that, whenever $(p_0,q_0)\in W_0^{1,4}(0,\ell_0)\times W_0^{1,4}(\ell_0,L)$ and

$$\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}}+|\ell_0-\ell_T|\leq\delta,$$

then $\Lambda_\varepsilon(\mathcal{A}_R)\subset\mathcal{A}_R$. Indeed, it follows easily from (4.35) that there exists $C>0$ (depending on $\ell_l,\ell_r,\omega_l,\omega_r,T$ and R) such that

$$|\mathcal{L}(t)-\ell_0|\leq CT\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}}+|\ell_0-\ell_T|\right)\quad\forall t\in[0,T],$$

and

$$|\mathcal{L}'(t)|\leq C\left(\|(p_0,q_0)\|_{W_0^{1,4}\times W_0^{1,4}}+|\ell_0-\ell_T|\right)\quad\forall t\in[0,T].$$

Thus, we get the result by taking $\delta\leq\min\left\{\dfrac{R}{C},\dfrac{\ell_0-\tilde{\ell}_l}{CT},\dfrac{\tilde{\ell}_r-\ell_0}{CT}\right\}$.

Consequently, for initial data p_0,q_0 and ℓ_0 satisfying the above conditions, Schauder's Fixed-Point Theorem guarantees that there exists $\ell_\varepsilon\in\mathcal{A}_R$ such that $\Lambda_\varepsilon(\ell_\varepsilon)=\ell_\varepsilon$. It is easy to see that this is suffices to achieve the proof of the result.

Now, we are in conditions to prove Theorem 4.2. Indeed, since the fixed-points ℓ_ε and controls $(h_l^\varepsilon,h_r^\varepsilon)$ furnished by the Theorem 4.3 are uniformly bounded, respectively, in $C^{1+1/8}([0,T])$ and $L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r)$, then there exist ℓ and (h_l,h_r) such that, at least for a subsequence, we have

$$\begin{cases}\ell_\varepsilon\to\ell\ \text{ strongly in }\ C^1([0,T])\ \text{ and}\\ (h_l^\varepsilon,h_r^\varepsilon)\to(h_l,h_r)\quad\text{weakly in }\ L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r).\end{cases}\tag{4.43}$$

Since the coefficients $(d_l^{\ell_\varepsilon}, b_l^{\ell_\varepsilon})$ and $(d_r^{\ell_\varepsilon}, b_r^{\ell_\varepsilon})$ are uniformly bounded, respectively, in the spaces $L^\infty(Q_{0,l}) \times L^\infty(Q_{0,l})$ and $L^\infty(Q_{0,r}) \times L^\infty(Q_{0,r})$, we can conclude from energy estimates and (4.43) that there exists (p,q) with

$$\begin{cases} p_\varepsilon \to p \quad \text{weakly in} \quad L^2(0,T;H^2(0,\ell_0)\cap H_0^1(0,\ell_0))\cap H^1(0,T;L^2(0,\ell_0)), \\ q_\varepsilon \rightharpoonup q \quad \text{weakly in} \quad L^2(0,T;H^2(0,\ell_0)\cap H_0^1(\ell_0,L))\cap H^1(0,T;L^2(\ell_0,L)), \end{cases} \tag{4.44}$$

where the $(p_\varepsilon, q_\varepsilon)$ are associated to the $(h_l^\varepsilon, h_r^\varepsilon)$. Then, (p,q) is the solution to (4.4), associated to (h_l, h_r). Moreover, from (4.36), it is clear that $\ell(T) = \ell_T$ and $(p(\cdot,T), q(\cdot,T)) = (0,0)$ on $(0,T)$.

Furthermore, as a consequence of (4.44) and the embeddings

$$H^2(0,\ell_0) \overset{c}{\hookrightarrow} C^1([0,\ell_0]) \hookrightarrow L^2(0,\ell_0) \quad \text{and} \quad H^2(\ell_0,L) \overset{c}{\hookrightarrow} C^1([\ell_0,L]) \hookrightarrow L^2(\ell_0,L),$$

we find that, for any given $t \in [0,T]$, the following holds:

$$\begin{aligned} \ell(t) = \lim_{\varepsilon\to 0} \ell_\varepsilon(t) &= \lim_{\varepsilon\to 0}\left(\ell_0 - \int_0^t [d_l p_{\varepsilon,\xi}(\ell_0,\tau) - d_r q_{\varepsilon,\xi}(\ell_0,\tau)]\,d\tau\right) \\ &= \ell_0 - \int_0^t [d_l p_\xi(\ell_0,\tau) - d_r q_\xi(\ell_0,\tau)]\,d\tau. \end{aligned}$$

This implies that the Stefan condition $(4.3)_7$ is satisfied by (ℓ, p, q) and ends the proof of Theorem 4.2.

4.5 ADDITIONAL COMMENTS

4.5.1 Lack of controllability with only one control

In the next result it is proved that, if h_l or h_r vanishes and the interface does not collapse to the boundary, then null controllability cannot hold.

Theorem 4.4 *Assume that $u_0 \in W_0^{1,4}(0,\ell_0)$ with $u_0 \geq 0$, $v_0 \in W_0^{1,4}(\ell_0,L)$ with $v_0 \leq 0$ and $v_0 \not\equiv 0$. Then, if $(h_l,h_r) \in L^2(\mathcal{O}_l)\times L^2(\mathcal{O}_r)$, $h_r \equiv 0$ and the associated strong solution to (4.1) satisfies $\ell(t) < L$ for all $t \in [0,T]$, we necessarily have*

$$v(\cdot,T) \not\equiv 0 \quad \textit{in} \quad (\ell(T),L).$$

Let us assume, by contradiction, that (4.1) is null-controllable with $h_r \equiv 0$, i.e. $u(\cdot,T)\equiv 0$ in $(0,\ell(T))$ and $v(\cdot,T)\equiv 0$ in $(\ell(T),L)$.

Then, taking account the diffeomorphism Φ and the function $q = v \circ \Phi^{-1}$, defined in the section 4.2, we get easily that q is the solution to

$$\begin{cases} q_t - d_r^\ell q_{\xi\xi} + b_r^\ell q_\xi = 0 & \text{in} \quad Q_{0,r}, \\ q(\ell_0, \cdot) = q(L, \cdot) = 0 & \text{on} \quad (0,T), \\ q(\cdot, 0) = q_0 & \text{in} \quad (\ell_0, L), \end{cases} \tag{4.45}$$

where $q_0 := v_0 \circ [G(\cdot, \ell_0)]^{-1} \in W_0^{1,4}(\ell_0, L)$ and, obviously, $q_0 \leq 0$ and $q_0 \not\equiv 0$. We also have that

$$q(\cdot, T) \equiv 0 \ \text{ in } \ (\ell_0, L). \tag{4.46}$$

Now, we consider a sequence $\ell_1, \ell_2, \ldots$ of functions in $C^\infty([0,T])$ converging strongly to ℓ in $C^1([0,T])$, as well as, a sequence $q_0^1, q_0^2, \ldots$ of functions in $C_0^\infty(\ell_0, L)$ so that $q_0^n \to q_0$ strongly in $W_0^{1,4}(\ell_0, L)$. Thus, there exists a sequence $\{q^n\}_{n=1}^\infty$ in $C^{2,1}(Q_{0,r}) \cap C^0(\overline{Q}_{0,r})$ formed by the solutions to (4.45), with initial data q_0^n and ℓ replaced by ℓ_n. Then, by applying the maximum principle for classical solutions of parabolic equations, we get

$$\max_{\overline{Q}_{0,r}} q^n = \max_{[\ell_0, L]} q_0^n, \quad \forall\, n \in N.$$

Then, using this equality and the fact that

$$q^n \to q \ \text{ strongly in } \ L^2(0,T; H^2(\ell_0, L)) \cap H^1(0,T; L^2(\ell_0, L))$$

we obtain

$$\max_{\overline{Q}_{0,r}} q = \max_{[\ell_0, L]} q_0,$$

what implies that $q \leq 0$ in $\overline{Q}_{0,r}$.

Thus, if (4.46) holds we get, from the strong maximum principle and from the fact $q \leq 0$ in $Q_{0,r}$, that $q \equiv 0$ in $\overline{Q}_{0,r}$, what contradicts $q_0 \not\equiv 0$.

Remark 4.4 *Note that the previous argument also shows that the null controllability for* (4.1) *cannot be achieved keeping the signs of the initial conditions in each phase region. In other words, in order to drive the solution to zero at time T, the liquid and solid state must penetrate each other before T.*

4.5.2 Boundary controllability and other extensions

We can prove local boundary controllability results similar to Theorem 4.1. Thus, let us introduce the system

$$\begin{cases} u_t - d_l u_{xx} = 0 & \text{in} \quad Q_l, \\ v_t - d_r v_{xx} = 0 & \text{in} \quad Q_r, \\ u(0,t) = k_l(t), \ v(L,t) = k_r(t) & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,\ell_0), \\ v(\cdot,0) = v_0 & \text{in} \quad (\ell_0,L), \\ u(\ell(t),t) = v(\ell(t),t) = 0 & \text{on} \quad (0,T), \\ -\ell'(t) = d_l u_x(\ell(t),t) - d_r v_x(\ell(t),t) & \text{on} \quad (0,T). \end{cases} \tag{4.47}$$

where (k_l, k_r) stands for the boundary control pair.

Then, using a domain extension technique and Theorem 4.1, it is easy to prove that, if u_0 and v_0 are sufficiently small, and ℓ_0 is sufficiently close to ℓ_T, there exist controls (k_l, k_r) and associated solutions to (4.47) that satisfy $\ell(T) = \ell_T$, $u(\cdot,T) = 0$ in $(0,\ell_T)$ and $v(\cdot,T) = 0$ in (ℓ_T, L).

Let us finally mention that the arguments and results in this chapter can also be used to solve other variants of the two-phase Stefan controllability problem. Thus, we can prove results similar to Theorem 4.1 when the controls are Neumann data, we can assume that the equations contain lower order terms or even appropriate nonlinearities, etc.

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APPENDIX A – ANALYTICITY OF THE ELLIPTIC OPERATOR

In this section we will prove that the mapping $\mu \mapsto T$, which appears in the proof of the Theorem 2.5, is analytic in a neighbourhood of the $\mu = 0$. In order to do that, we consider the Banach space

$$\mathcal{W}_*^{1,\infty}(R^N;R^N) := \{\mu \in W^{1,\infty}(R^N;R^N);\ \mu \equiv 0 \text{ in } \Omega\backslash\overline{D}^*\}.$$

It is not difficult to see that the set $\mathcal{W}_\epsilon$, defined in Section 2.3, is an open set of $\mathcal{W}_*^{1,\infty}(R^N;R^N)$ and, moreover, one consider the map

$$\begin{aligned} \eta :\ & \mathcal{W}_\epsilon \mapsto \mathcal{L}(H_0^1(\Omega_0)^2; H^{-1}(\Omega_0)^2) \\ & \mu \mapsto T, \end{aligned}$$

with

$$\begin{aligned} T(u,v) := & -(\nabla \cdot (\mathsf{Jac}(m)M^*M\nabla u), \nabla \cdot (\mathsf{Jac}(m)M^*M\nabla v) \\ & + ((au+bv)\mathsf{Jac}(m), (Au+Bv)\mathsf{Jac}(m)), \end{aligned}$$

for all $(u,v) \in H_0^1(\Omega_0)^2$, where $m = I + \mu$.

One can see from (ALVAREZ, 2005) that the mapping $\eta_1 : \mathcal{W}_\epsilon \mapsto \mathcal{L}(H_0^1(\Omega_0)^2; H^{-1}(\Omega_0)^2)$ given by

$$\eta_1(\mu) = -(\nabla \cdot (\mathsf{Jac}(m)M^*M\nabla \cdot), \nabla \cdot (\mathsf{Jac}(m)M^*M\nabla \cdot)$$

is analytic in a neighbourhood of $\mu = 0$.

Now, assuming that the coefficients a, b, A and B are constant, one can see that the mapping $\eta_2 : \mathcal{W}_\epsilon \mapsto \mathcal{L}(H_0^1(\Omega_0)^2; H^{-1}(\Omega_0)^2)$, given by

$$\eta_2(\mu)(u,v) = ((au+bv)\mathsf{Jac}(m), (Au+Bv)\mathsf{Jac}(m)), \quad \forall\ (u,v) \in H_0^1(\Omega_0)^2,$$

is a polynomial in the coordinates of μ. Therefore, it is analytic in a neighbourhood of origin.

To a better illustration, let us consider the particular case where $N = 2$ and, consequently, $\mathcal{W}_\epsilon \subset \mathcal{W}_*^{1,\infty}(R^2;R^2)$. Moreover, given $\mu_0 = (\mu_0^1, \mu_0^2) \in \mathcal{W}_\epsilon$ and $\mu = (\mu^1, \mu^2) \in \mathcal{W}_*^{1,\infty}(R^2, R^2)$, we have that

$$\eta_2'(\mu_0) \cdot \mu = \lim_{t \to 0} \frac{\eta_2(\mu_0 + t\mu) - \eta_2(\mu_0)}{t}.$$

Now, for $((u,v),(z,q)) \in H_0^1(\Omega_0)^2 \times H_0^1(\Omega_0)^2$, let us introduce

$$\zeta\left((u,v),(z,q)\right) := auz + bvz + Auq + Bvq.$$

Then, after some computations, the above limit leads to

$$\langle (\eta_2'(\mu_0)\cdot\mu)\,(u,v),(z,q)\rangle_{H^{-1};H_0^1} = \int_{\Omega_0} \zeta\,((u,v),(z,q))\left(\nabla\cdot\mu + \partial_2\mu^2\partial_1\mu_0^1 + \partial_1\mu^1\partial_2\mu_0^2\right)\,dx$$
$$-\int_{\Omega_0}\zeta\,((u,v),(z,q))\left(\partial_1\mu^2\partial_2\mu_0^1 + \partial_2\mu^1\partial_1\mu_0^2\right)\,dx,$$

for all $((u,v),(z,q)) \in H_0^1(\Omega_0)^2 \times H_0^1(\Omega_0)^2$.

Thus, above equality implies

$$\lim_{\mu\to 0}\frac{\|\eta_2(\mu_0+\mu)-\eta_2(\mu_0)-\eta_2'(\mu_0)\cdot\mu\|_{\mathcal{L}(H_0^1;H^{-1})}}{\|\mu\|_{W^{1,\infty}}} = 0.$$

This means that the mapping η_2' is the first Fréchet derivative of η_2.

Furthermore, by making similar computations we obtain the second Fréchet derivative $\eta_2''\ :\mathcal{W}_\epsilon \mapsto \mathcal{L}_2(\mathcal{W}_*^{1,\infty}(R^2,R^2);\mathcal{L}(H_0^1(\Omega_0)^2;H^{-1}(\Omega_0)^2)$. More precisely, for each $\mu_0\in\mathcal{W}_\epsilon$ and μ_1,μ_2 in $\mathcal{W}_*^{1,\infty}(R^2,R^2)$, we obtain:

$$\langle (\eta_2''(\mu_0)\cdot\mu_1\cdot\mu_2)\,(u,v),(z,q)\rangle_{H^{-1};H_0^1} = \int_{\Omega_0}\zeta\,((u,v),(z,q))\left(\partial_2\mu_1^2\partial_1\mu_2^1 + \partial_1\mu_1^1\partial_2\mu_2^2\right)\,dx$$
$$-\int_{\Omega_0}\zeta\,((u,v),(z,q))\left(\partial_1\mu_1^2\partial_2\mu_2^1 + \partial_2\mu_1^1\partial_1\mu_2^2\right)\,dx.$$

Finally, $\eta_2^{(k)}\equiv 0$, for all $k\geq 3$, which implies the analyticity of the mapping η_2 near the origin.

APPENDIX B – DIFFERENTIATION WITH RELATION DOMAINS

In this appendix we will prove Lemma 2.3 of Section 2.4.4. The idea is to use the arguments from (SIMON, 1980, Section 6.3) and (BELLO, 1997, Lemma 7). Let us now recall that, for each $\mu \in W_\epsilon$, the pair $(y_\mu, z_\mu) \in H^2(\Omega\backslash\overline{D+\mu})^2$ is the unique solution of the perturbed system:

$$\begin{cases} -\Delta y_\mu + a y_\mu + b z_\mu = 0 & \text{in} \quad \Omega\backslash(\overline{D+\mu}), \\ -\Delta z_\mu + A y_\mu + B z_\mu = 0 & \text{in} \quad \Omega\backslash(\overline{D+\mu}), \\ y_\mu = \varphi, \ z_\mu = \psi & \text{on} \quad \partial\Omega, \\ y_\mu = 0, \ z_\mu = 0 & \text{on} \quad \partial(D+\mu). \end{cases}$$

Then, defining $(Y_\mu, Z_\mu) := (y_\mu, z_\mu)\circ(I+\mu)$ and performing similar change of variables used in Section 2.3, we have that $(Y_\mu, Z_\mu) \in H^2(\Omega\backslash\overline{D})^2$ is the unique solution of the elliptic problem:

$$\begin{cases} -\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla Y_\mu) + (a_\mu Y_\mu + b_\mu Z_\mu)\,\mathsf{Jac}(m) = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ -\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla Z_\mu) + (A_\mu Y_\mu + B_\mu Z_\mu)\,\mathsf{Jac}(m) = 0 & \text{in} \quad \Omega\backslash\overline{D}, \\ Y_\mu = \varphi, \ Z_\mu = \psi & \text{on} \quad \partial\Omega, \\ Y_\mu = 0, \ Z_\mu = 0 & \text{on} \quad \partial D, \end{cases} \tag{B.1}$$

where $(a_\mu, b_\mu, A_\mu, B_\mu) := (a, b, A, B)\circ m$ and $m = I+\mu$.

Let us now introduce the new variables $(\widehat{Y}_\mu, \widehat{Z}_\mu) := (Y_\mu - \varphi^*, Z_\mu - \psi^*)$, where the pair $(\varphi^*, \psi^*) \in H^2(\Omega\backslash\overline{D})^2$ satisfies

$$\begin{cases} (\varphi^*, \psi^*) = (\varphi, \psi) & \text{on} \quad \partial\Omega, \\ (\varphi^*, \psi^*) = (0, 0) & \text{on} \quad \partial D. \end{cases}$$

Therefore, the system (B.1) can be rewritten as

$$H(\mu; \widehat{Y}_\mu, \widehat{Z}_\mu) = 0,$$

where the mapping $H : \mathcal{W}_\epsilon \times H_0^2(\Omega\backslash\overline{D})^2 \mapsto L^2(\Omega\backslash\overline{D})^2$ is given by

$$\begin{cases} H(\mu;\chi,\zeta) = (F(\mu;\chi,\zeta), G(\mu;\chi,\zeta)), \\ F(\mu;\chi,\zeta) = -\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla(\chi+\varphi^*)) + [a_\mu(\chi+\varphi^*) + b_\mu(\zeta+\psi^*)]\,\mathsf{Jac}(m), \\ G(\mu;\chi,\zeta) = -\nabla\cdot(\mathsf{Jac}(m)M^*M\nabla(\zeta+\psi^*)) + [A_\mu(\chi+\varphi^*) + B_\mu(\zeta+\psi^*)]\,\mathsf{Jac}(m). \end{cases}$$

Observe that the mapping H satisfies the conditions of the Implicit Function Theorem. Indeed, we can see in (SIMON, 1980) that the mapping $\mu \in \mathcal{W}_\epsilon \mapsto M(\mu) \in W^{1,\infty}(R^N; R^{N^2})$ and the mapping $\mu \in \mathcal{W}_\epsilon \mapsto \text{Jac}(I+\mu) \in W^{1,\infty}(R^N, R)$ are of class C^1 and, consequently, H is a continuous differentiable mapping in a neighbourhood of $(0, \widehat{Y}_0, \widehat{Z}_0)$, where $(\widehat{Y}_0, \widehat{Z}_0) = (Y_0 - \varphi^*, Z_0 - \psi^*)$.

On the other hand, one can see that the differential mapping $D_{\chi,\zeta}H(0; \widehat{Y}_0, \widehat{Z}_0)$ is a isomorphism. Indeed, it is a consequence of the fact the mapping $H(0;\cdot,\cdot) : H_0^2(\Omega\backslash\overline{D})^2 \mapsto L^2(\Omega\backslash\overline{D})^2$, given by

$$H(0;\chi,\zeta) = (-\Delta(\chi+\varphi^*) + a(\chi+\varphi^*) + b(\zeta+\psi^*), -\Delta(\zeta+\psi^*) + A(\chi+\varphi^*) + B(\zeta+\psi^*)),$$

is an isomorphism.

Thus, by the Implicit Function Theorem, there exist neighbourhoods $U \subset \mathcal{W}_\epsilon$ of $\mu = 0$, $V \subset H_0^2(\Omega\backslash\overline{D})^2$ of the $(\widehat{Y}_0, \widehat{Z}_0)$ and a differentiable mapping $\xi : U \mapsto V$ such that

$$H^{-1}(0) \cap [U \times V] = \{(\mu;\chi,\zeta); (\chi,\zeta) = \xi(\mu)\},$$

which proves the first item of the Lemma 2.3.

The second and third items of the Lemma 2.3 are immediate consequences of the Lemma 2.1, Theorem 3.1 and Theorem 3.2 of the (SIMON, 1980).

APPENDIX C – LOCAL PARABOLIC REGULARITY

In this appendix, let us prove a local regularity result for a $1d$ parabolic equation.

Proposition C.1 *Let $T > 0$ be a positive time, $I \subset R$ an open interval and consider the cylinder $Q := I \times (0,T)$. Moreover, let us consider functions $d, b : Q \mapsto R$ satisfying:*

$$\begin{cases} d \in C^0(\overline{Q}),\ b \in L^\infty(Q), \\ d(\xi,t) \geq \lambda > 0,\ \text{for all } (\xi,t) \in \overline{Q} \text{ and some } \lambda > 0, \end{cases}$$

and also consider $f \in L^2(0,T;L^2(I))$ and the unique strong solution y in the space $L^2(0,T;H^2(I)) \cap H^1(0,T;L^2(I))$, of the parabolic system

$$\begin{cases} y_t - dy_{\xi\xi} + by_\xi = f & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \partial I \times (0,T), \\ y(\cdot,0) = 0 & \text{in} \quad (0,T). \end{cases}$$

Then, the following holds: let $\mathcal{U} \subset\subset \mathcal{V} \subset I$ open sets and we assume $f \in L^4(0,T;L^4(\mathcal{V}))$. Then, $y \in X^4(0,T;\mathcal{U}) := L^4(0,T;W^{2,4}(\mathcal{U})) \cap W^{1,4}(0,T;L^4(\mathcal{U}))$ and there exists a positive constant C (depending on I, $\mathcal{V}$, $\mathcal{U}$, $\|d\|_\infty$, $\|b\|_\infty$), such that:

$$\|y\|_{X^4(0,T;\mathcal{U})} \leq C \left(\|f\|_{L^4(L^4(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap H^1(L^2)} \right).$$

Indeed, let us consider firstly the open set $\mathcal{U} \subset\subset \mathcal{U}' \subset\subset \mathcal{V}$ and a test function $\chi \in C_0^\infty(\mathcal{U}')$ so that $\chi \equiv 1$ in $\mathcal{U}$. Then, one can see that $w = \chi y$ is the strong solution of the problem:

$$\begin{cases} w_t - dw_{\xi\xi} = g & \text{in} \quad Q, \\ w = 0 & \text{on} \quad \partial I \times (0,T), \\ w(\cdot,0) = 0 & \text{in} \quad I, \end{cases}$$

where $g = \chi f - (\chi b y_\xi + 2d\chi_\xi y_\xi + d\chi_{\xi\xi} y)$.

As consequence of the hypothesis on f and y we get that $\chi f \in L^4(0,T;L^4(I))$ and $g_0 := \chi b y_\xi + 2d\chi_\xi y_\xi + d\chi_{\xi\xi} y$ is a function of $L^\infty(0,T;L^2(I)) \cap L^2(0,T;L^\infty(I))$. Now, by using standard interpolation results, we can assure that $g_0 \in L^4(0,T;L^4(I))$ and there exists a positive constant C (depending on I, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{U}$, $\|d\|_\infty$, $\|b\|_\infty$) so that

$$\|g_0\|_{L^4(L^4(I))} \leq C\|y\|_{L^2(H^2)\cap H^1(L^2)}.$$

Consequently, $g \in L^4(0,T;L^4(I))$ and, this way, we can use the estimate above and results of L^p-regularity for parabolic operators, see (WU, 2006, Theorems $9.2.3$ and $9.2.5$), to obtain $w \in X^4(0,T;I)$ and

$$\|w\|_{X^4(0,T;I)} \leq C \left(\|f\|_{L^4(L^4(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap H^1(L^2)} \right),$$

for some constant $C > 0$ (depending on I, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{U}$, $\|d\|_\infty$, $\|b\|_\infty$ and s). Therefore, since $w = y$ in $\mathcal{U}$ then we can conclude the proof of the result.