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LUCAS FELIPE BEZERRA DE MELO OLIVEIRA

ENLARGING THE CONCEPT OF ADDITIVITY OF QUANTUM RESOURCES

Recife

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LUCAS FELIPE BEZERRA DE MELO OLIVEIRA

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Dissertation presented with the postgraduate program in Physics at the **Universidade Federal de Pernambuco** as a partial requirement to obtain the title of **Master in Physics**.

Program of study: Theoretical and Computational Physics

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Dedico este trabalho à minha mãe, Francisca, e à minha afilhada, Sofia.

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"Acreditamos que a educação sozinha não transforma a sociedade, sem ela tampouco a sociedade muda. Se a nossa opção é progressiva, se estamos a favor da vida e não da morte, da equidade e não da injustiça, do direito e não do arbítrio, da convivência com o diferente e não de sua negação, não temos outro caminho se não viver a nossa opção. Encarná-la, diminuindo, assim, a distância entre o que dizemos e o que fazemos." (FREIRE, 2000).

ABSTRACT

For future quantum computation developments, it is important to address the issue of quantifying the amount of resources of a certain number of copies of quantum states. This is a hard problem, often involving optimizations over Hilbert spaces of large dimensions. We propose a way to circumvent the direct evaluation of such quantities, provided that the employed quantifiers satisfy a self-similarity property, which we call scalability. This property is in essence a constraint on the way certain quantities can scale with the number of copies of a given system. If analyticity is assumed, recursive relations can be derived for the Maclaurin series of $\mathcal{E}(\varrho^{\otimes N})$, which enables us to determine its possible functional forms. Our approach sets possible ways in which a broad class of quantum functions can scale with the number of copies of a quantum state, describing very simply the non-additivity of some relevant resources, such as quantum coherence and distillable entanglement. A generalization of linear scalable functions is formalized.

Keywords: Scalability. Non-Additivity. Many-qudit Resources. Quantum Coherence. Distillable Entanglement.

RESUMO

Para futuros desenvolvimentos em computação quântica, é importante abordar o problema em quantificar o total de recursos para um certo número de cópias de estados quânticos. Esse é um problema difícil, ocasionalmente envolvendo otimizações sobre espaços de Hilbert de alta dimensão. Nós propomos uma forma de contornar a avaliação direta dessas quantidades, desde que os quantificadores estudados satisfaçam uma relação de autossimilaridade que nós chamamos de escalabilidade. Essa propriedade é, em essência, uma restrição à maneira que certas quantidades escalam com o número de cópias de um certo sistema. Se assumirmos analiticidade, relações de recorrência podem ser derivadas para a série de Maclaurin de $\mathcal{E}(\varrho^{\otimes N})$, o que nos permite determinar as suas possíveis formas funcionais. Nossa abordagem fixa os possíveis comportamentos de uma classe extensa de funções quânticas com o número de cópias de um estado quântico, descrevendo de forma bastante simples a não aditividade de alguns recursos relevantes, como coerência quântica e emaranhamento destilável. Uma generalização para funções escaláveis lineares é formalizada.

Palavras-chaves: Escalabilidade. Não Aditividade. Recursos de Muitos Qudits. Coerência Quântica. Emaranhamento Destilável.

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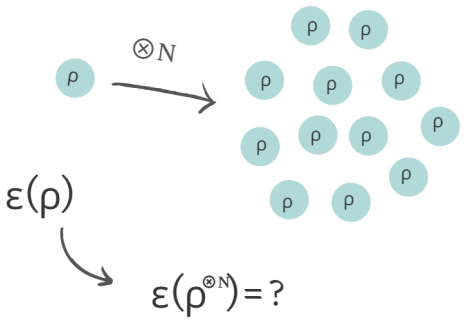
1 INTRODUCTION

If one takes N copies of a certain quantum state (maybe for doing some protocol of process) a relevant question is how to determine the amount of resources[1] for $\varrho^{\otimes N}$. In quantum mechanics this may be a impractical problem due to the exponential growth of the Hilbert space dimension as the number of copies increases linearly, which justifies computational optimizations (BUSCEMI; DATTA, 2010) over large dimensional Hilbert spaces. Naturally there are asymptotic results ($N \to \infty$) in the literature, however, in actual situations the number of copies is always finite and the asymptotic regime may become dominant only for an impractible number of copies (FANG X. WANG; DUAN, 2019). Therefore, an alternative way to addres the problem of quantifying quantum functions for a large but finite number of copies of a state could be theoretically important and computationally usefull.

The way a quantum function $\mathcal{E}(\varrho^{\otimes N})$ scales with the number of copies may be very simple, as the additivity of the squashed entanglement (BUSCEMI; DATTA, 2004) and the logarithmic negativity (VIDAL; WERNER, 2002); or may be more complicated, as the entanglement of formation (TOMAMICHEL; RENES, 1996), which has been ultimately shown to be *non-additive* (HASTINGS, 2009), or as the entanglement distillation (WATROUS, 2004), which has been shown to be "nonlinear with respect to the number of copies used in the distillation process". This is actually the case of most quantifiers (the relative entropy of entanglement (VEDRAL M. B. PLENIO; KNIGHT, 1997), the distillable entanglement (BUSCEMI; DATTA, 2010; RAINS, 1999), the geometric distance (BARNUM; LINDEN, 2001) and the geometric measure of entanglement (ZHU; HAYASHI, 2010)), nonlinear behaviors are quite common in nature. The main question of this work is: *is there a general property that these functions satisfy?*

Figure 1 – Illustration of the problem we address in this dissertation.



Source: the author (2021).

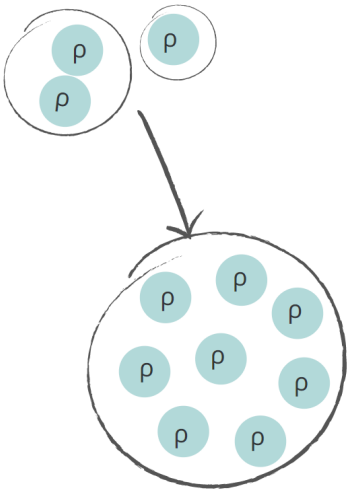
[1] The term "resource" here is employed in the general sense, since the results are actually general properties a quantum function might present and not necessarilly restrict to resource theories.

Instead of directly employing the calculation of some quantifiers for large Hilbert spaces (which can be a very hard problem) we will be interested in a different approach. If we assume that the quantum function of interest is analytical and supose that $\mathcal{E}(\varrho^{\otimes a})$ is a function of $\mathcal{E}(\varrho)$, a being a small number of copies, then we show that the series expansion $\mathcal{E}(\varrho^{\otimes a})$ and the series expansion $\mathcal{E}(\varrho^{\otimes N})$, for N copies, are connected in a specific way if the function is, by our definition, **scalable** (PARISIO, 2020). So the problem is restated as the calculation of the coefficients in a series expansion.

The aforementioned requirement that $\mathcal{E}(\varrho^{\otimes N})$ be a function of $\mathcal{E}(\varrho)$, which is inspired by the additive case $\mathcal{E}(\varrho^{\otimes N}) = N\mathcal{E}(\varrho)$ (but with linearity constraint lifted), is what we define as **1-extensibility**. This mathematical feature enables us to study a broad class of functions (the so called 1-extensible functions) with different behaviors in terms of the number of copies N.

In quantum theory, a dramatic manifestation of non-additivity is the phenomenon of **superactivation**, for which, given a particular quantifier $\mathcal{E}$, one may find states ϱ, such that $\mathcal{E}(\varrho) = 0$ and $\mathcal{E}(\varrho \otimes \varrho) > 0$, as is the case of the distillation of entanglement (WATROUS, 2004), the bound entanglement (SHOR; THAPLIYAL, 2003) and the quantum steering (QUINTINO; HUBER, 2016). So $\mathcal{E}(\varrho)$ *may not contain enough information to determine* $\mathcal{E}(\varrho^{\otimes N})$. Therefore, consider the question: how the amount of resources embodied by several copies of a state relates to those of *fewer* copies?

Figure 2 – Sketch of a 2-extensible function.



Source: the author (2021).

To get a more general approach, we ask what are the variables that possibly determine $\mathcal{E}(\varrho^{\otimes N})$ and how it depends on them. We can start enlarging the notion of additivity, $\mathcal{E}(\varrho^{\otimes N}) = N\mathcal{E}(\varrho)$, and relax this form to get a more general dependence: for example, $\mathcal{E}(\varrho^{\otimes N})$ could be a function of $\mathcal{E}(\varrho)$ and $\mathcal{E}(\varrho^{\otimes 2})$ (illustrated in Figure 2).

$$\mathcal{E}\left(\varrho^{\otimes N}\right) \approx E^{(N)}\left[\mathcal{E}(\varrho)\right] \ ? \quad \rightarrow \ \textit{1-extensible}$$

or

$$\mathcal{E}\left(\varrho^{\otimes N}\right) \approx E^{(N)}[\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes 2})] \ ? \ \rightarrow \ \textit{2-extensible}$$

$E^{(N)}$ is numerically equal to $\mathcal{E}(\varrho^{\otimes N})$, but with a different domain. Enlarging this possibility, we study the class of functions which satisfies $\mathcal{E}(\varrho^{\otimes N}) = E^{(N)}[\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes 2}), \ldots, \mathcal{E}(\varrho^{\otimes i_q})]$ (with $i_q < N$), which we denote as *q-extensible* functions, and with this approach we derive interesting properties that help the evaluation of non-additive behaviors of quantum functions. The mentioned concepts of Extensibility and Scalability are the central themes of the first chapter and the main tools for the investigation of this dissertation.

One case treated in this work is quantum coherence, which is an important feature in quantum theory and is useful for many fields of physics. We use some of the well accepted quantifiers of this resource (BAUMGRATZ; PLENIO, 2014) to ensure the validity of the method. Also it is an interesting example of how scalability works because we show that the way one quantifies coherence implies (or not) its non-additivity.

The structure of the dissertation is organized as follows: in chapter **2** we make preliminary definitions, like Extensibility, to define the concept of Scalability, and then we apply the theory to the l_1-norm of coherence and also to the squared Hilbert-Schmidt norm (BAUMGRATZ; PLENIO, 2014). In chapter **3** we extend the previous results to the 2-S case (functions that depend only on $\mathcal{E}(\varrho)$, $\mathcal{E}(\varrho^{\otimes 2})$ and N, for instance); here the one-shot-distillable entanglement of an N-fold mixture of Bell states (FANG X. WANG; DUAN, 2019) and the phenomenon of superactivation (SHOR; THAPLIYAL, 2003; QUINTINO; HUBER, 2016) are discussed as examples. Finally, chapter **4** is dedicated to formalize the general linear solution of a q-S function (here the 3-S form is derived and compared with computational data of a well known quantifier, the one-shot-distillable entanglement of an N-fold isotropic state (FANG X. WANG; DUAN, 2019)).

2 1-SCALABLE FUNCTIONS

> *"On the surface, there was always an impeccably realistic world, but underneath, behind the backdrop's cracked canvas, lurked something different, something mysterious or abstract."*
>
> – Milan Kundera, *The Unbearable Ligthness of Being*

The method presented here intends to circumvent the problem of quantifying a certain function of interest for many copies of a quantum state. The method, which can only be used if the quantum function under study satisfies a self-similarity property called *scalability*, reveals an intimate relation between a broad class of quantum functions.

2.1 PRELIMINARIES AND NOTATION

Let $\varrho \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the Hilbert-Schmidt space[1]. Consider the various arbitrary ways one can express the N-fold quantum state $\varrho^{\otimes N}$:

$$\begin{aligned} \varrho^{\otimes N} &= \varrho^{\otimes N/2} \otimes \varrho^{\otimes N/2} = \varrho^{\otimes N/4} \otimes \varrho^{\otimes N/4} \otimes \varrho^{\otimes N/4} \otimes \varrho^{\otimes N/4} = \dots \\ \dots &= \underbrace{\varrho \otimes \dots \otimes \varrho}_{N \text{ times}}, \end{aligned} \tag{2.1}$$

where we are assuming that $N = 2^n$. Note that one can take $N = 2^n$ and $K = 2^k$ with $k < n$ and group the N-fold density matrix $\varrho^{\otimes N}$ in a different way:

$$\varrho^{\otimes N} = (\varrho^{\otimes K})^{\otimes N/K} = \sigma^{\otimes N/K} \text{ with } \sigma = \varrho^{\otimes K}. \tag{2.2}$$

In general, is convenient to consider $N, K \in \mathbf{P}_a$, where $\mathbf{P}_a$ denotes the set of all integer powers of a:

$$\mathbf{P}_a = \{1, a, a^2 \dots\}. \tag{2.3}$$

We denote by *quantum function* $\mathcal{E}(\varrho)$ an operation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to R_+$. It is well known that we may have $\mathcal{E}(\varrho^{\otimes N}) = 0$ even when the state $\varrho^{\otimes N}$ does contain some finite amount of the considered quantity [2]. We reserve the term "zero-resource state" for those states that

[1] The space of all density matrices. It is constructed by the tensor product $\mathcal{H}^* \otimes \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space and $\mathcal{H}^*$ is its dual, the space of linear maps from $\mathcal{H}$ to the complex numbers (BENGTSSON; ZYCZKOWSKI, 2017).

[2] For instance, the negativity of bound entangled states ϱ_b vanishes, although ϱ_b is not separable (ZHU; HAYASHI, 2010)

indeed contain no resource, e. g., separable states for entanglement, incoherent states for coherence, and so on. For any quantifier we will assume that $\mathcal{E}(\varrho) = 0$ for zero-resource states ϱ, but not the other way around.

In particular, we assume that a quantum function $\mathcal{E}$ must be compatible with the tensor product structure (2.2), this is:

$$\mathcal{E}(\varrho^{\otimes N}) = \mathcal{E}(\sigma^{\otimes N/K}), \tag{2.4}$$

to be physically acceptable, e. g., the quantification of $\mathcal{E}$ for the states $\varrho^{\otimes N}$ and $\sigma^{\otimes N/K}$, as in (2.2), should be equivalent.

As discussed in the introduction, we intend to circumvent the problem of quantifying $\mathcal{E}(\varrho^{\otimes N})$ by studying the class of functions which can be written in terms of a set evaluated for *fewer copies* $\mathcal{E}(\varrho^{\otimes i_\ell})$, with $\{i_\ell\}_{\ell=1}^{q}$ being an arbitrary ordered set of q integers ($i_q < N$). The referred set is fixed and it does not depend on N for some state ϱ. In what follows we call $\mathcal{E}(\varrho^{\otimes i_\ell}) = e_{i_\ell}$ and represent an arbitrary subset of the natural numbers as $\mathcal{S}_{\mathbf{N}}$.

Definition 1: Let $\mathcal{E}(\varrho^{\otimes N}) : \mathcal{B}\left(\mathcal{H}^{\otimes N}\right) \to R_+$ be a quantum function that depends exclusively on ϱ and $N \in \mathcal{S}_{\mathbf{N}}$. If one can express $\mathcal{E}(\varrho^{\otimes N})$ as a function of the vector $(e_{i_1}, ..., e_{i_q}) \equiv \mathbf{e}$ and N, where $\{i_\ell\}_{\ell=1}^{q}$ is an arbitrary ordered set of q integers, then:

$$\mathcal{E}(\varrho^{\otimes N}) = E^{(N)}(e_{i_1}, ..., e_{i_q}) = E^{(N)}(\mathbf{e}). \tag{2.5}$$

We say that $\mathcal{E}(\varrho^{\otimes N})$ is q-extensible (q-E) with respect to ϱ and $\mathcal{S}_{\mathbf{N}}$ (PARISIO, 2020).

We call the vector $\mathbf{e}$ a *basis of functions*, with respect to ϱ and $\mathcal{E}$. Note that while $\mathcal{E}(\varrho^{\otimes N}) : \mathcal{B}\left(\mathcal{H}^{\otimes N}\right) \to R_+$, $E^{(N)}(\mathbf{e}) : \mathcal{S}_{\mathbf{N}} \times R_+^q \to R_+$, so that, typically, *the domain of the latter has a dimension which is much lower than that of the former.*

A simple example: consider 5 copies of an arbitrary qubit ϱ and a quantum function or some figure of merit concerning these qubits as $\mathcal{G}$, suppose that this quantifier is a 2-E function following definition (2.5). If $\{i_\ell\}_{\ell=1}^{2} = \{1, 2\}$, for instance, the basis of functions $\mathbf{g} = (g_1, g_2) = (\mathcal{G}(\varrho), \mathcal{G}(\varrho^{\otimes 2}))$ completely determines $\mathcal{G}(\varrho^{\otimes N}) = G^{(N)}(\mathbf{g})$. So while $\mathcal{B}\left(\mathcal{H}^{\otimes 5}\right)$ has dimension $d = 2^5$, the referred function $G^{(5)}(g_1, g_2)$ is determined by 2 numbers only.

Because of (2.2) and the requirement that $N, K \in \mathbf{P}_a$, in this dissertation we will simplify our object of study to the class of quantum functions which satisfies $\mathcal{E}(\varrho^{\otimes N}) = E[\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}), ..., \mathcal{E}(\varrho^{\otimes a^{q-1}})]$, with $a^q < N$. So we specify the set $\{i_\ell\}_{\ell=1}^{q}$ by $i_\ell = a^{\ell-1}$:

$$(\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}), ..., \mathcal{E}(\varrho^{\otimes a^{q-1}})) = (e_1, e_2, ..., e_q) \equiv \mathbf{e}, \tag{2.6}$$

or, in a compact form, $E^{(a^\ell)}(\mathbf{e}) = \mathbf{e}_{\ell+1}$.

1-extensible (1-E) functions (the case where the vector $\mathbf{e}$ has only one component $e_1 = E^{(1)}(e_1) = \mathcal{E}(\varrho)$), e. g., functions that satisfy:

$$\mathcal{E}(\varrho^{\otimes N}) = E^{(N)}(e_1), \tag{2.7}$$

are the object of study of this chapter. It is important to remark, as discussed in the introduction, that several quantifiers cannot be described by this class of functions, e. g., those that allow for superactivation (SHOR; THAPLIYAL, 2003; QUINTINO; HUBER, 2016).

2.2 1-SCALABILITY

Now we state that there is a simple, but non-trivial, constraint that follows from extensibility (2.5) and from compatibility with the tensor product structure (2.4). It is a self-similarity relation which we call **scalability** (PARISIO, 2020). For pedagogic purposes, we initially refer to 1-E functions.

Proposition 1: Let $\varrho \in \mathcal{B}(\mathcal{H})$ and $N, K \in \mathbf{P}_a$ with $K < N$. If $\mathcal{E}$ is a 1-E quantum function with respect to ϱ and $\mathbf{P}_a$, then, by definition (2.5):

$$E^{(N)}(e_1) = E^{(N/K)}\left[E^{(K)}(e_1)\right], \tag{2.8}$$

where $e_1 = \mathcal{E}(\varrho)$. We say that a 1-$E$ function that satisfies (2.8) is a 1-*scalable* (1-S) function (PARISIO, 2020).

Proof:

Because $\mathcal{E}(\varrho^{\otimes N})$ is a 1-extensible function, our notation and (2.2) implies:

$$E^{(N)}(e_1) = \mathcal{E}(\varrho^{\otimes N}) = \mathcal{E}(\sigma^{\otimes N/K}) = E^{(N/K)}\left[\mathcal{E}(\sigma)\right]. \tag{2.9}$$

We may use the notation again to rewrite:

$$\mathcal{E}(\sigma) = \mathcal{E}(\varrho^{\otimes K}) = E^{(K)}(e_1), \tag{2.10}$$

then (2.9) becomes:

$$E^{(N)}(e_1) = E^{(N/K)}\left[E^{(K)}(e_1)\right],$$

and the proof is finished.

A consequence of (2.8) is that the function $E^{(a)}(e_1)$ completely determines $E^{(N)}(e_1)$. Setting $K = a$ in (2.8), it is immediate via finite induction that:

$$\begin{aligned} E^{(N)}(e_1) = E^{(N/a)}\left(E^{(a)}(e_1)\right) &= E^{(N/a^2)}\left(E^{(a)}\left(E^{(a)}(e_1)\right)\right) = \dots \\ E^{(N)}(e_1) &= \underbrace{E^{(a)} \circ E^{(a)} \circ \cdots \circ E^{(a)} \circ E^{(a)}}_{\text{n times}}(e_1), \end{aligned} \tag{2.11}$$

where $\circ$ denotes composition and $n = \log_a N$. This means that if $a = 2$, for example, $\mathcal{E}(\varrho^{\otimes 4}), \mathcal{E}(\varrho^{\otimes 8})$, etc, are completely determined by the properties of $\mathcal{E}(\varrho^{\otimes 2})$. In other words, for 1-$E$ nonlinear functions, the way $E^{(a)}$ deviates from linearity completely determines the functions $E^{(a^n)}$.

To give a simple example of the kind of constraint the previous result imposes, consider a hypothetical 1-E function $E^{(N)}(e_1) = N\lambda^{N-1}e_1$ ($\lambda \neq 0$). At first glance it seems a natural candidate as a consistent quantifier of a physical quantity related to N copies of a certain system. Note that $E^{(0)}(e_1) = 0$ and $E^{(1)}(e_1) = e_1$, as it should be. However, according to (2.8):

$$\begin{aligned} E^{(N/K)}\left(K\lambda^{K-1}e_1\right) &= \frac{N}{\cancel{K}}\lambda^{N/K-1}\left(\cancel{K}\lambda^{K-1}e_1\right) \\ &= N\lambda^{N/K+K-2}e_1 \neq E^{(N)}(e_1). \end{aligned}$$

This result does not coincide with the left-hand side, therefore this function would not be compatible with the tensor product structure.

On the other hand, the *additive* case $\mathcal{E}(\varrho^{\otimes N}) = N\mathcal{E}(\varrho)$[3]:

$$E^{(N/K)}[E^{(K)}(e_1)] = \frac{N}{K}E^{(K)}(e_1) = \frac{N}{\cancel{K}}\cancel{K}e_1 = E^{(N)}(e_1), \tag{2.12}$$

where $e_1 = \mathcal{E}(\varrho)$, trivially satisfies relation (2.8). This is the case of some relevant quantum functions like the logaritmic negativity (VIDAL; WERNER, 2002) and the Von Newmann entropy (BENGTSSON; ZYCZKOWSKI, 2017).

Another simple example, which we call *multiplicative*, is the function $E^{(N)}(e_1) = e_1^N$ (the reader can easily check in (2.8)), this is the case of the pure-state entanglement measure (for N even) (LI XIANGRONG LI; LI, 2007; LI XIANGRONG LI; LI, 2009). However, (2.8) allows for more complicated depedencies as we will see in the following sections. For instance, this possibility can be enlarged to:

$$E^{(N)}(e_1) = \lambda(N)e_1^N, \tag{2.13}$$

with $\lambda(N)$ being a hypotetical function of N only. We can use the 1-scalability relation (2.8) to determine the condition for $\lambda(N)$:

$$\lambda(N) = \lambda(N/K)[\lambda(K)]^{\frac{N}{K}}. \tag{2.14}$$

For any compatible with the tensor product structure quantum function of the form (2.13), the relation (2.14) must be satisfied. Note that the function $\lambda(N) = \lambda^{N-1},$ with $\lambda \in \mathbb{R}_+$, is a solution of (2.14).

$$\lambda(N) = \lambda^{\frac{N}{K}-1}[\lambda^{K-1}]^{\frac{N}{K}} \quad = \quad \lambda^{K\frac{N}{K}-1} = \lambda^{N-1} \quad \checkmark \tag{2.15}$$

The solution $E^{(N)}(e_1) = \lambda^{N-1}e_1^N$ is a non-trivial possibility of (2.8) and it will be derived from the tools to be defined in the next section.

[3] A stronger notion of additivity is $\mathcal{E}(\varrho \otimes \sigma) = \mathcal{E}(\varrho) + \mathcal{E}(\sigma)$, for all pairs of states ϱ and σ, but in this dissertation whenever we refer to additivity we mean the weaker condition.

2.3 ANALYTIC SCALABLE FUNCTIONS

From this point we proceed by considering functions $E^{(N)}(\mathbf{e})$ which are analytic in the vicinity of $\mathbf{e} = 0$. In the 1-S case this corresponds to the existence of a power series that converges to $E^{(N)}(e_1)$ in some non-vanishing interval $[0, \epsilon_N)$, $\epsilon_N > 0$. In the general q-S case, analyticity amounts to functions $E^{(N)}(\mathbf{e})$ which have a power series that converges to $E^{(N)}(\mathbf{e})$ in some q-ball (q being the number of elements in the basis of functions $\mathbf{e}$) of finite radius ϵ_N (restricted to the positive hyperoctant) centered at $\mathbf{e} = 0$.

2.4 1-S RECURRENCE RELATIONS

For a 1-S function take $e_1 = \mathcal{E}(\varrho)$ and supose that $E^{(N)}(e_1)$ is analytic at $e_1 = 0$. More precisely, we will assume that the function $E^{(N)}(e_1)$ has a Maclaurin series that converges in the non-vanishing interval $[0, \epsilon_N)$ with $\epsilon_N > 0$:

$$E^{(N)}(e_1) = \sum_{k=1}^{\infty} d_k(N) e_1^k. \tag{2.16}$$

If we suppose that this series, instead of being infinite, has an upper limit which depends only on N, $E^{(N)}(e_1) = \sum_{k=1}^{L(N)} d_k(N) e_1^k$, we can find a simple equation for it. Expanding both sides of (2.8) with the assumption of analyticity (2.16) we get:

$$\begin{aligned} E^{(N)}(e_1) &= \sum_{l=1}^{L(N/K)} d_l(N/K) \left[E^{(K)}(e_1)\right]^l \\ \sum_{k=1}^{L(N)} d_k(N) e_1^k &= \sum_{l=1}^{L(N/K)} d_l(N/K) \left[\sum_{m=1}^{L(K)} d_m(K) e_1^m\right]^l . \end{aligned} \tag{2.17}$$

This means that **if** the series is limited, the function $L(N)$ must satisfy:

$$L(N) = L(N/K)L(K). \tag{2.18}$$

Choosing $K = a$ (remember that $N \in \mathbf{P}_a$) and changing the notation $L(N)$ to L_n we get the recurrence $L_n = L_{n-1}L_1$, which has a simple solution:

$$L(N) = [L(a)]^{\log_a N}, \tag{2.19}$$

where $L(a)$ is the upper limit of the series expansion of $E^{(a)}(e_1)$. Evidently, if $L(a) \to \infty$ then, of course, $L(N) \to \infty$.

Table 1 – Upper limit of the 1-S series.

a	2	3
$L(a) = 1$	$L(N) = 1$	$L(N) = 1$
$L(a) = 2$	$L(N) = N$	$L(N) = 2^{\log_3 N}$
$L(a) = 3$	$L(N) = 3^{\log_2 N}$	$L(N) = N$

Source: the author (2021).

Table 1 lists (2.19) for $a = 2, 3$ using the values $L(a) = 1, 2, 3$, as an example. In particular, note that if $L(a) = a$ the series for N copies ends at N. This will an important feature for determining the scalability properties of coherence related quantifiers in section 2.5.

The following results do not rely on the assumption of a finite series, being also valid for infinite expansions (2.16). Combining terms order by order in (2.17), we get (PARISIO, 2020):

$$\begin{aligned} d_1(N) &= d_1(N/K)d_1(K), \\ d_2(N) &= d_1(N/K)d_2(K) + d_2(N/K)[d_1(K)]^2. \end{aligned} \tag{2.20}$$

Theorem 1.1: The general recursive relation for the Maclaurin series coefficients of a 1-S analytic function is given by:

$$d_j(N) = \sum_{l=1}^{j} d_l(N/K) \sum_{i=1}^{\binom{j-1}{l-1}} \pi_i(j, l; K), \tag{2.21}$$

where $\pi_i(j, l; K) = d_{\mu_1^i}(K)d_{\mu_2^i}(K)\dots d_{\mu_l^i}(K)$, with $(\mu_1^i, \dots, \mu_l^i)$ being the i-th composition of j into l parts (PARISIO, 2020).

Recalling the definition of *composition*: A composition of an integer j in l parts is an ordered sum $j = \mu_1 + \mu_2 + \cdots + \mu_l$, of strictly positive integers. A well-known result in enumerative combinatorics is that there are $\binom{j-1}{l-1}$ such compositions (MIER, 2004). For instance, there are 2 compositions of the integer $j = 3$ in $l = 2$ parts: $1 + 2$ and $2 + 1$. This result was previously proven and discussed in (PARISIO, 2020).

Note that each recurrence $d_j(N)$ depends on the previous relations $d_{l<j}(N)$. We may continue the list (2.20) to get more coefficients. The third one, for example:

$$\begin{aligned} d_3(N) &= d_1(N/K)\pi_1(3,1,K) + d_2(N/K)\sum_{i=1}^{2}\pi_i(3,2,K) + d_3(N/K)\pi_1(3,3,K) \\ &= d_1(N/K)d_3(K) + d_2(N/K)[\pi_1(3,2,K) + \pi_2(3,2,K)] + d_3(N/K)[d_1(K)]^3. \end{aligned}$$

As the number 3 can be composed as the pairs 1+2 and 2+1, $\pi_1(3,2,K) = \pi_2(3,2,K) = d_1(K)d_2(K)$ and then we have the following recurrence:

$$d_3(N) = d_1(N/K)d_3(K) + 2d_2(N/K)d_1(K)d_2(K) + d_3(N/K)[d_1(K)]^3.$$

These coupled recurrence relations may be solved with computational softwares, but we will calculate the first ones for pedagogic purposes. Because $N = a^n$ we can make $K = a$, the integer that defines the set of powers $\mathbf{P}_a$, to simplify our relations. For d_1:

$$\begin{aligned} d_1(N) &= d_1(a^{n-1})d_1(a) = \dots \\ \dots &= [d_1(a)]^n = N^\nu \;\rightarrow\; \nu = \log_a d_1(a). \end{aligned} \tag{2.22}$$

With the change of variables (2.22) (valid only for $d_1(a) \neq 0$) we can rewrite $d_1(a) = a^\nu$ and iterate the d_2 recurrence relation as:

$$\begin{aligned} d_2(a^n) &= d_1(a^{n-1})d_2(a) + d_2(a^{n-1})[d_1(a)]^2 = \dots \\ \dots &= d_2(a)[d_1(a)]^{n-1}\sum_{k=0}^{n-1} a^{\nu k} \\ d_2(N) &= d_2(a)\left(\frac{N}{a}\right)^\nu \left(\frac{N^\nu - 1}{a^\nu - 1}\right). \end{aligned} \tag{2.23}$$

Equations (2.22) and (2.23) were derived in reference (PARISIO, 2020). Adding the the third order coefficient (whose demonstration is in appendix **A**) we write the Maclaurin series for a 1-scalable function, up to 3rd order:

$$\begin{aligned} E^{(N)}(e_1) &= N^\nu e_1 + d_2(a)\left(\frac{N}{a}\right)^\nu \left(\frac{N^\nu - 1}{a^\nu - 1}\right) e_1^2 \\ &+ \left\{2[d_2(a)]^2\left(\frac{N}{a^2}\right)^\nu \frac{(N^\nu - 1)(N^\nu - a^\nu)}{(a^\nu - 1)(a^{2\nu} - 1)} + d_3(a)\left(\frac{N}{a}\right)^\nu \left(\frac{N^{2\nu} - 1}{a^{2\nu} - 1}\right)\right\} e_1^3 \end{aligned} \tag{2.24}$$

All 1-S analytic functions whose coefficient $d_1(a)$ is non zero must satisfy (2.24). So if one knows the expansion for a copies $E^{(a)}(e_1) = d_1(a)e_1 + d_2(a)e_1^2 + \ldots$ the coefficients $d_k(N)$ in (2.24) are well determined.

Note that 1-S analytic quantifiers whose regularized counterparts $E_{reg} = \lim_{N\to\infty} \frac{E^{(N)}(e_1)}{N}$ are finite and non-zero are necessarily additive, and, thus $E_{reg} = e_1$, for e_1 sufficiently small (evidently this is not necessarily true for 2-S functions, as we will see). In (2.24) this possibility means that $d_j(a) \neq 0$ only for $j = 1$ and $\nu = 1$ [4].

Note also that if a 1-S function is additive for 2 copies consequently it is additive for N copies. So it is not possible for a 1-S function to present $\mathcal{E}(\varrho^{\otimes 2}) = 2\mathcal{E}(\varrho)$ but $\mathcal{E}(\varrho^{\otimes 4}) \neq 4\mathcal{E}(\varrho)$, for example (what we will be refering to as *superactivation of non-additivity*).

In the next sections we study the cases where the expansion of $E^{(a)}(e_1)$ has only one ($L(a) = 1$) or only two ($L(a) = 2$) coefficients through the recurrence relation (2.21).

[4] If $\nu < 1$ the function is **subadditive**.

2.4.1 One-coefficient Case

In this section we study cases where the series expansion for a copies has only one non-zero coefficient and show that recurrence relations (2.21) predicts the previously studied functional forms *additive* and *multiplicative*. First note that by (2.19) if $L(a) = 1 \rightarrow L(N) = 1^n = 1$, so the series to be studied here only have 1 term for $N \in \mathbf{P}_a$.

- **Additive**

  As $d_k(a) = a\delta_{k1}$ then we have to consider in (2.21) only compositions of j into $l = j$ equal parts, this means that $\pi_i(j, l; a) = \pi_1(j, l; a) = \delta_{l,j}[d_1(a)]^j$ and then:

  $$d_j(N) \;=\; \sum_{l=1}^{j} \delta_{jl} d_l(N/a) a^j = d_j(N/a) a^j.$$

  Changing notation by $d_j(N) = d_j^n$ (so $d_j^0 = \delta_{j1}$ by our requirements) we get $d_j^n = a^j d_j^{n-1} = a^{2j} d_j^{n-2} = \cdots = a^{nj}\delta_{j1}$. Which is our well known result: $d_j(N) = N\delta_{j1}$.

- **Multiplicative**

  Now as $d_k(a) = d_a(a)\delta_{ka}$ we have to consider in (2.21) only compositions of j into $l = \frac{j}{a}$ equal parts, so $j \in \mathbf{P}_a$ so that l is an integer. This means that $\pi_i(j, l; a) = \pi_1(j, l; a) = \delta_{l\frac{j}{a}}[d_a(a)]^{j/a}$ and then:

  $$d_j(N) \;=\; \sum_{l=1}^{j} d_l(N/a)\delta_{l\frac{j}{a}}[d_a(a)]^{j/a} = d_{j/a}(N/a)[d_a(a)]^{j/a}.$$

  We change notation again to $d_j^n = d_{j/a}^{n-1}[d_a(a)]^{j/a}$ and we have to iterate this recurrence in the two indexes (n and j):

  $$d_j^n \;=\; d_{j/a^2}^{n-2}[d_a(a)]^{j/a^2}[d_a(a)]^{j/a} = \cdots = d_1^{n-p}[d_a(a)]^{1+a+a^2+\cdots+a^{p-1}},$$

  where we made $j = a^p$, as $j \in \mathbf{P}_a$. The geometric series can be rewritten as $\sum_{l=0}^{p-1} a^l = \frac{1-a^p}{1-a}$ and also, by our requirements, $d_1^{n-p} = d_1(a^{n-p}) = \delta_{np}$ (which means that $j = N$). Therefore:

  $$d_N(N) = [d_a(a)]^{\frac{N-1}{a-1}}. \tag{2.25}$$

  It is easy to see that the solution (2.15) fits perfectly.

These two types of quantifiers are examples of a symmetry in scalability relation (2.8) refering to the number of elements in the series expansion (2.19). In other words: if a 1-S function is additive (multiplicative) for a copies, it will be additive (multiplicative) for $N \in \mathbf{P}_a$ copies.

2.4.2 Two-coefficient Case

If a quantum function $\mathcal{E}(\varrho^{\otimes a})$ is 1-S and has the form $E^{(a)}(e_1) = d_1(a)e_1 + d_2(a)e_1^2$ (only the first two coefficients are non-zero), e. g., $L(a) = 2$, then the series for N copies ends at $L(N) = 2^{\log_a N}$ (see equation (2.19) and Table 1). The higher order recurrence relations become:

$$\begin{aligned} d_3(N) &= 2d_2(N/a)d_1(a)d_2(a) + d_3(N/a)[d_1(a)]^3, \\ d_4(N) &= d_2(N/a)[d_2(a)]^2 + 3d_3(N/a)[d_1(a)]^2 d_2(a) + d_4(N/a)[d_1(a)]^4. \end{aligned}$$

The reader may find the unsolved general fourth coefficient recurrence relation in appendix **A**. By making $K = a$ and taking $d_3(a) = d_4(a) = 0$ one gets to the relations above. Solving the 4th coefficient with *Mathematica* software we get:

$$\begin{aligned} d_1(N) &= N^\nu \\ d_2(N) &= d_2(a)\left(\frac{N}{a}\right)^\nu \left(\frac{N^\nu - 1}{a^\nu - 1}\right) \\ d_3(N) &= 2[d_2(a)]^2 \left(\frac{N}{a^2}\right)^\nu \left(\frac{N^\nu - 1}{a^\nu - 1}\right)\left(\frac{N^\nu - a^\nu}{a^{2\nu} - 1}\right) \\ d_4(N) &= [d_2(a)]^3 \left(\frac{N}{a^3}\right)^\nu \left(\frac{N^\nu - 1}{(a^\nu - 1)^2}\right)\left(\frac{N^\nu - a^\nu}{a^{2\nu} - 1}\right)\left(\frac{N^\nu(5 + a^\nu) - 1 - 5a^{2\nu}}{1 + a^\nu + a^{2\nu}}\right). \end{aligned} \tag{2.26}$$

Note that there is no clear pattern (at least not a simple one) determining how $d_k(N)$ looks like. However if one takes $\nu = 1$ and $a = 2$ one gets the following series:

$$\begin{aligned} E^{(N)}(e_1) = Ne_1 &+ [d_2(a)]\frac{N}{2}(N-1)e_1^2 + [d_2(2)]^2 N\frac{(N-1)(N-2)}{1.2.3}e_1^3 \\ &+ [d_2(2)]^3 N\frac{(N-1)(N-2)(N-3)}{1.2.3.4}e_1^4 + O(e_1^5), \end{aligned} \tag{2.27}$$

ending at $L(N) = 2^{\log_2 N} = N$, as we see in Table 1 (equation (2.19)).

Observing this pattern we can induce that the series (2.27) is constructed in terms of Newton coefficients $d_k(N) = [d_2(a)]^{k-1}\begin{pmatrix} N \\ k \end{pmatrix}$. We summarize this result in the theorem bellow, which has a more difficult proof.

Theorem 1.2: Let $\varrho \in \mathcal{B}(\mathcal{H})$, $N \in \mathbf{P}_2$ and $\mathcal{E}$ be an analytic 1-S function with respect to ϱ and $\mathbf{P}_2$. If $\mathcal{E}(\varrho^{\otimes 2})$ can be expanded as the maclaurin series $E^{(2)}(e_1) = 2e_1 + [d_2(2)]e_1^2$, then:

$$E^{(N)}(e_1) = \sum_{k=1}^{N} \left(\begin{array}{c} N \\ k \end{array} \right) [d_2(2)]^{k-1} e_1^k, \tag{2.28}$$

where $e_1 = \mathcal{E}(\varrho)$.

Proof:

We start with (2.21) for $K = 2$ (here the index l is rewritten as $j - k$):

$$d_j(N) = \sum_{k=0}^{j-1} d_{j-k}(N/2) \sum_{i=1}^{\left(\begin{array}{c} j-1 \\ j-k-1 \end{array} \right)} \pi_i(j, j-k; 2).$$

The hypotesis that only $d_1(2)$ and $d_2(2)$ are not zero means that $\pi_i(j, l; 2)$ will be a product of combinations of these two variables only (all other contributions will vanish), this means that we need to consider the composition of the number j using only the numbers 1 and 2. If we compose j with a sum of j positive integers then each one must be equal to 1, but if we compose j with a sum of $j - 1$ positive integers we need to take $j - 2$ of the elements equal to 1 and one element equal to 2 to complete the sum.

$$j \quad \rightarrow \quad \underbrace{1 + \cdots + 1}_{j-2k \text{ times}} + \underbrace{2 + \cdots + 2}_{k \text{ times}}$$

So actually the sum in k has an upper limit equal to $\lfloor \frac{j}{2} \rfloor$ (if j is even the sum stops at $j/2$ and if j is odd the sum stops at $(j-1)/2$). To summarize, the composition of the number j into $(j - k)$ parts using only the numbers 1 and 2 has $\frac{(j-k)!}{k!(j-2k)!}$ elements, being k the number of repetitions of the number 2:

$$\sum_{i=1}^{\left(\begin{array}{c} j-1 \\ j-k-1 \end{array} \right)} \pi_i(j, j-k; 2) = \frac{(j-k)!}{k!(j-2k)!} [d_1(2)]^{j-2k} [d_2(2)]^k, \tag{2.29}$$

where, as before, we will use $d_1(2) = 2^\nu$. For the proof of **theorem 1.2**, take $\nu = 1$:

$$d_j(N) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} d_{j-k}(N/2) \left(\begin{array}{c} j-k \\ k \end{array} \right) 2^{j-2k} [d_2(2)]^k.$$

Now we test expression $d_j(N) = [d_2(2)]^{j-1}\begin{pmatrix} N \\ j \end{pmatrix}$, induced in expansion (2.27):

$$\begin{aligned} d_j(N) &= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} [d_2(2)]^{j-k-1} \begin{pmatrix} N/2 \\ j-k \end{pmatrix} \begin{pmatrix} j-k \\ k \end{pmatrix} 2^{j-2k} \cancel{[d_2(2)]^k} \\ &= [d_2(2)]^{j-1} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \begin{pmatrix} N/2 \\ k \end{pmatrix} \begin{pmatrix} N/2-k \\ j-2k \end{pmatrix} 2^{j-2k}. \end{aligned} \tag{2.30}$$

In the last step we used the *subset-of-a-subset property* (GROSS, 2007) for the product of binomials: $(n,m)(m,k) \to (n,k)(n-k,m-k)$. Remember that because of $N \in \mathbf{P}_2$, $\frac{N}{2}$ is an integer; and because of the binomial, it is also the maximum possible value for k, so $j \leq N$.

For the evaluation of (2.30) we can use the integral representation of the binomial coefficient $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{1}{2\pi i} \oint_\Gamma \frac{(1+z)^n}{z^{m+1}} dz$ to solve the problem using the Egorychev method (EGORYCHEV, 1984; RIEDEL, 2021), where z is a complex variable and Γ is a small ($|z| < 1$) closed contour surrounding $z = 0$. Rewritting the terms in (2.30), we get:

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \begin{pmatrix} N/2 \\ k \end{pmatrix} \begin{pmatrix} N/2-k \\ j-2k \end{pmatrix} 2^{j-2k} &= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{2\pi i} \oint_\Gamma dz \frac{(1+z)^{N/2-k}}{z^{j-2k+1}} \begin{pmatrix} N/2 \\ k \end{pmatrix} 2^{j-2k} \\ &= \frac{1}{2\pi i} \oint_\Gamma dz \frac{(1+z)^{N/2}}{z^{j+1}} 2^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \begin{pmatrix} N/2 \\ k \end{pmatrix} \left(\frac{z^2}{4(1+z)} \right)^k. \end{aligned}$$

The integrand is proportional to $\frac{(1+z)^{N/2-k}}{z^{j+1-2k}}$ and is easy to see that for $\lfloor \frac{j}{2} \rfloor < k < \frac{N}{2}$ the integral vanishes. Make $k = \lfloor \frac{j}{2} \rfloor + l$ (so $l \leq \frac{N}{2} - \lfloor \frac{j}{2} \rfloor$), then the integrand is proportional to:

$$\begin{cases} j \text{ even} & \to \quad z^{2l-1}(1+z)^{\frac{(N-j)}{2}-l} \\ j \text{ odd} & \to \quad z^{2(l-1)}(1+z)^{\frac{(N-j+1)}{2}-l} \end{cases} \tag{2.31}$$

These functions do not have residue (DENNERY; KRZYWICKI, 1996) at $z = 0$ for any value of $l \in (1, \frac{N}{2} - \lfloor \frac{j}{2} \rfloor)$. Therefore, we can rewritte the sum without loss of generality with upper limit equal to $\frac{N}{2}$.

Now we can use the binomial theorem to rewrite the sum:

$$\sum_{k=0}^{\frac{N}{2}} \begin{pmatrix} N/2 \\ k \end{pmatrix} \left(\frac{z^2}{4(1+z)}\right)^k = \left[1 + \frac{z^2}{4(1+z)}\right]^{\frac{N}{2}} = \left[\frac{(z+2)^2}{4(1+z)}\right]^{\frac{N}{2}}, \tag{2.32}$$

and substitute (2.32) in the Cauchy integral:

$$\frac{1}{2\pi i}\oint_\Gamma dz \frac{\cancel{(1+z)^{N/2}}}{z^{j+1}} 2^j \left[\frac{(z+2)^2}{\cancel{4(1+z)}}\right]^{\frac{N}{2}} = \frac{1}{2\pi i}\oint_\Gamma dz \frac{(2+z)^N}{z^{j+1}} 2^{j-N}. \tag{2.33}$$

Now all we have to do is a change of variables $z \rightarrow 2z'$ ($\Gamma \rightarrow \Gamma'$) and the integral becomes:

$$\frac{1}{2\pi i}\oint_\Gamma dz \frac{(2+z)^N}{z^{j+1}} 2^{j-N} \rightarrow \frac{1}{2\pi i}\oint_{\Gamma'} dz' \frac{(1+z')^N}{z'^{j+1}}. \tag{2.34}$$

The right hand side of equation (2.34) is exactly the Cauchy integral representation of the binomial coefficient:

$$d_j(N) = [d_2(2)]^{j-1} \begin{pmatrix} N \\ j \end{pmatrix}, \tag{2.35}$$

and the theorem is proven.

2.5 SCALABILITY OF SOME COHERENCE RELATED QUANTIFIERS

Coherence is one of the most important features of quantum mechanics and it is the central concept in optics and quantum information (BAUMGRATZ; PLENIO, 2014; XI; YUWEN, 2019; MALEKI; AHANSAZ, 2020). It is the heart of quantum phenomena, such as entanglement and interference, and is present even in biological systems (LLOYD, 2011). In this section we study the scalability properties of this resource.

One well accepted coherence quantifier (BAUMGRATZ; PLENIO, 2014) is the *relative entropy of coherence* (REC):

$$\mathcal{C}_S(\varrho) = \mathcal{S}(\varrho) - \mathcal{S}(\varrho_{diag}), \tag{2.36}$$

where $\mathcal{S}(\varrho)$ is the Von Newmann entropy[5] and ϱ_{diag} is a matrix constructed by ϱ's diagonal elements only. It is well known that entropy is an additive quantity[6] and it does not depend on the reference basis (it is invariant under $U\varrho U^+$, U being an unitary transformation), so if we take a matrix constructed only by the diagonal elements of ϱ additivity remains:

$$\begin{aligned} \mathcal{C}_S(\varrho^{\otimes N}) = \mathcal{S}(\varrho^{\otimes N}) - \mathcal{S}(\varrho_{diag}^{\otimes N}) = N\left(\mathcal{S}(\varrho) - \mathcal{S}(\varrho_{diag})\right) \\ C_S^{(N)}(c_1) = Nc_1 \end{aligned} \tag{2.37}$$

Therefore, the relative entropy of coherence is a 1-S function, where we used $\mathcal{C}_S$ for the function whose domain is the set $\mathcal{B}\left(\mathcal{H}^{\otimes N}\right)$ and $C_S^{(N)}$ for the function whose domain is the non-negative number c_1 (numerically equal to $\mathcal{C}_S(\varrho)$). The same is not true for the much more intuitive l_1-*norm of coherence*, where we sum the modulus of all non-diagonal elements of ϱ:

$$\mathcal{C}_{l_1}(\varrho) = \sum_{i\neq j}^{d} |\varrho_{ij}| = \sum_{i,j}^{d} |\varrho_{ij}| - 1. \tag{2.38}$$

The goal of the next section is to derive the scalability property of (2.38). We also study the squared Hilbert-Schmidt norm $\mathcal{C}_{l_2}(\varrho) = \sum_{i\neq j} |\varrho_{ij}|^2$, which *is not a good candidate to quantify coherence* because it may increase under some incoherent operations (BAUMGRATZ; PLENIO, 2014), but we will determine its scalability properties in section **2.5.2** as a non-trivial example.

[5] In quantum statistical mechanics, the Von Newmann entropy plays the role of the classical Gibbs entropy for the microcanonical ensemble and it vanishes if, and only if, ϱ represents a pure state.

[6] There are other coherence quantifiers which are also additive quantities, like the logarithm coherence number (XI; YUWEN, 2019) and the purity of coherence.

2.5.1 l_1-norm of Coherence

Now we calculate the coherence of the non-interacting pair $\varrho \otimes \varrho$ using the l_1-norm:

$$\mathcal{C}_{l_1}(\varrho^{\otimes 2}) = \sum_{i,h,k,l} |\varrho_{ij}\varrho_{lk}| - 1. \tag{2.39}$$

We can take all possibilities $i = j, i \neq j$ and $k = l, k \neq l$ and take the trace of ϱ:

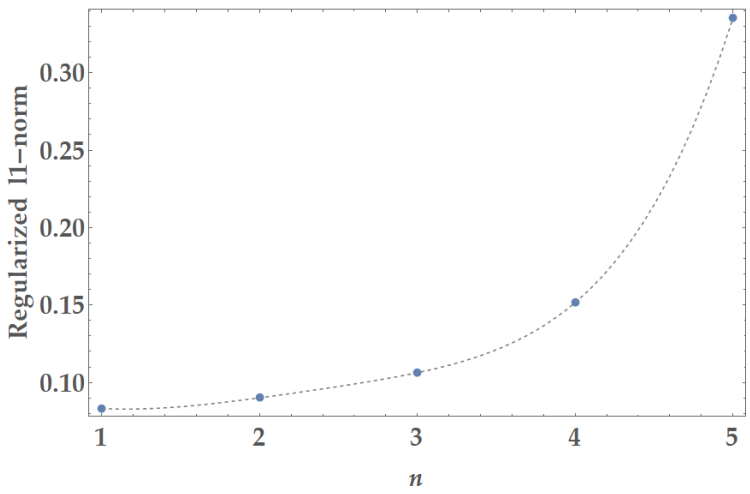
$$\begin{aligned} \mathcal{C}_{l_1}(\varrho^{\otimes 2}) &= Tr\varrho \left(\sum_{k\neq l} |\varrho_{kl}|\right) + \left(\sum_{i\neq j} |\varrho_{ij}|\right) Tr\varrho + \left(\sum_{i\neq j} |\varrho_{ij}|\right)\left(\sum_{k\neq l} |\varrho_{kl}|\right), \\ C_{l_1}^{(2)}(c_1) &= 2c_1 + c_1^2 = (1+c_1)^2 - 1, \end{aligned} \tag{2.40}$$

where, using the latter example notation, $c_1 = \mathcal{C}_{l_1}(\varrho)$. Note that **if** the l_1-norm of coherence is a 1-S function (proof in appendix **B**), then this is the two-coefficient case with $a = 2$ (2.28)! As the coefficient $d_2(2) = 1$, we have the following solution:

$$C_{l_1}^{(N)}(c_1) = \sum_{k}^{N} \binom{N}{k} c_1^k = (1+c_1)^N - 1, \tag{2.41}$$

for $N \in \mathbf{P}_2$. This result is valid for qudits and it does not depend on the parameters of ϱ.

Figure 3 – Regularized l_1-norm of coherence of 2^n qudits.



Source: the author (2020).

Figure 3 shows the regularized l_1-norm, $\frac{C_{l_1}^{(N)}(c_1)}{N}$, in terms of $n = \log_2 N$ with $c_1 = 0.08$.

Note that this quantifier is non-additive, differently from the relative entropy of coherence (2.36), and it diverges for large N. So the way one quantifies quantum coherence implies, or not, its non-additivity, determining its behavior for a large number of copies.

Expression (2.40) was already deduced in (MAZIERO, 2017).

2.5.2 Squared Hilbert-Schmidt Norm

As it was already pointed out, the squared Hilbert-Schmidt norm (also called by "l_2-norm of coherence"),

$$\mathcal{C}_{l_2}(\varrho) = \sum_{i \neq j}^{d} |\varrho_{ij}|^2, \tag{2.42}$$

was discarded (BAUMGRATZ; PLENIO, 2014) for *not being a reasonable coherence quantifier* because it may increase under some incoherent operations, thus failing the quantum resource theory requirements. But it is interesting to see how it scales with the number of copies. For a qubit:

$$\varrho \;=\; \begin{pmatrix} a & b \\ b^* & (1-a) \end{pmatrix} \quad \rightarrow \quad \mathcal{C}_{l_2}(\varrho) = 2|b|^2.$$

By making the tensor product $\varrho \otimes \varrho$, we find the following relation:

$$\varrho \otimes \varrho \;=\; \begin{pmatrix} a^2 & ab & ba & b^2 \\ ab^* & a(1-a) & |b|^2 & b(1-a) \\ b^*a & |b|^2 & (1-a)a & (1-a)b \\ b^{*2} & (1-a)b^* & (1-a)b^* & (1-a)^2 \end{pmatrix},$$

$$\begin{aligned} \mathcal{C}_{l_2}(\varrho \otimes \varrho) &= 4\left[a^2|b|^2 + |b|^4 + |b|^2(1-a)^2\right] \\ &= \bar{c}_1^2 + 2[a^2 + (1-a)^2]\bar{c}_1, \end{aligned} \tag{2.43}$$

and now we use $c_1 = \mathcal{C}_{l_2}(\varrho)$. We may use the constraint imposed by $Tr\varrho^2$ to rewrite (2.43):

$$Tr\varrho^2 \;=\; a^2 + (1-a)^2 + 2|b|^2 \quad \rightarrow \quad a^2 + (1-a)^2 = Tr\varrho^2 - c_1,$$

note that the difference $Tr\varrho^2 - c_1 \leq 1$ must be satisfied. Differently from the l_1-norm, the expansion of the l_2-norm depends on the parameters of the quantum state ϱ, more specifically on ϱ's purity:

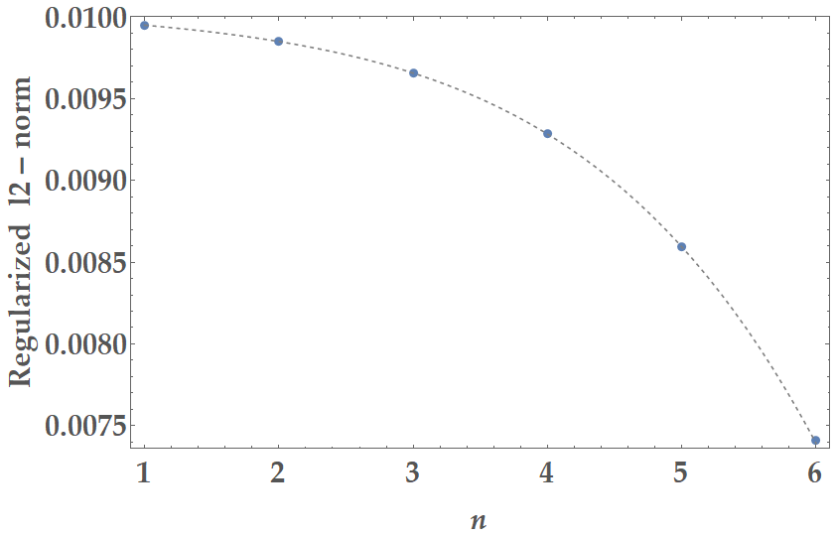
$$C_{l_1}^{(2)}(c_1) = (2Tr\varrho^2)c_1 - c_1^2. \tag{2.44}$$

So, **if** the Hilbert-Schmidt norm is a 1-S function then we would have the two-coefficient case discussed in section **2.4.2**. If the state is pure ($Tr\varrho^2 = 1$) we get $\nu = 1$ and the result is (2.28) with $d_2(a) = -1$:

$$C_{l_2}^{(N),\,\text{pure}}(c_1) = \sum_{k}^{N} \begin{pmatrix} N \\ k \end{pmatrix} (-1)^{k-1} c_1^k = 1 - (1 - c_1)^N, \tag{2.45}$$

where $c_1 = \mathcal{C}_{l_2}(\varrho)$ and $N \in \mathbf{P}_2$. In this case if N goes to infinity the l_2-norm saturates at 1 and thus its regularized part $\frac{C_{l_2}^{(N)}(c_1)}{N}$ vanishes (as shown in Figure 4 for $c_1 = 0.01$), a very different behavior from the l_1-norm of coherence, which diverges.

Figure 4 – Regularized squared Hilbert-Schmidt norm of 2^n pure qubits.



Source: the author (2020).

In appendix **B** we show that the squared Hilbert-Schmidt norm of an arbitrary qubit ϱ grows with the number of copies N as $C_{l_2}^{(N)}(c_1) = (Tr\varrho^2)^N - (Tr\varrho^2 - c_1)^N$. So $C_{l_2}^{(N)}(c_1)$ depends on the purity of ϱ, as we observed in (2.44). The scalability test on this formula for a pure state and for a non-pure state yields different results.

Pure states:

$$\begin{aligned} C_{l_2}^{(N/K)}[C_{l_2}^{(K)}(c_1)] &= 1 - \left(1 - C_{l_2}^{(K)}(c_1)\right)^{N/K} \\ &= 1 - [\cancel{1} - (\cancel{1} - (1 - c_1)^K)]^{N/K} \\ &= 1 - (1 - c_1)^N = C_{l_2}^{(N)}(c_1) \quad \checkmark \end{aligned}$$

Non-pure states:

$$C_{l_2}^{(N/K)}[C_{l_2}^{(K)}(c_1)] = (Tr\varrho^2)^{N/K} - \left(Tr\varrho^2 - C_{l_2}^{(K)}(c_1)\right)^{N/K}$$

$$= (Tr\varrho^2)^{N/K} - [Tr\varrho^2 - ((Tr\varrho^2)^K - (Tr\varrho^2 - c_1)^K)]^{N/K} \neq C_{l_2}^{(N)}(c_1) \quad \times$$

If the state is non-pure one could think that by making $\nu = 1 + \log_2(Tr\varrho^2)$ in (2.26) one would solve the problem order by order, but actually the l_2-norm is *not scalable for non-pure states*. This difference between the l_1-norm of coherence and the squared Hilbert-Schmidt norm can be seen as another evidence that the latter cannot be used to quantify coherence, because the purity of ϱ can change throught incoherent operations.

3 2-SCALABILITY

"The simple geometrical figure had been quickly constructed, line by line, self-reflexive..."

– Carl Sagan, *Contact*

We saw in the first chapter the way a 1-S function can scale with the number of copies N and its limitations: (i) It does not allow for superactivation nor does it allow for superactivation of non-additivity[1] and (ii) its regularized part can be bounded only if the function is additive. We define here the concept of q-scalability to enlarge our description to more complex classes of quantum functions, like the one-shot-distillable entanglement (FANG X. WANG; DUAN, 2019), an important example in this chapter.

3.1 q-SCALABILITY

Now we provide the generalization of the previous results for physically consistent q-scalable functions, which follows from the definition of q-extensibility. The following theorem was demonstrated in (PARISIO, 2020).

Theorem 2.1: Let $\varrho \in \mathcal{B}(\mathcal{H})$ and $N, K \in \mathbf{P}_a$ with $K < N$. If $\mathcal{E}$ can be written as a function of the vector $\mathbf{e} \equiv (\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}), \ldots, \mathcal{E}(\varrho^{\otimes a^{q-1}}))$, e. g., a q-E function with respect to ϱ and $\mathbf{P}_a$, then:

$$E^{(N)}(\mathbf{e}) = E^{(N/K)}(E^{(K)}(\mathbf{e}), E^{(aK)}(\mathbf{e}) \ldots, E^{(a^{q-1}K)}(\mathbf{e})). \tag{3.1}$$

A more complete form of (3.1) is in (PARISIO, 2020).

Proof:

As before, we start assuming compatibility with the tensor product structure:

$$\mathcal{E}(\varrho^{\otimes N}) = \mathcal{E}(\sigma^{\otimes N/K}). \tag{3.2}$$

[1] Remember that by *superactivation of non-additivity* we mean the possibility of $\mathcal{E}(\varrho^{\otimes 2}) = 2\mathcal{E}(\varrho)$ but $\mathcal{E}(\varrho^{\otimes 4}) \neq 4\mathcal{E}(\varrho)$.

Since $\mathcal{E}$ is q-E for a basis of functions $\mathbf{e}$, by hypotesis:

$$\begin{aligned} E^{(N)}(\mathbf{e}) &= E^{(N/K)}\left[\mathcal{E}(\sigma), \mathcal{E}(\sigma^{\otimes a}), \ldots, \mathcal{E}(\sigma^{\otimes a^{q-1}})\right] \\ &= E^{(N/K)}\left[\mathcal{E}(\varrho^{\otimes K}), \ldots, \mathcal{E}(\varrho^{\otimes a^{q-1}K})\right] = E^{(N/K)}\left(E^{(K)}(\mathbf{e}), \ldots, E^{(a^{q-1}K)}(\mathbf{e})\right), \end{aligned}$$

and the proof is finished. Now we enlarge **definition 1** to:

> **Definition 2:** A q-extensible (q-E) quantum function with respect to ϱ and $\mathbf{P}_a$ that satisfy condition (3.1) is *q-scalable* (q-S) with respect to ϱ and $\mathbf{P}_a$ (PARISIO, 2020).

Physically consistent q-extensible functions must be q-scalable, otherwise $\mathcal{E}(\varrho^{\otimes N}) \neq \mathcal{E}(\sigma^{(N/K)})$ with $\sigma = \varrho^{\otimes K}$. We now say that, for example, $E^{(2)}, E^{(4)}$ completely determine $E^{(N)}(e_1, e_2)$, with $e_1 = \mathcal{E}(\varrho^{\otimes 2})$ and $e_2 = \mathcal{E}(\varrho^{\otimes 4})$, and this is thus a 2-scalable function. And also for nonlinear 2-E functions we say that the way $E^{(2)}(\mathbf{e}), E^{(4)}(\mathbf{e})$ deviates from linearity determines completely the behavior of $E^{(2^n)}$. All valid for a general q-extensible case with respect to some state ϱ.

3.2 2-S RECURRENCE RELATIONS

Consider $e_1 = \mathcal{E}(\varrho)$ and $e_2 = \mathcal{E}(\varrho^{\otimes a})$ as the values of $\mathcal{E}$ for 1 and a copies of ϱ respectively. Assuming analyticity around $\mathbf{e} = (e_1, e_2) = (0, 0)$, and that $\mathcal{E}$ is a 2-E function, we may write $E^{(N)}(e_1, e_2)$ as a Maclaurin series:

$$\begin{aligned} E^{(N)}(e_1, e_2) &= \sum_{k,l}^{\infty} d_{kl}(N) e_1^k e_2^l \qquad (3.3) \\ &= d_{10}(N)e_1 + d_{01}(N)e_2 + d_{20}(N)e_1^2 + d_{11}(N)e_1e_2 + d_{02}(N)e_2^2 + \ldots \end{aligned}$$

The function now depends on both e_1 and e_2 and we have to compute the combinations of these terms in the expansion. In this case, the scalability relation is simply:

$$E^{(N)}(\mathbf{e}) = E^{(N/K)}\left[E^{(K)}(\mathbf{e}), E^{(aK)}(\mathbf{e})\right], \qquad (3.4)$$

where $\mathbf{e} = (e_1, e_2)$ and $O(2)$ denotes second order contributions, that is, terms proportional to e_1^2, e_2^2 and e_1e_2. So the question is: by knowing the result for a^2 copies:

$$E^{(a^2)}(\mathbf{e}) = x\ e_1 + y\ e_2 + O(2),$$

can we determine the series for N copies $E^{(N)}(\mathbf{e})$? How the coefficients of this series scale with N? Does this scalable quantity allow superactivation of non-additivity?

3.2.1 First Order: The Fibonacci Polynomials

If we expand $\mathcal{E}(\varrho^{\otimes N})$ as a Maclaurin series (3.3) in terms of $\mathbf{e} = (e_1, e_2)$ we first have to meet the requirements $\mathcal{E}(\varrho) = e_1$ and $\mathcal{E}(\varrho^{\otimes a}) = e_2$ on coefficients $d_{kl}(1)$ and $d_{kl}(a)$, which are thus translated into $d_{kl}(1) = \delta_{k1}\delta_{l0}$ and $d_{kl}(a) = \delta_{k0}\delta_{l1}$. For N we call the first order coefficients $d_{10}(N) = A(N)$ and $d_{01}(N) = B(N)$ and now we match them on both sides of equation (3.4) by combining only first order terms:

$$\begin{aligned} A(N)e_1 + B(N)e_2 &= A(N/K)E^{(K)}(e_1, e_2) + B(N/K)E^{(aK)}(e_1, e_2) \\ &= A(N/K)\left[A(K)e_1 + B(K)e_2\right] + B(N/K)\left[A(aK)e_1 + B(aK)e_2\right]. \end{aligned} \tag{3.5}$$

As before, we can take $K = a$ and we will have $A(a) = 0$ and $B(a) = 1$ and, by calling $A(a^2) = x$ and $B(a^2) = y$, we use $n = \log_a N$ to simplify the notation to $A(N) = A_n$ (same for $B(N)$). Combining the terms of same type we get a system of coupled equations:

$$\begin{cases} A(N) &= B(N/a)A(a^2) \\ B(N) &= A(N/a) + B(N/a)B(a^2) \end{cases} \rightarrow \begin{cases} A_n &= x\ B_{n-1} \\ B_n &= A_{n-1} + y\ B_{n-1} \end{cases}$$

By decoupling the above equations we get linear homogeneous recurrence relations with constant coefficients $B_n = y\ B_{n-1} + x\ B_{n-2}$ (same for A_n). The solution of this recurrence relation is the *generalized hybrid Fibonacci polynomials* $B_n = F_n^{(H)}(x, y)$ (RANGARAJAN; P., 2017), in two variables x and y and degree n, which have the following explicit form:

$$\begin{aligned} F_n^{(H)}(x, y) &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} x^{n-1-2k} y^k \\ &= \frac{1}{\sqrt{4x + y^2}} \left[\left(\frac{y + \sqrt{4x + y^2}}{2} \right)^n - \left(\frac{y - \sqrt{4x + y^2}}{2} \right)^n \right]. \end{aligned} \tag{3.6}$$

Coefficients (3.6) are called the *Fibonacci Polynomials* (which become the famous *Fibonacci Numbers* if $x = y = 1$ (RANGARAJAN; P., 2017)) and the first five iterations of the sequence are represented in Table 2. This result was previously achieved in reference (PARISIO, 2020):

Table 2 – Table of Fibonacci polynomials and Fibonacci numbers.

n	$F_n^{(H)}(x,y)$	$F_n(1,1)$
1	1	1
2	1	1
3	$x+y^2$	2
4	$2xy+y^3$	3
5	$x^4+3x^2y+y^2$	5

Source: (RANGARAJAN; P., 2017).

Theorem 2.2: Let $\varrho \in \mathcal{B}(\mathcal{H})$ and n be an arbitrary integer. If $\mathcal{E}$ is a 2-S function such that $E^{(N)}(\mathbf{e})$ depends on $\mathbf{e} = (e_1, e_2) = (\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}))$ and on n, where $E^{(a^2)}(\mathbf{e}) = xe_1 + ye_2 + O(2)$, with x and y known, then, for e_1 and e_2 sufficiently small, we have:

$$E^{(a^n)}(\mathbf{e}) = xF_{n-1}^{(H)}(x,y)e_1 + F_n^{(H)}(x,y)e_2 + O(2), \tag{3.7}$$

where $F_n^{(H)}(x,y)$ are the generalized hybrid Fibonacci polynomials (PARISIO, 2020).

The first terms of the series are:

$$\begin{aligned} E^{(a^2)}(\mathbf{e}) &= x\ e_1 + y\ e_2 + O(2), \\ E^{(a^3)}(\mathbf{e}) &= xy\ e_1 + \left(x + y^2\right) e_2 + O(2), \\ E^{(a^4)}(\mathbf{e}) &= x\left(x + y^2\right) e_1 + y\left(2x + y^2\right) e_2 + O(2), \\ &\vdots \end{aligned} \tag{3.8}$$

Note that superactivation of non-additivity may happen in scenarios involving q-S functions. In the present case we may have $\mathcal{E}(\varrho^{\otimes 2}) = 2\mathcal{E}(\varrho)$, that is, $e_2 = 2e_1$ (in this case, $a = 2$), however, with $\mathcal{E}(\varrho^{\otimes 4}) = (x+2y)\ e_1 \neq 4\mathcal{E}(\varrho)$ whenever $x \neq 4-2y$ (PARISIO, 2020). This behavior would not be possible for a 1-S function (2.24).

Substituting (3.6) in the recurrence of coefficients A_n and B_n in expansion $E^{(a^n)}(\mathbf{e}) = A_n e_1 + B_n e_2 + O(2)$ we get:

$$A_n = \frac{x}{\sqrt{4x+y^2}}\left[\left(\frac{y+\sqrt{4x+y^2}}{2}\right)^{n-1} - \left(\frac{y-\sqrt{4x+y^2}}{2}\right)^{n-1}\right], \tag{3.9}$$

$$B_n = \frac{1}{\sqrt{4x+y^2}}\left[\left(\frac{y+\sqrt{4x+y^2}}{2}\right)^{n} - \left(\frac{y-\sqrt{4x+y^2}}{2}\right)^{n}\right]. \tag{3.10}$$

Note that **not any values of x and y are permited**, the inequality $y^2 > -4x$ must be satisfied. Now, simirlaly to (2.22), we can make a change of variables to rewrite coefficients A_n and B_n throught:

$$\frac{y+\sqrt{4x+y^2}}{2} = a^{\nu_1} \qquad \text{and} \qquad \frac{y-\sqrt{4x+y^2}}{2} = a^{\nu_2}. \tag{3.11}$$

As $a^n = N$, by substituting (3.11) into (3.9) and (3.10) (As $\sqrt{4x+y^2} = a^{\nu_1} - a^{\nu_2}$) we get the following expressions in terms of the number of copies N:

$$A(N) = \frac{-a^{\nu_1+\nu_2}}{a^{\nu_1}-a^{\nu_2}}\left[\left(\frac{N}{a}\right)^{\nu_1} - \left(\frac{N}{a}\right)^{\nu_2}\right], \tag{3.12}$$

$$B(N) = \left(\frac{N^{\nu_1}-N^{\nu_2}}{a^{\nu_1}-a^{\nu_2}}\right). \tag{3.13}$$

With these new representations a 2-S analytic function must be of the form:

$$E^{(N)}(\mathbf{e}) = \left(\frac{N^{\nu_2}a^{\nu_1} - N^{\nu_1}a^{\nu_2}}{a^{\nu_1}-a^{\nu_2}}\right)e_1 + \left(\frac{N^{\nu_1}-N^{\nu_2}}{a^{\nu_1}-a^{\nu_2}}\right)e_2 + O(2). \tag{3.14}$$

Equation (3.14) is symmetric by exchange of ν_1 and ν_2 and they must satisfy $\nu_1 \neq \nu_2$ ($y^2 > -4x$) for the coefficients not to diverge. Note in (3.14) that, differently from a 1-S function, whose regularized part is bounded only if the quantum function is additive, *a regularized 2-S function can be bounded in many different ways*: if parameters satisfy $\nu_1, \nu_2 < 1$ the function vanishes as n increases, while if $\nu_1 = 1$ and $\nu_2 \leq 1$ the function is limited by a finite value. Of course, if $\nu_i > 1$, $i = 1, 2$, the function diverges for a large number of copies.

We will see in next subsections some simple examples of 2-S functions with selected values of ν_1 and ν_2 but, of course, in a general physical situation these parameters (As long as they are different) may be arbitrary numbers, not even rational ones; and there may be much more interesting properties in these scenarios.

3.2.1.1 About x and y

The change of variables (3.11) determines ν_1 and ν_2 in terms of x and y for a certain number of copies a^2. By inverting the expressions (3.11) we get:

$$\begin{cases} x &= -a^{\nu_1+\nu_2} \\ y &= a^{\nu_1} + a^{\nu_2} \end{cases} . \tag{3.15}$$

Note that x is *always* a negative number. Also, x and y are intrinsically dependent of each other and the relation between them is governed by the number of copies a, which can be essentially determinated by the state ϱ and the quantifier of interest.

Because of equations (3.9) and (3.10), these coefficients must satifsy $y^2 \geq 4|x|$ for (3.14) to be a physically acceptable quantifier: this means that $a^{2\nu_1} + a^{2\nu_2} \geq 2a^{\nu_1+\nu_2}$ must be satisfied. According to equations (3.8), $E^{(a^2)}(\mathbf{e}) = x\ e_1 + y\ e_2$, up to first other, so:

$$x = \frac{\mathcal{E}(\varrho^{\otimes a^2}) - y\ e_2}{e_1}. \tag{3.16}$$

The statement that x is negative is equivalent to:

$$\frac{\mathcal{E}(\varrho^{\otimes a^2})}{\mathcal{E}(\varrho^{\otimes a})} < y. \tag{3.17}$$

This inequality sets an upper bound for the growth of the amount of resources, if this quantity is well described by a 2-S function, when we go from a to a^2 copies.

3.2.2 Vanishing Second Order

Now, we are looking for possible contributions $E^{2^o}(\mathbf{e})$, e. g., terms proportional to e_1^2, e_2^2 and e_1e_2. Defining the functions $d_{20}(N) = W(N), d_{11}(N) = P(N)$ and $d_{02}(N) = Q(N)$ as the second order components of the series we get the expression:

$$\begin{gathered} W(N)e_1^2 + P(N)e_1e_2 + Q(N)e_2^2 = \\ W_{(N/K)}[E^{(K)}(\mathbf{e})]^2 + P_{(N/K)}[E^{(K)}(\mathbf{e})E^{(aK)}(\mathbf{e})] + Q_{(N/K)}[E^{(aK)}(\mathbf{e})]^2, \end{gathered} \tag{3.18}$$

and then:

$$\begin{aligned} E^{2^o}(\mathbf{e}) &= W(N/K)\left[[d_{10}(K)]^2e_1^2 + 2d_{10}(K)d_{01}(K)e_1e_2 + [d_{01}(K)]^2e_2^2\right] \\ &+ P(N/K)\left[d_{10}(K)e_1 + d_{01}(K)e_2\right]\left[d_{10}(aK)e_1 + d_{01}(aK)e_2\right] \\ &+ Q(N/K)\left[[d_{10}(aK)]^2e_1^2 + 2d_{10}(aK)d_{01}(aK)e_1e_2 + [d_{01}(aK)]^2e_2^2\right]. \end{aligned} \tag{3.19}$$

Combining same order terms in (3.19) we will get coupled recurrence relations for $W(N), P(N)$ and $Q(N)$. Choosing $K = a$ and using the requirements for $d_{10}(N)$ and $d_{01}(N)$ we get rid of some terms (because $d_{10}(a) = 0$ and $d_{01}(1) = 0$):

$$\begin{aligned} W(N) &= \cancel{W(N/a)[d_{10}(a)]^2} + \cancel{P(N/a)d_{10}(a)d_{01}(a^2)} + Q(N/a)[d_{10}(a^2)]^2, \\ P(N) &= \cancel{2W(N/a)d_{10}(a)d_{01}(a)} + P(N/a)\left[\cancel{d_{10}(a)d_{01}(a^2)} + d_{10}(a^2)\right] + 2Q(N/a)d_{10}(a^2)d_{01}(a^2), \\ Q(N) &= W(N/a) + P(N/a)d_{01}(a^2) + Q(N/a)[d_{01}(a^2)]^2. \end{aligned}$$

As before, we define $d_{10}(a^2) = x$ and $d_{01}(a^2) = y$ and change notation to $P(N) = P_n$:

$$\begin{cases} W_n &= x^2Q_{n-1} \\ P_n &= xP_{n-1} + 2x\ Q_{n-1} \\ Q_n &= W_{n-1} + yP_{n-1} + y^2Q_{n-1} \end{cases},$$

where $n = \log_a N$. Substituting W_{n-1} in the third equation and making algebric manipulations one derives the same recurrence relation for P_n and Q_n (consequently $W_n = x^2Q_{n-1}$ obeys the same recurrence):

$$P_{n+1} = (x + y^2)P_n + (xy + x^2)P_{n-1} - x^3P_{n-2}. \tag{3.20}$$

So the three coefficients obey a recurrence that depends on the last 3 steps. This means that as $W_0 = W_1 = 0$ and $W_2 = x^2Q_1 = 0$ then $W_n = 0$ for any n. This leads to a sequential vanishing of all second order coefficients. So we have an empty solution for all 2nd order coefficients $d_{20}(N) \to 0,\ d_{11}(N) \to 0,\ d_{02}(N) \to 0$ for any value of $N = a^n$.

3.3 SOME EXAMPLES AND PLOTS

3.3.1 Example I: $\nu_1 = 0$ and $\nu_2 = 1$

In this case, coefficients (3.12) and (3.13) are simpler and the 2-S function becomes:

$$E^{(N)}(\mathbf{e}) = \left(\frac{N-a}{1-a}\right) e_1 + \left(\frac{1-N}{1-a}\right) e_2 + O(3). \tag{3.21}$$

So the asymptotic limit grows linearly with the difference between e_1 and e_2 (fixed a):

$$\lim_{n\to\infty} \frac{E^{(a^n)}(\mathbf{e})}{a^n} = \frac{e_2 - e_1}{a-1}.$$

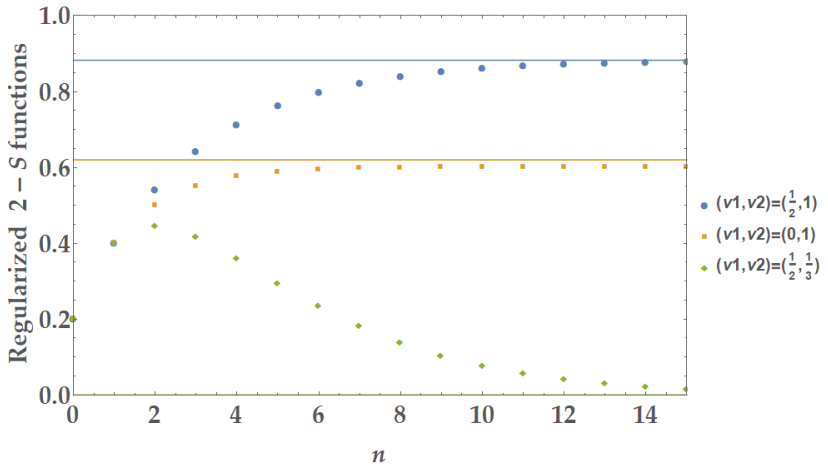
3.3.2 Example II: $\nu_1 = \frac{1}{2}$ and $\nu_2 = 1$

If $(\nu_1, \nu_2) = \left(\frac{1}{2}, 1\right)$, we have in (3.14) the following 2-S function and asymptotic limit:

$$E^{(N)}(\mathbf{e}) = \left(\frac{N\sqrt{a} - \sqrt{N}a}{\sqrt{a}-a}\right) e_1 + \left(\frac{\sqrt{N}-N}{\sqrt{a}-a}\right) e_2 + O(3), \tag{3.22}$$

$$\lim_{n\to\infty} \frac{E^{(a^n)}(\mathbf{e})}{a^n} = \frac{\sqrt{a}e_1 - e_2}{\sqrt{a}-a}.$$

Figure 5 - Asymptotic limits of different 2-S functions.



Source: the author (2020).

Figure 5 shows a 2-S function with $\mathbf{e} = (0.2, 0.8)$ and $a = 2$ for different pairs of parameters (ν_1, ν_2) and $n = \log_2 N$. The asymptotic limits for $(\nu_1, \nu_2) = (0, 1)$ (represented by the blue line) and $(\nu_1, \nu_2) = (\frac{1}{2}, 1)$ (represented by the yellow line) are the ones predicted in **example I** and in **exemple II**. Also we see that if $\nu_1, \nu_2 < 1$ the regularized series vanishes in the large number of copies limit.

3.3.3 Example III: Superactivation

Superactivation is an interesting phenomenon in which $\mathcal{E}(\varrho) = 0$ and $\mathcal{E}(\varrho^{\otimes a}) > 0$, for a certain number of copies a. Evidently, the number of states $\varrho^{\otimes a}$ necessary for the superactivation to occur depends on the quantifier of interest $\mathcal{E}(\varrho)$, on the quantum state ϱ (WATROUS, 2004; SHOR; THAPLIYAL, 2003), and may also depend on the measurement procedure as well (QUINTINO; HUBER, 2016) [2]. In this example we visualize the behavior of a 2-S function which presents superactivation for a copies. In our framework, $\mathbf{e} = (0, e_2)$, so:

$$E^{(N)}(\mathbf{e}) = \left(\frac{N^{\nu_1} - N^{\nu_2}}{a^{\nu_1} - a^{\nu_2}}\right) e_2 + O(3) \tag{3.23}$$

where $N = a^n$. Note that in this case $E^{(a^2)}(0, e_2) = y\ e_2$ is actually *not enough information* to determine parameters ν_1 and ν_2, because of $e_1 = 0$ the quantification of $\mathcal{E}$ for a^2 copies do not offer any information about x. So, additionally, one would need the quantity $E^{(a^3)}(0, e_2) = (x + y^2)e_2$, up to first order (3.8), to determine the number x; this way, equations (3.11) give the values of parameters ν_1 and ν_2 and the series is well determined.

3.3.4 One-Shot-Distillable Entanglement of a Mixture of Bell States

Entanglement is the most striking phenomenon in quantum theory and there are many ways to quantify it. The one shot distillation of entanglement of a bipartite state ϱ_{AB} (BUSCEMI; DATTA, 2010; FANG X. WANG; DUAN, 2019), for example, is related to the maximal dimension k of the maximally entangled state $|\psi\rangle = \frac{\sum |kk\rangle}{\sqrt{k}}$ that can be obtained from $\varrho^{\otimes N}$ via non-entangling operations for N finite (for details see (BUSCEMI; DATTA, 2010)). This quantity, here we denote it by $\mathcal{E}_{OSD}(\varrho_{Bell}^{\otimes N})$, has been analytically determined for the family of mixtures of Bell diagonal states ($\varrho_{\text{Bell}} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|\Psi^-\rangle\langle\Psi^-|$ with $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, the maximally entangled states in 2 dimensions):

$$\mathcal{E}_{OSD}(\varrho_{Bell}^{\otimes N}) = N(1 - h(p)) + \sqrt{Np(1-p)}\left|\ln\left(\frac{1-p}{p}\right)\right|\Phi^{-1}(\epsilon) + O(\ln N) \tag{3.24}$$

where ϵ is the associated error and Φ^{-1} is the cumulative distribution (FANG X. WANG; DUAN, 2019). Considering $N > M > L$ large numbers we can neglect the $O(\ln N)$ contribution and it is possible to express (3.24) in terms of quantities e_1 and e_2 as:

[2] For example, the quantum steering, throught projective measurements, is superactivated with $a = 7$ copies for dimension $d = 2$ while it is superactivated with only $a = 2$ copies for dimension $d \geq 6$ (QUINTINO; HUBER, 2016).

$$E_{OSD}^{(N)}(e_1, e_2) = \frac{\sqrt{N}}{\sqrt{M} - \sqrt{L}} \left[\left(\frac{\sqrt{M} - \sqrt{N}}{\sqrt{L}} \right) e_1 + \left(\frac{\sqrt{N} - \sqrt{L}}{\sqrt{M}} \right) e_2 \right] \tag{3.25}$$

with $\mathcal{E}_{OSD}(\varrho_{Bell}^{\otimes L}) = e_1$ and $\mathcal{E}_{OSD}(\varrho_{Bell}^{\otimes M}) = e_2$ (note that if we take $N = L$ and $N = M$ we get e_1 and e_2 respectively, as it should be), for any fixed M and L and arbitrary, but large, N. It has been proven that "any quantum measure that can be expressed as $\mathcal{E}(\varrho^{\otimes N}) = FN + G\sqrt{N} + O(\ln N)$, where F and G depend on the state ϱ and on fixed parameters, is a 2-S function up to logarithmic order in the limit of a large but finite number of copies N"[3] (PARISIO, 2020).

To get equation (3.25) from the 2-S relation (3.4) one would have to consider $e_1 = \mathcal{E}(\varrho^{\otimes L})$ and $M = aK = aL$ (with $L \in \mathbf{P}_a$), but actualy (PARISIO, 2020) states that the q-scalability relation (3.1) is also valid for the set of integer powers of a multiplied by an integer b:

$$\mathbf{P}_a^b = \{b, ba, ba^2, ...\}. \tag{3.26}$$

This is easy to understand because what realy matters for the mathematical derivation of q-S relations is the ratio between these numbers and b will always be canceled out. With (3.26) one can simply make $b = L$ and $M = aL$ (so $M, N \in \mathbf{P}_a^L$), so that the ratio $\frac{a}{1} = \frac{M}{L}$ is maintained.

$$E^{(N)}(\mathbf{e}) = E^{(N/K)}[E^{(K)}(\mathbf{e}), E^{(aK)}(\mathbf{e})]$$

$$\downarrow$$

$$E^{(N)}(\mathbf{e}) \;\; = \;\; E^{(N/K)}[E^{(LK)}(\mathbf{e}), E^{(MK)}(\mathbf{e})]$$

As demonstrated in (PARISIO, 2020), (3.25) satisfies this constraint (evidently, if L is large M is also). Equation (3.25) is exactly our solution (3.22) for a 2-S function with parameters $(\nu_1, \nu_2) = (\frac{1}{2}, 1)$, thus the coefficient that multiplies e_1 (e_2) is the variable x (y). Also, since $N > M$, the term proportional to e_1 gives a negative contribution to $\mathcal{E}_{OSD}(\varrho_{Bell}^{\otimes N})$, which is exactly the mandatory negativity of the variable x (3.15).

[3] This constitutes a proof of principle, showing that a complex figure of merit as the OSD entanglement is a 2-S quantity for the whole family of Bell-diagonal states in the large-number regime. In fact, the details of the factors multiplying the terms in N and $\sqrt{N}$ are irrelevant to the proof (completely analogous results hold for arbitrary pure states (FANG X. WANG; DUAN, 2019)).

4 GENERAL q-S LINEAR SOLUTION

"A equação me propõe
Computador me resolve"

– Os Mutantes / Tom Zé, *Dois Mil e Um*

Is there a general form for a q-scalable function? In this chapter we show that it is possible to generalize the 1st order recurrence relations structure for the general q-S case. We present here a method based on linear algebra concepts to solve the general form of the first order structure for a q-scalable quantity under some circunstances determinated by our framework.

The definition of scalability can be seen as a generalization of additivity in two independent ways. For additive functions we simply have $E^{(N)}(e_1) = Ne_1$, that is, a dependence on the single real variable $e_1 = \mathcal{E}(\varrho)$ and linearity. For a general q-scalable quantity, we may have a dependence with several real variables $(e_1, e_2, \ldots, e_q) = \mathbf{e}$ and also nonlinear behaviors. It is instructive to lift each of these constraints separately, i. e., (i) to allow for nonlinear dependences in the single variable e_1, as we did in chapter 1, and (ii) to consider only linear functions of the several variables $(e_1, e_2, \ldots, e_q)$:

$$E^{(N)}(\mathbf{e}) = \eta^1(N)e_1 + \cdots + \eta^q(N)e_q = \boldsymbol{\eta}_N \cdot \mathbf{e}.$$

This would constitute a direct generalization of the primitive notion of additivity ($E^{(N)}(e_1) = Ne_1$), where $\boldsymbol{\eta}_N \equiv (\eta^1(N), \eta^2(N), \ldots, \eta^q(N))$ and $\mathbf{e}$ are cartesian vectors in $\mathbb{R}^q$. Therefore, a q-S function has the form:

$$E^{(N)}(\mathbf{e}) = \sum_{\ell=1}^{q} \eta^\ell(N)e_\ell, \tag{4.1}$$

up to linear order, and the goal of this chapter is to determine the coefficients $\eta^\ell(N)$.

4.1 GENERALIZING FIRST ORDER RECURRENCES

Does (4.1) constitute a q-scalable function? This question leads to the kind of recurrence relations that we referred to. Remember that, by our requirements in section **2.1**, $E^{(a^\ell)}(\mathbf{e}) = \mathbf{e}_{\ell+1}$ so, up to linear order (4.1), theorem 2.1 reads:

$$\begin{aligned} E^{(N)}(\mathbf{e}) &= \sum_{\ell=1}^{q} \eta^\ell(N/K) E^{(a^{\ell-1}K)}(\mathbf{e}) \\ &= \sum_{\ell=1}^{q} \eta^\ell(N/K) \sum_{j=1}^{q} \eta^j(a^{\ell-1}K) e_j, \end{aligned} \tag{4.2}$$

which is equivalent to:

$$E^{(N)}(\mathbf{e}) = \sum_{j=1}^{q} \left(\eta^j(a^{\ell-1}K) \sum_{\ell=1}^{q} \eta^\ell(N/K) \right) e_j. \tag{4.3}$$

Comparing equations (4.1) and (4.3) we get:

$$\eta^j(N) = \eta^j(a^{\ell-1}K) \sum_{\ell=1}^{q} \eta^\ell(N/K). \tag{4.4}$$

Therefore, expression (4.1) represents a q-scalable function if (4.4) is satisfied. In the simplest case we have $q = 1$ with $j = 1$ and then $\eta^1(N) = \eta^1(K)\eta^1(N/K)$.

In order to obtain the general recurrence it suffices to set $K = a$. As before, we change notation to $\eta^j(N) = \eta^j_n$, where $n = \log_a N$, and then we obtain:

$$\eta^j_n = \sum_{\ell=1}^{q} \eta^j_\ell \eta^\ell_{n-1}. \tag{4.5}$$

Now note that, by our requirements, $\eta^1_\ell = 0$ except for $\ell = q$ (which we called x in previous sections) and $\eta^2_\ell = 0$ except for $\ell = 1$ (a copies) and for $\ell = q$ (which we have called y in previous sections). This will go on and on and we may write:

$$\eta^j_\ell = \underbrace{\delta^{j-1}_\ell}_{\text{for } \ell < q} + \delta^q_\ell \eta^j_q. \tag{4.6}$$

Recurrence relations (4.6) states that every coefficient is null except if it fulfills our requirements that $E^{(a^\ell)}(\mathbf{e}) = \mathbf{e}_{\ell+1}$, and that for a^q copies we have the new values $\eta^1_q(a^q) = x$, $\eta^2_q(a^q) = y$ etc: so the delta symbols are labeled by an index $\ell < q$ or for $\ell = q$. Substituting (4.6) into (4.5), we find the general 1st order recurrences:

$$\begin{aligned} \eta^j_n &= \sum_{\ell=1}^{q} \left(\underbrace{\delta^{j-1}_\ell}_{\text{for } \ell < q} + \delta^q_\ell \eta^j_q \right) \eta^\ell_{n-1} \\ &= \eta^{j-1}_{n-1} + \eta^q_{n-1} \eta^j_q. \end{aligned} \tag{4.7}$$

4.1.1 Decoupled q-S Recurrence Relations

Recurrences (4.7) can be easily decoupled and solved with some software. Note that the 1st equation in (4.7) must be substituted on the 2nd and the latter will be substituted in the 3rd and so on. The only way to decouple all equations is to begin from the last one η_n^q, so by making $m = q$:

$$\begin{aligned}
\eta_n^q &= \eta_{n-2}^{q-2} + \eta_{n-2}^q \eta_q^{q-1} + \eta_{n-1}^q \eta_q^q \\
&= \eta_{n-3}^{q-3} + \eta_{n-3}^q \eta_q^{q-2} + \eta_{n-2}^q \eta_q^{q-1} + \eta_{n-1}^q \eta_q^q \\
&\vdots \\
\eta_n^q &= \sum_{k=1}^{q} \eta_q^{q+1-k} \eta_{n-k}^q. \qquad (4.8)
\end{aligned}$$

The pattern forms a sum which ends at q, where η_q^{q+1-k} are the 1st order coefficients in $\mathcal{E}(\varrho^{\otimes q})$. With the decoupled recurrence for the higher-order term η_n^q (4.8), we can see that all coefficients in the 1st order expansion of a q-S function obeys the same relation. Once we know that the higher-order term η_n^q obeys (4.8), we can "go downstairs" (the procedure can be made for η_n^2, η_n^3 and so on):

$$\begin{aligned}
\eta_n^1 &= \eta_q^1 \eta_{n-1}^q = \eta_q^1 \sum_{k=1}^{q} \eta_q^{q+1-k} \eta_{n-1-k}^q \\
&= \sum_{k=1}^{q} \eta_q^{q+1-k} \left(\eta_q^1 \eta_{n-1-k}^q \right) = \sum_{k=1}^{q} \eta_q^{q+1-k} \eta_{n-k}^1.
\end{aligned}$$

So we can state that *the recurrence relations for 1st order coefficients of a q-S function expansion depends on the last q steps*:

$$\eta_n^l = \sum_{k=1}^{q} \eta_q^{q+1-k} \eta_{n-k}^l, \qquad (4.9)$$

with $l \leq q$. Evidently, for $q = 1$ we get the recurrence $\eta_n^1 = \eta_1^1 \eta_{n-1}^1 = x\eta_{n-1}^1$, where $x = d_1(a)$ as in (2.22). While for $q = 2$ we have a sum of 2 terms: $\eta_n^l = \eta_2^2 \eta_{n-1}^l + \eta_2^1 \eta_{n-2}^l = y\eta_{n-1}^l + x\eta_{n-2}^l$ (with $x = d_{10}(a^2)$ and $y = d_{01}(a^2)$), giving the hybrid Fibonacci polynomials recurrence for each coefficient $l = 1, 2$.

- 3-Scalability: we can easily find the 1st order recurrences for the 3-S case:

$$\eta_n^l = \eta_3^3 \eta_{n-1}^l + \eta_3^2 \eta_{n-2}^l + \eta_3^1 \eta_{n-3}^l = z\eta_{n-1}^l + y\eta_{n-2}^l + x\eta_{n-3}^l, \qquad (4.10)$$

  for $l = 1, 2, 3$.

4.2 GENERAL SOLUTION

Equations (4.6) can also be transcripted into a matrix structure (4.11):

$$\begin{pmatrix} \eta_n^1 \\ \vdots \\ \eta_n^q \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \eta_q^1 \\ 1 & 0 & \dots & 0 & \eta_q^2 \\ 0 & 1 & \dots & 0 & \eta_q^3 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & \eta_q^q \end{pmatrix} \begin{pmatrix} \eta_{n-1}^1 \\ \vdots \\ \eta_{n-1}^q \end{pmatrix}. \tag{4.11}$$

Note that the first line is null except for the last term η_q^1, while the following lines form an $(q-1)\times(q-1)$ identity on the lower-left block and the last columm is constructed by the numbers η_q^m. Using $\boldsymbol{\eta}_n$ to represent the q-vector in (4.11), we have in a compact form:

$$\boldsymbol{\eta}_n = \mathcal{Q}\boldsymbol{\eta}_{n-1}. \tag{4.12}$$

- We may check this structure for the 2-S case and we will find the same coupled system of recurrence relations as in **3.2.1**:

$$\begin{pmatrix} \eta_n^1 \\ \eta_n^2 \end{pmatrix} = \begin{pmatrix} \eta_1^1 & \eta_2^1 \\ \eta_1^2 & \eta_2^2 \end{pmatrix} \begin{pmatrix} \eta_{n-1}^1 \\ \eta_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix} \begin{pmatrix} \eta_{n-1}^1 \\ \eta_{n-1}^2 \end{pmatrix}, \tag{4.13}$$

  which generates the hybrid Fibonacci polynomials (3.6).

We can find a basis that diagonalizes $\mathcal{Q}$ throught the operation $\boldsymbol{\eta}_n = D^{-1}\boldsymbol{\zeta}_n$ (the matrix D has dimension $q \times q$):

$$D^{-1}\boldsymbol{\zeta}_n = \mathcal{Q}D^{-1}\boldsymbol{\zeta}_{n-1},$$
$$\underbrace{DD^{-1}}_{\mathcal{I}}\boldsymbol{\zeta}_n = D\mathcal{Q}D^{-1}\boldsymbol{\zeta}_{n-1},$$

where $\boldsymbol{\eta}_n = D^{-1}\boldsymbol{\zeta}_n$ can be expanded as a linear combination $\sum_{k=1}^q C_k^m \zeta_n^k$ and the coefficients C_k^m can be easily determined. In this basis the solution is simply:

$$\zeta_n^k = (\lambda_k)^n, \tag{4.14}$$

where λ_k is the k-th eigenvalue of the diagonalized matrix $[D\mathcal{Q}D^{-1}]_{kl} = \lambda_k\delta_{kl}$. Then we have that the solution of recurrences (4.7) is:

$$\eta_n^m = \sum_{k=1}^q C_k^m \zeta_n^k = \sum_{k=1}^q C_k^m (\lambda_k)^n. \tag{4.15}$$

4.2.1 Eigenvalues and Coefficients Equations

The eigenvalues of the matrix (4.11) are the key ingredients of the method. Note that the eigenvalues of (4.13) are exactly the values we used in the change of variables (3.11). So we can generalize this step to:

$$\nu_k = \log_a \lambda_k, \tag{4.16}$$

where λ_k is the k-th eigenvalue of the matrix (4.11) (so k goes from 1 to q). Then the general solution, by making the change of variables (4.16) is, because $a^n = N$:

$$\eta_n^m = \sum_{k=1}^{q} C_k^m(\lambda_1, ..., \lambda_q)(\lambda_k)^n \qquad \rightarrow \qquad \eta^m(N) = \sum_{k=1}^{q} C_k^m(a^{\nu_1}, ...a^{\nu_q})N^{\nu_k}, \tag{4.17}$$

where the coefficients C_k^m are functions of the eigenvalues of (4.11) and they must satisfy the requirements $E^{(a^\ell)}(\mathbf{e}) = \mathbf{e}_{\ell+1}$. As each coefficient $\eta^m(a^l)$ is a sum (4.17) with coefficients $C_k^m(a^{\nu_1}, ..., a^{\nu_q})$ these conditions give rise to q systems of q coupled equations to be solved. After one solves these systems of equations the problem is finished.

Theorem 3.1: Let $\varrho \in \mathcal{B}(\mathcal{H})$, $N \in \mathbf{P}_a$ and $\mathcal{E}$ be a q-S function with respect to ϱ and $\mathbf{P}_a$. If $\mathbf{e} = (\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}), \ldots, \mathcal{E}(\varrho^{\otimes a^{q-1}}))$, then for $\varrho^{\otimes N}$:

$$E^{(N)}(\mathbf{e}) \;\; = \;\; \sum_{m=1}^{q}\sum_{k=1}^{q} C_k^m N^{\nu_k} e_m + O(2),$$

where $\nu_k = \log_a \lambda_k$, with λ_k being the k-th eigenvalue of the matrix (4.11) and C_k^m are coefficients determined by the systems of equations:

$$\sum_{k=1}^{q} C_k^m a^{\ell\nu_k} = \delta_\ell^{m-1} \quad , \quad \text{with} \;\; \ell = 0, ..., q-1.$$

We see that the form of a q-scalable function gets much more complicated as we increase q, the fact that there is a general solution to 1st order and a practical simple method is the main result of this chapter. The solutions of the system of equations determining coefficients C_k^m can be easily found with computational softwares.

In the next subsection we apply this method to revisit the 2-S case under this new mathematical framework and after that we solve the 3-S case (which would be an almost impossible task without this formalism).

4.2.1.1 2-Scalability Revisited

The eigenvalues of (4.11) for the 2-S case:

$$\mathcal{Q} = \begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix} \tag{4.18}$$

are $\lambda_1 = \frac{y+\sqrt{4x+y^2}}{2}$ and $\lambda_2 = \frac{y-\sqrt{4x+y^2}}{2}$. Then:

$$A^m(N) = C_1^m N^{\nu_1} + C_2^m N^{\nu_2}, \tag{4.19}$$

where $\nu_1 = \log_a \lambda_1$ and $\nu_2 = \log_a \lambda_2$. With the requirements that $\eta^1(1) = 1, \eta^1(a) = 0$ and $\eta^2(1) = 0, \eta^2(a) = 1$, we will have two systems of equations to solve:

$$\begin{cases} C_1^1 + C_2^1 &= 1 \\ C_1^1 a^{\nu_1} + C_2^1 a^{\nu_2} &= 0 \end{cases} \quad \rightarrow \quad C_1^1(a^{\nu_1}, a^{\nu_2}) = \frac{-a^{\nu_2}}{a^{\nu_1} - a^{\nu_2}} = 1 - C_2^1(a^{\nu_1}, a^{\nu_2}),$$

and:

$$\begin{cases} C_1^2 + C_2^2 &= 0 \\ C_1^2 a^{\nu_1} + C_2^2 a^{\nu_2} &= 1 \end{cases} \quad \rightarrow \quad C_1^2(a^{\nu_1}, a^{\nu_2}) = \frac{1}{a^{\nu_1} - a^{\nu_2}} = -C_2^2(a^{\nu_1}, a^{\nu_2}).$$

This solution is exactly the one we have found in (3.13).

4.3 3-SCALABILITY

Now we are looking for the general form of a scalable function with $\mathbf{e} = (e_1, e_2, e_3)$. Up to 1st order, this means that coefficients $d_{100}(N)$, $d_{010}(N)$ and $d_{001}(N)$ are to be determined:

$$E^{(N)}(\mathbf{e}) = \sum_{k,l,m} d_{klm}(N) e_1^k e_2^l e_3^m. \tag{4.20}$$

In this case the matrix (4.11) is:

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & x \\ 1 & 0 & y \\ 0 & 1 & z \end{pmatrix}. \tag{4.21}$$

One can find the three eigenvalues of this matrix and make the change of variables $\nu_k = \log_a \lambda_k$, $k = 1, 2, 3$ (these parameters are determined by the values $\eta^1(a^3) = x, \eta^2(a^3) = y$ and $\eta^3(a^3) = z$ respectively) and then write:

$$A^m(N) = C_1^m N^{\nu_1} + C_2^m N^{\nu_2} + C_3^m N^{\nu_3}. \tag{4.22}$$

Now we consider the requirements $\eta^1(1) = 1, \eta^1(a) = \eta^1(a^2) = 0$, $\eta^2(1) = \eta^2(a^2) = 0, \eta^2(a) = 1$ and $\eta^3(1) = \eta^3(a^2) = 0, \eta^3(a^2) = 1$. So there are three systems of equations to be solved:

$$\begin{cases} C_1^m + C_2^m + C_3^m &= \delta_0^{m-1} \\ C_1^m a^{\nu_1} + C_2^m a^{\nu_2} + C_3^m a^{\nu_3} &= \delta_1^{m-1} \\ C_1^m a^{2\nu_1} + C_2^m a^{2\nu_2} + C_3^m a^{2\nu_3} &= \delta_2^{m-1} \end{cases}.$$

These systems can be easily solved with some software, like mathematica. In these solutions we see that all three exponents must be different from each other $\nu_1 \neq \nu_2 \neq \nu_3$ and $\nu_3 \neq \nu_1$, for none of the coefficients to diverge.

$$\begin{cases} C_1^1 &= \frac{a^{\nu_2+\nu_3}}{(a^{\nu_1}-a^{\nu_2})(a^{\nu_1}-a^{\nu_3})} \\ C_2^1 &= \frac{a^{\nu_1+\nu_3}}{(a^{\nu_2}-a^{\nu_1})(a^{\nu_2}-a^{\nu_3})} \\ C_3^1 &= \frac{a^{\nu_1+\nu_2}}{(a^{\nu_3}-a^{\nu_1})(a^{\nu_3}-a^{\nu_2})} \end{cases} \quad \begin{cases} C_1^2 &= \frac{-(a^{\nu_2}+a^{\nu_3})}{(a^{\nu_1}-a^{\nu_2})(a^{\nu_1}-a^{\nu_3})} \\ C_2^2 &= \frac{-(a^{\nu_1}+a^{\nu_3})}{(a^{\nu_2}-a^{\nu_1})(a^{\nu_2}-a^{\nu_3})} \\ C_3^2 &= \frac{-(a^{\nu_1}+a^{\nu_2})}{(a^{\nu_3}-a^{\nu_1})(a^{\nu_3}-a^{\nu_2})} \end{cases} \quad \begin{cases} C_1^3 &= \frac{1}{(a^{\nu_1}-a^{\nu_2})(a^{\nu_1}-a^{\nu_3})} \\ C_2^3 &= \frac{1}{(a^{\nu_2}-a^{\nu_1})(a^{\nu_2}-a^{\nu_3})} \\ C_3^3 &= \frac{1}{(a^{\nu_3}-a^{\nu_1})(a^{\nu_3}-a^{\nu_2})} \end{cases}$$

Recombining the terms in (4.17), we can write the scaled function as $E^{(N)}(\mathbf{e}) = (C_1^1 e_1 + C_1^2 e_2 + C_1^3 e_3)N^{\nu_1} + (C_2^1 e_1 + C_2^2 e_2 + C_2^3 e_3)N^{\nu_2} + (C_3^1 e_1 + C_3^2 e_2 + C_3^3 e_3)N^{\nu_3}$. This simplifies the final expression because the coefficients C_l^m have the same denominator for the same l.

So we can state the following theorem:

Theorem 3.2: Let $\varrho \in \mathcal{B}(\mathcal{H})$ and $N \in \mathbf{P}_a$. If $\mathcal{E}$ is a 3-S function such that $\mathcal{E}(\varrho^{\otimes N}) = E^{(N)}$ depends on $\mathbf{e} = (e_1, e_2, e_3) = (\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes a}), \mathcal{E}(\varrho^{\otimes a^2}))$ and on n, where $E^{(a^3)}(\mathbf{e}) = x\ e_1 + y\ e_2 + z\ e_3 + O(2)$, with x, y and z known, then, for $\mathbf{e}$ sufficiently small, we have, up to 1st order:

$$E^{(N)}(\mathbf{e}) = \frac{a^{\nu_2+\nu_3}e_1 - (a^{\nu_2} + a^{\nu_3})e_2 + e_3}{(a^{\nu_1} - a^{\nu_2})(a^{\nu_1} - a^{\nu_3})}N^{\nu_1} + \frac{a^{\nu_1+\nu_3}e_1 - (a^{\nu_1} + a^{\nu_3})e_2 + e_3}{(a^{\nu_2} - a^{\nu_1})(a^{\nu_2} - a^{\nu_3})}N^{\nu_2} + \frac{a^{\nu_1+\nu_2}e_1 - (a^{\nu_1} + a^{\nu_2})e_2 + e_3}{(a^{\nu_3} - a^{\nu_1})(a^{\nu_3} - a^{\nu_2})}N^{\nu_3},$$

where $\nu_k = \log_a \lambda_k$, $k = 1, 2, 3$, with λ_k being the k-th eigenvalue of matrix (4.21).

Of course, the regularized 3-S function $\frac{E^{(N)}(\mathbf{e})}{N}$ is bounded if $\nu_1, \nu_2, \nu_3 \leq 1$ and vanishes if $\nu_1, \nu_2, \nu_3 < 1$ (again we have symmetry throught $\nu_1 \leftrightarrow \nu_2$, $\nu_2 \leftrightarrow \nu_3$ or $\nu_1 \leftrightarrow \nu_3$).

In the next section we compare a 3-S function with computational data of a well known quantifier in literature, wich is the one-shot-distillable entanglement, for an N-fold isotropic state.

4.3.1 Comparison with Computational Data

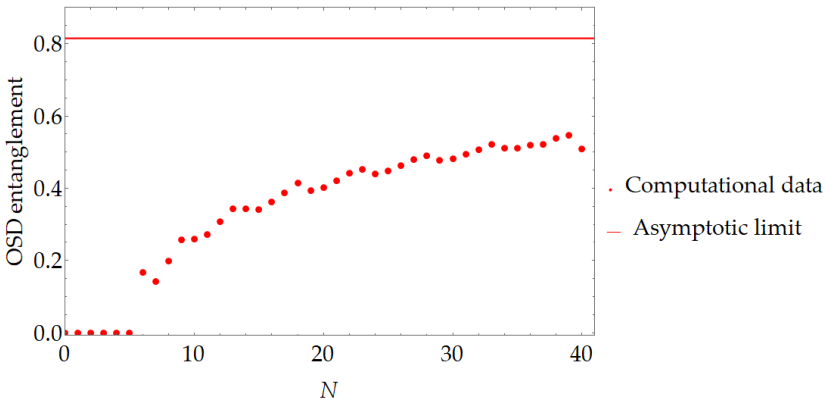
In this section we compare a 3-S function with the one-shot-distillable entanglement of an N-fold d dimensional isotropic state $\varrho_F^{\otimes N}$. The quantum state ϱ_F is defined as:

$$\varrho_F = F\Psi_d + (1 - F)\frac{\mathbf{I} - \Psi_d}{d^2 - 1}, \quad \text{with} \ \ 0 \leq F \leq 1, \tag{4.23}$$

where F is the fidelity of the quantum state ϱ_F with Ψ_d, the maximally entangled state in d dimensions. Differently from the case in section **3.3.4**, the one-shot-distillable entanglement of a d-dimensional isotropic state (4.23), which we denote as $\mathcal{E}_{OSD}(\varrho_F^{\otimes N})$, *does*

not have an analytical expression. However, the unitary symmetry of this family of states facilitates the optimization process on softwares for the avaliation of this figure of merit.

Figure 6 – One-shot-distillable entanglement of an N-fold 3D isotropic state $\varrho_F^{\otimes N}$ with $F = 0.9$.



Source: the author and Thiago Melo (2021).

It is mentioned in reference (FANG X. WANG; DUAN, 2019) that a suitable fit for this function contains terms in N, $\sqrt{N}$, $\ln N$ and an additive constant, so we study a 3-S function with parameters $(\nu_1, \nu_2, \nu_3) = (1, \frac{1}{2}, 0)$ to compare the accuracy of the scalability method in describing the quantifier $\mathcal{E}_{OSD}(\varrho_F^{\otimes N})$.

The first result of the computational code is that this quantity is superactivated with $a = 6$ copies when one takes $F = 0.9$ and $d = 3$ (for details about the optimization program and the computational data see appendix **C**). A 3-S function as in **theorem 3.2** which superactivates at a copies (so $\mathbf{e} = (0, e_2, e_3)$) with the aforementioned parameters ν_1, ν_2, ν_2 has the form:

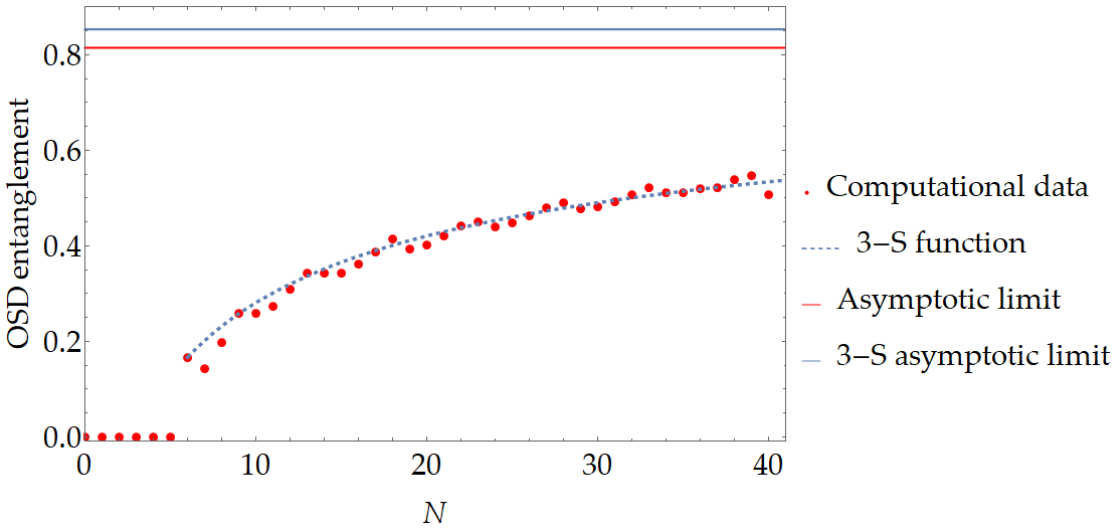
$$E^{(N)}(\mathbf{e}) = -\left[\frac{(\sqrt{a}+1)N}{(a-\sqrt{a})(a-1)} + \frac{(a+1)\sqrt{N}}{(\sqrt{a}-a)(\sqrt{a}-1)} + \frac{(a+\sqrt{a})}{(1-a)(1-\sqrt{a})}\right] e_2 \\ \left[\frac{N}{(a-\sqrt{a})(a-1)} + \frac{\sqrt{N}}{(\sqrt{a}-a)(\sqrt{a}-1)} + \frac{1}{(1-a)(1-\sqrt{a})}\right] e_3, \quad (4.24)$$

with $e_2 = \mathcal{E}(\varrho^{\otimes a})$ and $e_3 = \mathcal{E}(\varrho^{\otimes a^2})$. Naturally, we will use expression (4.24) for arbitrary N *as a fit curve of the points* $N \in \boldsymbol{P}_6$, the ones that actually satisfies recurrences (4.10). The asymptotic limit of the regularized part becomes simply:

$$\lim_{n\to\infty} \frac{E^{(a^n)}(\mathbf{e})}{a^n} = \frac{-(\sqrt{a}+1)e_2 + e_3}{(a-\sqrt{a})(a-1)}. \quad (4.25)$$

The computational data for the one-shot-distillable entanglement of an N-fold isotropic state is listed on Table 3 (see appendix **C**), where the values of $\mathcal{E}(\varrho^{\otimes 6})$ and $\mathcal{E}(\varrho^{\otimes 36})$ were used to plot (4.24) in Figure 7.

Figure 7 – Comparison of a 3-S function with the one-shot-distillable entanglement of an N-fold 3D isotropic state $\varrho_F^{\otimes N}$ with $F = 0.9$.



Source: the author and Thiago Melo (2021).

The asymptotic limit predicted by the 3-S form is ~ 0.85, while the actual result is ~ 0.81, an accuracy of $\sim 95\%$. This means that at some point, according to some error tolerance, the 3-S funtion will disagree with the computational data and thus one would need to consider an approximative iteration to correct this error. This can be an pathway to the development of an approximative method for quantum figures of merit in terms of the number of copies using q-scalability as an ingredient in some computational framework.

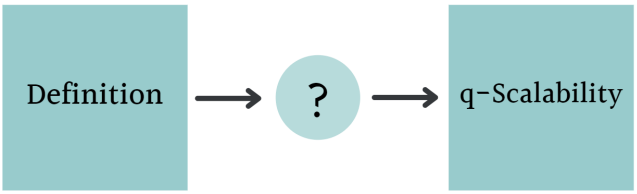
Note in this example that, as it was discussed in the introduction, the asymptotic regime may become dominant only for $N \to \infty$, which is impractical, so is important to be able to predict results for N large but finite.

5 CONCLUSIONS AND PERSPECTIVES

In this dissertation we have addressed the concept of scalability (PARISIO, 2020), which enabled us to study an interesting class of quantum functions by using combinatorics and elementary tools of analysis. The derived results can naturally be seen as a *generalization of the restrictive notion of additivity*. In the case of quantum coherence, for example, we showed that the way one quantifies this resource for $\varrho^{\otimes N}$ may, or not, imply a non-additive behavior for this quantity with the number of copies N (the relative entropy of coherence is additive while the l_1-norm is not).

The main results of this work are: (i) more orders of the 1-S series expansion, (ii) scalability properties of some coherence related quantifiers, (iii) the general linear solution of a q-S function for arbitrary q, and (iv) an approximation of the one-shot-distilable entanglement of an N-fold isotropic state by a scalable function.

Figure 8 – Can we demonstrate, at least in some cases of interest, that a quantum function is scalable without calculating its explicit form?



Source: Fernando Parisio (2021).

We point out that there is a limitation of the method concerning the certification that some function is scalable. One can calculate series coefficients and determine the expansion for N copies *once the scalability of the quantififer is shown to hold*. In the current stage of the development of the method, it would be an important advance to infer a quantifier's scalability only by knowing its definition.

Moreover, the fact that many different behaviors of quantum functions can be derived from one simple relation is an interesting description from a mathematical point of view. We saw that many types of quantifiers (additive, multiplicative, with binomial coefficients, with bounded coefficients) can have the same origin in a single equation.

A curious observation is that some results provided by the theory, which, at first, take place for $N \in \mathbf{P}_a$, are actually valid for arbitrary N (as is the case of the l_1-norm of coherence and the squared Hilbert-Schmidt norm of pure states). In particular, the comparison of a scalable function with the one-shot-distillable entanglement of an N-fold isotropic state (see Figure 7) is quite accurate.

Also we think that these concepts may be usefull for a computational procedure that could test functions for a certain number of copies of ϱ and extend it to many copies through an iterative procedure. This iterative task would search for a matching pattern to determine the quantum function's scalability. For example, if $\mathcal{E}(\varrho^{\otimes 2}) = 2\mathcal{E}(\varrho)$, the evaluation of $\mathcal{E}(\varrho^{\otimes 4})$ would fix the quantififer as a 1-S function if $\mathcal{E}(\varrho^{\otimes 4}) = 2\mathcal{E}(\varrho^{\otimes 2})$ and $x \neq 4 - 2y$ or would infer it as a $q > 1$ scalable function instead (defining $\mathcal{E}(\varrho^{\otimes 4})$ as a new element of a bigger vector $\mathbf{e} = (\mathcal{E}(\varrho), \mathcal{E}(\varrho^{\otimes 2}))$ for the following tests). Of course this is just a simple glance of how such computational procedure would work: as it was demonstrated, there are more ways for a quantum function to be 1-S beyond additivity and evidently this must be considered.

This computational method could be usefull to another potentially interesting approach, which is to consider the description of physical quantities via scalable functions as an approximative method. *Given that a certain quantifier is not scalable, how well can we approximate it via a scalable expression?* The squared Hilbert-Schmidt norm, not scalable on specific domains, is a good candidate for the development of an approximative method.

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APPENDIX A - ADDITIONAL RESULTS

A.1 3TH ORDER COEFFICIENTS SOLUTION

The recurrence relation for $d_3(N)$ is (2.20):

$$d_3(N) = d_1(N/a)d_3(a) + 2d_2(N/a)d_1(a)d_2(a) + d_3(N/a)[d_1(a)]^3.$$

Appyling the already known recurrence relations for $d_1(a^n)$ and $d_2(a^n)$:

$$d_3(a^n) = a^{\nu(n-1)}d_3(a) + 2[d_2(a)]^2 a^{\nu(n-1)}\left(\frac{a^{\nu(n-1)}-1}{a^\nu - 1}\right) + d_3(a^{n-1})a^{3\nu},$$

where we have to make the recursive substitution of $d_3(a^n)$ into itself and iterate $l = n-1$ times:

$$\begin{aligned} d_3(N) &= d_3(a)\left[a^{\nu(n-1)} + a^{\nu(n-2)}a^{3\nu} + \dots\right] + 2[d_2(a)]^2 \sum_{l=1}^{n-1}\left\{a^{\nu(n-l)}\left(\frac{a^{\nu(n-l)}-1}{a^\nu-1}\right)a^{3\nu(l-1)}\right\} \\ \dots &= d_3(a)\sum_{l=1}^{n} a^{\nu(n-l)}a^{3\nu(l-1)} + \frac{2[d_2(a)]^2}{a^\nu - 1}\left(\frac{N}{a^3}\right)^\nu \sum_{l=1}^{n-1} a^{2\nu l}\left(N^\nu a^{-\nu l} - 1\right). \end{aligned}$$

By substituting $a^n = N$. Redefining all sums to begin in $l = 0$:

$$d_3(N) = d_3(a)\left(\frac{N}{a^3}\right)^\nu\left(\sum_{l=0}^{n} a^{2\nu l} - 1\right) + \frac{2[d_2(a)]^2}{a^\nu - 1}\left(\frac{N}{a^3}\right)^\nu\left[N^\nu\left(\sum_{l=0}^{n-1} a^{\nu l} - 1\right) - \left(\sum_{l=0}^{n-1} a^{2\nu l} - 1\right)\right].$$

We may use the geometric series $\sum_{l=0}^{n-1} x^l = \frac{1-x^n}{1-x}$ to rewrite all the sums and then we get the third coefficient in expansion (2.24):

$$d_3(N) = d_3(a)\left(\frac{N}{a}\right)^\nu\left(\frac{1-N^{2\nu}}{1-a^{2\nu}}\right) + 2[d_2(a)]^2\left(\frac{N}{a^2}\right)^\nu \frac{(N^\nu-1)(N^\nu-a^\nu)}{(a^\nu-1)(a^{2\nu}-1)}.$$

A.2 4TH ORDER RECURRENCE RELATION

As for the 4th coefficient, we can determine it's recurrence relation and solve it by programing on mathematica language. Expanding (2.21):

$$\begin{aligned} d_4(N) &= d_1(N/K)\sum_i^1 \pi_i(4,1;K) + d_2(N/K)\sum_i^3 \pi_i(4,2;K) \\ &+ d_3(N/K)\sum_i^3 \pi_i(4,3;K) + d_4(N/K)\sum_i^1 \pi_i(4,4;K), \end{aligned}$$

and then, following the definition of $\pi_i(j,l;K)$, we get:

$$\begin{aligned} d_4(N) &= d_1(N/K)\pi_1(4,1;K) + d_2(N/K)[\pi_1(4,2;K) + \pi_2(4,2;K) + \pi_3(4,2;K)] \\ &+ d_3(N/K)[\pi_1(4,3;K) + \pi_2(4,3;K) + \pi_3(4,3;K)] + d_4(N/K)\pi_1(4,4;K). \end{aligned}$$

As the compositions for the number 4 are (1+3, 2+2, 3+1), (1+1+2, 1+2+1, 2+1+1) and, for last, (1+1+1+1):

$$\begin{aligned} d_4(N) &= d_1(N/K)d_4(K) + d_2(N/K)\{2d_1(K)d_3(K) + [d_2(K)]^2\} \\ &+ 3d_3(N/K)[d_1(K)]^2 d_2(K) + d_4(N/K)[d_1(K)]^4. \end{aligned} \tag{5.1}$$

Then if $d_3(K) = 0$ and $d_4(K) = 0$ we get the fourth recurrence for the two-coefficient case, explored in section **2.4.2**.

APPENDIX B - DEMONSTRATIONS

B.1 SCALABILITY OF THE l_1-NORM OF COHERENCE

According to the quantifier (2.38), N tensor products will have the following $l1$-norm of coherence:

$$\mathcal{C}_{l_1}(\varrho^{\otimes N}) = \sum_{\substack{i_1,...,i_N \\ j_1,...,j_N}}^{d^N} |\varrho_{i_1 j_1} ... \varrho_{i_N j_N}| - 1, \tag{5.2}$$

where d is the dimension of ϱ. Taking all the combinations of the indices, the sum of the products is the product of the sums by taking the indice l to vary from 0 to N:

$$\mathcal{C}_{l_1}(\varrho^{\otimes N}) = \left(\sum_{i_1 j_1}^{d} |\varrho_{i_1 j_1}| \times ... \times \sum_{i_N j_N}^{d} |\varrho_{i_N j_N}| \right) - 1 = \prod_{l=0}^{N} \sum_{i_l j_l}^{d} |\varrho_{i_l j_l}| - 1. \tag{5.3}$$

The productory in (5.3) is equivalent to simply multiply the sum N times:

$$\mathcal{C}_{l_1}(\varrho^{\otimes N}) = \left(\sum_{i,j}^{d} |\varrho_{ij}| \right)^N - 1 = \left(\sum_{i=j}^{d} |\varrho_{ij}| + \sum_{i \neq j}^{d} |\varrho_{ij}| \right)^N - 1,$$

where we break the sum into two parts: the trace, hence the unity, and the other is our coherence quantifier (2.38):

$$C_{l_1}^{(N)}(c_1) = (1 + c)^N - 1 = \sum_{k=1}^{N} \begin{pmatrix} N \\ k \end{pmatrix} {c_1}^k, \tag{5.4}$$

where $c_1 = \mathcal{C}_{l_1}(\varrho)$.

B.2 SCALABILITY OF THE SQUARED HILBERT-SCHMIDT NORM

In this section we show that the squared Hilbert-Schmidt norm is scalable only if the state ϱ is pure. With d being the dimension of ϱ and N being the number of copies, we start splitting the sum $\sum_{l\neq k}^{d} |\varrho_{lk}|^2$ for $\varrho^{\otimes N}$ in two parts:

$$\begin{aligned}\mathcal{C}_{l_2}(\varrho^{\otimes N}) &= \sum_{\substack{i_1\ldots i_N \\ j_1\ldots j_N}}^{d^N} |\varrho_{i_1 j_1}\ldots\varrho_{i_N j_N}|^2 - \sum_{i_1\ldots i_N}^{d^N} |\varrho_{i_1 i_1}\ldots\varrho_{i_N i_N}|^2 \\ &= \prod_{l=1}^{N}\sum_{i_l j_l}^{d} |\varrho_{i_l j_l}|^2 - \prod_{l=1}^{N}\sum_{i_l}^{d} |\varrho_{i_l i_l}|^2.\end{aligned}$$

As all matrices are equal, the product of the sums can de rewrited as the sums elevated to the Nth power:

$$\mathcal{C}_{l_2}(\varrho^{\otimes N}) = \left(\sum_{ij}^{d} |\varrho_{ij}|^2\right)^N - \left(\sum_{l}^{d} |\varrho_{ll}|^2\right)^N. \tag{5.5}$$

For qubits, the sum of the squared modulus of all elements is equal to $a^2+(1-a)^2+2|b|^2$, which is the same as $Tr\varrho^2$ (easy to check). The second sum is simply $a^2+(1-a)^2$, the difference between the trace of ϱ^2 and $2|b|^2 = c_1$, so:

$$C_{l_2}^{(N)}(c_1) = \left(Tr\varrho^2\right)^N - \left(Tr\varrho^2 - c_1\right)^N,$$

where $c_1 = \mathcal{C}_{l_2}(\varrho)$. Then the scalability test in section **2.5.2** states that the squared Hilbert-Schmidt norm is scalable only in the domain $Tr\varrho^2 = 1$.

APPENDIX C - COMPUTATIONAL DATA

This section was written and planned in collaboration with Thiago Melo [1], who wrote and ran the computer code and generated the computational data.

According to reference (FANG X. WANG; DUAN, 2019), for any N-fold isotropic state $\varrho_F^{\otimes N}$, where ϱ_F is:

$$\varrho_F = F\Psi_d + (1-F)\frac{\mathbf{I}-\Psi_d}{d^2-1}, \quad \text{with } \ 0 \geq F \leq 1,$$

the one-shot-distillable entanglement (under PPT[2] operations) $\mathcal{E}^{\epsilon}_{OSD}(\varrho_F^{\otimes N})$, within some error tolerance ϵ, can be solved by the following non-linear optimization program:

$$\mathcal{E}^{\epsilon}_{OSD}(\varrho_F^{\otimes N}) = \log \ \max \lfloor \frac{1}{\eta} \rfloor \tag{5.6}$$

$$0 \leq m_i \leq 1, \quad \forall i = 0, 1, \ldots, N \tag{5.7}$$

$$\sum_{i=0}^{N} \left(\begin{array}{c} N \\ i \end{array} \right) F^i (1-F)^{N-i} m_i \geq 1-\epsilon, \tag{5.8}$$

$$-\eta \leq \sum_{i=0}^{N} x_{i,k} m_i \leq \eta, \ \ \forall \ k = 0, 1, \ldots, N \tag{5.9}$$

with coefficients $x_{i,k}$ equal to:

$$x_{i,k} = \frac{1}{d^2} \sum_{m=max\{0,i+k-N\}}^{mini,k} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\begin{array}{c} N-k \\ i-m \end{array} \right) (-1)^{i-m} (d-1)^{k-m} (d+1)^{N-k+m-i}.$$

Equations (5.7 - 5.9) are constraints to be satisfied by the program and (5.6) is the objective function of the optimization. For the computational calculations we used the PuLP library for *Python*, where we ran equations (5.6 - 5.9) for $\epsilon = 0.001$, $d = 3$ and $F = 0.9$. The computational data were made for $N \in [1, 40]$ and the results are listed in Table 3. Note that superactivation occurs for $a = 6$ copies and the components of the vector $\mathbf{e} = (0, e_2, e_3)$ are $e_2 = \mathcal{E}(\varrho^{\otimes 6}) = 1$ and $e_3 = \mathcal{E}(\varrho^{\otimes 36}) \approx 18.678$.

[1] Departamento de Física, Universidade Federal de Pernambuco.
[2] Completely positive partial transpose preserving.

Table 3 – Data for the one-shot-distillable entanglement of an N-fold 3D isotropic state $\varrho_F^{\otimes N}$ with $F = 0.9$.

N	$\frac{E_{OSD}(\varrho_F^{\otimes N})}{N}$	N	$\frac{E_{OSD}(\varrho_F^{\otimes N})}{N}$
1	0	21	$0,4201603448$
2	0	22	$0,4414002924$
3	0	23	$0,4512773203$
4	0	24	$0,4396545109$
5	0	25	$0,4474940829$
6	$0,1666666667$	26	$0,4618895688$
7	$0,1428571429$	27	$0,4788819698$
8	$0,1981203126$	28	$0,4904875327$
9	$0,2579920105$	29	$0,4774746072$
10	$0,2584962501$	30	$0,4825075593$
11	$0,2727272727$	31	$0,4934465977$
12	$0,3083699765$	32	$0,5070306698$
13	$0,3430332014$	33	$0,5217379365$
14	$0,3433824944$	34	$0,5112409818$
15	$0,3419522011$	35	$0,5113460791$
16	$0,3613349821$	36	$0,5188596493$
17	$0,3873507353$	37	$0,5224750652$
18	$0,4134968609$	38	$0,5387927054$
19	$0,3943080577$	39	$0,546191108$
20	$0,4024924275$	40	$0,5081069313$

Source: Thiago Melo (2021).