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MARIANA CRISTINA DE LIMA

PERTURBATIONS IN THE KERR SPACETIME

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Dissertation presented to the graduation program of the Physics Department of Universidade Federal de Pernambuco as part of the duties to obtain the degree of Master of Science in Physics.

Concentration area: Theoretical and computational physics.

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Espesso,
porque é mais espessa
a vida que se luta
cada dia,
o dia que se adquire
cada dia
(como uma ave
que vai cada segundo
conquistando seu voo).
João Cabral de Melo Neto, O rio [1].

Abstract

In the present dissertation we study s-spin weight field perturbations in the Kerr background. Such perturbations are described by an equation called Teukolsky master equation (TME). Here we show how to obtain the TME and study it. Then, in order to obtain the *quasi-normal modes*, that is, the possible frequencies of the field between the outer horizon and infinity, we discuss the continued fraction method (known as Leaver's method in such context), which consists in writing a series expansion for the angular and radial equations derived from the TME. It follows that the coefficients of the series expansions obey a *three-term recurrence relation* (TTRR). For this reason, we study properties of TTRR to make clear the Leaver's method, and then to obtain numerically the frequencies. In the last chapter we discuss the Kerr scattering problem and explain the Penrose process in order to make a link with black hole thermodynamics. Lastly, we address superradiance, which is directly related with the Hawking's area theorem.

Keywords: General relativity. Kerr black hole. Teukolsky master equation. Quasi-normal modes. Continued fraction method. Scattering.

Resumo

Nesta dissertação é feito um estudo sobre perturbações de campos de peso de spin s no background de Kerr. Tais perturbações são descritas por uma equação chamada equação master de Teukolsky (EMT). Aqui nós mostramos como obter a EMT e a estudamos. Em seguida, para obter os *modos quasi-normais*, isto é, as possíveis frequências do campo entre o horizonte externo e infinito, nós discutimos sobre o método de frações continuadas (conhecido como método de Leaver neste contexto), que consiste em escrever uma expansão em série para as equações angular e radial obtidas a partir da EMT. Por conseguinte, os coeficientes das expansões em série obedecem uma *relação de recorrência de três termos* (RRTT). Por isso nós estudamos propriedades das RRTT para deixar claro o método de Leaver e assim obter numericamente as frequências. No último capítulo nós discutimos o problema de espalhamento de Kerr e explicamos o processo de Penrose com o objetivo de relacionar com termodinâmica de buracos negros. Por fim, nós abordamos superradiância, que está diretamente relacionada com o teorema de área de Hawking.

Palavras-chave: Relatividade geral. Buraco negro de Kerr. Equação master de Teukolsky. Modos quasinormais. Método de frações continuadas. Espalhamento.

Contents

1 Introduction

In 1915, Albert Einstein published a series of four papers with the basis of the theory of *General Relativity* [2, 3, 4, 5]. In the second of these papers, he presented what we call today the Einstein's equation[1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.2}$$

The equation above represents a coupled system of second order partial differential equations for the components of the metric, then it is anything but straightforward to find solutions of (1.2). However, just a few months after the publication of Einstein's papers, making use of symmetries, Schwarzschild published the first exact solution of equation (1.2) [6]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.3}$$

The metric above does not depend on t, then we say it describes a static spacetime. Besides this, in the isometry group of such spacetime there exists a subgroup which is isomorphic to the $SO(3)$, and then we say that (1.3) is a spherically symmetric solution. Such symmetry can easily be seen by fixing t and r: the result is the metric of the sphere. The Birkhoff's theorem [11, 7] states that the Schwarzschild metric is the most general stationary spherically symmetric solution of Einstein's equation in vacuum, and then, any other solution with such characteristics corresponds to a portion of the Schwarzschild spacetime.

The first thing that stands out is that the first term of (1.3) diverges when $r = 0$, and the second term diverges when $r = 2M$. As it turns out to be $r = 0$ is a singularity of the spacetime described by (1.3), while $r = 2M$ is just a problem of the coordinate system, that is, the coordinate system which is being used is not useful to describe such region. We can check if these singularities are real or not by evaluating the scalar $R_{abcd}R^{abcd}$. Such scalar diverges for $r = 0$, but not for $r = 2M$. The $r = 2M$, called event horizon, is what defines such spacetime as a "black hole": once that something crosses the event horizon it can not come back, and nothing from inside get out, even light.

As it is well known, sometimes divergences are not well received in physics. In 1939, Einstein wrote in a paper [8]:

"The essential result of this investigation is a clear understanding as to why the 'Schwarzschild singularities' do not exist in physical reality."

That is, even Einstein did not accept (at least in the beginning) the existence of a singularity predicted by his own theory. But later the Schwarzschild solution showed to be much more than a "bizarre" solution of a complicated equation. Today we know that

[1] As a matter of fact, in Einstein's paper the form of the equation is

$$G_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{1.1}$$

black holes do exist. More than that, we know there is a black hole at the center of our galaxy and there are indications that may exist a supermassive black hole at the center of every galaxy which has a bulge component [9].

In 1963, the Kerr solution was published. Such solution is basically the spinning version of the Schwarzschild black hole. While the Schwarzschild solution can be used as an approximation for a nonspinning star, the Kerr solution can be used for a rotating massive body. Once that the Kerr solution represents something spinning, then there exists a preferential axis, the axis of rotation. Therefore, the Kerr solution is not spherically symmetric. Because the Schwarzschild spacetime possesses more symmetries, it is expected to be easier (but not necessarily easy) to deal with, and this is indeed what happens. While Schwarzschild spacetime is characterized by only one parameter, the mass M, the Kerr spacetime is characterized by two, the mass M, and the angular momentum J.

Why study black hole perturbations? Is it not enough to have a solution of Einstein's equation? The answer for these questions is very simple: the Kerr solution, for example, describes an isolated rotating black hole, it is like having a universe with a black hole and nothing else. But, in general, as it happens in our galaxy, there is a distribution of mass around a black hole (BH). Nevertheless, let us suppose there exists an isolated black hole in the sense that such BH is far enough from other bodies to be considered isolated. Then, even in this case, it would happen a perturbation because the BH would interact with the vacuum creating pairs of particles, and then it would evaporate due to Hawking radiation [10].

In the present thesis, our goal is to study perturbations of different types of fields in the Kerr background. Given a metric g which is a solution of Einstein's equation (1.2), how can we obtain the linearized perturbation equations for a field in such background? We can replace the metric $g_{\mu\nu}$ in Einstein's equation by itself plus a perturbative part $h_{\mu\nu}$, that is, make $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, keeping only terms up to first order in $h_{\mu\nu}$ [12]. This method was used by some authors to obtain linearized equations for field perturbation in the Schwarzschild background. Then, can we try to use the same method for the Kerr background? Yes, we can. Should we do it? No, we should not. When we try to approach the problem in this way, what follows is that, because the Kerr spacetime is not spherically symmetric, we arrived in some partial differential equations which are not separable in the variables r and θ.

However, there is an alternative way to deal with Kerr. In 1962, Newman and Penrose proposed a formalism which uses the *tetrad formalism* with a special choice[2]. This formalism turns out to be very interesting to be used for metrics which are of type D in the Petrov classification[3], as the Kerr metric is. And this was the approach used by Teukolsky. The Newman-Penrose formalism was crucial for Teukolsky to obtain an equation which describes the dynamics of a s-spin field perturbation in the Kerr background which is separable, the *Teukolsky master equation* (TME).

The main purpose of the first chapter is to show how to derive the Teukolsky master equation, explain what such equation describe, and to extract some information from it. Before doing that, in section 2.1 we are going to discuss some characteristics of the Kerr spacetime. In this first part of the chapter we are also going to discuss briefly black holes thermodynamics. Once that the main properties of the Kerr spacetime were exposed in the first section, then, in section 2.2, we explain the Newman-Penrose formalism and other

[2]The description of such formalism is given in the next chapter.

[3]The Petrov classification is an important way to classify solutions of Einstein's equation. Such classification will be briefly discussed in chapter 1.

tools in order to show how to derive the master equation. A big part of the process is made without choosing a particular tetrad frame, nor even the metric, but considering that the spacetime is of type D. In this section we will explain better some concepts which were exposed here without details, such as Petrov classification and tetrad formalism, for example. Finally, after having written the TME at the end of section 2.2, we finish the chapter with the section 2.3 making an analysis of the TME. In this last section we show that the Teukolsky master equation is separable, giving rise to the *Teukolsky angular equation* and the *Teukolsky radial equation*. These equations will be fully explored in the following chapter.

In the second chapter the primary purpose is to evaluate quasi-normal modes (QNMs) of different types of field perturbations in the Kerr background. In order to do that, we need to explore the angular and radial equations which were found in chapter 1. Because of the nature of the singular points of such equations, we know that we can put them in the form of a confluent Heun equation (CHE), a special second order differential equation. Then, in section 3.4 we present the CHE and study it by writing power series solutions. When we do it, we arrive at a recurrence relation with three terms for the coefficients of the series solutions. Then, we use the section 3.2 to discuss some properties of linear homogeneous *three-term recurrence relations* and introduce the so-called *continued fractions*. The main result of this section is the Pincherle theorem, which will ensure the convergence of the continued fractions. Thereafter, in section 3.3 we write the angular and the radial equations in the CHE form and apply the results from sections 3.1 and 3.2, and then we obtain two continued fractions. At this point we have all we need to obtain *numerically* the quasi-normal modes. Lastly, in section 3.4 we use all this to evaluate them.

To finish the present thesis, in chapter 4 we discuss the Kerr scattering problem, black hole thermodynamics, and superradiance. In section 4.1 we discuss the scattering problem focusing on bosonic fields. Thereafter, we discuss again black holes thermodynamics in section 4.2. However, this time we start explaining the Penrose process in order to make a connection with the first law of black holes thermodynamics. Then, we dedicate a short subsection to discuss the zeroth, second, and third laws. Finally, in the last part of section 4.2 we present superradiance as a direct consequence of the Hawking's area theorem.

2 The Teukolsky Master Equation

The Teukolsky master equation (TME) was first published by Teukolsky in 1973 in the first paper of a series of three [15, 16, 17]. There Teukolsky showed how to decouple and make separable the equations for three kinds of fields: electromagnetic, gravitational, and spin-1/2 fermionic field. The TME is a second order partial differential equation in four variables, t, r, θ, and ϕ, which describes the dynamics of a perturbation in a s-spin field in the Kerr background. In this chapter we will first briefly describe the Kerr metric, then we will use the Newman-Penrose formalism in order to show how to obtain the TME. Lastly, we will do a short analysis of the TME.

2.1 The Kerr Spacetime

"When I turned to Alfred Schild, who was still sitting in the armchair smoking away, and said 'Its rotating!' he was even more excited than I was. I do not remember how we celebrated, but celebrate we did!" Roy P. Kerr (2009)

Between the publication of the Einstein's equation and the publication of the Schwarzschild solution passed only two months. In contrast, the Kerr solution come only in 1963, that is, about 48 years later. In [19] one can find a pedagogical way to derive the Kerr metric.

The Kerr metric is a solution of Einstein's equation in vacuum, namely equation (1.2) for $T_{\mu\nu} = 0$. It is characterized by two parameters: the mass M, and the angular momentum J. In the original Kerr coordinates (v, r, θ, φ) the metric is written as

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)\left(dv + a\sin^2\theta d\varphi\right)^2 \\ &+2\left(dv + a\sin^2\theta d\varphi\right)\left(dr + a\sin^2\theta d\varphi\right) + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned} \tag{2.1}$$

where $\rho^2 = r^2 + (J/M)^2\cos^2\theta$, $a = J/M$ and we are using $c = G = 1$ [13]. And in terms of the *Boyer-Lindquist coordinates* we have

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \end{aligned} \tag{2.2}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$

and the coordinates t, r, θ, and ϕ are defined in the following intervals:

$$-\infty \leq t \leq \infty, \quad 0 \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad \text{and} \quad 0 \leq \phi \leq 2\pi.$$

The constants r_- and r_+ are the roots of $\Delta(r)$, $r_\pm = M \pm \sqrt{M^2 - a^2}$. They are the radial position of the so-called inner and outer horizons, respectively. The parameter a, defined by $a \equiv J/M$, is called *Kerr parameter* [20]. When $a \to 0$ we recover Schwarzschild solution.

The Kerr spacetime is asymptotically flat, stationary, and axisymmetric. It is easy to check the asymptotically flatness taking the limit $r \to \infty$ in (2.2): we obtain the Minkowski metric in spherical coordinates. It is stationary because (2.2) does not depend on t, and hence has a timelike Killing vector field[1] at infinity, ∂_t. The metric (2.2) does not depend on the cyclic coordinate ϕ, hence ∂_ϕ is a spacelike Killing vector at infinity whose integral curves are closed. To have this properties is the definition of a axisymmetric spacetime. We say that a spacetime is stationary and axisymmetric when it possesses these properties and the Killing vectors $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$ commute [12]. In addition to this, the Kerr metric is not invariant under time reversal, thus it is not a static spacetime. On the other hand, (2.2) does not change by the simultaneously transformations $t \to -t$ and $\phi \to -\phi$. Lastly, the metric is unchanged by the transformation $\theta \to \pi - \theta$, that is, by a reflection in the plane $\theta = \pi/2$.

When a metric seems to have singularities we need to analyze if such singularities are "real singularities" or, otherwise, a problem with the coordinate system which is being used. In the case of the Kerr metric in the Boyer-Lindiquist coordinates, what follows is that $r = 0$ (with $\theta = \pi/2$) is a true singularity, that is, a "point" of infinite curvature which does not disappear by any possible change of coordinate system, while r_- and r_+ are coordinate singularities. Note that the metric in the original Kerr coordinates (2.1) is regular at r_- and r_+, which means that these "points" are *coordinate singularities*.

The outer horizon is the one which, let us say, define the Kerr spacetime as a black hole. Once that something passes through r_+, it can not come back. From $r_+ = M + \sqrt{M^2 - a^2}$ we can see that r_+ exists whenever $M^2 \geq a^2$. The case $a = M$ is called *extreme Kerr black hole*.

The Kerr metric admits two Killing vectors, as it was mentioned before, and, in addition, it possesses a nontrivial rank-2 Killing tensor[2], that is, a Killing tensor which is not built from combinations of the Killing vectors and the metric tensor. Besides, the metric itself is another rank-2 Killing tensor and, therefore, the Kerr spacetime possesses enough symmetries to make the geodesic equations completely integrable, in the sense that we can obtain the geodesic equations.

To end this section, we are going to discuss a little bit about the thermodynamics of the Kerr black hole. In 1971, Penrose and Floyd noted that in the Penrose process[3] the surface area of the Kerr black hole increases. Later, in March 1971, Hawking published a paper with today's so-called *Hawking's area theorem*, which states that the surface area of any black hole never decreases.

The area of the outer horizon is given by

$$A = 8\pi M r_+,$$

[1] We say that v^μ is a Killing vector field if it satisfies

$$\nabla_{(\mu} v_{\nu)} = 0.$$

[2] Analogously, we say that T is a rank-k Killing tensor if it satisfies [11]

$$\nabla_{(\mu} T_{\nu_1 \nu_2 \ldots \nu_k)} = 0.$$

[3] For an explanation about the Penrose process see [11, 20, 12].

then, replacing r_+ by $M + \sqrt{M^2 - a^2}$, a by J/M, and differentiating, we obtain

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J, \tag{2.3}$$

where

$$\kappa = \frac{r_+ - M}{2Mr_+} \qquad \text{and} \qquad \Omega = \frac{a}{2Mr_+}$$

are the surface gravity[4] and the angular velocity of the outer horizon, respectively. The equation (2.3) and the first law of thermodynamics are similar, and this was noted by Bekenstein and Smarr. In 1972 the Bekenstein's professor, John Weeler, asked him what would happen if he dropped his cup of tea in a black hole [21]. Such question culminated with Bekenstein's proposal that a black hole should have an entropy of the form

$$S = \eta\frac{A}{\hbar G}. \tag{2.4}$$

In 1973, Bardeen, Carter and Hawking published a paper titled "The four laws of black holes mechanics" [22], where they stated what the title propose. However, in this paper they argued that $\kappa/8\pi$ and A are not the temperature and the entropy of the black hole because the temperature should be zero. How can something that does not irradiate have a nonzero temperature? The outlook changed when in 1974 Hawking published a paper [27] where he showed that when quantum effects are considered, a black hole can radiate with a black body temperature given by

$$T = \frac{\hbar\kappa}{2\pi}. \tag{2.5}$$

Now, using the Bekenstein proposal (2.4) and the expression for the temperature (2.5) found by Hawking, and assuming that the first term in right hand side in (2.3) must be $T\delta S$, one finds[5] $\eta = 1/4$. Therefore,

$$S = \frac{A}{4\hbar}.$$

We are going to discuss more about black holes thermodynamics in the last chapter.

2.2 How to derive the TME

In 1973, Saul A. Teukolsky was a second-year graduate student, under the guidance of Kip Thorne, studying the field perturbation in the Kerr background problem [19]. He used the Newman-Penrose formalism and was able to decouple some equations, but the variables r and θ did not separate. About six months later he had an idea and finally could obtain separated equations.

Here we are going to discuss about how Teukolsky achieved the TME.

In order to obtain the Teukolsky master equation we are going to use the so-called *tetrad formalism.* Such formalism consists into using a local coordinate basis and write

[4]The surface gravity is defined by

$$\nabla^a(\xi^b\xi_b) = -2\kappa\xi^a,$$

where ξ^a is a Killing field normal to a Killing horizon $\mathcal{K}$ [47].

[5]Obviously, this is not enough to be sure that η must be $\frac{1}{4}$.

the relevant quantities of the problem in this basis [23]. An important advantage in using it is that the resulting equations do not depend on covariant derivatives. How to choose a good *tetrad basis* will depend on the symmetries of the problem.

Then, firstly in each point of the spacetime we have a four dimensional basis $\{\boldsymbol{e}_{(a)}\}$, $a = 1,2,3,4$. The contravariant vectors we obtain via

$$e_{(a)}{}^{\mu} = g^{\mu\nu} e_{(a)\nu},$$

where g is the metric tensor. We automatically define the constant matrix

$$\eta_{ab} = e_{(a)}{}^{\mu} e_{(b)\mu}.$$

When $\{\boldsymbol{e}_{(a)}\}$ is an orthonormal basis, $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$.

Once we have a tetrad we can project the quantities of the problem in the tetrad frame. For example, a rank-2 tensor $T^{\mu\nu}$ is projected as

$$T_{ab} = e_{(a)}{}^{\mu} e_{(b)}{}^{\nu} T_{\mu\nu},$$

where $e_{(a)}{}^{\mu} = e_{(a)} dx^{\mu}$. Following Chandrasekhar [23] we can see how the Ricci and the Weyl tensors are decomposed, and how the Bianchi identities are written using a tetrad.

Now we use the Newman-Penrose (NP) formalism to evaluate the spin coefficients and the Weyl scalars. The definitions of these objects are coming soon. Let us first explain what is the NP formalism: the NP formalism consists in to writing the metric in a null basis, also called tetrad. So, we first write the metric in an orthonormal basis $\{\boldsymbol{e}_{(i)}\}$:

$$g = \eta^{ij}\, \boldsymbol{e}_{(i)} \otimes \boldsymbol{e}_{(j)},$$

$$\boldsymbol{e}_{(i)} \cdot \boldsymbol{e}_{(j)} = \boldsymbol{e}_{(i)}{}^{\mu} \boldsymbol{e}_{(j)}{}^{\nu} g_{\mu\nu} = \eta_{ij},$$
$$\eta_{ij} = \eta^{ij} = \mathrm{diag}(-,+,+,+),$$

where $g_{\mu\nu}$ is the metric in the coordinate basis $\{\partial_\mu\}$, and $\boldsymbol{e}_{(i)}$ means $\boldsymbol{e}_{(i)}{}^{\mu}\partial_\mu$.

Once that we choose an orthonormal basis, a possible way to construct a tetrad $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ is, for example,

$$\begin{aligned} \boldsymbol{l} &= \frac{\boldsymbol{e}_0 + \boldsymbol{e}_3}{\sqrt{2}}, & \boldsymbol{n} &= \frac{\boldsymbol{e}_0 - \boldsymbol{e}_3}{\sqrt{2}}, \\ \boldsymbol{m} &= \frac{\boldsymbol{e}_0 + i\boldsymbol{e}_2}{\sqrt{2}}, & \bar{\boldsymbol{m}} &= \frac{\boldsymbol{e}_0 - i\boldsymbol{e}_3}{\sqrt{2}}. \end{aligned} \tag{2.6}$$

By the definition above it is easy to check that

$$\boldsymbol{l} \cdot \boldsymbol{n} = -1, \qquad \boldsymbol{m} \cdot \bar{\boldsymbol{m}} = 1,$$

and all the other products between them are zero, as null basis must be. Then, the components of the metric in this frame are given by

$$\eta_{ij} = \eta^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

There exists an infinite number of possible null basis, however we know the Kerr family of solutions, that is, the stationary axisymmetric asymptotically flat solutions of

Einstein's equation in vaccum, are of type D[6]. This means that there exists a tetrad such that all the Weyl scalars vanish, but Ψ_2. Such basis is the one which has $\boldsymbol{l}$ and $\boldsymbol{n}$ aligned with the principal null directions of the Weyl tensor. Here we are not going to *look for* such basis. Instead, we will just provide it later. Then, from now on we will use our tetrad denoting

$$\boldsymbol{e}_1 = \boldsymbol{l}, \qquad \boldsymbol{e}_2 = \boldsymbol{n}, \qquad \boldsymbol{e}_3 = \boldsymbol{m}, \qquad \boldsymbol{e}_4 = \bar{\boldsymbol{m}},$$

where the $\boldsymbol{e}_i$ above are not the same as those in (2.6). Using η^{ij} we have

$$\begin{aligned} \boldsymbol{e}_1 &= \boldsymbol{l} = -\boldsymbol{e}^2, & \boldsymbol{e}_2 &= \boldsymbol{n} = -\boldsymbol{e}^1, \\ \boldsymbol{e}_3 &= \boldsymbol{m} = \boldsymbol{e}^4, & \boldsymbol{e}_4 &= \bar{\boldsymbol{m}} = \boldsymbol{e}^3. \end{aligned}$$

Then we can define

$$\begin{aligned} \boldsymbol{e}_1 &= \boldsymbol{e}^2 \equiv D, & \boldsymbol{e}_2 &= \boldsymbol{e}^1 \equiv \Delta, \\ \boldsymbol{e}_3 &= -\boldsymbol{e}^4 \equiv -\delta, & \boldsymbol{e}_4 &= -\boldsymbol{e}^3 \equiv -\delta^*. \end{aligned}$$

The Weyl tensor is defined by [23]

$$\begin{aligned} C_{ijkl} =& \frac{1}{n-2}\left(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}\right) \\ &+ \frac{1}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)R + R_{ijkl}, \end{aligned}$$

where R_{ijkl} is the Riemann tensor. The Weyl scalars, in turn, are ten real independent components of the Weyl tensor that can be represented by five complex scalars:

$$\begin{aligned} \Psi_0 &\equiv -C_{pqrs}l^p m^q l^r m^s, \\ \Psi_1 &\equiv -C_{pqrs}l^p n^q l^r m^s, \\ \Psi_2 &\equiv -C_{pqrs}l^p m^q \bar{m}^r n^s, \\ \Psi_3 &\equiv -C_{pqrs}l^p n^q \bar{m}^r n^s, \\ \Psi_4 &\equiv -C_{pqrs}n^p \bar{m}^q n^r \bar{m}^s, \end{aligned}$$

with

$$\begin{aligned} &C_{1334} = C_{1231} = \Psi_1 = C^*_{1241} = C^*_{1443}, \\ &C_{1212} = C_{3434} = -(\Psi_2 + \Psi_2^*), \\ &C_{1234} = \Psi_2 - \Psi_2^*, \\ &C_{2443} = -C_{1242} = \Psi_3 = -C^*_{1232} = -C^*_{2343}, \\ &C_{1314} = C_{2324} = C_{1332} = C_{1442} = 0. \end{aligned}$$

On the other hand, the Ricci tensor has ten linearly independent components R_{ij}, which can be represented by the following ten real scalars:

$$\begin{gathered} \Phi_{00} = \frac{R_{11}}{2}, \quad \Phi_{22} = \frac{R_{22}}{2}, \quad \Phi_{02} = -\frac{R_{33}}{2}, \quad \Phi_{20} = -\frac{R_{44}}{2}, \\ \Phi_{01} = -\frac{R_{13}}{2}, \quad \Phi_{10} = -\frac{R_{14}}{2}, \quad \Phi_{12} = -\frac{R_{23}}{2}, \quad \Phi_{21} = -\frac{R_{24}}{2}, \\ \Phi_{11} = -\frac{R_{12} + R_{34}}{4}, \quad \Lambda = \frac{R}{24} = -\frac{R_{12} - R_{34}}{12}, \end{gathered}$$

[6]There is a way to classify solutions of the Einstein's equation based on the Weyl tensor, called Petrov classification. By such classification the metrics are categorized in six types: I, II, III, D, N, and O. In particular, the Kerr spacetime is of type D. A metric is said to be of Petrov type D if there exists a tetrad such that in this frame the Weyl scalars Ψ_0,Ψ_1, Ψ_3, and Ψ_4 vanish, while only Ψ_2 remains not null [23].

where Φ and Λ are the tracefree and the pure trace parts of the Ricci tensor, respectively. The components of the Riemann tensor projected on the tetrad, R_{ijkl}, can be written in terms of the quantities above [24].

The connection coefficients are defined by

$$\gamma_{kij} = \boldsymbol{e}_k \cdot \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_i = -\gamma_{ijk},$$

and the spin coefficients, or Ricci rotation-coefficients, are defined in terms of the connection coefficients by

$$\begin{aligned}
\kappa &= \gamma_{311}, \quad \tau = \gamma_{312}, \quad \sigma = \gamma_{313}, \quad \rho = \gamma_{314}, \\
\pi &= \gamma_{241}, \quad \nu = \gamma_{242}, \quad \mu = \gamma_{243}, \quad \lambda = \gamma_{244}, \\
\varepsilon &= \frac{1}{2}(\gamma_{211} + \gamma_{341}), \quad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}), \\
\alpha &= \frac{1}{2}(\gamma_{214} + \gamma_{344}), \quad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}).
\end{aligned}$$

Now we can write the Lie brackets, $[e_i, e_j] = C^k{}_{ij} e_k$, in terms of the spin coefficients:

$$\begin{aligned}
[\Delta, D] &= (\gamma + \gamma^*)D + (\epsilon + \epsilon^*)\Delta - (\pi + \tau^*)\delta - (\tau + \pi^*)\delta^*, \\
[\delta, D] &= (\alpha^* + \beta - \pi^*)D + \kappa\Delta - (\epsilon - \epsilon^* + \rho^*)\delta - \sigma\delta^*, \\
[\delta, \Delta] &= -\nu^* D + (\tau - \alpha^* - \beta)\Delta + (\mu - \gamma + \gamma^*)\delta + \lambda^*\delta^*, \\
[\delta^*, \delta] &= (\mu^* - \mu)D + (\rho^* - \rho)\Delta + (\alpha - \beta^*)\delta + (\beta - \alpha^*)\delta^*.
\end{aligned}$$

These commutation relations are useful to eliminate second order derivatives in some equations.

In 1962, Newman and Penrose published a set of 18 equation involving the quantities we mentioned above. We can find these equations in the Chandrasekhar's book with the indication of from what component of the Riemann tensor the equation comes from. Using a set of symmetries of the problem and the Bianchi identities, we get a set of equations [23, 15] which will be referred as the Newman-Penrose (NP) equations.

The next step is to make a first order perturbation in the NP equations. We will use the upper index B for the perturbed part. That is, we will replace all the quantities mentioned before by what follows

$$\begin{aligned}
&\Psi_i \to \Psi_i + \varepsilon\Psi_i^B, \qquad i = 0, 1, 2, 3, 4, \\
&D \to D + \varepsilon D^B, \quad \Delta \to \Delta + \varepsilon\Delta^B, \quad \delta \to \delta + \varepsilon\delta^B, \\
&\kappa \to \kappa + \varepsilon\kappa^B, \quad \sigma \to \sigma + \varepsilon\sigma^B, \quad \text{and so on,}
\end{aligned}$$

with

$$\Psi_i = 0, \quad i = 0, 1, 3, 4, \qquad \kappa = \sigma = \nu = \lambda = 0,$$

and keep only terms up to first order in ε. In other words, every term without an upper index takes the original expression, while each one with the upper index B is the perturbative part, and ε is small. We already mentioned that we are going to work with a tetrad frame such that only Ψ_2 does not vanish, thus the first part of the equation above is justified. By the Goldberg-Sachs theorem [25], if the metric is a solution of the Einstein's equation in vacuum, given a tetrad $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, $\boldsymbol{l}$ is a repeated null direction if, and only if, $\boldsymbol{l}$ is geodesic and shearfree. To be geodesic means that $l^\mu \nabla_\mu l^\nu = 0$, consequently $\kappa = \nabla_\nu l^\mu m_\mu l^\nu = 0$. Besides this, $\boldsymbol{l}$ be shearfree means $-m^\mu m^\nu \nabla_\mu l_\nu = 0$, and by consequence $\sigma = \nabla_\nu l_\mu m^\mu m^\nu = 0$. The null vector $\boldsymbol{n}$ is also a repeated null direction, therefore the same applies and, analogously, we have $\nu = 0$ and $\lambda = 0$, explaining the second part of the equation above.

Three important identities, valid for type D metrics, are:

$$\begin{aligned} &D\Psi_2 = 3\rho\Psi_2, \qquad \delta\Psi_2 = 3\tau\Psi_2, \qquad \text{and} \\ &[D - (\boldsymbol{p}+1)\epsilon + \epsilon^* + \boldsymbol{q}\rho - \rho^*](\delta - \boldsymbol{p}\beta + \boldsymbol{q}\tau) \\ &\ - [\delta - (\boldsymbol{p}+1)\beta - \alpha^* + \pi^* + \boldsymbol{q}\tau](D - \boldsymbol{p}\epsilon + \boldsymbol{q}\rho) = 0, \end{aligned} \tag{2.7}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are any two constants.

In order to obtain the TME we need to make the perturbation mentioned for each kind of field, that is, for Maxwell's equations (vector field), for Dirac equations (spinor field), etc. And then we can summarize all the equations in only one, the Teukolsky master equation.

Here we are not going to do each s-spin field perturbation. Moreover, we will only give the instructions for the vector field case. In the NP formalism, the components of the Maxwell-tensor F are written in terms of three complex scalars [23]:

$$\begin{aligned} \phi_0 &= F_{13} = F_{ij} l^i m^j, \\ \phi_1 &= \frac{1}{2}(F_{12} + F_{43}) = \frac{1}{2} F_{ij}(l^i n^j + \bar{m}^i m^j), \\ \phi_2 &= F_{42} = F_{ij} \bar{m}^i n^j, \end{aligned}$$

and the Maxwell's equations are

$$F_{[ij|k]} = 0, \qquad \text{and} \qquad \eta^{lm} F_{il|m} = 0,$$

where the bar is a notation for *intrinsic derivative*[7].

After some manipulations we arrive at the following equations

$$(D - 2\rho)\phi_1 - (\delta^* + \pi - 2\alpha)\phi_0 = 2\pi J_l, \tag{2.8}$$

$$(\delta - 2\tau)\phi_1 - (\Delta + \mu - 2\gamma)\phi_0 = 2\pi J_m, \tag{2.9}$$

$$(D - \rho + 2\epsilon)\phi_2 - (\delta^* + 2\pi)\phi_1 = 2\pi J_{m^*}, \tag{2.10}$$

$$(\delta - \tau + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1 = 2\pi J_m, \tag{2.11}$$

where J_μ is the 4-current density and $J_l = J_\mu l^\mu$, $J_n = J_\mu n^\mu$, and so on. Now we perform the following three manipulations:

1) we apply $(\delta - \beta - \alpha^* - 2\tau + \pi^*)$ in (2.8), and $(D - \epsilon + \epsilon^* - 2\rho - \rho^*)$ in (2.9), and then subtract one from the other;

2) we use the identity (2.7) with $\boldsymbol{p} = 0$ and $\boldsymbol{q} = -2$ in order to simplify the result in step 1);

[7] The following definition can be seen in the chapter 1 of [23]:

$$A_{a|b} = e_a{}^\mu (\nabla_\nu A_\mu) e_b{}^\nu$$

3) we interchange $\boldsymbol{l}$ and $\boldsymbol{n}$, and $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$.

Before these 3 steps we obtain the following decoupled equations for the components ϕ_0 and ϕ_2:

$$\begin{cases} [(D-\epsilon+\epsilon^*-2\rho+\rho^*)(\Delta+\mu-2\gamma) \\ \qquad -(\delta-\beta-\alpha^*-2\tau+\pi^*)(\delta^*+\phi-2\alpha)]\phi_0 = 2\pi J_0, \\ [(\Delta+\gamma-\gamma^*+2\mu+\mu^*)(D-\rho+2\epsilon) \\ \qquad -(\delta^*+\alpha+\beta^*+2\pi-\tau^*)(\delta-\tau^*+2\beta)]\phi_2 = 2\pi J_2. \end{cases} \tag{2.12}$$

Note that, up to here, we did not use an explicit tetrad, even a metric, we only assumed it is of Petrov type D. We have just decoupled the equations for the components of the Maxwell's tensor, now it is time to make these equations separable. We are going to use the Kerr metric in the Boyer-Lindquist coordinates, and a "good basis" for the Kerr spacetime, that is, a basis such that only Ψ_2 does not vanish, is the so called Kinnersley tetrad:

$$\begin{aligned} l^\mu &= \frac{1}{\Delta}\left(r^2+a^2,\Delta,0,a\right), \\ n^\mu &= \frac{1}{2\Sigma}\left(r^2+a^2,-\Delta,0,a\right), \\ m^\mu &= \frac{-\bar{\rho}}{\sqrt{2}}\left(ia\sin\theta,0,1,\frac{i}{\sin\theta}\right), \end{aligned}$$

where $\rho = -1/(r-ia\cos\theta)$ is a "key variable" as was called by Teukolsky in [19], and we will see why.

The nonvanishing spin coefficients are α, β, γ, μ, ρ, π, and τ, and their expressions can be seen in [24]. Additionally, we have $\Psi_2 = M\rho^3$. The first equation in (2.12) is already separable, and the second one becomes separable when we replace ϕ_2 by $\psi_2 = \phi_2\rho^2$, and this is why ρ was called a key variable by Teukolsky.

The procedure to decouple and make separable the equations for gravitational perturbations can be seen in [15].

Finally, the equations can be summarized in only one, the Teukolsky Master Equation:

$$\begin{aligned} &\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi}+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} \\ &-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi} \\ &-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi}{\partial t}+(s^2\cot^2\theta-s)\psi = 4\pi\Sigma T, \end{aligned} \tag{2.13}$$

where s, the spin weight, can assume the values 0, $\pm 1/2$, ± 1, $\pm 3/2$[8], ± 2, and T is the source term. Then, for example, when we take $s=1$ we obtain the equation for ϕ_0, and with $s=-1$ we have a equation for $\rho^{-1}\phi_1$.

[8]See [18].

2.3 A Brief Study of the TME

In this section we are going to know the separated equations, they are known as *angular Teukolsky equation* and *radial Teukolsky equation*.

2.3.1 Description and Separability

The TME, as already was said, describes a perturbation in a s-spin field in the Kerr background. The scheme below shows what is ψ in equation (2.13) for each s:

s	type	ψ
0	scalar	Ψ, the KG field
$\pm 1/2$	spinor	χ_0 and $\rho^{-1}\chi_1$
± 1	vector	ϕ_0 and $\rho^{-2}\phi_2$, from the Maxwell tensor
$\pm 3/2$	vector-spinor	Ω_0 and $\rho^{-3}\Omega_3$
± 2	tensor	Ψ_0 and $\rho^{-4}\Psi_4$, from the Weyl tensor

Box 1: The corresponding kind of field for each s.

The vacuum case ($T = 0$) is separable in the following form:

$$\psi = e^{-i\omega t} e^{im\phi} R(r) S(\theta). \tag{2.14}$$

Using (2.14) we can easily find the Teukolsky Radial Equation (TRE) and the Teukolsky Angular Equation (TAE):

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - {}_s\lambda_{lm}\right) R = 0, \tag{2.15}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + s + {}_sA_{lm}\right) S = 0, \tag{2.16}$$

where $K \equiv (r^2+a^2)\omega - am$, ${}_sA_{lm}$ is the separation constant and ${}_s\lambda_{lm} \equiv {}_sA_{lm} + a^2\omega^2 - 2ma\omega$. The eigenfunctions ${}_sS_{lm}(\theta)$ become the so called spin-weighted spherical harmonics when $a\omega = 0$. Both equations have two regular singularities, r_- and r_+ for the TRE, 0 and π for the TAE, and one irregular singularity of Poincaré rank[9] 1 at ∞[10]. Therefore, they can be put in the confluent Heun equation[11] (CHE) form by a change of variables. The

[9] Consider a n-order differential equation:

$$\frac{d^n w}{dz^n} + \sum_{m=0}^{n-1} f_m(z)\frac{d^m w}{dz^m} = 0.$$

Let $z = z_0$ a singular point. The Poincaré rank of the singularity z_0 is the least integer r such that all $(z - z_0)^{n-m+nr} f_m(z)$ are analytic at $z = z_0$. [DLMF]

[10] While the irregular singularity of the TRE is $r = \infty$, for the TAE would be $\cos\theta = \infty$. However, the variables of the metric are real, then we can not have $\cos\theta = \infty$. In the next chapter we will see that the change of variables performed in order to put the TAE in a CHE form is $u = \cos\theta$. As a result, we obtain a ODE with two regular singularities, $u = -1$ and $u = 1$, and an irregular one, $u = \infty$. The points $u = 1$ and $u = -1$ correspond to $\theta = 0$ and $\theta = \pi$, respectively, while $u = \infty$ does not have a map to the real variable θ.

[11] A confluent Heun equation is a second order differential equation of the form:

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)} w = 0.$$

TAE, in particular, is a Sturn-Liouville problem with eigenvalues ${}_sA_{lm}$ and eigenfunctions ${}_sS_{lm}(\theta)$. For fixed s, m, and $a\omega$, the smallest ${}_sA_{lm}$ has $l = \max(|m|, |s|)$ [15].

The good news is that we do not need to solve the TRE and the TAE for both $s = |s|$ and $s = -|s|$. Once we have solved the equations for s, we can use the Teukolsky-Starobinsky identities (TSI) to obtain the solutions to $-s$. The TSI are [26]

$$\Delta^s(\mathcal{D}_0)^{2s}(\Delta^s R_s) = \mathcal{C}_s R_{-s}, \qquad \Delta^s(\mathcal{D}_0^\dagger)^{2s} R_{-s} = \mathcal{C}_s^*(\Delta^s R_s),$$
$$\mathcal{L}_{1-s}\mathcal{L}_{2-s}...\mathcal{L}_{s-1}\mathcal{L}_s\, S_s = \mathcal{B}_s S_{-s}, \qquad \mathcal{L}_{1-s}^\dagger\mathcal{L}_{2-s}^\dagger...\mathcal{L}_{s-1}^\dagger\mathcal{L}_s^\dagger\, S_{-s} = \mathcal{B}_s S_s,$$

with the operators $\mathcal{D}_n$, $\mathcal{L}_n$, $\mathcal{D}_n^\dagger$, and $\mathcal{L}_n^\dagger$ given by

$$\mathcal{D}_n = \partial_r + i\frac{K}{\Delta} + \frac{n}{\Delta}\partial_r\Delta, \qquad \mathcal{L}_n = \partial_\theta + \left(\frac{m}{\sin\theta} - a\omega\sin\theta\right) + n\cot\theta,$$
$$\mathcal{D}_n^\dagger = \partial_r - i\frac{K}{\Delta} + \frac{n}{\Delta}\partial_r\Delta, \qquad \mathcal{L}_n^\dagger = \partial_\theta - \left(\frac{m}{\sin\theta} - a\omega\sin\theta\right) + n\cot\theta.$$

Note that $\mathcal{D}_n$ and $\mathcal{L}_n$ act as lowering operators, while $\mathcal{D}_n^\dagger$ and $\mathcal{L}_n^\dagger$ act as raising ones.

Then, from the Maxwell equations, for example, we have

$$\Delta\mathcal{D}_0\mathcal{D}_0 R_{-1} = \mathcal{C}_1\Delta R_1, \qquad \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger(\Delta R_1) = \mathcal{C}_1 R_{-1},$$
$$\mathcal{L}_0\mathcal{L}_1 S_1 = \mathcal{C}_1 S_{-1}, \qquad \mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1} = \mathcal{C}_1 S_1,$$

with $\mathcal{C}_1 = \sqrt{(\lambda + 2 + 4ma\omega - 4a^2\omega^2)}$. The Maxwell tensor[12], F, depends on three complex scalar functions: ϕ_0, ϕ_1, and ϕ_2. And, as it was commented before, $R_{\pm1}(r)$ and $S_{\pm1}(\theta)$ are the radial and the angular part of the solutions for $\phi_0 = R_1(r)S_1(\theta)e^{-i\omega t + im\phi}$ and $\phi_2 = \rho^{-2}R_{-1}(r)S_{-1}(\theta)e^{-i\omega t + im\phi}$.

We can define

$$P_1 \equiv R_1 S_1, \qquad P_{-1} \equiv \rho^{-2}R_{-1}S_{-1},$$

and then

$$\phi_0 = P_1\, e^{i(-\omega t + m\phi)}, \qquad \phi_2 = P_{-1}\, e^{i(-\omega t + m\phi)}.$$

2.3.2 Asymptotic Behavior

In order to study the behavior of the solutions near the outer horizon, r_+, and infinity, we are going to use the *tortoise coordinate* r^* defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{2.17}$$

[DLMF]. More details about such ODE will be discussed in the next chapter.

[12] The matrix representation of the Maxwell tensor in terms of the tetrad is given by:

$$F_{ab} = F_{\mu\nu}(e_a)^\mu(e_b)^\nu = \begin{pmatrix} 0 & \phi_1 + \phi_1^* & \phi_0 & \phi_0^* \\ -\phi_1 - \phi_1^* & 0 & -\phi_2^* & -\phi_2 \\ -\phi_0 & \phi_2^* & 0 & \phi_1^* - \phi_1 \\ -\phi_0^* & \phi_2 & \phi_1 - \phi_1^* & 0 \end{pmatrix},$$

where

$$\phi_0 = F_{13} = F_{ij}l^i m^j, \quad \phi_1 = \frac{1}{2}(F_{12} + F_{43}) = \frac{1}{2}F_{ij}(l^i n^j + \bar{m}^i m^j), \quad \phi_2 = F_{42} = F_{ij}\bar{m}^i m^j.$$

Integrating the equation above, we obtain

$$r^*(r) = r + \frac{2M}{r_+ - r_-}\left[r_+ \log\left(\frac{r - r_+}{M}\right) - r_- \log\left(\frac{r - r_-}{M}\right)\right],$$

where the integration constant was chosen to make the argument of the logarithmic functions dimensionless. We have the correspondent limits

$$\begin{aligned} r \to r_+ \quad &\text{means} \quad r^* \to -\infty, \\ r \to \infty \quad &\text{means} \quad r^* \to \infty. \end{aligned}$$

Following Teukolsky [15], we will replace R by $Y = \Delta^{s/2}(r^2 + a^2)^{1/2}R$. In this way, the TRE becomes

$$\frac{d^2Y}{dr^{*2}} + f(r)Y = 0$$

with

$$f(r) = \frac{K^2 - 2is(r - M)K + \Delta(4is\omega r - \lambda)}{(r^2 + a^2)^2 - G^2 - dG/dr^*},$$

and $G = s(r - M)/(r^2 + a^2) + r\Delta/(r^2 + a^2)^2$.

When we take $r \to \infty$ in the equation (1.32) the result is

$$\frac{d^2Y(r^*)}{dr^{*2}} + \left(\omega^2 + \frac{2i\omega s}{r}\right)Y(r^*) \approx 0$$

and the solutions are $Y \sim r^{\pm s}e^{\mp i\omega r^*}$. Translating to R, the asymptotic solutions are $R \sim e^{-i\omega r^*}/r$ and $e^{i\omega r^*}/r^{2s+1}$.

Now when we take the limit $r^* \to -\infty$ in (1.32), the ODE becomes

$$\frac{d^2Y(r^*)}{dr^{*2}} + \left[k - \frac{is(r_+ - M)}{2Mr_+}\right]^2 Y(r^*) \approx 0,$$

with $k = \omega - am/(2Mr_+)$, and in this case the solutions are $Y(r^*) \sim e^{\pm\frac{s(r_+ - M)}{2Mr_+}r^*}e^{\pm ikr^*}$. Once that we are analyzing the solution when $r \to r_+$, we can replace r by $r_+ + \epsilon$. Note that, with this replacement, $\Delta = (r - r_+)(r - r_-) \to \epsilon(r_+ + \epsilon - r_-) = (r_+ - r_-)\epsilon + O(\epsilon^2)$. Therefore, keeping only up to the first order in ϵ and identifying $(r_+ - r_-)\epsilon$ as Δ, we can check that the first exponential simplifies as

$$e^{\frac{(r_+ - M)}{Mr_+}r^*} \propto \Delta + O(\epsilon^2).$$

Summarizing the asymptotic behaviors, we have

$$\begin{cases} R \sim e^{ikr^*}, & \text{ingoing wave when } r \to r_+, \\ R \sim \Delta^{-s}e^{-ikr^*}, & \text{outgoing wave when } r \to r_+, \\ R \sim e^{-i\omega r^*}/r, & \text{ingoing wave when } r \to \infty, \\ R \sim e^{i\omega r^*}/r^{2s+1}, & \text{outgoing wave when } r \to \infty. \end{cases}$$

3 Quasi-Normal Modes by The Continued Fraction Method

We call *normal modes* and *normal frequencies* the complete set of solutions and frequencies, Ψ_n and ω_n, respectively, of the following equation

$$\frac{d^2\Psi}{dr^{*2}} + [\omega^2 - f(r)]\Psi = 0. \tag{3.1}$$

The normal frequencies are real numbers and the normal modes can be written as

$$\Psi_n(t,r) = e^{-i\omega_n t}\psi_n(r). \tag{3.2}$$

Depending on the boundary conditions the label n can assume discrete values. An example is the propagation of stationary waves in a string with fixed endings [28]. However, the boundary conditions for a black hole perturbation are not the same as in the string example. In our case the frequencies ω_n will assume complex values and then they will be called *quasi-normal frequencies*, while the respective solutions will be called *quasi-normal modes*[1] [29].

In 1985, Leaver used the method of continued fractions in order to evaluate the QNMs for perturbations in the Kerr background [33]. In this chapter we will first introduce the confluent Heun equation, and then we will present and apply such method, which is often called *Leaver's method* in the black hole theory context because he was the first one to apply it to this problem[2].

3.1 The Confluent Heun Equation

Special functions (or special equations) as hypergeometric, hypergeometric confluent, Hermite polynomials, etc, commonly appear in problems of physics. In 1889, Karl Heun published a new special equation [14]. Since then, a large variety of applications of the Heun equation and its confluent versions has emerged. In [14] the author cites many examples of application of Heun's equations in physics.

3.1.1 Some forms of the Confluent Heun Equation

As discussed in the previous chapter, both the Teukolsky radial and angular equations have two regular singular points and one irregular singular point with Poincaré rank $r = 1$, and, for this reason, we can put both of them in the Confluent Heun Equation form, which is obtained from the General Heun Equation (GHE). The GHE is a second order linear differential equation with four regular singularities $\{0, 1, z_0, \infty\}$ [37],

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-z_0}\right)\frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-z_0)}w = 0, \tag{3.3}$$

[1] Sometimes we call the frequencies quasi-normal modes.

[2] In his paper, Leaver [33] refer to previous papers of Jaffé [30] and Baber & Hassé [31], where the continued fraction method was used.

where γ, δ, ϵ, α, β, q, and z_0 are complex constants. In particular, q is known as *accessory parameter* [35], and α, β, γ, δ, and ϵ obey $\alpha+\beta+1=\gamma+\delta+\epsilon$. The confluent limit consists in first make

$$\beta \to z_0\beta, \qquad \epsilon \to z_0\epsilon, \qquad q \to z_0 q$$

in (3.3), and then take the limit $z_0 \to \infty$. The result is the ODE below

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \left(\frac{q}{z} + \frac{\alpha - q}{z-1}\right) w = 0. \tag{3.4}$$

Now $z=0$ and $z=1$ are regular singular points, while $z=z_0=\infty$ is the irregular singular one. When the CHE is written in the format above, we say it is written in the *nonsymmetrical canonical form.*

Besides the form given above, other two commonly used forms of the CHE are found in the literature: the *Bôcher symmetrical form* and the *normal symmetric form.* The first one is given by

$$\frac{d}{dx}\left[(x^2-1)\frac{dH_B(x)}{dz}\right] - \left[\left(\frac{\epsilon}{4}\right)^2 (x^2-1) - \frac{\epsilon}{2}\beta x + \kappa + \frac{\mu^2+\nu^2+2\mu\nu x}{x^2-1}\right] H_B(x) = 0, \tag{3.5}$$

where

$$H_B(x) \equiv (x-1)^{\frac{\mu+\nu}{2}}(x+1)^{\frac{\mu-\nu}{2}} e^{-\frac{\epsilon z}{4}} w\left(\frac{1-x}{2}\right), \quad x = 1-2z.$$

Note that the change of variables $x=1-2z$ maps the singularities 0 into 1, and 1 into -1. The second one, in turn, is given by

$$(x^2-1)\frac{d^2H_N(x)}{dx^2} - \left(\frac{\epsilon^2}{16}(x^2-1) - \frac{\epsilon\beta}{2}x + \kappa + \frac{\mu^2\nu^2 - 1 + 2\mu\nu x}{x^2-1}\right) H_N(x) = 0, \tag{3.6}$$

where

$$H_N(z) = (1-x^2)^{1/2} H_B(z),$$

and the constants are

$$\kappa = \frac{1}{4}\left[-\left(\frac{2\alpha}{\epsilon} + \gamma - \delta\right)\epsilon + (\gamma+\delta)^2 - 2(\gamma+\delta-2q)\right],$$

$$\beta = \frac{1}{2}\left(-\frac{2\alpha}{\epsilon} + \gamma + \delta\right), \quad \mu = \frac{1}{2}(\gamma+\delta-2), \quad \nu = \frac{\gamma-\delta}{2}.$$

From (3.5) we can see that when $\epsilon\beta = 0$ the CHE reduces to the spheroidal wave equation[3].

[3]The spheroidal differential equation is

$$\frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right] + \left[\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right] w = 0.$$

The points $z=\pm 1$ are regular singularities and, if $\gamma \neq 0$, $z=\infty$ is an irregular singularity of rank 1.

3.1.2 Power series solutions of the Confluent Heun Equation

We represent a solution of (3.4) by $H_C(\gamma,\delta,\epsilon,\alpha,q;z)$. Firstly, let us observe the asymptotic behavior of $H_C(\gamma,\delta,\epsilon,\alpha,q;z)$, which is determined by the characteristic exponents. Writing

$$w_0(z)=\sum_{n=0}^{\infty}a_n z^{n+\mu_0}, \qquad w_1(z)=\sum_{n=0}^{\infty}b_n(z-1)^{n+\mu_1}, \tag{3.7}$$

when we insert (3.7) in (3.4), by the characteristic equation we find $\mu_0=\{0,1-\gamma\}$ and $\mu_1=\{0,1-\delta\}$, then

$$\lim_{z\to 0}H_C(\gamma,\delta,\epsilon,\alpha,q;z)\sim 1 \qquad \text{or} \qquad z^{1-\gamma}, \tag{3.8}$$

$$\lim_{z\to 1}H_C(\gamma,\delta,\epsilon,\alpha,q;z)\sim 1 \qquad \text{or} \qquad (z-1)^{1-\delta}. \tag{3.9}$$

And for the solution around $z\to\infty$ we obtain

$$\lim_{z\to\infty}H_C(\gamma,\delta,\epsilon,\alpha,q;z)\sim z^{-\alpha/\epsilon} \qquad \text{or} \qquad e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}. \tag{3.10}$$

The CHE has two types of solutions: they are usually called “angular”, denoted by $Hc^{(a)}(\gamma,\delta,\epsilon,\alpha,q;z)$, and “radial” solution, denoted by $Hc^{(r)}(\gamma,\delta,\epsilon,\alpha,q;z)$ [37]. The first one is a Frobenius solution around a regular singularity, $z=0$, with the condition $Hc^{(a)}(\gamma,\delta,\epsilon,\alpha,q;0)=1$. Then, in summary,

$$Hc^{(a)}(\gamma,\delta,\epsilon,\alpha,q;z)=\sum_{k=0}^{\infty}c_k^{(a)}z^k, \tag{3.11}$$

$$Hc^{(a)}(\gamma,\delta,\epsilon,\alpha,q;0)=1. \tag{3.12}$$

While the second one, sometimes called Tomé’s type solution [34], is defined by its behavior at infinity:

$$Hc^{(r)}(\gamma,\delta,\epsilon;\alpha,q;z)=\sum_{k=0}^{\infty}c_k^{(r)}z^{-k-\alpha/\epsilon}, \tag{3.13}$$

$$\lim_{z\to\infty}Hc^{(r)}(\gamma,\delta,\epsilon;\alpha,q;z)=z^{-\alpha/\epsilon}. \tag{3.14}$$

Because (3.4) is a second order differential equation, we know it has two linearly independent solutions. $Hc^{(a)}(\gamma,\delta,\epsilon;\alpha,q;z)$ satisfies the first behavior in (3.8), then, we need to find the second solution, that is, the one which satisfies the second behavior in (3.8).

There exists a set of transformations which does not change the form of the CHE (3.4), and by those transformations we can find other solutions. More details can be seen in [37, 34]. For example, (3.4) is invariant by the following transformation:

$$w(z)\mapsto v(z), \qquad v(z)=z^{\gamma-1}w(z),$$

and this one gives us the second solution. Then we obtain the two linearly independent solutions around $z=0$:

$$\begin{cases} Hc^{(a)}(\gamma,\delta,\epsilon;\alpha,q;z), \\ z^{1-\gamma}Hc^{(a)}\left(2-\gamma,\delta,\epsilon;\alpha+\frac{1-\gamma}{\epsilon},q+(1-\gamma)(\epsilon-\delta);z\right). \end{cases}$$

In order to obtain the linearly independent solutions local to $z = 1$, we use the transformation $z \mapsto 1 - z$. Such transformation interchange the singular points 0 and 1, and then we obtain [38]

$$\begin{cases} Hc^{(a)}(\gamma, \delta, -\epsilon; \alpha, q - \alpha; 1 - z), \\ (z-1)^{1-\delta} Hc^{(a)}(\gamma, 2-\delta, -\epsilon; \alpha - \delta + 1, q - (1-\delta)\gamma - \alpha - \epsilon(1-\delta); 1 - z). \end{cases}$$

In order to satisfy the conditions (3.12) and (3.14) we must have $c_0^{(a)}{=}1$ and $c_0^{(r)} = 1$.

Now, replacing w in (3.4) by the series (3.11), we obtain a three-term recurrence relation (TTRR)

$$f_k^{(a)} c_{k+1}^{(a)} + g_k^{(a)} c_k^{(a)} + h_k^{(a)} c_{k-1}^{(a)} = 0,$$

with

$$f_k^{(a)} = (1+k)(\gamma + k), \tag{3.15}$$

$$g_k^{(a)} = -k^2 - k(\gamma + \delta - \epsilon - 1) + q, \tag{3.16}$$

$$h_k^{(a)} = -\alpha - \epsilon(k-1). \tag{3.17}$$

Dividing the TTRR by $f_k^{(a)} c_k^{(a)}$ it is easy to check that

$$\lim_{k\to\infty} \frac{c_{k+1}^{(a)}}{c_k^{(a)}} = 1,$$

and then, by the test of convergence, we see that the radius of convergence of the series (3.11) is equal to 1, which is the distance to the next singular point.

Similarly, using the series given in (3.13) we obtain the following TTRR

$$f_k^{(r)} c_{k+1}^{(r)} + g_k^{(r)} c_k^{(r)} + h_k^{(r)} c_{k-1}^{(r)} = 0,$$

with

$$f_k^{(r)} = -\epsilon(k+1), \tag{3.18}$$

$$g_k^{(r)} = k^2 - k\left(1 - \gamma - \delta + \frac{2\alpha}{\epsilon} + \epsilon\right) - q + \frac{\alpha(\alpha - (\gamma + \delta - 1)\epsilon + \epsilon^2)}{\epsilon^2}, \tag{3.19}$$

$$h_k^{(r)} = \frac{\gamma(\alpha + (k-1)\epsilon)}{\epsilon}. \tag{3.20}$$

In this case, when we evaluate $\lim_{k\to\infty} c_{k+1}/c_k$, we obtain

$$\lim_{k\to\infty} \frac{c_{k+1}^{(r)}}{c_k^{(r)}} = \frac{k}{\epsilon} + O(1) \Rightarrow \lim_{k\to\infty} \frac{c_{k+1}^{(r)}}{c_k^{(r)}} = \frac{1}{\epsilon}\infty,$$

which means that the series (3.13) diverges [37].

Obviously, once that both the series expansions (3.11) and (3.13) start from $k = 0$, $c_{-n}^{(a)} = 0$ and $c_{-n}^{(r)} = 0$, $\forall\, n > 0$.

A special case of confluent Heun functions happens when the series terminates: the confluent Heun polynomials. Observe that, in equations (3.17) and (3.20), if $-\alpha/\epsilon = p$, with p being a positive integer, then $h_{p+1}^{(a,r)} = 0$. If, in addition, $c_{p+1}^{(a,r)} = 0$, then necessarily all the c_k with $k > p$ vanish and the series (3.11) and (3.13) terminate, becaming a $(p+1)$-th order polynomial. These two conditions can be summarized in the equation $\Delta_{p+1} = 0$, where Δ_{p+1} is defined by the determinant

$$\Delta_{p+1} = \begin{vmatrix} g_0 & f_0 & 0 & \dots & 0 \\ h_1 & g_1 & f_1 & \dots & 0 \\ 0 & h_2 & g_2 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots & h_q & g_q \end{vmatrix}.$$

3.2 Linear Homogeneous Three-Term Recurrence Relations

In the previous section appeared a *linear homogeneous three-term recurrence relation.* In order to be able to evaluate quasi-normal modes, it is important to know some characteristics of such kind of recurrence relation. For this reason, the goals of this short section are to present the linear homogeneous TTRR and shows some definitions and theorems which will be important later.

Linear homogeneous three-term recurrence relations have some similarity with second order differential equations. A TTRR of the form

$$a_n y_{n+1} + b_n y_n + c_n y_{n-1} = 0 \tag{3.21}$$

has two linearly independent solutions, let us denote them by $y_n^{(1)}$ and $y_n^{(2)}$. Also, we say that two solutions are LI if their *Casorati determinant*,

$$D_n = \begin{vmatrix} y_n^{(1)} & y_n^{(2)} \\ y_{n-1}^{(1)} & y_{n-1}^{(2)} \end{vmatrix}, \tag{3.22}$$

does not vanish [39]. Assuming that $a_n \neq 0$, from (3.21) and (3.22) we obtain

$$D_{n+1} = \frac{c_n}{a_n} D_n.$$

Assuming $c_n \neq 0$, we have that D_{n+1} vanish if and only if $D_n = 0$. Therefore, if $D_n \neq 0$ for a given n, then $D_n \neq 0\ \forall n$.

In what follows, we will present some results that will be of great importance when get back to discuss about the angular and radial equations.

Definition 3.2.1. Minimal and dominant solutions. *Let f_n be a solution of a TTRR. If there exists another solution linearly independent g_n such that*

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 0,$$

then f_n is said to be a minimal solution, while g_n is called dominant solution.

Theorem 1. *A TTRR admits a minimal solution if, and only if, the limit*

$$\lim_{n\to\infty} \frac{y_n^{(1)}}{y_n^{(2)}}$$

exists or diverges, where $y_n^{(1)}$ and $y_n^{(2)}$ are any two linearly independent solutions of this TTRR.

Proof. Let $\{y_n^{(1)}, y_n^{(2)}\}$ be a pair of linearly independent solutions of the TTRR. If the limit $\lim_{n\to\infty} y_n^{(1)}/y_n^{(2)} = c$, then $f_n = y_n^{(1)} - c\, y_n^{(2)}$ is a minimal solution, once

$$\lim_{n\to\infty} \frac{f_n}{y_n^{(2)}} = 0.$$

If, on the other hand, $\lim_{n\to\infty} y_n^{(1)}/y_n^{(2)} = \infty$, then $y_n^{(2)}$ is minimal, since the limit of the inverse, $y_n^{(2)}/y_n^{(1)}$, gives zero. If the recurrence relation admits a minimal solution then there exists a pair $\{f_n, g_n\}$ of minimal and dominant solutions, and then any other pair of linearly independent solutions can be written as a linear combination of f_n and g_n, and then the limit will exist or diverge. □

Minimal solutions are useful for backward recursion, that is, it gives a good result if we start from y_N and y_{N+1}, and use the minimal solution to obtain y_M, with $M < N$; but not for forward recursion, which means when we start from y_N and y_{N+1} to evaluate y_P, with $P > N$. For dominant solutions, the opposite works: such solutions are useful for forward recursion, bot not for backward recursion. To understand it better we are going to give an example that can be seen in [39]. Consider the TTRR

$$y_{n+1} - 2\cosh x\, y_n + y_{n-1} = 0, \quad n \geq 0,$$

which has the pair of minimal and dominant solutions $\{f_n, g_n\} = \{e^{-nx}, e^{nx}\}$. Now suppose we want to evaluate e^{-40}. Taking $x = 1$ and using the minimal solution, e^{-n}, in the TTRR above, starting from $f_0 = 1$ and $f_1 = e^{-1}$, we obtain $y_{40} \approx -1.0568583$, that is obviously a very wrong answer.

Now, it is convenient to introduce the definition of ratio of a solution, $r_n \equiv \frac{y_{n+1}}{y_n}$, and then we can rewrite equation (3.21) as

$$r_n = -\frac{1}{a_n}\left(b_n + \frac{c_n}{r_{n-1}}\right).$$

Then, making $n \to n+1$ in the above expression, we obtain

$$r_n = \frac{-c_{n+1}}{b_{n+1} + a_{n+1} r_{n+1}}, \qquad n = 0,\, 1,\, 2... \tag{3.23}$$

$$\Rightarrow r_n = \cfrac{-c_{n+1}}{b_{n+1} - a_{n+1}\cfrac{c_{n+2}}{b_{n+2} - a_{n+2}\cfrac{c_{n+3}}{b_{n+3} - \dots}}}. \tag{3.24}$$

A compact notation commonly used for the continued fraction (3.24) is

$$r_n = \frac{-c_{n+1}}{b_{n+1}-}\,\frac{a_{n+1}c_{n+2}}{b_{n+2}-}\,\frac{a_{n+2}c_{n+3}}{b_{n+3}-}\cdots \tag{3.25}$$

Now we are going to state an important theorem which relates the existence of a minimal solution and the convergence of the CF (3.25).

Theorem 2 (Pincherle)**.** *Given a TTRR, the continued fraction* (3.25) *converges if, and only if, the TTRR admits a minimal solution. Furthermore, if a minimal solution f_n exist, r_n converges to f_n/f_{n-1}.*

The proof of the Pincherle's theorem can be seen in [40].

3.3 The Teukolsky Angular and Radial Equations as a CHE

3.3.1 The Teukolsky Angular Equation

It is straightforward to put the TAE in the *Bôcher symmetrical form.* Indeed, performing the change of variable $u = \cos\theta$, the equation (2.16) becomes

$$\frac{d}{du}\left[(1-u^2)\frac{dS(u)}{du}\right] + \left[c^2u^2 - 2csu - \frac{(m+su)^2}{1-u^2} + s + {}_sA_{lm}\right]S(u) = 0, \qquad (3.26)$$

which is equivalent to equation (3.5) with the following identification of parameters:

$$\epsilon = 4c, \qquad \beta = s, \qquad \mu = \pm m, \qquad \nu = \pm s, \qquad \kappa = {}_sA_{lm} + s(s+1) + c^2, \qquad (3.27)$$

where $c \equiv a\omega$ is defined for convenience. Note that if we exchange μ and ν in (3.5), the equation remains the same, that is, the parameters μ and ν are symmetric. Thus, another solution for the parameters is $\mu = \pm s$ and $\nu = \pm m$.

Equation (3.26) is actually a complex Sturn-Liouville eigenvalue problem, where the eigenvalues are the separation constant ${}_sA_{lm}(c)$, which are complex, in general [15]. The solutions ${}_sS_{lm}(u;c)$ with the ortogonality condition

$$\int_0^\pi |{}_sS_{lm}|^2 \sin\theta d\theta = 1$$

are called Spin-Weighted Spheroidal Harmonics (SWSH) and, when $s = 0$, that is, for the scalar case, they reduce to the Scalar Spheroidal Harmonics [36]. Once that (3.26) is a Sturn-Liouville problem, for fixed s, m, and c, the eigenfuntions ${}_sS_{lm}(u;c)$ form a complete and orthogonal set of functions for $u \in [-1, 1]$ or, equivalently, for $\theta \in [0, \pi]$. Besides that, when $c = 0$, the SWSH becomes the familiar spin-weighted spherical harmonics, and the separation constant takes the form ${}_sA_{lm} = l(l+1) - s(s+1)$, that corresponds to the Schwarzchild case. In general, the parameter c is complex, so we can write $c = c_R + ic_I$. In the case when $c_I = 0$, we say that the eigenfunctions are *oblate*; when $c_R = 0$, we say that they are *prolate.*

The separation constant has two important symmetries:

$$\begin{cases} {}_{-s}A_{lm} = {}_sA_{lm} + 2s, \\ {}_sA_{l-m} = {}_sA^*_{lm}. \end{cases}$$

Because of these properties we can only consider, for example, $s \leq 0$ and $m \geq 0$. There is one more important symmetry: it is easy to see that the Kerr metric is invariant by the simultaneously change $t \to -t$ and $\phi \to -\phi$, and then the solution of the TME, equation (2.14), automatically inherits this symmetry. But in (2.14) exchange the signs of t and ϕ is equivalent to exchange the signs of ω and m. Thus we also have the following symmetry

$${}_sA_{-lm}(c) = {}_sA_{lm}(-c).$$

We usually write ${}_sS_{lm}(c;u)$ as

$${}_sS_{lm}(c;u) = e^{c\,u}(1+u)^{k_-}(1-u)^{k_+}\sum_{n=0}^{\infty} b_n(1+u)^n, \qquad (3.28)$$

where the solutions for the constants k_- and k_+ are $k_- = (s \pm m)/2$ and $k_+ = \pm(m+s)/2$. Besides substituting (3.28) in (3.26), we can also use directly the result found in section 3.1.2 for $Hc^{(a)}(\gamma, \delta, \epsilon, q, \alpha; z)$. The set of parameters $\{\gamma, \delta, q, \alpha\}$ from (3.4) is related with the set $\{\beta, \mu, \nu, \kappa\}$ via

$$\gamma = 1 + \mu + \nu, \qquad \delta = 1 + \mu - \nu, \qquad \alpha = \epsilon(1 - \beta + \mu),$$
$$q = \kappa - \mu(\mu + 1) + \frac{\epsilon}{2}(1 - \beta + \mu + \nu).$$

Therefore, the coefficients b_n obey the TTRR

$$\begin{aligned} &f_0^\theta b_1 + g_0^\theta b_0 = 0, \\ &f_n^\theta b_{n+1} + g_n^\theta b_n + h_n^\theta b_{n-1} = 0, \end{aligned} \tag{3.29}$$

with

$$\begin{aligned} f_n^\theta &= (n+1)(n+\gamma), \\ g_n^\theta &= -n^2 - d\,n + q, \\ h_n^\theta &= -\epsilon(n + 1 - p). \end{aligned} \tag{3.30}$$

In the above expressions we have used the following definitions: $p \equiv \alpha/\epsilon$ and $d \equiv \gamma + \delta - \epsilon - 1$. Hence, following the discussion in section 3.2, we find the following continued fraction

$$g_0^\theta - \frac{f_0^\theta h_1^\theta}{g_1^\theta -} \frac{f_1^\theta h_2^\theta}{g_2^\theta -} \frac{f_2^\theta h_3^\theta}{g_3^\theta -} \dots = 0. \tag{3.31}$$

Recall that, as we saw in section 3.2, by the Pincherle theorem, the continued fraction above will converge if, and only if, the TTRR (3.29) admits a minimal solution. Consequently, for fixed c, s, and m, the eigenvalue ${}_sA_{lm}$ is a root of (3.31).

3.3.2 Small-c Expansion

An special case occurs when c is small. In such case we can write the separation constant as a Taylor expansion around $c = 0$.

We already know that when $c = 0$ the separation constant must becomes $l(l+1) - s(s+1)$. Then, for small c it makes sense to write ${}_sA_{lm}$ as a Taylor expansion around $c = 0$ such that the order zero term is $l(l+1) - s(s+1)$. That is,

$${}_sA_{lm}(c) = \sum_{n=0}^{\infty} d_n c^n, \tag{3.32}$$

with $d_0 = l(l+1) - s(s+1)$. Now, observe that by equation (3.30) all h_n^θ vanish when $c = 0$, because h_n^θ is proportional to $\epsilon = 4c$. If we impose that $g_j^\theta = 0$ for some $j > 0$, we find $j = l - m$.Taking the nth inversion of (3.31) we obtain

$$\frac{f_{j-1}^\theta h_j^\theta}{g_{j-1}^\theta -} \frac{f_{j-2}^\theta h_{j-1}^\theta}{g_{j-2}^\theta -} \dots \frac{f_1^\theta h_2^\theta}{g_1^\theta -} \frac{f_0^\theta h_1^\theta}{g_0^\theta} = g_j^\theta - \frac{f_j^\theta h_{j+1}^\theta}{g_{j+1}^\theta -} \frac{f_{j+1}^\theta h_{j+2}^\theta}{g_{j+2}^\theta -} \dots \tag{3.33}$$

Substituting the expansion (3.32) in (3.33) we obtain the coefficients d_n:

$$\begin{aligned} d_0 &= l(l+1) - s(s+1), \\ d_1 &= -\frac{2ms^2}{l(l+1)}, \end{aligned}$$

and so on.

3.3.3 The Teukolsky Radial Equation

In order to write the TME in the *nonsymmetrical canonical form* of the CHE, we perform the following transformation

$$R(r) = (r - r_+)^{\xi}(r - r_-)^{\eta}e^{\zeta\, r}K(r).$$

Then, inserting the above expression to $R(r)$ in (2.15) find that the exponent must be

$$\xi = \frac{\pm(2i\sigma_+ + s)}{2} - \frac{s}{2} \equiv \xi_\pm, \qquad \eta = \frac{\pm(2i\sigma_- - s)}{2} - \frac{s}{2} \equiv \eta_\pm, \qquad \zeta = \pm i\omega \equiv \zeta_\pm,$$

with

$$\sigma_\pm = \frac{\omega - m\Omega_\pm}{4\pi T_\pm}, \qquad 2\pi T_\pm = \frac{r_+ - r_-}{4Mr_\pm}, \qquad \Omega_\pm = \frac{a}{2Mr_\pm},$$

where T_+ is the temperature at the outer horizon and $\Omega_\pm$ are the angular velocities at the horizons. Moreover, in order to map the singular points $\{r_-, r_+, \infty\}$ into $\{0,\, 1,\, \infty\}$, we do the transformation

$$r \mapsto (r_+ - r_-)z + r_-,$$

and hence $K(z)$ will obey equation (3.4) with

$$\begin{aligned}
&\gamma = 1 + s + 2\eta, \qquad \delta = 1 + s + 2\xi, \qquad \epsilon = 2(r_+ - r_-)\zeta,\\
&\alpha = \epsilon\left(1 + s + \xi + \eta - 2M\zeta + is\frac{\omega}{\zeta}\right),\\
&q = {}_sA_{lm} + a^2\omega^2 - 8(M\omega)^2 + \frac{\epsilon}{4}(2\alpha + \gamma - \delta) + \left(1 + s - \frac{\gamma - \delta}{2}\right)\left(s + \frac{\gamma + \delta}{2}\right),
\end{aligned}$$

for any combination of $\xi_\pm$, $\eta_\pm$, and $\zeta_\pm$. Then, we need to analyze which set of plus and minus signs upholds the desired asymptotic behavior. If we want only ingoing waves at the horizons we must choose $\xi = \xi_-$ and $\eta = \eta_-$. Likewise, for outgoing waves at infinity the correct choice is $\zeta = \zeta_+$.

Notice that, since the asymptotic behaviors of $K(z)$ are given by (3.8), (3.9), and (3.10), we know the asymptotic behaviors for $R(z)$:

$$R(z) = (r_+ - r_-)^{\xi+\eta}e^{\zeta\, r_-}z^{\eta}(z-1)^{\xi}e^{(r_+ - r_-)\zeta z}K(z),$$

$$\begin{cases}
\lim_{z\to 0} R(z) \propto z^{\eta} & \text{or} \quad z^{\eta-\gamma+1},\\
\lim_{z\to 1} R(z) \propto (z-1)^{\xi} & \text{or} \quad (z-1)^{\xi-\delta+1},\\
\lim_{z\to\infty} R(z) \propto z^{\xi+\eta-\frac{\alpha}{\epsilon}}e^{(r_+ - r_-)\zeta z} & \text{or} \quad z^{\xi+\eta-\gamma-\delta+\frac{\alpha}{\epsilon}}e^{-\epsilon z + (r_+ - r_-)\zeta z}.
\end{cases}$$

Now, using the results from section 3.1.2 for the Tomé's type solution, equations (3.13) and (3.14), we find the following TTRR for the radial equation:

$$\begin{aligned}
&f_0^r b_1 + g_0^r b_0 = 0,\\
&f_n^r b_{n+1} + g_n^r b_n + h_n^r b_{n-1} = 0,
\end{aligned} \tag{3.34}$$

with

$$\begin{aligned}
f_n^r &= \epsilon(n+1),\\
g_n^r &= -n^2 + (d - 2p)n + p(d - p) + q,\\
h_n^r &= n^2 + (2p - 1 - \gamma)n + (p - 1)(p - \gamma),
\end{aligned}$$

where the constants p and d are, again, defined as α/ϵ and $\gamma+\delta-\epsilon-1$, respectively, but, obviously, they take different values from the p and d defined before, because now α, ϵ, γ, and δ are different. However, as we saw in section 3.1.2, the Tomé's type power series solution does not converge. Nevertheless, doing the transformation $K(z) \rightarrow z^{-\alpha/\epsilon}W(z)$ and performing the change of variables $z \mapsto 1/(1-x)$, the function $W(x)$ obeys the following ODE:

$$\frac{d^2W}{dx^2} + \left(\frac{d_0}{x} + \frac{d_2-d_0}{x-1} + \frac{d_0+d_1+d_2}{(x-1)^2}\right)\frac{dW}{dx} + \left(\frac{d_3}{x} - \frac{d_3}{x-1} + \frac{d_3+d_4}{(x-1)^2}\right)W = 0,$$

with

$$\begin{aligned} d_0 &= \delta, \\ d_1 &= \epsilon - 2p + \gamma - \delta - 2, \\ d_2 &= 2p - \gamma + 2, \\ d_3 &= p(\epsilon-\delta) - q, \\ d_4 &= p(p-\gamma+1). \end{aligned}$$

With this substitution, we have mapped the points $\{0, 1, \infty\}$ into $\{-\infty, 0, 1\}$, which means that, in terms of the variable r, $r_- \rightarrow -\infty$, $r_+ \rightarrow 0$, and $r = \infty \rightarrow 1$. Thus, writing $W(x)$ as $W(x) = \sum_{n=0}^{\infty} a_n x^n$ we obtain the following recurrence rule:

$$f_n a_{n+1} + g_n a_n + h_n a_{n-1} = 0, \quad \text{with} \quad n \geq 0, \quad a_{-1} = 0, \quad \text{and,} \tag{3.35}$$

$$\begin{aligned} f_n &= n^2 + (d_0+1)n + d_0, \\ g_n &= -2n^2 + (d_1+2)n + d_3, \\ h_n &= n^2 + (d_2-3)n + 2 + d_4 - d_2. \end{aligned}$$

Once that $\lim_{z\rightarrow\infty} K(z) \sim z^{-\frac{\alpha}{\epsilon}}$, all the three limits, $x \rightarrow -\infty$, $x \rightarrow 0$, and $x \rightarrow 1$, give finite numbers for $W(x)$, and then the solution for a_n must provide it.

We know that the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n x^n|$ converges, and the convergence test states that a series $\sum_{n=0}^{\infty} S_n$ converges if the limit $\lim_{n\rightarrow\infty} |S_{n+1}/S_n|$ is a finite number $L < 1$, and the convergence radius is L^{-1}. Applying the convergence test for the series of $W(x)$, we obtain that it converges for $x < 1$. But $W(x)$ must converge for $x \rightarrow 1$ because, as it was said at the end of the last paragraph, the limit $\lim_{x\rightarrow 1} W(x)$ must be a number, and for this case we have

$$\lim_{x\rightarrow 1} W(x) = \sum_{n=0}^{\infty} a_n.$$

We define the inverse of the convergence radius as $\rho = \lim_{n\rightarrow\infty} a_{n+1}/a_n$. Dividing equation (3.35) by $a_n f_n$ and taking the limit when $n \rightarrow \infty$, we obtain a second order polynomial equation for ρ. The two roots of such equation are the inverse of the convergence radii of each of the possible solutions for the TTRR. When we do it we find the double root $\rho = 1$ and hence, in this case, the convergence test is inconclusive. In order to try to overcome this problem, we assume that ρ has higher order behavior in n [33, 38]:

$$\rho \equiv \lim_{n\rightarrow\infty} \frac{a_{n+1}}{a_n} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n^{\frac{n}{2}}} = 1 + \frac{c_1}{n^{\frac{1}{2}}} + \frac{c_2}{n} + \frac{c_3}{n^{\frac{3}{2}}} + \frac{c_4}{n^2} \cdots . \tag{3.36}$$

In the limit $n \to \infty$ we can write the derivative of a_n whith respect to n as

$$\lim_{n\to\infty} \frac{da_n}{dn} \approx \frac{a_{n+1} - a_n}{1}.$$

Then,

$$\lim_{n\to\infty} \frac{1}{a_n}\frac{da_n}{dn} \approx \lim_{n\to\infty} \frac{a_{n+1}}{a_n} - 1 = \sum_{n=1}^{\infty} \frac{c_n}{n^{n/2}}$$
$$\Rightarrow \lim_{n\to\infty} \log(a_n) \approx \int \left(\frac{c_1}{n^{\frac{1}{2}}} + \frac{c_2}{n} + \frac{c_3}{n^{\frac{3}{2}}} + \frac{c_4}{n^2} \cdots \right) dn.$$

Integrating the right hand side of the above expression up to second order, we obtain the following asymptotic behavior for a_n:

$$\lim_{n\to\infty} a_n \propto n^{c_2} e^{2c_1\sqrt{n}}. \tag{3.37}$$

Hence, substituting the expansion (3.36) into (3.35), and taking the limit $n \to \infty$, we find the coefficients

$$\begin{aligned}
c_1 &= \pm\sqrt{-\epsilon}, \\
c_2 &= -\frac{\epsilon + \gamma + \delta}{2} + p - 1, \\
c_3 &= \frac{1}{8\epsilon c_1}[-4\alpha + [4 + 4q - 8\alpha + (\gamma + \delta)^2)]\epsilon + 2(4 + \gamma + 3\delta)\epsilon^2 + \epsilon^3], \\
c_4 &= \frac{1}{2}\left[(1 + \delta)(\gamma + \delta + \epsilon) - \frac{\alpha}{\epsilon}(2 + \delta + \epsilon) + q + 2\right],
\end{aligned}$$

and so on.

In (3.37) the dominant term is the exponential and we have two solutions for c_1, each of them corresponding to one of the solutions of the recurrence relation (3.35). If we call y_n the solution which has $\mathrm{Re}(c_1) < 0$ and x_n the other one, we automatically have $\lim_{n\to\infty} y_n/x_n = 0$, which means that, if these solutions do exist, then y_n and x_n are the minimal and the dominant solutions, respectively. Remember, at the beginning of this section, we found $\epsilon = 2(r_+ - r_-)\zeta$, and before that we concluded ζ should be $\zeta_+ = i\omega$. Therefore, if $\mathrm{Re}(\omega) < 0$ and $\mathrm{Re}(\sqrt{-\epsilon}) < 0$, then the minimal solution is the one with $c_1 = \sqrt{-\epsilon}$.

Because $\lim_{z\to 1} K(z) = 1$ and $W(z) = z^{\alpha/\epsilon}K(z)$ we can see that with $a_0 = 1$ the boundary condition at the outer event horizon is obeyed. Then, from (3.35), setting $n = 0$, we have

$$f_0 a_1 + g_0 a_0 = 0 \quad \Rightarrow \quad \frac{a_1}{a_0} = -\frac{g_0}{f_0}.$$

Using (3.25) we obtain

$$r_n = \frac{-h_{n+1}}{g_{n+1}-}\,\frac{f_{n+1}h_{n+2}}{g_{n+2}-}\,\frac{f_{n+2}h_{n+3}}{g_{n+3}-}\cdots,$$

with $r_n = \frac{a_{n+1}}{a_n}$. In particular, for $n = 0$ we have the following continued fraction

$$0 = g_0 - \frac{f_0 h_1}{g_1-}\,\frac{f_1 h_2}{g_2-}\,\frac{f_2 h_3}{g_3-}\cdots. \tag{3.38}$$

Again, by the Pincherle theorem, the continued fraction above will converge if, and only if, the TTRR (3.35) admits a minimal solution and, if such solution do exist, r_0 must converge to y_1/y_0, where $a_n = y_n$ is the minimal solution. On the other hand, we already know (3.38) converges because $r_0 = \frac{a_1}{a_0} = -\frac{d_3}{d_0}$. Therefore, by the Pincherle theorem, the TTRR (3.35) admits a minimal solution. Besides that, as it was mentioned before, it is necessary that $\sum_{n=0}^{\infty} a_n$ converges absolutely, and this will happen when, for a given a, s, m, and ${}_sA_{lm}$, $\omega = \omega_n$ is a root of (3.38) or any of its inversions. The roots ω_n are the *quasi-normal frequencies*. The nth inversion of (3.38) gives us

$$\frac{f_{n-1}h_n}{g_{n-1}-}\frac{f_{n-2}h_{n-1}}{g_{n-2}-}\cdots\frac{f_1h_2}{g_1-}\frac{f_0h_1}{g_0} = g_n - \frac{f_nh_{n+1}}{g_{n+1}-}\frac{f_{n+1}h_{n+2}}{g_{n+2}-}\cdots, \quad n > 0.$$

3.4 The Quasi-Normal Modes

Finally, we have all we need to find the *quasi-normal modes* numerically by the continued fraction method. The angular Teukolsky equation gave us a continued fraction such that the eigenvalues ${}_sA_{lm}$ are roots for given s, l, m and ω. At the same time, the radial Teukolsky equation provided us another continued fraction where this time the quasi-normal modes are the roots for given s, l, m, and ${}_sA_{lm}$. Then, what we need is to find out ${}_sA_{lm}$ and ω which are simultaneously roots of (3.31) and (3.23) or any of their inversions. This is the continued fraction method.

In 3.3.3 we found an expansion for $\rho = \lim_{n\to\infty} r_n$, where $r_n = \frac{a_{n+1}}{a_n}$, and we concluded that the minimal solution for the TTRR (3.35) does exist and it is the one with $c_1 = \sqrt{-\epsilon}$. We can use such expansion[4] to approximate r_N for a large N,

$$r_N \approx \frac{(N+1)^{c_2} e^{2c_1\sqrt{N+1}}}{N^{c_2} e^{2c_1\sqrt{N}}},$$

and then look for the roots of the truncated continued fraction. In 3.3.1 we discussed about ${}_sA_{lm}$ be equal to $l(l+1) - s(s+1)$ when $a\omega \to 0$. Therefore, we already have a "guess" for ${}_sA_{lm}$, at least for small c. Also, we know that given s and m, the smallest eigenvalue has the label $l_{\min} = \max(|s|, |m|)$ [15]. We already discussed the symmetries of the problem, then if we know ${}_sA_{lm}$ for given s, l, and m, we automatically know ${}_{-s}A_{lm}$ and ${}_sA_{l-m}$ (see section 3.3.1). Besides this, if ω is a solution, so it is $-\omega^*$.

Remember, the solution of the TME has the form

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} f(r, \theta, \phi) = e^{-i\omega_R t} e^{\omega_I t} f(r, \theta, \phi), \tag{3.39}$$

where ω_R and ω_I are the real and imaginary parts of ω, respectively. Therefore, by (3.39) a condition for the solution be stable is $\omega_I \leq 0$, otherwise the solution diverges as $t \to \infty$.

It is useful to define a parameter ι by

$$\frac{r_+ - r_-}{r_+ + r_-} \equiv \sin(\iota)$$

and we parameterize

$$a = M\cos(\iota), \qquad r_- = M[1 - \sin(\iota)], \qquad r_+ = M[1 + \sin(\iota)].$$

For gravitational waves ($s = -2$), for example, for $l = 2$ we obtain

$\cos\iota = a/M$	$\text{Re}(\omega)$	$\text{Im}(\omega)$	$\text{Re}({}_{-2}A_{20})$	$\text{Im}({}_{-2}A_{20})$
0.01	0.74735055	-0.17792335	3.99999310	0.00000348
0.02	0.74737210	-0.17791952	3.99997240	0.00001393
0.03	0.74740803	-0.17791314	3.99993790	0.00003134
0.04	0.74745834	-0.17790419	3.99988960	0.00005572
0.05	0.74752306	-0.17789267	3.99982740	0.00008707
0.10	0.74806358	-0.17779618	3.99930860	0.00034837
0.20	0.75024789	-0.17740081	3.99721620	0.00139472
0.30	0.75397013	-0.17670656	3.99366660	0.00314249
0.40	0.75936317	-0.17565311	3.98855960	0.00559598
0.50	0.76663655	-0.17413810	3.98173840	0.00875742
0.60	0.77610784	-0.17198934	3.97296910	0.01262019
0.70	0.78825858	-0.16890524	3.96190150	0.01715280
0.80	0.80383470	-0.16431253	3.94799660	0.02225636
0.90	0.82400893	-0.15696539	3.93038430	0.02763286
0.97	0.84171178	-0.14861264	3.91511080	0.03109069
0.98	0.84450849	-0.14706484	3.91268770	0.03151647

Table 1: Quasi-normal modes obtained computationally for $s = -2$, $l = 2$, and $m = 0$.

$\cos\iota = a/M$	$\text{Re}(\omega)$	$\text{Im}(\omega)$	$\text{Re}({}_{-2}A_{21})$	$\text{Im}({}_{-2}A_{21})$
0.01	0.74861005	-0.17790345	3.99500160	0.00118987
0.02	0.74989494	-0.17787979	3.98997080	0.00238713
0.03	0.75119832	-0.17785363	3.98490690	0.00359179
0.04	0.75252048	-0.17782492	3.97980930	0.00480385
0.05	0.75386169	-0.17779361	3.97467760	0.00602330
0.10	0.76086451	-0.17759660	3.94848430	0.01223122
0.20	0.77649564	-0.17697704	3.89315180	0.02519706
0.30	0.79466079	-0.17599034	3.83320640	0.03888096
0.40	0.81595822	-0.17451429	3.76757440	0.05324058
0.50	0.84126479	-0.17234599	3.69474520	0.06817770
0.60	0.87193694	-0.16912836	3.61247340	0.08346989
0.70	0.91024297	-0.16417045	3.51715390	0.09859410
0.80	0.96046141	-0.15590996	3.40228200	0.11217341
0.90	1.03258280	-0.13960869	3.25344620	0.11951010
0.97	1.11265720	-0.11128712	3.10648450	0.10799748
0.98	1.12831020	-0.10328544	3.07965980	0.10215726

Table 2: Quasi-normal modes obtained computationally for $s = -2$, $l = 2$, and $m = 1$.

$\cos\iota = a/M$	$\mathrm{Re}(\omega)$	$\mathrm{Im}(\omega)$	$\mathrm{Re}({}_{-2}A_{22})$	$\mathrm{Im}({}_{-2}A_{22})$
0.01	0.74987314	-0.17788366	3.98999190	0.00237669
0.02	0.75243219	-0.17784055	3.97989580	0.00476211
0.03	0.75502112	-0.17779524	3.96971000	0.00715630
0.04	0.75764056	-0.17774768	3.95943280	0.00955925
0.05	0.76029115	-0.17769780	3.94906240	0.01197100
0.10	0.77403508	-0.17741140	3.89574880	0.02416212
0.20	0.80429065	-0.17662233	3.78096840	0.04920850
0.30	0.83905336	-0.17545854	3.65313720	0.07512744
0.40	0.87968384	-0.17376392	3.50867840	0.10185231
0.50	0.92824605	-0.17127767	3.34226030	0.12919566
0.60	0.98808956	-0.16753040	3.14538660	0.15669039
0.70	1.06520050	-0.16158575	2.90316700	0.18315825
0.80	1.17203390	-0.15125910	2.58529400	0.20529749
0.90	1.34322850	-0.12973847	2.10981960	0.21112358
0.97	1.58641650	-0.08958071	1.48920120	0.16919868
0.98	1.65085900	-0.07726039	1.33361870	0.14999429

Table 3: Quasi-normal modes obtained computationally for $s = -2$, $l = 2$, and $m = 2$.

These results can be compared with the Leaver's results [33] or with the Berti's results, which can be found in his page [41]. Plotting the points starting from $a/M = 0.01$ up to $a/M = 0.98$ in steps of 0.01, we obtain the curves below. The same can be done for other values of s.

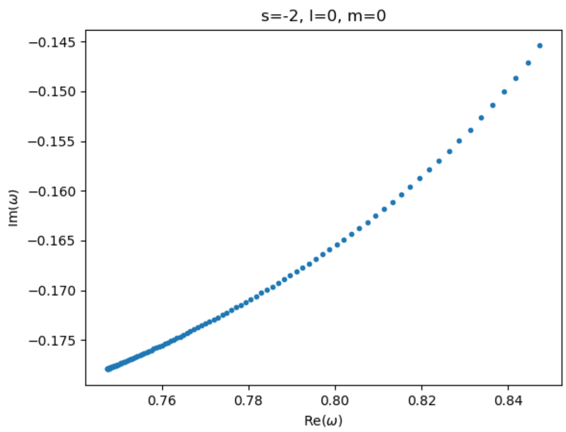

Figure 1: Numerical results for a gravitational perturbation with $l = 0$. In this case, the only allowed value for m is $m = 0$.

[4]We can use as much terms of the expansion (3.36) as we want.

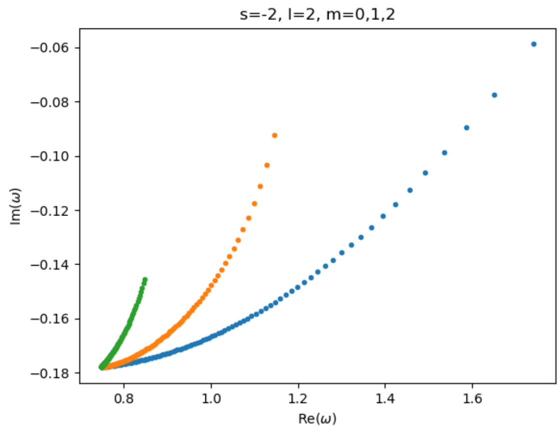

Figure 2: Numerical results for a gravitational perturbation with $l = 2$. The green dots are associated with $m = 0$, the orange with $m = 1$, and the blue with $m = 2$.

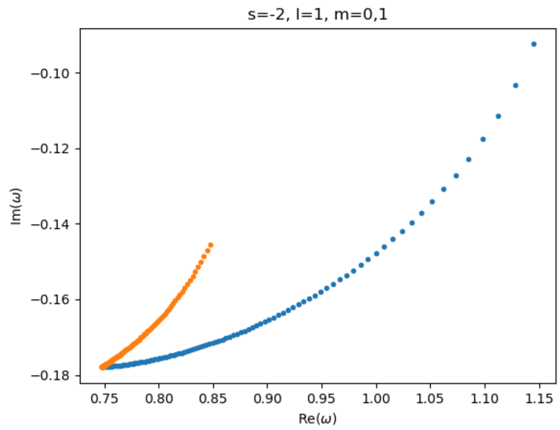

Figure 3: Numerical results for a gravitational perturbation with $l = 1$. The orange dots are associated with $m = 0$ and the blue dots with $m = 1$.

4 Scattering and Superradiance

The goals of the present chapter are to describe the scattering problem in the Kerr background, discuss black holes thermodynamics and superradiance.

4.1 Kerr Scattering

Let us start rewriting the boundary conditions in a little bit different way:

$$\psi \sim \mathcal{T}\Delta^{-s}e^{-ikr^*}e^{-i\omega t} + \mathcal{O}e^{ikr^*}e^{-i\omega t}, \qquad \text{for} \qquad r \to r_+, \tag{4.1}$$

$$\psi \sim \mathcal{I}\frac{e^{-i\omega r}}{r}e^{-i\omega t} \qquad + \mathcal{R}\frac{e^{i\omega r}}{r^{2s+1}}e^{-i\omega t}, \qquad \text{for} \qquad r \to \infty, \tag{4.2}$$

with

$$k = \omega - m\Omega_+ = \omega - \frac{am}{2Mr_+}.$$

A scheme representing such boundary conditions can be seen below.

Figure 4: Simple representation of the scattering problem.

The first term in (4.1) represents an ingoing wave at the event horizon r_+, while the second term represents an outgoing wave, where we are calling "ingoing" and "outgoing" as a local observer should refer to the directions of the waves. Analogously, in (4.2) the first term refers to ingoing, and the second to outgoing waves. Classically, one should require only ingoing waves at the event horizon, thus we should have $\mathcal{O} = 0$.

Remember the definition of tortoise coordinate used in section 2.3.2. For $r \to r_+$ we have

$$r^* = \int \frac{1}{f(r)}dr \sim \frac{1}{f'(r_+)}\log(r - r_+)$$

where

$$f(r) = \frac{r^2 + a^2}{(r - r_+)(r - r_-)}.$$

Now, taking $v = t + r^*$, the second term of (4.1) becomes

$$\begin{aligned} \mathcal{O}e^{-i\omega(-r^*+t)}e^{-im\Omega_+ r^*} &= \mathcal{O}e^{-im\Omega_+ r^*}e^{-i\omega v}e^{2i\omega r^*} \\ &\sim \mathcal{O}e^{-im\Omega_+ r^*}e^{-i\omega v}(r - r_+)^{\frac{2i\omega}{f'(r_+)}}. \end{aligned}$$

The function above is smooth only if $2i\omega/f'(r_+)$ is a positive integer. Thus, in principle, we should not consider this solution.

The conserved energy and angular momentum flux vectors are given in terms of the stress-energy tensor by [44]

$$\epsilon^\mu = -T^\mu{}_\nu(\partial_t)^\nu, \qquad l^\mu = T^\mu{}_\nu(\partial_\phi)^\nu,$$

respectively. And then, over a hypersurface $\mathbf{d}\Sigma_\mu$ we have

$$\delta E = \epsilon^\mu \mathbf{d}\Sigma_\mu, \qquad \delta J = l^\mu \mathbf{d}\Sigma_\mu,$$

where $\mathbf{d}\Sigma_\mu = n_\mu r^2 dt d\Omega$ and n_μ is an unit vector normal to the surface in the radial direction. Considering a scalar field of the form (2.14), that is,

$$\psi(t, r, \theta, \phi) = f(r, \theta)e^{-i\omega t + im\phi},$$

with the stress-energy tensor given by

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\psi^2, \tag{4.3}$$

one can find that for $s = 0$ and $s = \pm 1$, the energy flux at infinity per unit solid angle is given by

$$\frac{d^2E}{dt d\Omega} = \lim_{r\to\infty} r^2 T^r{}_t.$$

Using the asymptotic behavior (4.2), one finds for the bosonic fields $s = 0, \pm 1, \pm 2$ [17]:

$$\begin{aligned}
Y_0 &\equiv e^{-i\omega t+im\phi}P_0 = \Phi \sim e^{-i\omega t+im\phi}{}_0S_{lm}(\theta)\left(\mathcal{I}\frac{e^{-i\omega r^*}}{r} + \mathcal{R}\frac{e^{i\omega r^*}}{r}\right),\\
Y_1 &\equiv e^{-i\omega t+im\phi}P_1 = \phi_0 \sim e^{-i\omega t+im\phi}{}_1S_{lm}(\theta)\left({}_1Y_{\text{in}}\frac{e^{-i\omega r^*}}{r} + {}_1Y_{\text{out}}\frac{e^{i\omega r^*}}{r^3}\right),\\
Y_{-1} &\equiv e^{-i\omega t+im\phi}P_{-1} = \rho^{-2}\phi_2 \sim e^{-i\omega t+im\phi}{}_{-1}S_{lm}(\theta)\left(\mathcal{I}\frac{e^{-i\omega r^*}}{r} + \mathcal{R}e^{i\omega r^*}r\right),\\
Y_2 &\equiv e^{-i\omega t+im\phi}P_2 = \Psi_0 \sim e^{-i\omega t+im\phi}{}_2S_{lm}(\theta)\left({}_2Y_{\text{in}}\frac{e^{-i\omega r^*}}{r} + {}_2Y_{\text{out}}\frac{e^{i\omega r^*}}{r^5}\right),\\
Y_{-2} &\equiv e^{-i\omega t+im\phi}P_{-2} = \Psi_4\rho^{-4} \sim e^{-i\omega t+im\phi}{}_{-2}S_{lm}(\theta)\left(\mathcal{I}\frac{e^{-i\omega r^*}}{r} + \mathcal{R}e^{i\omega r^*}r\right),
\end{aligned}$$

where

$$\begin{aligned}
B({}_1Y_{\text{in}}) &= -8\omega^2\mathcal{I}, \qquad & -2\omega^2{}_1Y_{\text{out}} &= B\mathcal{R}\\
C({}_2Y_{\text{in}}) &= 64\omega^4\mathcal{I}, \qquad & \omega^4{}_2Y_{\text{out}} &= C^*\mathcal{R},
\end{aligned}$$

and $P_s \equiv R_s(r)S_s(\theta)$.

And then, the fluxes of energy at infinity for $s = 0, 1$ are related with the coefficients in (4.1) and (4.2) via

$$\begin{aligned}
\frac{dE_{out}}{dt} &= \frac{\omega^2}{2}|\mathcal{R}|^2, \qquad \frac{dE_{in}}{dt} = \frac{\omega^2}{2}|\mathcal{I}|^2, \qquad \text{for} \quad s = 0,\\
\frac{dE_{out}}{dt} &= \frac{4\omega^4}{B^2}|\mathcal{R}|^2, \qquad \frac{dE_{in}}{dt} = \frac{1}{4}|\mathcal{I}|^2, \qquad \text{for} \quad s = 1,
\end{aligned}$$

with $B^2 = [{}_sA_{lm} + a^2\omega^2 - 2ma\omega + s(s+1)]^2 + 4ma\omega - 4a^2\omega^2$. The expressions for $s = -1$ can be found using the TS identities.

For gravitational waves one finds [44, 42]

$$\frac{dE_{out}}{dt} = \frac{8\omega^6}{|C|^2}|\mathcal{R}|^2, \qquad \frac{dE_{in}}{dt} = \frac{1}{32\omega^2}|\mathcal{I}|^2, \qquad \text{for} \quad s = 2,$$

with $|C|^2 = B^2[({}_sA_{lm} + a^2\omega^2 - 2ma\omega + s(s+1))^2] + [2({}_sA_{lm} + a^2\omega^2 - 2ma\omega + s(s+1)) - 1](96a^2\omega^2 - 48ma\omega)$.

For an observer at infinity, the flux of energy and angular momentum at the horizon are given by

$$\delta E = -T_\mu{}^\nu(\partial_t)^\mu d^3\Sigma_\nu, \qquad \delta J = T_\mu{}^\nu(\partial_\phi)^\mu d^3\Sigma_\nu, \tag{4.4}$$

with the 3-surface element given by

$$d^3\Sigma_\mu = n_\mu\sqrt{-\tilde{g}}\, dtd\theta d\phi = n_\mu 2Mr_+ \sin\theta\, dtd\theta d\phi,$$

where $\tilde{g}$ is the determinant of the induced metric at the horizon r_+, and n^μ at the horizon is equal to the vector $-\chi^\mu$, as defined in the previous section. From (4.4) we obtain

$$\frac{d^2E}{dtd\Omega} = -2Mr_+T_\mu{}^\nu\partial_t{}^\mu n_\nu, \qquad \frac{d^2J}{dtd\Omega} = 2Mr_+T_\mu{}^\nu\partial_\phi{}^\mu n_\nu,$$

and thus we have that for any wave which enters into the black hole

$$\frac{d^2E}{dtd\Omega} - \Omega_H\frac{d^2J}{dtd\Omega} = 2Mr_+T^{\mu\nu}n_\mu n_\nu.$$

Besides this, for a stress-energy tensor of the form (4.3) one can find

$$\frac{\delta J}{\delta E} = -\frac{T^r{}_\phi}{T^r{}_t} = \frac{m}{\omega}. \tag{4.5}$$

Using the two equations above and $\delta E = \delta M$, we obtain

$$\frac{d^2E}{dtd\Omega} = \frac{\omega}{k_H}2Mr_+T^{\mu\nu}n_\mu n_\nu, \tag{4.6}$$

with $k_H = \omega - m\Omega_H$. Then, using the asymptotic behavior (4.2) we find

$$\frac{d^2E_{\text{hole}}}{dtd\Omega} = Mr_+\omega k_H\frac{{}_0S^2_{lm}(\theta)}{2\pi}|\mathcal{T}|^2, \qquad \text{for} \quad s = 0, \tag{4.7}$$

$$\frac{d^2E_{\text{hole}}}{dtd\Omega} = \frac{\omega}{8Mr_+k_H}\frac{{}_1S^2_{lm}(\theta)}{2\pi}|\mathcal{T}|^2, \qquad \text{for} \quad s = 1. \tag{4.8}$$

4.2 Thermodynamics and Superradiance

In section 2.1 we mentioned the Penrose process without an explanation. Now we are going to understand such process and its relation with the Hawking's area theorem.

4.2.1 The Penrose Process and Black Hole Thermodynamics

In order to understand the Penrose process it is important to know that in the Kerr geometry there is a special region called *ergoregion*. The ergoregion is the region between the outer horizon and the *ergosurface*. The ergosurface, in turn, is defined by $g_{tt} = 0$, that is, $r = M + \sqrt{M^2 - a^2 \cos^2 \theta}$. Hence it assumes the values $2M$ and r_+ in the equatorial plane and at the polos, respectively, for example. A representative picture of the ergoregion can be seen below. In such region it is not possible to have a static particle in the point of view of an observer at infinity. We can also define the ergoregion as the place where the Killing vector $\xi^\mu = (\partial_t)^\mu$, which is timelike at infinity, becomes spacelike. Outside the ergoregion ξ^μ is timelike, at the ergosurface it is null, and inside the ergoregion it becomes spacelike.

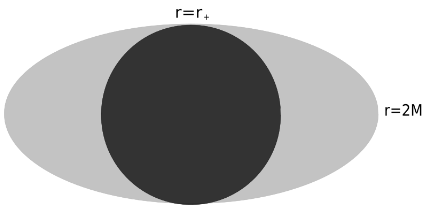

Figure 5: Ergoregion.

Penrose showed that because of the existence of the ergoregion it is possible to extract energy from a rotating black hole [43]. If p^μ is the four-momentum of a particle, its energy is given by $E = -\xi_\mu p^\mu$. At infinity both ξ^μ and p^μ are timelike (actually, p^μ is null for a massless particle and timelike otherwise), and then $E > 0$. However, in the ergoregion ξ^μ becomes spacelike, and thus an observer at infinity can measure a negative energy for a particle there.

The geodesic equation is given by

$$u^\mu \nabla_\mu u^\nu = 0, \tag{4.9}$$

where u^μ is the tangent vector, that is,

$$u^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu,$$

where λ is an affine parameter and, as it is clear by the equation above, the dot means derivative with respect to λ. Equation (4.9) is equivalent to

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{\partial \mathcal{L}}{\partial x^\alpha},$$

with

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu.$$

In section 2.1 we said that the Kerr metric has two Killing vectors, let us denote them by $k^\mu = (1, 0, 0, 0)$ and $m^\mu = (0, 0, 0, 1)$. Then, it follows that along the geodesic one has two conserved quantities:

$$E = -k_\mu u^\mu = -p_t,$$
$$L = m_\mu u^\mu = p_\phi,$$

where p_t and p_ϕ are components of the conjugate momenta

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu = \dot{x}_\mu.$$

For a massive particle, E and L are the energy at infinity per mass unit and the angular momentum per mass unit, respectively. While for a massless particle they represent the energy at infinity and the angular momentum.

The geodesic equations for a pointlike particle in the equatorial plane ($\theta = \pi/2$) are

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r^2}\right)E - \frac{2Ma}{r}L\right], \tag{4.10}$$

$$\dot{\phi} = \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)L + \frac{2Ma}{r}E\right] \tag{4.11}$$

$$r^2\dot{r} = \Delta(E\dot{t} - L\dot{\phi} + \kappa), \tag{4.12}$$

where $\kappa = g_{\mu\nu}u^\mu u^\nu$ is equal to -1 for timelike geodesics, 0 for null geodesics, and 1 for spacelike geodesics.

Now, consider a particle with rest mass μ_i at infinity of the Kerr spacetime following a geodesic in the equatorial plane. Let us say that in the turning point r_0 ($\dot{r}|_{r_0} = 0$) such particle decays in two identical particles with rest mass μ_f. We will denote by E_i the energy per mass unit of the first particle, and by E_{f1} and E_{f2} the energies per mass unit of the identical particles. In the same way, we will denote by L_i, L_{f1}, and L_{f2} the angular momentum per mass unit of them. We require that the energy and angular momentum E_i and L_i are such that the turning point occurs in the ergoregion, that is, $r_0 < 2M$. Besides, we write $E_i = \mathcal{E}_i/\mu_i = 1$, $E_{f1,f2} = \mathcal{E}_{f1,f2}/\mu_f$, $L_i = \mathcal{L}_i/\mu_i$, and $L_{f1,f2} = \mathcal{L}_{f1,f2}/\mu_f$.

Using (4.12) in r_0, we obtain:

$$L_i = \frac{2aM - \sqrt{2Mr_0\Delta}}{2M - r_0},$$

$$L_{f1,f2} = \frac{2aME_{f1,f2} \mp \sqrt{\Delta r_0[2M + (E_{f1,f2}^2 - 1)r_0]}}{2M - r_0}.$$

And by energy and angular momentum conservation

$$\mathcal{E}_{f1} + \mathcal{E}_{f2} = \mathcal{E}_i, \qquad \mathcal{L}_{f1} + \mathcal{L}_{f2} = \mathcal{L}_i,$$

we obtain

$$\mathcal{E}_{f1} = \frac{\mu_i}{2}\left[1 \pm \sqrt{\frac{2M}{r_0}\left(1 - 4\frac{\mu_f^2}{\mu_i^2}\right)}\right], \qquad \mathcal{E}_{f2} = \frac{\mu_i}{2}\left[1 \mp \sqrt{\frac{2M}{r_0}\left(1 - 4\frac{\mu_f^2}{\mu_i^2}\right)}\right].$$

Once we are talking about one particle decaying into two *identical* particles, the term $4\mu_f^2/\mu_i^2$ is smaller than 1. Besides, we already said that r_0 is inside the ergoregion, and thus $2M/r_0 > 1 \Rightarrow 2M(1 - 4\mu_f^2/\mu_i^2)/r_0 > 1$. Therefore, the energy which takes the minus sign is *negative* while the one which takes the plus sign is larger than the energy of the incoming particle. Let us associate $\mathcal{E}_{f1}$ to the negative energy and $\mathcal{E}_{f2}$ to the positive one. Hence if the particle "$f1$" falls into the black hole and "$f2$" escapes, in summary, from the point of view of an observer at infinity, one particle comes into the ergoregion and another one gets out with a larger energy. The energy must have come from somewhere and there is no other option besides the black hole.

How much energy can a particle extract from a black hole? We define η as the ratio[1]

$$\eta = \frac{\mathcal{E}_{f2}}{\mathcal{E}_0} = \frac{1}{2}\left[1 + \sqrt{\frac{2M}{r_0}\left(1 - 4\frac{\mu_f^2}{\mu_i^2}\right)}\right].$$

From the equation above we can see that η is maximum for the minimum value of r_0, that is, for $r_0 = r_+$. Besides, η is larger for $\mu_f \to 0$, then

$$\eta_{\max,\,\mu_f \to 0} = \frac{1}{2}\left(1 + \sqrt{\frac{2M}{r_+}}\right).$$

There are some ways to explain the Penrose process. The present explanation was strongly based on [44]. Moreover, in [43] Penrose and Floyd, for example, state the process as a particle which in some point of the ergoregion splits into two (not necessarily identical) particles. In [11] Carroll uses the idea of someone coming into the ergoregion with a big rock, there he throws the rock, which falls into the black hole, and then escapes to infinity with a gain of energy.

Now, what is the link between the Penrose process and black hole thermodynamics? We said that in the process occurs an extraction of energy from the black hole and soon we will understand how.

A stationary and not static spacetime has a Killing horizon, that is, a hypersurface Σ where the vector field $\chi^\mu = (\partial_t)^\mu + \Omega_H(\partial_\phi)^\mu$ is null, with Ω_H being a constant [11]. Besides, any event horizon of a stationary spacetime is a Killing horizon. For the Kerr spacetime the outer horizon is a Killing horizon with Ω_H being the angular velocity at the horizon:

$$\Omega_H = \Omega_+ = \frac{a}{2Mr_+}.$$

The momentum p^μ_{f1} of the particle which falls into the black hole is timelike, while the vector χ^μ is null at the horizon, then the inner product between them is negative at r_+. Therefore, at r_+

$$\begin{aligned} p^\mu_{f1}\chi_\mu < 0 \quad &\Rightarrow \quad p^\mu_{f1}(k_\mu + \Omega_H m_\mu) < 0 \quad \Rightarrow -E_{f1} + \Omega_H L_{f1} < 0 \\ &\Rightarrow \quad L_{f1} < \frac{E_{f1}}{\Omega_H}. \end{aligned} \tag{4.13}$$

Thus, once that $E_{f1} < 0$ and $\Omega_H > 0$, equation (4.13) implies $L_{f1} < 0$. In other words, the particle $f1$ has an angular momentum in the opposite direction of the angular momentum of the black hole.

[1]For a generic stationary axisymmetric spacetime, the result would be [44]

$$\eta = \frac{1}{2}\left[1 + \sqrt{(1 + g_{tt})\left(1 - 4\frac{\mu_f^2}{\mu_i^2}\right)}\right].$$

As a consequence, before the Penrose process, the mass and the angular momentum of the black hole changes by

$$\delta M = E_{f1}, \qquad \delta J = L_{f1}.$$

And then, by (4.13)

$$\delta J \leq \frac{\delta M}{\Omega_H}, \tag{4.14}$$

where the equality would happen for the limit case when p^{μ}_{f1} is null, that is, when the incoming particle decays into massless particles. In summary, at the end of the process, both M and J decrease.

On the other hand, Christodoulou showed that there is a quantity which can not decrease [45, 46]. Such quantity is called *irreducible mass* and for the Kerr black hole is given by

$$M_{\text{irr}} = \sqrt{\frac{Mr_+}{2}} = \sqrt{\frac{A}{16\pi}}, \tag{4.15}$$

where A is the area of the black hole, as was said in 2.1. We can do the same that we did in section 2.1, but now for the irreducible mass: substituting A by $4\pi(r_+^2 + a^2)$, r_+ by $M + \sqrt{M^2 - a^2}$, a by J/M, and differentiating M_{irr}^2, we obtain

$$\delta M_{\text{irr}} = \frac{a}{4M_{\text{irr}}\sqrt{M^2 - a^2}} \left(\frac{\delta M}{\Omega_H} - \delta J \right). \tag{4.16}$$

Then, it follows by (4.14) that, in fact,

$$\delta M_{\text{irr}} \geq 0.$$

By equation (4.15) one can conclude that if M_{irr} can not decrease, the same is true for the area A ($\delta A \geq 0$ is the Hawking's area theorem). Using equations (4.15) and (4.16) we recover the equation (2.3) showed in section 2.1. For a Kerr-Newman black hole, a rotating charged black hole, the equation is given by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q, \tag{4.17}$$

where κ is the surface gravity and Φ_H is the electrostatic potential at the horizon [44].

4.2.2 The Zeroth, Second, and Third Laws of Black Hole Thermodynamics

Equation (4.17) is already the "first law" of black holes thermodynamics for the case of a spinning charged BH. For the zeroth law there are two versions [47]. The first one states that κ, the surface gravity, must be constant at its event horizon. The second one states that κ must be constant at any Killing horizon when the dominant energy condition[2] is satisfied.

The second law of thermodynamics states that the entropy of a system never decreases. But such law does not seem to make much sense in the black hole context, because once

[2] Let v^a be either a timelike or null future directed vector. The dominant energy condition states that $w^a = -T^a{}_b v^b$ is also a future directed vector. Physically, it means that the observer v^a can not measure a speed of energy flow of matter faster then c [12].

that something crosses the event horizon, in principle, all the information about it is lost. However, we can define the *generalized entropy* as the sum of the BH entropy and the entropy of the matter distribution outside the BH. That is,

$$S' \equiv S_{bh} + S_{out}.$$

And then, the *generalized second law* (GSL), which was proposed by Bekenstein, states that the generalized entropy S' never decreases with time:

$$\Delta S' \geq 0.$$

Nevertheless, it is questionable if the GSL do hold. By energy conservation, because of the Hawking radiation a black hole should lose mass while irradiates, what means that the area of the BH will decrease and then the GSE should not hold. On the other hand, in the Hawking process we have particle creation. So there is hope for the GSL if the amount of entropy generated outside by the particle creation compensates the decrease of area. When the factor η in (2.4) is indeed $1/4$, and then $S_{bh} = A/4$ in Plank units, the generalized second law holds. A great discussion about the GSL can be seen in section 4 of [47].

The first way to think about the third law is to state that $S \to 0$ as $T \to 0$. (Remember that, as was said in section 2.1, the temperature and the entropy of a BH are given by $T = \kappa/2\pi$ and $S = A/4$.) However, extremal BHs have $\kappa = 0$ and a finite A. Another way to state the third law for BHs is: it is impossible, by a finite number of processes, to reduce κ to zero. Discussions about the third law and its possibilities can be seen in [47, 48].

4.2.3 Superradiance

The result (4.5) written in section 4.1 also holds for other kinds of fields [44]. Substituting (4.5), exchanging δE by δM, into the first law (4.17) with $Q = 0$, we obtain

$$\delta M = \frac{\omega\kappa}{8\pi} \frac{\delta A}{\omega - m\Omega_H}. \tag{4.18}$$

The Hawking area theorem states that $\delta A \geq 0$, as already was said. Therefore, the equation above tells us that for $\omega > 0$, *waves with* $\omega < m\Omega_H$ *extract energy from the black hole*, once that $\delta E = \delta M < 0$. We say that in such regime occur *superradiance*. In this sense, superradiance appears as a consequence of the Hawking's area theorem.

On the other hand, in section 4.1 we obtained an expression for the energy flux per solid angle unit, equation (4.6). By such equation, assuming $T^{\mu\nu} n_\mu n_\nu > 0$, we already have that the energy flux is negative if $0 < \omega < m\Omega_H$. This can be seen explicitly for $s = 0$ and $s = 1$ by equations (4.7) and (4.8). When we write $0 < \omega < m\Omega_H$ we are, obviously, considering $\omega \in \Re$.

We define the *amplification factor* as

$$Z_{slm\omega} \equiv \frac{dE_{out}}{dt} \left(\frac{dE_{in}}{dt} \right)^{-1} - 1.$$

In terms of $dE_{in,\,out}/dt$, the superradiance regime corresponds to

$$\frac{dE_{in}}{dt} < \frac{dE_{out}}{dt},$$

which follows from $\frac{dE_{\text{hole}}}{dt} < 0$. Then, clearly, $Z_{slm\omega} > 0$ in the superradiance regime. $Z_{slm\omega}$ inherit the symmetries of the TME, and then we have

$$Z_{slm\omega} = Z_{sl-m-\omega}.$$

As an example, for bosonic fields we have [44]

$$\begin{aligned}
Z_{slm\omega} &= \frac{|\mathcal{R}|^2}{|\mathcal{I}|^2} - 1, && \text{for} \quad s = 0,\\
Z_{slm\omega} &= \frac{|\mathcal{R}|^2}{|\mathcal{I}|^2}\left(\frac{16\omega^4}{B^2}\right)^{\pm 1} - 1, && \text{for} \quad s = \pm 1,\\
Z_{slm\omega} &= \frac{|\mathcal{R}|^2}{|\mathcal{I}|^2}\left(\frac{256\omega^8}{|C|^2}\right)^{\pm 1} - 1, && \text{for} \quad s = \pm 2.
\end{aligned}$$

Where B and C are the same as defined in section 4.1.

By equation (4.18) we can note that δM is larger as $\omega - m\Omega_H$ is smaller. In other words, the extraction of energy will be larger for small frequencies. In the low-frequency regime, $M\omega \ll 1$, it follows that [44, 49, 50]

$$Z_{slm\omega} = Z_{0lm\omega}\left[\frac{(l-s)!(l+s)!}{(l!)^2}\right]^2, \tag{4.19}$$

with

$$Z_{0lm\omega} = -8mr_+k\omega^{2l+1}(r_+ - r_-)^{2l}\left[\frac{(l-s)!(l+s)!}{(l!)^2}\right]^2 \prod_{j=1}^{l}\left[1 + \frac{M^2}{j^2}\left(\frac{k}{\pi r_+ T_+}\right)^2\right].$$

Expanding equation (4.19) for large l, we infer that in the low-frequency regime, for $l \gg 2s^2$ the amplification factor does not depend on the spin of the field.

The maximum amplification for electromagnetic waves occur when $l = m = 1$. Then, substituting $l = 1$ and $s = 1$ in (4.19) we obtain that, in the low-frequency regime, the maximum amplification factor for such waves is 4 times the amplification factor for scalar waves. Analogously, for gravitational waves the maximum amplification occur when $l = m = 2$, and when we substitute $l = 2$ and $s = 2$ in (4.19) the result is an amplification factor 36 times $Z_{0lm\omega}$. This shows that it is easiest to measure a sign resulting from a gravitational perturbation than one caused by scalar or electromagnetic perturbation.

5 Conclusion

The Kerr spacetime has a rich geometry; because the black hole possesses an angular momentum J, it is not spherically symmetric, which makes harder the study of perturbation fields in such background when compared with the Schwarzschild black hole. On the other hand, it is precisely this property that makes the Kerr black hole so interesting. In order to be able to discuss perturbations in the Kerr spacetime, we first studied such spacetime in chapter 2. There, in the first section, we discussed the symmetries of such solution of Einstein's equations in vacuum, some properties, and the nature of the singularities of the metric. Then we started to approach the problem. The equation which describes a s-spin perturbation in the Kerr background is the so-called *Teukolsky master equation.*

The TME is a partial differential equation in four variables, t, r, θ, and ϕ, the variables used to write the Kerr metric in the Boyer-Lindiquist coordinates, for the function $\psi_s(t, r, \theta, \phi)$. The TME also depends on the spin-weight s of the field. For example, when $s = \pm 1$ the functions $\psi_{\pm 1}(t, r, \theta, \phi)$ are related with the components ϕ_0 and ϕ_2 of the Maxwell tensor, and when $s = \pm 2$ the functions $\psi_{\pm 2}(t, r, \theta, \phi)$ are related with the components Ψ_0 and Ψ_4 of the Weyl tensor.

The formalism which was used to obtain the TME is the *tetrad formalism*, which consists of writing the metric in terms of a null basis. We used such formalism and then we made a first-order perturbation on the so-called Newman-Penrose equations. To a certain degree, it was not necessary to use explicitly the metric, it was only necessary to state that is a metric of type D. Therefore, the same mechanism applies for any other type D solution of Einstein's equations.

The great characteristic of the TME is its separability. The function $\psi_s(t, r, \theta, \phi)$ can be written as a product of four functions, each one depending on one of the variables. In particular, the t part is given by $e^{i\omega t}$. The possible values for ω in the region $r_+ < r < \infty$ are the so-called quasi-normal modes. We dedicated the chapter 2 to find them by the *continued fraction method* or, as it is referred sometimes, the *Leaver's method*, once that Leaver was the first one who applied the method in the black hole context.

Both angular and radial Teukolsky equations can be put in the confluent Heun equation form. The continued fraction method consists of after writing them in one of the forms of the CHE, write a series expansion for each of them. When we do this for the TAE we obtain a three-term recurrence relation, which gives a convergent continued fraction. When we do the same for the TRE, we first obtain a continued fraction that does not converge (the TRE has different asymptotic behavior and a different kind of series expansion). However, using a variable transformation, we obtain a new ODE such that the series expansion gives us another TTRR, which in turn gives us a convergent continued fraction. The convergence of the continued fraction is discussed in chapter 2. The Pincherle theorem is very important to ensure the convergence of both continued fractions: the one which comes from the angular part, and the one which comes from the radial part. Both the continued fractions depend on ω and ${}_sA_{lm}$, the separation constant. Once we have them we are able to obtain numerically ω and ${}_sA_{lm}$. We showed some results at the end of chapter 2.

The last chapter is dedicated to discussing black hole thermodynamics and superradiance. In order to do that, we first discussed the Kerr scattering problem, which is actually what we were doing since the first chapter. The way we choose to introduce black hole thermodynamics was by explaining the Penrose process. There exists a special region in the Kerr spacetime called ergoregion. Penrose noted that because of the existence of such region, it is possible to extract energy from the black hole. Exploring the Penrose process we found $\delta J \leqslant \delta M/\Omega_H$ (equation (4.14)), and because of the existence of the irreducible mass, showed by Christodoulou, we concluded that the area of the black hole can not decrease, which is the Hawking's area theorem. We also obtained an equation that is very similar to the first law of thermodynamics, and it is the *first law of black hole thermodynamis*. After doing that, we briefly discussed the *zeroth, second* and *third laws of BH thermodynamics*. Lastly, to end the chapter, we discussed superradiance as a consequence of BH thermodynamics. We observed, for example, that in the low-frequency regime the amplification factor is larger when $s = 2$, that is, for gravitational waves.

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