UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA

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An Alternative Approach to Open Quantum Systems

Recife

2021

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Trabalho apresentado ao Programa de Pós-graduação em Física do Centro de Ciências Exatas e da Natureza da Universidade Federal de Pernambuco, como requisito parcial para obtenção do grau de Mestre em Física.

Área de Concentração: Física Teórica e Computacional

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Recife

2021

Catalogação na fonte
Bibliotecária Fernanda Bernardo Ferreira, CRB4-2165

M777a Montenegro, José Bento Ferreira
An Alternative Approach to Open Quantum Systems / José Bento Ferreira Montenegro. – 2021.
78 f.: il., fig.

Orientadora: Nadja Kolb Bernardes.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CCEN, Física, Recife, 2021.
Inclui referências.

1. Física Teórica e Computacional. 2. Sistemas abertos. 3. Medição. 4. Spin. I. Bernardes, Nadja Kolb (orientadora). II. Título.

530.1 CDD (23. ed.) UFPE- CCEN 2021 - 105

JOSÉ BENTO FERREIRA MONTENEGRO

AN ALTERNATIVE APPROACH TO OPEN QUANTUM SYSTEMS

Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Física.

Aprovada em: 28/04/2021

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Dedicated to all of those who have a project beyond themselves.

ACKNOWLEDGEMENTS

I would like to express all of my gratitude to my mother, Danielle, and my grandmother, Rejane, whom I own not only all I could get so far, but also all who I am.

To my uncle Emiliano, by all of his love, support and productive conversations. To Valentin, Irina, Cora, João, Laís, Maurício and Enison. To my grandfather, prof. Carlos Dantas, who teached me how “un-human" sciences require the love, the inspiration and the motivation of the human spirit. To his wife, Ana.

To my precious friends Érica, Pedro Ivo, Euclides, Marton, Lea and Ricardo by all of their emotional support and love. To Jayme Castro and *in memorian* of my friend Paulo Teixeira, who advised me that philosophy and physics walk together, but sometimes may wait a little while for each other. To Leyllyanne, who listens to me and helps me since long.

To Nadja, who supported me day after day in the development of the present text, and who always added, as Descartes would say, lead feet to my wings. To Parisio, whose style of work and ideas inspire me to absorb them into my future as a researcher.

To UFPE, DF-UFPE and CNPq by giving me full institutional support and funding for developing my career so far. To Azadeh, Bruno, S. Rezende and Leonardo Cabral, who teached me that everyday dignity and professional life are indeed indissociable.

To all of my quarantine and personal-evolution companions of 2020, 2021 and always: Slavoj Žižek, Sigmund Freud, Jacques Lacan, Christian Dunker, Immanuel Kant, Friedrich Nietzsche, Georg Hegel, Maria Homem, Vladmir Safatle, Antônio Abujamra, Bertolt Brecht, Franz Kafka, Johann Wolfgang von Goethe, Milan Kundera, Fernando Pessoa, Mikhail Tal and Tim Ingold. To Niels Bohr, Werner Heisenberg, Richard Feynmann and Marcelo Gleiser by recalling me everyday that the mind of a physicist can never be limited by real problems only.

To Luis Inácio Lula da Silva, who, since long, represents hope and dignity to all of the so long marginalized and forgotten people of this big, strong, beautiful and amazing country where I am born. To Leonel Brizola and Miguel Arraes.

"Sempre o mistério do fundo tão certo como o sono de mistério da superfície." (CAMPOS, 2019, p. 165).

ABSTRACT

The dynamics of open quantum systems is described by a stochastic process that comes from the inevitable interaction between the system and its environment. More precisely, the environment is chosen as an external system which may have a finite or an infinite number of degrees of freedom. We analyse here how the orbital degrees of freedom (position, momentum) of a system represented by a coherent state of the quantum harmonic oscillator coupled with an in-homogeneous magnetic field can suffer transitions when a sequence of measurements if performed in its spin degree of freedom. In other words, the situation is as if the orbital degree of freedom represents a system of interest and the possible outcomes of such a measurement process represents a stochastic source. We shall explicitly show, using an analysis based on the concept of quantum trajectories, that it is a problem to which an analytical solution could be obtained to all relevant physical quantities. Moreover, we could derive an equal-probability rule for the state of our system: similar to the central *a priori* postulate of statistical mechanics. Through our analytical expressions and a numerical analysis, we could also determine that the continuous measurement process, in general, raises the mean energy of the system keeping its mean position and mean momentum invariant at the cost of making the distribution of their uncertainties broader.

Keywords: Opens systems. Measurement. Spin. Coherent states. Quantum trajectories. Random walks.

RESUMO

A dinâmica de sistemas quânticos abertos é descrita por um processo estocástico resultante da inevitável interação de um sistema de interesse com o seu ambiente. Mais precisamente, o ambiente é geralmente escolhido como um sistema externo possuindo um número finito ou infinito de graus de liberdade. Analisaremos aqui como os graus de liberdade orbitais (posição, momentum) de um sistema representado por um estado coerente do oscilador harmônico quântico em contato com um campo magnético não homogêneo podem sofrer transições quando uma sequência de medições do grau de liberdade de spin deste mesmo sistema é efetuada. Em outras palavras, tudo se passa como se os graus de liberdade orbitais representassem um sistema de interesse e os possíveis resultados destas medições representassem uma fonte de aleatoriedade. Mostraremos, utilizando uma análise baseada no conceito de trajetórias quânticas, que trata-se de um problema em que expressões analíticas fechadas puderam ser obtidas para todas as quantidades físicas de interesse. Além disso, descobrimos que o estado do sistema obedece a uma regra de iguais probabilidades, semelhante ao postulado central *a priori* da mecânica estatística. Utilizando nossas expressões analíticas, bem como uma análise numérica, descobrimos também que, em geral, o processo de medição de spin aumenta a energia média do sistema, mantendo sua posição média e seu momentum médio invariantes às custas de aumentar também a distribuição de suas respectivas variâncias.

Palavras-chaves: Sistemas abertos. Medição. Spin. Estados coerentes. Trajetórias quânticas. Caminhadas aleatórias.

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1 INTRODUCTION

In quantum mechanics, the time-evolution of the wavefunction of a closed system is described by the Schrödinger equation. It is well known that this statement is equivalent to say that the state of the system evolves according to the action of the unitary operator $\hat{U} = exp\{-i\hat{H}t/\hbar\}$, where $\hat{H}$ is the Hamiltonian operator associated with the system. However, in the situation where the system is *open*, that is, when the system is put in contact with an external system (or an *environment*) and no longer can be treated as closed, this unitary operator has to be replaced by other dynamical maps, described by the so called *Kraus operators* (see $\S 3.1$).

Now there are not only one, but many possibilities for how exactly the system may change in time due to the contact with the environment. Those possibilities are described by a probability distribution and, because of that, we say that the dynamical map induces what we call a *stochastic process*. In this way, the environment is said to be an *stochastic source* for the system in question.

In the present work we shall explore this idea of studying an open system in a different way. Rather than analysing the situation of an environment as a stochastic source, we will analyse the situation where continuous observations (or measurements) of a certain degree of freedom of a system may induce changes in *another* degree of freedom of the same system. Because in quantum mechanics measurements return random outcomes (except for the case where the system is in an eigenstate of the chosen observable), this observed degree of freedom can then be considered as a stochastic source for the degree of freedom that we want to study.

Let us be more precise here. We shall consider the initial orbital state of our system (a spin-1/2 particle) as a coherent state of the harmonic oscillator $|z\rangle$ and take its initial spin state as an eigenstate of the spin angular momentum operator in the x-direction, $\hat{S}_x$. In this way, the total initial state (orbital + spin) is given by $|\psi(0)\rangle = |z\rangle \otimes |s_0\rangle_x$, where $s_0 = \pm 1$ (the choice of an eigenstate of $\hat{S}_x$ instead of $\hat{S}_z$ will be clear in a minute). Furthermore, we consider that an external ihnomogeneous magnetic field pointing in the z-direction is present throughout the region of the space that the particle is put in. We choose this field as varying linearly with the position operator $\hat{X}$: $\hat{\boldsymbol{B}} \stackrel{def}{=} (\hat{B}_0/d)\hat{X}\hat{\boldsymbol{e}}_z$, where d is a constant with units of length. In this case, the spin degree of freedom interacts with the field so that the Hamiltonian of the system can be taken as $\hat{H} = \hat{H}_{HO} \otimes \hat{I} + \hat{H}_{int}$, where $\hat{H}_{HO} = \hat{P}^2/2m + m\omega^2/2\hat{X}^2$ is the well known

hamiltonian of the harmonic oscillator and $\hat{H}_{int} = -\gamma\hat{\boldsymbol{B}} \cdot \hat{\boldsymbol{S}} = -(\gamma\hbar B_0/2d)\hat{X} \otimes \hat{S}_z$ is the interaction energy between the spin an the field (of course $\hat{S}_z$ is the spin angular momentum operator in the z-direction).

The system is then let to evolve unitarily by a time δt, so that its state turns into a state of the form $|\psi(\delta t)\rangle = C[|\psi_+(\delta t)\rangle \otimes |+\rangle_x + s_0 |\psi_-(\delta t)\rangle \otimes |-\rangle_x]$, where C is a normalization constant. Now, according to the well known measuremnt postulate of quantum mechanics, if a spin measurement in the x-direction is performed, the state of the system must be immediately updated to either the state $(|\psi_+(\delta t)\rangle / || |\psi_+(\delta t)\rangle ||) \otimes |+\rangle\, x$ or the state $(|\psi_-(\delta t)\rangle / || |\psi_-(\delta t)\rangle ||) \otimes |-\rangle\, x$. In other words, the measurement performed on the spin degree of freedom induces a change in the orbital degree of freedom of the system.

If this process (time-evolution followed by a x-direction spin measurement) is repeated a finite number N of times from now on there will be a probability p_N that at the end the system will be in a certain state $|\psi_N(N\delta t)\rangle$. A single repetition like such of this whole N-step process (or protocol) is called a *trajectory*. Indeed, this terminology is not restricted to this specific case, but rather, comes from algorithmic approaches used to simulate the dynamics of open systems such as the one in (MøLMER; CASTIN; DALIBARD, 1993) (which shall be discussed briefly in §3.2).

After the realization of a large number (or an ensemble) of such trajectories we can obtain a set of all possible accessible states $\{\left|\psi_N^k\right\rangle\}$ for the system together with their respective probabilities (or frequencies) $\{p_k\}$ of occurrence. Finally, with this data in hands, we can construct a *density operator* for the orbital state of the system simply as $\hat{\rho}_S^N = p_k \sum_k \left|\psi_N^k\right\rangle \left\langle\psi_N^k\right|$ and use it to compute explicitly the expectation value of many physical quantities of interest, such as energy, position, momentum and so on.

This idea is somewhat analogous to the idea of a so-called *quantum random walk* (AHARONOV et al., 1993) (for a review, see (KEMPE, 2003), (VENEGAS-ANDRACA, 2012)). There we have the followig: a system is prepared in a certain initial orbital state $|\psi_{x_0}\rangle$ associated with a wave-function centered at $x = x_0$ and in a certain superposed spin state, say, $c_+ |+\rangle + c_- |-\rangle$, where $|\pm\rangle$ are the eigenstates of $\hat{S}_z$. Then the orbital state is displaced from x_0 by a certain distance l through the action of an effective displacement operator of the form $exp\{-i\hat{P} \otimes \hat{S}_z l/\hbar\}$, where $\hat{P}$ is the momentum operator associated with its orbital part. After that, the state of the system will be of the form $c_+ |\psi_{x_0+l}\rangle \otimes |+\rangle + |\psi_{x_0-l}\rangle \otimes |-\rangle$ and, only then, a spin measurement is performed, so that the orbital state colapses onto $|\psi_{x_0 \pm l}\rangle$ with probability $|c_\pm|^2$. Therefore, if we compare with a classical random walk, a spin measurement functions as the process of

"throwing a coin".

One of the differences between this approach of quantum random walks and our approach is also that, in the former, in order to displace the center of the new wavefunction obtained after each measurement by a further random step, we must "manually" reinitialize the spin state of the system to a superposition like $c_+ |+\rangle + c_- |-\rangle$; whist, in our case, it is the external magnetic field $\hat{\boldsymbol{B}}$ which tends to bring the spin of the system to realign itself with the z-direction (now it must be clear why we choose to perform measurements in the x-direction). Nevertheless, a similar (but in fact different) situation to ours can be found in (HOROWITZ, 2012), where a system of interest is coupled with an external two-level reservoir subject to measurement processes.

Our text is organized as follows: in chapter 2, we begin with an overview of the basic textbook topics of quantum mechanics, such as its postulates (formulated in its simplest form), the quasi-classical (coherent) states of the harmonic oscillator, spin and density operators. However, it must be understood that the only aim of the chapter is to cover the basics of such topics in order to fix notations, help the inexperienced reader to apprehend those fundamental tools and provide a somewhat "auto-consistent" formulation of our work.

In chapter 3, we formulate also a very basic (but somewhat sufficient for our purposes) overview of quantum open systems and analyse the so called *Monte Carlo Wave Function Approach* as an illustration of how to use quantum trajectories as a fundamental and algorithmic analysis tool. Finally, we shall finish with an exposition of a *measurement master equation*. Those two latter topics will be used as a motivation for what follows.

Chapter 4 is the main part of the present text. There we shall formulate in a precise way our (already outlined here) alternative approach to open quantum systems regarding all of its mathematical subtleties and comment all the results which we could obtain so far. Objectively, we address the following question: "how do random outcomes of continuous measurements performed on the spin degree of freedom of this system may induce transitions in its orbital degree of freedom?" It shall be shown that this is a problem to which we could obtain a closed analytical solution not only to the state of the system after the realization of an ensemble of quantum trajectories, but also to all relevant physical quantities associated with it (such as energy, position, momentum and so on). We shall see that this state satisfies an equal-probability rule, similarly to what happens in the context of usual statistical mechanics (without, of course taking this analogy too far, because in this latter topic we are dealing with a closed, not an open, system). At the end of the chapter, numerical plots are presented in order to give us

an wide view of how those quantities exactly behave and an appendix containing all lengthy calculations is included so that the interested reader may check precisely all of our results.

Finally, we conclude in chapter 5 with an overview of all results obtained and with some prospects which we ma intend to consider from now on.

Quantum measurements have been studied today in the context of heat exchanges and thermal machines, a major subject which fits in the research area of *quantum thermodynamics*. Therefore, we consider relevant to explore our approach to the dynamics of open quantum systems and refer to (ELOUARD et al., 2017b), (ELOUARD et al., 2017a), (JORDAN; ELOUARD; AUFFèVES, 2020) as a further motivation for the present work. It must be also pointed that we believe that a narrative more close to the context of "kicked harmonic oscillators" (part of the subject of the research area of quantum chaos and semi-classical methods) may be also possible and refer the interested reader to (BERMAN; ZASLAVSKY, 1991).

2 BASICS

As our work shall require a very basic picture of what is quantum mechanics, in this first chapter we intend to make a brief overview of specific theoretical aspects in order to clarify concepts, fix notation and make latter references. We focus precisely in two example systems, which shall be fundamental for us: the so called *coherent states* of the quantum harmonic oscillator and a spinning particle. This introduction follows the *Schrödinger picture* of quantum mechanics, where quantum states (rather than quantum operators) are the quantities which evolve in time. Our aim is to be as simple as possible for our purposes and for further reading, references will be given when necessary.

2.1 RULES OF QUANTUM MECHANICS

Here we state and briefly explain the basic rules used to make predictions in quantum mechanics.

2.1.1 Postulates

Quantum mechanics is a statistical theory. This means that it only makes predictions about *frequencies of outcomes* in the context of a large number of realizations of a specific experiment ((D'ESPAGNAT, 2019), Chap.3). For example, if we want to infer some property of a determined system, we must prepare N equal system of such and all that quantum mechanics call tell us is that the desired property will assume a certain specific value n times (or with frequency n/N). Because of that, every time we refer to a system in this text, we mean precisely an *ensemble* (collection) of equal systems as defined.

We begin by stating the standard rules used to make predictions and focus on the simple case of a single-particle system [1].

We call a *Hilbert space* $\mathcal{H}$ a finite or infinite inner product vector space defined on the field of the complex numbers $\mathbb{C}$. Particularly, if $|v\rangle$ is an arbitrary element of $\mathcal{H}$, the set of all possible maps $\mathcal{H} \to \mathcal{H}$, or *operators*, which can perform any change on $|v\rangle$ is denoted by $B(\mathcal{H})$. For example, if $\langle\phi|\psi\rangle$ represents an well defined inner product relation between two

[1] In the many-particle context, symmetrization rules must be regarded, depending wheter the particles we are treating are bosons or fermions.

arbitrary vectors $|\phi\rangle$, $|\psi\rangle \in \mathcal{H}$, the operator $\hat{P}(\{w_n\})$ represents the *projection* of $|v\rangle$ onto the subspace associated with the set of vectors $\{|w_n\rangle\}$; that is, $\hat{P}(\{w_n\})|v\rangle = \sum_n \langle w_n|v\rangle \ |w_n\rangle$ if the index n assumes discrete values, or $\hat{P}(\{w_n\})|v\rangle = \int_a^b dn \langle w_n|v\rangle \ |w_n\rangle$ if it assumes continuous values in the range $(a, b) \subseteq \mathbb{R}$.

To every operator $\hat{A} \in B(\mathcal{H})$ we may assign in a one-to-one correspondence an operator $\hat{A}^\dagger \in B(\mathcal{H})$ that satisfies the relation

$$\langle\phi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^* \ \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}. \tag{2.1}$$

$\hat{A}^\dagger$ is called the *Hermitian conjugate* of $\hat{A}$ and, for the particular case where $\hat{A} = \hat{A}^\dagger$, it is a well known result of linear algebra that (i) $\hat{A}$ has a real valued set of eigenvalues (or *spectrum*) $\{a_n\}$ and (ii) it is always possible to construct an orthonormal basis $\{|a_n\rangle\} \subset \mathcal{H}$ composed only by eigenvectors of $\hat{A}$ (see (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.II-D).

Note that if $\hat{A}$ is equal to the unity, i.e. $\hat{A} = \hat{I}$, (2.1) implies that we must have $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ for the corresponding inner product relation. We say that this relation has a *Hermitian structure*.

In general, two or more eigenvectors can be associated with a specific eigenvalue a_n, but when this is not the case (that is, when every eigenvalue a_n of $\hat{A}$ is associated with one and only one eigenvector $|a_n\rangle$) we say that $\hat{A}$ has a *nondegenerate* spectrum.

Let us fix this last situation and denote by $|a_n\rangle$ the eigenvector of $\hat{A}$ in a one-to-one correspondence with the eigenvalue a_n (for the whole picture, see (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.III).

1. **Definition of state.** the system is described by a *normalized* vector $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is to be specified by the particular physical situation.

2. **Definition of observable.** to every classical physical quantity a we assign an Hermitian operator $\hat{A} \in B(\mathcal{H})$.

3. **Unitary Schrödinger evolution.** if at $t = t_0$ the system is described by the state $|\psi(t_0)\rangle$, at an arbitrary time $t > t_0$ the system evolves to $|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}|\psi(t_0)\rangle$, where $\hat{H}$ is the time-independent *Hamiltonian operator* associated with the system [2].

[2] Usually, the exponential of an operator is defined as the power series $e^{\hat{A}} = \sum_n^\infty \frac{\hat{A}^n}{n!}$ or the limit $\lim_{n\to\infty}\left(\hat{I} + \frac{\hat{A}}{n}\right)^n$ (see, for ex. (ARNOLD, 1973) Chap.2, §14). Also, the assumption that $\hat{H}$ is time-independent is taken for the sake of simplicity, because throughout the whole text this will be always the case.

4. **Born rule.** if a measurement of the observable $\hat{A}$ is performed on the system, the outcome is certain to be an eigenvalue of $\hat{A}$, but all that we can say beforehand is that we have a probability

$$\mathcal{P}(a_n) = |\langle a_n|\psi\rangle|^2 \tag{2.2}$$

to obtain the *specific* outcome a_n.

5. **Von Neumann measurement scheme.** after a measurement of $\hat{A}$ in which we have obtained the outcome a_n, the state of the system must be updated according to

$$|\psi\rangle \xrightarrow{\text{measurement}} \frac{\langle a_n|\psi\rangle}{|\langle a_n|\psi\rangle|}|a_n\rangle\,. \tag{2.3}$$

Note, in particular, that in postulate 1, $|\psi\rangle$ is allowed to be any superposition (linear combination) of vectors in $\mathcal{H}$, because, by the definition of a vector space, every such superposition also lies within $\mathcal{H}$. This is the so called *superposition principle*.

We stress that this statement of the set of rules is not unique, but rather, specifically formulated for our purposes. For a more general picture (for ex. including the case of degenerate spectra of observables), we again refer to (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.III.

2.1.2 Wavefunction approach

It is also worth mentioning that it is possible to make predictions in an alternative manner to the formalism stated above.

This goes as follows: to every possible state $|\psi\rangle \in \mathcal{H}$ which may describe our physical system, we assign in a one-to-one correspondence a function, or *wavefunction*, $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ that satisfies the condition that $\int_{-\infty}^{\infty} d^3x|\psi(\boldsymbol{x},t)|^2$ remains finite (in general, this condition is satisfied by functions that decrease rapdly in space, such as a Gaussian or a negative exponential). In this case, we say that $\psi \in \mathcal{L}^2$, where $\mathcal{L}^2$ is the *space of square-integrable functions*.

Due to Max Born, $|\psi(\boldsymbol{x},t)|^2$ is to be interpreted as the probability of finding a particle in the spacial region within $\boldsymbol{x}$ and $\boldsymbol{x}+d\boldsymbol{x}$ at time t. Although ψ speads itself all over the space, because we must necessarily find the particle somewhere, the condition $\int_{-\infty}^{\infty} d^3x|\psi(\boldsymbol{x},t)|^2 = 1$ must be satisfied. $|\psi\rangle$ and ψ being assigned, this motivates the correspondence between the Hermitian inner product relation of $\mathcal{H}$ defined in 2.1.1 with the relation

$$\langle\psi|\phi\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} d^3x\psi^*\phi \tag{2.4}$$

defined in $\mathcal{L}^2$ (ψ and ϕ being arbitrary functions).

Observables are now represented by operators acting in $\mathcal{L}^2$ and, in analogy with the last subsection, we write $\hat{A} \in B(\mathcal{L}^2)$ for an arbitrary physical quantity $\hat{A}$. For example, the extrinsic (coordinate-system dependent) degrees of freedom of position and momentum are assigned to the operators $\hat{X}, \hat{P} \in B(\mathcal{L}^2)$ s.t. $\forall \psi \in \mathcal{L}^2$, $\hat{X}\psi = \boldsymbol{x}\psi$ and $\hat{P}\psi = -i\hbar\nabla\psi$ (using (2.4) it is easy to show that $\hat{X}$ and $\hat{P}$ are Hermitian) [3].

If the system starts in a state assigned to the wavefunction $\psi(\boldsymbol{x}, t_0)$, to predict the dynamics and find ψ at a latter time, we must solve the *Schrödinger equation*

$$\frac{\partial \psi}{\partial t} = \frac{i}{\hbar}\hat{H}\psi(\boldsymbol{x}, t) \tag{2.5}$$

where $\hat{H} = \hat{H}(\hat{X}, \hat{P}, t)$ is the classical Hamiltonian of the system [4] written with $\boldsymbol{x}$ and $\boldsymbol{p}$ promoted to $\hat{X}$ and $\hat{P}$ defined above (t remains only a real valued parameter). Of course, $\psi(\boldsymbol{x}, t_0)$ serves as an initial condition to the solution of this first order [5] partial differential equation and this construction assures that $\psi(\boldsymbol{x}, t)$ is in a one-to-one correspondence with the state $|\psi(t)\rangle$ obtained by postulate 3.

For a detailed treatment, we refer to (GRIFFITHS; SCHROETER, 2018) (Chap.1) and specific illustrations of how to use the two approaches we let to be given in what follows by the examination of the two systems mentioned at the begining of this chapter (coherent states of the quantum harmonic oscillator and spinning particles).

2.1.3 Expectation values of observables

Suppose that we can expand the state $|\psi\rangle$ of the system in terms of an orthonormal basis $\{|n\rangle\} \subset \mathcal{H}$. That is, we can write $\psi = \sum_n c_n |n\rangle$ for n being discrete, or $|\psi\rangle = \int_a^b dn c_n |n\rangle$ for n being continuous in $(a, b) \in \mathbb{R}$.

Fixing the former (discrete) situation, we note by Born rule that if $n \neq m$ and $|c_n|^2 \geq |c_m|^2$ (ensuring, of course, that $\sum_n |c_n|^2 = 1$), the system is "more likely" to be observed in the state $|n\rangle$. Still using statistical language, this motivates the definition of the *expectation value*

[3] The precise construction of a basis set $\{|\boldsymbol{x}\rangle\}$ and $\{|\boldsymbol{p}\rangle\}$ of eigenstates of those two operators in a one to one correspondence with functions in $\mathcal{L}^2$ is rather complicated and, as it shall not be useful for the development of our work, we leave the discussion as a reference to (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.2, §B-2.

[4] We could arrive at the same equation for $|\psi(t)\rangle$ in postulate 3 of 2.1.1 by taking its time-derivative.

[5] First order *in time*. In space it is usually second order (i.e. it depends on the *laplacian operator* ∇^2).

of some observable $\hat{A} \in B(\mathcal{H})$ as

$$\langle \hat{A} \rangle_\psi \stackrel{\text{def}}{=} \langle \psi | \, \hat{A} \, | \psi \rangle \tag{2.6}$$

or, in the case of the last subsection,

$$\langle \hat{A} \rangle_\psi \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} d^3\boldsymbol{x} \psi^* \hat{A} \psi \tag{2.7}$$

where, of course, $\psi \in \mathcal{L}^2$ is properly assigned to $|\psi\rangle \in \mathcal{H}$. These definitions have the precise meaning that if we take N equally prepared systems and measure $\hat{A}$ in each one of them, $\langle \hat{A} \rangle_\psi$ is the average result we will get (of course, the larger the N, the closer the inferred average is from $\langle \hat{A} \rangle$). Finally, we mention that expectation values are, indeed, the quantities which can be measured in laboratory.

2.1.4 Heisenberg uncertainty relation

The previous discussion suggests that because now we cannot determine the outcome of a measurement of $\hat{A}$ *a priori* and therefore must use statistical distributions (in other words, $\hat{A}$ is a *random variable* and we must work with the probabilities of a determined outcome to happen), we may be interested in knowing how broad would be the interval of those possible outcomes.

This is also done by statistical language. We define the *standard deviation* of $\hat{A}$ for a system in the state $|\psi\rangle$ to be

$$\Delta A_\psi \stackrel{\text{def}}{=} \sqrt{\langle \hat{A}^2 \rangle_\psi - \langle \hat{A} \rangle_\psi^2}. \tag{2.8}$$

Furthermore, it can be shown ((GRIFFITHS; SCHROETER, 2018), Chap.3, §3.5) that for any pair of observables $\hat{A}_1, \hat{A}_2$, we have

$$\Delta A_1 \Delta A_2 \geq -\frac{1}{4} \langle [\hat{A}_1, \hat{A}_2] \rangle^2; \tag{2.9}$$

the most important result being the case of a pair of *canonically conjugated* operators, such as $\hat{X}$ and $\hat{P}$, which by the definitions given in 2.1.2 implies the canonical commutation relation $[\hat{X}, \hat{P}] = i\hbar$, leading, therefore, to the famous *Heisenberg uncertainty principle*

$$\Delta X_\psi \Delta P_\psi \geq \frac{\hbar}{2}. \tag{2.10}$$

The precise meaning of this result is that the more we want to measure a certain definite value for, say, the position observable $\hat{X}$, the smallest we can make ΔX, but the price we pay

is that in this case ΔP will be very large. This means that we cannot measure a precise value for position and momentum at the same time and, therefore, these quantities are said to be *incompatible observables*.

2.1.5 Tensor-product states

If we have two independent systems or two independent degrees of freedom of a single system, each described by states lying in two distinct Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote the Hilbert space of the *composite* system by the symbolic notation $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{H}_1 \otimes \mathcal{H}_2$, where "$\otimes$" is called the *tensor product* between vector spaces.

Furthermore, we require this product to be *linear* with respect to each of its arguments; that is, if all set of vectors are discrete [6], given two orthonormal basis $\{\,|n_1\rangle\} \subseteq \mathcal{H}_1$ and $\{\,|n_2\rangle\} \subseteq \mathcal{H}_2$, we may expand $|v_1\rangle \in \mathcal{H}_1$ and $|v_2\rangle \in \mathcal{H}_2$ as $|v_1\rangle = \sum_{n_1} c_{n_1} |n_1\rangle$ and $|v_2\rangle = \sum_{n_2} d_{n_2} |n_2\rangle$ and write

$$|v_1\rangle \otimes |v_2\rangle \stackrel{\text{def}}{=} \sum_{n_1,n_2} c_{n_1} d_{n_2} |n_1\rangle \otimes |n_2\rangle \,. \tag{2.11}$$

Because linear operators are also vectors belonging to vector spaces, if we have $\hat{A}_1 \in B(\mathcal{H}_1)$ and $\hat{A}_2 \in B(\mathcal{H}_2)$ we may define $\hat{A} = \hat{A}_1 \otimes \hat{A}_2 \in B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ s.t.

$$\hat{A}\,|v_1\rangle \otimes |v_2\rangle = \hat{A}_1\,|v_1\rangle \otimes \hat{A}_2\,|v_2\rangle \,. \tag{2.12}$$

Finally, if there are well defined inner product relations in $\mathcal{H}_1$ and in $\mathcal{H}_2$, taking $|w_1\rangle \in \mathcal{H}_1$ and $|w_2\rangle \in \mathcal{H}_2$, we also demand

$$(\,\langle w_1| \otimes \langle w_2|)(\,|v_1\rangle \otimes |v_2\rangle) = \langle w_1|v_1\rangle\,\langle w_2|v_2\rangle \,. \tag{2.13}$$

From this definition we see that $\{\,|n_1\rangle \otimes |n_2\rangle\}$ as defined above is a basis set for $\mathcal{H}$, because $(\,\langle n_1| \otimes \langle n_2|)(\,|m_1\rangle \otimes |m_2\rangle) = \delta_{n_1,m_1}\delta_{n_2,m_2}$. Therefore, if $|n_1\rangle$ and $|n_2\rangle$ represent the eigenstates of two observables defined in $B(\mathcal{H}_1)$ and in $B(\mathcal{H}_2)$ with eigenvalues n_1 and n_2, respectively, and (2.11) represents the state of the system, Born rule tells us that we have a probability $|c_{n_1}|^2|d_{n_2}|^2$ of measuring n_1 *and* n_2 together. This last fact is consistent with the hypothesis that the two systems, or the two degrees of freedom, are, in this case, indeed independent (recall that for two independent events 1 and 2 which occur with probabilities p_1 and p_2, respectively, the probability of 1 *and* 2 to occur is just $p_1 p_2$).

[6] For the continuous case, it suffices to change the summations by integrals $\sum_n \to \int_a^b dn$.

Except when explicitly necessary for the sake of clarity, throughout the rest of this text we shall omit the $\otimes$ symbol from tensor products (for ex. $|v_1\rangle \otimes |v_2\rangle \stackrel{\text{def}}{=} |v_1\rangle\, |v_2\rangle$ or even $|v_1\rangle \otimes |v_2\rangle \stackrel{\text{def}}{=} |v_1, v_2\rangle$).

2.1.6 Entanglement

By the discussion of the above section, still in the same context, we now know that if we are given two states $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ for the uncorrelated systems 1 and 2, respectively, we can construct a state for the composite system $1+2$ simply as $|\psi_1\rangle \otimes |\psi_2\rangle$. This state, of course, lies within $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and this space can be said to be the Hilbert space of the composite system. Hence, we know from postulate 1 of 2.1.1 that all states within $\mathcal{H}$ are allowed to describe the composite system, *even linear combinations of two or more vectors* (superposition principle).

But now the converse question may arise: given a state $|\psi\rangle \in \mathcal{H}$ for the composite system, there always exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ s.t. $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$? The answer is *no*.

For example, suppose that the Hilbert spaces of both systems have dimension $N = 2$ and fix the basis sets $\{|0_1\rangle, |1_1\rangle\} \subset \mathcal{H}_1$ and $\{|0_2\rangle, |1_2\rangle\} \subset \mathcal{H}_2$. Of course a basis set for the composite system is simply $\{|0_1, 0_2\rangle, |0_1, 1_2\rangle, |1_1, 0_2\rangle, |1_1, 1_2\rangle\}$. Take now the composite system to be in a so called *Bell state* $|\mathbf{\Phi}^+\rangle = (|0_1, 0_2\rangle + |1_1, 1_2\rangle)/\sqrt{2}$ and try to write

$$|\mathbf{\Phi}^+\rangle = (c_1 |0_1\rangle + c_2 |0_2\rangle) \otimes (c_3 |0_2\rangle + c_4 |1_2\rangle) \tag{2.14}$$

for some set of coeficients $\{c_1, c_2, c_3, c_4\} \subset \mathbb{C}$.

It can be verified that the system

$$\begin{aligned} c_1 c_3 &= \frac{1}{\sqrt{2}} \\ c_1 c_4 &= 0 \\ c_2 c_3 &= 0 \\ c_2 c_4 &= \frac{1}{\sqrt{2}} \end{aligned} \tag{2.15}$$

has no solution!

Therefore, we conclude that the superposition principle implies that not every state accessible to the composite system can be decomposed into a tensor product of two states accessible to the individual subsystems. This phenomenon is called *entanglement* and is a

major feature of quantum systems with an almost infinite range of beautiful implications and applications. If there exists no $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ s.t. $|\psi\rangle \in \mathcal{H}$ can be decomposed as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, we say that $|\psi\rangle$ is an *entangled state*. In general, many operations can generate entanglement, such as time-evolutions with interactions and measurements. This happens because after those operations the state of the system can (in general) be turned into any state lying within $\mathcal{H}$ (and now we know that not every such state is separable).

Finally, we note that because operators also lie within vector spaces, the preceding analysis is also applicable to them. We shall come back to this important point when we talk about *density operators* at the end of the chapter.

2.2 QUANTUM HARMONIC OSCILLATOR

As a useful illustration, we analyse the well known one-dimensional quantum harmonic oscillator, described by the Hamiltonian

$$\hat{H}_{HO} = \frac{\hat{P}^2}{2m} + \frac{m\omega^2 \hat{X}^2}{2}. \tag{2.16}$$

2.2.1 Ladder operator approach

Based in the expositions in (PESKIN; SCHROEDER, 2007) (Chap.2, §2.3) and (MANDL; SHAW, 2010) (Chap.1, §1.2.2) we briefly follow the so called *ladder operator method* to construct explicitly the eigenvalues and the eigenstates of (2.16). This approach, first proposed by Dirac, finds an almost infinite range of applicability in the literature and serves as a ground for the subjects of quantum field theories and condensed matter theories where multiparticle systems are important [7].

We begin by noting that the definitions of $\hat{X}$ and $\hat{P}$ as stated in 2.1.2 implies the canonical commutation relation $[\hat{X}, \hat{P}] = \hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$. Therefore, defining the (non-Hermitian!) *destruction* or *lowering* ladder operator

$$a \stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{P} \tag{2.17}$$

we can show that

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{2.18}$$

[7] Indeed, it can be shown that simmetrization rules for bosons and fermions, instead of being postulated *a priori*, appear naturally in this approach.

and

$$\hat{H}_{HO} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{2.19}$$

This last expression implies that finding the eigenvalues and the eigenstates of $\hat{H}_{HO}$ is equivalent to finding the eigenvalues and the eigenstates of $\hat{a}^\dagger\hat{a} \stackrel{\text{def}}{=} \hat{N}$.

Because inner products must always satisfy the condition $\langle\psi|\psi\rangle \geq 0$ and because $(\hat{a}\,|\psi\rangle)^\dagger = \langle\psi|\,\hat{a}^\dagger$, we see that $\langle\psi|\,\hat{N}\,|\psi\rangle = \langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle \geq 0$; taking then the eigenvalue equation $\hat{N}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ we see that $\alpha \geq 0$ (of course we are supposing $|\alpha\rangle$ already normalized) and therefore that all eigenvalues of $\hat{N}$ are non-negative. $\hat{N}$ is said to be *positive semi-definite* and, as a consequence, it must have a lowest possible eigenvalue $\alpha_0 \geq 0$.

Still with the eigenvalue equation and with the aid of (2.18), we see that $\hat{N}\hat{a}\,|\alpha\rangle = (\alpha - 1)\hat{a}\,|\alpha\rangle$, so that $\hat{a}\,|\alpha\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $(\alpha - 1)$. Moreover, this relation also implies that $\alpha_0 = 0$ necessarily, because by hypothesis there is no eigenvalue lower than α_0.

Now, if $|0\rangle$ represents the eigenstate associated with the eigenvalue $\alpha_0 = 0$, the above discussion implies that $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Therefore, the states

$$|n\rangle \stackrel{\text{def}}{=} \frac{(\hat{a}^\dagger)^n\,|0\rangle}{\sqrt{n}} \tag{2.20}$$

can be verified to be the eigenstates of (2.16) with eigenvalues $\alpha = n = 0, 1, 2, 3, ...$ associated. Therefore, the energy spectrum of the harmonic oscillator is given simply by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{2.21}$$

This result is consistent with solving the Schrödinger equation (2.5) explicitly (see, for ex. (GRIFFITHS; SCHROETER, 2018) Chap.2, §2.3) if the states $|n\rangle$ are mapped in a one-to-one correspondence to the wavefunctions [8]

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{m\omega}{\hbar}x\right)e^{-m\omega x^2/2\hbar} \tag{2.22}$$

where H_n is the *Hermite polynomial* of degree n given by the Rodriguez formula

$$H_n(\lambda) = (-1)^n e^{\mathcal{Z}^2}\left(\frac{d}{d\lambda}\right)^n e^{-\mathcal{Z}^2}. \tag{2.23}$$

Because the Hamiltonian is time independent, we have simply

$$\psi(x,t) = \psi_n(x)e^{-i\omega(t-t_0)}. \tag{2.24}$$

[8] This can be proven indeed by inverting (2.17), promoting $\hat{x}$ and $\hat{P}$ to operators in $B(\mathcal{L}^2)$ as in 2.1.2 and solving the differential equation $(\hat{a}^\dagger)^n\psi_0 = \alpha_n\psi_n$.

For the three-dimensional case where the particle oscillates with the same frequency ω in all independent degrees of freedom, we simply use the tensor product construction of 2.1.5 and write $|n\rangle \rightarrow |n_y\rangle\ |n_z\rangle\ |n_x\rangle$ and $\psi_{n_x n_y n_z}(\boldsymbol{x}) \rightarrow \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$.

The calculations of ΔX and ΔP for those states yield

$$\begin{aligned} \Delta X_n &= \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} \\ \Delta P_n &= \sqrt{m\hbar\omega\left(n+\frac{1}{2}\right)} \end{aligned} \tag{2.25}$$

implying the uncertainty relation

$$\Delta X_n \Delta P_n = \left(n+\frac{1}{2}\right)\hbar. \tag{2.26}$$

Equation (2.25) is in accordance with (2.10) and means that the higher the n, the broader the distribution of both $\hat{X}$ and $\hat{P}$.

Finally, we note that (2.17) implies $\langle\hat{X}\rangle_n = \langle\hat{P}\rangle_n = 0$. This means that the states $|n\rangle$ do not describe directly the behaviour of a classical harmonic oscillator, where x and p are zero only when the energy of the motion is zero (see the topic of the next section). Because of that, we pose the following question: is it possible so far to use the developed formalism to construct states that lead to predictions close to classical mechanics? Indeed, we expect that (in principle) quantum mechanics should yield the same results of classical mechanics in the limit of large quantum numbers (for example in the limit of energies much higher than $\hbar\omega$).

The so posed question serves as a motivation for the treatment of the next section.

2.3 COHERENT STATES

Proposed by Schrödinger in 1926 and first called *coherent states* by Glauber in 1963, those states are of wide applicability in many subjects of modern science, such as quantum information processing, quantum optics, quantum superselection principles and mathematical physics (DEY; FRING, 2018). We follow here the simple approach of defining them as eigenvectors of the ladder operator $\hat{a}$ [9], i.e.

$$\hat{a}\,|z\rangle = z\,|z\rangle\,. \tag{2.27}$$

[9] We could choose also to define the so called *displacement operator* $D(z) = e^{z^*\hat{a} - z\hat{a}^\dagger}$ and apply it to the ground state $|0\rangle$. We would get the same results.

Because $\hat{a}$ is not Hermitian, z is, in general, a complex number. If we write $|z\rangle = \sum_n c_n |n\rangle$ and use $\langle 0|0\rangle = 1$, it is easy to show, by (2.27) that

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{n=\infty} \frac{z^n}{\sqrt{n!}} |n\rangle . \tag{2.28}$$

Note that *two coherent states are not orthogonal*.

Furthermore, this expansion together with postulate 3 in 2.1.1 implies that the time evolution of $|z(0)\rangle = |z\rangle$ is simply

$$|z(t)\rangle = e^{-\frac{i\omega(t-t_0)}{2}} |z(0)e^{-i\omega(t-t_0)}\rangle . \tag{2.29}$$

This is remarkable, because, the global phase factor $e^{-\frac{i\omega(t-t_0)}{2}}$ being dropped (this does not change physical predictions, since this exponential is a *global phase* and, consequently, does not change any expectation value), we see that the state *remains an eigenstate of $\hat{a}$ throughout time*. This has very neat consequences, which will be discussed in the following.

Let us first review [10] an useful method to describe the motion of the classical harmonic oscillator by introducing the dimensioless parameters $x_c(t) = \sqrt{\frac{m\omega}{\hbar}} x(t)$ and $p_c(t) = \frac{1}{\sqrt{m\hbar\omega}} p(t)$. In that case, Newton's equations become

$$\begin{aligned} \frac{dx}{dt} = \frac{p(t)}{m} &\rightarrow \frac{dx_c}{dt} = \omega p_c(t) \\ \frac{dp}{dt} = -m\omega^2 x(t) &\rightarrow \frac{dp_c}{dt} = \omega x_c(t). \end{aligned} \tag{2.30}$$

These two equations can be combined into a single complex-valued differential equation:

$$\frac{dz_c}{dt} = -i\omega z_c(t) \tag{2.31}$$

with $z_c(t) = \frac{1}{\sqrt{2}}[x_c(t) + ip_c(t)]$. In this case, the position and the momentum of the particle are specified by the single complex parameter $z_c(t)$.

The solution of (2.31) is easy to obtain and is simply

$$z_c(t) = z_c(0)e^{-i\omega(t-t_0)}. \tag{2.32}$$

It means that, if we set $t_0 = 0$,

$$\begin{aligned} x(t) &= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{z_c(t)\} = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t \\ p(t) &= \sqrt{2m\hbar\omega}\operatorname{Im}\{z_c(t)\} = -m\omega x(0)\sin\omega t + p(0)\cos\omega t \end{aligned} \tag{2.33}$$

[10] Here we follow the exposition in (COHEN-TANNOUDJI; DIU; LALOë, 2019) (Complement G_V) in a very brief way.

which are exactly the classical results (note how similar are (2.32) and the evolution of z in (2.29)!). Moreover, using (2.33) the energy of the system can be obtained immediately:

$$E_c = \hbar\omega |z_c(0)|^2 \tag{2.34}$$

where, of course, we must have the condition $|z_c(0)|^2 \gg 1$, because we are analyzing a classical system.

Now, it is a simple matter to show, using Eqs.(2.27), (2.29), that $\langle \hat{H} \rangle_z = \hbar\omega(|z(t)|^2 + 1/2)$, $\langle \hat{X} \rangle_z = \sqrt{2\hbar/m\omega}\, \mathrm{Re}\{z(t)\}$ and $\langle \hat{P} \rangle_z = \sqrt{2m\hbar\omega}\, \mathrm{Im}\{z(t)\}$; or even,

$$\langle \hat{X} \rangle_z(t) = \langle \hat{X} \rangle_z(0) \cos\omega t + \frac{\langle \hat{P} \rangle_z(0)}{m\omega} \sin\omega t \tag{2.35a}$$

$$\langle \hat{P} \rangle_z(t) = -m\omega \langle \hat{X} \rangle_z(0) \sin\omega t + \langle \hat{P} \rangle_z(0) \cos\omega t \tag{2.35b}$$

$$\langle \hat{H} \rangle_z(t) = \hbar\omega \left(|z(0)|^2 + \frac{1}{2} \right) \tag{2.35c}$$

which are exactly [11] the classical results (2.33) with $\langle \hat{X} \rangle_z(0) = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\{z(0)\}$ and $\langle \hat{P} \rangle_z(0) = \sqrt{2m\hbar\omega}\, \mathrm{Im}\{z(0)\}$. This is a remarkable point.

This exposition makes it clear that (i) stationary states of the classical harmonic oscillator (i.e. states for which the oscillator does not move, and, consequently $x(t) = p(t) = 0$) are connected with stationary states of the quantum harmonic oscillator (where $\langle \hat{X} \rangle_n = \langle \hat{P} \rangle_n = 0$) and (ii) the quantum nonstationary states that recover the classical motion are the states $|z\rangle$.

As said before, those states have a set of very compeling properties, such as (2.29); we end the discussion by listing two more of them.

1. (*The uncertainty relation is minimum.*) A simple calculation, using

$$\begin{aligned} \langle \hat{X}^2 \rangle_z &= \frac{2\hbar}{m\omega} \left(\mathrm{Re}^2 \left\{ z \right\} + \frac{1}{4} \right) \\ \langle \hat{P}^2 \rangle_z &= 2m\hbar\omega \left(\mathrm{Im}^2 \left\{ z \right\} + \frac{1}{4} \right) \end{aligned} \tag{2.36}$$

shows that $\Delta X_z = \sqrt{\frac{\hbar}{2m\omega}}$, $\Delta P_z = \sqrt{\frac{m\hbar\omega}{2}}$ and, consequently,

$$\Delta X_z \Delta P_z = \frac{\hbar}{2}. \tag{2.37}$$

[11] Of course there is an extra factor in $\langle \hat{H} \rangle_z$ due to the zero point energy. But we know that an additive constant in the energy does not change any physical prediction, since all that can be measured are energy *differences*.

This is nice because, following the discussion in 2.1.4, to obtain a certain definite outcome of, say, $\hat{X}$, the price we shall pay in making the distribution of $\hat{P}$ broader is the lesser possible.

2. (*The wavepacket remains Gaussian.*) Because of (2.29), we see that the wavefunction of the states $|z\rangle$ shall not vary its shape in time. Indeed, in (COHEN-TANNOUDJI; DIU; LALOë, 2019) (Complement G_V) it is shown that

$$\psi_z(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} e^{\frac{z^{*2}+z^2}{4}} e^{\frac{ix}{\hbar}\langle\hat{P}\rangle_z(t)} e^{-[\frac{x-\langle\hat{X}\rangle_z(t)}{2\Delta X}]^2} \tag{2.38}$$

so that the shape of the probability density is simply

$$|\psi_z(x,t)|^2 = |\psi_0[x - \langle\hat{X}\rangle(t)]|^2. \tag{2.39}$$

That means that the wavepacket (another name for the probability density) oscillates back and forth arround $x = 0$ without changing its shape (contrary to what occurs, for example, with free particles, whose wavepackets just spread in space with time). We note that this behaviour is very similar to the behaviour of a classical particle under the action of a parabolic potential, further confirming the connection between the states $|z\rangle$ and the nonstationary states of a classical harmonic oscillator.

2.4 SPIN

The approach of section 2.1.2 is particularly suited for *extrinsic* degrees of freedom, which are the degrees of freedom that can be described in terms of the position and the momentum observables. However, there are degrees of freedom which cannot be described in this way, and, therefore, are said to be *intrinsic* to the system. We shall, in this section, study the so called *spin angular momentum* degree of freedom in order to illustrate how the general approach of 2.1.1 may be, indeed, more handful, as it allows the description of such intrinsic quantities without making explicit reference to algebraic functions in $\mathcal{L}^2$.

Let us begin with a brief introduction. In classical mechanics, if we imagine an spherical body that moves in space and rotates about its own axis (such as a planet like the Earth, which rotates around the sun and also around “itself") we can define two tipes of angular momenta: the *orbital* angular momentum $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$ and the *spin* angular momentum $\boldsymbol{s} = I\boldsymbol{\omega}$, where I represents the moment of inertia of the particle (the rest of the notation should be clear in the classical picture). Nevertheless, both of those degrees of freedom are said to be extrinsic,

because, classically, a macroscopic body can be decomposed as a set of smaller constituents, and, therefore, the calculation of $\boldsymbol{s}$ involves the motion of those constituents around the axis of rotation (parallel to $\boldsymbol{\omega}$).

In quantum mechanics, we know that electrons, for example, are said to be *elementary particles*, and, therefore, cannot be decomposed into a set of smaller constituents. Precisely, the electron cannot be described as for ex. a spinning sphere, but the very existence of an angular momentum analogous to $\boldsymbol{s}$ is experimentally confirmed, and that is why we inherit the name *spin angular momentum*.

Spin is an essential part of the description of quantum behaviour and it allows us to understand many properties of matter, such as ferromagnetism in metals. Its existence is experimentally confirmed in a broad rage of phenomena, such as the fine structure of spectral lines and the Zeeman effect (see (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.IX); however, it can only be deduced theoretically by relativistic approaches to quantum mechanics, where the equation of motion is no longer the Schrödinger equation. [12]. In nonrelativistic contexts (our case), where velocities are much smaller than the velocity of light, it has to be nonetheless *postulated*.

The analysis of the Stern-Gerlach apparatus (which is another experiment that confirms the existence of spin), where silver atoms are deflected from their classical trajectories [13] when passing through a region where an inhomogeneous magnetic field is $\boldsymbol{B}$ present, suggests that this intrinsic angular momentum is proportional to the magnetic moment $\boldsymbol{\mu}$ of a quantum particle. Furthermore, this same deflection, regardless of the fact that $\boldsymbol{\mu}$ can have any spatial orientation, can only occur in *two possible directions*, which are aligned with the direction of $\boldsymbol{B}$. Because of that, we see that spin is also a *quantized* quantity, such as the energy of the harmonic oscillator (2.21) [14].

Following this last discussion and postulate 2 of section 2.1.1, we promote $\boldsymbol{s}$ for quantum particles to an operator $\hat{\boldsymbol{S}}$ acting in a two dimensional Hilbert space $\mathcal{H}_{spin}$. If in a Stern-Gerlach apparatus we choose $\boldsymbol{B}$ to point in the z direction, a basis for $\mathcal{H}_{spin}$ is denoted by the two states $|+\rangle$ (particle deflected to $+z$ direction [15]) and $|-\rangle$ (particle deflected to $-z$ direction). Those states are constructed as being normalized eigenvectors of $\hat{S}_z$ with respective experimentally

[12] For fermions (half-spin perticles) it is, for example, the so called *Dirac equation* (see (THOMSON, 2013), Chap.4, §4.4)

[13] They are heavy enough to be considered to behave so.

[14] For a detailed discussion of the Stern-Gerlach experiment, see (COHEN-TANNOUDJI; DIU; LALOë, 2019), Chap.IV.

[15] A more precise word, in the context of mathematics, would be "orientation".

measured eigenvalues $\pm\frac{\hbar}{2}$ [16]. Because $\hat{\boldsymbol{S}}$ represents a physical quantity (Hermitian operator), the set $\{\,|\pm\rangle\}$ is an *orthogonal* set ($\left\langle s_1^k \middle| s_2^k \right\rangle = \delta_{s_1^k, s_2^k}$, where $s_1^k, s_2^k = \pm$) and its operator representation, together with $\hat{S}_x$ and $\hat{S}_y$ are given by

$$\begin{aligned}
\hat{S}_x &= \frac{\hbar}{2}(\,|+\rangle\,\langle -| + |-\rangle\,\langle +|) \\
\hat{S}_y &= -\frac{i\hbar}{2}(\,|+\rangle\,\langle -| - |-\rangle\,\langle +|) \\
\hat{S}_z &= \frac{\hbar}{2}(\,|+\rangle\,\langle +| - |-\rangle\,\langle -|).
\end{aligned} \tag{2.40}$$

The set of eigenvectors of $\hat{S}_x$ and $\hat{S}_y$ can be verified to be

$$\begin{aligned}
\{\,|\pm\rangle_x\} &= \{\frac{1}{\sqrt{2}}(\,|+\rangle \pm |-\rangle)\} \\
\{\,|\pm\rangle_y\} &= \{\frac{1}{\sqrt{2}}(\,|+\rangle \pm i\,|-\rangle)\}
\end{aligned} \tag{2.41}$$

with respective eigenvalues $\pm\frac{\hbar}{2}$.

It can also be verified that those definitions imply the so called *angular momentum algebra*:

$$[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k \tag{2.42}$$

where ϵ_{ijk} is the usual Levy-Civita total antisymetric tensor, which is equal to 1 if ijk is a cyclic permutation of xyz (such as yzx and zxy), -1 for noncyclic permutations (such as yxz and zyx) and 0 if two indexes are equal. Note that those commutation relations together with (2.9) imply that spins in different directions are incompatible observables.

Finally, still in the Stern-Gerlach context, if we write $\mu_z = \gamma\hat{S}_z$, the interaction energy between the particle and the external magnetic field $\boldsymbol{B} \overset{\text{def}}{=} B_z\boldsymbol{e_z}$ can be imported from classical electrodynamics ($\boldsymbol{U} = -\boldsymbol{\mu}.\boldsymbol{B}$) and, the composite hilbert space being symbolized by, $\mathcal{H}_{orbital}\otimes\mathcal{H}_{spin}$ (sec 2.1.5), regarded as

$$\hat{U}_{int} = -\gamma\hat{B}_z \otimes \hat{S}_z \tag{2.43}$$

where the proportionality constant γ is the so called *gyromagnetic ratio* of the particle and B_z, to represent an inhomogeneous field in space, must be a function of the position operator, i.e. $\hat{B}_z = \hat{B}_z(\hat{X}, \hat{Y}, \hat{Z})$ (ensuring, of course, that $\boldsymbol{\nabla}.\boldsymbol{B} = 0$).

[16] We are treating here electrons, which are *fermions*, and, therefore, have half-integer values for its spin. At the other hand, Bosons have integer values, such as 0, $\hbar$, $2\hbar$, etc.

2.5 THE DENSITY OPERATOR

We have formulated quantum mechanics in a way that a system S is characterized by a state vector $|\psi\rangle$ which evolves in time and can be used to compute significant physical quantities related to that system. However, this characterization is not the most general possible, because it can only specify *quantum* uncertainties; that is, uncertainties which are inherent to S and cannot be controlled by any means as, say, sharpening our measurement apparatus. In order to include in our description *classical* uncertainties (that is, uncertainties which can be controlled by, say, sharpening our measurement apparatus) we need a more general object, such as the so called *density operator*, usually denoted by $\hat{\rho}$.

Indeed, in "real life" we face the situation of not exactly knowing even if $|\psi\rangle$ alone can characterize S. Perhaps all information we may have beforehand is only that there is a probability p_α that S can be characterized by the normalized state vector $|\psi_\alpha\rangle$ from a given set $\{|\psi_\alpha\rangle\}$. Fixing then this scenario, if the states accessible to S lie in a Hilbert space $\mathcal{H}_S$, we construct $\hat{\rho} \in B(\mathcal{H}_S)$ simply as

$$\hat{\rho} \stackrel{\text{def}}{=} \sum_\alpha p_\alpha |\psi_\alpha\rangle \langle\psi_\alpha| . \tag{2.44}$$

It was said before that quantum mechanics can only make predictions relative to an *ensemble* of equally prepared systems and that when we speak about the object $|\psi\rangle$ we refer precisely to $|\psi\rangle$ determined *via an ensemble as such*. In this way, we immediately see that *the characterization through $\hat{\rho}$ already includes the very nature of this "ensemble-determination"*.

Of course $\hat{\rho}$ must attend to some specific conditions in order to characterize S properly. Direct from (2.44) we see that (i) $\hat{\rho}^\dagger = \hat{\rho}$ ($\hat{\rho}$ is Hermitian) and that (ii) $\langle\psi|\hat{\rho}|\psi\rangle = \sum_\alpha p_\alpha |\langle\psi|\psi_\alpha\rangle|^2 \geq 0 \; \forall \, |\psi\rangle \in \mathcal{H}_S$ ($\hat{\rho}$ is *positive semi-definite*). If $\{|a_n\rangle\}$ is a normalized eigenbasis of the operator $\hat{A} \in B(\mathcal{H}_S)$, because of Born rule (§2.1.1, postulate 4) and because $\sum_\alpha p_\alpha = 1$, we must necessarily have (iii) $\mathrm{Tr}\,\hat{\rho} = \sum_\alpha p_\alpha \sum_n |\langle a_n|\psi_\alpha\rangle|^2 = 1$ ($\hat{\rho}$ has unitary trace).

If there is only one possible value for α we say that S is in a *pure* state. In this case, $\hat{\rho}$ is a single projector and we have $\hat{\rho}^2 = \rho$. If there is more than one possible value for α we say that S is in a *mixed* state. For this latter case, in general, $\rho^2 \neq \rho$. Moreover, the quantity

$$P \stackrel{\text{def}}{=} \mathrm{Tr}\left\{\hat{\rho}^2\right\} \leq 1 \tag{2.45}$$

can measure how far $\hat{\rho}$ is from a pure state (for which $P = 1$) and, because of that, is called *purity*.

Now, consider an arbitrary orthonormal basis set $\{|n\rangle\} \subset \mathcal{H}$. We can define the so called *matrix elements* of $\hat{\rho}$ relative to the basis $\{|n\rangle\}$ as the quantities $\rho_{m,n} = \langle m|\hat{\rho}|n\rangle$. Furthermore, if we make the expansion $|\psi_\alpha\rangle = \sum_n |c_{\alpha,n}|e^{i\phi_n}|n\rangle$, $\phi_n \in \mathbb{R}$, we have $\rho_{m,n} = \sum_\alpha p_\alpha |c_{\alpha,m}||c_{\alpha,n}|e^{i(\phi_m-\phi_n)}$. The *diagonal* elements $\rho_{n,n} = \sum_{\alpha,n} p_\alpha |c_{\alpha,n}|^2$ are called the *populations* of $\hat{\rho}$ because they are simply the probability that we will find the system in the state $|n\rangle$. On the other hand, the *non-diagonal* elements $\rho_{m,n}$, $m \neq n$, are called the *coherences* of $\hat{\rho}$ because they give us information about the phase difference between the states $|m\rangle$ and $|n\rangle$. Note that this is a basis-dependent nomenclature and for the special case where $\{|n\rangle\}$ is a normalized eigenbasis which diagonalizes $\hat{\rho}$, we see that only populations are nonzero. Naturally, choosing this basis may simplify a lot specific analysis.

In the same context, it can be verified by expanding $|\psi_\alpha\rangle$ in (2.44) in terms of $\{|n\rangle\}$ that the expectation value of the arbitrary observable $\hat{A}$ is given simply by

$$\begin{aligned}\langle\hat{A}\rangle_{\hat{\rho}} &= \mathrm{Tr}\{\hat{\rho}\hat{A}\} \\ &= \sum_\alpha p_\alpha \langle\psi_\alpha|\hat{A}|\psi_\alpha\rangle\end{aligned} \tag{2.46}$$

If we choose to characterize S using $\hat{\rho}$, we must also specify how the dynamics works. Suppose that the Hamiltonian $\hat{H}$ of S is time-independent and that at $t = 0$ the system is in the state (2.44). Writing $\hat{U}_t \overset{\text{def}}{=} e^{-i\hat{H}t/\hbar}$, postulate 3 of §2.1.1 implies that $|\psi_\alpha\rangle$ evolves in time to $|\psi_\alpha(t)\rangle = \hat{U}_t|\psi_\alpha\rangle$. Consequently,

$$\hat{\rho}(t) = \hat{U}_t\hat{\rho}\hat{U}_t^\dagger. \tag{2.47}$$

Differentiating this last equation, we arrive then at the so called *von Neumann master equation*

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] \tag{2.48}$$

which shall be very important to our subsequent analysis.

To solve this equation with initial condition $\hat{\rho}(0) = \hat{\rho}$ is, of course, equivalent to calculate (2.47) explicitly and we shall use both approaches in due course, depending on the specific situation.

In the next chapter we will show how the description of quantum systems in terms of density matrices can be useful to analyze a very important and powerful context: that of an *open system*. This is basically the context where a system of interest is no more isolated, but coupled to another system in a way that they may interact.

3 OPEN QUANTUM SYSTEMS

In this chapter, a generalization of the construction considered in the previous chapter will be presented. Using the description of quantum system in terms of density operators, introduced in §2.5, we shall construct a scenario which is closer to experimental realizations: that of open quantum systems and master equations. We do so in order to set up the context and the terminology to be used in our project (Chap.4).

3.1 DEFINITION AND TIME-EVOLUTION OF AN OPEN SYSTEM

Suppose that at $t = 0$ we have a system of interest S which is set in interaction with another system E (not necessarily large or having infinite degrees of freedom). We call S the *reduced system* or simply the *system*, E the *environment* and $S + E$ the *composite system*.

Suppose that the states of S and E lie within the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively. We know from §2.1.5 that the Hilbert space of the composite system can be constructed simply as $\mathcal{H}_S \otimes \mathcal{H}_E$. Furthermore, if S and E are characterized at $t = 0$ by the states $\hat{\rho}_S(0) = \hat{\rho}_S$ and $\hat{\rho}_E(0) = \hat{\rho}_E$, respectively, and if these states are initially *uncorrelated*, the initial *composite state* can be constructed as

$$\hat{\rho}_{SE}(0) = \hat{\rho}_S \otimes \hat{\rho}_E. \tag{3.1}$$

Consequently, noting that, despite the fact that S and E can interact as times passes, $S + E$ will be always isolated, the discussion of the above section implies that the state

$$\hat{\rho}_{SE}(t) = \hat{U}_t \hat{\rho}_S \otimes \hat{\rho}_E \hat{U}_t^\dagger \tag{3.2}$$

is not necessarily a product-form state as (3.1). This means that, in general, the interaction may *entangle* S and E. Of course, to obtain $\hat{U}_t$ we must specify how the composite Hamiltonian is constructed. If S and E when isolated have Hamiltonians $\hat{H}_S \in B(\mathcal{H}_S)$ and $\hat{H}_E \in B(\mathcal{H}_E)$, respectively, the standard form of the composite Hamiltonian taken in literature is

$$\hat{H} = \hat{H}_S \otimes \hat{I} + \hat{I} \otimes \hat{H}_E + \hat{H}_{SE} \tag{3.3}$$

where the "additional" term $\hat{H}_{SE}$ represents the interaction energy between S and E.

Now, suppose that the state $\hat{\rho}_{SE}(t)$ is specified. Because S is our system of interest, we can define a *partial trace operation* $\mathcal{E} : \mathcal{H}_{S+E} \to \mathcal{H}_S$ with respect only to the degrees of

freedom of E to know exactly what happened to S after the interaction:

$$\hat{\rho}_S(t) \stackrel{\text{def}}{=} \mathcal{E}(\hat{\rho}_S) = \mathrm{Tr}_E \left\{ \hat{U}_t \hat{\rho}_S \otimes \hat{\rho}_E \hat{U}_t^\dagger \right\}. \tag{3.4}$$

This state is called the *reduced state* of S. For example, if $S+E$ is in the Bell state $\hat{\rho}_{SE} = |\mathbf{\Phi}^+\rangle \langle \mathbf{\Phi}^+| = (|0_S,0_E\rangle \langle 0_S,0_E| + |0_S,0_E\rangle \langle 1_S,1_E| + |1_S,1_E\rangle \langle 0_S,0_E| + |1_S,1_E\rangle \langle 1_S,1_E|)/2$, we would have $\hat{\rho}_S = (|0_S\rangle \langle 0_S| + |1_S\rangle \langle 1_S|)/2$.

It can be verified that the map $\mathcal{E}$ satisfies (i) $\mathcal{E}(c\hat{\rho}) = c\mathcal{E}(\hat{\rho}), c \in \mathbb{C}$; (ii) $\mathcal{E}(\hat{\rho}_1 + \hat{\rho}_2) = \mathcal{E}(\hat{\rho}_1) + \mathcal{E}(\hat{\rho}_2)$; (iii) $\mathrm{Tr}\{\mathcal{E}(\hat{\rho})\} = \mathrm{Tr}\{\hat{\rho}\}$ and (iv) $\mathcal{E}$ takes positive operators into positive operators (§2.5 above). That is, *$\mathcal{E}$ is a linear positive-definite trace-preserving map*. In the same way, we could also define a reduced state for E simply as $\hat{\rho}_E(t) = \mathrm{Tr}_S \left\{ \hat{U}_t \hat{\rho}_S \otimes \hat{\rho}_E \hat{U}_t^\dagger \right\}$.

Finally, the following important question arises: we know, in principle, how to treat the dynamics of an open system; but, for instance, would it possible to describe this dynamics in some alternative way? Specifically, in a way s.t. we may refer only to quantities associated with the system of interest S? Fortunately, the answer is positive. We treat that in a simple context in the next section based in the exposition of ref. (SCHUMACHER; WESTMORELAND, 2010).

3.1.1 Kraus decomposition and the Lindblad master equation

Let $\{|v_j\rangle\} \subset \mathcal{H}_E$ be an orthonormal basis set and suppose that E is initially in a pure state $\hat{\rho}_E = |\chi\rangle \langle\chi|$ [1]. If we define a set of operators $\{\hat{A}_j : \mathcal{H}_S \to \mathcal{H}_S\}$ s.t.

$$\hat{A}_j |\phi\rangle_S = \langle v_j| \hat{U}_t |\phi,\chi\rangle \ \forall |\phi\rangle_S \in \mathcal{H}_S \tag{3.5}$$

the following relation can be verified [2]:

$$\mathcal{E}(\hat{\rho}_S) = \sum_j \hat{A}_j \hat{\rho}_S \hat{A}_j^\dagger. \tag{3.6}$$

That is, in this case to specify the set $\{\hat{A}_j\}$ is equivalent to specify the action of $\mathcal{E}$. The operators $\hat{A}_j$ are known as *Kraus operators* and (3.6) as the *Kraus decomposition* of the map $\mathcal{E}$. If $\mathcal{H}_E$ has dimension N, we choose by convention $j = 0, 1, 2, ..., N-1$. Moreover, it can also be verified that the trace-preserving condition of $\mathcal{E}$ implies a normalization condition for $\{\hat{A}_j\}$, namely $\sum_j \hat{A}_j^\dagger \hat{A}_j = \hat{I}$.

[1] If it is not the case, we can always couple E to another system E' in a way that the total state is pure. This method called *purification*.

[2] This more easily achievable if we consider without loss of generality the diagonal representation (or the *spectral decomposition*) of $\hat{\rho}_S$.

For example, for a general *qubit state* $\hat{\rho}_S = (\hat{I} + \boldsymbol{s}\cdot\boldsymbol{\sigma})/2$, $\boldsymbol{s} = (s_x, s_y, s_z) \in \mathbb{R}^3$, $\boldsymbol{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$, interacting with another (environment) qubit described by the state $|\chi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ via the CNOT evolution operator (see (SCHUMACHER; WESTMORELAND, 2010), chap. 18) $\hat{U}_c = \hat{I}\otimes|0\rangle\langle 0| + \hat{\sigma}_x|1\rangle\langle 1|$, we have $\hat{A}_0 = \hat{I}/\sqrt{2}$ and $\hat{A}_1 = \hat{\sigma}_x/\sqrt{2}$. In this case, S evolves to the state $\hat{\rho}_S(t) = (\hat{I} + s_x\hat{\sigma}_x)/\sqrt{2}$, which is just the projection of $\hat{\rho}_S$ onto the x-axis.

This approach is very useful because it allows us to develop a master equation to describe the dynamics of S "alone" without referring specifically to E. Although the most general deduction of such a master equation requires a discussion about dynamical semi-groups in Markovian dynamics and Liouville operators, we choose to follow the simpler procedure in (SCHUMACHER; WESTMORELAND, 2010) and deduce it using a set of assumptions which shall suffice for us in due course. For the most general case, we refer to (BREUER; PETRUCCIONE, 2007), Chap. 3.

First, we suppose that after a short time interval δt, we have

$$\begin{aligned}\mathcal{E}(\hat{\rho}_S) &= \hat{\rho}_S + \delta\hat{\rho}_S \\ &= \sum_j \hat{A}_j\hat{\rho}_S\hat{A}_j^\dagger\end{aligned} \tag{3.7}$$

where the $\hat{A}_j$'s are written in an appropriate form to describe an infinitesimal change:

$$\begin{aligned}\hat{A}_0 &= \hat{I} + \delta t(\hat{L}_0 - \frac{i}{\hbar}\hat{H}) \\ \hat{A}_j &= \sqrt{\delta t}\hat{L}_j \ (j \neq 0).\end{aligned} \tag{3.8}$$

Note that this choice recovers the infinitesimal dynamics of a closed system in the limit $\hat{L}_j \to 0$. Indeed, if we apply $\hat{A}_0$, in this case, a large number of times we recover the exponential representation $\hat{A}_0 = \hat{U}_{\delta t} = e^{-i\hat{H}\delta t/\hbar}$ for a finite (non-infinitesimal) time-evolution (see footnote 2 in §2.1.1).

Now, substituting (3.8) into (3.7), we get

$$\begin{aligned}\hat{A}_0\hat{\rho}_S\hat{A}_0^\dagger &= \hat{\rho}_S + \delta t(\hat{L}_0\hat{\rho}_S + \hat{\rho}_S\hat{L}_0 - \frac{i}{\hbar}\hat{H}\hat{\rho}_S + \frac{i}{\hbar}\hat{\rho}_S\hat{H}) + \mathcal{O}(\delta t^2) \\ \hat{A}_j\hat{\rho}_S\hat{A}_j^\dagger &= \delta t\hat{L}_j\hat{\rho}_S\hat{L}^\dagger.\end{aligned} \tag{3.9}$$

Therefore, neglecting second order terms in δt,

$$\frac{\delta\hat{\rho}}{\delta t} = \left(\{\hat{L}_0, \hat{\rho}_S\} - \frac{i}{\hbar}[\hat{H}, \hat{\rho}_S] + \sum_j \hat{L}_j\hat{\rho}_S\hat{L}_j^\dagger\right) \tag{3.10}$$

where $\{\hat{L}_0, \hat{\rho}_S\} \stackrel{\text{def}}{=} \hat{L}_0\hat{\rho}_S + \hat{\rho}_S\hat{L}_0$ is the usual *anticommutator* symbolic notation. Taking then explicitly the limit $\delta t \to 0$, we have

$$\frac{d\hat{\rho}_S}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}_S] + \sum_j \hat{L}_j\hat{\rho}_S\hat{L}_j^\dagger + \{\hat{L}_0, \hat{\rho}_S\}. \tag{3.11}$$

Finally, because $\mathrm{Tr}\{\hat{\rho}\} = 1 \Rightarrow \mathrm{Tr}\{d\hat{\rho}/dt\} = 0$ for arbitrary $\hat{\rho}$, we get the restriction $\hat{L}_0 = -\frac{1}{2}\sum_j \hat{L}_j^\dagger \hat{L}_j$, $j \neq 0$, so that we arrive at

$$\frac{d\hat{\rho}_S}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}_S] + \sum_j \left(\hat{L}_j \hat{\rho}_S \hat{L}_j - \frac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}_S\} \right). \tag{3.12}$$

This is the so called *Lindblad equation*, which represents the time-evolution of an open system only in terms of quantities related to the system of interest S (recall that $\hat{A}_j : \mathcal{H}_S \to \mathcal{H}_S$. Consequently, $\hat{L} : \mathcal{H}_S \to \mathcal{H}_S$).

It can be seen imediately by (3.12) that in the limit $\hat{L}_j \to 0$ the Lindblad equation reduces to the von Neumann equation (2.48), but there are some subtleties that shall be pointed out in the following.

Recall that we are assuming since the very beginning that the state of E is pure. Thus, we are also considering that the initial composite state is separable. This means that we cannot simply evolve the system from, say, $t = 0$ to $t = 2\delta t$ just applying the map (3.6) two times. After the first evolution the state of the composite system could become an entangled state and, consequently, the second evolution using the same map would not be justified. Therefore, we see that the exact regime where we can apply the Lindblad equation so deduced is that of $\delta t \ll T_S$ and $\delta t \gg T_E$, where T_S is the time scale over which the state of S does not change significantly (although it may indeed change) and T_E is the timescale over which the entanglement between S and E is no longer present.

As an example of those subtleties, consider the case of a two-level atom S in the surroundings of an electromagnetic field E in its *vacuum state* $|0\rangle_E$ (a state where no photon is present). The interaction between S and E may induce a transition in the atom making it emit a photon to E whose state can be entangled to the state of S. Therefore, as the photon pertains now to E, we can say that the state of $S + E$ became entangled. However, the photon quickly propagates away, making the whole process to happen *as if the state of* E *have remained* $|0\rangle_E$. Therefore, T_S would be the timescale over which the atom emits the photon and, consequently, has its state altered, and T_E would be the timescale over which the photon is gone away (making the state of E become $|0\rangle_E$ again).

We shall regard the Lindblad equation as an *effective model* to describe the dynamics of open quantum systems and, consequently, the specification of a set of *Lindblad operators* $\{\hat{L}\}$ must take this into account.

As a plausible illustration, we could model the transition of a two-level atom accompanied by the emission of a photon taking a single Lindblad operator $\hat{L} = \Lambda |0\rangle \langle 1|$, where Λ is

a constant with units of $\sqrt{\text{time}}$, and writing $\hat{\rho} = \rho_{00} |0\rangle \langle 0| + \rho_{01} |0\rangle \langle 1| + \rho_{10} |1\rangle \langle 0| + \rho_{11} |1\rangle \langle 1|$. Plugging that into the Lindblad equation we get a simple system of ordinary differential equations for the components $\rho_{i,j}$ whose solution is $\rho_{0,0}(t) = 1 - \rho_{1,1}(0)e^{-t^2}$, $\rho_{0,1}(t) = \rho_{0,1}(0)e^{-\mathcal{Z}^2 t/2}$, $\rho_{1,0}(t) = \rho_{1,0}(0)e^{-\mathcal{Z}^2 t/2}$, $\rho_{1,1}(t) = \rho_{1,1}(0)e^{-\mathcal{Z}^2 t}$. Note then that given any initial state, for $t \to \infty$ the system approaches the ground state $|0\rangle \langle 0|$.

3.2 THE MONTE CARLO WAVEFUNTION APPROACH

After the discussion about the deduction of the Lindblad equation, whose objective is to describe the evolution of the system S "alone" without making explicit reference to its environment E, we shall discuss now the alternative algorithmic approach given in (MøLMER; CASTIN; DALIBARD, 1993). It may serve to shed some more light between the analysis of open systems formulated in terms of density operators and in terms of state vectors. The equivalence between the two methods shall be explicitly shown.

3.2.1 General presentation

Suppose that our system of interest S in contact with an environment E can be described at time t by the normalized vector state $|\psi_S(t)\rangle$. We can predict its time-evolution basically in two steps:

1. We define a new non-Hermitian Hamiltonian

$$\hat{H} = \hat{H}_S - \frac{i\hbar}{2} \sum_j \hat{L}_j^\dagger \hat{L}_j \tag{3.13}$$

where $\hat{H}_S$ is the Hamiltonian of the system S (when isolated) and $\{\hat{L}_j\}$ is a set of Lindblad operators constructed as in the previous section. Then, for a sufficiently small δt we write

$$|\psi_S(t+\delta t)\rangle = \left(\hat{I} - \frac{i}{\hbar}\hat{H}\delta t\right) |\psi_S(t)\rangle \tag{3.14}$$

for an infinitesimal time-evolution of the state $|\psi_S(t)\rangle$. But because $\hat{H}$ is not Hermitian, we cannot guarantee that this evolved state ket will be normalized. Its norm will be

$$\begin{aligned} \langle \psi_S(t+\delta t)|\psi_S(t+\delta t)\rangle &= \langle \psi_S(t)| \left(\hat{I} + \frac{i}{\hbar}\hat{H}^\dagger \delta t\right)\left(\hat{I} - \frac{i}{\hbar}\hat{H}\delta t\right) |\psi_S(t)\rangle \\ &\stackrel{\text{def}}{=} 1 - \delta p \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \delta p =& \frac{i}{\hbar} \langle \psi_S(t)| \left(\hat{H} - \hat{H}^\dagger\right) |\psi_S(t)\rangle \, \delta t \stackrel{\text{def}}{=} \sum_j \delta p_j \\ \delta p_j =& \, \delta t \, \langle \psi_S(t)| \, \hat{L}_j^\dagger \hat{L}_j \, |\psi_S(t)\rangle \geq 0. \end{aligned} \tag{3.16}$$

The magnitude of δt is adjusted so that the calculation is valid to first order and $\delta p \ll 1$.

2. Now we simulate the effect of E which may cause in S a possible *quantum jump*, such as a transition between accesible states of S or even a projection of its state associated with a possible measurement process (postulate 5 in §2.1.1). Because such change of the state of S may or may not happen, we decide it by *throwing a coin* ϵ; i.e. we generate a random number ϵ between 0 and 1 and compare the outcome with the value of δp.

   If $\delta p < \epsilon$, which is more likely to occur, we update the state of S to

$$|\psi'_S(t+\delta t)\rangle = \frac{|\psi_S(t+\delta t)\rangle}{\sqrt{1-\delta p}}. \tag{3.17}$$

   But if $\delta p > \epsilon$, we choose the new state of S as $\hat{L}_j \, |\psi_S(t)\rangle$ with probability $p_j = \delta p_j / \delta p$ and normalize it:

$$\begin{aligned} |\psi'_S(t+\delta t)\rangle &= \frac{\hat{L}_j \, |\psi_S(t)\rangle}{\left|\hat{L}_j \, |\psi_S(t)\rangle\right|} \\ &= \frac{\hat{L}_j \, |\psi_S(t)\rangle}{\sqrt{\delta p_j / \delta t}} \end{aligned} \tag{3.18}$$

   (Note that (3.16) implies $\sum_j p_j = 1$.)

This algorithm is called the *Monte Carlo wavefunction* (MCWF) approach and, as an illustration, we take a situation very similar to the one analyzed in the last paragraph of § 3.1.1. Consider a quibit state of the Harmonic oscillator (see §2.2)

$$|\psi_S(0)\rangle = \alpha_0 \, |0\rangle + \beta_0 \, |1\rangle \, . \tag{3.19}$$

Suppose that we know that by interacting with some environment, the system will relax into the state $|0\rangle$. This process may occur accompanied by the emission of a photon (probability $|\alpha_0|^2$) or without emiting any photon (probability $|\beta_0|^2$). (Note also that this is very similar to the situation of a measurement process of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$, whose eigenstate set is $\{ \, |n\rangle \}$).

If we fix the energy reference point as the energy of the ground state $|0\rangle$, $E_0 = \hbar\omega/2$, we may use the Hamiltonian $\hat{H}_S = \hbar\omega\hat{N}$ instead of (2.21). Choosing a single Lindblad operator $\hat{L}_1 = \sqrt{\Gamma}\hat{a}$, the first step of the algorithm leads to

$$|\psi'_S(\delta t)\rangle = \alpha_0 |0\rangle + e^{-i\omega\delta t} e^{\frac{-\Gamma\delta t}{2}} \beta_0 |1\rangle \tag{3.20}$$

where we used that $x \ll 1 \Rightarrow 1 + x \approx e^x$. The probability for a quantum jump emitting a photon between $t = 0$ and $t = \delta t$ to happen is then $\delta p = \Gamma|\beta_0|^2\delta t$. After that, by (3.18), the system relax into the state $|0\rangle$ and remains there.

This last situation corresponds to the case where $\delta p > \epsilon$. For the case $\delta p < \epsilon$ the state of the system would simply evolve to

$$|\psi'_S(\delta t)\rangle = \alpha_0 \left(1 + \frac{\Gamma\delta t}{2}|\beta_0|^2\right)|0\rangle + \beta_0 \left(1 - \frac{\Gamma\delta t}{2}|\alpha_0|^2\right) e^{-i\omega\delta t}|1\rangle . \tag{3.21}$$

Note that the probability of being in the state $|0\rangle$ has increased, whilst the probability of being in the state $|1\rangle$ has decreased. This indicates that the system indeed evolves into relaxation (in this case emitting no photon).

3.2.2 Equivalence between the MCWF approach and the Lindblad equation approach

Before proceeding, we show explicitly that the two presented methods for describing the effective evolution of a quantum open system are actually equivalent.

First, because in the MCWF approach we are using the state ket formalism of §2.1.1, we must look at an ensemble of copies of the same system in order to determine its state at time t. More precisely, to construct the density matrix which describe the system at time t, we must average the projectors $|\psi_S(t)\rangle \langle\psi_S(t)|$ over all possible states $|\psi_S(t)\rangle$ in which the system may be at time t.

At $t + \delta t$ we have a probability δp of being in the state (3.18) and a probability $1 - \delta p$ of being in the state (3.17). Therefore, we get

$$\begin{aligned} \hat{\rho}_S(t + \delta t) = \delta p \sum_j p_j \frac{\hat{L}|\psi_S(t)\rangle \langle\psi_S(t)| \hat{L}^\dagger}{\delta p_j/\delta t} + \\ + (1 - \delta p)\frac{|\psi_S(t + \delta t)\rangle \langle\psi_S(t + \delta t)|}{1 - \delta p}. \end{aligned} \tag{3.22}$$

Now, if we use (3.14) this last equation reads

$$\begin{aligned}\hat{\rho}_S(t+\delta t) = & \left|\psi_S(t)\right\rangle \left\langle\psi_S(t)\right| + \frac{i\delta t}{\hbar}[\left|\psi_S(t)\right\rangle \left\langle\psi_S(t)\right|, \hat{H}_S] + \\ & + \delta t \sum_j \left(\hat{L}_j \left|\psi_S(t)\right\rangle \left\langle\psi_S(t)\right| \hat{L}_j - \frac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j,\ \left|\psi_S(t)\right\rangle \left\langle\psi_S(t)\right|\} \right).\end{aligned} \tag{3.23}$$

Taking then the average of this last equation over all possible states $\left|\psi_S(t)\right\rangle \left\langle\psi_S(t)\right|$ and making $\delta t \to 0$ we arrive at the Lindblad equation (3.12). **Q.E.D.**

We point that although the two methods lead to the same physical predictions, there are many computational advantages in using the MCWF approach instead of the Lindblad equation. Many examples of such are given in (MøLMER; CASTIN; DALIBARD, 1993).

3.2.3 Expectation values of observables

Because after repeating the MCWF approach to an ensemble of equally prepared systems we shall have at time t a set of $N \to \infty$ state vectors $\{\left|\psi_S^n(t)\right\rangle\}$, $n = 0, 1, 2, ..., N$, we can define the expectation value of some observable $\hat{A} \in B(\mathcal{H}_S)$ at time t simply as the mean value

$$\langle \hat{A} \rangle(t) = \frac{1}{N} \sum_n \left\langle\psi_S^n(t)\right| \hat{A} \left|\psi_S^n(t)\right\rangle . \tag{3.24}$$

Because N is supposed to be large, this shall give us a correct value for the expectation value defined in §2.1.3. We shall come back to this point in Chap.4.

3.3 MEASUREMENT MASTER EQUATION

In §3 we presented the map $\mathcal{E}$ defined by (3.4) and (3.6) in the context of time-evolutions, but we made no restriction about what it can represent. In particular, it may also represent the effective action of a sequence of measurements made in some *isolated* system described by the state $\hat{\rho}(t)$ at time t (just as if the system was the reduced system S and the experimental apparatus was the environment E). This process is also called *continuous monitoring process* of the system and we shall see here, following (CRESSER et al., 2006), that a simple master equation of Lindblad-form can also be derived in this context.

Let $\mathcal{H}$ be the Hilbert space of our system and $\{a_n\}$ be the spectrum of the observable $\hat{A} \in B(\mathcal{H})$. Suppose that the eingenvalue a_n is associated with the eingenvector $\left|a_n\right\rangle$ of $\hat{A}$. In terms of density operators, the von Neumann measurement scheme (§2.1.1, postulate 5)

can be represented simply by

$$\hat{\rho} \xrightarrow{\text{measurement}} \frac{\hat{P}_{a_n}\hat{\rho}\hat{P}_{a_n}}{\mathrm{Tr}\left\{\hat{P}_{a_n}\hat{\rho}\right\}} \tag{3.25}$$

where $\hat{P}_{a_n} \stackrel{\text{def}}{=} |a_n\rangle \langle a_n| = \hat{P}_{a_n}^\dagger$ is the projector onto the subspace asosciated with $|a_n\rangle$. In this case, note that if $\hat{\rho} = |\psi\rangle \langle\psi|$, $\mathrm{Tr}\left\{\hat{P}_{a_n}\hat{\rho}\right\} = |\langle a_n|\psi\rangle|^2$ is the probability that the outcome will be a_n.

If the result of the measurement is not known (or, in other words, is not recorded), we must also incorporate that into the density operator (§2.5):

$$\hat{\rho} \to \sum_n |\langle a_n|\psi\rangle|^2 \frac{\hat{P}_{a_n}\hat{\rho}\hat{P}_{a_n}}{\mathrm{Tr}\left\{\hat{P}_{a_n}\hat{\rho}\right\}} = \sum_n \hat{P}_{a_n}\hat{\rho}\hat{P}_{a_n}. \tag{3.26}$$

The set $\{\hat{P}_{a_n}\}$ will be then the Kraus decomposition of the evolution caused by the monitoring process.

Let $\hat{H}$ be the Hamiltonian of the system as usual and suppose that now we make a sequence of measurements of the observable $\hat{A}$ at a *measurement rate* R. After a small time interval δt there will be a probability $R\delta t$ that the state of the system will be given by the above equation (that is, that a measurement occurs in the time interval δt) and a probability $1 - R\delta t$ that the state of the system will be

$$\hat{\rho}(t + \delta t) = \hat{\rho}(t) - \frac{i}{\hbar}[\hat{H}, \hat{\rho}]\delta t \tag{3.27}$$

according to the von Neumann master equation (2.48) (this is the case where a measurement does not occur in the time interval δt). We know that the density operator must also incorporate those two possibilities at $t + \delta t$. Therefore, we have

$$\hat{\rho}(t + \delta t) = (1 - R\delta t)\hat{\rho}(t) - \frac{i}{\hbar}[\hat{H}, \hat{\rho}]\delta t + R\delta t \sum_n \hat{P}_{a_n}\hat{\rho}\hat{P}_{a_n} \tag{3.28}$$

at first order in δt. Finally, taking the limit $\delta t \to 0$ and using that $\hat{P}_{a_n}^2 = \hat{P}_{a_n}$, we arrive at the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + R\left[\sum_n \hat{P}_{a_n}\hat{\rho}\hat{P}_{a_n} - \hat{\rho}\right] \tag{3.29}$$

which is clearly of Lindblad form (3.12) with Lindblad operators $\{\hat{L}_{a_n} \stackrel{\text{def}}{=} \sqrt{R}\hat{P}_{a_n}\}$.

As an example, suppose that we have a two-level atom whose resonant interaction with a driven laser field is described by the Hamiltonian $\hat{H} = -\hbar\Omega\hat{\sigma}_x/2$, where Ω is a constant called *Rabi frequency* of the atom. If at time time t the atom is subject to a monitoring process as

described above and its state may be projected to $|0\rangle$ or $|1\rangle$, we take the elements of the set $\{\hat{P}_{a_n}\}$ to be

$$\begin{aligned} \hat{P}_0 &= |0\rangle \langle 0| \\ \hat{P}_1 &= |1\rangle \langle 1| . \end{aligned} \tag{3.30}$$

In this case, the master equation (3.29) takes the form (see §2.4)

$$\frac{d\hat{\rho}}{dt} = \frac{i\Omega}{2}[\hat{\sigma}_x, \hat{\rho}] + \frac{R}{2}(\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \hat{\rho}). \tag{3.31}$$

Writing then $\hat{\rho}$ as a general time-dependent qubit state,

$$\hat{\rho}(t) = \frac{1}{2}\Big(\hat{I} + u(t)\hat{\sigma}_x + v(t)\hat{\sigma}_y + w(t)\hat{\sigma}_z\Big) \tag{3.32}$$

we arrive at the following system of linear differential equations

$$\begin{aligned} \frac{du}{dt} &= -Ru \\ \frac{dv}{dt} &= \Omega w - Rv \\ \frac{dw}{dt} &= -\Omega v. \end{aligned} \tag{3.33}$$

The solution can verified to be:

$$\begin{aligned} u(t) &= u(0)e^{-Rt} \\ v(t) &= v(0)e^{-Rt/2}\left(\cos\Omega' t - \frac{R}{2\Omega'}t\right) + w(0)e^{-Rt/2}\frac{\Omega}{\Omega'}\sin\Omega' t \\ w(t) &= w(0)e^{-Rt/2}\left(\cos\Omega' t + \frac{R}{2\Omega'}t\right) + v(0)e^{-Rt/2}\frac{\Omega}{\Omega'}\sin\Omega' t \end{aligned} \tag{3.34}$$

with $\Omega' \stackrel{\text{def}}{=} \sqrt{\Omega^2 - \gamma^2}$. Note that at $t \to \infty$ the atom reaches the steady state $\hat{\rho} = \hat{I}/2$, which is a fully mixed state as described in §2.5.

4 SIMULATING AN OPEN SYSTEM VIA CONTINUOUS SPIN MEASUREMENTS

This chapter contains the main results of our project. We shall investigate precisely how continuous observations of the spin degree of freedom of an harmonic oscillator put in a region of space where an inhomogeneous magnetic field is present may affect its orbital degree of freedom. In analogy with the notation of the previous chapter, we shall refer to this orbital degree of freedom of the system as S and to the spin degree of freedom coupled with S through the inhomogeneous magnetic field as E. The situation can be pictured "imaginatively" as if E was a "diathermal wall" mediating the interaction between the system and an effective environment (the measurement apparatus). Particularly, this is a problem to which we could find a detailed closed analytical solution not only for the state of the system, but also for all relevant physical quantities (something unusual given the complexity intuitively expected by its statement). Numerical simulations shall be presented in order to double-check or clarify those results and an appendix containing lengthily calculations is also included at the end for the sake of fluidity and cleanness of the discussion.

4.1 PRESENTATION OF THE SYSTEM

Suppose that we put a spin-$1/2$ oscillating particle ($\S 2.4$) initially described by a coherent state $|z\rangle$ in a region where an inhomogeneous magnetic field $\mathbf{B}$ is present. If we fix our coordinate system (x, y, z) in a way that the particle oscillates in the x-direction and $\mathbf{B}$ points effectively in the z-direction, we may write $\mathbf{B} = B_0 f(x)\mathbf{e}_z$, where f is, for now, an arbitrary dimensionless function of the coordinate x.

Following Postulate 2 of $\S 2.1.1$ and taking Eqs.(2.16),(2.43) into account, we may, therefore, write the Hamiltonian of the system simply as

$$\hat{H} = \hat{H}_{HO} \otimes \hat{I} - \alpha f(\hat{X}) \otimes \hat{\sigma_z} \tag{4.1}$$

where $\alpha = \frac{\gamma B_0 \hbar}{2}$ is the *coupling constant* that describes the strength of interaction between the system and the magnetic field $\mathbf{B}$ (or, in other words, how strongly $\mathbf{B}$ may affect the spin of the particle). Note that this last equation is already in the form of Eq.(3.3) (with $\hat{H}_E \stackrel{\text{def}}{=} 0$), indicating that we may study, indeed, how changes in E may affect S (as if E was a "diathermal wall" through which energy coming from the external world may pass).

4.1.1 Time-evolution of a superposition of coherent states

As mentioned, our aim is to study in the following sections how continuous measurements performed on E may affect S. However, before that, we need to address the question of how a state of the form

$$|\psi(0)\rangle \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} c_i |z_i\rangle \right) \otimes |s\rangle_x \tag{4.2}$$

unitarily evolves in time (§2.1.1, postulate 3) given the Hamiltonian (4.1). In this formula, the states $|z_i\rangle$, $i = 1, 2, 3, ..., n$, are all eigenstates of the operator $\hat{a}$ (coherent states), the c_i's are, for now, arbitrary complex coefficients, and $s = \pm 1$. The evolution of the initial state $|z\rangle$ of S would correspond naturally to the case where $i = 1, c_1 = 1$. Latter on, when we discuss our spin measurement protocol (§4.2), it will be clarified why we need to evolve a state of this specific type and why we are considering the state of E written in terms of the eigenstates of $\hat{\sigma}_x$, not of $\hat{\sigma}_z$, as would be more natural to regard if we look at our Hamiltonian [1].

First, we begin by noting that, by Eqs.(2.41), we have

$$|s\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + s|-\rangle) \tag{4.3}$$

so that, writing for now $|\phi\rangle \stackrel{\text{def}}{=} \sum_i c_i |z_i\rangle$,

$$\begin{aligned} |\psi(\delta t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}\delta t}|\psi(0)\rangle \\ &= \left[\sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\delta t \right)^n \frac{\hat{H}^n}{n!} \right] |\phi\rangle \frac{1}{\sqrt{2}}(|+\rangle + s|-\rangle). \end{aligned} \tag{4.4}$$

But,

$$\begin{aligned} \hat{H}|\phi\rangle\,|\pm\rangle &= \{[\hat{H}_{HO} \mp \alpha f(\hat{X})]|\phi\rangle\}|\pm\rangle \\ &= (\hat{H}_{\pm}|\phi\rangle)|\pm\rangle \\ \Rightarrow \hat{H}^n|\phi\rangle\,|\pm\rangle &= (\hat{H}_{\pm}^n|\phi\rangle)|\pm\rangle \end{aligned} \tag{4.5}$$

where $\hat{H}_{HO} = \hat{P}^2/2m + m\omega^2\hat{X}^2/2$ is the Hamiltonian of the harmonic oscillator [Eq.(2.16)] and $\hat{H}_{\pm} \stackrel{\text{def}}{=} \hat{H}_{HO} \mp \alpha f(\hat{X})$.

Therefore, writing $\hat{U}_{\pm}^{\delta t}$, we arrive at

$$\begin{aligned} |\psi(\delta t)\rangle &= \frac{1}{\sqrt{2}}[(e^{-\frac{i}{\hbar}\hat{H}_+\delta t})|\phi\rangle\,|+\rangle + s(e^{-\frac{i}{\hbar}\hat{H}_-\delta t})|\phi\rangle\,|-\rangle] \\ &= \frac{1}{\sqrt{2}}[(\hat{U}_+^{\delta t}|\phi\rangle)|+\rangle + s(\hat{U}_-^{\delta t}|\phi\rangle)|-\rangle] \\ &= \frac{1}{2}\{[(\hat{U}_+^{\delta t} + s\hat{U}_-^{\delta t})|\phi\rangle]|+\rangle_x + [(\hat{U}_+^{\delta t} - s\hat{U}_-^{\delta t})|\phi\rangle]|-\rangle_x\}. \end{aligned} \tag{4.6}$$

[1] Nonetheless, some information about this choice can be already obtained in the Introduction of this present text.

Now, in order to finish the calculation we must explicitly obtain $\hat{U}_\pm |\phi\rangle = e^{-i\hat{H}_\pm \delta t/\hbar} |\phi\rangle$. This is not (analytically) easy in general because it may depend on the specific dependence of $f(\hat{X})$ on $\hat{X}$. However, this is achievable, for example, for the case $f(\hat{X}) \stackrel{\text{def}}{=} \hat{X}/d$, where $d = \sqrt{\hbar/2m\omega}$ [2]. We believe that this is also true for $f(\hat{X}) \propto \hat{X}^2$, but in this project we shall focus in the former case.

4.1.2 Linear external magnetic field

Let us begin by writing the Hamiltonians $\hat{H}_\pm$ using the representation of Eq.(2.16) in terms of $\hat{X}$ and $\hat{P}$, the position and the momentum operators of the harmonic oscillator. We have

$$\hat{H}_\pm = \frac{\hat{P}^2}{2m} + \frac{m\omega^2 \hat{X}^2}{2} \mp \alpha \sqrt{\frac{2m\omega}{\hbar}} \hat{X}. \tag{4.7}$$

Completing the square, this gives

$$\hat{H}_\pm = \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}(\hat{X} \mp X_\alpha)^2 - \frac{m\omega^2}{2} X_\alpha^2 \tag{4.8}$$

where $X_\alpha \stackrel{\text{def}}{=} \frac{\alpha}{m\omega^2}\sqrt{\frac{2m\omega}{\hbar}}$. Defining a further parameter $z_\alpha \stackrel{\text{def}}{=} X_\alpha \sqrt{m\omega/2\hbar}$, this last equation can be rewritten in terms of the non-Hermitian operators

$$\begin{aligned} \hat{a}_\pm &\stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}(\hat{X} \mp X_\alpha) + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{P}^2 \\ &= \hat{a} \mp z_\alpha. \end{aligned} \tag{4.9}$$

That is,

$$\hat{H}_\pm = \hbar\omega\left(\hat{a}_\pm^\dagger \hat{a}_\pm + \frac{1}{2}\right) - \hbar\omega z_\alpha^2. \tag{4.10}$$

This is handful, because now we note the following fact: *if the state $|z_i\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue z_i, so it is an eigenstate of $\hat{a}_\pm$ with eigenvalue $z_i \mp z_\alpha$*. Hence, following the discussion of §2.2, if $|n_\pm\rangle$ is an eigenstate of $\hat{N}_\pm \stackrel{\text{def}}{=} \hat{a}_\pm^\dagger \hat{a}_\pm$ (and, consequently, of $\hat{H}_\pm$) with eigenvalue $n_\pm$, the following expansion is possible:

$$|z_i\rangle = e^{-\frac{|z_i \mp z_\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(z_i \mp z_\alpha)^n}{\sqrt{n!}} |n_\pm\rangle . \tag{4.11}$$

[2] This expression for d is a convenient choice. Because the purpose of this parameter is only to make f dimensionless, any parameter with units of length would in principle suffice.

Using this last equation together with Eq.(2.29), now it becomes very easy to obtain $\hat{U}_{\pm}^{\delta t}\left|\phi\right\rangle$ explicitly [3]:

$$\begin{aligned}\hat{U}_{\pm}^{\delta t}\left|\phi\right\rangle &= \hat{U}_{\pm}^{\delta t}\left(\sum_i c_i\left|z_i\right\rangle\right)\\ &= \sum_i c_i \hat{U}_{\pm}^{\delta t}\left|z_i\right\rangle\\ &= e^{-i\omega(\frac{1}{2}-z_\alpha^2)}\sum_i c_i\left|(z_i \mp z_\alpha)e^{-i\omega\delta t}\right\rangle_{\pm}.\end{aligned} \tag{4.12}$$

where the subscript "$\pm$" indicates that we are expanding the coherent state $\left|(z_i \mp z_\alpha)e^{-i\omega\delta t}\right\rangle$ using the basis $\{\left|n_\pm\right\rangle\}$. We may come back to the eigenstate basis $\{\left|n\right\rangle\}$ of $\hat{N} = \hat{a}^\dagger\hat{a}$ by noting that, conversely to the previous fact, *if* $\left|z_\pm\right\rangle$ *is an eigenstate of* $\hat{a}_\pm$ *with eigenvalue* $z_\pm$*, so it is an eigenstate of* $\hat{a} = \hat{a}_\pm \pm z_\alpha$ *with eigenvalue* $z_\pm \pm z_\alpha$. That is,

$$\hat{U}_{\pm}^{\delta t}\left|\phi\right\rangle = e^{-i\omega(\frac{1}{2}-z_i^2{}_\alpha)\delta t}\left|(z_i \mp z_\alpha)e^{-i\omega\delta t} \pm z_\alpha\right\rangle. \tag{4.13}$$

Finally, for a fixed value of δt, we may define the set of maps

$$\left\{\mathcal{Z}_\pm : \mathbb{R}\times\mathbb{C}\to\mathbb{C}\;\middle|\;\mathcal{Z}_\pm(z_\alpha, z; \delta t) \overset{\text{def}}{=} (z \mp z_\alpha)e^{-i\omega\delta t} \pm z_\alpha\right\} \tag{4.14}$$

and drop the unimportant global phase factor $e^{-i(1/2-z_\alpha^2)\delta t}$ [4] to write (4.6) simply as

$$\begin{aligned}\left|\psi(\delta t)\right\rangle = \frac{1}{2}\Bigg\{&\sum_i c_i\left[\left|\mathcal{Z}_+(z_\alpha, z_i)\right\rangle + s\left|\mathcal{Z}_-(z_\alpha, z_i)\right\rangle\right]\left|+\right\rangle_x +\\ &+\sum_i c_i\left[\left|\mathcal{Z}_+(z_\alpha, z_i)\right\rangle - s\left|\mathcal{Z}_-(z_\alpha, z_i)\right\rangle\right]\left|-\right\rangle_x\Bigg\}\end{aligned} \tag{4.15}$$

For the special case where $i = 1, c_1 = 1, z_1 = z$, we have

$$\begin{aligned}\left|\psi(\delta t)\right\rangle = \frac{1}{2}\Big\{&\left[\left|\mathcal{Z}_+(z_\alpha, z)\right\rangle + s\left|\mathcal{Z}_-(z_\alpha, z)\right\rangle\right]\left|+\right\rangle_x +\\ &+\left[\left|\mathcal{Z}_+(z_\alpha, z)\right\rangle - s\left|\mathcal{Z}_-(z_\alpha, z)\right\rangle\right]\left|-\right\rangle_x\Big\}.\end{aligned} \tag{4.16}$$

The importance of those mathematical results shall become clear in the next section, where we will define our *measurement protocol*. For the sake of latter reference, we state also here the general form of N compositions of the map (4.14):

$$\mathcal{Z}_{I_N}(z_\alpha, z; \delta t) \overset{\text{def}}{=} (z - i_1 z_\alpha)e^{-iN\omega\delta t} + z_\alpha\left[\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t} + i_N\right]. \tag{4.17}$$

[3] Note only the additional phase factor $e^{-i\omega z_\alpha^2}$ in comparison to Eq.(2.29). It comes from the definition of $\hat{H}_\pm$ (Eq.(4.10)), which is basically the Hamiltonian of the Harmonic oscillator with an additive constant. However, this phase factor shall not be important for us.

[4] This is possible, as mentioned before, because expectation values are not changed by a global phase factor.

In this notation, $I_N = \{i_N, i_{N-1}, i_{N-2}, ..., i_2, i_1\}$ denotes an specific arrangement of N elements $i_k = \pm 1$. It is to be understood that the map $\mathcal{Z}_{i_1}$ is applied first, followed the maps $\mathcal{Z}_{i_2}$, $\mathcal{Z}_{i_3}$, $\mathcal{Z}_{i_4}$, ..., $\mathcal{Z}_{i_N}$ (notice that the ordering of the set I_N runs from i_N to i_1, not from i_1 to i_N). This last result can be proved straightforwardly by finite induction.

We further fix that *all* capital subscripts (such as I_N) will denote specific arrangements, such as, say, for $N = 3$, $\{+,+,+\}$, $\{+,-,+\}$, $\{-,-,-\}$, and so on. For the sake of clarity, we anticipate that latter on we shall perform summations over *all possibilities* of such arrangements; that is, we shall write, for example, expressions like

$$\sum_{I_N} |\mathcal{Z}_{I_N}\rangle \tag{4.18}$$

or even

$$\sum_{I_N, J_N} |\mathcal{Z}_{I_N}\rangle \langle \mathcal{Z}_{J_N}| \,. \tag{4.19}$$

(From now on let us also omit the arguments z_α, z and δt from the maps $\mathcal{Z}_I^N$ in order to clean the notation even further.)

We are now in a position to begin our analysis.

4.2 SPIN MEASUREMENT PROTOCOL

Let us return to our first question of how continuous observations (measurements) of the spin degree of freedom E can affect the orbital degree of freedom S. More precisely, we want to predict what happens to physical quantities related to S when the following simple procedure is repeated a certain number N of times:

1. The state of the system $(S+E)$ is let to evolve unitarily in time by δt units: $|\psi_{S+E}(t)\rangle \to |\psi_{S+E}(t+\delta t)\rangle$.

2. A von Neumann measurement is performed on E returning a random outcome and projecting its state onto some state $|\chi_E\rangle$.

According to our Hamiltonian (Eq.(4.1)), the presence of the magnetic field may induce interactions between S and E as time passes, so that the projection of E in the second step may induce transitions in S. This shall become fully clear in the next section.

This is a so called *stochastic process*, because it involves a set of *random* outcomes. In other words, if the procedure is repeated a certain number N of times for a given initial state

and, then, repeated again the same number N of times for the same given initial state, the state of the system after the second set of repetitions will not, in general be equal to the state of the system after the first set of repetitions. In the literature it is common to call a *step* a single realization of 1 and 2 above and a *trajectory*, usually symbolized by γ^N, a set of N steps repeated in sequence.

As discussed briefly in Chap.2, quantum mechanics is an ensemble theory. Because of that, a large enough set $\{\gamma_k^N\}_{k=1}^n$ of N-steps trajectories must be realized so that we obtain a set $\{\,|\psi_N^k\rangle\}_{k=1}^J$ of *all possible states* which may describe the system at step N together with their respective probabilities p_k to occur (or their *frequencies of occurrence*). More precisely, if we realize a large set of n N-steps trajectories, we may determine with accuracy that there are J possible states accessible to the system at step N and that, if one of such state, say, $\left|\psi_N^k\right\rangle$ occurs n_k times, its probability of occurence is simply $p_k = n_k/n$. In this way, the quantity

$$\langle \hat{A} \rangle = \sum_{k=1}^{J} p_k \, \langle \psi_N^k | \, \hat{A} \, | \psi_N^k \rangle \tag{4.20}$$

coincides with the expectation value of some observable $\hat{A} \in B(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of the system. This is very similar to the well known context of statistical mechanics, where a system let to evolve sufficiently long in time access all of its possible microstates, and those microstates, given their respective probabilities to occur as time passes, are used to compute mean values of physical quantities related to the system.

This treatment for our specific case (that is, of a single coherent state whose spin is subject to continuous observations in a region of space where an inhomogeneous linear magnetic field is present) shall be explicitly developed in the following two subsections.

4.2.1 Determination of a single trajectory

Let us analyse what happens in a single trajectory γ_k^N if the initial state of the system is prepared as

$$|\psi(0)\rangle = \, |z\rangle \, |s_0\rangle_x \tag{4.21}$$

and we subject E to continuous observations of $\hat{S}_x$.

But, why $\hat{S}_x$? Indeed, we could have chosen to perform measurements of $\hat{S}_n$, where n is an arbitrary direction. However, not of $\hat{S}_z$, as Eq.(4.5) implies that after the first measurement the state of E would be projected onto $|+\rangle$ or $|-\rangle$ and remain there even throughout a

time-evolution. In other words, in the case where we choose to measure $\hat{S}_z$, if the outcome of the first measurement happens to be, say, $+\hbar/2$, the outcome of all the subsequent ones would be simply $+\hbar/2$ with probability 1, meaning that the process would not be random at all. Hence, it would not be possible to simulate the contact of a system of interest with an environment.

This choice of $\hat{S}_x$ (or $\hat{S}_n$, $n \neq z$) is also physically interesting, because the magnetic field $\mathbf{B}$ points in the z-direction and we know (as mentioned in §2.4) that the magnetic moment $\mu \propto \mathbf{S}$ of the system will tend to align with it after some time. This means that if a measurement of $\hat{S}_x$ is performed, the state of E is projected onto $|+\rangle_x$ or $|-\rangle_x$ and the time-evolution tends to bring it back to the z-direction. This is more or less as if we were "hitting E" onto the x-direction whilst the field tries to realign it with the z-direction.

That said, we turn now to the protocol.

STEP 0.

The state of the system is given simply by Eq.(4.21) and after a time δt, it evolves to Eq.(4.16):

$$\begin{aligned} |\psi^0(\delta t)\rangle &= \frac{1}{2}\{[(\hat{U}_+^{\delta t} + s_0\hat{U}_-^{\delta t})\,|z\rangle]\,|+\rangle_x + [(\hat{U}_+^{\delta t} - s_0\hat{U}_-^{\delta t})\,|z\rangle]\,|-\rangle_x\} \\ &= \frac{1}{2}[(\,|\mathcal{Z}_+\rangle + s_0\,|\mathcal{Z}_-\rangle)\,|+\rangle_x + (\,|\mathcal{Z}_+\rangle - s_0\,|\mathcal{Z}_-\rangle)\,|-\rangle_x)]. \end{aligned}$$

If now a measurement of $\hat{S}_x$ is performed on E, Born rule (§2.1.1, postulate 4) implies that we have a probability

$$\begin{aligned} p_1(s_1^k|s_0) &= \frac{1}{4}||(\hat{U}_+^{\delta t} + s_1^k s_0\hat{U}_-^{\delta t})\,|z\rangle\,||^2 \\ &= \frac{1}{4}||\,|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle\,||^2 \end{aligned} \tag{4.22}$$

that the outcome will be $s_1^k\hbar/2 = \pm\hbar/2$. Consequently, immediately after that, E will be projected onto $|s_1^k = \pm\rangle_x$.

STEP 1.

A measurement of $\hat{S}_x$ was performed on E, the outcome was $s_1^k\hbar/2$ and, according to the previous mathematical analysis, the state of $S + E$ is now

$$\begin{aligned} |\psi_1^k(\delta t)\rangle &= \left[\frac{(\hat{U}_+^{\delta t} + s_1^k s_0\hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_1^k s_0\hat{U}_-^{\delta t})\,|z\rangle\,||}\right]|s_1^k\rangle_x \\ &= \left[\frac{|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle}{||\,|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle\,||}\right]|s_1^k\rangle_x\,. \end{aligned} \tag{4.23}$$

Note that the projection of the state of E effectively changes the state of S, meaning that something happened to S after an observation of E. More precisely, an observation of E has turned now S into a normalized linear combination of coherent states.

It is useful to make a table associating the label of each state of the linear combination in the numerator of Eq.(4.23) with its respective associated coefficient:

Z_+	$(\mathsf{s}_1^k s_0)^0$
Z_-	$(\mathsf{s}_1^k s_0)^1$

Proceeding further, after a time δt, the state of the system evolves to

$$\begin{aligned}
|\psi_1^k(2\delta t)\rangle &= \frac{1}{2}\Bigg\{(\hat{U}_+^{\delta t} + s_1^k \hat{U}_-^{\delta t})\left[\frac{(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||}\right]|+\rangle_x + \\
&\quad + (\hat{U}_+^{\delta t} - s_1^k \hat{U}_-^{\delta t})\left[\frac{(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||}\right]|-\rangle_x\Bigg\} \\
&= \frac{1}{2}\Bigg\{\left[\frac{|\mathcal{Z}_{+,+}\rangle + s_1^k s_0\,|\mathcal{Z}_{+,-}\rangle + s_1^k\,|\mathcal{Z}_{-,+}\rangle + (s_1^k)^2 s_0\,|\mathcal{Z}_{-,-}\rangle}{||\,|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle\,||}\right]|+\rangle_x + \\
&\quad + \left[\frac{|\mathcal{Z}_{+,+}\rangle - s_1^k s_0\,|\mathcal{Z}_{+,-}\rangle - s_1^k\,|\mathcal{Z}_{-,+}\rangle + (s_1^k)^2 s_0\,|\mathcal{Z}_{-,-}\rangle}{||\,|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle\,||}\right]|-\rangle_x\Bigg\}
\end{aligned} \tag{4.24}$$

where we have used Eq. (4.15). (Now it should be fully clear why we have evolved an state of the form (4.2) in §4.1.1.)

If a measurement of $\hat{S}_x$ is again to be performed, we have a probability

$$\begin{aligned}
p_2(s_2^k|s_1^k, s_0) &= \frac{1}{4}\frac{||(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||^2}{||(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||^2} \\
&= \frac{1}{4}\frac{||\,|\mathcal{Z}_{+,+}\rangle + s_1^k s_0\,|\mathcal{Z}_{+,-}\rangle + s_2^k s_1^k\,|\mathcal{Z}_{-,+}\rangle + s_2^k (s_1^k)^2 s_0\,|\mathcal{Z}_{-,-}\rangle\,||^2}{||\,|\mathcal{Z}_+\rangle + s_1^k s_0\,|\mathcal{Z}_-\rangle\,||^2}
\end{aligned} \tag{4.25}$$

that the outcome will be $s_2^k\hbar/2 = \pm\hbar/2$. Naturally, immediately after that, E will be projected onto $|s_2^k = \pm\rangle_x$.

STEP 2.

The outcome of the last measurement performed on E was s_2^k and $S+E$ is now described by the state

$$\begin{aligned}
|\psi_2^k(2\delta t)\rangle &= \left[\frac{(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||}\right]|s_2^k\rangle_x \\
&= \left[\frac{|\mathcal{Z}_{+,+}\rangle + s_1^k s_0\,|\mathcal{Z}_{+,-}\rangle + s_2^k s_1^k\,|\mathcal{Z}_{-,+}\rangle + s_2^k (s_1^k)^2 s_0\,|\mathcal{Z}_{-,-}\rangle}{||\,|\mathcal{Z}_{+,+}\rangle + s_1^k s_0\,|\mathcal{Z}_{+,-}\rangle + s_2^k s_1^k\,|\mathcal{Z}_{-,+}\rangle + s_2^k (s_1^k)^2 s_0\,|\mathcal{Z}_{-,-}\rangle\,||}\right].
\end{aligned} \tag{4.26}$$

It is also useful to construct a table associating each label of the states in the linear combination of the numerator of this last expression and its respective associated coefficient, exactly as we did in the previous step:

$$\begin{array}{c|l} \mathsf{Z}_{+,+} & (\mathsf{s}_2^k s_1^k)^0 (s_1^k s_0)^0 \\ \mathsf{Z}_{+,-} & (\mathsf{s}_2^k s_1^k)^0 (s_1^k s_0)^1 \\ \mathsf{Z}_{-,+} & (\mathsf{s}_2^k s_1^k)^1 (s_1^k s_0)^0 \\ \mathsf{Z}_{-,-} & (\mathsf{s}_2^k s_1^k)^1 (s_1^k s_0)^1 \end{array}$$

Notice that for each index $+(-)$ on the left side we have a power $0(1)$ on the right side.

Now, the state of the system evolves to

$$\begin{aligned} |\psi_2^k(3\delta t)\rangle = \frac{1}{2}\Bigg\{ & (\hat{U}_+^{\delta t} + s_2^k \hat{U}_-^{\delta t}) \times \\ & \times \left[\frac{(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||} \right] |+\rangle_x + \\ & + (\hat{U}_+^{\delta t} - s_2^k \hat{U}_-^{\delta t}) \times \\ & \times \left[\frac{(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle}{||(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||} \right] |-\rangle_x \Bigg\}. \end{aligned} \tag{4.27}$$

(Let us not write this and the next expression explicitly in terms of the set $\{\,|\mathcal{Z}_{I_3}\rangle\}$ in order to avoid even longer equations.)

Consequently, there is a probability

$$p_3(s_3^k | s_2^k, s_{1,0}^k) = \frac{1}{4} \frac{||(\hat{U}_+^{\delta t} + s_3^k s_2^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||^2}{||(\hat{U}_+^{\delta t} + s_2^k s_1^k \hat{U}_-^{\delta t})(\hat{U}_+^{\delta t} + s_1^k s_0 \hat{U}_-^{\delta t})\,|z\rangle\,||^2} \tag{4.28}$$

that the outcome of a further measurement of $\hat{S}_x$ will be $s_3^k \hbar/2$.

STEP 3. It can be demonstrated that, if we write the state of the system now (before evolving it in time), we would have the following table

$$
\begin{array}{c|l}
\mathsf{Z}_{+,+,+} & (\mathsf{s}_3^k s_2^k)^0 (s_2^k s_1^k)^0 (s_1^k s_0)^0 \\
\mathsf{Z}_{+,+,-} & (\mathsf{s}_3^k s_2^k)^0 (s_2^k s_1^k)^0 (s_1^k s_0)^1 \\
\mathsf{Z}_{+,-,+} & (\mathsf{s}_3^k s_2^k)^0 (s_2^k s_1^k)^1 (s_1^k s_0)^0 \\
\mathsf{Z}_{+,-,-} & (\mathsf{s}_3^k s_2^k)^0 (s_2^k s_1^k)^1 (s_1^k s_0)^1 \\
\mathsf{Z}_{-,+,+} & (\mathsf{s}_3^k s_2^k)^1 (s_2^k s_1^k)^0 (s_1^k s_0)^0 \\
\mathsf{Z}_{-,+,-} & (\mathsf{s}_3^k s_2^k)^1 (s_2^k s_1^k)^0 (s_1^k s_0)^1 \\
\mathsf{Z}_{-,-,+} & (\mathsf{s}_3^k s_2^k)^1 (s_2^k s_1^k)^1 (s_1^k s_0)^0 \\
\mathsf{Z}_{-,-,-} & (\mathsf{s}_3^k s_2^k)^1 (s_2^k s_1^k)^1 (s_1^k s_0)^1
\end{array}
$$

(We restrict ourselves here to just construct the table because the explicit expressions for the state of the system are too long and because all of the relevant information which will allow us to generalize the procedure is contained in it.)

Notice that, again, for each subscript $+(-)$ on the left we have an associated power $0(1)$ on the right.

...

Now, it may be verified that at the N-th step we would have the following.

STEP N.

The state of the system is projected onto

$$
\begin{aligned}
|\psi_N^k(N\delta t)\rangle &= \left[\frac{\prod_{m=1}^N (\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle}{||\prod_{m=1}^N (\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||}\right] |s_N^k\rangle_x \\
&\stackrel{\text{def}}{=} \left[\frac{\sum_{I_N} a_{I_N}^k\,|\mathcal{Z}_{I_N}\rangle}{||\sum_{I_N} a_{I_N}^k\,|\mathcal{Z}_{I_N}\rangle\,||}\right] |s_N^k\rangle_x
\end{aligned}
\tag{4.29}
$$

where

$$
\left\{ a_{I_N}^k \in \mathbb{R} \middle| a_{I_N}^k \stackrel{\text{def}}{=} \prod_{j=1}^N (s_m^k s_{m-1}^k)^{\delta_{i_m,-1}} = \pm 1 \right\}. \tag{4.30}
$$

In this formula, $\delta_{i,j}$ is simply the Kronecker delta symbol, which is equal to 1 if $i = j$ and 0 otherwise [5].

[5] This is not the only possible representation for the coefficients. Indeed, any "binary" function which assumes only an even and an odd value would in principle suffice. We have chosen the Kronecker delta because it is well known.

According to the above analysis, the state (4.29) evolves to

$$
\begin{aligned}
|\psi_N^k((N+1)\delta t)\rangle = \frac{1}{2}\Bigg\{ & (\hat{U}_+^{\delta t} + s_N^k \hat{U}_-^{\delta t}) \times \\
& \times \left[\frac{\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle}{||\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||} \right] |+\rangle_x + \\
& + (\hat{U}_+^{\delta t} - s_N^k \hat{U}_-^{\delta t}) \times \\
& \times \left[\frac{\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle}{||\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||} \right] |-\rangle_x \Bigg\}.
\end{aligned}
\tag{4.31}
$$

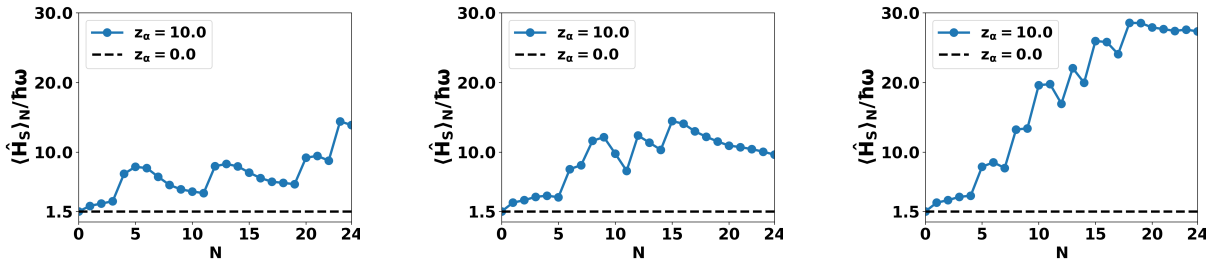
Finally, it implies that we have a probability

$$
\begin{aligned}
p_N(s_{N+1}^k | s_N^k, ..., s_0) &= \frac{1}{4} \frac{||(\hat{U}_+^{\delta t} + s_{N+1}^k s_N^k \hat{U}_-^{\delta t}) \prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||^2}{||\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||^2} \\
&= \frac{1}{4} \frac{||\prod_{m=1}^{N+1}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||^2}{||\prod_{m=1}^{N}(\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||^2} \\
&= \frac{1}{4} \frac{||\sum_{I_{N+1}} a_{I_{N+1}}^k\,|\mathcal{Z}_{I_{N+1}}\rangle\,||^2}{||\sum_{I_N} a_{I_N}^k\,|\mathcal{Z}_{I_N}\rangle\,||^2}
\end{aligned}
\tag{4.32}
$$

that the outcome of a further measurement of $\hat{S}_x$ will be $s_{N+1}^k \hbar/2$, and so on.

In Fig.1 we use a matrix-representation algorithm to implement this measurement protocol and plot the energy of the system for three different trajectories. We see that random *fluctuations* arise during each process, meaning that the process is indeed stochastic. In the next section we shall analyse the behaviour of S if we consider not only one, two or three, but an ensemble of such trajectories.

Figure 1 – Energy of the system for three different implementations of our measurement protocol. Here $z = 1/\sqrt{2} + i/\sqrt{2}$, $\omega\delta t = 0.1$. In each plot the dashed line indicates the energy of the system for $z_\alpha = 0$ as a reference.



(a) (b) (c)

Source: personal archive.

4.2.2 Full ensemble: emergence of an equal-probability rule for orbital microstates

Now that we have determined the general form of the state of the system for the k-th trajectory γ_k^N, we apply the discussion in §2.5 to calculate the average of those results over an entire ensemble of N-steps trajectories $\{\gamma_k^N\}_{k=1}^n$, $n \to \infty$. Note that, given an initial state $|s_0\rangle_x$ for E, the k-th trajectory is uniquely characterized by the set $\{s_1^k, s_2^k, s_3^k, ..., s_N^k\}$ of N specific outcomes.

First, in expression (4.29), because each i_m assumes only two values, all the 2^N possible arrangements of the set $I_N = \{i_N, i_{N-1}, .., i_1\}$ take part in the labeling of the indexes in the state of the system at step N (for example, at the second step there is a state with label $\mathcal{Z}_{+,+}$, another with label $\mathcal{Z}_{+,-}$, another with label $\mathcal{Z}_{-,+}$ and, finally, another with label $\mathcal{Z}_{-,-}$). This implies that the state of S after the trajectory γ_k^N differs from the state of S after the trajectory γ_j^N, $k \neq j$, only through the values assumed by the coefficients used to construct the linear combinations of the coherent states $\{|\mathcal{Z}_{I_N}\rangle\}$ (Eq.(4.30)). Moreover, because s_j^k also assumes only two possible values, like i_m, at step N there are 2^N possible states (or linear combinations of coherent states) which the system can reach after the trajectory γ_k^N.

Note that Eq.(4.32) implies that the probability for the trajectory γ_k^N to occur is given by

$$\begin{aligned} p_N[\gamma_k^N] &= p_1(s_1^k|s_0)p_2(s_2^k|s_1^k, s_0)...p_N(s_N^k|s_{N-1}^k, ..., s_0) \\ &= \frac{||\prod_{m=1}^N (\hat{U}_+^{\delta t} + s_m^k s_{m-1}^k \hat{U}_-^{\delta t})\,|z\rangle\,||^2}{4^N} \\ &= \frac{||\sum_{I_N} a_{I_N}^k\,|\mathcal{Z}_{I_N}\rangle\,||^2}{4^N}. \end{aligned} \tag{4.33}$$

Therefore, we can construct the density operator of the system $S+E$ at step N as

$$\begin{aligned}
\hat{\rho}^N_{S+E} &= \sum_{k=1}^{2^N} p_N[\gamma^N_k]\,|\psi^k_N(N\delta t)\rangle\,\langle\psi^k_N(N\delta t)| \\
&= \left[\sum_{k=1}^{2^N} \frac{\left|\left|\sum_{I_N} a^k_{I_N}\,|\mathcal{Z}_{I_N}\rangle\right|\right|^2}{4^N}\times\right. \\
&\left.\times\left(\frac{\sum_{J_N} a^k_{J_N}\,|\mathcal{Z}_{J_N}\rangle}{||\sum_{I_N} a^k_{I_N}\,|\mathcal{Z}_{I_N}\rangle\,||}\right)\left(\frac{\sum_{L_N} a^k_{L_N}\,\langle\mathcal{Z}_{L_N}|}{||\sum_{I_N} a^k_{I_N}\,|\mathcal{Z}_{I_N}\rangle\,||}\right)\right]\otimes \\
&\otimes\,|s^k_N\rangle_{x\,x}\,\langle s^k_N| \\
&= \frac{1}{4^N}\left[\sum_{k=1}^{2^N}\left(\sum_{I_N} a^k_{I_N}\,|\mathcal{Z}_{I_N}\rangle\right)\left(\sum_{L_N} a^k_{L_N}\,\langle\mathcal{Z}_{L_N}|\right)\right]\otimes\,|s^k_N\rangle_{x\,x}\,\langle s^k_N| \\
&= \frac{1}{4^N}\left(\sum_{k,I_N,L_N} a^k_{I_N}a^k_{L_N}\,|\mathcal{Z}_{I_N}\rangle\,\langle\mathcal{Z}_{L_N}|\right)\otimes\,|s^k_N\rangle_{x\,x}\,\langle s^k_N| \\
&= \frac{1}{4^N}\left(\sum_{k,I_N}|\mathcal{Z}_{I_N}\rangle\,\langle\mathcal{Z}_{I_N}| + \right. \\
&\left. + \sum_{k,I_N\neq L_N} a^k_{I_N}a^k_{L_N}\,|\mathcal{Z}_{I_N}\rangle\,\langle\mathcal{Z}_{L_N}|\right)\otimes\,|s^k_N\rangle_{x\,x}\,\langle s^k_N|
\end{aligned} \tag{4.34}$$

where we have used that $\left(a^k_{I_N}\right)^2 = 1$.

Let us concentrate now in the second term inside the parenthesis. Expanding the summation over, say, I_N, we have

$$\begin{aligned}
\sum_{k,I_N\neq L_N} a^k_{I_N}a^k_{L_N}\,|\mathcal{Z}_{I_N}\rangle\,\langle\mathcal{Z}_{L_N}| = \sum_{k=1}^{2^N}\left[a^k_{I^1_N}\,|\mathcal{Z}_{I^1_N}\rangle\left(\sum_{L_N\neq I^1_N} a^k_{L_N}\,\langle\mathcal{Z}_{L_N}|\right)+\right. \\
+\,a^k_{I^2_N}\,|\mathcal{Z}_{I^2_N}\rangle\left(\sum_{L_N\neq I^2_N} a^k_{L_N}\,\langle\mathcal{Z}_{L_N}|\right)+...+ \\
\left.+\,a^k_{I^{2^N}_N}\,|\mathcal{Z}_{I^{2^N}_N}\rangle\left(\sum_{L_N\neq I^{2^N}_N} a^k_{L_N}\,\langle\mathcal{Z}_{L_N}|\right)\right]
\end{aligned} \tag{4.35}$$

where I^j_N, $j = 1, 2, 3, ..., 2^N$, is used to indicate all specific "values" (or arrangements of "$+$"-es and "$-$"-es) that I_N may assume (it does not matter which arrangement is associated with each I^j_N). For example, for $N = 2$, we could choose $I^1_2 = \{+,+\}$, $I^2_2 = \{+,-\}$, $I^3_2 = \{-,+\}$ and $I^1_2 = \{-,-\}$. Now, looking at the definition of the coefficients $a^k_{I_N}$ (Eq.(4.30)) note that for each possible trajectory γ^N_k where $a^k_{I^j_N} = 1$, $I^j_N \neq \{+,+,+,...,+\}$, there always exist a "reciprocal" trajectory γ^p_N where $a^p_{I^j_N} = -1$. This implies that *when the summation*

over all possible trajectories is performed, this last quantity will simply vanish [6].

$$\sum_{k,I_N \neq L_N} a^k_{I_N} a^k_{L_N} \left|\mathcal{Z}_{I_N}\right\rangle \left\langle \mathcal{Z}_{L_N}\right| = 0. \tag{4.36}$$

This happens because for each trajectory γ^N_k where at step N the outcome is $s^k_N = 1$, there corresponds a "reciprocal" trajectory γ^N_p where $s^p_N = -1$ (and the coefficients $a^k_{I_N}$ are defined in terms of the s^k_j's, $j = 1, 2, 3, ..., N$, so that this fact occurs —again, it can be verified by checking Eq.(4.30)).

Finally, noting that the first term inside the parenthesis is a summation of 2^N terms which *do not depend on the index* k, and tracing off E from $\hat{\rho}^N_{S+E}$, we have finally

$$\hat{\rho}^N_S = \frac{1}{2^N} \sum_{I_N} \left|\mathcal{Z}_{I_N}\right\rangle \left\langle \mathcal{Z}_{I_N}\right|. \tag{4.37}$$

That is, given the definitions in Eqs.(4.17),(4.30) for the composite maps $\mathcal{Z}_{I_N}$ and the coefficients $a^k_{I_N}$, respectively, we could construct an analytic expression for the state of the system after the realization of all possible trajectories $\{\gamma^N_k\}^{2^N}_{n=1}$ (taking into account, of course, their respective probabilities to occur).

However, what makes Eq.(4.37) interesting is that, regarding the discussion of $\S 2.5$, it can be interpreted as follows: *each of all* 2^N *possible microstates accessible to the system* $\{\left|\mathcal{Z}_{I_N}\right\rangle\}$ *occurs with probability* $1/2^N$. In other words, this is an equal-probability rule, such as we have, for example, in the well known case postulated *a priori* in statistical mechanics (of course there we are dealing with a closed system, whilst here we are dealing with an open system). Nevertheless, this is a somewhat simple result, given the expected complexity of our problem. Finally, we point that, because of the mathematical arguments given right above, such a fact would occur for any choice of the function $f(\hat{X})$ that defines the magnetic field $\mathbf{B}$.

4.2.3 Expectation values of physical quantities

With Eq.(4.37) in hands, we may use Eq.(2.46) to explicitly compute the expectation value of some observable $\hat{A} \in B(\mathcal{H}_S)$, where $\mathcal{H}_S$ is the Hilbert space of S. We have simply

$$\langle \hat{A} \rangle_N = \frac{1}{2^N} \sum_{I_N} \left\langle \mathcal{Z}_{I_N}\right| \hat{A} \left|\mathcal{Z}_{I_N}\right\rangle. \tag{4.38}$$

[6] The fact that the coefficient $a^k_{I^j_N=\{+,+,+,...,+\}}$ is always equal to 1 is not relevant, because in the summations of Eq.(4.35) we are considering only terms $L_N \neq I^j_N$.

Nevertheless, we have also arrived at a somewhat simple expression for the expectation value of *any* observable $\hat{A} \in B(\mathcal{H}_S)$. Note the similarity of the above equation with Eq.(3.24) in our discussion about the MCWF approach ($\S 3.2$).

Because the states $|\mathcal{Z}_{I_N}\rangle$ are all coherent states, we may use Eqs.(2.35), (2.36) to write

$$\langle \hat{H}_S \rangle_N = \frac{1}{2^N} \hbar\omega \sum_{I_N} \left(|\mathcal{Z}_{I_N}|^2 + \frac{1}{2} \right) \tag{4.39a}$$

$$\langle \hat{X} \rangle_N = \frac{1}{2^N} \sqrt{\frac{2\hbar}{m\omega}} \sum_{I_N} \mathrm{Re}\{\mathcal{Z}_{I_N}\} \tag{4.39b}$$

$$\langle \hat{P} \rangle_N = \frac{1}{2^N} \sqrt{2m\hbar\omega} \sum_{I_N} \mathrm{Im}\{\mathcal{Z}_{I_N}\} \tag{4.39c}$$

$$\langle \hat{X}^2 \rangle_N = \frac{1}{2^N} \frac{2\hbar}{m\omega} \sum_{I_N} \left(\mathrm{Re}^2 \left\{ \mathcal{Z}_{I_N} \right\} + \frac{1}{4} \right) \tag{4.39d}$$

$$\langle \hat{P}^2 \rangle_N = \frac{1}{2^N} 2m\hbar\omega \sum_{I_N} \left(\mathrm{Im}^2 \left\{ \mathcal{Z}_{I_N} \right\} + \frac{1}{4} \right) \tag{4.39e}$$

Although the above formulae may seem compact enough, it is possible to use Eq.(4.17) to simplify it even further in terms of the parameters $\{z, z_\alpha, N, \delta t\}$. Because this shall elucidate considerably the analysis of what precisely happens to S due to continuous observations of E, we describe the explicit calculations (which are somewhat heavy for the purpose of exposition) in the Appendix $\S A$.

4.2.4 Energy

It can be demonstrated ($\S A.1$) that the expectation value of the energy of S at the N-th step is given by

$$\langle \hat{H}_S \rangle_N = \hbar\omega \left(|z|^2 + \frac{1}{2} \right) + 4N\hbar\omega z_\alpha^2 \sin^2 \left(\frac{\omega\delta t}{2} \right). \tag{4.40}$$

Because the first term is just the energy of the initial state, $|z\rangle$, we see that *every observation of* E *raises the expectation value of the energy of the system by an ammount* $4\hbar\omega z_\alpha^2 \sin^2(\omega\delta t/2)$. The maximum and the minimum energy gain at each measurement occurs when we choose $\omega\delta t = n\pi$: the maximum being $4\hbar\omega z_\alpha^2$, for n *odd*, and the minimum being 0, for n *even*.

If the continuous measurement process occurs between the instants $t_0 = 0$ and t, and we take $\delta t \ll 1$ (so that $\langle \hat{H}_S \rangle_N$ is a continuous function), we may write $N = t/\delta t$ and state the

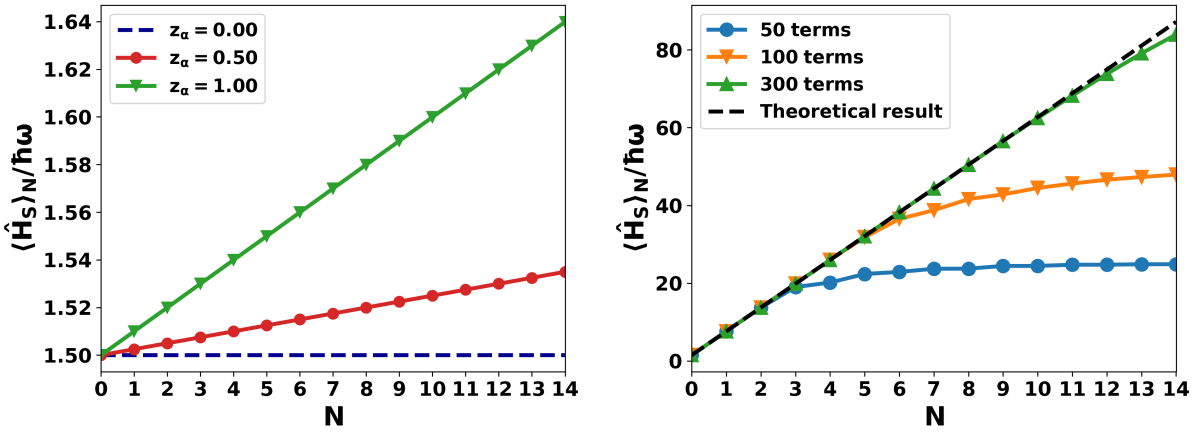
rate of change of the expectation value of the energy at step N simply as

$$\frac{\partial\langle\hat{H}_S\rangle_N}{\partial t} = \frac{4\hbar\omega z_\alpha^2}{\delta t}\sin^2\left(\frac{\omega\delta t}{2}\right) \geq 0. \tag{4.41}$$

(Note that the total time elapsed t and the time-step δt have different meanings.)

Furthermore, we see that in the limit $\alpha \to 0 \Rightarrow z_\alpha \to 0$ (that is, in the limit where the influence of the magnetic field is no longer present) the energy of S remains simply the energy of its initial state. This happens, of course, because E never leaves the $|s_0\rangle_x$ state, so that the process is no longer random.

Figure 2 – (a) Energy of S through direct implementation of Eq.(4.39a) (not (4.40)). (b) Comparison between a matrix representation algorithm and Eq.(4.40). The higher the number of terms we keep in the expansion of the initial coherent state, the better the curve approaches our theoretical model. Here we considered a set of 10^5 trajectories such as the one described in our protocol ($\S 4.2.1$).



(a) $z = 1/\sqrt{2} + i/\sqrt{2}$, $\omega\delta t = 0.1$. (b) $z = 1/\sqrt{2} + i/\sqrt{2}$, $\omega\delta t = 0.5$, $z_\alpha = 5.0$

Source: personal archive.

We can also further confirm this result for the energy with a numerical analysis. In Fig.2a we implement Eq.(4.39a) using a simple algorithm to calculate all $\mathcal{Z}_{I_N}$'s so that, comparing it with Eq.(4.40), we see that it confirms the linear dependence of the energy on N. Moreover, in Fig.2b we implemented also a very basic algorithm using the matrix representations of all relevant quantities (operators and state vectors) and directly applied steps 1 and 2 of our protocol ($\S 4.2$) for 10^5 trajectories. In this last algorithm we used the *qutip* python package for the study of open quantum systems, which is a handful and well known tool for this kind of implementation (J.R.JOHANSSONA1; NATIONB; NORI, 2013),(J.R.JOHANSSONA1; NATIONB; NORI, 2012). However, it must be pointed that, in this latter case, the finite representation of the infinite sum in the expansion of the coherent state $|z\rangle$ (Eq.(2.28)) may lead us to wrong

conclusions if we are not careful [7]. It is explicitly shown in that figure that, the higher the number of terms we keep in the expansion of $|z\rangle$, the better the curve plotted by the algorithm approaches our analytic closed expression.

We comment finally that, at first, we intended to study the problem numerically, but noted after some time that this "finite-representation difficulty" was not so easy to overcome. Then we decided to invest in the present analytical solution, which, for instance (and luckily!), could be obtained in a very neat and closed form.

4.2.5 Position and momentum

Following, the expectation value of the position and the momentum at step N are obtained in $\S A.2$. They are given, respectively, by

$$\langle \hat{X} \rangle_N = \sqrt{\frac{2\hbar}{m\omega}} \Big[\mathrm{Re}\{z\} \cos(N\omega\delta t) + \mathrm{Im}\{z\} \sin(N\omega\delta t) \Big] \tag{4.42}$$

$$\langle \hat{P} \rangle_N = \sqrt{2m\hbar\omega} \Big[-\mathrm{Re}\{z\} \sin(N\omega\delta t) + \mathrm{Im}\{z\} \cos(N\omega\delta t) \Big]. \tag{4.43}$$

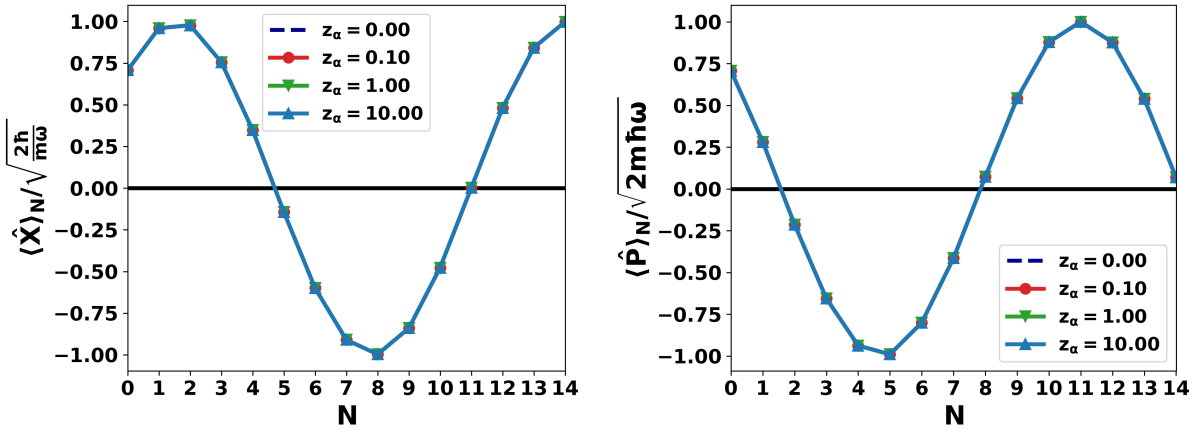
Comparing those expressions with the previous results for a single coherent state (Eqs.(2.35)), we see that they are exactly the same expressions we would obtain by a simple unitary time evolution of the initial state by $N\delta t$ units. Therefore, we conclude that *continuous observations of E do not affect the expectation values of the position and the momentum of S throughout time*. Nevertheless, we shall see soon that the *uncertainties* (ΔX and ΔP) related to those quantities change drastically with respect of those of a Simple harmonic oscillator.

Note that the above results are independent of the coupling constant α, meaning that it does not matter how strong the field drives the system, the behaviour of S is exactly the same (Figs 3a and 3b). This latter result together with the expression that we obtained for the expectation value of the energy shows that, just though observations of E, we could obtain a very effective way of raising the expectation value of the energy of S without changing the expectation value of its position or its momentum.

In Fig.4a we, again, confirm our analytical results with a simple matrix-representation algorithm, exactly as we did for the expectation value of the energy. Finally, in Fig.4b we make a phase-space plot of the position and the momentum of the system through direct implementation of our analytic expressions. The figure is exactly equal to the standard "textbook"

[7] It is not possible to implement an infinite sum in a numerical (not symbolic) algorithm.

Figure 3 – Expectation values of the position (a) and the momentum (b) of S for $z = 1/\sqrt{2} + i/\sqrt{2}$ and $\omega\delta t = 0.5$ through direct implementation of Eqs.(4.42),(4.39c). In both plots all curves are superposed so that there is no dependence on z_α, as indicated by our theoretical model (Eqs.(4.42), (4.43)). Although we would prefer to make those plots for a lower value of $\omega\delta t$ (so that the measurement process is closer to what we could call "continuous"), this choice would make the oscillation frequencies too low, forcing us to use more steps and making the algorithm too heavy. (For example, the dimension of the set I_N would be $2^{25} = 33554432$ for 25 steps.)

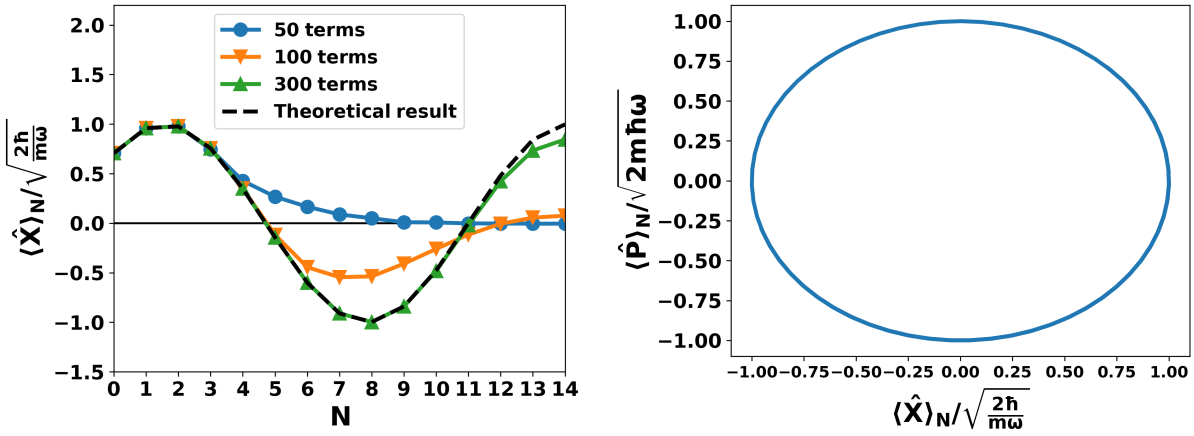


(a) (b)

Source: personal archive.

phase-space plot of a simple (isolated) harmonic oscillator and shall be used in the next section for the sake of comparison.

Figure 4 – (a) A new comparison between a matrix-representation algorithm which directly implements our protocol and our theoretical results. (b) A Phase space plot for $N = 100$. The result is the same as that of a simple harmonic oscillator. In both figures $z = 1/\sqrt{2} + i/\sqrt{2}$.



(a) $\omega\delta t = 0.5$, $z_\alpha = 5.0$ (b) $\omega\delta t = 0.1$, $z_\alpha = 1.0$

Source: personal archive.

4.2.6 Uncertainty relation

Finally, we would like to shed some light on the behaviour of the position-momentum uncertainty relation throughout the continuous measurement process. In order to do this we need the expressions for the expectation values of the position squared and the momentum squared. It can be shown ($\S A.3$) that they are given by

$$\begin{aligned}\langle \hat{X}^2 \rangle_N = \langle \hat{X} \rangle_N^2 + \frac{2\hbar}{m\omega}\Bigg\{ z_\alpha^2 \Bigg[2(N-1)\sin^2\left(\frac{\omega\delta t}{2}\right) - \\ - \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Big(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t)\Big) + \\ + 1 + \cos^2(N\omega\delta t) - 2\cos(\omega\delta t)\Bigg] + \frac{1}{4}\Bigg\}\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\langle \hat{P}^2 \rangle_N = \langle \hat{P} \rangle_N^2 + 2m\hbar\omega\Bigg\{ z_\alpha^2 \Bigg[2(N-1)\sin^2\left(\frac{\omega\delta t}{2}\right) + \\ + \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Big(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t)\Big) + \\ + \sin^2(N\omega\delta t)\Bigg] + \frac{1}{4}\Bigg\}\end{aligned} \tag{4.45}$$

for $\omega\delta t \neq n\pi$, respectively. For $\omega\delta t = n\pi$, we have simply

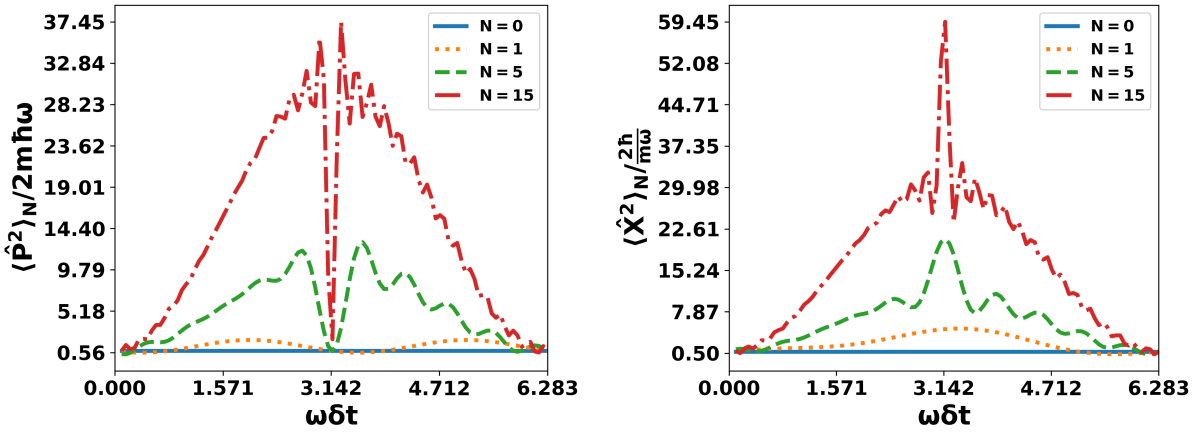
$$\langle \hat{X}^2 \rangle_N = \frac{2\hbar}{m\omega}\left(\mathrm{Re}^2\{z\} + 4Nz_\alpha^2 + \frac{1}{4}\right) \tag{4.46}$$

and

$$\langle \hat{P}^2 \rangle_N = 2m\hbar\omega\left(\mathrm{Im}^2\{z\} + \frac{1}{4}\right). \tag{4.47}$$

Considering the Hamiltonian of the harmonic oscillator (Eq.(2.16)), we see that with these equations in hands we may also study the expectation values of the kinetic and the potential energy of S. Fig.**??** indicates that the amount of energy gained by S at each step can be stored both as kinetic or potential energy because the particle is oscillating. Note that, naturally, when one type of such is increased, the other type is decreased. Moreover, Eqs.(4.46),(4.47) indicate that when the energy increment rate is a maximum, the energy given to the system at each measurement is purely potential. This is further confirmed by Fig.5.

Figure 5 – Position squared (a) and momentum squared (b) as a function of $\omega\delta t$ (not as a function of time! The parameter that represents time is, essentially, N). Because the expectation value of the kinetic energy is proportional to $\langle\hat{P}^2\rangle$ and the expectation value of the potential energy is proportional to $\langle\hat{X}^2\rangle$, for $\omega\delta t = \pi$ we see that all the mean energy gained by S is purely potential. For $\omega\delta t \neq \pi$, we see that the gain is partially potential and partially kinetic. Nevertheless, it is never purely kinetic. In both plots $z_\alpha = 1.0$ and $z = 1/\sqrt{2} + i/\sqrt{2}$.



(a) (b)

Source: personal archive.

From the definitions of ΔX and ΔP (§2.1.4) we see that the uncertainty relation for the state of the system at the N-th step is given by

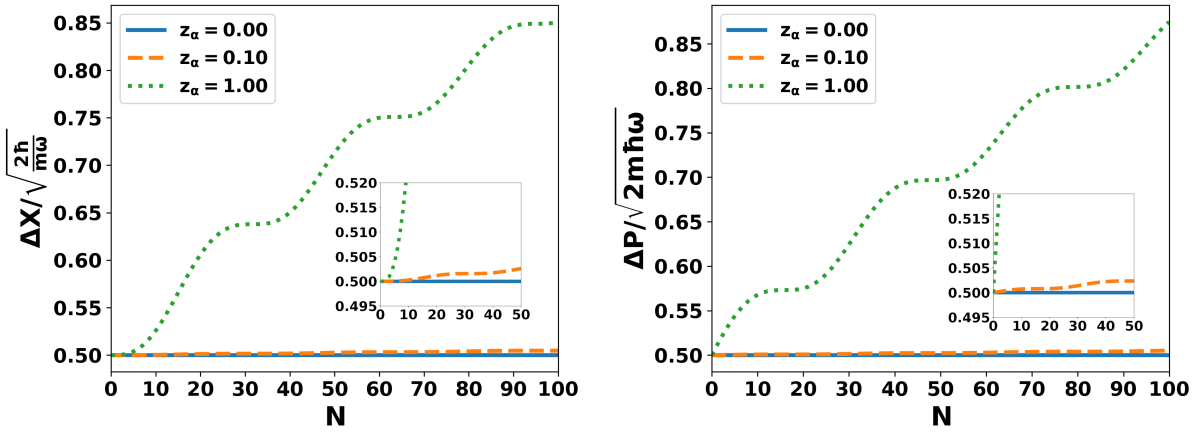
$$\Delta X_N \Delta P_N = \frac{\hbar}{2}\left\{4z_\alpha^2\left[2(N-1)\sin^2(\omega\delta t/2) - \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Big(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t)\Big) + 1 + \right.\right.$$
$$\left.\left. + \cos^2(N\omega\delta t) - 2\cos(\omega\delta t)\right] + 1\right\}^{1/2}\left\{z_\alpha^2\left[2(N-1)\sin^2(\omega\delta t/2) + \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\times\right.\right.$$
$$\left.\left.\times\Big(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t)\Big) + \sin^2(N\omega\delta t)\right] + 1\right\}^{1/2} \tag{4.48}$$

for $\omega\delta t \neq n\pi$. For $\omega\delta t = n\pi$, we have simply

$$\Delta X_N \Delta P_N = \frac{\hbar}{2}\sqrt{1 + 16Nz_\alpha^2}. \tag{4.49}$$

Although the expression (4.48) may not seem so elegant as those for the energy, the position and the momentum, we note that in the limit $\alpha \to 0$ we recover the minimum relation of Eq.(2.37). Before analysing the uncertainty relation, in Fig.6 we have a plot of ΔX and ΔP as a function of N for the sake of clarity. As mentioned in the previous section, the plot indicates that those quantities grow as further measurements are performed on E.

Figure 6 – Uncertainties of the position (a) and the momentum (b) of S. In both plots $\omega\delta t = 0.1$ and $z = 1/\sqrt{2} + i/\sqrt{2}$.



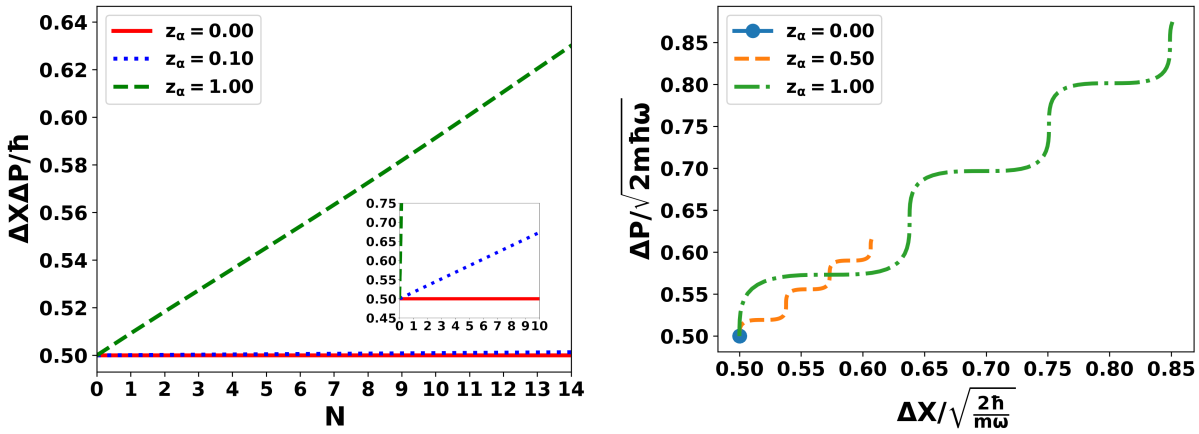
(a) (b)

Source: personal archive.

This last result, together with the previous ones, indicates that there is indeed a "cost" in using the measurement process on E to raise the energy of S maintaining the expectation value of its position and its momentum "invariant". For example, as further and further measurements are performed on E, in order to obtain a certain definite value for, say, the position of S, we must make the distribution of all possible values which can be obtained for the momentum broader and broader (and vice-versa).

Finally, in Fig.7a we see that, for $z_\alpha \neq 0$, the uncertainty relation always grows as further measurements are performed on the system. Also, in Fig.7b we make a phase-space plot for ΔX and ΔP. If $z_\alpha = 0$, the uncertainty remains a point (exactly as the case of an isolated harmonic oscillator), bur if $z_\alpha \neq 0$, the uncertainty of the position (momentum) always grows as the uncertainty of the momentum (position) grows.

Figure 7 – (7a). Uncertainty relation for various orders of magnitude of z_α according to our analytic expressions (Eqs.(4.48),(4.49)). (7b). Phase-space plot for the uncertainties ΔX and ΔP. We see that, in comparison with the simple (isolated) harmonic oscillator, the uncertainties do not remain only a point throughout time. In both plots $z = 1/\sqrt{2} + i/\sqrt{2}$.



(a) $\omega\delta t = 0.1$

(b) $\omega\delta t = 0.1$, $N = 100$.

Source: personal archive.

5 CONCLUSION AND PERSPECTIVES

In chapter 2 we introduced the mathematical tools that were essential for the development of our project (postulates of quantum mechanics, tensor products, density operators, etc) and in chapter 3 we used those tools in order to study the (non-unitary) dynamics of systems that are no longer closed, but in contact with an external environment. In this way, we could situate to which context our project represents an alternative approach.

We considered our proposal in chapter 4, and it consists basically in studying no longer the situation of an external environment as an stochastic source to some system of interest as in chapter 3. Rather, we are interested in studying random outcomes of measurements performed in one degree of freedom of some system as an stochastic source for another degree of freedom of the same system. This latter degree of freedom was indeed the one in which we had interest and we even made the analogy that the observed degree of freedom may be seen as a diathermal wall for it. In this way, we could formulate a closed analytical solution for the problem of a quantum harmonic oscillator whose spin degree of freedom is subject to a continuous measurement process in a region of space where the influence of an external inhomogeneous magnetic field which depends linearly on the position of the oscillator is present.

More precisely, considering an ensemble of quantum trajectories (defined earlier) for this spin observation process, we concluded that the state of the system in question obeys an *equal-probability rule*, similarly to what happens in statistical mechanics. From there on, we studied in detail the expectation values of all physical quantities related to the system and discovered that continuous observations of its spin degree of freedom induce a mean-energy gain in its orbital degree of freedom, leaving the expectation value of its position and momentum invariant at the cost of raising the distribution of their uncertainties. We also determined, using numerical plots and algorithm comparisons, that the position-momentum uncertainty relation always grows as a higher number of spin measurements are considered.

As prospects for the present work, we intend to study the case of the same system (coherent state + spin) under the influence not of a magnetic field which depends linearly on the position of the oscillator, but of a magnetic field which depends *quadratically* on the position of the oscillator (as said earlier). We also intend to search for measurement regimes where, for example, a thermalization process could be achieved (thermalization in the sense that the

energy of the system can become time-independent at some stage). In a certain manner, the idea of the present project walks in the pathway of the so called *quantum thermodynamics* research area, a very promising scientific perspective nowadays. This motivates us also to address questions relative to *entropy production*, *reversibility* and *fluctuation theorems* (major characteristics of this subject) to the study of our alternative approach to open quantum system.

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APPENDIX A – EXPLICIT CALCULATIONS FOR CHAPTER 4

Here we describe in details how we may obtain an analytic closed form to Eqs.(4.38).

A.1 EXPLICIT CALCULATION OF THE ENERGY

First we need to obtain $|\mathcal{Z}_{I_N}|^2$. Indicating by $T(I_N)$ terms that depend linearly on any of the $i_k = \pm 1$ (such as, for example, $i_4 i_9 z_\alpha^2$, $i_1 z_\alpha e^{iN\omega\delta t}$, etc.), using that $z_\alpha \in \mathbb{R}$ and the identity $|\sum_{i=1}^n z_i|^2 = \sum_{i=1}^n |z_i|^2 + 2\sum_{m<n} \mathrm{Re}\{z_m^* z_n\}$, where z_i, $i = 1, ..., n$, are arbitrary complex numbers, we have

$$
\begin{aligned}
|\mathcal{Z}_{I_N}|^2 &= \{(z - i_1 z_\alpha)e^{-iN\omega\delta t} + z_\alpha[\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t} + i_N]\}\times \\
&\times \{(z^* - i_1 z_\alpha)e^{iN\omega\delta t} + z_\alpha[\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{i(N-j)\omega\delta t} + i_N]\} \\
&= |z - i_1 z_\alpha|^2 + z\alpha^2\left|\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t} + i_N\right|^2 + \\
&+ 2z_\alpha \,\mathrm{Re}\left\{(z^* - i_1 z_\alpha)e^{iN\omega\delta t}\left[\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t} + i_N\right]\right\} + T(I_N) \\
&= |z|^2 + z_\alpha^2 + z_\alpha^2\left[\left|\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t}\right|^2 + 1 - 2\cos\omega\delta t\right] + T(I_N).
\end{aligned}
$$

But

$$
\begin{aligned}
\left|\sum_{j=1}^{N-1}(i_j - i_{j+1})e^{-i(N-j)\omega\delta t}\right|^2 &= \sum_{j=1}^{N-1}|i_j - i_{j+1}|^2 + \\
&+ 2\sum_{j<k}(i_j - i_{j+1})(i_k - i_{k+1})\,\mathrm{Re}\left\{e^{-i(j-k)\omega\delta t}\right\} \\
&= \sum_{j=1}^{N-1}[1+1] + 2\Bigg[(i_2 - i_3)(i_1 - i_2)e^{-i(1-2)\omega\delta t} + \\
&+ (i_3 - i_4)\sum_{j=1}^{N-1}(i_j - i_{j+1})\,\mathrm{Re}\left\{e^{-i(j-3)\omega\delta t}\right\} + ... + \\
&+ (i_{N-1} - i_N)\sum_{j=1}^{N-2}(i_j - i_{j+1})\,\mathrm{Re}\left\{e^{-i(j-N+1)\omega\delta t}\right\}\Bigg] + T(I_N) \\
&= 2(N-1) + 2[\overbrace{-\cos\omega\delta t - \cos\omega\delta t - ... - \cos\omega\delta t}^{N-2 \text{ times}}] + T(I_N) \\
&= 2N - 2(N-2)\cos\omega\delta t - 2 + T(I_N) \\
&= 4N\sin^2\left(\frac{\omega\delta t}{2}\right) - 2 + 4\cos\omega\delta t + T(I_N)
\end{aligned}
$$

so that

$$\begin{aligned}|\mathcal{Z}_{I_N}|^2 &= |z|^2 + 2z_\alpha^2 - 4z_\alpha^2\cos\omega\delta t + \\ &+ z_\alpha^2\left[4N\sin^2\left(\frac{\omega\delta t}{2}\right) - 2 + 4\cos\omega\delta t\right] + T(I_N) \\ &= |z|^2 + 4Nz_\alpha^2\sin^2\left(\frac{\omega\delta t}{2}\right) + T(I_N).\end{aligned} \tag{A.1}$$

Note that in each step we are simply using that $i_k^2 = 1$ to select all terms which do not depend on any i_k and "absorbing" all the remaining terms into $T(I_N)$. That will be so also in all subsequent calculations of this type.

Now we note that if we fix all i_k's except but one, say, i_j, there will be a term in the expansion of the summation $\sum_{I_N}|\mathcal{Z}_{I_N}|^2$ where $i_j = 1$ and a term *exactly equal* but with $i_j = -1$. Because those two terms shall cancel out and the same argument is valid for each i_k, we see that $\sum_{I_N} T(I_N) = 0$. This is basically the same argument that led us to (4.38) (there we have explained it with more details). Therefore,

$$\langle\hat{H}_S\rangle_N = \hbar\omega\left(|z|^2 + \frac{1}{2}\right) + 4N\hbar\omega z_\alpha^2\sin^2\left(\frac{\omega\delta t}{2}\right). \tag{A.2}$$

A.2 EXPLICIT CALCULATION OF THE POSITION AND THE MOMENTUM

Repeating the arguments of the above section, those expectation values are simpler to obtain. First we compute $\mathrm{Re}\{\mathcal{Z}_{I_N}\}$ and $\mathrm{Im}\{\mathcal{Z}_{I_N}\}$ using that, for two arbitrary complex numbers z_1 and z_2, $\mathrm{Re}\{z_1z_2\} = \mathrm{Re}\{z_1\}\,\mathrm{Re}\{z_2\} - \mathrm{Im}\{z_1\}\,\mathrm{Im}\{z_2\}$ and $\mathrm{Im}\{z_1z_2\} = \mathrm{Re}\{z_1\}\,\mathrm{Im}\{z_2\} + \mathrm{Im}\{z_1\}\,\mathrm{Re}\{z_2\}$.

$$\begin{aligned}\mathrm{Re}\{\{\mathcal{Z}_{I_N}\}\} &= \mathrm{Re}\Big\{(z - i_1z\alpha)e^{-iN\omega\delta t}\Big\} + T(I_N) \\ &= \mathrm{Re}\{(z - i_1z_\alpha)\}\cos(N\omega\delta t) - \mathrm{Im}\{(z - i_1z_\alpha)\}\sin(N\omega\delta t) + T(I_N) \\ &= \mathrm{Re}\{z\}\cos(N\omega\delta t) + \mathrm{Im}\{z\}\sin(N\omega\delta t) + T(I_N)\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\{\{\mathcal{Z}_{I_N}\}\} &= \mathrm{Im}\Big\{(z - i_1z\alpha)e^{-iN\omega\delta t}\Big\} + T(I_N) \\ &= \mathrm{Re}\{(z - i_1z_\alpha)\}\sin(N\omega\delta t) + \mathrm{Im}\{(z - i_1z_\alpha)\}\cos(N\omega\delta t) + T(I_N) \\ &= -\mathrm{Re}\{z\}\sin(N\omega\delta t) + \mathrm{Im}\{z\}\cos(N\omega\delta t) + T(I_N).\end{aligned}$$

Therefore, using once again that $\sum_{I_N} T(I_N) = 0$, we have

$$\langle\hat{X}\rangle_N = \sqrt{\frac{2\hbar}{m\omega}}\Big(\mathrm{Re}\{z\}\cos(N\omega\delta t) + \mathrm{Im}\{z\}\sin(N\omega\delta t\Big) \tag{A.3}$$

$$\langle \hat{P} \rangle_N = \sqrt{2m\hbar\omega}\Big(- \mathrm{Re}\{z\} \sin\left(N\omega\delta t\right) + \mathrm{Im}\{z\} \cos\left(N\omega\delta t\right)\Big). \tag{A.4}$$

A.3 EXPLICIT CALCULATION OF THE POSITION AND THE MOMENTUM SQUARED

Regarding the two previous subsections, we proceed here straightforwardly to obtain $\mathrm{Re}^2\left\{\mathcal{Z}_{I_N}\right\}$ and $\mathrm{Im}^2\left\{\mathcal{Z}_{I_N}\right\}$.

$$\begin{aligned}
\mathrm{Re}^2\left\{\mathcal{Z}_{I_N}\right\} &= \left\{ \mathrm{Re}\Big\{(z - i_1 z_\alpha) e^{-iN\omega\delta t}\Big\} + z\alpha \left[\sum_{j=1}^{N-1} (i_j - i_{j+1}) \cos\left((N-j)\omega\delta t\right) + i_N \right] \right\}^2 \\
&= \mathrm{Re}\Big\{(z - i_1 z\alpha) e^{-iN\omega\delta t}\Big\} + \\
&+ z_\alpha^2 \left\{ \left[\sum_{j=1}^{N-1} (i_j - i_{j+1}) \cos\left((N-j)\omega\delta t\right) \right]^2 + 1 - 2\cos\omega\delta t \right\} - \\
&- 2 z_\alpha^2 \cos\left(N\omega\delta t\right) \cos\left(N-1\right)\omega\delta t + T(I_N)
\end{aligned}$$

But, using that $(\sum_{i=1}^n z_i)^2 = \sum_{i=1}^n z_i^2 + 2\sum_{i<j} \mathrm{Re}\{z_i z_j\}$ for arbitrary complex numbers z_i, $i = 1, ..., n$, we have that

$$\left[\sum_{j=1}^{N-1}(i_j - i_{j+1})\cos\left((N-j)\omega\delta t\right)\right]^2 = \sum_{j=1}^{N-1}(i_j - i_{j+1})^2\cos^2\left((N-j)\omega\delta t\right)+$$

$$+\sum_{j<k}\Bigg[(i_j - i_{j+1})\cos\left((N-j)\omega\delta t\right)]\times$$

$$\times\left[(i_k - i_{k+1})\cos\left((N-k)\omega\delta t\right)\right]$$

$$= 2\sum_{j=1}^{N-1}\cos^2(j\omega\delta t) + 2\Big[(i_2 - i_3)\cos\left((N-2)\right)\times$$

$$\times(i_1 - i_2)\cos\left((N-1)\right) + (i_3 - i_4)\cos\left((N-3)\right)\times$$

$$\times\sum_{k=1}^{2}(i_k - i_{k+1})\cos\left((N-k)\omega\delta t\right) + ...+$$

$$+(i_{N-1} - i_N)\cos(\omega\delta t)\times$$

$$\times\sum_{k=1}^{N-2}(i_k - i_{k+1})\cos\left((N-k)\omega\delta t\right)\Big]+$$

$$+T(I_N)$$

$$= 2\sum_{j=1}^{N-1}\cos^2(j\omega\delta t) + 2\Bigg[-\cos\left((N-2)\omega\delta t\right)\times$$

$$\times\cos\left((N-1)\omega\delta t\right) - \cos\left((N-3)\omega\delta t\right)\times$$

$$\times\cos\left((N-2)\omega\delta t\right) - ... - \cos(\omega\delta t)\cos(2\omega\delta t)\Bigg] + T(I_N)$$

$$= 2\sum_{j=1}^{N-1}\cos^2(j\omega\delta t)-$$

$$-2\sum_{j=1}^{N-2}\cos(j\omega\delta t)\cos\left((j+1)\omega\delta t\right) + T(I_N).$$

Hence,

$$\mathrm{Re}^2\left\{\mathcal{Z}_{I_N}\right\} = \mathrm{Re}^2\{z\}\cos^2(N\omega\delta t) + \mathrm{Im}^2\{(N\omega\delta t)\}\sin^2(N\omega\delta t)+$$

$$+2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\cos(N\omega\delta t)\sin(N\omega\delta t)+$$

$$+z_\alpha^2\Bigg[]2\sum_{j=1}^{N-1}\cos^2(j\omega\delta t) - 2\sum_{j=1}^{N-1}\cos(j\omega\delta t)\cos\left((j+1)\omega\delta t\right) + 1+$$

$$+\cos^2(N\omega\delta t) - 2\cos(N\omega\delta t)\Bigg] + T(I_N).$$

In the same way,

$$\begin{aligned}\mathrm{Im}^2\left\{\mathcal{Z}_{I_N}\right\} &= \left\{\mathrm{Im}\Big\{(z-i_1z_\alpha)e^{-iN\omega\delta t}\Big\} - z\alpha\left[\sum_{j=1}^{N-1}(i_j-i_{j+1})\sin\left((N-j)\omega\delta t\right)\right]\right\}^2 \\ &= \mathrm{Im}^2\left(z-i_1z_\alpha\right)e^{-iN\omega\delta t} + z_\alpha^2\left[\sum_{j=1}^{N-1}(i_j-i_{j+1})\sin\left((N-j)\omega\delta t\right)\right]^2 - \\ &- 2z_\alpha^2\sin\left(N\omega\delta t\right)\sin\left((N-1)\omega\delta t\right) + T(I_N).\end{aligned}$$

Now, by a calculation similar to the above one, it can be shown that

$$\begin{aligned}\left[\sum_{j=1}^{N-1}(i_j-i_{j+1})\sin\left((N-j)\omega\delta t\right)\right]^2 &= 2\sum_{j=1}^{N-1}\sin^2\left(j\omega\delta t\right) - \\ &- 2\sum_{j=1}^{N-2}\sin\left(j\omega\delta t\right)\sin\left((j+1)\omega\delta t\right) + T(I_N).\end{aligned}$$

This leads to

$$\begin{aligned}\mathrm{Im}^2\left\{\mathcal{Z}_{I_N}\right\} &= \mathrm{Re}^2\left\{z\right\}\sin^2\left(N\omega\delta t\right) + \mathrm{Im}^2\left\{z\right\}\cos^2\left(N\omega\delta t\right) - \\ &- 2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\sin\left(N\omega\delta t\right)\cos\left(N\omega\delta t\right) + z_\alpha^2\times \\ &\times\left[2\sum_{j=1}^{N-1}\sin^2\left(j\omega\delta t\right) - \right. \\ &\left. - 2\sum_{j=1}^{N-1}\sin\left(j\omega\delta t\right)\sin\left((j+1)\omega\delta t\right) + \sin^2\left(N\omega\delta t\right)\right] + T(I_N).\end{aligned}$$

(It may be explicitly verified that $\mathrm{Re}^2\left\{\mathcal{Z}_{I_N}\right\} + \mathrm{Im}^2\left\{\mathcal{Z}_{I_N}\right\}$ concides with Eq. (A.1)).

Now, using Eqs.(4.38) and effecting the summation over I_N, we have

$$\begin{aligned}\langle\hat{X}^2\rangle_N = \frac{2\hbar}{m\omega}\Bigg\{ &\mathrm{Re}^2\left\{z\right\}\cos^2\left(N\omega\delta t\right) + \mathrm{Im}^2\left\{(N\omega\delta t)\right\}\sin^2\left(N\omega\delta t\right) + \\ &+ 2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\cos\left(N\omega\delta t\right)\sin\left(N\omega\delta t\right) + \\ &+ z_\alpha^2\Bigg[2\sum_{j=1}^{N-1}\cos^2\left(j\omega\delta t\right) - 2\sum_{j=1}^{N-1}\cos\left(j\omega\delta t\right)\cos\left((j+1)\omega\delta t\right) + 1 + \\ &+ \cos^2\left(N\omega\delta t\right) - 2\cos\left(\omega\delta t\right)\Bigg] + \frac{1}{4}\Bigg\}\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}\langle \hat{P}^2 \rangle_N = 2m\hbar\omega \Bigg\{ & \operatorname{Re}^2\{z\} \sin^2(N\omega\delta t) + \operatorname{Im}^2\{z\} \cos^2(N\omega\delta t) - \\ & - 2\operatorname{Re}\{z\}\operatorname{Im}\{z\} \sin(N\omega\delta t)\cos(N\omega\delta t) + \\ & + z_\alpha^2 \Bigg[2\sum_{j=1}^{N-1} \sin^2(j\omega\delta t) - \\ & - 2\sum_{j=1}^{N-1} \sin(j\omega\delta t)\sin((j+1)\omega\delta t) + \sin^2(N\omega\delta t) \Bigg] + \frac{1}{4} \Bigg\}.\end{aligned} \tag{A.6}$$

But, using the partial sum of the geometric series $\sum_{n=1}^{J} ar^n = a(1-r^J)/(1-r)$, $r \neq 1$, we may write

$$\begin{aligned}\sum_{n=1}^{J} \sin(nx) &= \frac{\sin(Jx/2)}{\sin(x/2)} \sin\left(\frac{(J+1)x}{2}\right) \\ \sum_{n=1}^{J} \cos(nx) &= \frac{\sin(Jx/2)}{\sin(x/2)} \cos\left(\frac{(J+1)x}{2}\right) \\ \sum_{n=1}^{J} \sin^2(nx) &= \frac{1}{2}\left[J - \frac{\sin(Jx)}{\sin(x)} \cos\Big((J+1)x\Big) \right] \\ \sum_{n=1}^{J} \cos^2(nx) &= \frac{1}{2}\left[J + \frac{\sin(Jx)}{\sin(x)} \cos\Big((J+1)x\Big) \right] \\ \sum_{n=1}^{J} \cos(nx)\cos((n+1)x) &= \frac{1}{2}\left[J\cos(x) + \frac{\sin(Jx)}{\sin(x)} \cos\Big((J+2)x\Big) \right] \\ \sum_{n=1}^{J} \sin(nx)\sin((n+1)x) &= \frac{1}{2}\left[J\cos(x) - \frac{\sin(Jx)}{\sin(x)} \cos\Big((J+2)x\Big) \right]\end{aligned} \tag{A.7}$$

so that

$$\begin{aligned}
\sum_{j=1}^{N-1}\Big[\cos^2(j\omega\delta t)-\cos(j\omega\delta t)\cos((j+1)\omega\delta t)\Big] &= \frac{1}{2}\times \\
&\times\frac{1}{2}\Bigg[2(N-1)\sin^2(\omega\delta t/2)- \\
&-\frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Bigg(\cos\left((N+1)\omega\delta t\right)- \\
&-\cos\left(N\omega\delta t\right)\Bigg)\Bigg] \\
\sum_{j=1}^{N-1}\Big[\sin^2(j\omega\delta t)-\sin(j\omega\delta t)\cos((j+1)\omega\delta t)\Big] &= \frac{1}{2}\times \\
&\times\frac{1}{2}\Bigg[2(N-1)\sin^2(\omega\delta t/2)+ \\
&-\frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Bigg(\cos\left((N+1)\omega\delta t\right)- \\
&-\cos\left(N\omega\delta t\right)\Bigg)\Bigg].
\end{aligned} \tag{A.8}$$

Therefore, for $\omega\delta t \neq n\pi$ (that is, $\sin(\omega\delta t)\neq 0$) we have

$$\begin{aligned}
\langle\hat{X}^2\rangle_N = \frac{2\hbar}{m\omega}\Bigg\{ &\mathrm{Re}^2\left\{z\right\}\cos^2\left(N\omega\delta t\right)+\mathrm{Im}^2\left\{(N\omega\delta t)\right\}\sin^2\left(N\omega\delta t\right)+ \\
&+2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\cos\left(N\omega\delta t\right)\sin\left(N\omega\delta t\right)+ \\
&+z_\alpha^2\Bigg[2(N-1)\sin^2(\omega\delta t/2)- \\
&-\frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Bigg(\cos((N+1)\omega\delta t)-\cos(N\omega\delta t)\Bigg)+1+ \\
&+\cos^2\left(N\omega\delta t\right)-2\cos\left(\omega\delta t\right)\Bigg]+\frac{1}{4}\Bigg\}
\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}\langle\hat{P}^2\rangle_N = 2m\hbar\omega\Bigg\{ & \mathrm{Re}^2\{z\}\sin^2(N\omega\delta t) + \mathrm{Im}^2\{z\}\cos^2(N\omega\delta t) - \\ & - 2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\sin(N\omega\delta t)\cos(N\omega\delta t) + \\ & + z_\alpha^2\Bigg[2(N-1)\sin^2(\omega\delta t/2) + \\ & + \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)}\Bigg(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t)\Bigg) + \\ & + \sin^2(N\omega\delta t)\Bigg] + \frac{1}{4}\Bigg\}.\end{aligned} \tag{A.10}$$

And for the case $\omega\delta t = \pi$,

$$\langle\hat{X}^2\rangle_N = \frac{2\hbar}{m\omega}\left[\mathrm{Re}^2\{z\} + 4Nz_\alpha^2 + \frac{1}{4}\right] \tag{A.11}$$

$$\langle\hat{P}^2\rangle_N = 2m\hbar\omega\left[\mathrm{Im}^2\{z\} + \frac{1}{4}\right]. \tag{A.12}$$

It may be verified explicitly that, indeed, $\langle\hat{H}\rangle_N = \langle\hat{P}^2\rangle_N/2m + m\omega^2\langle\hat{X}\rangle_N/2$.

A.4 EXPLICIT CALCULATION OF THE UNCERTAINTY RELATION

With Eqs.(A.3), (A.4), (A.5) and (A.10) in hands we may compute $\Delta X_N^2 = \langle\hat{X}^2\rangle_N - \langle\hat{X}\rangle_N^2$ and $\Delta P_N^2 = \langle\hat{P}^2\rangle_N - \langle\hat{P}\rangle_N^2$ explicitly.

We have

$$\begin{aligned}\langle\hat{X}\rangle_N^2 = \frac{2\hbar}{m\omega}\Bigg\{ & \mathrm{Re}^2\{z\}\cos^2(N\omega\delta t) + \mathrm{Im}^2\{(N\omega\delta t)\}\sin^2(N\omega\delta t) + \\ & + 2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\cos(N\omega\delta t)\sin(N\omega\delta t)\end{aligned} \tag{A.13}$$

and

$$\begin{aligned}\langle\hat{P}\rangle_N^2 = 2m\hbar\omega\Bigg\{ & \mathrm{Re}^2\{z\}\sin^2(N\omega\delta t) + \mathrm{Im}^2\{z\}\cos^2(N\omega\delta t) - \\ & - 2\,\mathrm{Re}\{z\}\,\mathrm{Im}\{z\}\sin(N\omega\delta t)\cos(N\omega\delta t)\end{aligned} \tag{A.14}$$

so that

$$\Delta X_N^2 = \frac{2\hbar}{m\omega}\Bigg\{ z_\alpha^2 \Bigg[2(N-1)\sin^2(\omega\delta t/2) - \\ - \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)} \Bigg(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t) \Bigg) + 1 + \\ + \cos^2(N\omega\delta t) - 2\cos(N\omega\delta t) \Bigg] + \frac{1}{4} \Bigg\} \tag{A.15}$$

and

$$\Delta P_N^2 = 2m\hbar\omega \Bigg\{ z_\alpha^2 \Bigg[2(N-1)\sin^2(\omega\delta t/2) + \\ + \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)} \Bigg(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t) \Bigg) + \\ + \sin^2(N\omega\delta t) \Bigg] + \frac{1}{4} \Bigg\}. \tag{A.16}$$

Therefore,

$$\Delta X_N \Delta P_N = \frac{\hbar}{2}\Bigg\{ 4z_\alpha^2 \Bigg[2(N-1)\sin^2(\omega\delta t/2) - \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)} \Bigg(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t) \Bigg) + 1 + \\ + \cos^2(N\omega\delta t) - 2\cos(N\omega\delta t) \Bigg] + 1 \Bigg\}^{1/2} \times \Bigg\{ 4z_\alpha^2 \Bigg[2(N-1)\sin^2(\omega\delta t/2) + \\ + \frac{\sin((N-1)\omega\delta t)}{\sin(\omega\delta t)} \Bigg(\cos((N+1)\omega\delta t) - \cos(N\omega\delta t) \Bigg) + \sin^2(N\omega\delta t) \Bigg] + 1 \Bigg\}^{1/2}. \tag{A.17}$$

for $\omega\delta t \neq n\pi$ and

$$\Delta X_N \Delta P_N = \frac{\hbar}{2}\sqrt{1 + 16Nz_\alpha^2} \tag{A.18}$$

for $\omega\delta t = n\pi$.