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ÉRITON ARAUJO ANTONIO JÚNIOR

IMPROVING THE COMPUTATION OF THE τVI PAINLEVÉ FUNCTION USING THE QUADRATURE METHOD FOR THE FREDHOLM DETERMINANT

Recife
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Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Física.

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BANCA EXAMINADORA

Prof. Bruno Geraldo Carneiro da Cunha
Orientador
Universidade Federal de Pernambuco

Prof. Antônio Murilo Santos Macêdo
Examinador Interno
Universidade Federal de Pernambuco

PARTICIPAÇÃO VIA VIDEOCONFERÊNCIA

Prof. Giovani Lopes Vasconcelos
Examinador Externo
Universidade Federal do Paraná

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See the microcosm
In macro vision
Our bodies moving
With pure precision
One universal celebration
One evolution
One creation
(Depeche Mode, 2005) [1]

ABSTRACT

The Painlevé transcendent functions are important tools in theoretical physics, they appear in a variety of physical systems going from quantum integrable systems to random matrix theory. The accessory parameter problem for ODEs, which has connections to black hole scattering problem, can be solved by using the connection between the Painlevé VI transcendent with isomonodromic deformations of a linear ordinary differential equation. In this case, the isomonodromic τ_{VI} function plays a major role, and finding its roots is equivalent to solving the accessory parameter problem. The τ_{VI} function can be expressed as a function of a Fredholm determinant. In this dissertation, we will discuss the two main different methods of calculation of the τ_{VI} in the Fredholm determinant form. We will also present how to construct codes for both methods and analyze them in order to understand which one is the most numerically efficient to find the roots of the τ_{VI} function.

Keywords: Painlevé VI. Fredholm Determinant. Riemann-Hilbert Problem. Accessory Parameter Problem. Gaussian Quadrature.

RESUMO

As funções transcendentais de Painlevé são ferramentas importantes dentro da física teórica, elas aparecem em uma variedade de sistemas físicos indo de sistemas quânticos integráveis à teoria de matrizes aleatórias. O problema dos parâmetros acessórios, que tem conexões com o problema de espalhamento em buracos negros, pode ser resolvido usando a conexão entre as funções transcendentais de Painlevé com as transformações isomonodrômicas em uma equação diferencial ordinária linear. Neste caso a função isomonodrômica τ_{VI} é de grande importância, e encontrar as raízes de tal função é equivalente a resolver o problema de parâmetro acessório. A função τ_{VI} pode ser expressada em termos de um determinante de Fredholm. Nesta dissertação serão discutidos os dois principais métodos de se calcular a função τ_{VI} na sua forma de determinante de Fredholm. Também será apresentado como construir códigos utilizando ambos os métodos e tais códigos serão analisados de maneira a se entender qual dos dois é mais numericamente eficiente para o cálculo das raízes da função τ_{VI}

Palavras-chave: Painlevé VI. Determinante de Fredholm. Problema de Riemann-Hilbert. Problema de Parâmetro Acessório. Quadratura Gaussiana

CONTENTS

1 INTRODUCTION

In theoretical physics, it is common to find out special functions when trying to understand the properties of a specific system. In many undergraduate courses, one stumbles upon these functions [2]. The Gamma function, the zeta function, the theta function, the hypergeometric function, the Bessel function, the Hermite function, and Airy function, all of then developed by mathematicians and physicists at the necessity of solving the problems of each epoch. The Painlevé transcendents are believed to be one of the next to enter such a group, as said by Iwasaki et al.[3].

The Painlevé transcendents are the solutions to a set of six second order ordinary differential equations(ODEs) whose singularities have the Painlevé property which means that the only movable singularities are poles. The six equations can be written as

$$P_I : \quad \frac{d^2y}{dt^2} = 6y^2 + t \tag{1.1}$$

$$P_{II} : \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha \tag{1.2}$$

$$P_{III} : \quad ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4 \tag{1.3}$$

$$P_{IV} : \quad y\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2-\alpha)y^2 + 4ty^3 + \frac{3}{4}y^4 \tag{1.4}$$

$$P_V : \quad \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y} + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}\right) \tag{1.5}$$

$$P_{VI} : \quad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \tag{1.6}$$

where the parameters $\alpha, \beta, \gamma, \delta$ are complex constants. In physics those equations have appeared in many problems, going from integrable quantum systems to random matrix theory [4], [5], [6], [7], [8],[9], [10], [11], [12]. In this dissertation, we will focus on the sixth Painlevé transcendent(PVI), because this transcendent has connections to the theory of isomonodromic deformations in a Heun equation.

A Heun equation is a second-order ODE with four regular singular points, such differential equation can be written as

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_1}{w-1} + \frac{1-\theta_{t_0}}{w-t_0}\right)y'(w) + \left[\frac{q_+q_-}{w(w-1)} - \frac{t_0(t_0-1)K_0}{w(w-1)(w-t_0)}\right]y(w) = 0 \tag{1.7}$$

where $q_{\pm} = 1 - \frac{1}{2}(\theta_0 + \theta_{t_0} + \theta_1 \pm \theta_{\infty_0})$, the θ_i are called single monodromy parameters and t_0 and K_0 are the accessory parameters. The accessory parameters of a Heun equation can be written in terms of the monodromy parameters, although this task is complicated. Finding this relation between the parameters is what we call *the accessory parameter problem*. Such a problem can be seen as one version of the 21^{st} *Riemann-Hilbert problem*. Which in the words of Hilbert is "To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group", where the monodromy parameters in (1.7) are a representation of the monodromy group, and the Fuchsian class ODE is a differential equation where all the singular points are regular singular points. In this context, the transformations in the parameters of a Heun equation that maintain the monodromy representation, the so-called isomonodromic deformations, are the tools that we use to solve this problem. And is here that arises the connection between this problem and the PVI transcendent. The transformation induced in one of the parameters by the isomonodromic deformations acts according to the Painlevé VI ODE, therefore the accessory parameters can be cast as functions of the monodromy parameters using this relation.

A good way to see how all those concepts come together is by examining some physical problems as an example. Let us recall the problem of two-dimensional flow in fluid dynamics. When the fluid is incompressible we have that it's velocity obeys $\vec{\nabla} \cdot \vec{v} = 0$, and if the fluid in question is also irrotational, $\vec{\nabla} \times \vec{v} = 0$, we can express the velocity as a scalar potential ϕ where $\vec{v} = \vec{\nabla}\phi$. For such a special case in two dimensions since $\nabla^2\phi = 0$, which is the classical Laplace equation, we can use complex coordinates to describe the system and write the analytic function $\Omega(w) = \phi(x,y) + i\psi(x,y)$, where ϕ is the potential for the velocity and ψ is the Stokes stream function. The ψ function has the property of giving the streamlines of the flow. Since those two functions, ϕ and ψ, are analytic and obey the Laplace's equation, a change in the coordinates of the problem through some conformal mapping have a known formula given by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\psi(u(x,y), v(x,y)) = 0 \tag{1.8}$$

So if we are trying to analyze the streamlines of some flow with boundaries that are not easy to deal with, we can use the solution for a case we already know, like the uniform flow in the upper half-plane, and then use some conformal transformation and equation (1.8) to obtain the streamlines for the problem. This use of conformal transformations to help to solve problems in 2D is a field of great interest in fluid mechanics and, for an introductory view in the subject, see [13].

In the context of the last paragraph, knowing how to perform a conformal mapping to the upper half-plane from a region of interest is equivalent to solving the problem of obtaining the streamlines of the flow in such a region. Let us examine them the problem of finding the transformation from a channel with a half-disk barrier to the UHP (upper

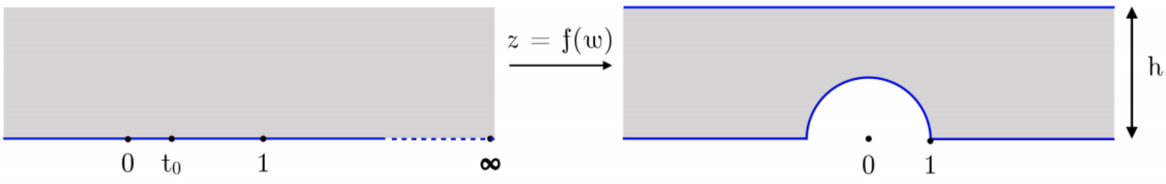


Figure 1 – Conformal mapping from the upper-half w-plane to a channel with a semi-disk barrier in the z-plane. By T. Anselmo in [16]

half-plane), as it is exemplified in Figure 1. This problem is similar to the well known Schwarz-Christoffel mapping, which consists of finding the transformation that goes from the interior of some polygon to a standard domain such as the UHP or the unity disk. For a detailed analysis of Schwarz-Christoffel transformations see [14]. Back to Figure 1 one can see it as some kind of generalization of the Schwarz-Christoffel problem where one of the sides of the polygon is now a circular arc. Those kinds of domains are called *Polycircular arc domains*, and in order to find the conformal map from the UHP to it, one needs to solve an ODE. The so-called *Schwarzian differential equation* [15] can be written as

$$\{f(w), w\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{n}\left[\frac{1-\theta_i^2}{2(w-w_i)^2} + \frac{\beta_i}{w-w_i}\right] \tag{1.9}$$

and solving it will give us the function f(w), responsible for the transformation to the semi-disk barrier domain. The w_i in equation 1.9 are the four pre-vertices points $(0, t_0, 1, \infty)$ in our domain. The parameters θ_i can be obtained from the geometry of the problem while the parameters β_i obey $\sum_{i=1}^{n}\beta_i = \sum_{i=1}^{n}(2w_i\beta_i + 1 - \theta_i^2) = \sum_{i=1}^{n}(\beta_i w_i^2 + w_i(1-\theta_i^2)) = 0$.

The Schwarzian equation (1.9), according to [17], can be written as $f(w) = \frac{y_1(w)}{y_2(w)}$ where y_1 and y_2 are two linearly independent solutions of the second order ODE

$$\tilde{y}''(w) = \sum_{i=1}^{n}\left[\frac{1-\theta_i^2}{4(w-w_i)^2} + \frac{\beta_i}{2(w-w_i)}\right]\tilde{y}(w). \tag{1.10}$$

With the introduction of equation 1.10 one can see that the problem of finding a conformal transformation from the UHP to the semi-disk barrier domain can be written as the problem of finding the two linearly independent solutions of a Fuchsian class ODE. In our case, since we have four pre-vertex points, the Fuchsian equation can be written in the form of a Heun ODE. As is developed in [18], one can find the parameters θ_i using the geometry of the domain, then it only remains to express who are the β_i in term of the θ_i and the w_i. This is the same as solving the accessory parameter problem for a Heun equation with some given monodromy parameters.

In conclusion, the problem of finding streamlines in the semi-disk barrier domain resumes to the problem of finding the conformal transformation described in Figure 1. This conformal transformation can be written as a second-order Fuchsian class ODE, with some of its parameters obtained by the geometry of the domain. The remaining parameters are obtained by linking this Fuchsian ODE to a Heun equation and solving the accessory parameter problem. The accessory parameter problem of such a Heun equation is solved by using the connection between the isomonodromic transformations performed in the ODE and the sixth Painlevé transcendent.

There are other examples of physical problems that make use of the PVI property, one can solve the so-called *reverse Riemann-Hilbert problem* in the context of black hole physics to find scattering amplitudes for scalar fields in a Kerr-de Sitter space-time. For more details see [19].

Thus, there are some good motivations in theoretical physics for knowing the properties of the PVI transcendent function. Some works in this field show us that the best way to express the sixth transcendent function is in terms of the isomonodromic tau function, or τ_{VI} function [20]. In the context of the accessory parameter problem, the solutions to the parameters can be given as an initial value problem for the tau function, where one of such parameters can be obtained as the root of the τ_{VI}. Since one can connect the capability to calculate the τ_{VI} with the solution to the accessory parameters problem, it is important to express such a function in an efficient numerical way.

The τ_{VI} function can be completely determined by the monodromy parameters in the Painlevé VI ODE, in particular, it can be written as an expansion near one of its singular points in terms of conformal blocks in conformal field theory, as stated in [21] and [22]. Such expansion can be used to generate numerical values for the tau function, although it is not the most efficient method known. Based on [23], one can be able to express the τ_{VI} function as a Fredholm determinant. In that representation, the series expansion for the tau can be computed in a relatively fast way.

By itself, the Fredholm determinant is a powerful tool in operator theory and mathematical theory, [24], [25]. In physics, it was found significant applications for this determinant, for example, atomic collision n theory, [26], or as the two-point correlation function of the two-dimensional Ising model [27]. In our problem, the expansion for the determinant is calculated by expanding the operators inside it in a Fourier basis. And recently one general numerical method for the calculation of Fredholm's determinant was developed by [28]. In his paper, the quadrature method is said to be the most efficient for the computation of this determinant. The purpose of this dissertation is to explain and analyze both methods and to ascertain which of the two approaches introduced above is the best suited for our application to the accessory parameters problem. To this end, codes for the Fourier and quadrature methods were constructed, the latter as a new development, using as standard the Julia Language.

This dissertation is divided into four chapters. In chapter 2 we will, reviewing the work of [16], present the concepts of monodromy, the 21*st* Riemann-Hilbert problem(RHp), the accessory parameter problem, the isomonodromic transformations of the Heun equation and its connection to the Painlevé VI equation. It will also be shown how the Toda equation can be connected to the accessory parameters to obtain one of the conditions for them. Also showing the role of the τ_{VI} function, and its root, in the solution of this problem.

In chapter 3 it will be introduced the general form of the isomonodromic τ function for a Fuchsian system with n singular points. Describing the general Riemann-Hilbert problem defined in a n-punctured Riemann sphere, how to break such a problem into $n-2$ pairs of pants that can be described by auxiliary 3-point RHps. Then, we introduce the Plemelj integral operators and construct a general τ function for n singular points in N dimensions expressed in terms of a Fredholm determinant. After that, this general form of the tau function is applied to the specific case of $n = 4, N = 2$, which gives us the Painlevé τ_{VI} function.

In Chapter 4 we proceed to define the two main methods by which we can calculate the Fredholm determinant, the Fourier, and the quadrature approach. We will show how to construct algorithms for both methods and analyze its efficiency. We will also use the formalism introduced here to solve one example of the accessory parameter problem as a demonstration of the capacities of the new method

At last, in Chapter 5 it will be presented a short review of the advances made through this dissertation and will be proposed directions in which the work can be expanded into new problems.

2 THE PAINLEVÉ τ_{VI} FUNCTION AND THE ACCESSORY PARAMETER PROBLEM

In this chapter, we aim to explain the connection between the sixth Painlevé Transcendent (PVI) to the isomonodromic deformations of a Heun ODE. To this end, we will need to introduce some concepts. First, we will enunciate the Riemann-Hilbert problem (RHp). Then we will see how this relates to the so-called accessory parameter problem for the Heun equation, tying its solution to the isomonodromic transformations of the parameters. We will review how those transformations for one of the parameters resume to the PVI, and then we will introduce the concept of the isomonodromic tau function (τ_{VI}). At last, it will be shown that the solution for the accessory parameters can be written as an initial value problem for the τ_{VI}.

2.1 The Riemann-Hilbert Problem

The Riemann-Hilbert problem, also known as the twenty-first Hilbert problem, is one of the twenty-three problems presented by Hilbert in 1900. It concerns the relation between differential equations of the Fuchsian class and the monodromy group. To have a better grasp at the RHp it will be necessary to introduce the concepts of the Fuchsian equation and the Monodromy group. The next two subsections are devoted to that.

2.1.1 Fuchsian Equations

A Fuchsian equation, according to [3], is an ordinary differential equation (ODE) defined in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, having the form of

$$D^2 y(w) + P_1(w) D y(w) + P_2(w) y(w) = 0 \quad , \quad D = \frac{d}{dw}. \tag{2.1}$$

Where P_1 and P_2 are rational functions, the set $S = \{w_1, w_2, ..., w_i, w_{i+1} = \infty\}$ is a subset of $\mathbb{P}^1$ and w_i are the singularities of the equation. The points in the set S are all regular singularities of equation (2.1). In the context of the RHp the best way to express the Fuchsian ODE is by its matricial form

$$\frac{d\mathbf{Y}}{dw} = \sum_{m=1}^{i} \frac{\mathbf{A_m}}{(w - w_m)} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y_1(w) & y_2(w) \\ u_1(w) & u_2(w) \end{pmatrix}. \tag{2.2}$$

This is also known as the Fuchsian system, because to every element of the matrix $\mathbf{Y}$ we can associate a Fuchsian equation of the form (2.1) ([29]). The matrices $\mathbf{A_m}$ are nonzero and independent of w. The matrix solution $\mathbf{Y}$ is better suited to understand the properties of the monodromy group in such solutions. A particular and more mathematical study of those matrix ODEs can be found in ([30]).

2.1.2 Monodromy Group

According to the definition written in [31], we start with the punctured Riemann sphere as $D = \mathbb{P}^1 \setminus S$ with $S = \{w_1, w_2, ..., w_i, w_{i+1} = \infty\}$. And then we can define a loop in D with base point in x as a curve

$$\gamma : I = [0, 1] \to D \tag{2.3}$$

starting and ending in x. Now, let $L(D, x)$ be the set of all loops that have x as base point. Two loops γ_1 and γ_2 are said to be (homotopy-)equivalent if and only if γ_1 can be continuously deformed into γ_2 in D with x fixed.We represent this equivalence relation as $\gamma_1 \simeq \gamma_2$. Therefore the set of equivalence classes of loops in $L(D, x)$ is called the fundamental group of D and is denoted as $\pi_1(D, x) = L(D, x)/\simeq$. The monodromy group is a $GL(2, \mathbb{C})$ matricial representation of the $\pi_1(D, x)$.

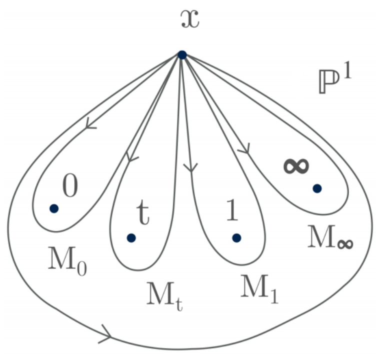


Figure 2 – Loops and monodromy matrices of the 4-point punctured Riemann sphere. By T. Anselmo in [16]

2.1.2.1 *Fuchsian System*

Given some representation of the monodromy group, one can interpret its action by looking at how the solutions of a Fuchsian system behave. When we make analytic continuations around a singular point one can see that the solution does not return to itself, instead, it will be given by some linear combination of the solutions.

First let us start with some simpler case, a first-order Fuchsian equation, as written bellow

$$\frac{dy}{dw} - \frac{\alpha}{w - w_i} y = 0, \quad w_i, \alpha \in \mathbb{C}. \tag{2.4}$$

Assuming $\alpha \notin \mathbb{Z}$, the solution becomes

$$y(w) = a(w - w_i)^\alpha, \quad a \in \mathbb{C} \tag{2.5}$$

The function $y(w)$ has some interesting properties. Namely if we calculate the change in the function value when we perform a loop over the singular point w_i we get

$$y((w - w_i)e^{2\pi i} + w_i) = e^{2\pi i\alpha} y(w). \tag{2.6}$$

Hence, the pole in (2.5) represents a branch point in the solution (2.6). This property is common on Fuchsian equations: the presence of regular singularities in its coefficients induces the existence of branch points in the solutions ([32]). For this particular case, a first-order ODE, we see that an analytic continuation around the singularity brings the solution to itself multiplied by a complex constant.

The general case is a little more complex. As said in 2.1.1 we can relate a Fuchsian equation to a Fuchsian system of the type (2.2). In this context, the behavior of the solution around a singularity is not given by a function, but by a 2×2 matrix $\mathbf{Y}$. For such a system, an analytic continuation around one of its singularities brings the solution to the form

$$\mathbf{\Phi}_{\gamma_i}(w) = \mathbf{\Phi}(w) M_{\gamma_i}. \tag{2.7}$$

Where $M_{\gamma_i} \in GL(2, \mathbb{C})$ is the *monodromy matrix* associated with γ_i. So, as highlighted before, the solution is brought to a linear combination of itself given by the monodromy representation of the group π_1. In figure 2 there is an example of the loops, singularities and respective monodromy matrices for a 4-punctured sphere. We can see from the figure that a loop that encircles all four points can be contractible to a point, this is true for any number of points in the set S so we can write

$$M_n \cdots M_2 M_1 = \mathbb{1}. \tag{2.8}$$

The 4-punctured sphere that appears in figure 2 is a very special case that will be dealt with in later sections, then we will introduce a notation that is best suited for such domain. We define the set $S = (0, t, 1, \infty)$ and the trace coordinates

$$p_i = 2\pi \cos\theta_i = Tr(M_i), \quad p_{ij} = 2\pi \cos\sigma_{ij} = Tr(M_i M_j). \tag{2.9}$$

The parameters θ_i and σ_{ij} will be called monodromy parameters and they will be used to represent the monodromy matrices M_i in the next sections. These trace coordinates generate an algebra [33] and also satisfy the so called Fricke-Jimbo relation [34]:

$$\begin{aligned} p_0 p_t p_1 p_\infty + p_{0t} p_{1t} p_{01} - p_{0t}(p_0 p_t + p_1 p_\infty) - p_{1t}(p_0 p_\infty + p_t p_1) - p_{01}(p_0 p_1 + p_\infty p_t) \\ + p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 = 4 \end{aligned} \tag{2.10}$$

The equation (2.10) will play the role of a consistency check for the monodromy parameters. This relation will become more relevant in chapter 4.

2.1.2.2 *Monodromy Matrices from the Geometry of Complex Domains*

Since the concepts of the monodromy group and its representation are very abstract, in this short subsection we will present an example of how the monodromy information can be related to some physical problems. In particular, we will be dealing with the same

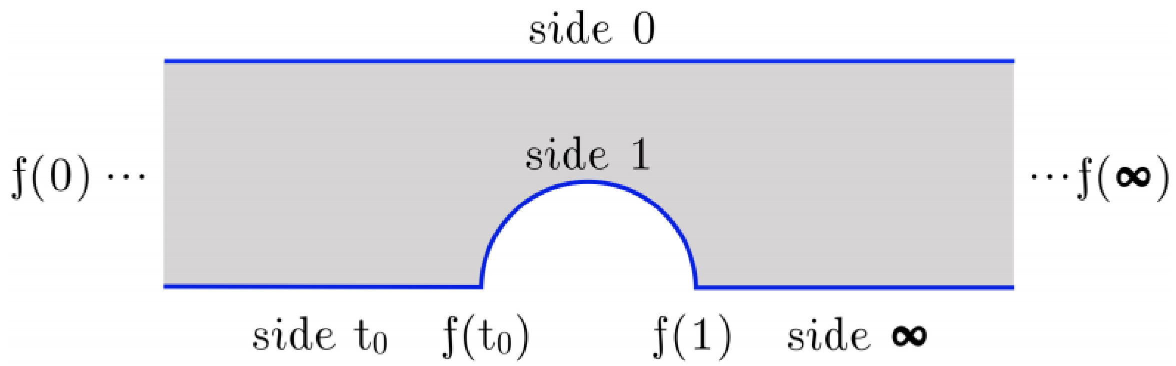


Figure 3 – – The sides of the quadrangle are labeled according to the identification $-\infty = f(0)$, $-1 = f(t0)$, $1 = f(1)$ and $\infty = f(\infty)$, By T. Anselmo in [16]

system shown in the introduction, the conformal mapping between a polycircular arc domain and the upper half-plane.

In the discussion at the introduction, it was mentioned that the conformal transformation $f(w)$ should obey a Schwarzian ODE of the type (1.9). From this point one could proceed with a linearization of the problem by expressing the solution f as a ratio

$$f(w) = \frac{y_1(w)}{y_2(w)} \tag{2.11}$$

where solutions $y_i(w)$ would obey a Fuchsian type ODE, given by (1.10). The mapping we are going to use here is the one showed in Figure 1, were the domain is the channel with a half-disk barrier. In Figure 3 we have a more detailed sketch of the such domain. The set $(0, t_0, 1, \infty)$ with images $f(0) = -\infty$, $f(t_0) = -1$, $f(1) = 1$ and $f(\infty) = \infty$, will be our singular points. Now we will express the solutions as the row vector $Y(w) = (y_1(w)\ y_2(w))$ and try to find expressions for the monodromy matrices by making use of the (2.7). The formalism in this subsection, and all the results written in the next paragraphs were introduced in the paper [18], by Anselmo *et al.*.

We seek to make an analytic continuation in a loop enclosing one of the singularities. To that end, we will make use of the so-called *Schwarz function*

$$S(z(w)) := \overline{z(w)} = z(w), \tag{2.12}$$

that will be responsible for expressing the changes in the vector Y when an analytic continuation is performed through one of the boundary lines.

Let us take in consideration the side t_0, when making an analytic extension of $f(w)$ near this side one obtains the $\tilde{f}(w) = \overline{S_{t_0}(f(\overline{w}))} := \overline{f}(w) = \overline{f(\overline{w})}$. Where S_{t_0} represents the Schwarz function of the curve that is defined by t_0 and $\overline{f}(w)$ is the Schwarz conjugate of an analytic function. Expressing this function using the row vector Y one can write

$$(\overline{y_1(w)} \quad \overline{y_2(w)}) = (y_1(w) \quad y_2(w))S_{t_0}, \quad where \quad S_{t_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.13}$$

When such a continuation occurs from bellow t_0 to side 1, one can write

$$S_1(w) = \frac{1}{z(w)}, \tag{2.14}$$

which comes from the equation for the unit circle $z\bar{z} = zS(z) = 1$. In this case, the analytic continuation is written as

$$(\overline{y_1(w)} \quad \overline{y_2(w)}) = (y_1(w) \quad y_2(w))S_1, \quad where \quad S_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.15}$$

Where the factor i was included in S_1 to enforce the condition $\det S_1 = \mathbb{1}$. This kind of overall factor does not change the corresponding $S(z)$.

Going through with this kind of analysis one can obtain the full set of matrices S

$$S_0 = \begin{pmatrix} 1 & 0 \\ -2hi & 1 \end{pmatrix} \quad S_{t_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.16}$$

where h is the parameter that represents the distance between the walls of the channel. With those matrices in hands the process of obtaining the monodromy matrices is straightforward. Let us examine how it goes for the point t_0. We start in a point above the side t_0, then proceed to the part bellow the real line where the function pass through a branch change due to the singularity at t_0, at this region we will have

$$(\overline{y_1(w)} \quad \overline{y_2(w)}) = (y_1(w) \quad y_2(w))\overline{S_{t_0}}. \tag{2.17}$$

Now, from the lower half-plane we seek to return to the starting point, but since we are in a positive oriented loop we should cross side 1. This crossing makes a change in our row function given by

$$Y(w) = \overline{Y}(w)\overline{S_1} \tag{2.18}$$

In the end, the change in our solutions after the continuation was performed in a loop around t_0 should take into account those two equations. Therefore, the monodromy matrix is written as

$$Y_{\gamma t0}(w) = Y(w)M_{t_0}, \quad M_{t_0} = S_1\overline{S_{t_0}} \tag{2.19}$$

Where it was used that $\overline{Y}(w) = Y(w)\overline{S}^1 = Y(w)S_1$. Repeating this algorithm for the other singular points give us the remaining three monodromy matrices of the domain. A general formula for those monodromy matrices, as long as the full set of M_{γ_i} is given below

$$M_{\gamma_i} = S_{i+1}\overline{S_i} \tag{2.20}$$

and

$$M_0 = \begin{pmatrix} 1 & 0 \\ 2hi & 1 \end{pmatrix} \quad M_{t_0} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_\infty = \begin{pmatrix} 1 & 0 \\ -2hi & 1 \end{pmatrix} \tag{2.21}$$

Here we could use the monodromy matrices to generate the trace coordinates, that in this particular problem are related to the angles of the polycircular-arc domain. Another good example of a physical concept that can be associated with the monodromy matrices and the trace coordinates is given by [19], where the scattering amplitudes of a scalar field in a Kerr-de Sitter spacetime are written as functions of the monodromy parameters θ_i and σ_{ij}.

Now that we have a better grasp of the concepts of monodromy and how it relates to the Fuchsian systems, we can go further and define the Riemann-Hilbert problem, in both its reverse and original form.

2.1.3 The 21^{st} Riemann-Hilbert Problem

In the last section, we saw that the solutions of a Fuchsian equation are multivalued and that analytic continuation around one of its singularities brings the solutions vector to a linear combination of itself, given by the monodromy matrices. That property can be used to associate a Fuchsian class ODE with the monodromy group $\pi_1(\mathbb{P}^1 \setminus S, x)$ where $S = \{w_1, w_2, ..., w_i, w_{i+1} = \infty\}$ this leaves us with the *direct monodromy problem*:

> *Given a differential equation with n regular singular points, find an $SL(2, \mathbb{C})$ representation associated with an equivalence class of loops around its singular points.*

From this same logic we can define the so-called *reverse monodromy problem*, which is also known as the $21st$ *Riemann-Hilbert problem* (RHp):

> *Given an irreducible $SL(n, \mathbb{C})$ representation ρ of the fundamental group of the n-punctured Riemann sphere, find a Fuchsian differential equation which has ρ as its monodromy representation*

Or as it was said by Hilbert [35]: "To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group."

While the two problems may seem to be the mirror of one another a question must be made clear: "Is the map between the space of $2nd$ order Fuchsian equations and the space of representations of $\pi_1(\mathbb{P}^1 \setminus S, x)$ bijective ?" To answer that let $\mathcal{M}(S)$ be the space of conjugacy classes of irreducible representations of $\pi_1(\mathbb{P}^1 \setminus S, x)$ of rank 2 and $\mathcal{E}(S)$ be the space of second-order irreducible Fuchsian differential equations with singular points in at most S, then

$$m(S) = dim(\mathcal{M}(S)), \quad e(s) = dim(\mathcal{E}(S)) \tag{2.22}$$

where $dim(X)$ is the complex dimension of some space X. From [3] one can show that

$$m(S) - e(S) = n - 3 \tag{2.23}$$

in that case, the only bijective example happens when $n = 3$ which is the hypergeometric case of the Fuchsian equation. For $n > 3$ the $dim(\mathcal{M}(S)) > dim(\mathcal{E}(S))$ and so we have that one representation of $\pi_1(\mathbb{P}^1 \setminus S, x)$ can be related to more than one Fuchsian equation at the same time. This fact leads to the theory of *isomonodromic deformations*, changes in the ODE that leave the monodromy of the system preserved. This is a rich topic and it will be explored with more details in the next sections.

2.1.4 The n=3 reverse Riemann-Hilbert problem

To have a better grasp of how the Riemann-Hilbert problem presents itself, we will examine the case with 3 singular points in this section. We will start by remembering that, with $n = 3$, the dimension of the Fuchsian space and the monodromy representation is the same. This means that we will not face problems with accessory parameters as it happens to the Heun case, neither we will need to make use of isomonodromic transformations.

In this problem, all information about the ODE can be given if we know the monodromy matrices. Therefore, it is more interesting to look into the point of view of the direct monodromy problem or the reverse Riemann-Hilbert problem as its called by some authors. Therefore, we now seek to obtain the monodromy representation from the Fuchsian differential equation.

First, we note that in the complex plane a Fuchsian type ODE with three singularities can always have its parameters changed to become

$$w(1-w)\frac{d^2u}{dw^2} + \{\gamma - (\alpha + \beta + 1)w\}\frac{du}{dw} - \alpha\beta u = 0, \tag{2.24}$$

which is the celebrated Gauss hypergeometric ODE. So our reverse RHp turns out to be the problem of determining the monodromy representation of the hypergeometric differential equation. This makes our task easier now since the properties of the ODE and its solutions are well known. There are a few different ways of determining the monodromy matrices. Here we will make use of one of the integral representation for the function, the Euler integrals.

Integral representations can be used to express some solutions of a given differential equation, this kind of form is useful when trying to understand global properties of some solutions. The Euler integral representation for hypergeometric can be derived from the power series of the function and it has the following form

$$F_{pq}(w) = \int_p^q x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}(x-w)^{-\alpha}dx. \tag{2.25}$$

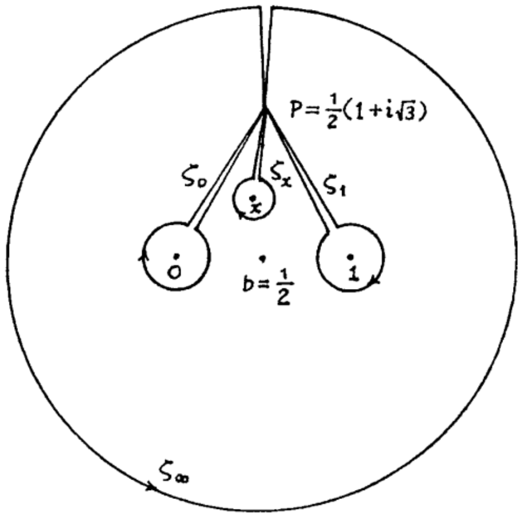


Figure 4 – Loops encircling the singularities of the domain of the hypergeometric ODE, by K. Iwasaki et al. from [3]

Where $q \to p$ is a path in the complex plane. Following the discussions in [3] we will use the Euler integrals over double loops, instead of the usual arcs. Let T be an equilateral triangular domain with vertices $0, 1$ and $p := \frac{(1+\sqrt{3})i}{2}$, for w within T we set $Y_w = \mathbb{P}^1 \setminus \{0, 1, \infty, w\}$ and take p as base point for the fundamental group of this sphere. We proceed by introducing the double loop notation

$$\langle \xi_1, \xi_2 \rangle = \int_{[\gamma_1, \gamma_2]} x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}(x-w)^{-\alpha}dx \tag{2.26}$$

with the commutator $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$. Where this path of integration represents two subsequent loops starting and ending in point p. In Figure 4 there are examples of loops that could be perform around the singularities in this setting.

The commutator of equation (2.26) can be used to derive an expression for the integrals around double loops that encircle the singular points. Here the concept of analytic continuation encompasses deforming one point inside the double loops to perform a path. In particular, the loops which encircles the singularities of the hypergeometric $\langle \gamma_0, \gamma_\infty \rangle$ and $\langle \gamma_1, \gamma_\infty \rangle$ can be shown to obey the relations bellow when an analytic continuation is performed in such solutions.

$$\langle \gamma_0, \gamma_\infty \rangle_{\gamma_1} = \langle \gamma_0, \gamma_\infty \rangle \tag{2.27}$$

$$\langle \gamma_1, \gamma_\infty \rangle_{\gamma_0} = \langle \gamma_1, \gamma_\infty \rangle \tag{2.28}$$

If we put some constraints in the parameters, such as

$$\begin{cases} \alpha \neq -1, -2, 3, ... \\ \quad \alpha - \beta \notin \mathbb{Z} \\ \gamma - \beta \neq 1, 2, 3... \quad or \quad \alpha - \gamma \neq 1, 2, 3... \end{cases} \tag{2.29}$$

it can be proven that the solutions $\langle\gamma_0,\gamma_\infty\rangle$ and $\langle\gamma_1,\gamma_\infty\rangle$ are linearly independent, by using their power series around 1 and 0 respectively.

$$\langle\gamma_0,\gamma_\infty\rangle = \sum_{m=0}^{\infty} c_m^{(0)}(w-1)^m \tag{2.30}$$

$$\langle\gamma_1,\gamma_\infty\rangle = \sum_{m=0}^{\infty} c_m^{(1)}w^m \tag{2.31}$$

where

$$c_m^{(0)} = c_0(\alpha)_m(1-e^{2\pi i(\alpha-\gamma)})(1-e^{2\pi i(\beta-\alpha)})\frac{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\alpha+m)}{\Gamma(\beta-\gamma+m+1)} \tag{2.32}$$

$$c_m^{(1)} = c_1(\alpha)_m(1-e^{2\pi i(\beta-\gamma)})(1-e^{2\pi i(\gamma-\beta)})\frac{\Gamma(\beta-\alpha+m)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha+m)} \tag{2.33}$$

For a hypergeometric that obeys (2.29), we can make use of the linear independence of the solutions stated above and say that $F = (\langle\gamma_1,\gamma_\infty\rangle \quad \langle\gamma_0,\gamma_\infty\rangle)$ form as fundamental system of solutions. With this and the properties enforced by the commutations of the loops one can arrive at the result that for $\nu = (0,1)$ an analytic continuation of F gives

$$F_{\gamma_\nu} = FM_\nu \tag{2.34}$$

where

$$M_0 = \begin{pmatrix} 1 & e^{2\pi i(-\alpha)} - e^{2\pi i(-\gamma)} \\ 0 & 1 + e^{2\pi i(\gamma-\alpha)} - e^{2\pi i(-\alpha)} \end{pmatrix} \quad M_1 = \begin{pmatrix} e^{2\pi i(\alpha+\beta-\gamma)} & 0 \\ e^{2\pi i(\beta)} - e^{2\pi i(\alpha)} & 1 \end{pmatrix} \tag{2.35}$$

For proofs and detailed calculations of the results stated above one can look into [3]. This example show us two important things, the first one is the natural connection between the 3-point RHp and the hypergeometric equation, which will be explored later in chapter 3. The second is the verification of the statement given in the last section, for the 3-point problem, one could construct the matrices M_γ only by knowing the parameters of the ODE and vice versa. In the next section we will go further and construct the 4-point Riemann-Hilbert problem, which comes with the aforementioned accessory parameters problem.

2.2 Heun Equation

As stated above, the accessory parameter problem for the Heun equation is the representation of the Riemann-Hilbert problem with four singular points. The Heun equation itself appears in a wide range of physical problems, in particular in the conformal mapping of polycircular arcs and the scattering problem of black holes. For those two, the accessory parameter determination in terms of monodromy coefficients is crucial to solving the problems. In this section, we introduce the Heun ODE and try to obtain the accessory parameters for such an equation connecting it to a Fuchsian system formalism. This provides us with the tools we need to solve the accessory parameter problem.

The Heun equation is a 2^{nd} order ODE defined in the Riemann sphere $\mathbb{P}^1$ with four regular singular points, with one of them being the ∞. We can see from the discussions on [36] that every ODE of second order that has the same numbers of singularities can be transformed into a Heun equation by a coordinate transformation.Therefore we will work with the so-called canonical form of such ODE:

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_1}{w-1} + \frac{1-\theta_{t_0}}{w-t_0}\right) y'(w) + \left[\frac{q_+ q_-}{w(w-1)} - \frac{t_0(t_0-1)K_0}{w(w-1)(w-t_0)}\right] y(w) = 0 \tag{2.36}$$

with $q_\pm = 1 - \frac{1}{2}(\theta_0 + \theta_{t_0} + \theta_1 \pm \theta_{\infty_0})$. The parameters t_0 and K_0 are what we call accessory parameters of the equation, and the set $\theta_i \in \{\theta_0, \theta_{t_0}, \theta_1, \theta_{\infty_0}\}$ are the single monodromy coefficients. Those are the same that were defined in equation (2.9), so the θ_i are derived from the monodromy representation of the fundamental group associated to this ODE. We now seek to develop the solution of the Riemann-Hilbert problem, given the monodromy θ_i how can we construct the ODE above, more specifically, how are the accessory parameters t_0 and K_0 related to the composite monodromy of the system in question. The next sections will introduce the formalism we need to answer this question.

2.2.1 Fuchsian Approach

From the subsection 2.1.2 we remember that a Fuchsian system of the type of (2.2) can be related to a matrix representation of the monodromy group $\pi_1(\mathbb{P}^1 \setminus S, x)$, so it would be advantageous to associate the Heun equation with a Fuchsian system.

Let us introduce the appropriate system for this problem. First we write a matricial ODE with four singularities at the points 0,t, 1, ∞ as

$$\frac{d\mathbf{\Phi}}{dw} = A(w)\mathbf{\Phi}, \quad \mathbf{\Phi} = \begin{pmatrix} y_1(w) & y_2(w) \\ u_1(w) & u_2(w) \end{pmatrix} \tag{2.37}$$

with the A matrix being defined as:

$$A(w) = \frac{A_0}{w} + \frac{A_t}{w-t} + \frac{A_1}{w-1} \tag{2.38}$$

The 2×2 matrices A_i being not dependent on w and the residue of $A(w)$ at infinity implying $A_0 + A_1 + A_t = -A_\infty$, where A_∞ can be diagonalized by a transformation $\mathbf{\Phi} \to G\mathbf{\Phi}$.

Now, using equation (2.38) we can arrive at a 2^{nd} order ODE for $y_1(w)$:

$$y_1''(w) - (\text{Tr}A + (\log A_{12})')y_1'(w) + (\det A + A_{11}' + A_{11}(\log A_{12})')y_1(w) = 0 \tag{2.39}$$

where A_{ij} corresponds to the ij-entry of the $A(w)$ matrix (2.36). In a straightforward way one can find a similar ODE for any of the elements of $\mathbf{\Phi}$. If the matrix A is invertible, and $A_{12} \neq 0$, it can be shown that the solutions y_1, y_2, u_1 and u_2 are linearly independent.

We now seek to transform equation (2.39) into (2.36) This requirement states some impositions on the number of free parameters of A. Making A_∞ diagonal leads to the assumption that the term A_{12} needs be of $\mathcal{O}(-w^2)$ as w approaches infinity, and looking into (2.38) we can only conclude that:

$$A_{12} = \frac{k(w-\lambda)}{w(w-1)(w-t)}, \quad k \in \mathbb{C} \tag{2.40}$$

In this way, the off diagonal element now have a single zero called λ, and after some algebra and the comparison with (2.36) we arrive at the conclusion that $\mathrm{Tr}(A_i) = \theta_i$ and $\det A_i = 0$. In the end we obtain the ODE bellow:

$$y'' + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_1}{w-1} + \frac{1-\theta_t}{w-t} - \frac{1}{w-\lambda}\right) y' + \\ \left[\frac{\kappa_-(1+\kappa_+)}{w(w-1)} - \frac{t(t-1)K}{w(w-1)(w-t)} + \frac{\lambda(\lambda-1)\mu}{w(w-1)(w-\lambda)}\right] y = 0 \tag{2.41}$$

where μ is the residue of A_{11} at $w = \lambda$, $A_\infty = \mathrm{diag}(\kappa_-, \kappa_+)$ with $\kappa_\pm = -\frac{1}{2}(\theta_0+\theta_t+\theta_1 \pm \theta_\infty)$, and K is given by:

$$K(\lambda,\mu,t) = \frac{\lambda(\lambda-t)(\lambda-1)}{t(t-1)}\left[\mu^2 - \left(\frac{\theta_0}{\lambda} + \frac{\theta_1}{\lambda-1} + \frac{\theta_t - 1}{\lambda - t}\right)\mu + \frac{\kappa_-(1+\kappa_+)}{\lambda(\lambda-1)}\right] \tag{2.42}$$

From equation (2.42) we see that there is a relation between the parameters K, μ and λ, that relation shows that the singularity λ at (2.41) is in fact an *apparent singularity*. Which means that its indicial exponents are $(0, 2)$ and also (2.42) guarantee us that there is no logarithmic behavior. This means that a loop encircling λ will have trivial monodromy, therefore $M_\lambda = \mathbb{1}$. With this set of parameters (K, μ, λ, t), we now wish to relate then to the accessory parameters of the Heun equation. In order to do this, one needs to make use of the aforementioned isomonodromic transformations, we seek to make deformations into equation (2.41) and force the parameters to become of the form of (2.36). Next section is devoted to explain the details behind such deformations in the parameters.

2.2.2 Isomonodromic Transformations

The concept of the isomonodromic deformations becomes possible when the dimension of the monodromy group exceeds the dimension of the space of Fuchsian ODEs. In this way, some transformations can be performed in the set of parameters for such equations that maintain the monodromy representation. According to equation (2.23) for the Heun case, where there are four singularities, the monodromy space has one extra complex dimension. Hence, we can associate an isomonodromic transformation with the change of one parameter in the set (K, μ, λ, t), which will be chosen as t.

It was stated by Garnier in [37] and [38], that a change in t would give rise to a change in the parameters according to a Hamiltonian system:

$$\frac{d\lambda}{dt} = \{K, \lambda\}, \quad \frac{d\mu}{dt} = \{K, \mu\}, \quad \{f, g\} = \frac{df}{d\mu}\frac{dg}{d\lambda} - \frac{df}{d\lambda}\frac{dg}{d\mu} \tag{2.43}$$

Or, precisely,

$$\dot{\lambda} = \frac{\partial K}{\partial \mu}, \quad \dot{\mu} = -\frac{\partial K}{\partial \lambda} \tag{2.44}$$

We can see that the parameter K takes part as a Hamiltonian of the system in this formalism. Although this Hamiltonian cannot be interpreted in a physical way, as we are used to, it is interesting to see that the parameters are described by such equation. However, this is not the only known approach to deal with isomonodromic deformations, one could use the changes in the matrix $A(w)$ to characterize such transformation.

Let us see how we could express those transformation from the point of view of the A matrix. It was shown by Schlesinger that a deformation of a Fuchsian system of the form (2.2), with n singular points in $S = w_i$, is isomonodromic if $\mathbf{Y}(w)$ satisfies a system of linear partial differential equations or if the matrices A_i as functions of the same deformation parameters satisfies a completely integrable nonlinear differential system.

Theorem 1 *(Schlesinger [39]). The deformation equations of the system of linear differential equations*

$$\frac{\partial \mathbf{Y}(w,S)}{\partial w} = \sum_{i=1}^{n} \frac{A_i}{(w-w_i)} \mathbf{Y}(w,S), \quad S = \{w_i\}_{i=1,\cdots,n} \tag{2.45}$$

are isomonodromic if and only if either $\mathbf{Y}(w,S)$ *satisfies the following set of linear partial differential equations*

$$\frac{\partial \mathbf{Y}(w,S)}{\partial w_i} = -\frac{A_i}{(w-w_i)} \mathbf{Y}(w,S) \tag{2.46}$$

or the matrices $A_i(S)$ *satisfy the integrability conditions of (2.45) and (2.46) given by the completely integrable set of nonlinear equations:*

$$\frac{\partial A_j}{\partial w_i} = \frac{[A_i, A_j]}{w-w_i}, \quad (i \neq j), \quad \frac{\partial A_i}{\partial w_i} = -\sum_{i\neq j, i=1}^{n} \frac{[A_i, A_j]}{w_i - w_j} \tag{2.47}$$

which is known as the Schlesinger Equations.

Now let us write how would be the Schlesinger equations for the Fuchsian system (2.37)

$$\frac{\partial \hat{A}_0}{\partial t} = \frac{[\hat{A}_t, \hat{A}_0]}{t}, \quad \frac{\partial \hat{A}_1}{\partial t} = \frac{[\hat{A}_t, \hat{A}_1]}{t-1}, \quad \frac{\partial \hat{A}_t}{\partial t} = \frac{[\hat{A}_0, \hat{A}_t]}{t} + \frac{[\hat{A}_1, \hat{A}_t]}{t-1} \tag{2.48}$$

Since it is a monodromy preserving transformation, the eigenvalues of $\hat{A}_i$, related to the parameters θ_i, are conserved under the effect of (2.48). For a more detailed review into isomonodromic deformations one could go to [40].

Now, departing from both possible approaches, equation (2.43) or equation (2.48), we can arrive at an ODE for the parameter λ as a function of the monodromy parameters

θ_i and t [41]. The ODE has the following form

$$\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)(\dot{\lambda})^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\dot{\lambda} + \\ + \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(t-1)^2}\left[(\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{\lambda^2} + \frac{\theta_1^2(t-1)}{(\lambda-1)^2} - \frac{(\theta_t^2-1)t(t-1)}{(\lambda-t)^2}\right]. \quad (2.49)$$

This differential equation is one of the forms of the Painlevé VI (PVI) equation. Then, the aforementioned connection between the theory of isomonodromic deformations and the Painlevé sixth transcendent is given by (2.49). For more on the relation between those two concepts one can go to ([42]). Now we are able to use the whole theory behind the PVI equation to help us solve the accessory parameter problem.

2.2.3 Painleve VI ODE and τ function

In the last section we saw how the Painlevé VI equation appears in the context of isomonodromic transformations, now let us take a look into some properties of such ODE. In the seminal work did by Jimbo, Miwa and Ueno ([20]) it was stated that for any solution of the Schlesinger equations the 1-form $w = \sum_{i>j} \mathrm{Tr}(\hat{A}_i\hat{A}_j)\mathrm{d}\log(w_i - w_j)$ is closed. Which allow us to introduce the concept of the τ function as being $w = \mathrm{d}\log\hat{\tau}$. Then we have:

$$\frac{d}{dt}\log\hat{\tau} = \frac{1}{t}\mathrm{Tr}\hat{A}_0\hat{A}_t + \frac{1}{t-1}\mathrm{Tr}\hat{A}_1\hat{A}_t \quad (2.50)$$

The relation between the τ function and the parameters in (2.41) is given by:

$$\frac{d}{dt}\log\hat{\tau} = K + \frac{\theta_0\theta_t}{t} + \frac{\theta_1\theta_t}{t-1} - \frac{\kappa_-(\lambda-t)}{t(t-1)} - \frac{\lambda(\lambda-1)\mu}{t(t-1)} \quad (2.51)$$

Now, using the Schlesinger equations (2.48), it can be shown that $\frac{d}{dt}\log\hat{\tau}$ obeys a differential equation. Let us consider the $\hat{\zeta}(t)$ function bellow.

$$\hat{\zeta}(t) := t(t-1)\frac{d}{dt}\log\hat{\tau}, \quad \hat{\zeta}'(t) = \mathrm{Tr}\hat{A}_0\hat{A}_t + \mathrm{Tr}\hat{A}_1\hat{A}_t, \quad \hat{\zeta}''(t) = \frac{\mathrm{Tr}([\hat{A}_0,\hat{A}_t]\hat{A}_1)}{t(t-1)}. \quad (2.52)$$

It is known that any triplet of traceless matrices obeys:

$$(\mathrm{Tr}([\hat{A}_0,\hat{A}_t]\hat{A}_1))^2 = -2\det\begin{pmatrix} \mathrm{Tr}\hat{A}_0^{\,2} & \mathrm{Tr}\hat{A}_0\hat{A}_t & \mathrm{Tr}\hat{A}_0\hat{A}_1 \\ \mathrm{Tr}\hat{A}_t\hat{A}_0 & \mathrm{Tr}\hat{A}_t^{\,2} & \mathrm{Tr}\hat{A}_t\hat{A}_1 \\ \mathrm{Tr}\hat{A}_1\hat{A}_0 & \mathrm{Tr}\hat{A}_1\hat{A}_t & \mathrm{Tr}\hat{A}_1^{\,2} \end{pmatrix} \quad (2.53)$$

The formula above can be used to determine the differential equation for $\hat{\zeta}(t)$. The only issue is that the matrices defined in (2.37) are not traceless. To solve this we work with a modified τ function: $\tau := t^{\frac{\theta_0\theta_t}{2}}(t-1)^{\frac{\theta_1\theta_t}{2}}\hat{\tau}(t)$, such a transformation induces changes in the traces of the A_i matrices, they will be explored in more detail in chapter 3. Then with this new form of the tau function is straightforward to arrive at

$$t(t-1)\frac{d}{dt}\log\tau(t) = (t-1)\mathrm{Tr}A_0A_t + t\mathrm{Tr}A_tA_1 = \hat{\zeta}(t) + (t-1)\frac{(\theta_0\theta_t)}{2} + t\frac{(\theta_1\theta_t)}{2} \quad (2.54)$$

Now we can use $A_0 + A_t + A_1 = -A_\infty$ to write equation (2.53) in terms of $\hat{\zeta}(t)$:

$$(t(t-1)\hat{\zeta}''(t))^2 = -2\det\begin{pmatrix} \frac{\theta_0^2}{2} & t\hat{\zeta}'(t) - \hat{\zeta}(t) & \hat{\zeta}(t) - \frac{\theta_0^2+\theta_t^2\theta_1^2+\theta_\infty^2}{4} \\ t\hat{\zeta}'(t) - \hat{\zeta}(t) & \frac{\theta_t^2}{2} & (t-1)\hat{\zeta}'(t) - \hat{\zeta}(t) \\ \hat{\zeta}(t) - \frac{\theta_0^2+\theta_t^2\theta_1^2+\theta_\infty^2}{4} & (t-1)\hat{\zeta}'(t) - \hat{\zeta}(t) & \frac{\theta_t^2}{2} \end{pmatrix} \tag{2.55}$$

The equation (2.55) above is known as the σ-form of the Painlevé VI equation (σ-PVI). We can now think of the solution of (2.55), which is the same as thinking in the solution for (2.48), as a representation of a class of differential equations of the type of (2.37), this also implies that it is a representation for solutions of (2.41) that have the same monodromy parameters. In other words, the solution for the PVI, which can be expressed in terms of the τ function, can be used to characterize the isomonodromic deformations performed in the Fuchsian system. The set is parameterized by the position of the singularity $w = t$ and it will be called the isomonodromic deformation of the Heun equation. The tau function introduced above will play a central role in the solution of the accessory parameter problem.

2.3 Equations for the Accessory Parameters

With all the formalism introduced in the last section we can proceed with our problem of finding the accessory parameters. Since now we are looking into the Heun equation as an element of a family of isomonodromically deformed system, if we take a second look at (2.41) and (2.42) and re-enforce the equality with (2.36) one can conclude that:

$$\lambda(t_0) = t_0, \quad \mu(t_0) = \frac{-K_0}{\theta_{t_0} - 1} \tag{2.56}$$

By doing that we are putting the Heun equation in the form (2.36), the one that does not have the extra singularity, as a smooth limit of the isomonodromic family. The conditions above can be seen as initial condition for solving the Schlesinger equation. To this end one can set $\theta_t = \theta_{t_0} - 1$ and $\theta_\infty = \theta_{\infty_0} + 1$, that implies $q_- q_+ = \kappa_-(1+\kappa_+)$, then we can recover (2.36) exactly from (2.41).

The conditions above can be written using the tau function as an initial value problem for (2.55):

$$\begin{aligned} t(t-1)\frac{d}{dt}\log\tau(\theta_i,\sigma_{ij},t)\bigg|_{t=t_0} &= t_0\frac{\theta_1\theta_t}{2} + (t_0-1)\frac{\theta_0\theta_t}{2} + t_0(t_0-1)K_0 \\ \frac{d}{dt}\left[t(t-1)\frac{d}{dt}\log\tau(\theta_i,\sigma_{ij},t)\right]\bigg|_{t=t_0} &= (\theta_\infty - \theta_0)\frac{\theta_t}{2} \end{aligned} \tag{2.57}$$

where the hat symbol has been dropped to make notation lighter. With these equations is possible to obtain the accessory parameters of the Heun equation (2.36) completely in terms of the monodromy data. Using the τ function to calculate the accessory parameters brings more advantages to our approach. To begin with, it can be shown that the τ is an analytic function of t in every point except when $t = 0$, $t = 1$ or $t = \infty$ ([43]). It is a function of the invariant monodromy data and its existence can be seen by the conditions (2.57) from standard theorems of the existence of solutions of ODEs such as (2.55). The full set of arguments for the τ

$$\theta_i \in \{\theta_0, \theta_{t_0} - 1, \theta_1, \theta_{\infty_0} + 1\}, \quad \sigma_{ij} \in \{\sigma_{0t_0} - 1, \sigma_{1t_0} - 1, \sigma_{01}\} \tag{2.58}$$

are the monodromy data that derives from the trace coordinates (2.9). The equations in (2.57) are generic and can be used to relate the monodromy data to the accessory parameters in any system involving a Heun equation. As far as we know, those equations appeared for the first time in [19] and [44]. For a more recent discussion on the many different connections and applications see [45].

One of the possible interpretations of the equations (2.57) is that the first one establishes the τ function as the generating function for the canonical transformation relating the monodromy data with the accessory parameters, moreover, the second condition can be understood from the *Toda equation* for the τ function [46]:

$$\frac{d}{dt}\left[t(t-1)\frac{d}{dt}\log\tau(\theta_i, \sigma_{ij}, t)\right] - (\theta_\infty - \theta_0)\frac{\theta_t}{2} = c\frac{\tau^+(t)\tau^-(t)}{\tau^2(t)} \tag{2.59}$$

where $c \in \mathbb{C}$ is a t-independent constant, where $\tau^\pm$ are defined as the same τ function but for systems with modified monodromies:

$$\theta_i^\pm = \{\theta_0, \theta_1, \theta_t \pm 1, \theta_{\infty_0} \mp 1\}, \quad \sigma_{ij}^\pm \in \{\sigma_{0t} \pm 1, \sigma_{1t} \pm 1, \sigma_{01}\} \tag{2.60}$$

The Toda equation belongs to a rich branch of theoretical physics, and for our case can be obtained by direct construction from the Fuchsian system by acting on the solution $\phi(w)$ of (2.37) with a Schlesinger transformation. This connection with such equation makes possible for us to interpret that one of the $\tau^\pm$ goes to zero when the limit $t \to t_0$ is applied. That statement is equivalent to say that t_0 would be the root of either τ^+ or τ^-. The next subsection is devoted to explore how the Toda equation arises in this formalism and which one of the two $\tau^\pm$ functions has t_0 as its roots.

2.3.1 The Toda equation

In Physics the Toda equation derives from a simple model in one-dimensional solid state physics, where the interaction between the nearest neighbors in a lattice was given by the Hamiltonian

$$H(p, q) = \sum_{n\in\mathbb{Z}}\left(\frac{p(n,t)^2}{2} + V(q(n+1,t) - q(n,t))\right) \tag{2.61}$$

With p and q being the generalized momentum and coordinates respectively, and V assuming the Toda potential form $V(r) = e^{-r} + r - 1$. From this starting point, the theory around the Toda lattice expanded into a whole branch of mathematics and theoretical physics. In particular, the τ function introduced in the last sections can be written as a Toda equation for some special Hamiltonian systems, as can be seen in [47]. In the remainder of this section, we will examine how, departing from a particular choice of basis for the matrices A_i, one can make use of the properties of the tau function to write it in the form of a Toda equation.

We should start by showing that when the A_i composing the system are not traceless, the form of the Toda equation change a bit from (2.60). The steps ahead follow the analysis of [48]. When using matrices A_i with $\mathrm{Tr}A_i = \theta_i$, we define $\tau(t) = t^{-\frac{\theta_0\theta_t}{2}}(t-1)^{-\frac{\theta_1\theta_t}{2}}\hat{\tau}(t)$ where :

$$\frac{d}{dt}\log\hat{\tau} = \frac{1}{t}\mathrm{Tr}A_0A_t + \frac{1}{t-1}\mathrm{Tr}A_1A_t \tag{2.62}$$

therefore

$$\frac{d}{dt}\left[t(t-1)\frac{d}{dt}\log\hat{\tau}\right] - \frac{\theta_t(\theta_t-\theta_0-\theta_1-\theta_\infty)}{2} = c\frac{\hat{\tau^+}(t)\hat{\tau^-}(t)}{\hat{\tau^2}(t)} \tag{2.63}$$

It can be seen that $\frac{(\theta_t-\theta_0-\theta_1-\theta_\infty)}{2} = \theta_t + \kappa_+$ and $\kappa_+ = \frac{-(\theta_t+\theta_0+\theta_1+\theta_\infty)}{2}$. We seek to show how to obtain the equation in this form.

Now we will use a specific parameterization to the Fuchsian system (2.38), in order to go from the Schlesinger equations and arrive at equation 2.63. We start by writing $A(w)$ in a basis that diagonalizes A_t

$$A(w) = \frac{A_0}{w} + \frac{A_1}{w-1} + \frac{1}{w-t}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \tag{2.64}$$

where $\beta = -\alpha + \theta_t$. Then one could introduce the following transformation in order to obtain a definition for τ^+ (known as Schlesinger Transformation by the Japanese school [46] [20] [**?**]):

$$\Phi^+(w) \equiv L^+(w)\Phi(w), \quad L^+ \equiv \begin{pmatrix}1 & 0\\ p^+ & 1\end{pmatrix}\begin{pmatrix}w-t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & q^+\\ 0 & 1\end{pmatrix} \tag{2.65}$$

where p^+ and q^+ are variables that are still to be specified. The Fuchsian system for $\Phi(w)$ implies:

$$\frac{\partial\Phi^+(w)}{\partial w}\left[\Phi^+(w)\right]^{-1} = A^+(w) := L^+A(w)[L^+]^{-1} + \frac{\partial L^+}{\partial w}(L^+)^{-1} \tag{2.66}$$

Now we will look for the conditions on p^+ and q^+ of (2.65) in which the transformation that they impose leaves the monodromy of all finite singular points of the new system $\Phi^+(w)$ unchanged, except for $w = t$. We make this choice because we want A^+ to be in the form of:

$$A^+(w) = \frac{A_0^+}{w} + \frac{A_1^+}{w-1} + \frac{1}{w-t}\begin{pmatrix}\alpha+1 & 0\\ 0 & \beta\end{pmatrix} \tag{2.67}$$

Proceeding, we make the index i that can assume values of 0 and 1, then define:

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \tag{2.68}$$

When comparing equations (2.66) and (2.67) we get the conditions

$$q^+(\alpha - \beta) = \sum_i b_i - (a_i - d_i)q^+ - c_i(q^+)^2 \tag{2.69}$$

$$p^+(\alpha + 1 - \beta) = -\sum_i \frac{c_i}{t - i} \tag{2.70}$$

which guarantee equation (2.67), therefore, we can define H^+ and τ^+ as

$$H^+ = \frac{d}{dt}\log\tau^+ = \frac{1}{t}\mathrm{Tr}A_0^+A_t^+ + \frac{1}{t-1}\mathrm{Tr}A_1^+A_t^+ \tag{2.71}$$

We proceed in the same way to define τ^-, by introducing a transformation that decreases α. The calculation goes analogously to the increasing case, we define

$$\Phi^-(w) = \frac{1}{w-t}\begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & w-t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q^- & 1 \end{pmatrix}\Phi(w) \tag{2.72}$$

which satisfies

$$\frac{\partial\Phi^-(w)}{\partial w}\left[\Phi^-(w)\right]^{-1} = \frac{A_0^-}{w} + \frac{A_1^-}{w-1} + \frac{1}{w-t}\begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} \tag{2.73}$$

with $A_i^-(w) := L_i^- A_i(w)[L_i^-]^{-1}$ and

$$L^- = \begin{pmatrix} 1 & p^- \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (w_i - t)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q^- & 1 \end{pmatrix} \tag{2.74}$$

The equation

$$\frac{A_0^-}{w} + \frac{A_1^-}{w-1} + \frac{1}{w-t}\begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} = L^-A(L^-)^{-1} + \frac{\partial L^-}{\partial w}(L^-)^{-1} \tag{2.75}$$

implies this following relations to p^- and q^-

$$q^-(\beta - \alpha) = \sum_i c_i - (a_i - d_i)q^- - b_i(q^-)^2 \tag{2.76}$$

$$p^-(\beta + 1 - \alpha) = -\sum_i \frac{b_i}{t - w_i} \tag{2.77}$$

Then it could be defined

$$H^- = \frac{d}{dt}\log\tau^- = \frac{1}{t}\mathrm{Tr}A_0^-A_t^- + \frac{1}{t-1}\mathrm{Tr}A_1^-A_t^- \tag{2.78}$$

With these new definitions in hand, we can try to write equation (2.60). It should be noted that one of the terms in the Toda equation is proportional to:

$$\exp\left(H^+ + H^- - 2H\right) = \frac{\hat{\tau}^+\hat{\tau}^-}{\hat{\tau}^2} \tag{2.79}$$

And in order to calculate (2.79) in terms of the coefficients of A_i, we are going to need

$$[L^+(w_i)]^{-1}A_t^+L^+(w_i) = \begin{pmatrix}\alpha+1 & 0\\ 0 & \beta\end{pmatrix} - (\alpha-\beta+1)\left[\begin{pmatrix}0 & q^+\\ 0 & 0\end{pmatrix} - p^+(w_i-t)\begin{pmatrix}-q^+ & -(q^+)^2\\ 1 & q^+\end{pmatrix}\right] \tag{2.80}$$

and

$$[L^-(w_i)]^{-1}A_t^-L^-(w_i) = \begin{pmatrix}\alpha-1 & 0\\ 0 & \beta\end{pmatrix} - (\alpha-\beta-1)\left[\begin{pmatrix}0 & 0\\ q^- & 0\end{pmatrix} - p^-(w_i-t)\begin{pmatrix}q^- & 1\\ -(q^-)^2 & -q^-\end{pmatrix}\right] \tag{2.81}$$

After some algebra we get the results:

$$H^+ - H = \frac{a_0}{t} + \frac{a_1}{t-1} + \left(\frac{c_0}{t}\frac{c_1}{t-1}\right)q^+ \tag{2.82}$$

$$H^- - H = -\frac{a_0}{t} - \frac{a_1}{t-1} + \left(\frac{b_0}{t}\frac{b_1}{t-1}\right)q^- \tag{2.83}$$

summing those equations one obtain

$$\ln\frac{\hat{\tau}^+\hat{\tau}^-}{\hat{\tau}^2} = \left(\frac{c_0}{t} + \frac{c_1}{t-1}\right)q^+ + \left(\frac{b_0}{t} + \frac{b_1}{t-1}\right)q^- \tag{2.84}$$

Our goal now is trying to write the right hand side of in terms of the τ function. Therefor it is necessary to know more about the parameters $q^\pm$, let us start by defining A_∞. In our choice of basis $A_\infty = -A_0 - A_t - A_1$ becomes

$$A_\infty = -\begin{pmatrix}a_0+a_1+\alpha & b_0+b_1\\ c_0+c_1 & d_0+d_1+\beta\end{pmatrix}. \tag{2.85}$$

Remembering that for any basis $\det A_\infty = \kappa_+\kappa_-$ and $\mathrm{Tr}A_\infty = k_+ + k_-$, we can come up with equations for $q^\pm$

$$(b_0+b_1)(q^-)^2 - (a_0-d_0+a_1-d_1+\alpha-\beta)q^- - (c_0+c_1) = 0 \tag{2.86}$$

$$(c_0+c_1)(q^+)^2 + (a_0-d_0+a_1-d_1+\alpha-\beta)q^+ - (b_0+b_1) = 0 \tag{2.87}$$

$$(\kappa_\pm)^2 + (a_0+d_0+a_1+d_1+\alpha+\beta)\kappa_\pm + \det A_\infty = 0 \tag{2.88}$$

The equations above have the same discriminant $\Delta = \kappa_+ - \kappa_-$ and then one can isolate Δ for each equation and obtain the following relations

$$q^+ = \frac{\kappa_+ + d_0 + d_1 + \beta}{c_0+c_1}, \quad q^- = -\frac{\kappa_+ + d_0 + d_1 + \beta}{b_0+b_1} \tag{2.89}$$

We could use the equation above to rewrite (2.84) and thus

$$\frac{d}{dt}\log\frac{\hat{\tau}^{+}\hat{\tau}^{-}}{2\hat{\tau}^{2}}=-\frac{\kappa_{+}+d_0+d_1+\beta}{t(t-1)}\frac{c_0b_1-c_1b_0}{(b_0+b_1)(c_0+c_1)} \tag{2.90}$$

Using the Schlesinger equations and the parameterization of the matrices A_i and A_t, one can find:

$$\begin{aligned}\frac{d}{dt}t(t-1)\frac{d}{dt}\log\hat{\tau}&=\mathrm{Tr}((A_0+A_1)A_t)=\alpha(a_0+a_1)+\beta(d_0+d_1)\\ \frac{d^2}{dt^2}t(t-1)\frac{d}{dt}t(t-1)\frac{d}{dt}\log\hat{\tau}&=\frac{1}{t(t-1)}\mathrm{Tr}(A_0[A_1,A_t])=-\frac{(\alpha-\beta)(c_0b_1-b_0c_1)}{t(t-1)}\end{aligned} \tag{2.91}$$

Now we use the relation $A_0+A_t+A_1=-A_\infty$ to manipulate the expression

$$(\kappa_{+}+d_0+\beta+d_1)(\kappa_{-}+d_0+\beta+d_1)=\det A_\infty-\mathrm{Tr}(A_\infty)(d_0+\beta+d_1)+(d_0+\beta+d_1)^2 \tag{2.92}$$

and obtain the equation

$$\frac{(\alpha-\beta)(b_0+b_1)(c_0+c_1)}{\kappa_{+}+d_0+\beta+d_1}=-(\alpha-\beta)(\kappa_{-}+d_0+\beta+d_1) \tag{2.93}$$

$$=\frac{d}{dt}t(t-1)\frac{d}{dt}\log\hat{\tau}+\alpha(\alpha+\kappa_{+})+\beta(\beta+\kappa_{-}) \tag{2.94}$$

where we used the first equation in (2.91) and $\kappa_{+}+\kappa_{-}=-\sum(a_i+d_i)-\alpha-\beta$. Therefore, using (2.90) and the second equation of (2.91), after some algebra, we finally establish

$$\frac{d}{dt}t(t-1)\frac{d}{dt}\log\hat{\tau}+\alpha(\alpha+\kappa_{+})+\beta(\beta+\kappa_{-})=C\frac{\hat{\tau}^{+}\hat{\tau}^{-}}{\hat{\tau}^{2}} \tag{2.95}$$

This last equation is invariant by a change of basis, since the eigenvalues of the matrices A_i and A_t do not change by this kind of transformation.

It is possible to rewrite (2.95) in the basis where $\alpha=\theta_t$ and $\beta=0$, in this case we would have:

$$\frac{d}{dt}t(t-1)\frac{d}{dt}\log\hat{\tau}+\theta_t(\theta_t+\kappa_{+})=C\frac{\hat{\tau}^{+}\hat{\tau}^{-}}{\hat{\tau}^{2}} \tag{2.96}$$

and thus we demonstrated how one can write the Toda equation (2.63) in terms of the τ function and its parameters. In the next subsection it will be evaluated the consistency of this choice of basis and it will be shown how we can arrive at the second equation in 2.57 when $t\to t_0$.

2.3.1.1 *Jimbo and Miwa parameterization*

In the previous discussion, we parameterized the Fuchsian system to get A_t diagonal. However, this is not the most common parameterization. It is usual to use the one by [?], in which one could analyze the limit of equation (2.97) when we make $t\to t_0$. Let us start by

$$\tilde{A}_i=\begin{pmatrix}\tilde{a}_i & \tilde{b}_i\\ \tilde{c}_i & \tilde{d}_i\end{pmatrix}=\begin{pmatrix}p_i+\theta_i & -q_ip_i\\ \frac{1}{q_i}(p_i+\theta_i) & -p_i\end{pmatrix},\quad i=0,1,t \tag{2.97}$$

where

$$\tilde{A_\infty} = -\tilde{A}_0 - \tilde{A}_t - \tilde{A}_1 = \begin{pmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{pmatrix}, \quad \theta_i = \mathrm{Tr} A_i, \quad \theta_i^2 = \mathrm{Tr} A_i^2, \tag{2.98}$$

the tilde is used to remind us that in this basis A_t is not diagonal. If we need to diagonalize A_t we do

$$A_t = G_t^{-1} \tilde{A}_t G_t, \quad G_t = \begin{pmatrix} q_t & 1 \\ 1 & \frac{p_t+\theta_t}{q_t p_t} \end{pmatrix} \tag{2.99}$$

In fact the existence of G_t - which is true by construction - completes the demonstration of (2.63)

The matrix G_t can be used to express all A_i in the basis with diagonal A_t. In that way we will be able to express (2.97) in terms of θ_i. According to (2.97) we can find the initial condition for the derivative of the τ by solving $\tau^{\pm}(t_0) = 0$ for one of the shifted functions. It can be noticed that if $\tau(t)$ goes to infinity at some t in $0 < t < 1$ the right side of equation (2.97) becomes zero, while the left side goes on to $\pm\infty$, which implies by contradiction that $\tau(t)$ is always finite in the interval $t \in (0, 1)$. Nonetheless, $\tau(t)$ is holomorphic except at the fixed singular points $0, 1, \infty$ [43].

Then,equations (2.82) and $\tilde{A}_i = G_i A_i G_i^{-1}$ yield:

$$H^- - H = -\frac{\tilde{a}_0}{t} - \frac{\tilde{a}_1}{t-1} - \left(\frac{\tilde{c}_0}{t} \frac{\tilde{c}_1}{t-1}\right) \frac{p_t q_t}{p_t + \theta_t} \tag{2.100}$$

$$H^+ - H = \frac{\tilde{a}_0}{t} + \frac{\tilde{a}_1}{t-1} + \left(\frac{\tilde{b}_0}{t} \frac{\tilde{b}_1}{t-1}\right) \frac{1}{q_t} \tag{2.101}$$

from (2.97)

$$\frac{1}{q_t} = \frac{p_t}{p_t q_t} = \frac{\theta_t - \tilde{a}_t}{\tilde{b}_t}, \quad \frac{p_t q_t}{p_t + \theta_t} = -\frac{\tilde{d}_t}{\tilde{c}_t} \tag{2.102}$$

From the previous derivation of the initial conditions we can analyze the behavior of H^+ and H^- when $t \to t_0$. For the case of H^+ we start by noticing that in this limit a_t and b_t should go to zero. In the left equation of (2.102) the limit implies that $\frac{1}{q_t} \to \infty$, which makes $H^+ \to \infty$ and therefore τ^+ should go to zero. In principle, this statement answer the question posed in the last section, but it is important that we also check the limit for τ^-. We know that if $a_t \to 0$ them $\lim_{\lambda \to t_0} p_t = -\theta_t$, also since $b_t \to 0$ we have $\lim_{\lambda \to t_0} q_t = 0$. Which means that in order to understand the limit for $\frac{p_t q_t}{p_t + \theta_t}$ another approach is going to be necessary.

Now, we will find $\tilde{c}_t$ using the expression for K_0 (in terms of the traces of the matrices A_i), equation (2.97), and the relation $\tilde{b_0} = -\tilde{b_1} = k$, with arbitrary k, that comes from

$$A_{12}(\lambda = t) = \left. \frac{k(w-\lambda)}{w(w-1)(w-t)} \right|_{\lambda=t} = \frac{b_0}{w} + \frac{b_1}{w-1} = -\frac{k}{w} + \frac{k}{w-1} \tag{2.103}$$

First, we state the following relations, that are valid for $\lambda = t = t_0$:

$$p_0 + p_1 = \Theta := \frac{1}{2}(\theta_t - \theta_0 - \theta_1 - \theta_\infty) \tag{2.104}$$

$$p_0 q_0 = -p_1 q_1 = -k := 1, \quad \tilde{c}_t = -\frac{p_0 + \theta_0}{q_0} - \frac{p_1 + \theta_1}{q_1} \tag{2.105}$$

$$-t_0\theta_t\Theta - \tilde{c}_t + \theta_t p_0 = t_0(t_0 - 1)K_0 + t_0\theta_t\theta_1 + (t_0 - 1)\theta_0\theta_t \tag{2.106}$$

Notice that in the second line, k was chosen to be equal to -1. A different choice would provide different parameterization for the entries of A_i, but this means no harm for the Fuchsian differential equation, leave alone the value of the τ function at this point. The freedom to choose the value of k only means that there is an extra degree of freedom when we diagonalize the matrix $\tilde{A}_\infty$ - this matrix can be conjugated by any non-vanishing diagonal $\mathrm{SL}(2, \mathbb{C})$ matrix without a problem. Using (2.104) we find:

$$\tilde{c}_t|_{\lambda = t_0} = p_0(\theta_t + \theta_\infty) - \Theta^2 - \theta_1\Theta \tag{2.107}$$

$$p_0 = \frac{\Theta(\Theta + \theta_1 - t_0\theta_t)}{\theta_\infty} + \frac{t_0(t_0 - 1)}{\theta_\infty}\left[K_0 + \frac{\theta_0\theta_t}{t_0} + \frac{\theta_1\theta_t}{t_0 - 1}\right] \tag{2.108}$$

Thus, at $\lambda = t_0$, $\tilde{c}_t$ is nonzero, because if it is zero then K_0 can be trivially written in terms of θ_i and t_0, which is not the case in general.

Now, by using what we constructed in this section, one sees that the second equation in (2.55) can be interpreted as a limit for $t \to t_0$ of the Toda equation. In this context, we reviewed how one can use the properties of the τ function and the $A(w)$ matrix to express such a Toda equation and analyzed its limits when $t \to t_0$. It was found that one of the two tau functions with shifted parameters $\tau_\pm$ should have a root in t_0. After making use of some known relations we concluded that this condition is fulfilled by τ^+. Now we have one simpler equation to determine the t_0 accessory parameter $\tau^+(t_0) = 0$. In the next section, we will summarize this result in the context of the Riemann-Hilbert problem.

2.3.2 Solving the Riemann-Hilbert Problem

In the end, the Riemann-Hilbert problem, or the accessory parameter problem , can be solved by this approach: we start with the monodromy data of the ODE, and by making use of the theory of isomonodromic deformations and its connection to the Painlevé VI transcendent we can write the solution of the problem as

$$\tau^+(t_0) = 0, \quad K_0 = K(t_0), \quad K(t) := \frac{d}{dt}\log\tau(\theta_i, \sigma_{ij}, t) - \frac{(\theta_{t_0} - 1)\theta_1}{2(t-1)} - \frac{(\theta_{t_0} - 1)\theta_0}{2t} \tag{2.109}$$

Remembering that the arguments for the τ function, the monodromy data, are those that came from the Fuchsian system

$$\rho = \{\theta_0, \theta_t = \theta_{t_0} - 1, \theta_1, \theta_\infty = \theta_{\infty_0} + 1, \sigma_{0t} = \sigma_{0t_0} + 1, \sigma_{1t} = \sigma_{1t_0} - 1, \sigma_{01}\} \tag{2.110}$$

this guarantees that equation (2.41) reduces to (2.36) when $\lambda = t$. On the other hand, the monodromy data used for τ^+ is related to ρ by a shift:

$$\rho^+ = \{\theta_0, \theta_{t_0}, \theta_1, \theta_{\infty_0}, \sigma_{0t_0}, \sigma_{1t_0}, \sigma_{01}\} \tag{2.111}$$

which are, in fact, the parameters for the solution of (2.36).

Therefore, we can interpret that knowing the τ function for a given set of monodromy parameters is equivalent to solve the RHp. There are a few ways to express this function, the most usual is by expanding the τ in the form of a Nekrasov sum, first proposed by [22]. Since we need to find its roots to solve one of the equations for the accessory parameters, we must take into account the numerical efficiency when choosing the method of calculation. We have three main methods for the computation of the isomonodromic τ_{VI} function:

- The first one is solving the Painlevé VI equation numerically. The dependence of the solutions on monodromy data is computed from the asymptotic expressions given by Jimbo. ([34])
- The next method is by using the algebraic evaluation of the Nekrasov sum. This one excels at generating approximate analytical expressions for the τ function. Although this kind of computation involves a lot of operations, which makes it slower than it should be for an algorithm that needs to calculate the zeros of a function.
- The last is based on the fact that the τ function can be expressed in terms of a *Fredholm Determinant* ([23]). There are two main ways of calculating the Fredholm determinant numerically, using a Fourier expansion or the Quadrature method. And those two are simpler than the Nekrasov sum cited in the last paragraph.

Then, the most logical choice is to use the Fredholm determinant form of the tau function to tackle this kind of problem. In the next chapter, we will discuss how to express such functions in terms of the Fredholm determinant, first by taking a look into the theory for a general isomonodromic tau function and then going from the general formula to the Painlevé VI case.

3 CALCULATING THE τ_{VI} FUNCTION

In the last chapter, we showed that there is a relation between the accessory parameters of a Heun equation and the monodromy data of the system. This relation is expressed as an initial value problem for the τ function. To calculate this function in its most numerically efficient way, we need to express it as a Fredholm determinant. This chapter focus on how we can solve a general Riemann-Hilbert problem and obtain the isomonodromic τ function in dimension N, for n regular singularities as a Fredholm determinant. After that, we will explore the specific case of $N = 2$ and $n = 4$, the Painelvé case. To avoid confusion will be referred to as τ_{VI} function through the rest of this chapter.

3.1 General isomonodromic τ function

In the work by Gavrylenko and Lisovyy's, [23] a generalized expression for the isomonodromic τ function is constructed. There it was used a Riemann-Hilbert formalism to obtain the τ function in terms of a Fredholm determinant. Based on this discussion, the next subsections focus on explaining the process by which one can obtain such representation.

3.1.1 General Riemann-Hilbert problem

The original concept of the Riemann-Hilbert problem was explained in subsection 2.1.3, here we present a generalized form of it. Let us consider a n-punctured Riemann sphere $\mathbb{P}^1 \setminus a$, with a being the set of points:

$$a := \{a_0 = 0, a_1, ..., a_{n-2} = 1, a_{n-1} = \infty\} \tag{3.1}$$

where the points are radial ordered $0 < |a_1| < ... < |a_{n-3}| < 1$. We can define in such a sphere a contour Γ that encircles the n-points, as exemplified in Figure 5, we can also define some jump matrices J such as $J : \Gamma \to \mathrm{GL}(N, \mathbb{C})$. With those initial definition we state the Riemann-Hilbert problem as the task of finding a function $\Psi : \mathbb{P}^1 \setminus \Gamma \to \mathrm{GL}(N, \mathbb{C})$ such as the boundary conditions of that function, $\Psi_\pm$, obey $\Psi_+ = J\Psi_-$.

Furthermore we can define n matrices $\Theta_k = \mathrm{diag}\{\theta_{k,1}, \theta_{k,2}, ..., \theta_{k,N}\} \in \mathbb{C}^{N\times N}$ that satisfy the relations: $\sum_{k=1}^{n-1} \mathrm{Tr}\Theta_k = 0$ and $\theta_{k,\alpha} - \theta_{k,\beta} \notin \mathbb{Z}$, where $(k = 1, 2.., n-1)$. And use then to produce a set of $2n$ matrices $C_\pm$ written as

$$M_{0\to k} := C_{k,-}e^{2\pi i\Theta_k}C_{k,+}^{-1} = C_{k+1,-}C_{k+1,+}^{-1}, \quad k = 0, ..., n-3 \tag{3.2}$$

$$M_{0\to n-2} := C_{n-2,-}e^{2\pi i\Theta_{n-2}}C_{n-2,+}^{-1} = C_{n-1,-}e^{-2\pi i\Theta_{n-1}}C_{n-1}^{-1}, \tag{3.3}$$

$$M_{0\to n-1} := \mathbb{1} = C_{n-1,-}C_{n+1,+} = C_{0,-}C_{0,+}^{-1}. \tag{3.4}$$

Then it is possible to express the jump matrices J as

$$J(z)|_{l_k} = M_{0\to k}^{-1}, \quad k = 0, ..., n-2, \tag{3.5}$$

$$J(z)|_{\gamma_k} = (a_k - z)^{-\Theta_k} C_{k,\pm}^{-1}, \quad \Im z \lessgtr 0, \quad k = 0, ..., n-2 \tag{3.6}$$

$$J(z)|_{\gamma_{n-1}} = (-z)^{\Theta_{n-1}} C_{n-1,\pm}^{-1}, \quad \Im z \lessgtr 0. \tag{3.7}$$

the γ_k and l_k are the labels of the circles and segments that belong to contour Γ, as can be seen in Figure 5

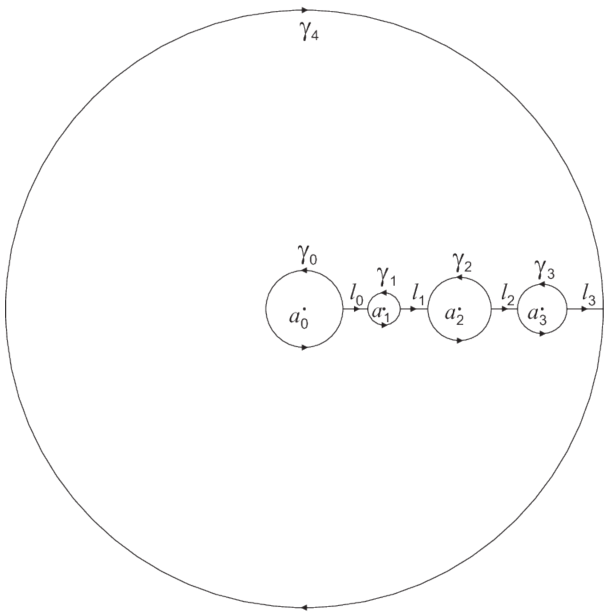


Figure 5 – Example of a Γ contour for $n = 5$. By P. Gavrylenko and O. Lisovyy from [23].

As it was introduced in chapter 2 there is also a connection between this general RHp and a Fuchsian type equation. In fact for the matricial ODE bellow:

$$\partial_z \Phi = \Phi A(z), \quad A(z) = \sum_{k-0}^{n-2} \frac{A_k}{z - a_k} \tag{3.8}$$

where the matrix function Φ defined as

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_{0,...,n-1}, \\ C_k (a_k - z)^{\Theta_k} \Psi(z), & z \text{ inside } \gamma_k,\ k = 0, ..., n-2 \\ C_{n-1}(-z)^{-\Theta_{n-1}} \Psi(z), & z \text{ inside } \gamma_{n-1} \end{cases} \tag{3.9}$$

is a solution of the system 3.8, with monodromy exponents Θ_k. The monodromy representation associated with Φ, $\rho : \pi_1(\mathbb{P}^1 \setminus a) \to \mathrm{GL}(N, \mathbb{C})$, is generated by the matrices

$$M_0 = M_{0\to 0}, \quad M_{k+1} = M_{0\to k}^{-1} M_{0\to k+1} \tag{3.10}$$

The monodromy matrices $M_{0\to k}$ can be better dealt with when diagonalized, so we introduce

$$M_{0\to k} = S_k e^{2\pi i \mathfrak{S}_k} S_k^{-1}, \quad \mathfrak{S}_k = \mathrm{diag}(\sigma_{k,1}, \sigma_{k,2}, ..., \sigma_{k,N}) \tag{3.11}$$

where we impose that $\mathrm{Tr}(\mathfrak{S}_k) = \sum_{j=0}^{k} \mathrm{Tr}(\Theta_k)$, $|\Re(\sigma_{k,\alpha} - \sigma_{k,\beta})| \leq 1$ and $\sigma_{k,\alpha} - \sigma_{k,\beta} \neq \pm 1$. At last we have that $\mathfrak{S}_0 \equiv \Theta_0$ and $\mathfrak{S}_{n-2} \equiv -\Theta_{n-1}$. Therefore we have a problem similar to those treated in the last chapter but in a more general way for n singularities and rank N, in the next subsection we will address a way of treating this problem.

3.1.1.1 *Auxiliary Riemann-Hilbert problems*

The RHp described above is considerably more complex than the one introduced in chapter 2 and to solve it one must make use of a mathematical trick. The trick consists in dividing the n-punctured Riemann sphere from last section into $n-2$ pair of pants(or trinions). Those pairs will be written as $\mathfrak{T}^{[1]}, ..., \mathfrak{T}^{[n-2]}$ and connected with $n-3$ annuli, $\mathscr{A}_1, ..., \mathscr{A}_{n-3}$. We will label the boundary components of an annulus $\mathscr{A}^{[k]}$, belonging to trinions $\mathfrak{T}^{[k]}$ and $\mathfrak{T}^{[k+1]}$, as $\mathscr{C}_{out}^{[k]}$ and $\mathscr{C}_{in}^{[k+1]}$ respectively. In Figure 6 there is an example of such division of the Riemann sphere.

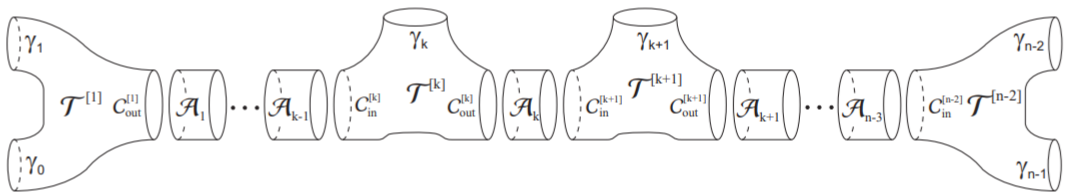


Figure 6 – Labels for the trinions, annuli and boundary curves. By P. Gavrylenko and O. Lisovyy from [23].

Now, those trinions can be associated with a simpler 3-point RHp, then solving the n-point Riemann-Hilbert problem can be seen as solving $n-2$ auxiliary 3-point problems. In the boundary, $\mathscr{C}_{out}^{[k]}$ and $\mathscr{C}_{in}^{[k+1]}$, of the auxiliary 3-point RHps the form of the jump matrices is changed to

$$J^{[k]}\Big|_{\mathscr{C}_{out}^{[k]}} = (-z)^{-\mathfrak{S}_k} S_k^{-1}, \quad J^{k+1}\Big|_{\mathscr{C}_{in}^{[k+1]}} = (-z)^{-\mathfrak{S}_k} S_k^{-1}, \quad k = 1, ..., n-3 \tag{3.12}$$

while we have new solutions $\Psi^{[k]}$ that are associated with $\Phi^{[k]}$ in an analogous way to (3.9) and those $\Phi^{[k]}$ are solutions of a Fuchsian system, with three regular singular points $(0, a_k, \infty)$ and monodromies $M_{0\to k-1}$, M_k and $M_{0\to k}$, written as:

$$\partial_z \Phi^{[k]} = \Phi^{[k]} A^{[k]}(z), \quad A^{[k]}(z) = \frac{A_0^{[k]}}{z} + \frac{A_1^{[k]}}{z - a_k} \tag{3.13}$$

In Figure 7 one can see an example of the resulting anulli after dividing the problem into $n-2$ auxiliary 3-point RHps. The utilization of this technique makes possible for us to make use of special operators, named Cauchy-Plemlj operators, to define a τ function for such a problem.

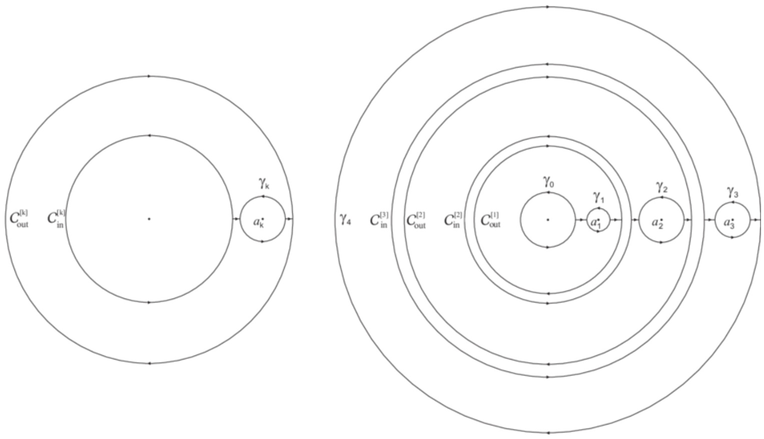


Figure 7 – Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for n = 5 (right). By P. Gavrylenko and O. Lisovyy from [23].

3.1.2 The τ function

By breaking the general RHp into the auxiliary problems one can be able to express the generalized τ function for such a system in terms of a Fredholm determinant, as written bellow

$$\tau(a) = \Upsilon(a) \cdot \det(\mathbb{1} - K) \tag{3.14}$$

where

$$\Upsilon(a) = \prod_{k=1}^{n-3} a_k^{\frac{1}{2}\mathrm{Tr}\mathfrak{S}_k^2 - \frac{1}{2}\mathrm{Tr}\mathfrak{S}_{k-1}^2 - \frac{1}{2}\mathrm{Tr}\Theta_k^2} \tag{3.15}$$

and the kernel K inside the determinant is expressed by

$$U_k = \begin{pmatrix} 0 & a^{[k+1]} \\ d^{[k]} & 0 \end{pmatrix}, \quad k = 1, \ldots n-3, \tag{3.16a}$$

$$V_k = \begin{pmatrix} b^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_k = \begin{pmatrix} 0 & 0 \\ 0 & c^{[k+1]} \end{pmatrix}, \quad k = 1, \ldots, n-4, \tag{3.16b}$$

$$K = \begin{pmatrix} U_1 & V_1 0 & . & 0 & \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & 0 \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix} \tag{3.16c}$$

The operators $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}$, are the *Plemelj Operators*. Those are integral operators defined in the boundaries of the auxiliary 3-point RHp, they are written as functions of the solutions $\Phi^{[k]}$

$$\left(a^{[k]}g\right) = \frac{1}{2\pi i}\oint_{\mathscr{C}^{[k]}_{in}z} \frac{\left[\Psi^{[k]}_{+}(z)(\Psi^{[k]}_{+}(z'))^{-1} - \mathbb{1}\right] g(z')dz'}{z'-z}, \quad z \in \mathscr{C}^{[k]}_{in}, \tag{3.17a}$$

$$\left(b^{[k]}g\right) = \frac{1}{2\pi i}\oint_{\mathscr{C}^{[k]}_{out}z} \frac{\Psi^{[k]}_{+}(z)(\Psi^{[k]}_{+}(z'))^{-1} g(z')dz'}{z'-z}, \quad z \in \mathscr{C}^{[k]}_{in}, \tag{3.17b}$$

$$\left(c^{[k]}g\right) = \frac{1}{2\pi i}\oint_{\mathscr{C}^{[k]}_{in}z} \frac{\Psi^{[k]}_{+}(z)(\Psi^{[k]}_{+}(z'))^{-1} g(z')dz'}{z'-z}, \quad z \in \mathscr{C}^{[k]}_{out}, \tag{3.17c}$$

$$\left(d^{[k]}g\right) = \frac{1}{2\pi i}\oint_{\mathscr{C}^{[k]}_{out}z} \frac{\left[\Psi^{[k]}_{+}(z)(\Psi^{[k]}_{+}(z'))^{-1} - \mathbb{1}\right] g(z')dz'}{z'-z}, \quad z \in \mathscr{C}^{[k]}_{out}. \tag{3.17d}$$

3.2 The 4-point problem

The general solution obtained above can be used in our case of interest. We first treat the with four singular points, $n = 4$. Now, this is the simplest non-trivial case of Fuchsian systems. Three of those points are already fixed at $a_0 = 0$, $a_2 = 1$, $a_3 = \infty$, so it remains only one time variable $a_1 = t$. We will assume, for the sake of being able to apply the results of the last few sections, that $0 < t < 1$. Also to make the connection with Chapter 2 easier we will use the notation with indices $0, t, 1, \infty$ instead of $0, 1, 2, 3$.

The monodromy data are given by the 4 diagonal matrices $\Theta_{0,t,1,\infty}$ of the local monodromy exponents and by the connection matrices $C_0, C_{t,\pm}, C_{1,\pm}, C_\infty$, satisfying the relations

$$M_0 \equiv C_0 e^{2\pi i\Theta_0} M_0^{-1} = C_{t,-}C_{t,+}^{-1}, \quad e^{2\pi i\mathfrak{S}} = C_{t,-}e^{2\pi t\Theta_i}C_{t,+}^{-1} = C_{1,-}C_{1,+}^{-1} \tag{3.18}$$

Since for $n = 4$ there is only one non-trivial matrix $M_{0\to 1}$ it becomes convenient to work in a distinguished basis where $M_{0\to 1}$ is given by a diagonal matrix $e^{2\pi i\mathfrak{S}}$ with $\mathrm{Tr}\mathfrak{S} = \mathrm{Tr}(\Theta_0 + \Theta_t) = -\mathrm{Tr}(\Theta_1 + \Theta_\infty)$. In terms of the previous notation it corresponds to setting $\mathfrak{S}_1 = \mathfrak{S}$ and $S_1 = \mathbb{1}$.

For this case we have a 4-punctured sphere, which will be divided into two trinions and one ring. To lighten notation we will express the trinions by left $[L]$ and right $[R]$, instead of 1 and 2. In Figure 8 there is a sketch of the contour two trinions and the respective jump matrices. Now, equation (3.14) becomes

$$\tau(t) = t^{\frac{1}{2}\mathrm{Tr}(\mathfrak{S}^2-\Theta_0^2-\Theta_t^2)} \det\left(\mathbb{1} - U\right), \quad U = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix} \in \mathrm{End}(\mathscr{H}_{\mathscr{C}}), \tag{3.19}$$

where the operators $a \equiv a^{[R]} \equiv a^{[2]}$ and $d \equiv d^{[L]} \equiv d^{[1]}$ are given by

$$(ag)(z) = \frac{1}{2\pi i}\oint_{\mathscr{C}} a(z,z')g(z')dz', \quad a(z,z') = \frac{\Psi^{[R]}(z)(\Psi^{[R]}(z'))^{-1} - \mathbb{1}}{z-z'}, \tag{3.20a}$$

$$(dg)(z) = \frac{1}{2\pi i}\oint_{\mathscr{C}} d(z,z')g(z')dz', \quad d(z,z') = \frac{\mathbb{1} - \Psi^{[L]}(z)(\Psi^{[L]}(z'))^{-1}}{z-z'}, \tag{3.20b}$$

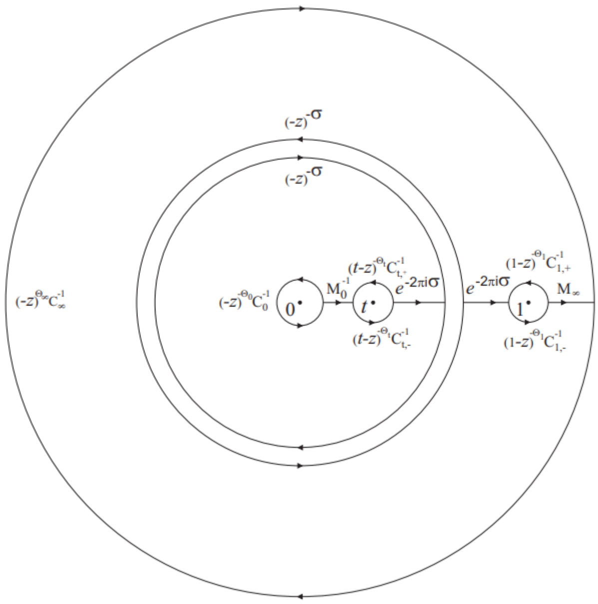


Figure 8 – Contour $\hat{\Gamma}$ and jump matrices for $n = 4$. By P. Gavrylenko and O. Lisovyy from [23].

The contour $\mathscr{C}$ is oriented counterclockwise, which is the origin of the sign difference in the expression for d (see equations (3.17)).

The matrix functions $\Psi^{[L]}(z)$, $\Psi^{[R]}(z)$ appearing in the integral kernels of a an d solve the 3-point RHps associated to Fuchsian systems with regular singularities at $0, t, \infty$ and $0, 1, \infty$, respectively. In order to understand the dependence of the 4-point τ function on the time variable t, let us re-scale the fundamental solution of the first system by setting

$$\Phi^{[L]}(z) = \tilde{\Phi}^{[L]}\left(\frac{z}{t}\right). \tag{3.21}$$

Te re-scaled matrix $\tilde{\Phi}^{[L]}(z)$ solves a Fuchsian system characterized by the same monodromy as $\Phi^{[L]}(z)$ but the corresponding singular points are located at $0, 1, \infty$. Denote by $\tilde{\Psi}^{[L]}$ the solution of the RHp associated to $\tilde{\Phi}^{[L]}$. In figure 9 it is explicitly indicated the contours and jump matrices for the two RHp, note the independence of jumps on t. In particular, inside the disk around ∞ we have $\tilde{\Phi}^{[L]}(z) = (-z)^{\mathfrak{S}}\tilde{\Psi}^{[L]}(z)$. Since the annulus $\mathscr{A}$ belongs to the disk around ∞ in the RHp for $\Psi^{[L]}$, the formula (3.21) yields the following expression for $\Psi^{[L]}$ inside $\mathscr{A}$:

$$\Psi^{[L]}(z)\Big|_{\mathscr{A}} = (-z)^{-\mathfrak{S}}\Phi^{[L]}(z) = t^{-\mathfrak{S}}\tilde{\Psi}^{[L]}\left(\frac{z}{t}\right) \tag{3.22}$$

3.2.1 Rank 2 solutions

Now we proceed to solve the auxiliary 3-point RHp for the special case of rank 2 ($N = 2$). We can express the solution in this rank in terms of Gauss hypergeometric functions. This relation was explored in subsection 2.1.4, when we reviewed the 3-point reverse RHp. And for this case, the Fredholm determinant will have an analytic form. The

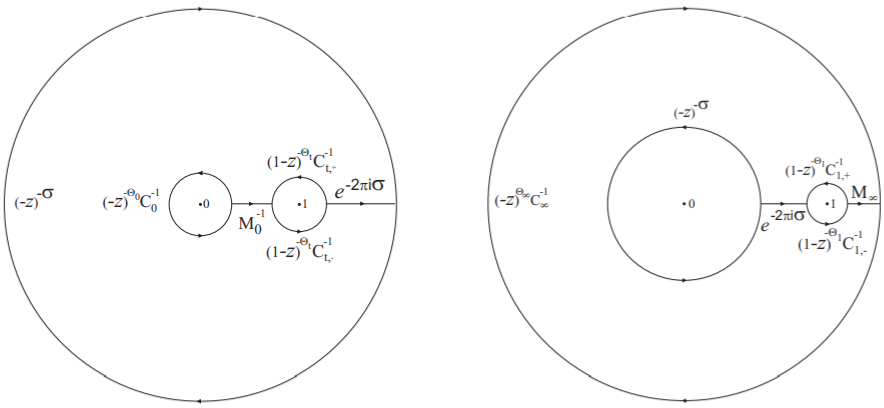


Figure 9 – Contours and jump matrices for $\tilde{\Psi}^{[L]}$ (left) and $\Psi^{[R]}$ (right). By P. Gavrylenko and O. Lisovyy from [23].

formalism in this section is for the τ function with n singular points. After we understand how to write the 3-point solution we will address the Painlevé case ($n = 4$, $N = 2$). Before we write the solutions, we will need to introduce some important concepts.

The form of the Fuchsian system (3.8) is preserved by the following non-constant scalar gauge transformation of the fundamental solution and coefficient matrices:

$$\Phi(z) \to \hat{\Phi}(z) \prod_{l=0}^{n-2} (z - a_l)^{\kappa_l} \tag{3.23}$$

$$A_l \to \hat{A}_l + \kappa_l \mathbb{1}, \quad l = 0, ..., n-2 \tag{3.24}$$

Under this transformation, the monodromy matrices M_l are multiplied by $e^{-2\pi\kappa_l}$, and the associated Jimbo-Miwa-Ueno tau function transforms as

$$\tau(a) \to \hat{\tau}(a) \prod_{0\leq k<l\leq n-2} (a_l - a_k)^{-N\kappa_l\kappa_k + \kappa_k \mathrm{Tr}\Theta_l + \kappa_l \mathrm{Tr}\Theta_k} \tag{3.25}$$

This means that if necessary one can make transformations such as (3.23) to obtain a set of monodromy matrices,$\Theta_0, ...\Theta_{n-2}$, that have one of the eigenvalues equal to 0.

With that in mind, we could change to a notation that better suits our problem

- The color indices will take values in the set $\{+, -\}$ and will be denoted by ϵ, ϵ'
- According to the last paragraph, the diagonal matrix Θ has a zero eigenvalue for $k = 0, ..., n-2$. Its second eigenvalue will be denoted by $-2\theta_k$. The eigenvalues of the remaining local monodromy exponent Θ_{n-1} may be parameterized as

$$\theta_{n-1,\epsilon} = \sum_{k=0}^{n-2} \theta_k + \epsilon\theta_{n-1}, \quad \epsilon = \pm \tag{3.26}$$

- Also, since $\mathrm{Tr}\mathfrak{S}_k = \sum_{j=0}^{k} \mathrm{Tr}\Theta_j$, we may write the eigenvalues of $\mathfrak{S}_k$ as

$$\sigma_{k,\epsilon} = -\sum_{j=0}^{k} \theta_j + \epsilon\sigma_k, \quad \epsilon = \pm, \quad k = 0, ..., n-2, \tag{3.27}$$

where $\sigma_0 \equiv \theta_0$ and $\sigma_{n-2} \equiv -\theta_{n-1}$ The parameters $\sigma_1, ..., \sigma_{n-}$ are associated to annuli $\mathscr{A}_1, ..., \mathscr{A}_{n-3}$.

For the remaining of this subsection we will assume that

$$2\theta_k \notin \mathbb{Z} \setminus \{0\}, \quad k = 0, ..., n-1, \tag{3.28}$$

$$|\Re\sigma_k| \leq \frac{1}{2}, \quad \sigma_k \neq \pm\frac{1}{2}, \quad k = 1, ..., n-3 \tag{3.29}$$

$$\sigma_{k-1} + \sigma_k \pm \theta \notin \mathbb{Z}, \quad \sigma_{k-1} - \sigma_k \pm \theta \notin \mathbb{Z}, \quad k = 1, ..., n-2 \tag{3.30}$$

Now we proceed in defining the 3-point solution $\Psi^{[k]}$, our freedom in the normalization gives us the opportunity to choose any representation in the conjugacy class $\left[A_0^{[k]}, A_1^{[k]}\right]$. In constructing the Fuchsian system (3.13) one makes use of the fact that in the $N = 2$ the conjugacy class is fixed by the local monodromy $\mathfrak{S}_{k-1}$,Θ_k and $\mathfrak{S}$. Then

$$A_0^{[k]} = \text{diag}\{\sigma_{k-1,+}, \sigma_{k-1,-}\}, \quad a_k A_1^{[k]} = -u^{[k]} \otimes v^{[k]} \tag{3.31}$$

with $\sigma_{k-1,\pm}$ parameterized as in (3.58) and

$$u_\pm^{[k]} = \frac{(\sigma_{k-1} \pm \theta_k)^2 - \sigma_k^2}{2\sigma_{k-1}} a_k, \quad v_\pm^{[k]} = \pm. \tag{3.32}$$

As it was demonstrated in section 3.2, one should first construct the solution $\tilde{\Phi}^{[k]}$ of the re-scaled system

$$\partial_z \tilde{\Phi}^{[k]} = \tilde{\Phi}^{[k]} A^{[k]}(z), \quad A^{[k]}(z) = \frac{A_0^{[k]}}{z} + \frac{A_1^{[k]}}{z-1} \tag{3.33}$$

having the same monodromy around $0, 1, \infty$ as the solution $\Phi^{[k]}$ of the original system (3.13) has around $0, a_k, \infty$. To write it explicitly in terms of the Gauss hypergeometric function ${}_2F_1 \begin{bmatrix} a, b \\ c \end{bmatrix}; z$ we introduce the notation

$$\chi \begin{bmatrix} b \\ a \ c \end{bmatrix}; z := {}_2 F_1 \begin{bmatrix} a+b+c, a+b-c \\ 2a \end{bmatrix}; z, \tag{3.34}$$

$$\phi \begin{bmatrix} b \\ a \ c \end{bmatrix}; z := \frac{c^2 - (a+b)^2}{2a(1+2a)} z {}_2F_1 \begin{bmatrix} 1+a+b+c, 1+a+b-c \\ 2+2a \end{bmatrix}; z. \tag{3.35}$$

Therefore, the solution of (3.33) can be written as

$$\tilde{\Phi}^{[k]}(z) = S_{k-1}(-z)^{\mathfrak{S}_{k-1}} \tilde{\Psi}_{in}^{[k]}(z) \tag{3.36}$$

where S_{k-1} is a constant matrix encoding the monodromy (see 3.11), and $\tilde{\Psi}_{in}^{[k]}(z)$ is given by

$$(\tilde{\Psi}_{in}^{[k]})_{\pm\pm}(z) = \chi \begin{bmatrix} \theta_k \\ \pm\sigma_{k-1} \ \sigma_k \end{bmatrix}; z, \tag{3.37}$$

$$(\tilde{\Psi}_{in}^{[k]})_{\pm\mp}(z) = \phi \begin{bmatrix} \theta_k \\ \pm\sigma_{k-1} \ \sigma_k \end{bmatrix}; z. \tag{3.38}$$

It follows that $\Phi^{[k]}(z) = \tilde{\Phi}^{[k]}(\frac{z}{a_k})$ and

$$\Psi_+^{[k]}(z) = a_k^{-\mathfrak{S}_k - 1} \tilde{\Psi}^{[k]}\left(\frac{z}{a_k}\right), \quad z \in \mathscr{C}_{in}^{[k]} \tag{3.39}$$

Also it can be noted that $\det \tilde{\Phi}^{[k]}(z) = const \cdot (-z)^{\mathrm{Tr} A_0^{[k]}}(1-z)^{\mathrm{Tr} A_1^{[k]}}$ implies that $\det \tilde{\Phi}_{in}^{[k]}(z) = (1-z)^{2\theta_k}$, which in turns yields a simple representation for the inverse matrix

$$(\Psi_+^{[k]}(z))^{-1} = \left(1 - \frac{z}{a_k}\right)^{2\theta_k} \begin{pmatrix} (\tilde{\Psi}_{in}^{[k]})_{--}(\frac{z}{a_k}) & -(\tilde{\Psi}_{in}^{[k]})_{+-}(\frac{z}{a_k}) \\ -(\tilde{\Psi}_{in}^{[k]})_{-+}(\frac{z}{a_k}) & (\tilde{\Psi}_{in}^{[k]})_{++}(\frac{z}{a_k}) \end{pmatrix} a_k^{\mathfrak{S}_{k-1}}, \quad z \in \mathscr{C}_k^{[k]} \tag{3.40}$$

The equations (3.37)-(3.39) are adapted for the description of local behavior of $\Psi^{[k]}(z)$ inside the disk around 0 bounded by the circle $\mathscr{C}_{in}^{[k]}$,check the left part of figure 9. To calculate $\Psi_+^{[k]}(z)$ inside the disk around ∞ bounded by $\mathscr{C}_{out}^{[k]}$, let us first rewrite (3.39) using the well-known ${}_2F_1$ transformations formulas. Then we get

$$\tilde{\Phi}^{[k]}(z) = S_{k-1} C_\infty^{[k]} (-z)^{\mathfrak{S}_k} \tilde{\Psi}_{out}^{[k]}(z) G_\infty^{[k]} \tag{3.41}$$

where

$$(\tilde{\Psi}_{out}^{[k]})_{\pm\pm}(z) = \chi \begin{bmatrix} \theta_k & \\ & ; z \\ \mp\sigma_k \ \sigma_k & \end{bmatrix}, \tag{3.42}$$

$$(\tilde{\Psi}_{out}^{[k]})_{\pm\mp}(z) = \phi \begin{bmatrix} \theta_k & \\ & ; z \\ \mp\sigma_k \ \sigma_k & \end{bmatrix}. \tag{3.43}$$

and

$$G_\infty^{[k]} = \frac{1}{2\sigma_k} \begin{pmatrix} -\theta_k + \sigma_{k-1} + \sigma_+ & \theta_k + \sigma_{k-1} - \sigma_+ \\ -\theta_k + \sigma_{k-1} - \sigma_+ & \theta_k + \sigma_{k-1} + \sigma_+ \end{pmatrix} \tag{3.44}$$

$$C_\infty^{[k]} = \begin{pmatrix} \frac{\Gamma(2\sigma_{k-1}\Gamma(1+2\sigma_k)}{\Gamma(1+\sigma_{k-1}+\sigma_k-\theta_k)\Gamma(\sigma_{k-1}+\sigma_k+\theta_k)} & -\frac{\Gamma(2\sigma_{k-1}\Gamma(1-2\sigma_k)}{\Gamma(1+\sigma_{k-1}-\sigma_k-\theta_k)\Gamma(\sigma_{k-1}-\sigma_k+\theta_k)} < \\ -\frac{\Gamma(-2\sigma_{k-1>}\Gamma(1+2\sigma_k)}{\Gamma(1+\sigma_{k-1}+\sigma_k-\theta_k)\Gamma(\sigma_{k-1}+\sigma_k+\theta_k)} & \frac{\Gamma(2\sigma_{k-1}\Gamma(1+2\sigma_k)}{\Gamma(1+\sigma_{k-1}+\sigma_k-\theta_k)\Gamma(\sigma_{k-1}+\sigma_k+\theta_k)} \end{pmatrix}. \tag{3.45}$$

Where Γ is the well known Gamma function $\Gamma(x) = (x-1)!$. And, as consequence of it

$$\Psi_+^{[k]} = D_\infty^{[k]} a_k^{\mathfrak{S}_k} \tilde{\Psi}_{out}^{[k]} G_\infty^{[k]}, \quad z \in \mathscr{C}_{out}^{[k]}, \tag{3.46}$$

where $D_\infty^{[k]} = \mathrm{diag}\{d_{\infty,+}^{[k]}, d_{\infty,-}^{[k]}\}$ is a diagonal matrix expressed in terms of monodromy as

$$D_\infty^{[k]} = S_k^{-1} S_{k-1} C_\infty^{[k]}. \tag{3.47}$$

Analogously to what we did in (3.40), one can arrive at

$$(\Psi_+^{[k]}(z))^{-1} = \left(1 - \frac{z}{a_k}\right)^{2\theta_k} (G_\infty^{[k]})^{-1} \begin{pmatrix} (\tilde{\Psi}_{out}^{[k]})_{--}(\frac{z}{a_k}) & -(\tilde{\Psi}_{out}^{[k]})_{+-}(\frac{z}{a_k}) \\ -(\tilde{\Psi}_{out}^{[k]})_{-+}(\frac{z}{a_k}) & (\tilde{\Psi}_{out}^{[k]})_{++}(\frac{z}{a_k}) \end{pmatrix} a_k^{\mathfrak{S}_{k-1}} (D_\infty^{[k]})^{-1} \tag{3.48}$$

With those solutions in hand we have what we needed to express the explicit form of the integral kernels $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}$ in the Fredholm determinant representation of the Jimbo-Miwa-Ueno tau function (3.14)

$$\left(a^{[k]}g\right) = a_k^{-\mathfrak{S}_{k-1}} \frac{\left(1-\frac{z'}{a_k}\right)^{2\theta_k} \begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z) \end{pmatrix} \begin{pmatrix} K_{--}(z') & -K_{-+}(z') \\ -K_{+-}(z') & K_{++}(z') \end{pmatrix} - \mathbb{1}}{z-z'} a_k^{\mathfrak{S}_{k-1}} \tag{3.49a}$$

$$\left(b^{[k]}g\right) = a_k^{-\mathfrak{S}_{k-1}} \frac{\left(1-\frac{z'}{a_k}\right)^{2\theta_k} \begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z) \end{pmatrix} (G_\infty^{[k]})^{-1} \begin{pmatrix} K_{--}(z') & -K_{-+}(z') \\ -K_{+-}(z') & K_{++}(z') \end{pmatrix}}{z-z'} a_k^{\mathfrak{S}_k} (D_\infty^{[k]})^{-1} \tag{3.49b}$$

$$\left(c^{[k]}g\right) = D_\infty^{[k]} a_k^{-\mathfrak{S}_k} \frac{\left(1-\frac{z'}{a_k}\right)^{2\theta_k} \begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z) \end{pmatrix} G_\infty^{[k]} \begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{-+}(z') \\ -\bar{K}_{+-}(z') & \bar{K}_{++}(z') \end{pmatrix}}{z-z'} a_k^{\mathfrak{S}_{k-1}} \tag{3.49c}$$

$$\left(d^{[k]}g\right) = D_\infty^{[k]} a_k^{-\mathfrak{S}_k} \frac{\mathbb{1} - \left(1-\frac{z'}{a_k}\right)^{2\theta_k} \begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z) \end{pmatrix} \begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{-+}(z') \\ -\bar{K}_{+-}(z') & \bar{K}_{++}(z') \end{pmatrix}}{z-z'} a_k^{\mathfrak{S}_k} (D_\infty^{[k]})^{-1} \tag{3.49d}$$

Where it was used a shorthand notation $K(z) = \Psi_{in}^{[k]}(\frac{z}{a_k})$ and $\bar{K}(z) = \Psi_{out}^{[k]}(\frac{z}{a_k})$.

One last important thing to introduce before going to the Painlevé case is the space of conjugacy classes of monodromy representations of the fundamental group for this general case

$$\mathscr{M}_\Theta = \left\{[M_0, ..., M_{n-1}] \in (GL(N,\mathbb{C}))^n / \; |M_0 ... M_{n-1} = \mathbb{1}, M_k \in [e^{2\pi i \Theta_k}], \quad k = 0, ..., n-1\right\} \tag{3.50}$$

This space has $\dim \mathscr{M}_\Theta = 2n-6$, and the parameters $\vec{\sigma} := (\sigma_1, ..., \sigma_{n-3}) \in \mathbb{C}^{n-3}$ are responsible for filling $n-3$ of those dimension. It remains half of the dimensions of the space of the conjugacy classes to be defined, and so we introduce

$$\vec{\eta} := (\eta_1, ..., \eta_{n-3}), \quad e^{i\eta_k} := \frac{d_{\infty,-}^{[k]}}{d_{\infty,+}^{[k]}}, \tag{3.51}$$

η provides the remaining $n-3$ local coordinates on the space $\mathscr{M}_\Theta$ of monodromy data. The η parameter is of great importance for the PVI case as we will see in the next section.

3.2.2 The Painlevé case

The result here presented was first derived by ([23]). Let the independent variable t of Painlevé VI equation vary inside the real interval $]0,1[$ and let $\mathscr{C} = \{\; z \in \mathbb{C} : |z| = R, t <$

$R < 1\}$ be a counter-clockwise oriented circle. Let σ, η be a pair of complex parameters satisfying the conditions

$$|\Re\sigma| \leq \frac{1}{2}, \quad \sigma \neq 0, \pm\frac{1}{2}, \tag{3.52}$$

$$\theta_0 \pm \theta_t + \sigma \notin \mathbb{Z}, \quad \theta_0 \pm \theta_t - \sigma \notin \mathbb{Z}, \quad \theta_1 \pm \theta_\infty + \sigma \notin \mathbb{Z}, \quad \theta_1 \pm \theta_\infty - \sigma \notin \mathbb{Z}. \tag{3.53}$$

The isomonodromic τ_{VI} function of the Painlevé VI equation (2.55) admits the following Fredholm determinant representation:

$$\tau_{VI} = const.t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{-2\theta_t\theta_1}\det(\mathbb{1}-U), \quad U = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix}, \tag{3.54}$$

where the operators a and d are:

$$(ag)(z) = \frac{1}{2\pi i}\oint_{\mathscr{C}} a(z,z')g(z')dz', \quad (dg)(z) = \frac{1}{2\pi i}\oint_{\mathscr{C}} d(z,z')g(z')dz' \tag{3.55}$$

and their kernels are explicitly given by

$$a(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z) \end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{-+}(z') \\ -K_{+-}(z') & K_{++}(z') \end{pmatrix} - \mathbb{1}}{z-z'} \tag{3.56a}$$

$$d(z,z') = \frac{\mathbb{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z) \end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{-+}(z') \\ -\bar{K}_{+-}(z') & \bar{K}_{++}(z') \end{pmatrix}}{z-z'} \tag{3.56b}$$

with

$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma \end{matrix}; z\right], \tag{3.57a}$$

$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)} z\, {}_2F_1\left[\begin{matrix} 1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma \end{matrix}; z\right], \tag{3.57b}$$

$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma \end{matrix}; \frac{t}{z}\right] \tag{3.57c}$$

$$\bar{K}_{\pm\mp}(z) = \mp t^{\mp 2\sigma}e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\frac{t}{z}\, {}_2F_1\left[\begin{matrix} 1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma \end{matrix}; \frac{t}{z}\right] \tag{3.57d}$$

The η in the expressions above is the remaining local coordinate of the Painlevé $\mathscr{M}_\Theta$, and it can be written in terms of monodromy data as [49]

$$e^{i\eta} = \frac{[p'_{1t} - p_{1t} - (p'_{01} - p_{01})e^{2\pi i\sigma_{0t}}]}{16\sin 2\pi(\frac{\sigma_{0t}+\theta_0-\theta_t}{2})\sin 2\pi(\frac{\sigma_{0t}+\theta_1-\theta_\infty}{2})} \tag{3.58}$$

where we used the trace coordinates introduced in equation (2.9), and also introduced the notation $p'_{0t} = p_0p_t + p_tp_\infty - p_{0t} - p_{01}p_{1t}$, and $p'_{01} = p_0p_1 + p_1p_\infty - p_{01} - p_{0t}p_{1t}$ ([22]) to make the equation more concise.

4 COMPUTATIONAL METHODS FOR CALCULATION OF THE FREDHOLM DETERMINANT

Through the past chapters, we have shown that the accessory parameters problem for the Heun equation can be solved by using the τ_{VI} Painlevé function. We also have shown how one of those parameters is expressed as a zero of the τ_{VI}. With this in mind, it is necessary to have a numerically efficient method for the calculation of such function. In chapter 3 we have seen how one can write the tau function as a Fredholm determinant. By expressing the τ_{VI} in such form we can construct faster algorithms for the function. In this chapter, we aim to show the two possible numerical methods to calculate the tau function in the Fredholm form. The first was already developed in the paper ([50]), which consists in expanding the operators a and d, from (3.54), in a Fourier basis. The second method is our main result, it is an extension of the algorithm proposed by [28] in which the Fredholm determinant could generally be expanded using the quadrature formalism. The first section is devoted to explaining both approaches. Then it will be shown how to construct algorithms for the two methods. After that, we will proceed with an analysis of the efficiency of both codes to understand which one is the best suited for our problem. At last, we will show the calculations for the accessory parameters of a Heun equation with a given set of monodromy data.

The codes presented in this chapter were written using the Julia Language. This choice was made because of the fast environment and of the multi-threading support that this language provides([51]).

4.1 Calculating the Fredholm determinant

4.1.1 Fourier method

In equation (3.54) we saw how the τ_{VI} can be written in its Fredholm determinant form. Now we will analyze what happens when we expand the operators a and d in (3.56) on a Fourier basis. First we write

$$a(z,z') = \sum_{p,q\in\mathbb{Z}'} a^{p}_{-q} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \quad d(z,z') = \sum_{p,q\in\mathbb{Z}'} d^{-p}_{q} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q} \tag{4.1}$$

Using equation (4.1) we can build expressions for the solutions of the 3-point RHP, $\Psi^{[L]}$ and $\Psi^{[R]}$

$$\Psi^{[L]}(z)\Big|_{\mathscr{A}} = t^{-\mathfrak{S}} \left(\mathbb{1} + \sum_{k=1}^{\infty} g^{[L]}_k t^k z^{-k} \right) G^{[L]}_{\infty}, \tag{4.2a}$$

$$\Psi^{[R]}(z)\Big|_{\mathscr{A}} = \left(\mathbb{1} + \sum_{k=1}^{\infty} g^{[R]}_k z^k \right) G^{[R]}_0 \tag{4.2b}$$

Where the $N \times N$ matrix coefficients $g_k^{[L]}, g_k^{[R]}$ are independent of t. These formulas allow us to extract from the determinant representation the asymptotics of the 4-point Jimbo-Miwa-Ueno tau function, $\tau_{VI}(t)$, as $t \to 0$ to any desired order.

We start by rewriting the integral kernel $d(z, z')$ as

$$d(z, z') = t^{-\mathfrak{S}} \frac{\mathbb{1} - \tilde{\Psi}^{[L]}(\frac{z}{t})(\tilde{\Psi}^{[L]}(\frac{z'}{t}))^{-1}}{z - z'} t^{\mathfrak{S}}. \tag{4.3}$$

Then, the block matrix elements d in the Fourier basis become

$$d_q^{-p} = t^{-\mathfrak{S}} \cdot \tilde{d}_q^{-p} \cdot t^{p+q}, \quad p, q \in \mathbb{Z}'_+ \tag{4.4}$$

and the $N \times N$ matrix coefficients $\tilde{d}_q^{-p}$ are independent of t. They can be extracted from the Fourier series

$$\frac{\mathbb{1} - \tilde{\Psi}^{[L]}(\frac{z}{t})(\tilde{\Psi}^{[L]}(\frac{z'}{t}))^{-1}}{z - z'} = \sum_{p,q \in \mathbb{Z}'_+} \tilde{d}_q^{-p} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q}, \tag{4.5}$$

and are, therefore, expressed in terms of the coefficients of local expansion of the 3-point solution $\tilde{\Phi}^{[L]}(z)$ around $z = \infty$ by straightforward algebra. For example , the first few coefficients can by written as

$$\tilde{d}_{\frac{1}{2}}^{-\frac{1}{2}} = g_1^{[L]}, \tag{4.6a}$$

$$\tilde{d}_{\frac{3}{2}}^{-\frac{1}{2}} = g_2^{[L]} - g_1^{[L]^2}, \quad \tilde{d}_{\frac{1}{2}}^{-\frac{1}{2}} = g_2^{[L]} \tag{4.6b}$$

$$\tilde{d}_{\frac{5}{2}}^{-\frac{1}{2}} = g_3^{[L]} - g_2^{[L]} g_1^{[L]} - g_1^{[L]} g_2^{[L]} + g_1^{[L]}, \quad \tilde{d}_{\frac{3}{2}}^{-\frac{3}{2}} = g_3^{[L]} - g_2^{[L]} g_1^{[L]}, \quad \tilde{d}_{\frac{1}{2}}^{-\frac{5}{2}} = g_3^{[L]} \tag{4.6c}$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \tag{4.6d}$$

Different lines above contain the coefficients of fixed degree $p + q \in \mathbb{Z}_{>0}$ which appears in the power of t in (4.4). Similarly, we can derive the formulas for the matrix elements of $a(z, z')$:

$$a_{-\frac{1}{2}}^{\frac{1}{2}} = g_1^{[R]}, \quad a_{-\frac{3}{2}}^{\frac{1}{2}} = g_2^{[R]} - g_1^{[R]^2}, \quad a_{-\frac{1}{2}}^{\frac{3}{2}} = g_2^{[R]}, \quad \ldots \tag{4.7}$$

Then, we can use those expansions to build the approximate formula for τ_{JMU}, let us set the number Q to be the order of approximation that we desire. To obtain a uniform approximation it suffices to account for Fourier coefficients $\tilde{d}_q^{-p}$ and a_{-q}^p with $p + q \le Q$, since $p, q \in \mathbb{Z}'_+$ the total number of relevant coefficients is finite and equal to $\frac{Q(Q-1)}{2}$.

We start imposing $Q \in \mathbb{Z}_{>0}$. The 4-point tau function $\tau_{VI}(t)$ has the following asymptotics as $t \to 0$:

$$\tau_{JMU} \simeq t^{\frac{1}{2}\mathrm{Tr}(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2)} \left[\det\left(\mathbb{1} - U_Q\right) + \mathcal{O}(t^Q)\right], \quad U_Q = \begin{pmatrix} 0 & a_Q \\ d_Q & 0 \end{pmatrix} \tag{4.8}$$

Here U_Q denotes a $2NQ \times 2NQ$ finite matrix whose $NQ \times NQ$-dimensional blocks a_Q and d_Q are themselves block lower and block upper triangular matrices of the form

$$a_Q = \begin{pmatrix} a_{-\frac{1}{2}}^{Q-\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & a_{-\frac{3}{2}}^{Q-\frac{3}{2}} & \cdot & \vdots \\ a_{-\frac{1}{2}}^{\frac{3}{2}} & \cdot & \ddots & 0 \\ a_{-\frac{1}{2}}^{\frac{1}{2}} & a_{-\frac{3}{2}}^{\frac{1}{2}} & \cdots & a_{\frac{1}{2}-Q}^{\frac{1}{2}} \end{pmatrix}, \quad d_Q = t^{-\mathfrak{S}} \begin{pmatrix} \tilde{d}_{Q-\frac{1}{2}}^{-\frac{1}{2}} t^Q & \cdots & \tilde{d}_{\frac{3}{2}}^{-\frac{1}{2}} t^2 & \tilde{d}_{\frac{1}{2}}^{-\frac{1}{2}} t \\ 0 & \ddots & \cdot & \tilde{d}_{\frac{1}{2}}^{-\frac{3}{2}} t^2 \\ \vdots & \cdot & \tilde{d}_{\frac{3}{2}}^{\frac{3}{2}-Q} & \vdots \\ 0 & \cdots & 0 & \tilde{d}_{\frac{1}{2}}^{-\frac{1}{2}-Q} t^Q \end{pmatrix} t^{\mathfrak{S}}, \tag{4.9}$$

where a_{-q}^{p}, $\tilde{d}_q^{-p}$ are determined by (3.20), (4.1) and (4.5) and the conjugation by $t^{\mathfrak{S}}$ in the expression for d^Q is understood to act on each $N \times N$ block of the interior matrix. Moreover, strengthening the condition $|\Re(\sigma_\alpha - \sigma_\beta)| \leq 1$(introduced in chapter 3) to strict inequality $|\Re(\sigma_\alpha - \sigma_\beta)| < 1$ improves the error estimate in (4.8) to $\mathcal{O}(t^Q)$. This approach was first outlined in [23]. For our case, the rank will be 2, which will leave us with two upper and lower triangular matrices with 2×2 blocks.

4.1.2 Quadrature method

In the last subsection, we made use of some properties of the kernel (3.56) to obtain an approximate expression for the τ_{VI} function, while using the Fredholm determinant form. Now we turn into a more general way to treat this determinant, using what we will call the quadrature method. The discussion below is heavily based on Bornemann's paper [28]. We start with showing Fredholm's initial problem [52], this one is now called *Fredholm equation of the second kind*:

$$u(x) + z\int_a^b K(x,y)u(y)dy = f(x) \quad (x \in (a,b)), \tag{4.10}$$

Where $f(x)$ and $K(x,y)$ are both assumed to be known continuous functions, Fredholm was interested in the solvability of this equation and explicit formulas for the solution. Then, he introduced the determinant

$$d(z) = \sum_{k=0}^{\infty} \frac{z^n}{n!} \int_b^a \cdots \int_b^a \det(K(t_p,t_q))|_{p,q=1}^{n} \, dt_1...dt_n \tag{4.11}$$

which is an entire function of $z \in \mathbb{C}$, and succeeded in showing that the integral equation is uniquely solvable if and only if $d(z) \neq 0$.

Furthermore, in Hilbert's work [53] the determinant was put into its generalized form

$$u + zAu = f, \quad d(z) = \det(\mathbb{1} + zA), \tag{4.12}$$

Where u and f are vector valued functions, and A is a compact operator in some Hilbert space. The numerical method for solving such equation was developed over the classical quadrature method by Nyström([54]). It is exceptionally efficient for smooth kernels, yielding small *absolute* errors (small to the scale of $\det(\mathbb{1}) = 1$). Then, given the quadrature rule

$$Q(f) = \sum_{j=1}^{m} w_j f(x_i) \approx \int_b^a f(x)dx, \tag{4.13}$$

Nystöm discretized (4.10) as the linear system

$$u_i + z\sum_{j=1}^{m} w_j K(x_i, x_j)u_j = f(x_i), \quad i = 1, ..., m; \tag{4.14}$$

which has to be solved for $u_i \approx u(x_i)(i = 1, ..., m)$. By keeping this conceptual simplicity, Bornemann introduces the approximation of the Fredholm determinant by the determinant of an $m \times m$-matrix that is applied to the vector (u_i):

$$d_Q(z) = \det(\delta_{ij} + zw_i K(x_i, x_j))|_{i,j=1}^{m}. \tag{4.15}$$

In the special cases when we expect the weights of quadrature rule to be all positive, we can switch to a more symmetric formula given by:

$$d_Q(z) = \det(\delta_{ij} + zw_i^{\frac{1}{2}} K(x_i, x_j)w_j^{\frac{1}{2}})\Big|_{i,j=1}^{m}. \tag{4.16}$$

So to make the computation of equation (3.54) using this method one only just needs to calculate the quadrature weights and input the values in (4.16) with the kernel (3.56), and calculate the determinant of it. In [28] the *Gauss-Legendre* or *Curtis-Clenshaw* quadrature rules are recommended. For our case, we have a complex 2×2 matrix kernel, which motivated us to use a formalism based in *Riemann's sums* over a circular path.

4.2 Algorithms for the τ_{VI} function

This section is devoted to explain the details on how to use expressions (4.8) and (4.16) in a code to calculate the tau function. The next subsection shows the similarities between the two possible ways of calculation, since both approaches seek to obtain (3.54) we start with the same input, the monodromy data $\vec{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty), \vec{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$ and t.

4.2.1 Common grounds

Starting with the monodromy parameters described above, they are not completely independent of each other. As we saw in subsection 2.1.2, the monodromy data has to obey the Fricke-Jimbo relation:

$$p_0p_tp_1p_\infty + p_{0t}p_{1t}p_{01} - p_{0t}(p_0p_t + p_1p_\infty) - p_{1t}(p_0p_\infty + p_tp_1) - p_{01}(p_0p_1 + p_\infty p_t) \\ + p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 = 4 \tag{4.17}$$

which can be rearranged in a quadratic equation for p_{01} as

$$p_{01}^2 + (p_{0t}p_{1t} - (p_0p_1 + p_\infty p_t))p_{01} + (p_0p_tp_1p_\infty - p_{0t}(p_0p_t + p_1p_\infty) - p_{1t}(p_0p_\infty + p_tp_1) + p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 - 4) = 0 \quad (4.18)$$

Then, the algorithm should take this into account and run a consistency check in the parameters before doing anything else.

After the confirmation of relation (4.17), the code should generate the last parameter necessary for the calculation of the tau function, the η defined in (3.58). At this point both approaches will have to construct the matrices a and d, which appear in the operator U in (3.54). The process by which one can do that depends on what method was chosen to do the Fredholm determinant calculation. However, the steps *after* we obtain the matrices are the same. Using the fact that $\mathbb{1}$ and U are block-diagonal, one can write:

$$\det(\mathbb{1} - U) = \det(\mathbb{1} - (a \times d)) \quad (4.19)$$

then the expression for τ_{VI} reads:

$$\tau_{VI} = t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{-2\theta_t\theta_1}\det(\mathbb{1} - (a \times d)) \quad (4.20)$$

Therefore, the usual script would start calculating η and enforcing the Fricke-Jimbo relation. Then it would calculate a and d and making use of (4.20) obtain the full value of τ_{VI}. It should be noted that some re-parameterization of the monodromy matrices could end up changing the value of the monodromy data used. Now the calculation of the matrices a and d can be done by the Fourier or the quadrature method, logical steps for both of them are provided in the next subsections.

4.2.2 Constructing the Fourier approach

The Fourier code, developed in [50], was the standard approach to interpret the Fredholm representation of the tau function. It relies on expressions (4.1) to build the 2×2 blocks of (4.9). Computationally one can take advantage of the similarities between equations (3.34) and (3.35) to write a faster code. Since both equations are hypergeometrics, one can construct an algorithm that generates the Fourier expansion of such function. With this in hand, solutions χ and ϕ can be generated by putting the right parameters into the expansion.

In other words, the algorithm for the Fourier case would need to generate the expansion of the hypergeometric solutions ϕ and χ, use them to calculate the 2×2 matrices a_{-q}^{p} and d_{q}^{-p}. Then, it would organize those values in two $2N_p \times 2N_p$ lower and upper triangular matrices, a and d respectively, where here N_p represents the order of approximation for the code, similar to Q in (4.9) . An example of such a code for matrix a can be seen below (the full Fourier code is in appendix A):

```
 function invseries(ser) #ser only being a 2x2 matrix vector
    inverse = zeros(ComplexF64, 2, 2, Np+1);
    inverse[:,:,1] = [ 1 0 ; 0 1 ];
    Threads.@threads for i = 2:(Np+1)
        A = -ser[:,:,i];
        Threads.@threads for j = 2:(i-1)
            A += - (ser[:,:,j]*inverse[:,:,(1+i-j)]);
        end
        inverse[:,:,i] = A
    end
    return inverse
end

function gee(sig,th1,th2,t0=1.0)
    psi = zeros(ComplexF64,2,2, Np+1);
    # wrong sign of th1 to recover Nekrasov expansion
    a = (sig-th1+th2)/2;
    b = (sig-th1-th2)/2;
    c = sig
    psi[:,:,1] = [ 1 0 ; 0 1 ];
    psi[:,:,2] = [ ((a*b)/c*t0) (-a*b/c/(1+c)*t0) ;
                   ((a-c)*(b-c)/c/(1-c)*t0) ((a-c)*(b-c)/(-c)*t0) ];
    Threads.@threads for p = 3:(Np+1)
        psi[1,1,p] = ((a+p-2)*(b+p-2)/((p-1)*(c+p-2))*psi[1,1,p-1]*t0)
        psi[1,2,p] = ((a+p-2)*(b+p-2)/((p-2)*(c+p-1))*psi[1,2,p-1]*t0)
        psi[2,1,p] = ((a-c+p-2)*(b-c+p-2)/((p-2)*(-c+p-1))*psi[2,1,p-1]*t0)
        psi[2,2,p] = ((a-c+p-2)*(b-c+p-2)/((p-1)*(-c+p-2))*psi[2,2,p-1]*t0)
    end
    return psi
end

function BuildA(sig,th1,th2)
    vecg = gee(sig,th1,th2)
    vecginv = invseries(vecg)
    bigA = zeros(ComplexF64,2Np,2Np)
    Threads.@threads for p = 1:Np
        Threads.@threads for q = 1:Np
            result = zeros(ComplexF64,2,2)
            if q + p <= Np+1
```

```
                for r = 1:q
                    result += vecg[:,:,p+r]*vecginv[:,:,q-r+1]
                end
            end
            bigA[(2p-1):(2p), (2q-1):(2q)] = result
        end
    end
    return bigA
end
```

Where gee is the function responsible for generating the fourier expansion terms and invseries is an algorithm that calculate the inverse of a series. The function BuildA is responsible for putting together the right terms, generating $(a^{[k]})^{p;\alpha}_{-q;\beta}$ and organizing the results into a lower triangular matrix. The matrix d can be constructed using a similar function, as one can see in Appendix A.

4.2.3 Kernel of the Quadrature approach

The main result of this dissertation is the adaptation of the Quadrature approach explained in section 4.1.2 to our particular kernel. In order to use this method with the kernel described in (3.56) and (3.57) a few changes had to be performed. First, as said in subsection 4.1.2, we used a Riemann sum formalism for the quadrature points and weights. Second, due to the form of our kernel, the finite matrix generated by the quadrature method is undefined when $i = j$ in 4.16. This happens because of the fact that when $z = z'$ the equations (3.56) become $\frac{0}{0}$. One way to circumvent this problem is by applying the L'Hôpital rule and operating with ∂_z in both terms of the fractions.Also, we can make use of equation (3.33) to express the derivative of the solution $\phi(z)$ in a most efficient way. At last, we discretized the variables z, z' by making a transformation defined as $z = Re^{\frac{2\pi in}{N_p}}$, where $n = (1, 2, ...)$ and N_p represents the order of approximation (equivalent to m in 4.15). By operating with this change of variables we will have a transformation in the differential of 3.55 as well, which will give rise to a multiplicative factor of $\frac{2\pi i}{N_p}(Re^{\frac{2\pi in}{N_p}})$ in the expression for the kernels.

Summarizing, the quadrature code would need to create the transformation of z, z', build the functions for the solutions Ψ, Ψ^{-1} and the derivative of Ψ, then it would need to use those functions to construct a and d using a Riemann sum. The piece of code below shows how that calculation can be written for the a matrix (the full quadrature code is available in appendix A):

```
function ze(R,n,i)
    R*exp(2pi*im*n[i]/Np)
```

```
end

function psi(th1, th2, th3,z)
    result = zeros(Complex{Float64},2,2)
    a = (th1-th2+th3)/2
    b = (th1-th2-th3)/2
    c = th1
    result[1,1] = _2F1(a,b,c,z)
    result[1,2] = -a*b/(c*(1+c))*z*_2F1(1+a,1+b,2+c,z)
    result[2,1] = ((a-c)*(b-c))/(c*(1-c))*z*_2F1(1+a-c,1+b-c,2-c,z)
    result[2,2] = _2F1(a-c,b-c,-c,z)
    return result
end

function psiinv(th1, th2, th3,z)
    result = zeros(Complex{Float64},2,2)
    a = (th1-th2+th3)/2
    b = (th1-th2-th3)/2
    c = th1
    result[1,1] = _2F1(a-c,b-c,-c,z)
    result[1,2] = a*b/(c*(1+c))*z*_2F1(1+a,1+b,2+c,z)
    result[2,1] = -((a-c)*(b-c))/(c*(1-c))*z*_2F1(1+a-c,1+b-c,2-c,z)
    result[2,2] = _2F1(a,b,c,z)
    return (1-z)^(-th2).*result
end

function dpsi(th1,th2,th3,z)
    result = zeros(Complex{Float64},2,2)
    sigma = [ 0 0 ; 0 th1 ]
    A1 = [ ((th3^2-(th1-th2)^2)/(4*th1)) (-(th3^2-(th1-th2)^2)/(4*th1)) ;
           ((th3^2-(th1+th2)^2)/(4*th1)) (-(th3^2-(th1+th2)^2)/(4*th1)) ]
    param = psi(th1,th2,th3,z)
    result = (sigma*param-param*sigma)/z+param*A1/(z-1)
    return result
end

function BuildNA(sig, th1, th2)
    bigA = zeros(ComplexF64,2Np,2Np)
    n = [(i) for i in range(1,stop=Np)]
```

```
    id = Matrix{ComplexF64}(I,2,2)
    Threads.@threads for i = 1:Np
        Threads.@threads for j = 1:Np
            z = ze(R,n, i)
            zp = ze(R,n,j)
            psiz =psi(sig, th1, th2, z)
            psiinvzp = psiinv(sig,th1,th2,zp)
            if (i==j)
                dpsiz = dpsi(sig,th1,th2,z)
                bigA[(2i-1):(2i),(2i-1):(2i)] =(z/Np).*(dpsiz*psiinvzp)
            else
                bigA[(2i-1):(2i),(2j-1):(2j)] =(sqrt(zp)*sqrt(z)/Np).*
                ( (psiz*psiinvzp)-id )./(z-zp)
            end
        end
    end
    return bigA
end
```

Where `ze` performs the transformation $z = Re^{\frac{2\pi i n}{N_p}}$, `psi`, `psiinv` and `dpsi` generate Ψ, Ψ^{-1} and $\partial_z\Psi$ respectively. The function `BuildNA` is responsible for generating the truncated matrix for the quadrature method. To make the interpretation of this method easier for the reader, there is below a schematic representation of the matrix $a(x_i, x_j)$ from the code above, where the a_{ij} are constructed using the solutions Ψ as in (3.17):

$$\begin{bmatrix} (w_1)a_{11}(x_1,x_1) & (w_1)a_{12}(x_1,x_1) & \dots & (w_1 w_{Np})^{\frac{1}{2}}a_{11}(x_1,x_{Np}) & (w_1 w_{Np})^{\frac{1}{2}}a_{12}(x_1,x_{Np}) \\ (w_1)a_{21}(x_1,x_1) & (w_1)a_{22}(x_1,x_1) & \dots & (w_1 w_{Np})^{\frac{1}{2}}a_{21}(x_1,x_{Np}) & (w_1 w_{Np})^{\frac{1}{2}}a_{22}(x_1,x_{Np}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (w_{Np}w_1)^{\frac{1}{2}}a_{12}(x_{Np},x_1) & (w_{Np}w_1)^{\frac{1}{2}}a_{11}(x_{Np},x_1) & \dots & (w_{Np})a_{11}(x_{Np},x_{Np}) & (w_{Np})a_{12}(x_{Np},x_{Np}) \\ (w_{Np}w_1)^{\frac{1}{2}}a_{21}(x_{Np},x_1) & (w_{Np}w_1)^{\frac{1}{2}}a_{22}(x_{Np},x_1) & \dots & (w_{Np})a_{21}(x_{Np},x_{Np}) & (w_{Np})a_{22}(x_{Np},x_{Np}) \end{bmatrix} \tag{4.21}$$

4.3 Comparing the methods

This section is devoted to analyze and compare the efficiency of both codes described in the last section. The objective of constructing the quadrature method was to obtain a faster way to calculate the τ_{VI} function. In the next paragraphs, taking the Fourier code as a standard, we will compare the precision and efficiency of the quadrature method.

The first thing we need to look into is the precision. The quadrature code must be able to replicate the results given by the Fourier method to a certain degree of numerical tolerance. Since both codes come from the Fredholm determinant form of tau function it is expected that they present at least similar behaviors. In Figure 10 one can see graphs for the real and imaginary parts of the τ_{VI} for both codes.

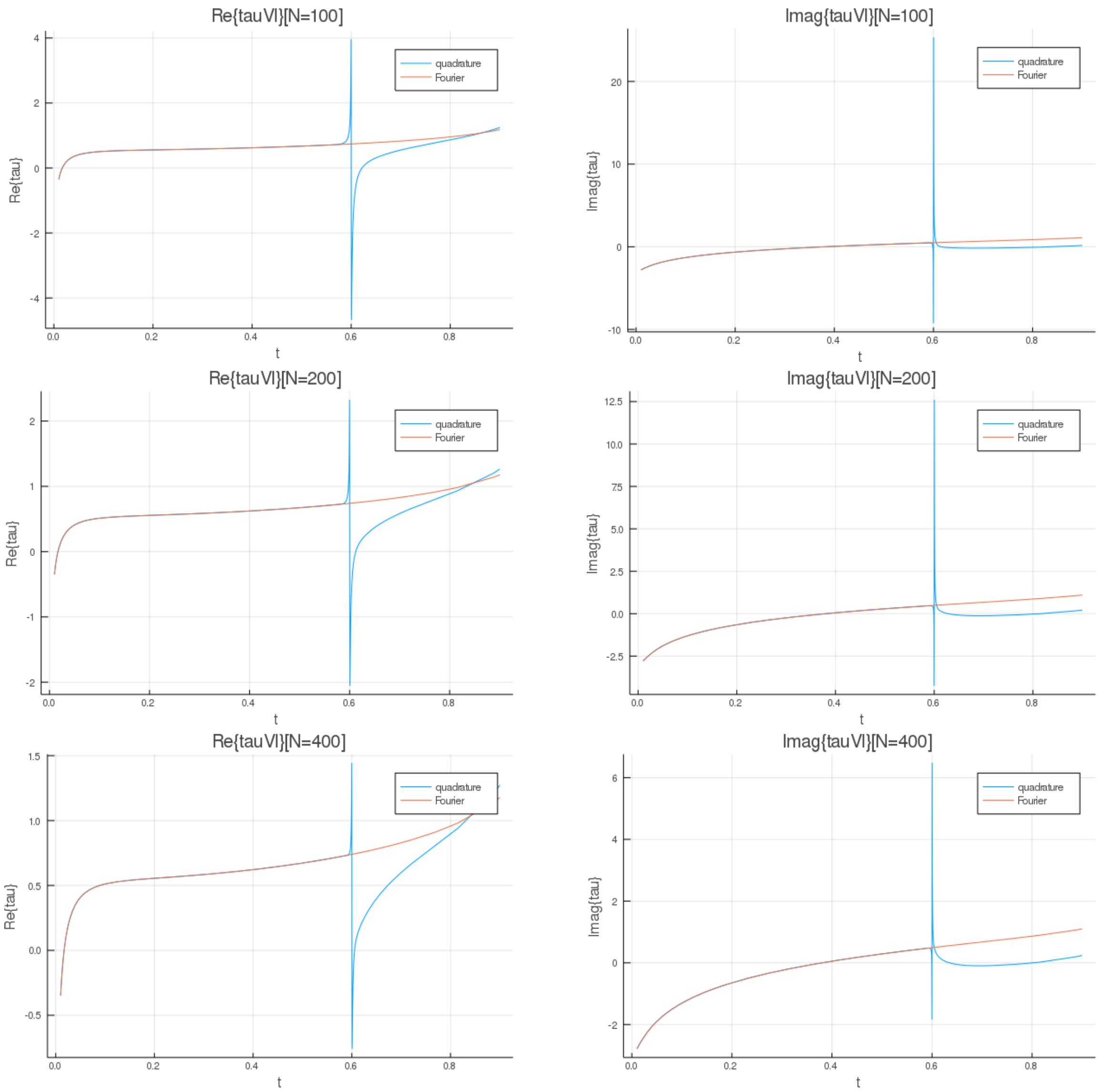
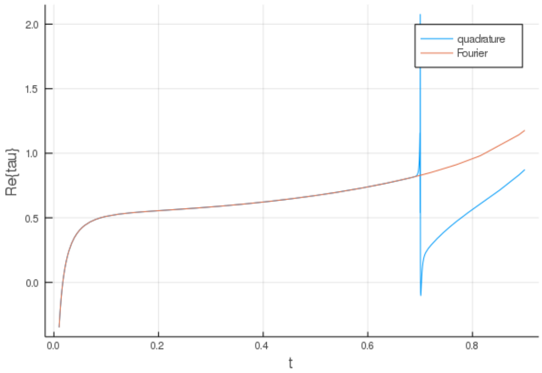


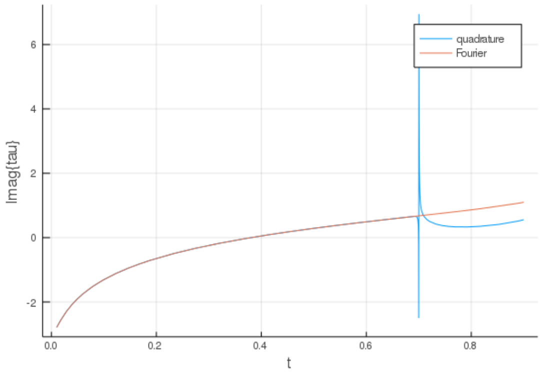
Figure 10 – Plots for three different orders of approximation of the τ_{VI} function, using $\vec{\theta} = (0.3804 + 0.6212i, 0.8364 - 0.4218i, 0.6858 + 0.6628i, 0.0326 + 0.3610i)$ and $\vec{\sigma} = (-0.6544 - 0.9622i, 0.1916 + 0.6336i, 0.9524 + 0.2056i)$

We observe that the two methods present the same behavior when $t < 0.5$. One can notice that, for the quadrature method, there is a strange formation of spikes close to $t = 0.6$, this happens because of our choice of the parameter R defined in the last section.

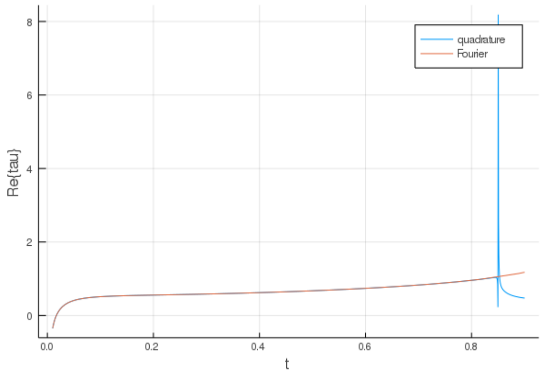
The parameter R can be associated with the definition of the tau function in its Fredholm form, see subsection 3.2.2, and it must obey $t < R < 1$. For our code, we choose $R = 0.6$, which means that the method starts to lose accuracy close to this point. Since we wish to compare the two methods in its entirety, we need to analyze how changes in the values of R affect the accuracy of the quadrature approach. In Figure 11 we can see



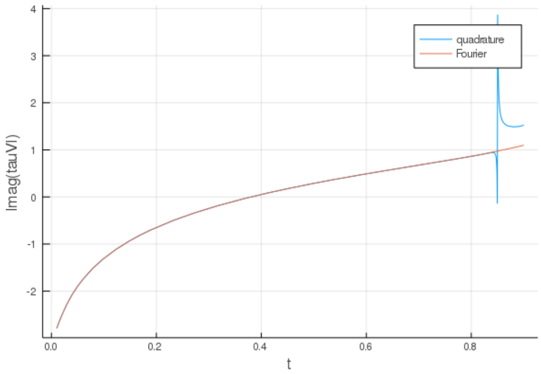
(a) Re(τ_{VI}) R=0.7



(b) Imag(τ_{VI}) R=0.7



(c) Re(τ_{VI}) R=0.85



(d) Imag(τ_{VI}) R=0.85

Figure 11 – Plots of the real and imaginary parts of the τ_{VI} function for different values of R, with $N = 400$ and using $\vec{\theta} = (0.3804 + 0.6212i, 0.8364 - 0.4218i, 0.6858 + 0.6628i, 0.0326+0.3610i)$ and $\vec{\sigma} = (-0.6544-0.9622i, 0.1916+0.6336i, 0.9524+ 0.2056i)$

two graphs for the τ_{VI} function with different values of R. The first two are plots for $R = 0.7$, in this case, we see that there is a very good agreement not only between the quadrature and Fourier method but also with the quadrature graph for $R = 0.6$. The graphs for $R = 0.85$ present a good agreement between the Fourier and the quadrature methods. However, the quadrature method starts to lose accuracy at this point, which can be seen by comparing the graphs for $R = 0.6$ and $R = 0.7$ with the one for $R = 0.85$. From our analysis, we concluded that for $0.6 < R < 0.7$ the accuracy of the quadrature code remains good, for $0.7 < R < 0.85$ there is a little loss in precision, and for $R > 0.85$, the method becomes unstable, with little changes in N_p making big changes in τ_{VI}. Now that we understand the role of the R parameter in the quadrature method, we can finally assert that, up to R, this approach has a very good agreement with the Fourier one. It is also worth pointing out that both methods are approximations of the τ_{VI} for t close to 0. This means that the precision of the two codes when $t > 0.5$ is not guaranteed. If necessary, an expansion of the τ_{VI} function close to $t = 1$ can be found by making

$$t \leftrightarrow (1-t), \quad \theta_0 \leftrightarrow \theta_1, \quad \sigma_{0t} \leftrightarrow \sigma 1t, \quad p_{01} \leftrightarrow p'_{01}, \tag{4.22}$$

where $p_\mu = 2cos(2\pi\theta_\mu)$, $p_{\mu\nu} = 2cos(2\pi\sigma_{\mu\nu})$ and $p'_{01} = p_0 p_t + p_1 p_\infty - p_{0t} - p_{01}p_{1t}$, see [22].

Now we seek to analyze which one of the codes has the best numerical efficiency. To start, we should remember that both codes present the same operations to obtain the tau function aside from the calculation of the matrices a and d. So we could understand which one of the methods will be faster by making a quick inspection of the functions responsible for generating the operator.

For the Fourier code, we construct a and d by using a double loop to place the 2×2 block matrices into the desired $2N_p \times 2N_p$ triangular form. Those blocks are constructed by a sum of 2×2 matrices generated through the Fourier expansion of the 3-point solution. On the other hand, the quadrature algorithm constructs the same matrices by placing 2×2 blocks into a truncated form of the operator a and d. The block matrices are generated through direct evaluation of the functions Ψ, Ψ^{-1}, and $\partial_z \Psi$ using Gauss hypergeometric function. We can observe from this superficial analysis that the Fourier code seems to be more complex than the quadrature one. To clarify, more details within the algorithms could impact the efficiency of each method and this last analysis was made only to give a hint of what we could expect. For example, in the quadrature code, the method used for the calculation of the hypergeometric functions could have a great impact on the time of compilation. To determine the most efficient code, we performed a test using a macro of the Julia language to calculate the time of compilation. In Figure 12 there is a graph comparing the times for the two algorithms as a function of N_p.

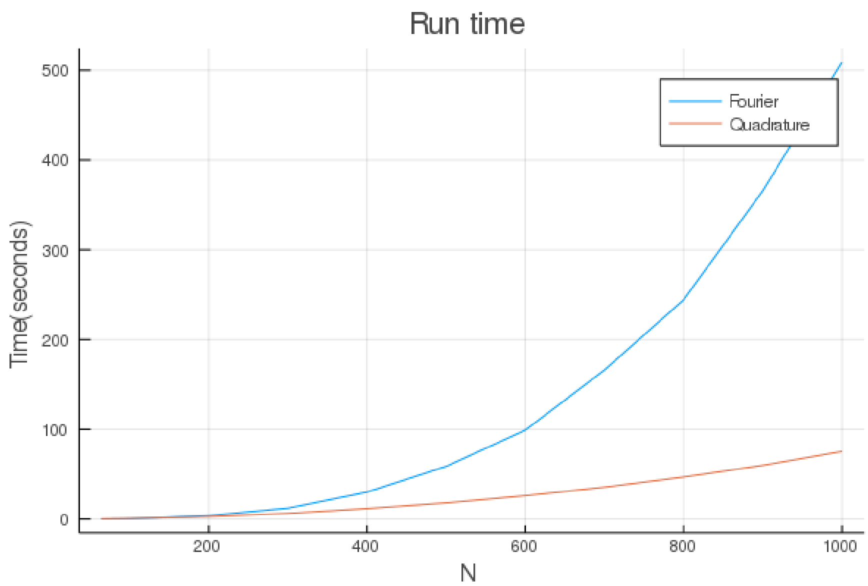


Figure 12 – Run time for both approaches, using $\vec{\theta} = (0.3804 + 0.6212i, 0.8364 - 0.4218i, 0.6858+0.6628i, 0.0326+0.3610i)$ and $\vec{\sigma} = (-0.6544-0.9622i, 0.1916+0.6336i, 0.9524 + 0.2056i)$ and $t = 0.1$

From the graph, we can arrive at the following conclusions. For low values of N_p ($N_p < 200$), the two approaches present almost the same efficiency. Also, for $N_p > 200$, we begin to see that the run time of the Fourier code exceeds the times of the quadrature method. We see that the quadrature code is faster when we work with better approximations, bigger N_p. However, we must perform one final test. Through this dissertation, we addressed the Riemann-Hilbert problem and the equivalent Accessory Parameter problem. We also showed how the τ_{VI} Painlevé function could be used to solve such a problem. The main reason for the construction of the quadrature code was to have a fast and accurate method to obtain the zeros of the tau function. Furthermore, we should find out if the quadrature method is the best suited for this task. Since τ_{VI} is a complex-valued function, it was developed a two-dimensional algorithm to find the roots, based on the well-known Newton-Raphson method. We used the same time macro to determine the run time of the 2D Newton method applied to the Fourier and quadrature codes. The data obtained is portrayed in Figure 13. (A copy of the 2D Newton-Raphson algorithm can be found in Appendix A).

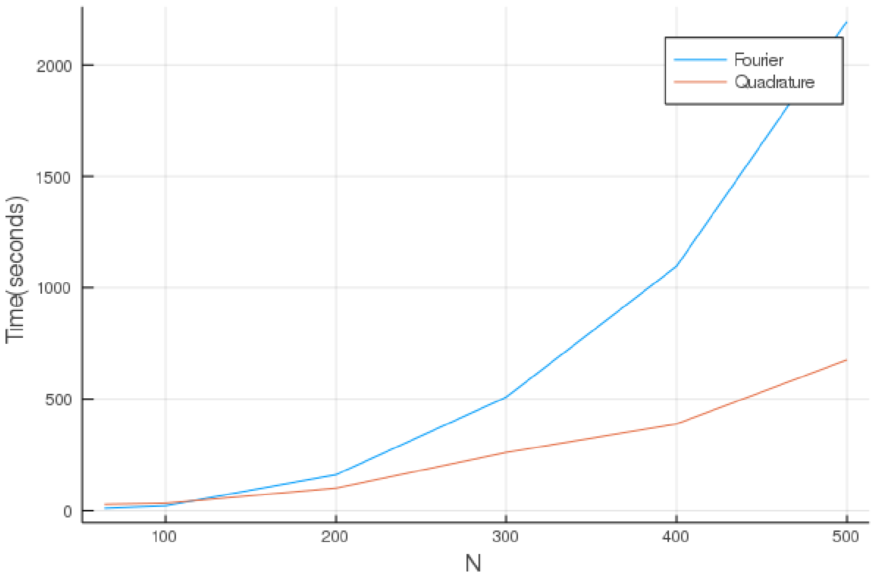


Figure 13 – Run time to find t_0 for both methods, using $\vec{\theta} = (0.3804 + 0.6212i, 0.8364-0.4218i, 0.6858+0.6628i, 0.0326+0.3610i)$ and $\vec{\sigma} = (-0.6544-0.9622i, 0.1916+0.6336i, 0.9524+0.2056i)$, where $t_0 = 0.30833646036160434+0.1908672621287217i$ is the result

Inspecting the graph, we can finally conclude that the quadrature method is, indeed, the fastest way to calculate the τ_{VI} function and its roots. From this point, we can go further and use the quadrature code to obtain high accuracy values for the tau function. As an example and consistency check, in the next section, we reproduce calculations of the accessory parameters from the literature comparing the results with values obtained by the quadrature method.

4.4 Obtaining the accessory parameters

As said in the last sections, the accessory parameter problem was one of the main motivations for the development of the quadrature code. In this section, we will test the consistency of the quadrature method. We will compute the accessory parameter of some specific examples and compare those values with the results in [18]. In Anselmo *et al.*'s paper, the accessory parameters were obtained using the expansion of the tau function, from [22]. We will present two examples in the next subsections, the generic polycircular arc domain and the half-disk barrier in an infinite channel.

A quick review of the topic is necessary to better understand the examples. We start with a generic polycircular domain (the half-disk barrier is a special case) and we wish to find a mapping between this domain and the upper half plane. As reviewed in the introduction, one needs to solve an ODE to find such map. The Schwarzian differential equation is a nonlinear ODE describing the mapping $f(w)$ that we desire to find. Writing $f(w) = \frac{y_1(w)}{y_2(w)}$ the y_i can be found by solving a second order differential equation of the form

$$\tilde{y}''(w) = \sum_{i=1}^{n} \left[\frac{1-\theta_i^2}{4(w-w_i)^2} + \frac{\beta_i}{2(w-w_i)} \right] \tilde{y}(w). \tag{4.23}$$

Where the parameters w_i are associated with the vertices of the domain, and θ_i represents the monodromy of the problem. We can obtain the parameters β_i, by using θ_i and w_i, through complex expressions. We will restrict ourselves to the case where the polycircular domain has four vertices. In this particular form, one can write (4.23) as a Heun equation and use all the formalism introduced in chapter 2. With four vertices, the parameters β_i become the accessory parameters of the Heun equation, and we can determine then using the expressions introduced in the end of chapter 2.

$$\tau^+(t_0) = 0, \quad K_0 = K(t_0), \quad K(t) := \frac{d}{dt} \log \tau(\theta_i, \sigma_{ij}, t) - \frac{(\theta_{t_0}-1)\theta_1}{2(t-1)} - \frac{(\theta_{t_0}-1)\theta_0}{2t} \tag{4.24}$$

Where the parameters for the τ and τ^+ functions are, respectively

$$\rho = \{\theta_0, \theta_t = \theta_{t_0} - 1, \theta_1, \theta_\infty = \theta_{\infty_0} + 1, \sigma_{0t} = \sigma_{0t_0} + 1, \sigma_{1t} = \sigma_{1t_0} - 1, \sigma_{01}\} \tag{4.25}$$

$$\rho^+ = \{\theta_0, \theta_{t_0}, \theta_1, \theta_{\infty_0}, \sigma_{0t_0}, \sigma_{1t_0}, \sigma_{01}\}. \tag{4.26}$$

Next two subsections will treat examples in which we use (4.24) to find the accessory parameter t_0 and K_0.

4.4.1 Generic polycircular arc domain

A generic polycicular domain with four vertices is formed by the region between four intersecting circles, as we can see in Figure 14. From [18], we are able to obtain the monodromy data ρ^+ using the geometry of the polycircular domain, the matrices for the

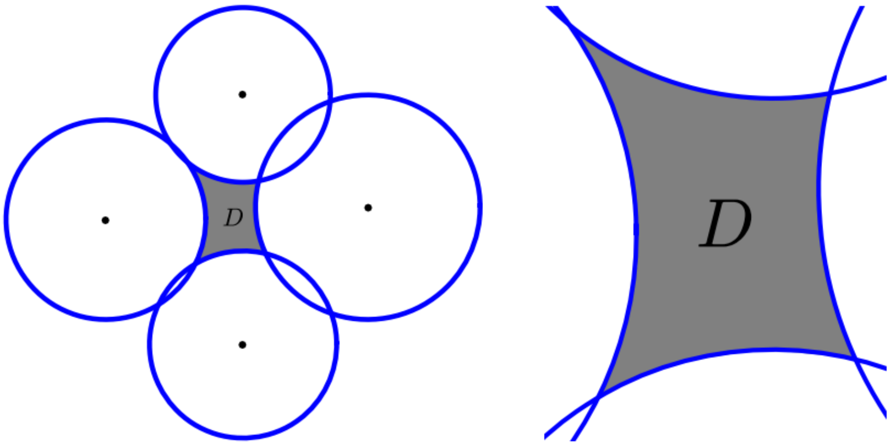


Figure 14 – Example of polycircular arc domain with four vertices. The circles forming region D are centered $-1.1, -i, 1+0.1i, i$ with respective radii $0.8, 0.75, 0.9, 0.7$. By Anselmo *et al* in [18]

general case are written as

$$M_i = \frac{1}{r_i r_{i+1}} \begin{pmatrix} z_i \bar{z}_{i+1} + r_i^2 - |z_i|^2 & \bar{z}_{i+1} - \bar{z}_i \\ z_i(r_{i+1}^2 - |z_{i+1}|^2) - z_{i+1}(r_i^2 - |z_i|^2) & \bar{z}_i z_{i+1} + r_{i+1}^2 - |z_{i+1}|^2 \end{pmatrix}. \tag{4.27}$$

Where r_i and z_i are the radius and position of the center of each circle. The monodromy parameters derive from the trace coordinates reviewed in chapter 2, $2\cos(\pi\theta_i) = -\text{Tr}(M_i)$ and $2\cos(\pi\sigma_{ij}) = \text{Tr}(M_i M_j)$. With those formulas we are able to obtain the set ρ for the domain defined in Figure 14. We will use the numbers with precision up to 10 digits both for the values of [18] and the ones generated by our method.The monodromy values will be given by

$\theta_0 = 0.1827991846$	$\sigma_{0t_0} = 1 - 0.4304546489i$
$\theta_{t_0} = 0.2869823004$	$\sigma_{1t_0} = 1 - 0.5385684561i$
$\theta_1 = 0.3673544015$	$\sigma_{01} = 0.9631297769 + 0.7221017400i$
$\theta_\infty = 0.0853271421$	

. The parameters t_0 and K_0 of the original paper and the ones that we calculated using the quadrature method, and the 2D Newton code, are displayed in the table bellow.

	quadrature	expansion
t_0	0.2086468768	0.2086468690
K_0	-0.4365581454 + 8.3433 $\times 10^{-6} i$	-0.4364792362

We can see that the values generated by the quadrature method have relatively good agreement with the previous ones from the literature. One of the outstanding facts here is that for the quadrature code a K_0 with a small imaginary part was generated, while the values of the expansion method are purely real. We know, from Anselmo *et al*'s paper, that the value of K_0 and t_0 should both be real, which means that there is some source of error in the method. By observing the values for t_0, we see that the agreement between the two methods can be expected to hold up to $6 - 7$ digits. Then, the small imaginary part of K_0 can be interpreted as an effect of the intrinsic approximation errors in the method. An example of what could have caused this low accuracy in K_0 is the fact that we used differentiation through finite differences, a more sophisticated method would be necessary to find this parameter with better precision. The optimization of the code as a whole, including more precise methods for computing K_0, will be addressed in the future. In the next section, we will treat an especial case where some of the arcs are straight lines.

4.4.2 Semi-disk barrier in an infinite channel

Reviewed in the introduction of this dissertation, the problem of finding the transformation from the upper-half plane to the semi-disk barrier domain is well known and has several applications ([55], [56], [57]). In the introduction, we have shown how one can generate the streamlines for the flow in this domain, using these transformations. The problem of finding the monodromy data associated with this semi-disk barrier was reviewed in subsection 2.1.2, where it was shown how to use the Schwarz function to obtain the M_i matrices. Applying the formulas for the trace coordinates to such matrices, we obtain $\vec{\theta} = (0.0, 0.5, 0.5, 0.0)$ and $\vec{\sigma} = (\frac{\cos^{-1}(-h)}{\pi}, 0.0, \frac{\cos^{-1}(h)}{\pi})$, where h is the channel width. Here we will use, as an example, the monodromy obtained when $h = 2$, which give us the following values:

	quadrature	expansion
t_0	3.904625×10^{-4}	3.904625×10^{-4}
K_0	$- 2.725525 \times 10^{2} - 2.991370 \times 10^{-5} i$	-2.725462×10^{2}

Here, we can see that the value of t_0 for the quadrature method has a stunning agreement with the result from Anselmo *et al.*. We can explain this agreement by the fact that the series expansion method, used in [18], excels when the values are closer to zero, which is the case in the semi-disk barrier. On the other hand, the accessory parameter K_0 only agrees with the expansion method up to $4 - 5$ digits. As in the polycircular domain from the last subsection, there is a small imaginary part associated with K_0. The problem here is the same, the precision is smaller than expected. Since the agreement for K_0 is only up to the fourth digit, we can argue that this imaginary term can be neglected. The relatively low precision in this example is possibly explained by the fact that t_0 is so close from zero, remembering that the τ_{VI} is not well defined when $t = 0$ (see subsection 3.2.2). In fact, for this particular case, the proximity with zero creates the possibility of working using an approximate expression for the parameter t_0. This expression is constructed with the first terms of the expansion for the tau function using conformal blocks from [34].

$$t_0^{1-\sigma_{0t_0}} \simeq \frac{1+\sin(\pi\sigma_{0t_0})}{1-\sin(\pi\sigma_{0t_0})}\frac{\Gamma^4(\frac{1}{4}+\frac{1}{2}\sigma_{0t_0})}{\Gamma^4(\frac{5}{4}-\frac{1}{2}\sigma_{0t_0})}\frac{\Gamma^2(1-\sigma_{0t_0})}{\Gamma^2(\sigma_{0t_0}-1)}, \quad h=-\cos\pi\sigma_{0t_0} \tag{4.28}$$

Using this expression, the value obtained for the accessory parameter is $t_0 \simeq 3.905353 \times 10^{-4}$(for $h = 2$). Comparing this value with the one obtained by the quadrature method, we see that the accuracy goes up to $3-4$ digits. This approximation serves as another consistency check for the code. This formula can be used to generate the parameter K_0, as

$$K_0 = \frac{d}{dt}\log\tau(\theta_i,\sigma_{ij},t)\bigg|_{t_0} - \frac{(\theta_{t_0}-1)\theta_1}{2(t_0-1)} - \frac{(\theta_{t_0}-1)\theta_0}{2t_0} \simeq \frac{(\sigma_{0t}-1)^2-(\theta_0+\theta_{t_0}-1)^2}{4t_0}. \tag{4.29}$$

Using this expression, we obtain $K_0 \simeq -2.725292\times10^2$, which agrees up to $3-4$ digits with the quadrature values. With this, we reiterate the argument used for the imaginary part of K_0. Even with the intrinsic errors of the actual quadrature code, the values agree relatively good with the previous ones, and we can assume that the imaginary part generated is a result of the fact that the code is not completely optimized yet.

Those two examples show us how the quadrature method behaves in comparison with other approaches. There is a good agreement between the method and the previous results from the literature. With this in hand, we can assume that the quadrature method is the best for the computation of the τ_{VI}. However, before finishing this discussion, it is important to stress the cases for $N_p \leq 200$ in Figure 12. In the graph, one can see that, for a low N_p, the Fourier and the quadrature methods are almost equivalent in run time, with a few points where the Fourier code is faster. This better efficiency for the Fourier code arises from the previous discussions about the complexity of the method. We expect that the quadrature method is more efficient, but a few processes in the code for a low N_p are responsible for this increase in the run time. This means that the quadrature code is not fully optimized, and further improvements will be necessary to achieve complete efficiency. With that in mind, the quadrature approach should be taken as the standard method for the computation of the τ_{VI}, since it behaves well for all values of the approximation constant N_p. We are looking forward to using the efficiency and precision of the quadrature code to unravel new physical problems in the future.

5 CONCLUSION

The Painlevé Transcendents are functions defined as solutions of the six Painlevé ODEs, a set of nonlinear second-order differential equations. Those functions appear in a considerable number of physical problems, ranging from integrable quantum systems to random matrix theory. In this dissertation, we explored the properties of the Painlevé sixth transcendent and its connection to the theory of isomonodromic transformations. The τ_{VI} function is associated with the solution of the sixth Painlevé ODE and can be used to represent such a solution in some specific problems.

Through the first chapters, we introduced the concept of the Riemann-Hilbert problem(RHp), which is the task to associate a Fuchsian type ODE to a given representation of the monodromy group. The accessory parameter problem of the Heun ODE is an example of the RHp, with four singular points. The Heun equation accessory parameter problem has applications in theoretical physics, such as in conformally invariant boundary dynamics in two dimensions and the scattering theory of black holes. This problem is described as the task to find two parameters t_0 and K_0 as functions of the monodromy data $\vec{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$ and $\vec{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$, which are connected to the monodromy representation associated to the ODE. We showed how one can use the theory of isomonodromic deformations, transformations that keep the monodromy representation of the ODE unchanged, to solve the accessory parameter problem with the help of the τ_{VI} function.

Then, we reviewed the generalized RHp, for n singular points and rank N, and showed how we can associate an isomonodromic tau function to this kind of problem. We can solve the generalized RHp by dividing the domain into $(n-2)$ auxiliary 3-point problems. By doing this, one can write the tau function in a Fredholm determinant form using Plemelj operators. When $n = 4$ and $N = 2$, we recover the τ_{VI} Painlevé function as Fredholm determinant with the operators being given in terms of Gauss hypergeometric functions. This form for the τ_{VI} is the most efficient way to calculate the function numerically.

Afterward, two possible computational methods for the calculation of the τ_{VI} in the determinant form were analyzed and compared. The first one, the Fourier method, was the current standard method for this kind of computation. It relies on expanding the Plemelj operator inside the determinant in the Fourier basis, enabling the matrix to be written in an approximated truncated form. The second, the quadrature method, was developed in this work based on an algorithm already proposed on how to approximate the Fredholm determinant in a more general way. This method uses the well-known quadrature rule for integrals to approximate the operator inside the determinant as truncated matrices with values being given by the evaluation of the kernel in the quadrature points. The adaptation of the method to the kernel of the Plemelj operators brought different aspects to the algorithm. Instead of using the Gauss-Legendre quadrature rule, we used a Riemann sum formalism, since the operators were defined in the complex plane. Some new terms were introduced in the formalism by the transformation that discretized the coordinates of the operator. And, at last, the singular behavior of the kernel was circumvented through the L'Hôpital rule in the diagonal of the matrices.

The quadrature method was developed to be more efficient than the Fourier one. Through some analyzes of the compilation time and precision of both codes, we concluded that, indeed, the quadrature approach has superior efficiency. Also, we made tests to see if this method had a good agreement with previous results from the literature. The theory of conformal mappings from a polycircular arc domains to the upper-half plane is governed by the Schwarz differential equation, which solution can be written as a ratio of two independent solutions of a Heun equation. We can find the monodromy representation for such a Hoen equation by an analysis of the geometry for the polycircular arc domain. And the accessory parameters of this ODE are obtained by making use of the τ_{VI} function. In this context, a recent work by Anselmo *et al* ([18]) has generated numerical results for some specific polycircular configurations. Particularly, one interesting special case of such a polycircular domain, the half-disk barrier, have also been analyzed. Due to the possible application in some theoretical physics problems, such as the theory of potential flow in fluid dynamics, this problem was also addressed in the tests made.

The improved accuracy of the quadrature method in the calculation of the tau function opens new paths to understand some problems. For example, from general relativity, the quasinormal fluctuations of black holes is a topic of great interest. For the specific background of the Kerr-AdS$_5$ black hole, the quasinormal modes depend on the connection problem of the solutions of a Fuchsian ODE. The connection problem can be solved by using the τ_{VI} function. In [50], the quasinormal modes are obtained using the Fourier method. Applying the quadrature approach to this problem might give rise to new results.

Also in the theory of black hole quasinormal modes, the Kerr background presents an interesting case. For this metric, one solves the connection problem by making use of the Painlevé V transcendent [58]. We can extend the quadrature method introduced in this dissertation to the Painlevé τ_V. The development of a quadrature code for the τ_V and its applications to such contexts might be the goal of future works.

Another possible utilization of the quadrature method used here is related to the tau function associated with more than four singular points. The formula on [23] is generalized for any set n, N, which means that it is possible to develop a code with the $n = 5$, rank 2 case. This kind of formalism can be used, for example, to find accessory parameters for the mapping of a polycircular arc with five vertices to the upper-half plane. The development of a quadrature method to the 5-point τ will be addressed in the future.

BIBLIOGRAPHY

1 DEPECHE Mode. **Macro**. 2005. Available in: https://www.youtube.com/watch?v=YYIWvx-EvxY. Accessed in: June 03, 2020. Cited in page 5.

2 HASSANI, S. *Mathematical Physics: A Modern Introduction to Its Foundations*. 2. ed. [S.l.]: Springer, 2013. ISBN 3319011944. Cited in page 10.

3 IWASAKI, K. et al. *From Gauss to Painlevé: A modern theory of Special Functions*. [S.l.]: vieweg, 1991. ISBN 978-3-322-90165-1. Cited 5 times in pages 10, 15, 21, 22, and 23.

4 FENDLEY, P.; SALEUR, H. $N = 2$ supersymmetry, painlevé iii and exact scaling functions in 2d polymers. *Nuclear Physics B*, Elsevier BV, v. 388, n. 3, p. 609–626, Dec 1992. ISSN 0550-3213. Disponível em: <http://dx.doi.org/10.1016/0550-3213(92)90556-Q>. Cited in page 10.

5 JIMBO, M. et al. Density matrix of an impenetrable bose gas and the fifth painlevé transcendent. *Physica D: Nonlinear Phenomena*, v. 1, n. 1, p. 80 – 158, 1980. ISSN 0167-2789. Disponível em: <http://www.sciencedirect.com/science/article/pii/0167278980900068>. Cited in page 10.

6 LUKYANOV, S. L. Critical values of the yang–yang functional in the quantum sine-gordon model. *Nuclear Physics B*, Elsevier BV, v. 853, n. 2, p. 475–507, Dec 2011. ISSN 0550-3213. Disponível em: <http://dx.doi.org/10.1016/j.nuclphysb.2011.07.028>. Cited in page 10.

7 SATO, M.; MIWA, T.; JIMBO, M. Holonomic quantum fields i–v. *Publ. RIMS Kyoto Univ. 14*, p. 223–267; 15 and 201–278; 15 and 577–629; 15 and 871–972; 16 and 531–584., 1978 and 1979 and 1979 and 1979 and 1980. Cited in page 10.

8 ZAMOLODCHIKOV, A. Painlevé iii and 2d polymers. *Nuclear Physics B*, Elsevier BV, v. 432, n. 3, p. 427–456, Dec 1994. ISSN 0550-3213. Disponível em: <http://dx.doi.org/10.1016/0550-3213(94)90029-9>. Cited in page 10.

9 TRACY, C. A.; WIDOM, H. Level-spacing distributions and the airy kernel. *Comm. Math. Phys.*, Springer, v. 159, n. 1, p. 151–174, 1994. Disponível em: <https://projecteuclid.org:443/euclid.cmp/1104254495>. Cited in page 10.

10 TRACY, C. A.; WIDOM, H. Fredholm determinants, differential equations and matrix models. *Communications in Mathematical Physics*, Springer Science and Business Media LLC, v. 163, n. 1, p. 33–72, Jun 1994. ISSN 1432-0916. Disponível em: <http://dx.doi.org/10.1007/BF02101734>. Cited in page 10.

11 TRACY, C.; WIDOM, H. Painlevé functions in statistical physics. *Publications of the Research Institute for Mathematical Sciences*, European Mathematical Society Publishing House, p. 361–374, 2011. ISSN 0034-5318. Disponível em: <http://dx.doi.org/10.2977/PRIMS/38>. Cited in page 10.

12 WU, T. T. et al. Spin spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. *Phys. Rev. B*, v. 13, p. 316–374, 1976. Cited in page 10.

13 KUNDU, P. K.; COHEN, I. M.; DOWLING, D. R. *Fluid Mechanics*. [S.l.]: Elsevier, 2012. Cited in page 11.

14 ABLOWITZ, M. J.; FOKAS, A. S. *Complex Variables*: Introduction and applications. 2. ed. [S.l.]: Cambridge University Press, 2003. Cited in page 12.

15 SCHWARZ, H. Ueber einige abbildungsaufgaben. *Journal für die reine und angewandte Mathematik*, De Gruyter, Berlin, Boston, v. 1869, n. 70, p. 105 – 120, 1869. Disponível em: <https://www.degruyter.com/view/journals/crll/1869/70/article-p105.xml>. Cited in page 12.

16 ANSELMO, T. *Accessory Parameters in Conformal Mapping: Exploiting the isomonodromic tau functions*. Tese (Doutorado) — Universidade Federal de Pernambuco, 2018. Cited 4 times in pages 12, 13, 16, and 18.

17 NEHARI, Z. *Conformal Mapping*. [S.l.]: Dover, 1975. Cited in page 12.

18 ANSELMO, T. et al. Accessory parameters in conformal mapping: Exploiting the isomonodromic tau function for painlevé vi. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, v. 474, p. 20180080, 08 2018. Cited 6 times in pages 12, 18, 61, 62, 63, and 66.

19 CUNHA, B. Carneiro da; NOVAES, F. Kerr–de sitter greybody factors via isomonodromy. *Physical Review D*, American Physical Society (APS), v. 93, n. 2, Jan 2016. ISSN 2470-0029. Disponível em: <http://dx.doi.org/10.1103/PhysRevD.93.024045>. Cited 3 times in pages 13, 20, and 29.

20 JIMBO, M.; MIWA, T. Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. ii. *Physica D: Nonlinear Phenomena*, v. 2, n. 3, p. 407 – 448, 1981. ISSN 0167-2789. Disponível em: <http://www.sciencedirect.com/science/article/pii/016727898190021X>. Cited 3 times in pages 13, 27, and 30.

21 GAMAYUN, O.; IORGOV, N.; LISOVYY, O. Conformal field theory of painlevé vi. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2012, n. 10, Oct 2012. ISSN 1029-8479. Disponível em: <http://dx.doi.org/10.1007/JHEP10(2012)038>. Cited in page 13.

22 GAMAYUN, O.; IORGOV, N.; LISOVYY, O. How instanton combinatorics solves painlevé vi, v and iiis. *Journal of Physics A: Mathematical and Theoretical*, IOP Publishing, v. 46, n. 33, p. 335203, Jul 2013. ISSN 1751-8121. Disponível em: <http://dx.doi.org/10.1088/1751-8113/46/33/335203>. Cited 5 times in pages 13, 36, 47, 58, and 61.

23 GAVRYLENKO, P.; LISOVYY, O. Fredholm determinant and nekrasov sum representations of isomonodromic tau functions. *Communications in Mathematical Physics*, Springer Science and Business Media LLC, v. 363, n. 1, p. 1–58, Aug 2018. ISSN 1432-0916. Disponível em: <http://dx.doi.org/10.1007/s00220-018-3224-7>. Cited 11 times in pages 13, 36, 37, 38, 39, 40, 42, 43, 46, 50, and 66.

24 GOHBERG, I.; GOLDBERG, S.; KRUPNIK, N. *Traces and determinants of linear operators*. [S.l.]: Birkhäuser Verlag, Basel, 2000. Cited in page 13.

25 SIMON, B. *Trace ideals and their applications*. 2. ed. [S.l.]: American Mathematical Society, Providence, 2005. Cited in page 13.

26 JOST, R.; PAIS, A. On the scattering of a particle by a static potential. *Phys. Rev.*, American Physical Society, v. 82, p. 840–851, Jun 1951. Disponível em: <https://link.aps.org/doi/10.1103/PhysRev.82.840>. Cited in page 13.

27 WILKINSON, D. Continuum derivation of the ising model two-point function. *Physical Review D - PHYS REV D*, v. 17, p. 1629–1636, 03 1978. Cited in page 13.

28 BORNEMANN, F. On the numerical evaluation of fredholm determinants. *Mathematics of Computation*, American Mathematical Society (AMS), v. 79, n. 270, p. 871–915, Sep 2009. ISSN 0025-5718. Disponível em: <http://dx.doi.org/10.1090/s0025-5718-09-02280-7>. Cited 4 times in pages 13, 48, 50, and 51.

29 FOKAS, A. S. et al. *Painlevé Transcendents: The Riemann-Hilbert Approach*. [S.l.]: United States of America: American Mathematical Society, 2006. Cited in page 15.

30 HILLE, E. *Ordinary Differential Equations in the Complex Domain*. [S.l.]: Jhon Wiley and Sons, 1976. Cited in page 15.

31 ZOLADEK, H. *The Monodromy Group*. [S.l.]: Germany: Birkhäuser, (Monografie Matematyczne), 2006. Cited in page 16.

32 SLAVYANOV, S.; LAY, W. *Special Functions: A Unified Theory Based on Singularities*. [S.l.]: United States: Oxford University Press, 2000. Cited in page 17.

33 GOLDMAN, W. Invariant functions on lie groups and hamiltonian flows of surface group representations. *Inventiones mathematicae*, v. 85, p. 263–302, 06 1986. Cited in page 17.

34 JIMBO, M. Monodromy problem and the boundary condition for some painleveacute; equations. *Publications of the Research Institute for Mathematical Sciences*, v. 18, n. 3, p. 1137–1161, 1982. Cited 3 times in pages 17, 36, and 64.

35 HILBERT, D. Mathematical problems. lecture delivered before the international congress of mathematicians at paris in 1900. translated by mary winston newson. *Bulletin of the American Mathematical Society*, v. 8, 01 2000. Cited in page 20.

36 BATEMAN, H. *Higher Transcendental Functions, volume III*. [S.l.]: Robert E. Krieger Publishing Company Malabar, FLORIDA, 1955. ISBN 0-89874-207- 2 (v. 111). Cited in page 24.

37 GARNIER, R. Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. *Annales scientifiques de l'École Normale Supérieure*, Elsevier, v. 3e série, 29, p. 1–126, 1912. Disponível em: <http://www.numdam.org/item/ASENS_1912_3_29___1_0>. Cited in page 25.

38 GARNIER, R. étude de l'intégrale générale de l'équation vi de m. painlevé dans le voisinage de ses singularités transcendantes. *Annales scientifiques de l'École Normale Supérieure*, Elsevier, v. 3e série, 34, p. 239–353, 1917. Disponível em: <http://www.numdam.org/item/ASENS_1917_3_34___239_0>. Cited in page 25.

39 SCHLESINGER, L. Über eine klasse von differentialsystemen beliebiger ordnung mit festen kritischen punkten. *Journal für die reine und angewandte Mathematik*, De Gruyter, Berlin, Boston, v. 1912, n. 141, p. 96 − 145, 1912. Disponível em: <https://www.degruyter.com/view/journals/crll/1912/141/article-p96.xml>. Cited in page 26.

40 FILIPUK, G. Isomonodromic deformations. 2012. Disponível em: <https://perso.math.univtoulouse.fr/jisom/files/2012/06/Filipuk_IsomDefs1.pdf.> Cited in page 26.

41 FUCHS, R. Über lineare homogene differentialgleichungen zweiter ordnung mit drei im endlichen gelegenen wesentlich singulären stellen. *Mathematische Annalen*, v. 63, p. 301–321, 1907. Disponível em: <http://eudml.org/doc/158289>. Cited in page 27.

42 OKAMOTO, K. *Isomonodromic Deformation and Painlevé Equations, and the Garnier System*. Université Louis Pasteur, Institut de Recherche Mathématique Avancée, 1981. (Institut de Recherche Mathématique Avancée Strasbourg: Publication). Disponível em: <https://books.google.com.br/books?id=1tHvOgAACAAJ>. Cited in page 27.

43 MIWA, T. Painleve Property of Monodromy Preserving Deformation Equations and the Analyticity of τ Functions. 10 1980. Cited 2 times in pages 29 and 34.

44 NOVAES, F.; CUNHA, B. C. da. Isomonodromy, painlevé transcendents and scattering off of black holes. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2014, n. 7, Jul 2014. ISSN 1029-8479. Disponível em: <http://dx.doi.org/10.1007/JHEP07(2014)132>. Cited in page 29.

45 PIATEK, M.; PIETRYKOWSKI, A. R. *Solving Heun's equation using conformal blocks*. 2017. Cited in page 29.

46 OKAMOTO, K. Studies on the painlevé equations. *Annali di Matematica Pura ed Applicata*, v. 146, p. 337–381, 1986. Cited 2 times in pages 29 and 30.

47 TSUDA, T. *Toda equation and special polynomials associated with the Garnier system*. 2004. Cited in page 30.

48 CUNHA, B. C. *Notes on Toda Chains. (Private Communication)*. 2017. Cited in page 30.

49 ITS, A. R.; LISOVYY, O.; PROKHOROV, A. Monodromy dependence and connection formulae for isomonodromic tau functions. *Duke Mathematical Journal*, Duke University Press, v. 167, n. 7, p. 1347–1432, May 2018. ISSN 0012-7094. Disponível em: <http://dx.doi.org/10.1215/00127094-2017-0055>. Cited in page 47.

50 AMADO, J. B.; CUNHA, B. Carneiro da; PALLANTE, E. Scalar quasinormal modes of kerr−ads5. *Physical Review D*, American Physical Society (APS), v. 99, n. 10, May 2019. ISSN 2470-0029. Disponível em: <http://dx.doi.org/10.1103/PhysRevD.99.105006>. Cited 3 times in pages 48, 52, and 66.

51 JULIA: A Fresh Approach to Numerical Computing. *SIAM Review*, v. 59, 2017. Disponível em: <https://julialang.org/research/julia-fresh-approach-BEKS.pdf>. Cited in page 48.

52 FREDHOLM, I. Sur une classe d'équations fonctionnelles. *Acta Math.*, Institut Mittag-Leffler, v. 27, p. 365–390, 1903. Disponível em: <https://doi.org/10.1007/BF02421317>. Cited in page 50.

53 HILBERT, D. Grundzüge einer allgemeinen theorie der linearen integralgleichungen. In: ______. *Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten.* Wiesbaden: Vieweg+Teubner Verlag, 1989. p. 8–171. ISBN 978-3-322-84410-1. Disponível em: <https://doi.org/10.1007/978-3-322-84410-1_1>. Cited in page 50.

54 NYSTRöM, E. J. Über die praktische auflösung von integralgleichungen mit anwendungen auf randwertaufgaben. *Acta Math.*, Institut Mittag-Leffler, v. 54, p. 185–204, 1930. Disponível em: <https://doi.org/10.1007/BF02547521>. Cited in page 50.

55 RICHMOND, H. W. On the Electrostatic Field of a Plane or Circular Grating Formed of Thick Rounded Bars. *Proceedings of the London Mathematical Society*, s2-22, n. 1, p. 389–403, 01 1924. ISSN 0024-6115. Disponível em: <https://doi.org/10.1112/plms/s2-22.1.389>. Cited in page 63.

56 PORITSKY, H. Potential of a charged cylinder between two parallel grounded planes. *Journal of Mathematics and Physics*, v. 39, n. 1-4, p. 35–48, 1960. Disponível em: <https://onlinelibrary.wiley.com/doi/abs/10.1002/sapm196039135>. Cited in page 63.

57 CROWDY, D. G. Uniform flow past a periodic array of cylinders. *European Journal of Mechanics - B/Fluids*, v. 56, p. 120 – 129, 2016. ISSN 0997-7546. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0997754615200490>. Cited in page 63.

58 CUNHA, B. C. da; CAVALCANTE, J. P. *Confluent conformal blocks and the Teukolsky master equation.* 2019. Cited in page 66.

APPENDIX A – FREDHOLM DETERMINANT CODES IN JULIA

A.0.1 Fourier method

```
using LinearAlgebra
using SpecialFunctions
using Plots

JULIA_NUM_THREADS=8
Np = 400

function frickejimbo(Pμ, Pμ)
    b = Pμ[1]*Pμ[2] - Pμ[1]*Pμ[3] - Pμ[2]*Pμ[4];
    c = (Pμ[1])^2 + (Pμ[2])^2 + (Pμ[1])^2 + (Pμ[2])^2 + (Pμ[3])^2 +
        (Pμ[4])^2 +(Pμ[1]*Pμ[2]*Pμ[3]*Pμ[4])-Pμ[1]*(Pμ[1]*Pμ[2]+Pμ[3]*Pμ[4]) -
        Pμ[2]*(Pμ[1]*Pμ[4]+Pμ[2]*Pμ[3])-4.0;
     return (-b - sqrt((b^2 - 4c)))/2
end

function pii(th1,th2,th3,th4,sig)
    return (gamma(1-sig)^2/gamma(1+sig)^2 *
            gamma(1+(th3+th4+sig)/2)/gamma(1+(th3+th4-sig)/2) *
            gamma(1+(th3-th4+sig)/2)/gamma(1+(th3-th4-sig)/2) *
            gamma(1+(th2+th1+sig)/2)/gamma(1+(th2+th1-sig)/2) *
            gamma(1+(th2-th1+sig)/2)/gamma(1+(th2-th1-sig)/2))
end

function ess(th1,th2,th3,th4,sig1,sig2,sig3)
    w1t = cospi(th2)*cospi(th3)+cospi(th1)*cospi(th4)
    w01 = cospi(th1)*cospi(th3)+cospi(th2)*cospi(th4)
    num = (w1t-cospi(sig2)-cospi(sig1)*cospi(sig3)) -
          (w01-cospi(sig3)-cospi(sig1)*cospi(sig2)) *exp(1im*pi*sig1)
    den = (cospi(th4)-cospi(th3-sig1)) *
          (cospi(th1)-cospi(th2-sig1))
    return num/den
end

function invseries(ser) #ser only being a 2x2 matrix vector
```

```
    inverse = zeros(ComplexF64, 2, 2, Np+1);
    inverse[:,:,1] = [ 1 0 ; 0 1 ];
    Threads.@threads for i = 2:(Np+1)
        A = -ser[:,:,i];
        Threads.@threads for j = 2:(i-1)
            A += - (ser[:,:,j]*inverse[:,:,(1+i-j)]);
        end
        inverse[:,:,i] = A
    end
    return inverse
end

function gee(sig,th1,th2,t0=1.0)
    psi = zeros(ComplexF64,2,2, Np+1);
    # wrong sign of th1 to recover Nekrasov expansion
    a = (sig-th1+th2)/2;
    b = (sig-th1-th2)/2;
    c = sig
    psi[:,:,1] = [ 1 0 ; 0 1 ];
    psi[:,:,2] = [ ((a*b)/c*t0) (-a*b/c/(1+c)*t0) ;
                   ((a-c)*(b-c)/c/(1-c)*t0) ((a-c)*(b-c)/(-c)*t0) ];
    Threads.@threads for p = 3:(Np+1)
        psi[1,1,p] = ((a+p-2)*(b+p-2)/((p-1)*(c+p-2))*psi[1,1,p-1]*t0)
        psi[1,2,p] = ((a+p-2)*(b+p-2)/((p-2)*(c+p-1))*psi[1,2,p-1]*t0)
        psi[2,1,p] = ((a-c+p-2)*(b-c+p-2)/((p-2)*(-c+p-1))*psi[2,1,p-1]*t0)
        psi[2,2,p] = ((a-c+p-2)*(b-c+p-2)/((p-1)*(-c+p-2))*psi[2,2,p-1]*t0)
    end
    return psi
end

function BuildA(sig,th1,th2)
    vecg = gee(sig,th1,th2)
    vecginv = invseries(vecg)
    bigA = zeros(ComplexF64,2Np,2Np)
    Threads.@threads for p = 1:Np
        Threads.@threads for q = 1:Np
            result = zeros(ComplexF64,2,2)
            if q + p <= Nf+1
                for r = 1:q
```

```
                result += vecg[:,:,p+r]*vecginv[:,:,q-r+1]
            end
        end
        bigA[(2p-1):(2p), (2q-1):(2q)] = result
    end
  end
  return bigA
end

function BuildD(sig,th1,th2,x,t)
    vecg = gee(sig,th1,th2,t);
    vecginv = invseries(vecg)
    bigD = zeros(ComplexF64,2Np,2Np)
    left = [ (1/x) 0 ; 0 1 ]
    right = [ (x) 0 ; 0 1 ]
    Threads.@threads for p = 1:Np
        Threads.@threads for q = 1:Np
            result = zeros(ComplexF64,2,2)
            if q + p <= Np+1
                for r = 1:p
                    result += -vecg[:,:,p-r+1]*vecginv[:,:,q+r]
                end
            end
            bigD[(2p-1):(2p), (2q-1):(2q)] = left*result*right
        end
    end
    return bigD
end

function tauhat(th,sig,esse,t)
    x = esse*pii(th[1],th[2],th[3],th[4],sig)*t^sig
    OpA = BuildA(sig,th[3],th[4])
    OpD = BuildD(-sig,th[2],th[1],x,t)
    id = Matrix{ComplexF64}(I,2Np,2Np)
    Fred =(id-OpD*OpA)
    factor = (t^(((sig[1]^2) - (th[1]^2) - (th[2]^2))/4))*((1-t)^(-th[2]*th[3]/2))
    return factor*det(Fred)
end
```

A.0.2 Quadrature method

```
using SpecialFunctions
using LinearAlgebra
using Base.Threads
using FastGaussQuadrature
using SingularIntegralEquations.HypergeometricFunctions

JULIA_NUM_THREADS=8
Np = 400
function frickejimbo(Pμ, Pμ)
    b = Pμ[1]*Pμ[2] - Pμ[1]*Pμ[3] - Pμ[2]*Pμ[4];
    c = (Pμ[1])^2 +(Pμ[2])^2 +(Pμ[1])^2 +(Pμ[2])^2 +(Pμ[3])^2 +(Pμ[4])^2 +
        (Pμ[1]*Pμ[2]*Pμ[3]*Pμ[4])-Pμ[1]*(Pμ[1]*Pμ[2]+Pμ[3]*Pμ[4])-
        Pμ[2]*(Pμ[1]*Pμ[4]+Pμ[2]*Pμ[3]);

     return (-b - sqrt((b^2 - 4c)))/2
end

function pii(th1,th2,th3,th4,sig)
    return (gamma(1-sig)^2/gamma(1+sig)^2 *
            gamma(1+(th3+th4+sig)/2)/gamma(1+(th3+th4-sig)/2) *
            gamma(1+(th3-th4+sig)/2)/gamma(1+(th3-th4-sig)/2) *
            gamma(1+(th2+th1+sig)/2)/gamma(1+(th2+th1-sig)/2) *
            gamma(1+(th2-th1+sig)/2)/gamma(1+(th2-th1-sig)/2))
end

function ess(th1,th2,th3,th4,sig1,sig2,sig3)
    w1t = cospi(th2)*cospi(th3)+cospi(th1)*cospi(th4)
    w01 = cospi(th1)*cospi(th3)+cospi(th2)*cospi(th4)
    num = (w1t-cospi(sig2)-cospi(sig1)*cospi(sig3)) -
          (w01-cospi(sig3)-cospi(sig1)*cospi(sig2)) *exp(1im*pi*sig1)
    den = (cospi(th4)-cospi(th3-sig1)) *
          (cospi(th1)-cospi(th2-sig1))
    return num/den
end

function ze(R,n,i)
    R*exp(2pi*im*n[i]/Np)
end
```

```
function psi(th1, th2, th3,z)
    result = zeros(Complex{Float64},2,2)
    a = (th1-th2+th3)/2
    b = (th1-th2-th3)/2
    c = th1
    result[1,1] = _2F1(a,b,c,z)
    result[1,2] = -a*b/(c*(1+c))*z*_2F1(1+a,1+b,2+c,z)
    result[2,1] = ((a-c)*(b-c))/(c*(1-c))*z*_2F1(1+a-c,1+b-c,2-c,z)
    result[2,2] = _2F1(a-c,b-c,-c,z)
    return result
end

function psiinv(th1, th2, th3,z)
    result = zeros(Complex{Float64},2,2)
    a = (th1-th2+th3)/2
    b = (th1-th2-th3)/2
    c = th1
    result[1,1] = _2F1(a-c,b-c,-c,z)
    result[1,2] = a*b/(c*(1+c))*z*_2F1(1+a,1+b,2+c,z)
    result[2,1] = -((a-c)*(b-c))/(c*(1-c))*z*_2F1(1+a-c,1+b-c,2-c,z)
    result[2,2] = _2F1(a,b,c,z)
    return (1-z)^(-th2).*result
end

function dpsi(th1,th2,th3,z)
    result = zeros(Complex{Float64},2,2)
    sigma = [ 0 0 ; 0 th1 ]
    A1 = [ ((th3^2-(th1-th2)^2)/(4*th1)) (-(th3^2-(th1-th2)^2)/(4*th1)) ;
           ((th3^2-(th1+th2)^2)/(4*th1)) (-(th3^2-(th1+th2)^2)/(4*th1)) ]
    param = psi(th1,th2,th3,z)
    result = (sigma*param-param*sigma)/z+param*A1/(z-1)
    return result
end

function BuildNA(sig, th1, th2)
    bigA = zeros(ComplexF64,2Np,2Np)
    n = [(i) for i in range(1,stop=Np)]
    id = Matrix{ComplexF64}(I,2,2)
```

```
    Threads.@threads for i = 1:Np
        Threads.@threads for j = 1:Np
            z = ze(R,n, i)
            zp = ze(R,n,j)
            psiz =psi(sig, th1, th2, z)
            psiinvzp = psiinv(sig,th1,th2,zp)
            if (i==j)
                dpsiz = dpsi(sig,th1,th2,z)
                bigA[(2i-1):(2i),(2i-1):(2i)] =(z/Nf).*(dpsiz*psiinvzp)
            else
                bigA[(2i-1):(2i),(2j-1):(2j)] =(sqrt(zp)*sqrt(z)/Np).*
                ( (psiz*psiinvzp)-id )./(z-zp)
            end
        end
    end
    return bigA
end

function BuildND(sig,th1,th2,x,t)
    bigD = zeros(ComplexF64,2Np,2Np)
    n = [(i) for i in range(1,stop=Np)]
    id = Matrix{ComplexF64}(I,2,2)
    left = [ (1/x) 0 ; 0 1 ]
    right = [ (x) 0 ; 0 1 ]
    Threads.@threads for i = 1:Np
        Threads.@threads for j = 1:Np
            z = ze(R,n, i)
            zp = ze(R,n,j)
            psiz =psi(sig, th1, th2, (t/z))
            psiinvzp = psiinv(sig,th1,th2,(t/zp))
            if (i==j)
                dpsiz = dpsi(sig,th1,th2,(t/z))
                bigD[(2i-1):(2i),(2i-1):(2i)] =( ((z/Nf)*(t/z^2)).*
                (left*dpsiz*psiinvzp*right) )
            else
                bigD[(2i-1):(2i) , (2j-1):(2j)] = ((sqrt(zp)*sqrt(z)/Nf)).*
                ( id -(left*psiz*psiinvzp*right) )./(z-zp)
            end
        end
```

```
    end
    return bigD
end

function tauhat(th,sig,esse,t)
    x = esse*pii(th[1],th[2],th[3],th[4],sig)*t^(sig)
    OpA = BuildNA(sig,th[3],th[4])
    OpD = BuildND(-sig,th[2],th[1],x,t)
    id = Matrix{ComplexF64}(I,2Np,2Np)
    Fred =(id-OpD*OpA)
    factor = (t^(((sig[1]^2) - (th[1]^2) - (th[2]^2))/4))*((1-t)^(-th[2]*th[3]/2))
    return factor*det(Fred)
end
```

A.0.3 2D Newton-Raphson Method

```
function newton2d(f,x,verb,tol=1e-15::Float64,maxsteps=100::Int64)
    h = 1e-6
    local counter = 0
    xnew = transpose(x)
    while true
        jac = zeros(ComplexF64,2,2)
        xold = xnew
        f0 = f(xold)
        fx = f(xold + [ h 0 ])
        fy = f(xold + [ 0 h ])
        jac[1,1] = (fx[1] - f0[1])/h
        jac[1,2] = (fy[1] - f0[1])/h
        jac[2,1] = (fx[2] - f0[2])/h
        jac[2,2] = (fy[2] - f0[2])/h
        step = inv(jac)*f0
        #xnew = xold-step
        xnew[1] = (xold[1] - step[1])[1]
        xnew[2] = (xold[2] - step[2])[1]
        counter += 1
        println( counter," ",Complex{Float64}(xnew[1]),"
        ",Complex{Float64}(xnew[2])," ",Float64(abs2d(f0)) )
        ( (counter > maxsteps) || ( abs2d(step) < tol)
        || (abs2d(f0) < tol ) ) && break
    end
```

```
        if counter >= maxsteps
            error("Did not converge in ", string(maxsteps), " steps")
    else
            xnew, counter
    end
end
```