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INTRODUCTION TO SUPERCONDUCTIVITY AND SELF-DUALITY AS A COOPERATION MECHANISM TO COMPLEXITY EMERGENCE

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INTRODUCTION TO SUPERCONDUCTIVITY AND SELF-DUALITY AS A COOPERATION MECHANISM TO COMPLEXITY EMERGENCE

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To my family of blood and heart, those who live and those who have said ‘see you soon’, Luiz, Austrina, Natanael, Marjorie, Lucas, Mariana, Arthur, Dionary, Carol, Joaquim, Joana, Charles, Raquel, Leonid, Lioudmila, Arkady, Alexei, Rita, Fr. Pedro, Serafim, the faithful of the Orthodox Church; and those yet to come.

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ABSTRACT

Initially we conduct a review of superconductivity and examine a variety of topics, including the Fermi-Landau theory, the generic Landau theory of phase transition with a focus on Ginzburg-Landau, the Fhrölich model, Bardeen-Cooper-Schrieffer, and Bogoliubov theories, as well as their relation to the coherent Glauber states. Next, we establish the connection between microscopic theories and GL, a result pioneered by Gor'kov, and recent developments in the Extended Ginzburg-Landau theory by A.Shanenko and A.Vagov *et al.* - a step beyond Gor'kov, providing a self-consistent expansion valid further away from the critical temperature. These results are reproduced by formulating an alternative time-saving method for computing higher-order Landau theories of superfluid phase transition (in the absence of the induction-field coupling). This is accomplished through the formulation of a diagrammatic dictionary and a concise collection of rules. The primary original contribution of this work, though, is the description of novel semi-analytic solutions to the self-dual superconducting solutions at the Bogomol'nyi point ($\kappa = 1/\sqrt{2}$) and their correspondence to the appearance of patterns similar to those in U.Krägeloh's (1969) pioneering measurement in "Flux line lattices in the intermediate state of superconductors near $\kappa = 1/\sqrt{2}$". The semi-analytic solutions are coined stripe, bubble and donut. They exhibit stable thermodynamics beyond $\kappa = 1/\sqrt{2}$, in the 'intertype' domain, as we predict from the Extended Ginzburg Landau theory. We observe the results in the time-dependent Ginzburg-Landau model starting from configurations similar to the semi-analytic solutions as *ab initio ansatz*. The time-evolved solutions qualitatively coincide with Krägeloh's experimental results. The obtained results allow us to cast doubt on a widely accepted view of how complexity develops. We present a phenomenology in which 'cooperation' rather than 'competition' is the appropriate keyword for justifying the complexity emergence.

Keywords: superconductivity; complexity emergence; self-duality; extended Ginzburg-Landau; Bogomol'nyi; Krägeloh.

RESUMO

Inicialmente conduzimos uma revisão da supercondutividade e examinamos uma variedade de tópicos, incluindo a teoria de Fermi-Landau, a teoria genérica de Landau de transição de fase com foco em Ginzburg-Landau, o modelo Fhrölich, as teorias de Bardeen-Cooper-Schrieffer e Bogoliubov e sua relação com os estados coerentes de Glauber. Em seguida, estabelecemos a conexão entre teorias microscópicas e GL, resultado pioneiro de Gor'kov, e desenvolvimentos recentes na teoria Extended Ginzburg-Landau (EGL) por A.Shanenko e A.Vagov *et al.* - um passo além de Gor'kov, fornecendo uma expansão auto-consistente válida mais longe da temperatura crítica. O resultado é reproduzido pela formulação de um método eficiente para calcular teorias de Landau de ordem mais alta para transição de fase superfluida (na ausência do acoplamento de campo de indução). Isso é realizado pela construção de um dicionário diagramático e uma coleção concisa de regras. A principal contribuição original deste trabalho, no entanto, é a descrição de novas soluções semi-analíticas para as soluções supercondutoras auto-duais no ponto Bogomol'nyi ($\kappa = 1/\sqrt{2}$) e sua correspondência com o aparecimento de padrões semelhantes aos da medição pioneira de U.Krägeloh (1967) em "Flux line lattices in the middle state of superconductors near $\kappa = 1/\sqrt{2}$". As soluções semi-analíticas são denominadas listra, bolha e rosca. Elas exibem termodinâmica estável além de $\kappa = 1/\sqrt{2}$, no domínio 'intertype', como prevemos a partir da teoria Extended Ginzburg Landau. As simulações no modelo de Ginzburg-Landau dependente do tempo são executadas a partir de configurações semelhantes às soluções semi-analíticas como *ab initio ansatz*, a solução evoluída no tempo coincide qualitativamente com os resultados experimentais de Krägeloh. Os resultados obtidos permitem questionar uma visão amplamente aceita de como a complexidade se desenvolve. Apresentamos uma fenomenologia em que 'colaboração' ao invés de 'competição' é a palavra-chave mais adequada para justificar o surgimento da complexidade.

Palavras-chave: superconductividade; emergência de complexidade; auto-dualidade; Ginzburg-Landau extendido; Bogomol'nyi; Krägeloh.

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1 INTRODUCTION

At the turn of the nineteenth century, there was a rush to develop liquids capable of being cooled to extremely low temperatures. Karmelingh Onnes and coworkers were the first to control this technique (ONNES, 1911), lowering the temperature of metals (initially, mercury) to a point where unexpected behavior, the 'superconductivity phase' of matter, could be observed in 1911 at Leiden University. The disappearance of resistance, or more precisely, a significant decrease in resistance relative to the preceding phase, was the first feature reported in the discovery of this novel phase of matter. Prior to the Meissner-Oschsenfeld discovery (MEISSNER; OCHSENFELD, 1933), the idea of a superconductor as equivalent to an ideal conductor had not been disproved. An ideal conductor does not expel the magnetic field from within when the temperature is lowered (after the magnetic field is turned on), which is a direct result of electromagnetic laws. Contrary to this, Meissner-Oschsenfeld experiments demonstrated that the magnetic field can be ejected even after the metals have been cooled. These materials with 'super' conductance were suggested to represent a state of matter, as their thermodynamic 'history' along a $H-T$ diagram would be irrelevant. The 'superconductor' state of matter has been the subject of a century of scientific investigation. This dissertation aims to present some of the techniques developed over the years for treating the 'superconductivity' phenomenology, as well as to share contributions that have not been reported previously - to our best knowledge.

When a magnetic field exceeds a certain threshold, it penetrates the superconductor in the form of vortices. Each vortex is a polar-symmetric configuration that describes the transition within the superconductor from the normal (vortex center) to the superconducting phase (vortex outwards). The material is still considered to be in a superconductor state, albeit with properties distinct from those of the Meissner-Ochsenfeld state. Abrikosov proposed the existence of the vortex solution and went beyond pioneering the study of its stability and optimal configuration (ABRIKOSOV, 1952; ABRIKOSOV, 1957). He recognized the existence of two superconducting phases, depending on the ratio of the spatial characteristic lengths governing scales the induction (λ) and condensation (ξ). The characteristic length of condensation is the linearly approximate distance for which the coherence of the collective macroscopic wave-function is kept. The charateristic length of the induced field is the linearly approximate distance for which the induction is sustained. These quantities are expressed in the Ginzburg-Landau parameter $\kappa = \lambda/\xi$. If $\xi >> \lambda$, the condensation rules the sample, and any magnetic effects are restricted to the edges of the superconductor. When the Meissner-Ochsenfeld state - the perfect superconducting state, no magnetic field - is present in the bulk of one of the phases, the superconductor is designated as of type I. In type I, a domain wall separates the normal from the superconducting domain in the boundaries of the sample - sufficing $\kappa < 1/\sqrt{2}$ in the GL theory. If the $\lambda >> \xi$, the induction is better sustained than the condensation; this regime describes the penetration of currents within the sample. In the second kind of superconductivity, or of type II, the current enters in the system in the form of vortices, tending to form an 'Abrikosov' lattice (for a review, (ABRIKOSOV, 2003),(ABRIKOSOV, 2004)) - sufficing $\kappa > 1/\sqrt{2}$ in the GL theory. Previously, it was accepted that a superconductor would only exist in one form or another, leaving little room for experimentation. Experimental techniques have demonstrated that materials can be doped, the effect of this being the harness of the κ value (BRANDT; DAS, 2011). The theory of GL appeared to be complete, but there was still an unjustified phenomenology unnoticed by the time, the appearance of exotic quasi-1d chains of vortices not classified in either type I or type II, such as those reported in U. Krägeloh experiment in single-band materials (KRAGELOH, 1969).

The appearance of stable exotic configurations that do not fall into either type I or type II superconductivity has been somewhat imprecisely justified in what has been dubbed 'type 1.5' superconductivity. This theory requires the presence of two-band material, in addition to the pertinent criticism about its uncontrolled accuracy, which results in the theory's failure to safeguard its results ((KOGAN; SCHMALIAN, 2011)). The criticism prompted A.Shanenko and A.Vagov *et al.* to develop a method for providing a reliable expansion, either for single or multi-band material, in a material-independent theory, which consists of a self-consistently expansion of the microscopic theory further away from the critical temperature, (SHANENKO et al., 2011; VAGOV; SHANENKO A, 2012; VAGOV et al., 2016), in the 'Extended Ginzburg-Landau Theory' (EGL). This approach established the stability of exotic patterns in between types I and II in a region of the phase diagram dubbed 'intertype', existent in both single-band and multi-band materials (VAGOV et al., 2016). The developments of A.Shanenko and A.Vagov *et al.* undoubtedly resulted in a paradigm shift in our understanding of superconductivity on a material-independent level.

Perhaps the manuscript's most significant technical contribution is the formulation of novel semi-analytic solutions to the Ginzburg-Landau theory at the Bogomol'nyi point ($\kappa = 1/\sqrt{2}$, B-point). These are believed to be the first solutions since A.Abrikosov's vortex proposal in his seminal work. We do not assert that the solutions are as fundamental as the vortex solution in the sense that they can be constructed by vortices within; however, they are representative of the patterns emerging in the intertype region. Additionally, the semi-analytic solutions are demonstrated to be stable (thermodynamically) in the phase diagram; they are more favorable structures than the Meissner-Ochsenfeld phase, as justified by EGL theory, in a large region of the phase diagram. The majority of introductory textbooks on superconductivity focus on the subject's treatment between types I and II. We suggest the possibility of a review of this literature to include a description of the intertype domain, which provides a richer material-independent phenomenology. To accomplish this, we believe it is prudent to present representative solutions to the variety of complexity found in the intertype domain, bubble, stripe and donut. When simulations in the time-dependent Ginzburg-Landau model are run, these *ab initio* solutions retain their essential properties and justify the appearance of patterns quite similar to those in the original U.Krägeloh (1967) experiment near the B-point. Additionally, we observe the cohexistence of at least three phases of matter. In the paragraph to follow we define complexity and explain how the technical contribution may be connected to a fundamental shift on the understanding of complexity emergence.

Complexity is a term that refers to the behavior of a system whose components interact in a variety of ways while adhering to a set of rules without external modification. The concept of organization within complexity refers to the presence of naturally occurring correlations between the system's components, solely as a result of the evolution of the rules, justifying the alternative term 'self-organization'. Self-duality is a generic term for describing the first-order derivative relations coupling two variables, as they feed each other recursively. Though the Ginzburg-Landau is governed by two coupled second-order differential equations, in a very specific but relevant scenario ($\kappa = \frac{1}{\sqrt{2}}$), the solution can be reduced to self-dual relations, at the 'Self-dual' or 'Bogomol'nyi' point. The evidence of the complexity appearance in the vicinity of the self-dual point is reported in the EGL theory followed by a κ expansion (VAGOV et al., 2016), suggesting the stability of solutions not falling in either types I or II, as well as with accurate numerical treatments (VAGOV et al., 2016; CÓRDOBA-CAMACHO et al., 2016; CÓRDOBA-CAMACHO et al., 2018; VAGOV et al., 2020).

The emergence of complexity in nature raises a critical question about how spontaneous self-organized patterns emerge (TURING, 1990; CROSS; HOHENBERG, 1993; SEUL; ANDELMAN, 1995). It is now widely accepted that spontaneous structures form as a result of length-scale competition (SEUL; ANDELMAN, 1995). We present a qualitatively distinct framework in which the key word is 'collaboration'. We consider the usual equations for the superconducting system, consisting of coupled scalar and gauge fields with a self-dual critical point. The self-dual state is infinitely degenerate, including patterns with equal scalar and gauge spatial lengths (TARANTELLO, 2008; VAGOV et al., 2016; VAGOV et al., 2020). Two entities acting in cooperation generate an array of exotic superstructures composed of bubbles, stripes, and donuts of the scalar condensate. We, therefore, suggest a shift on the requirements for the emergence of complexity.

Another original contribution of this work is the formulation of a time-saving route for computing higher-order phase transition theories - based on the EGL theory. In general, deriving the representation of higher-order theories would require a significant amount of time for both computing and identifying the appropriate terms according to the scheme proposed by the authors of the EGL formalism. In contrast, we significantly reduce the effort by using a diagrammatic dictionary and a concise collection of rules. We were particularly inspired by Feynman's extremely effective dictionary and diagrams, though our diagrammatic approach bears little resemblance to his diagrams. We emphasize that the strategy we develop has the significant limitation of not being able to account for the presence of a magnetic field - up to the time this dissertation was submitted. Thus, the set of rules and dictionary are valid for E$^{(n)}$L theory (an acronym introduced to address the n-order Extended-Landau theory) which departs from microscopic superconductivity theory. That is, such a theory would, in principle, describe superfluidity in the absence of magnetic induction. We end up conjecturing if the theory is not representative of other systems that do not have a coupling between the order parameter and magnetic induction, such as theories of phase transition in pure magnetic systems.

In the first part of this dissertation we review aspects of superconductivity (chap. 2 to 4). In the second part of this dissertation (chap. 5 and 6), we provide the technical and fundamental contributions of this writing. As with any written text, we have chosen a few topics. We do not intend to diminish the importance of many of the pioneering works in the field, which will not be discussed in detail in the subsequent writings but are already covered in a number of current textbooks on superconductivity. For example, we owe thanks to the brothers London (LONDON; LONDON, 1935) seminal work for providing the first explanation for the Meissner-Ochsenfeld effect while introducing a wave-function description in superconductivity, a brilliant and successful modeling of the superconducting state; their seminal work was critical for the field's development and is still used by modern scientists. However, techniques in superconductivity have evolved over time, and these advancements have enabled the development of more sophisticated tools for understanding the physics of superconductors. On one hand, we do not wish to reconstruct every detail on the historical evolution of superconductivity; on the other hand, we do not wish to begin writing at the end of the history, from a very narrow and technical perspective. This would completely obscure the advancements of critical concepts that remain central to contemporary science comprehension. Thus, we chose the subjects that we credit to be the most pertinent in introducing both the concepts and language necessary for comprehending modern techniques in superconductivity.

The second chapter discusses the Fermi-Landau theory, successfully explaining the superfluidity phenomena observed in liquid helium. This is a significant advance in condensate physics, not only because it models the discrepancy between a system of interacting and non-interacting particles, but since it provides concepts that pervade every branch of contemporary physics. In this chapter, we provide an overview of Feynman diagrams, an intuitive language through which physics is frequently expressed or referred to. Apart from the remarkable success of the theory and its concepts, this can be viewed as the mathematical foundation for the description of superconductivity in Bogoliubov's seminal work developed decades later. This justifies this manuscript's starting point to be the Fermi-Landau theory.

The third chapter discusses another Landau theory, which is based on phenomenological general principles regarding how a theory should behave near the phase transition, regardless of its particular microscopic features. In this chapter, we introduce the concept of U(1) symmetry breaking (Bogoliubov), a mechanism that enables the appearance of nonzero anomalous averages and thus the superconducting state, in connection with the results discussed in chap.3. We establish a link between U(1) symmetry breaking and the emergence of collective coherent Glauber states, thereby establishing superconductivity as a link between quantum and classical behavior. Additionally, we present a recurrent criterion for the thermodynamic stability of solutions.

In the fourth chapter, we consider Fhrölich's contribution to justifying the lattice's vibration as mediating an attractive force between two electrons. At low temperatures, this theory is shown to approach the Landau-Fermi theory, which is an exemplar of the Bardeen-Pines-like Hamiltonian, a theory preceding the Bardeen-Cooper-Schriefer (BCS) theory. Cooper used Fhrölich's results to analyze the stability of two electrons together, which was calculated to be stable. The Hamiltonian produced by Bardeen, Cooper, and Schrieffer justified the appearance of the Cooper pair as the ground state on the basis of the variational principle. However, using Bogoliubov's elegant treatment, the BCS result can be recovered in the mean-field approximation. We demonstrate it for a specific case: the correspondence between BCS's with the Heaviside-cutoff modeling and Bogoliubov theory with an uniform gap. Bogoliubov's results are obtained from the Landau-Fermi liquid theory. Additionally, we present an alternative formulation of superconductivity, incorporating the path integral treatment via the Hubbard-Stratonovitch transformation. This chapter is primarily concerned with the presentation of the uniform gap as the simplest way to introduce the techniques applied in the chapter to follow.

In the fifth chapter, we consider the case of a spatially dependent gap, not constrained to the limitations of chapter 4. Through a simple diagrammatic scheme, we retrieve the authors of EGL's results with an alternative representation to the expansion of the theory. This provides the path to the computation of higher orders in the absence of a magnetic field (E$^{(n)}$L theory) with ease in comparison to the usual technique. In the sixth chapter, we present novel semi-analytic self-dual solutions ($\kappa = \frac{1}{\sqrt{2}}$) representative of the intertype domain, and demonstrate their stability in the intertype region. In this chapter we compare the simulations to Krägeloh's experiment, with simulations at the Time Dependent Ginzburg-Landau TDGL equation (at the B-point) and the novel solutions as *ab initio ansatz*. It is in this chapter that we suggest a fundamental shift in the understanding of complexity emergence. The emergence of complexity is attributed to the presence of more than one competing length-scale. We, on the other hand, provide a phenomenology in which infinitely degenerate complex states may appear from the 'cooperation' of order-parameters, acting in 'unison' as if in a single entity.

2 LANDAU-FERMI LIQUID THEORY: THE BIRTH OF CONCEPTS

2.1 PROLOGUE

The study of the Landau-Fermi liquid theory is particularly useful since it applies the concept of 'adiabaticity' and introduces others such as 'quasi-particle', 'holes', 'renormalization', which permeates every branch of contemporary condensate-matter physics. It provides a successful description of a nontrivial many-body system with interactions included. Due to his contribution (for a review on his contribution, (HAAR, 1965), to mention a few of the original writings, (LANDAU, 1936; LANDAU; GINZBURG, 1950; LANDAU, 1959)), Landau has generated a spring of joy in condensate matter physics in the many years that have followed. From his successful description of superfluidity, he was awarded with the Nobel prize. The purpose of this chapter is to provide the content of the aforementioned concepts, most of which Landau has himself forged [1]. The concepts will be applied throughout this manuscript.

The non-interacting Sommerfeld model (SOMMERFELD, 1928) was the only theory available for a degenerate Fermi liquid. It predicts the ratio between the specific heat per temperature and the magnetic susceptibility to be

$$W = \frac{\chi_s}{\gamma} = 3\frac{\mu_F}{\pi k_B} \tag{2.1}$$

also known as the Wilson ratio (COLEMAN, 2006). For a gas of a non-interacting (noble) particles at large distances, with no forces of repulsion, the Sommerfeld model would be expected to hold. However, for liquid helium, ^{3}He, the ratio was shown to be about three times larger. The reason for this is that interactions between molecules occur due to collisions of the electronic clouds, short-range interactions, which exists even in neutral molecules. To understand this, it is necessary to go beyond the free theory, to turn on adiabatically small local interactions exciting the electrons in the uttermost shells. In developing Landau theory, it becomes clear that the main force contribution is linked to spin exchange. Molecules with non-null net spin tend to have its Wilson ratio increased.

A quasi-state is the eigenstate of the adiabatically (Appendix A.1) excited system. The concept of adiabaticity assures the life-time of the excitation ($\tau_{1/2}$) to be superior to the time-scale involved in the interaction switch on (τ). Landau has shown, in an heuristic derivation

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[1] Apart from the adiabaticity concept, whose origin is attributed to Murray-Gell-Man and Francis-Low in developing their theorem (GELL-MANN; LOW, 1951) (Appendix A.4)

that $\tau_{1/2} \propto 1/\varepsilon^2$. Since $\tau \propto 1/\varepsilon$ (Appendix A.1), he concluded $\tau_{1/2}/\tau \propto 1/\varepsilon$, with $\varepsilon = E - \mu_F$ the grand-canonical energy at zero temperature (ABRIKOSOV; KHALATNIKOV, 1959; LANDAU, 1959). Therefore, there are always excited particles placed near enough the Fermi-surface that develop a life-time superior to the time of the switch-on of the interaction, whatever the (adiabatic) switch-on time. The number of particles in the excited state near the Fermi-surface is kept constant. Hence, following from the Heisenberg equation, the particle-number operator commutes with the Hamiltonian; any tentative Hamiltonian describing the interaction between particles in the vicinity of the Fermi-surface is to be considered as a function of the particle-number operator. The other particles further away from the Fermi-energy level are immaterial, they do not contribute to the overall interaction, as these rapidly decay to the ground-state. This fact is also in excellent agreement with the chemistry we know, where the uttermost shells are the key in the description. One of the many Landau-theory merits is to treat the vicinity of the Fermi-surface as representative of the entire system configuration.

Before proceeding we will establish the quasi-particle and quasi-hole concepts which are particular quasi-states of the system. These are the elementary states on which more general quasi-states of the Landau-Fermi theory are built upon. Preceding the turning on of interactions,

$$\Psi_0 \equiv |\{n_{\boldsymbol{p}\sigma}\}\rangle \text{ such that } n_{\boldsymbol{p}\sigma} = \begin{cases} 1,\, p < p_F \\ 0, \text{ otherwise} \end{cases} \tag{2.2}$$

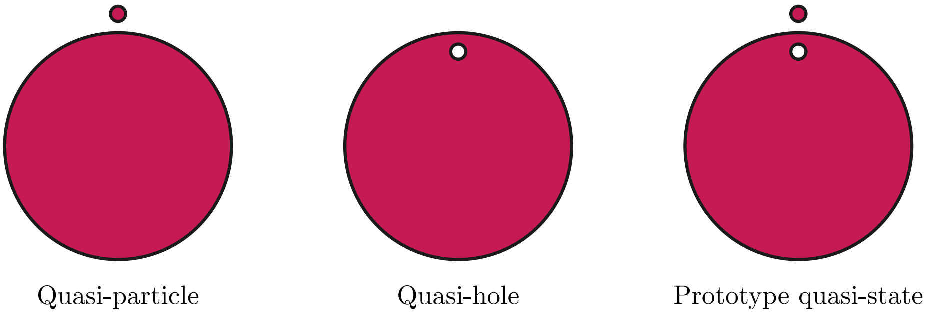
at zero temperature, where $n_{\boldsymbol{p}\sigma}$ denotes the number of particles with momentum $\boldsymbol{p}$ and spin orientation σ; p_F defines the Fermi-level momentum. We ponder on the state

$$\Psi_{\boldsymbol{p}_0,\sigma_0} \equiv |\{n_{\boldsymbol{p}\sigma}\}\rangle \text{ , such that } n_{\boldsymbol{p}\sigma} = \begin{cases} 1, \text{ if } p < p_F \text{ or } p = p_0,\ \sigma = \sigma_0 \text{ , with } p_0 > p_F \\ 0, \text{ otherwise} \end{cases} \tag{2.3}$$

which is the equivalent of adding a particle to the system. Due to Landau observation, if a particle remains close enough to the Fermi-surface, it tends to form a particle with arbitrary long life-time. Therefore, these are prototype eigenstates of the adiabatically excited Hamiltonian.

The quasi-particle state consists in adding an electron above the Fermi-shell. Such fact is stable indeed, as $\varepsilon_{\boldsymbol{p}_0} = E_{\boldsymbol{p}_0} - \mu > 0$ (μ, is the nuclei-electron bounding energy, or chemical potential, $E_{\boldsymbol{p}_0}$ defines the electronic energy in the absence of the nuclei bounding; $\varepsilon_{\boldsymbol{p}_0}$ defines the thermodynamic grand-potential at zero temperature). The converse, adding an electron below the Fermi-shell is not a stable procedure in a free theory. It is often convenient to treat the absence of an electron in terms of the 'hole' concept. If we remove an electron below the Fermi-sea we are said to have created a hole. To remove an electron below the Fermi-surface is a stable procedure, as we are removing a negative energy contribution, contrary to the removal above the Fermi-surface. With this observation, in a free theory, the locus the hole survive is below the Fermi-energy, whereas excited particles survive above the Fermi-shell. By allowing for pair-wise interaction, though, the image of restricted regions for particles and holes above and below the Fermi surface is blurred. This may be understood more seriously in chap.4, section 5.

Figure 1 – Particle and hole excitations, and a prototype excitation mixing both - sketch with the Fermi-sphere.



Source: The author

2.2 FREE ENERGY

For the quasi-particle to remain an eigenstate, we consider the deviation of the Hamiltonian from its free form, which is a function of the conserved quantity. Landau considered small deviations in the number of particles. (LANDAU, 1957; BAYM; PETHICK, 2008; COLEMAN, 2015)

$$\delta\varepsilon(\{n_{(\sigma,\mathbf{p})}\}) = \frac{\delta\varepsilon}{\delta n_{\mathbf{p}\sigma}}|_{\mathbf{p}=\mathbf{p}_F}\delta n_{\mathbf{p}\sigma} + \frac{1}{2}\frac{\delta^2\varepsilon}{\delta n_{\mathbf{p}\sigma}\delta n_{\mathbf{p}'\sigma'}}|_{\mathbf{p}=\mathbf{p}_F,\mathbf{p}'=\mathbf{p}'_F}\delta n_{\mathbf{p}\sigma}\delta n_{\mathbf{p}'\sigma'} \tag{2.4}$$

The first term is identified as the contribution due to the isolated excitation of a given quasi-particle. When one excites the quasi-particle or quasi-hole, the energy modifies approximately as

$$\frac{\delta\varepsilon}{\delta n_{\mathbf{p}\sigma}}|_{\mathbf{p}=\mathbf{p}_F} = E_{\mathbf{p}\sigma} - \mu \tag{2.5}$$

where $E_{\mathbf{p}\sigma}$ accounts for the energy of the excited-particle above the Fermi-surface for a positive particle-number variation, below the Fermi-surface for a negative variation (or positive hole-number variation), such that the state are forced to be stable. The second term accounts for the interaction between particles in a thin shell around the Fermi-surface. It is this term which is the cornerstone of the Landau theory, presenting non-null contributions when there is the variation of two-particle at different states.

The interaction part of the Hamiltonian is

$$H_I = \frac{1}{2}\sum_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}\delta n_{\boldsymbol{p}\sigma}\delta n_{\boldsymbol{p}'\sigma} \tag{2.6}$$

$$f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} = \frac{1}{2}\frac{\delta^2\varepsilon}{\delta n_{\boldsymbol{p}\sigma}\delta n_{\boldsymbol{p}'\sigma'}}|_{\boldsymbol{p}=\boldsymbol{p}_F,\boldsymbol{p}'=\boldsymbol{p}'_F} \tag{2.7}$$

To account for the grand-potential at finite temperature, $\varepsilon \to F \equiv \varepsilon - TS$ (T defines the temperature and S the entropy)

$$\delta F = \sum_{\boldsymbol{p}\sigma}(E_{\boldsymbol{p}\sigma} - \mu)\delta n_{\boldsymbol{p}\sigma} + \frac{1}{2}\sum_{\boldsymbol{p}\sigma\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma\boldsymbol{p}'\sigma'}\delta n_{\boldsymbol{p}\sigma}\delta n_{\boldsymbol{p}'\sigma'} + T\frac{\delta S}{\delta n_{\boldsymbol{p}\sigma}}\delta n_{\boldsymbol{p}\sigma} \tag{2.8}$$

by regarding the chemical potential as independent of the number of excited particles added or removed and the approximation ,

$$S = \overbrace{k_B\sum_{\boldsymbol{p}\sigma}[n_{\boldsymbol{p}\sigma}\ln n_{\boldsymbol{p}\sigma} + (1-n_{\boldsymbol{p}\sigma})\ln(1-n_{\boldsymbol{p}\sigma})]}^{\text{Non-interacting Fermi-gas entropy}} \tag{2.9}$$

The entropy can be proved to evolve smoothly as it is proportional to the logarithmic function of the partition function; which in turn, is the trace of the expression defining the adiabaticity (appendices A.1 and A.5). Hence it follows the approximate consideration that the entropy with low-lying interactions equals itself in the absence of interactions. [2]

2.3 FEED-BACK EFFECTS AND SELF-ENERGY

To compute the density of states, we consider the Fermi-Dirac distribution to account for the distribution of particles. This is indeed what is obtained by minimizing the free energy with respect to $\delta n_{\boldsymbol{p}\sigma}$. The difference is that the energy comprises an interaction part beyond the kinetic term, as to include the interaction between particles.

$$n_{\boldsymbol{p}\sigma} = (1 + \exp[\beta\varepsilon_{\boldsymbol{p}\sigma}])^{-1}$$

$$\varepsilon_{\boldsymbol{p}\sigma} = \overbrace{E_{\boldsymbol{p}\sigma} - \mu}^{\equiv\varepsilon^{(0)}_{\boldsymbol{p}\sigma}} + \overbrace{\sum_{\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}\delta n_{\boldsymbol{p}'\sigma'}}^{\equiv\delta\varepsilon^{(I)}_{\boldsymbol{p}\sigma}} \tag{2.10}$$

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[2] From appendices A.5 and A.1, $\frac{Z}{Z_0} \sim \exp\left[-\frac{\varepsilon}{\tau}\right] \sim \exp\left[-\varepsilon^2\right]$. Hence, $TS - TS_0 \propto \varepsilon^2$. But the expression (2.8) is of the order of ϵ, such that we may assume $S \sim S_0$ for producing a reliable equation with terms of the same accuracy order.

Hence,

$$n_{\boldsymbol{p}\sigma}(\varepsilon_{\boldsymbol{p}\sigma}) = n_{\boldsymbol{p}\sigma}(\varepsilon^{(0)}_{\boldsymbol{p}\sigma} + \delta\varepsilon^{(I)}_{\boldsymbol{p}\sigma}) = n_{\boldsymbol{p}\sigma}(\varepsilon^{(0)}_{\boldsymbol{p}\sigma}) + n'_{\boldsymbol{p}\sigma}(\varepsilon^{(0)}_{\boldsymbol{p}\sigma})\delta\varepsilon_{\boldsymbol{p}\sigma} \text{ Therefore, } \delta n_{\boldsymbol{p}\sigma} = n'_{\boldsymbol{p}\sigma}(\varepsilon^{(0)}_{\boldsymbol{p}\sigma})\delta\varepsilon_{\boldsymbol{p}\sigma} \tag{2.11}$$

For which we obtain a recursion rule

$$\varepsilon_{\boldsymbol{p}\sigma} = \varepsilon^{(0)}_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} n'_{\boldsymbol{p}\sigma}(\varepsilon^{(0)}_{\boldsymbol{p}\sigma})\delta\varepsilon_{\boldsymbol{p}'\sigma'} \tag{2.12}$$

One may state this in words by saying that a modification in the energy of the system causes a modification in the interaction, which in converse causes a change in the Fermi-energy, and hence on. We may produce an arbitrary order of correction. This is often said to be a 'feedback effect' or 'self-energy', for the particle changes its energy indirectly. The manifestation of this process owns its existence to the pair-wise interaction.

In particular, near the zero temperature, the paramagnetic effects can be neglected for reasonably small magnetic fields, and $n_{\boldsymbol{p}\sigma} = \theta(-\varepsilon_{\boldsymbol{p}})$ for any σ. Hence,

$$\delta\varepsilon_{\boldsymbol{p}\sigma} = \delta\varepsilon^{(0)}_{\boldsymbol{p}\sigma} - \sum_{\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} \delta(\varepsilon^{(0)}_{\boldsymbol{p}'})\delta\varepsilon_{\boldsymbol{p}'\sigma'} \tag{2.13}$$

Every particle is accounted for in a thin shell comprising the Fermi-surface,

$$N = 2\sum_{\boldsymbol{p}} \theta(-\varepsilon_{\boldsymbol{p}}) = \frac{2V}{(2\pi\hbar)^3}\int (4\pi p_F^2) d\varepsilon |\frac{dp}{d\varepsilon}|(\varepsilon)\theta(-\varepsilon) = V\int d\varepsilon \frac{p_F^2}{\pi^2\hbar^3}|\frac{dp}{d\varepsilon}|(\varepsilon)\theta(-\varepsilon) \tag{2.14}$$

Therefore,

$$N(\varepsilon) = \frac{p_F^2}{\pi^2\hbar^3}\frac{1}{|\frac{d\varepsilon}{dp}|} = 2\sum_{\boldsymbol{p}} \delta(\varepsilon - \varepsilon_{\boldsymbol{p}}) \tag{2.15}$$

The definition for an effective mass is provided

$$\frac{\boldsymbol{p}}{m^*} = \nabla_{\boldsymbol{p}}\varepsilon^*_{\boldsymbol{p}} \text{ with } \boldsymbol{p} \in \{\text{F.S}\} \tag{2.16}$$

The acronym F.S denoting the Fermi-Surface. In particular, for the exact limit of $\varepsilon \to 0$ ($\boldsymbol{p} \in \{\text{F.S}\}$), it it follows an effective density of state

$$N^*(0) = \frac{p_F m^*}{\pi^2\hbar^3} = 2\sum_{\boldsymbol{p}} \delta(\varepsilon_{\boldsymbol{p}}) \tag{2.17}$$

Before proceeding we state the identity

$$2\sum_{\boldsymbol{p}} w(\boldsymbol{p})\delta(\varepsilon_{\boldsymbol{p}}) = N^*(0)\int \frac{d\Omega_{\boldsymbol{p}}}{4\pi} w(\boldsymbol{p}) \tag{2.18}$$

which is verified from (2.17). In particular, if it were not for the pair-wise contribution, $m^* = m$, since, in the absence of the interaction, the linearization of the kinetic term near the Fermi-surface yields $\varepsilon_{\boldsymbol{p}} = \boldsymbol{p}(\boldsymbol{p} - \boldsymbol{p}_F)/2m = \boldsymbol{p}_F^2(1 - \cos\theta_{\boldsymbol{p},\boldsymbol{p}_F})$, with $\theta_{\boldsymbol{p},\boldsymbol{p}_F}$ the scattering angle, or angle shift, due to the switch-on of the interaction.

Landau Parameters

The interaction is, in general, considered in its simpler invariant form under spin rotation,[3]

$$f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} = f^s_{\boldsymbol{p},\boldsymbol{p}'} + f^a_{\boldsymbol{p},\boldsymbol{p}'}\sigma\sigma' \tag{2.19}$$

In the Landau theory, we are sitting in a thin shell in the vicinity of the Fermi level, the magnitude of $\boldsymbol{p}$ and $\boldsymbol{p}'$ being about the same. The dependence of the interaction is on the angle between the vectors lying in the Fermi surface, $f^{\{s,a\}}_{\boldsymbol{p},\boldsymbol{p}'} = f^{\{s,a\}}(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'})$. It is usual to define

$$F^{\{s,a\}}(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'}) = N^*(0)f^{\{s,a\}}(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'}) \tag{2.20}$$

The Landau parameters are defined as the coefficients $F^{\{s,a\}}_l$ of the expansion in Legendre polynomials (or spherical harmonics) (HASSANI, 2013; COLEMAN, 2015),

$$F^{\{s,a\}}(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'}) = \sum_{l=0}^{\infty}(2l+1)F^{\{s,a\}}_l P_l(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'}) = \sum_{lm} 4\pi F^{\{s,a\}}_l Y_{lm}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}})Y^*_{lm}(\theta_{\boldsymbol{p}'},\phi_{\boldsymbol{p}'}) \tag{2.21}$$

In which the identities are used, the second being important for future reference

$$P_l(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'}) = \frac{1}{2l+1}\sum_{m=-l}^{m=l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{2.22}$$

$$\frac{1}{4\pi}\int Y_{lm}(\Omega_{\boldsymbol{p}})Y^*_{l'm'}(\Omega_{\boldsymbol{p}})d\Omega_{\boldsymbol{p}} = \delta_{ll'}\delta_{mm'} \tag{2.23}$$

Symmetry and Strength Responses: the Screening of the Bare Perturbation

We consider a deformation in the 'bare' energy - in the absence of feed-back interaction - to be of the form

$$\delta\varepsilon^{(0)}_{\boldsymbol{p}\sigma} = u_l(\sigma)Y_{lm}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}}) \tag{2.24}$$

The displacement of the 'bare' particle relatively to the Fermi-surface must not be of high momentum transfer, otherwise the life-time would be short; in this way it is practically only dependent on the vector magnitude at the Fermi-surface. As an example, we recall the linear

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[3] $\uparrow\longleftrightarrow\sigma = 1$ and $\downarrow\longleftrightarrow\sigma = -1$, if we rotate the system by π, or reflect it, $\uparrow\longleftrightarrow\downarrow$, and the relation is manifestly invariant. The next-leading order of the expansion would contain $\sigma\times\sigma'\times\sigma''\times\sigma'''$, always an even number of terms. A more general justification consists in observing that the product of $\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\beta\gamma}$ is invariant under rotation.

modification of the kinetic energy near the Fermi-surface, $\delta\varepsilon^{(0)}_{\boldsymbol{p}\sigma} = p_F^2(1-\cos\theta_{\boldsymbol{p}})$ with the angles measured with respect to the initial reference vector of the initial vector describing the particle lying in the Fermi-surface, $\boldsymbol{p}_F$. This is only a particular where we have just modified the kinetic term. Since the orthogonality of spherical harmonics hold, each term may be treated independently, sufficing to consider a single channel l at a time, as in the above model.

We suppose the normalized response of the quasi-particle energy preserves the geometrical symmetry of the perturbation on the bare particle. This is what we expect when particles surround the bare particle shields or screen part of the interaction. The effect of screening is similar to effects of shielding in classical electrodynamics. For instance, in the response to an external electric field, particles in a dielectric material reorganize to reduce the net field while keeping the symmetry of the perturbation. No explicit long-range Coulomb potential was evoked here, but the same principle seem to hold to short-range quantum-electrodynamics interactions in between electronic clouds (it holds for neutral atoms). The quasi-particle is said to be 'dressed' or 'clothed' by the existence of surrounding particles creating a layer reducing the interaction strength between excited particles.(MATTUCK, 1992)

For now, we proceed with our treatment of short-range interactions,

$$\delta\varepsilon_{\boldsymbol{p}\sigma} = t_l(\sigma)Y_{lm}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}}) \tag{2.25}$$

From the feedback equation at near-zero temperature,

$$\begin{aligned}(t_l(\sigma') - u_l(\sigma))Y_{lm}(\boldsymbol{p}) = &-\sum_{\boldsymbol{p}'\sigma'} f^{(s)}_{\boldsymbol{p}\boldsymbol{p}'} t_l(\sigma')Y_{lm}(\theta_{\boldsymbol{p}'},\phi_{\boldsymbol{p}'})\delta(\varepsilon^{(0)}_{\boldsymbol{p}'}) \\ &-\sum_{\boldsymbol{p}'} \sigma\sigma' f^{(a)}_{\boldsymbol{p}\boldsymbol{p}'} t_l(\sigma')Y_{lm}(\theta_{\boldsymbol{p}'},\phi_{\boldsymbol{p}'})\delta(\varepsilon^{(0)}_{\boldsymbol{p}'})\end{aligned} \tag{2.26}$$

Considering a 'bare' energy modification independent of the spin, the values of u_l depends only on the channel l (also, t_l). The second term in (2.26), dependent on the product of spins, clearly sums to zero, as one of the spin indexes is not paired - cancelling off the terms with equal contribution and opposite signs. From the identity (2.18),

$$(t_l - u_l)Y_{lm}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}}) = -N^*(0)t_l \int \tfrac{d\Omega_{\boldsymbol{p}'}}{4\pi} f^{(s)}(\cos\theta_{\boldsymbol{p},\boldsymbol{p}'})Y_{lm}(\theta_{p'},\phi_{\boldsymbol{p}'})$$

$$(t_l - u_l)Y_{lm}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}}) = -t_l \textstyle\sum_{l'm'} \int \tfrac{d\Omega_{\boldsymbol{p}'}}{4\pi} Y_{l'm'}(\theta_{\boldsymbol{p}},\phi_{\boldsymbol{p}})F^s_{l'}Y^*_{l'm'}(\theta_{\boldsymbol{p}'},\phi_{\boldsymbol{p}'})Y_{lm}(\theta_{\boldsymbol{p}'},\phi_{\boldsymbol{p}'}) \tag{2.27}$$

Therefore, from the orthogonality relation,

$$t_l = \frac{u_l}{1+F^{(s)}_l} \tag{2.28}$$

In case we choose a bare energy modification dependent on the spin, specifically linear dependence, as that resulting from varying the magnetic field ($\delta\varepsilon_{\boldsymbol{p}\sigma} = -\sigma\mu_F B$). Hence, the first term, independent of the spin, is the one vanishing. The second term will contain a non vanishing sum in σ'^2. The conclusion is the same. Due to the linearity, by setting

$$u_l = u_l^s + \sigma u_l^a \tag{2.29}$$

$$t_l = t_l^s + \sigma t_l^a \tag{2.30}$$

It follows

$$t_l^{(s)} = \frac{u_l^s}{1+F_l^{(s)}} \tag{2.31}$$

$$t_l^{(a)} = \frac{\sigma u_l^a}{1+F_l^{(a)}} \tag{2.32}$$

For repulsion forces, if $F_l^{(s)} > 0$ (or $F_l^a > 0$), the effect of the perturbation is reduced or 'screened'. On the contrary, if $F_l^{(s)} \to -1$, we have the Pomeranchuk instability - the Fermi surface becomes unstable and Landau formalism is no longer applicable (COLEMAN, 2015; CHUBUKOV; KLEIN; MASLOV, 2018; METZNER; ROHE; ANDERGASSEN, 2003). If $F_l^{(a)} \to -1$ have the Stoner instability resulting in ferromagnetism.

2.4 RENORMALIZED THERMODYNAMICS

Renormalization of Mass

The introduction of an infinitesimal displacement in the magnetic field $\delta\boldsymbol{A} = \boldsymbol{A}$ yields the lowering of the energy of the system, for either the bare particle or the dressed particle

$$\delta\varepsilon_{\boldsymbol{p}\sigma}^{(0)} = -\frac{\boldsymbol{p}_F \cdot \boldsymbol{A}}{m} \tag{2.33}$$

$$\delta\varepsilon_{\boldsymbol{p}\sigma} = -\frac{\boldsymbol{p}_F \cdot \boldsymbol{A}}{m^*} \tag{2.34}$$

The symmetry of interaction is of dipolar kind ($\cos\theta$). The channel $l = 1$ is the one for the bare excitation and the dressed particle. Therefore, from (2.28),

$$m^* = m(1 + F_1^s) \tag{2.35}$$

The effective mass increases due to the presence of the interaction. Since $N(0) \propto m$ and $C_v = \frac{d\varepsilon}{dT} \propto N(0)$, and

$$C_v^* = (1 + F_1^{(s)})C_v \tag{2.36}$$

Renormalization of Magnetic Susceptibility and the Wilson Ratio

In a magnetic system, besides the displacement in the last section, there is a a displacement coupled to the spins,

$$\delta\varepsilon^{(0)}_{\boldsymbol{p}\sigma} = -\sigma\mu_F B \tag{2.37}$$

as $\delta B = B$ is the small deviation from $B = 0$. Following that,

$$\begin{aligned} u^s_l &= 0 \longrightarrow t^s_l = 0 \\ u^a_l &= -\mu_F B \longrightarrow t^a_l = \frac{1}{1+F^a_0} \end{aligned} \tag{2.38}$$

I.e,

$$\delta\varepsilon_{\boldsymbol{p}\sigma} = -\frac{\sigma\mu_F}{1+F^a_0}B \tag{2.39}$$

The total energy displacement for electrons is

$$\frac{\delta U}{V} = n_\uparrow\delta\varepsilon_\uparrow + n_\downarrow\delta\varepsilon_\downarrow = (n_\uparrow - n_\downarrow)\frac{\mu_F B}{2(1+F^a_0)} \tag{2.40}$$

But the number (density) of up and down quasi-particles difference in a magnetic field is $n_\uparrow - n_\downarrow = N^*(0)\mu_F B$. Thus,

$$\frac{1}{V}\frac{\delta U}{\delta B} = \frac{M}{V} = \frac{N^*(0)\mu_F^2 B}{1+F^a_0} \tag{2.41}$$

and since

$$\chi^*_s = \frac{\partial(M/V)}{\partial B} = \frac{N^*(0)\mu_F^2}{1+F^a_0} \tag{2.42}$$

But as both $N(0)^*$ and $C^*_v \propto N(0)^*$, the difference between the old and new Wilson ratio is provided through

$$W^* = \frac{N^*(0)\mu_F^2}{(1+F^{(a)}_0)C^*_v} \tag{2.43}$$

But as $C^*_v \propto N^*(0)$,

$$W^* = \frac{1}{1+F^{(a)}_0}, \tag{2.44}$$

explaining the enhancement of the Pauli susceptibility in liquid helium and different materials, as F^a_0 is negative in materials with ferromagnetic exchange interactions. In H_3, $F^{(a)}_0 \sim -2/3$, in palladium, $F^{(a)}_0 \sim -9/10$.

2.5 THE LANDAU-SILIN THEORY

In Landau theory, the long-range effect of electrodynamics in the level of the Gauss law has been neglected. These would indeed pose a fundamental problem to the theory. First of all, the energy scale would dramatically increase, and therefore, the switch-on time scale would approach zero. We, however, recall there is always arbitrariness in the proximity of the excited states to the Fermi-surface, such that we may find an eigenstate with life-time superior to the adiabatic time. This hand-waving argument may justify the applicability of the Landau theory when incorporating the long-range effect, as proposed by Silin (SILIN, 1957).

Despite the appearance of divergent effects, Silin went further in examining the effects of these divergences in the relevant thermodynamic quantities. He observed that the inclusion of the long-range coulombic interaction would lead to a modification only in the $l = 0$ channels of the symmetric Landau parameters. Neither the latent heat nor many other physical quantities of interest would modify.

Changing the interacting part of the Landau-Fermi liquid theory to the position representation, we have,

$$H_I = \frac{1}{2} \sum_{\sigma\sigma' \boldsymbol{x},\boldsymbol{x}'} f_{\boldsymbol{x}\sigma,\boldsymbol{x}'\sigma'} \delta n_{\sigma\boldsymbol{x}} \delta n_{\sigma'\boldsymbol{x}'} \tag{2.45}$$

In order to include long-range electrodynamics effects,

$$f_{\boldsymbol{x}\boldsymbol{x}'\sigma\sigma'} = \overbrace{f_{0;\boldsymbol{x},\boldsymbol{x}'\sigma\sigma'}}^{\text{Non-coulombic effects}} + \overbrace{\frac{e^2}{4\pi\varepsilon_0|\boldsymbol{x}-\boldsymbol{x}'|}}^{\equiv V_C,\text{Coulomb effects}} \tag{2.46}$$

The Coulomb effect has an isotropic feature of translational invariance. The second term is also refereed commonly as the 'polarization term'. We consider $\delta n_{\sigma\boldsymbol{x}} = n_{\sigma\boldsymbol{x}} = \Psi^\dagger_{\sigma\boldsymbol{x}}\Psi_{\sigma\boldsymbol{x}}$ to account for the number of extra particles added or removed above the Fermi-sea, in an effective Hamiltonian interaction term. In the center of mass (or Wigner) coordinate system,

$$H_{I,\text{Coulomb}} = \frac{1}{2} \sum_{\sigma\sigma'} \int d^3\boldsymbol{r} d^3\boldsymbol{R} \int V_{C;\boldsymbol{r}} \Psi^\dagger_{\sigma(\boldsymbol{r}+\boldsymbol{R}/2)} \Psi_{\sigma(\boldsymbol{r}+\boldsymbol{R}/2)} \Psi^\dagger_{\sigma'(\boldsymbol{r}-\boldsymbol{R}/2)} \Psi_{\sigma'(\boldsymbol{r}-\boldsymbol{R}/2)} \tag{2.47}$$

A Fourier transforming yields

$$H_I = \frac{1}{2} \sum_{\sigma\sigma'} \Big[\sum_{\boldsymbol{p},\boldsymbol{p}'} f_{0\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} n_{\sigma\boldsymbol{p}} n_{\sigma'\boldsymbol{p}'} + \sum_{\boldsymbol{q}} V_{\text{C}\boldsymbol{q}} n_{\boldsymbol{q}\sigma} n_{-\boldsymbol{q}\sigma'} \Big] \tag{2.48}$$

where the second is an expression of the isotropic form of the coulombic interaction - the fact that it depends only on $\boldsymbol{r}$ the momentum $\boldsymbol{q}$ and $-\boldsymbol{q}$ only. We often consider the translational invariant case of the non-coulombic interactions, which yields

$$H_I = \frac{1}{2}\sum_{\sigma\sigma'}\Big[\sum_{\boldsymbol{q}} f_{0\boldsymbol{q}} n_{\boldsymbol{q}\sigma} n_{-\boldsymbol{q}\sigma'} + \sum_{\boldsymbol{q}} V_{C\boldsymbol{q}} n_{\boldsymbol{q}\sigma} n_{-\boldsymbol{q}\sigma'}\Big] \tag{2.49}$$

In producing the perturbation theory, the terms $V_{C\boldsymbol{q}} = \frac{e^2}{\varepsilon_0 q^2}$ and $f_{0\boldsymbol{q}}$ must be small enough for convergence (although redundant, not to infinity) to hold. But when $|\boldsymbol{q}| < \Lambda$, the momentum transfer is small (planck-scales), the first term is large, while the second plays the whole of the quantum contribution for the interaction between the electronic clouds. Either we normalize the interaction in the region of small momentum or large momentum transfer, since it is a technical difficulty to normalize the entire domain consisting of small and large momentum transfer. In this way,

$$f_{\boldsymbol{q}} = \overbrace{\frac{e^2}{\varepsilon_0 q^2}}^{\text{Isotropic and spin independent}} + f^s_{0\boldsymbol{q}} + f^a_{0\boldsymbol{q}}\sigma\sigma' \tag{2.50}$$

The first term is direction-independent, therefore it only produces contributions to the channel $l = 0$ and spin-symmetric perturbations. That is, the inclusion of the Coulomb interaction is equivalent to

$$F^s_{l\boldsymbol{q}} \to \tilde{F}^s_{\boldsymbol{q}l} = \frac{e^2 N^*(0)}{\varepsilon_0 q^2}\delta_{l0} + F^s_{l\boldsymbol{q}} \tag{2.51}$$

Polarization Effects in the Susceptibility of Charge

In the absence of the Coulomb interaction

$$\chi_{c,0} = \frac{1}{V}\frac{\partial N}{\partial \mu} = \frac{N^*(0)}{1 + F^s_0} \tag{2.52}$$

But due to the Coulomb interaction

$$\chi_c = \frac{\chi_{c,0}}{1 + \frac{\kappa^2}{q^2}} \text{ with } \kappa^2 = \frac{e^2}{\varepsilon_0}\Big(\frac{N^*(0)}{1 + F^s_0}\Big) \tag{2.53}$$

The characteristic length k^{-1} defines the Thomas-Fermi length. The bulk modulus is related to the susceptibility such that

$$\frac{dP}{dV} = -\frac{\rho^2}{V\chi_c}\ ,\ \rho = \frac{N}{V} \tag{2.54}$$

Therefore, the presence of the long-range Coulomb interaction makes the fluid rigid. In the limit $|\boldsymbol{x} - \boldsymbol{x}'| << k^{-1}$ it behaves as a solid.

2.6 SELF-ENERGY AND FEYNMAN DIAGRAMS IN A NUTSHELL

This section aims to present the Feynman diagrams and their connection with the concept of self-energy. We usually denote in the context of the modern Green's function treatment, $\varepsilon_p^* = \varepsilon_p + \Sigma_{\boldsymbol{p}}^*$ where ε_p is the energy of particles in the absence of feed-back effects. ε_p^* denotes the 'normalized' energy. The general scheme for obtaining the self-energy is, as before, by recursively considering the above relation in higher-orders of accuracy in $\Sigma_{\boldsymbol{p}}$, provided convergence.

An Overview of Green's Function and Related Quantities

An important quantity is the zero-temperature two-particle Green's function, defined as the probability amplitude for the system to receive an additional particle; the average being on an eigenstate of the system. $\mathcal{T}$ denotes the time-ordering operator (see Appendix A.2).

$$\mathcal{G}_{\boldsymbol{p'p}}(t-t') = -i\langle \mathcal{T}\Psi_{\boldsymbol{p}}(t)\Psi^\dagger_{\boldsymbol{p'}}(t')\rangle = -i\langle \mathcal{T}\exp\Big[-i\int dt\boldsymbol{H}_I(t)\Big]\Psi_{\boldsymbol{p}}(t)\Psi^\dagger_{\boldsymbol{p'}}(t')\rangle \equiv =\!\!\blacktriangleright\!\!= \tag{2.55}$$

The second equality follows from the Murray-Gell-Man theorem (Appendix A.4) with the operators evolving as in a non-interacting environment. Alternatively, it is the amplitude that we add and remove a particle from a given state and end up in the same state. This is null for $\boldsymbol{p} \neq \boldsymbol{p'}$, which agrees with the expectation. Such interpretation is the reason for the term 'propagator' and the diagrammatic view of this consisting of source and sink. The reason for the two-line traces is to differ it from the "free particle propagator', the green's function of a non-interacting system. In the statement before we did not consider the ordering in which we add or remove a particle. Sometimes this is not needed indeed, but more precisely, we may divide the above diagram in two representative diagrams,

$$\mathcal{G}_{\boldsymbol{p'p}}(t-t') = \begin{cases} G_{\boldsymbol{p'p}}(t-t') \equiv =\!\!\blacktriangleright\!\!= \text{ if } t > t' \\ \tilde{G}_{\boldsymbol{p'p}}(t-t') \equiv =\!\!\blacktriangleleft\!\!= \text{ if } t < t' \end{cases} \tag{2.56}$$

If $t > t'$, the particle is first created and just after that it is annihilated. If $t < t'$, the particle is removed and then added. But to remove a particle is equivalent to creating and removing a hole. Explicitly distinguishing particles and holes, we denote the holes as particles moving backwards. This is in accordance with the Feynman interpretation (Appendix A.6).

For either a free system ($H_I = 0$) of fermions or bosons (Appendix A.6), the particle and hole propagators are defined as

$$G^{(0)}_{\boldsymbol{p}}(\omega) \equiv \frac{1}{\omega - \varepsilon_{\boldsymbol{p}}} = -\!\!-\!\!-\!\!\blacktriangleright\!\!-\!\!-\!\!- \ , \ \tilde{G}^{(0)}_{\boldsymbol{p}}(\omega) \equiv \frac{1}{\omega + \varepsilon_{\boldsymbol{p}}} = -\!\!-\!\!-\!\!\blacktriangleleft\!\!-\!\!-\!\!- \tag{2.57}$$

It is frequently convenient to define differently the Green's function in the temperature-dependent case, in which the average is on possible states for the system, following the observation due to Matsubara (MATSUBARA, 1955) (Appendix A.5), that consists of continually extending the real-time to the imaginary time, $\tau = it$, in order to produce the analogous of the S-matrix in the finite-temperature physics, the partition function. In finite-temperature physics, it is convenient to define

$$\mathcal{G}_{\boldsymbol{p}'\boldsymbol{p}}(\tau - \tau') = \langle \mathcal{T} \Psi_{\boldsymbol{p}}(\tau)\Psi^{\dagger}_{\boldsymbol{p}'}(\tau')\rangle = \langle \mathcal{T} \exp\left[-\int_0^{\beta} d\tau \boldsymbol{H}_I(\tau) d\tau\right] \Psi_{\boldsymbol{p}}(\tau)\Psi^{\dagger}_{\boldsymbol{p}'}(\tau')\rangle \tag{2.58}$$

The graphical notation to represent both zero-temperature and non-zero-temperature physics is the same. In general, when dealing with the finite-temperature case we will express the corresponding green's function in terms of the finite-temperature Green's function, $G^{(0)}_{\boldsymbol{p}}(i\omega)$ (Appendix A.6)). At the heart of the definition of the finite-temperature Green's function is the important partition function, the analytic continuation (Appendix A.3) of the S-matrix (Appendix A.4)

$$Z = Z_0 \langle \mathcal{T} \exp\left[-\int_0^{\beta} \boldsymbol{H}_I(\tau) d\tau\right] \rangle_0 \ , \tag{2.59}$$

where the average is over an ensemble of particles evolving in the absence of interaction.

Connection to the Self-Energy Concept

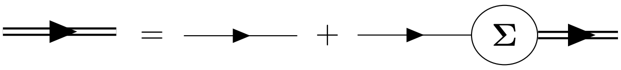
We investigate how the green's function modifies as we alter the energy driven by the feed-back effects. In the view of the discussion on the Fermi-liquid self-energy, by turning on the interaction adiabatically, the green's function modify such that

$$\begin{aligned} G_{\boldsymbol{p}}(\omega) = \frac{1}{\omega - \varepsilon^*_p} = \frac{1}{(\omega - \varepsilon_p)(1 - \frac{\Sigma^*_{\boldsymbol{p}}}{\omega - \varepsilon_p})} = \frac{1}{\omega - \varepsilon_p} + \frac{1}{\omega - \varepsilon_p}\Sigma^*_{\boldsymbol{p}}\frac{1}{\omega - \varepsilon_p} + \\ \frac{1}{\omega - \varepsilon_p}\Sigma^*_{\boldsymbol{p}}\frac{1}{\omega - \varepsilon_p}\Sigma^*_{\boldsymbol{p}}\frac{1}{\omega - \varepsilon_p} + ... \end{aligned} \tag{2.60}$$

or stated differently,

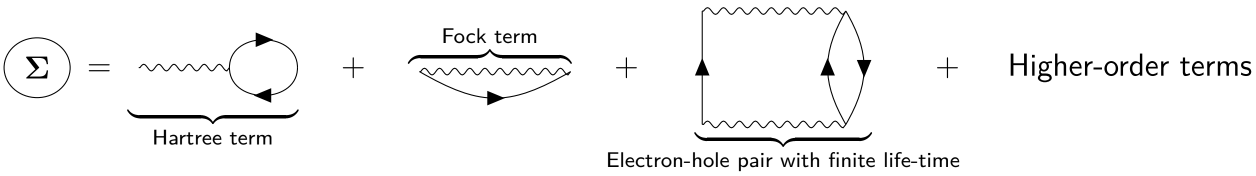
$$\begin{aligned} G_{\boldsymbol{p}}(\omega) &= \frac{1}{G^{-1}_{0\boldsymbol{p}} - \Sigma^*_{\boldsymbol{p}}} = G_{0\boldsymbol{p}} + G_{0\boldsymbol{p}}\Sigma^*_{\boldsymbol{p}}G_{0\boldsymbol{p}} + G_{0\boldsymbol{p}}\Sigma^*_{\boldsymbol{p}}G_{0\boldsymbol{p}}\Sigma^*_{\boldsymbol{p}}G_{0\boldsymbol{p}} + ... \\ &= G_{0\boldsymbol{p}}(\omega) + G_{0\boldsymbol{p}}(\omega)\Sigma^*_{\boldsymbol{p}}G_{\boldsymbol{p}}(\omega) \end{aligned} \tag{2.61}$$

which is an expression of the Dyson equation (DYSON, 1949). Σ_p^* accounts for the many virtual process each representing different orders of the pair-wise interaction. Often, in the diagrammatic notation, Σ_p^* is denoted Σ_p, with the same meaning - not restricted to a particular correction, but a result of all of the virtual pair-wise processes. In the diagrammatic form,



(2.62)

Analogously, a similar graphical expression, with the arrows reversed works out just as well for the hole propagator. As Σ is related to the interaction with other particles, the total propagator is the sum of different amplitudes with a source and interactions in between. On the other hand, the Dyson expansion is precisely the consequence of applying consistently the Murray-Gell-Man Low theorem (Appendix A.4) and identifying the contractions along the way. Each contraction leads to a particular contribution to the self-energy in the zoo,



(2.63)

The language of diagrams is useful since it allows for the terms under consideration to be tracked down to possibly nonphysical but tangible reality; the true physical process being the sum over (possibly nonphysical) virtual processes. We summarize the assertion by Mattuck (MATTUCK, 1992) on this topic: a word alone has little or no meaning, these being pretty much a convention, however, words together may produce a description of the reality surpassing the arbitrariness of conventions. With the concepts gathered in this chapter, we are in a position to move to the next chapter, on a systematic study of the microscopic superconductivity theory.

3 LANDAU THEORY OF PHASE TRANSITION AND THE (U(1)) SPONTANEOUS BREAK-DOWN OF SYMMETRY

3.1 PROLOGUE

L.D. Landau was the first to provide a unified treatment for the phase transition process (LANDAU, 1936). When the macroscopic properties of matter change, or to put it in another way, when the system's ground state does not evolve adiabatically, a phase transition occurs. Numerous examples include ferromagnetic material undergoing spin alignment transition) (NÉEL, 1948), superfluidity/superconductivity caused by the condensation of bosons or fermionic pairs into a single state – Bose-Einstein condensation (PENROSE; ONSAGER, 1956; BOGOLIUBOV, 1970). Adding to this list, we may consider the crystallization processes in nature, such as that of water (LIBBRECHT, 2005). Additionally, in the context of spin waves, the quasi-particles known as magnons or spinons has been the source of increasing attention (REZENDE, 2020; REZENDE, 2009; MALOMED et al., 2010) and produced what is now recognized as a phase transition, with all spin-waves having a well-defined wave number and frequency during a characteristic time period large enough to be considered an equilibrium thermodynamic state within a controlled window of time. It is self-evident that each process has a unique microscopic physics. However, these processes have one thing in common: they all involve an abrupt change in the ground state when an external parameter is changed. N.N Bogoliubov is responsible for a crucial remark on this notorious fact.

The order parameter refers to a thermodynamic property of the physical system in question (average over a microscopic field). For example, in the case of ferromagnetism phenomena, the order parameter is frequently assumed to be magnetization. Indeed, the choice of the order parameter is arbitrary - any microscopic average capable of developing a non-zero value is a possible order parameter benefiting from Landau's phase transition theory.

To understand phase transitions, we must first consider an abrupt change in a thermodynamic quantity. The simplest way to illustrate this is through Landau's initial proposal.

$$|\psi| = \begin{cases} 0 \text{ if } T > T_c \\ |\psi_0| > 0 \text{ if } T < T_c \end{cases} \tag{3.1}$$

where ψ is the relevant order parameter. As is well known, T_c denotes the critical temperature, $T > T_c$ is referred to as the 'normal' phase, and $T < T_c$ the 'ordered' phase.

The abrupt change in the order parameter is said to be caused by a current breaking the system's symmetry. Emmy Noether (NOETHER, 1971) is credited with coining the concept of symmetries and associated currents in physics. N.N.Bogoliubov proposed a mechanism of symmetry breaking to account for the abrupt changes in the process of ground-state selection based on the general microscopic theories (BOGOLIUBOV, 1970). As demonstrated in this chapter, the mechanism of superconductivity's symmetry breaking is related to the $U(1)$ symmetry, which is manifested by an instantaneous deformation of the potential energy and is responsible for the selection of a preferable coherent state. The process of symmetry breaking justifies both the presence of non-zero anomalous averages (Appendix B.1). We see candidate choices for an order-parameter of the theory, such as the Cooper-Pairing density. This choice is precisely understood in chap.5, when the Ginzburg-Landau equation is recovered from the microscopic theory, naturally providing the gap - proportional to the anomalous average- as the relevant order-parameter.

The Bogoliubov proposal for the break of the symmetry is the inclusion of a 'current term' (BOGOLIUBOV, 1970). Throughout this chapter, we will consider the Grassman algebra (GRASSMANN, 1844; HASSANI, 2013), an expression of the anti-commutation properties of numbers and operators (Appendix B.2). The first to notice the usefulness of this algebra within the realm of quantum field theory was Julian Schwinger (SCHWINGER, 1954). A reduced list of contributions using the anti-commutation algebra in field theory, include the subsequent works of F.A.Berezin and J.L.Martin., between which (MARTIN, 1959; BEREZIN; MARINOV, 1977; BEREZIN, 1980). We will show that each electron-pair in a superconducting system evolves to a coherent state, and the entire system is in a 'macroscopic coherent state' or 'macroscopic quantum state' through a mixture of concepts due to both Bogoliubov (BOGOLIUBOV, 1970) and J.Schwinger (SCHWINGER, 1954). In this way we connect the N.N.Bogoliubov's formulation of the symmetry breaking to the R.Glauber formulation (GLAUBER, 1963). As is well known, the coherent state most closely resembles the classical limit (SAKURAI; NAPOLITANO, 2014), providing the wave-packet with the least uncertainty. Hence, we begin our journey toward the study of the $U(1)$ symmetry breakdown.

(U(1)) SPONTANEOUS BREAK-DOWN OF THE SYMMETRY AND ANOMALOUS AVERAGES

A theory presents $U(1)$ symmetry if there is no energetic cost in twisting the phase of its solutions. We will concentrate in the concept of break-down of the $U(1)$ symmetry, i.e, the sudden loss of the phase degeneracy, causing the system to be driven to a particular choice with less energetic cost.

$$[\boldsymbol{H}, \boldsymbol{N}] = \boldsymbol{0} \tag{3.2}$$

The eigenstates of the particle number $\boldsymbol{N}$ are eigenstates of the Hamiltonian. Therefore, it is trivial to identify that ($\boldsymbol{y} \equiv (\boldsymbol{x}, \sigma)$)

$$\langle N| \, c^{\dagger}_{\boldsymbol{y}}(t) c^{\dagger}_{\boldsymbol{y}'}(t) \, |N\rangle = 0 \qquad , \qquad \langle N| \, c_{\boldsymbol{y}}(t) c_{\boldsymbol{y}'}(t) \, |N\rangle = 0 \tag{3.3}$$

hold at zero-temperature physics. In general, Bogoliubov noticed that if the Hamiltonian of the system is invariant under the unitary transformation $\boldsymbol{U} = e^{i\phi \boldsymbol{N}}$,

$$\boldsymbol{H} \to \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^{\dagger} \; , \tag{3.4}$$

then, the finite-temperature average cancels

$$\langle c^{\dagger}_{\boldsymbol{y}} c^{\dagger}_{\boldsymbol{y}'} \rangle = \frac{\mathrm{Tr}\Big[c^{\dagger}_{\boldsymbol{y}} c^{\dagger}_{\boldsymbol{y}'} \exp[-\beta \boldsymbol{H}] \Big]}{\mathrm{Tr}[\exp[-\beta \boldsymbol{H}]]} = 0 \tag{3.5}$$

It is a known fact that this is true for the Bogoliubov-Hamiltonian formulation of superconductivity (SHANENKO, 2000). In contrast to the conclusions above, we anticipate the presence of nonzero averages of the type $\langle c_{\boldsymbol{y}} c_{\boldsymbol{y}'} \rangle$ and $\langle c^{\dagger}_{\boldsymbol{y}} c^{\dagger}_{\boldsymbol{y}'} \rangle$ (Appendix B.1). Bogoliubov justified the existence of non-zero averages in the context of mean-field theory (next chapter) by introducing a phenomenological inclusion that would violate $U(1)$ symmetry. Other symmetries may also be violated in the presence of small fields. For example, in the Heisenberg model, particles can be aligned in any direction at high temperatures, but tend to align along a preferred ground state direction at low temperatures. This infinitesimal term is what breaks the rotational symmetry when a particular direction is chosen. We will concentrate on the $U(1)$ symmetry breaking in this section. Bogoliubov proposes that the symmetry may be broken by the inclusion of a 'current term'.

Mixing the ideas due to Bogoliubov and Schwinger, we consider

$$\boldsymbol{H_J} = \frac{\delta_{\tau,\tau_0}}{N_{\tau_0}} \int \boldsymbol{dy}[c(\boldsymbol{y})c_{\boldsymbol{y}} + c^{\dagger}_{\boldsymbol{y}}c^{*}(\boldsymbol{y})] \tag{3.6}$$

$$\int \boldsymbol{dy} \equiv \sum_{\sigma} \int \boldsymbol{dx} \tag{3.7}$$

in which we modeled Bogoliubov's small term δ as a Dirac-delta in time, as this allows for interesting conclusions and captures the essence of the Bogoliubov concept in relation to the emergence of quantum coherence.

We consider the effect of 'Bogoliubov's current' term is to change the number of particles in the system slightly apart from when $t = 0$ is the dominant term. The dominant term in the Dirac-delta model for the term δ is not the interaction, but the current term responsible for the system's symmetry breaking. However, infinitesimally after that, other terms become significant, despite the fact that the system has already chosen its preferred eigenstate, which is also the Hamiltonian's eigenstate in the absence of the 'Bogoliubov current'.

If the Hamiltonian's dominant term is the current term, the ground state tends to evolve in this direction around $\tau = \tau_0$. After this period, the system's eigenstate remains a minimally uncertain eigenstate of the entire Hamiltonian, a localized wave-packet.

$$\begin{aligned} S(0, \tau \geq \tau_0) &= \mathcal{T} \exp\Big[-\int_0^{\tau} d\tau \boldsymbol{H_J}(\tau)\Big] \\ |0\rangle \rightarrow |\Psi_0\rangle &= \exp\Big[-\frac{1}{N_{\tau_0}} \int \boldsymbol{dy}[c(\boldsymbol{y}, \tau_0)c_{\boldsymbol{y}\tau_0} + c^{\dagger}_{\boldsymbol{y}\tau_0}c^{*}(\boldsymbol{y}, \tau_0)]\Big] |0\rangle \end{aligned} \tag{3.8}$$

with $|\psi_0\rangle$ the symmetry broken state. This is written within the Grassman algebra. The conjugate wave function with a phase rotation of π produces a bosonic-like pair in the Hilbert space,

$$\boldsymbol{\Psi} = \frac{1}{N_s} \int \boldsymbol{dy} c(\boldsymbol{y})c_{\boldsymbol{y}} \tag{3.9}$$

$$\boldsymbol{\Psi}^{\dagger} = \frac{1}{N_s} \int \boldsymbol{dy} c^{\dagger}_{\boldsymbol{y}}c^{*}(\boldsymbol{y}) \tag{3.10}$$

By demanding

$$\boldsymbol{\Psi}^{\dagger}\boldsymbol{\Psi} = \boldsymbol{N_s} \tag{3.11}$$

It follows the consistency relation

$$\int \boldsymbol{dy}\, \underbrace{c^{*}(\boldsymbol{y})c(\boldsymbol{y})}_{\rho(\boldsymbol{y})} \tag{3.12}$$

- the order of the Grassman numbers is relevant.

It is convenient to define the density operators

$$\boldsymbol{\psi}(\boldsymbol{y}) = c(\boldsymbol{y})c_{\boldsymbol{y}}\ ,\ \boldsymbol{\psi}^{\dagger}(\boldsymbol{y}) = c_{\boldsymbol{y}}^{\dagger}c^{*}(\boldsymbol{y}) \tag{3.13}$$

$$\Big[\boldsymbol{\psi}(\boldsymbol{y}), -\boldsymbol{\psi}^{\dagger}(\boldsymbol{y})\Big] = c^{*}(\boldsymbol{y})c(\boldsymbol{y})\{c_{\boldsymbol{y}}, c_{\boldsymbol{y}}^{\dagger}\} = \rho(\boldsymbol{y})\ \underbrace{\delta_{\boldsymbol{y}\boldsymbol{y}'}}_{\delta_{\boldsymbol{x}\boldsymbol{x}'}\delta_{\sigma\sigma'}} \tag{3.14}$$

Therefore,

$$\Big[\boldsymbol{\Psi}, -\boldsymbol{\Psi}^{\dagger}\Big] = \boldsymbol{N_s} \tag{3.15}$$

Thus, the pair $(\boldsymbol{\Psi}, -\boldsymbol{\Psi}^{\dagger})$ behave as a many-body pair of bosonic operators. Hence, we evaluate

$$|\Psi_0\rangle \equiv \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]\,|0\rangle \tag{3.16}$$

$$\boldsymbol{\Psi}\,|\Psi_0\rangle = [\boldsymbol{\Psi}, \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]]\,|0\rangle = \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]\,|0\rangle \tag{3.17}$$

Since $\boldsymbol{\Psi}\,|0\rangle = 0$. Then, due to

$$[\boldsymbol{\Psi}, \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]]\,|0\rangle = \sum_{n}\frac{1}{n!}[\boldsymbol{\Psi}, (-\boldsymbol{\Psi}^{\dagger})^{n}]\,|0\rangle = \sum_{n=1}^{\infty}\frac{(-\boldsymbol{\Psi}^{\dagger})^{n-1}}{(n-1)!}\,|0\rangle = $$
$$= \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]\,|0\rangle = |\Psi_0\rangle \tag{3.18}$$

Therefore, $|\Psi_0\rangle$ is eigenstate of $\boldsymbol{\Psi}$ with eigenvalue 1. By expanding and applying the anti-comutation properties of Grassman numbers, the above result is reached, provided

$$[\boldsymbol{\psi}(\boldsymbol{y}), -\boldsymbol{\Psi}^{\dagger}] = c^{*}(\boldsymbol{y}) \tag{3.19}$$

Now, lets evaluate

$$\boldsymbol{\psi}(\boldsymbol{y})\,|\Psi_0\rangle = [\boldsymbol{\psi}(\boldsymbol{y}), \exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]]\,|0\rangle = c^{*}(\boldsymbol{y})\,|\Psi_0\rangle \tag{3.20}$$

since $\boldsymbol{\psi}\,|0\rangle = 0$. Therefore, the system is threw to a Glauber coherent state eigenstate of $\boldsymbol{\psi}$

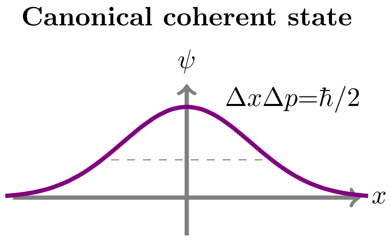
$$|\Psi_0\rangle = \exp\Big[-\Big(\boldsymbol{\Psi}^{\dagger} + \boldsymbol{\Psi}\Big)\Big]\,|0\rangle \tag{3.21}$$

due to the instant Hamiltonian

$$\boldsymbol{H_J} = \frac{\delta_{\tau-\tau_0}}{N_{\tau_0}}\int \boldsymbol{dy}[\boldsymbol{\psi}(\boldsymbol{y}) + \boldsymbol{\psi}^{\dagger}(\boldsymbol{y})] \tag{3.22}$$

As $[\boldsymbol{\psi}(\boldsymbol{y}), -\boldsymbol{\Psi}] = 0$ (as may be readily checked), the only relevant quantity relative to the operator $\boldsymbol{\psi}$ action is that related to $\exp\Big[-\boldsymbol{\Psi}^{\dagger}\Big]$, and this is also a Glauber eigenstate of the

Figure 2 – A sketch of Gaussian coherence, the most ‘classical’ quantum state.



Source: The author

operator $\boldsymbol{\psi}(\boldsymbol{y})$. As $|\Psi_0\rangle$ is an eigenstate of the operator $\boldsymbol{\psi}(\boldsymbol{y})$, it is also eigenstate of the operator $c_{\boldsymbol{y}}$ (see (3.13)).

Hence we are speculating on an order-parameter for the theory and look for its meaning. The average of the operator $\psi_{\boldsymbol{y}}$ over the ground-state yields, even at approximately zero temperature,

$$\langle \psi_{\boldsymbol{y}} \rangle = \langle 0 | \, \psi_{\boldsymbol{y}} \exp[-\boldsymbol{H_J}] \, |0\rangle = c(\boldsymbol{y}) \qquad \underbrace{\langle 0 | \Psi_0 \rangle}_{\text{transition amplitude}} \tag{3.23}$$

Another choice attempting involving pairing could be

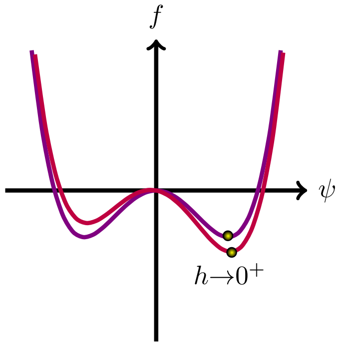
$$\langle \psi_{\boldsymbol{y}} \psi_{\boldsymbol{y}} \rangle = \langle 0 | \, \psi_{\boldsymbol{y}} \psi_{\boldsymbol{y}} \exp[-\boldsymbol{H_J}] \, |0\rangle = c^2(\boldsymbol{y}) \, \langle 0 | \Psi_0 \rangle = 0 \tag{3.24}$$

which fails to be a candidate order-parameter, as it is kept null. On the other hand, the choice

$$\langle \psi_{\boldsymbol{y}} \psi_{\boldsymbol{y}'} \rangle = \langle 0 | \, \psi_{\boldsymbol{y}} \psi_{\boldsymbol{y}'} \exp[-\boldsymbol{H_J}] \, |0\rangle = c(\boldsymbol{y}) c(\boldsymbol{y}') \, \langle 0 | \Psi_0 \rangle \tag{3.25}$$

does not forcefully vanish due to the Grassman algebra, and the electrons may even occupy the same position if their spin is contrary - this clearly represents the Cooper-pairing density. The inclusion of a finite-temperature Hamiltonian term such that $|\Psi_0\rangle$ remains an eigenstate after a non-negligible time has passed, may be understood in any theory (such as the Fermi-Landau theory) dependent on particle-number density operators $n_{\boldsymbol{y}} = \psi^{\dagger}(\boldsymbol{y})\psi(\boldsymbol{y})$. With this choice, the state $|\Psi_0\rangle$ diagonalizes the hamiltonian. In summary, it is the ‘Bogoliubov current’ (3.6) that is responsible for throwing the system in a coherent eigenstate, and it remains an eigenstate of the system long after the symmetry has broken. Coherent states are the closest quantum representation of classical systems.

Figure 3 – The break of Z_2 symmetry.



Source: The author, based on (COLEMAN, 2015)

3.2 LANDAU FREE-ENERGY

Considering the free energy density in the thermodynamic limit

$$f[\psi] = f_n(T,N) + \frac{\alpha_0}{2}(T - T_c)\psi^2 + \frac{b}{4}\psi^4 - g(h\psi) \tag{3.26}$$

Where $g(h,\psi)$ is a modification of the internal energy to take account for the field's interaction with the order parameter. Therefore, it must be such that $g(h \to 0, \psi) \to 0$, i.e, an analytic function in products of powers in its arguments. We readily identify that

$$|\psi_0| = \sqrt{\frac{\alpha_0(T - T_c)}{b}} \tag{3.27}$$

minimizes functional requirements. This results in a continuous order parameter at the critical temperature. The case in which $\psi_0 < 0$ is understood to be sign-dependent on the external field. Even though it is infinitesimal, the direction of this external field breaks the system's degeneracy, allowing for the selection of a preferred magnetization. Thus, even though we consider the limit $h \to 0$, the direction in which $h \to 0$ is approached, whether positive or negative, is relevant in determining the correct solution. Prior to h reaching zero, one of the possible solutions is preferred; this preference continues once h reaches zero. As $h \to 0^+$ increases, the potential is tipped to the right, and the particle must remain between the two initially symmetric states. Due to the field's history, stability is reached in the right valley. If $h \to 0^-$, on the other hand, the tipping occurs on the left side and stability is reached in the left valley.

Thermodynamics Near the Critical Point and the Z_2 Symmetry- Breaking

The free energy density in the vicinity of phase transition is

$$f_L = \begin{cases} f_n(T_c, N) \text{ , if } T > T_c \\ f_n(T_c, N) - \frac{\alpha_0^2}{4b}(T - T_c)^2 \text{ , if } T < T_c \end{cases} \tag{3.28}$$

It is no surprise the free energy is continuous at the critical temperature. As for the specific heat $c_v = -T\frac{\partial^2 F}{\partial T^2}$, we readily identify a jump at the critical temperature.

$$\Delta c_v = \frac{\alpha_0^2}{2b} T_c \tag{3.29}$$

We consider the linear coupling regime, $g(h\psi) = h\psi$, therefore,

$$\frac{\delta f}{d\psi}|_{\psi=\psi_0} = \alpha_0(T - T_c)\psi_0 + \psi_0^3 b - h = 0 \tag{3.30}$$

For the susceptibility, a measure of order-disorder variation with respect to varying an external parameter.

$$\chi(T) \equiv \frac{d\psi}{dh}|_{\psi=\psi_0} = [\frac{dh}{d\psi}]|_{\psi=\psi_0}^{-1} = \frac{1}{\alpha_0(T - T_c) + 3b\psi_0^2} \tag{3.31}$$

For $T > T_c$, $\psi_0 = 0$, and

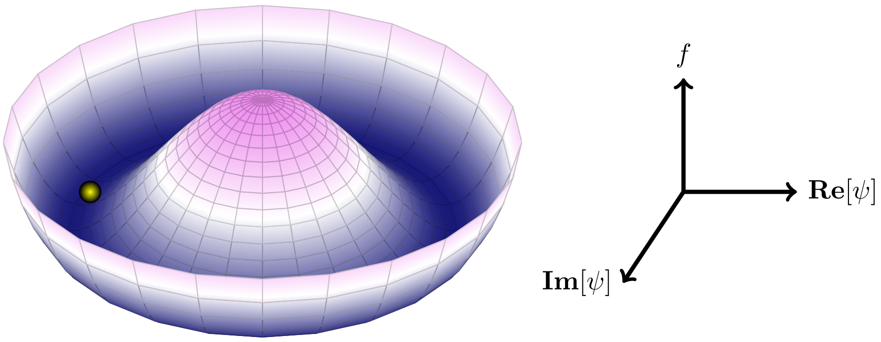
$$\chi(T > T_c) = \frac{1}{\alpha_0|T - T_c|} \tag{3.32}$$

For $T < T_c$, but in the vicinity of T_c, provided eq. (3),

$$\chi(T < T_c) = \frac{1}{2}\chi(T < T_c) \tag{3.33}$$

Thus, the smallest 'stray field' h produces an enormous change in the order parameter in the vicinity of the critical temperature from above or below. This is explained by the fact that the free energy is tipped with the introduction of the coupling - it was initially completely symmetric for either positive or negative ψ solutions (Z_2 symmetry), but the system chose a particular state when a small field is introduced. We will be most receptive to grasping the concept of $U(1)$ symmetry breaking. If we promote the order-parameter to a complex number, the Z_2 symmetry is equivalent to an invariant theory under a phase shift of π (reflection); the $U(1)$ symmetry extends it to any continuous phase shift.

Figure 4 – Sketch of the Mexican-hat potential. If the potential of the Mexican-hat is distorted, the system chooses a particular phase with less energetic cost.



Source: The author

The Landau Free Energy: the Complex Case

We demonstrated the effect of a real order parameter on an inversion symmetry in space in the preceding section. The complex case is a natural extension of the Landau free energy. If we wish to demand $U(1)$ symmetry, it suffices to demand

$$f[\psi,\psi^*] = \alpha_0(T-T_c)|\psi|^2 + \frac{b}{2}|\psi|^4 \tag{3.34}$$

Then the minimization yields (w.r.t ψ or ψ^*)

$$\alpha_0(T-T_c) + b|\psi_0|^2 = 0 \tag{3.35}$$

Hence, the minimum does not depend on the phase. In particular, the inversion symmetry is comprised in the $U(1)$ continuous symmetry. No particular phase is preferable,

$$\psi = \sqrt{\frac{\alpha_0(T-T_c)}{b}}e^{i\phi} \tag{3.36}$$

The introduction of an external coupled field in this case breaks the continuous $U(1)$ symmetry and forces the system to choose a phase. This is ultimately the source of the most general aspects of the emergence of superconductivity.

3.3 SUPERFLUIDITY

In a pioneering work much ahead of its time, Ginzburg and Landau introduced the phenomenological equation

$$f_{GL} = \int \boldsymbol{dx}[\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 + \alpha_0(T-T_c)|\psi|^2 + \frac{b}{2}|\psi|^4] = \int \boldsymbol{dx}[\underbrace{\frac{\hbar^2}{2m}|\psi|^2(\boldsymbol{\nabla}\phi)^2}_{\text{phase rigidity}} +$$

$$+ \underbrace{\frac{\hbar^2}{2m}(\boldsymbol{\nabla}|\psi|)^2 + \alpha_0(T-T_c)|\psi|^2 + \frac{b}{2}|\psi|^4]}_{\text{variation of amplitude}} \tag{3.37}$$

which can now be understood as a macroscopic Schrodinger equation for the Cooper pairs. This yields the same equations of the real case for both ψ and ψ^*, symmetrically. This theory is not $U(1)$ symmetric, as there is a cost for twisting the phase. The characteristic length for the variation in the amplitude defines the coherence length

$$\xi = \xi_0(1-T/T_c)^{-1/2} \text{ , } \xi_0 = (\frac{\hbar^2}{2m\alpha_0 T_c})^{1/2} \tag{3.38}$$

For a spacial scale beyond the coherence length, $|\psi|^2$ uniform, and the physics is controlled entirely by the phase,

$$f_{GL} = \int dx \frac{\rho_\phi}{2}(\boldsymbol{\nabla}\phi)^2 \text{ with } \rho_\phi = \frac{\hbar^2 n_s}{2m} \tag{3.39}$$

We may identify a current term by fixing $|\psi|$ and varying the phase in the boundary, ($\delta\psi = i\psi\delta\phi$).

$$\frac{\hbar^2}{2m}\int \boldsymbol{\nabla}\cdot[\delta\psi\boldsymbol{\nabla}\psi^* + \delta\psi^*\boldsymbol{\nabla}\psi]dV = \rho_\phi \int \boldsymbol{\nabla}\cdot[\delta\phi\nabla\phi] \tag{3.40}$$

Noether's theorem deals with the relationship between broken symmetry and a conserved quantity by means of a current not flowing out of the system. Because the system is not $U(1)$ symmetric, one may not infer the appearance of such current. In our problem, as ϕ is not a symmetry, it can not be made arbitrary. In fact, its boundary conditions are fixed by the boundary value problem. A careless thought could have led to

$$\boldsymbol{J}_s = \rho_\phi \boldsymbol{\nabla}\phi \text{ .} \tag{3.41}$$

Though this is strictly incorrect for the theory under consideration, it may be viewed as a current limiting-case of a $U(1)$ symmetric theory in which the fluid's charges interact with (small) electromagnetic fields. The case of a $U(1)$ symmetric theory is treated in the following section.

3.4 SUPERCONDUCTIVITY

We seek to develop a theory in which magnetic fields are coupled to the order parameter and is $U(1)$ invariant. Without coherence, $\psi = 0$ and there is no surface current. When this current attains a non-zero value, the $U(1)$ symmetry is broken, which is precisely the instability that results in the appearance of the collective macroscopic quantum state.

The electromagnetic theory (Maxwell equations) is unambiguously invariant under

$$\boldsymbol{A} \to \boldsymbol{A} + \frac{\hbar}{e^*}\nabla\alpha \ , \tag{3.42}$$

$$\Phi \to \Phi - \frac{h}{e^*}\partial_t\alpha \ , \tag{3.43}$$

, for an arbitrary scalar field α. The measurable quantities are the magnetic induction $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and the electric field $\boldsymbol{E} = -\nabla\boldsymbol{\Phi} - \frac{\partial\boldsymbol{A}}{\partial t}$. [1] e and h are the universal constants denoting the electron-charge and the Planck constant, respectively; $e^* \equiv 2e$ is justified as due to the two-electron condensation described by the Cooper-pairing. It is remarkable that this was indeed a first-principled consideration due to Landau and Ginzburg (LANDAU; GINZBURG, 1950), much before the microscopic theory (see chap.4) had been formulated. Considering $\alpha = \phi$, the theory becomes $U(1)$ invariant with the replacement

$$\boldsymbol{\nabla} \to \boldsymbol{\mathcal{D}} = \boldsymbol{\nabla} - \frac{ie^*}{\hbar}\boldsymbol{A} \tag{3.44}$$

Therefore,

$$F = \int \frac{\hbar^2}{2m^*}|(\boldsymbol{\nabla} - \frac{ie^*}{\hbar}\boldsymbol{A})\psi|^2 + \alpha_0(T-T_c)|\psi|^2 + b|\psi|^4 + \frac{\boldsymbol{B}^2}{2\mu_0} \tag{3.45}$$

The first GL equation is provided with the variation with respect to ψ,

$$\frac{\delta f}{\delta\psi^*} = -\frac{\hbar^2}{2m}(\boldsymbol{\nabla} - i\frac{e^*}{\hbar}\boldsymbol{A})^2\psi + \alpha_0(T-T_c)\psi + b|\psi|^2\psi = 0 \ . \tag{3.46}$$

In looking for variations with respect to ψ with a varying phase and fixed modulus, $\delta\psi = i\psi\delta\phi$, an associated Noether's current appear (it is equivalent to set $\boldsymbol{\nabla} \to \boldsymbol{\mathcal{D}}$ in (3.41)),

$$\boldsymbol{J}(\boldsymbol{x}) = \rho_\phi\Big[\nabla\phi - \frac{e^*}{\hbar}\boldsymbol{A}\Big] \tag{3.47}$$

In this case, we can really tell, due to the arbitrariness of $\delta\phi$, that this is a quantity whose divergence shall vanish for the equations of motion to hold - Noether's theorem. Thus,

$$\int \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{dS} = 0 \ , \ \oint \boldsymbol{\nabla}\times\boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{dx} = \rho_\phi\oint\Big[\boldsymbol{\nabla}\phi - \frac{e^*}{\hbar}\boldsymbol{A}\Big]\cdot\boldsymbol{dx} = 0 \tag{3.48}$$

---

[1] We slip the Aharonov effect under the carpet.

The single-valuedness of the theory relies on the necessity that the phase changes only by a multiple of 2π in any circuit path, thus,

$$\int \boldsymbol{B} \cdot \boldsymbol{dS} = n\Phi_0 \ , \ \Phi_0 = \frac{h}{e^*} \ , \tag{3.49}$$

which is the realization of Osanger on the quantization of the flux. By considering a variation of the functional w.r.t the vector potential,

$$\frac{\delta F}{\delta \boldsymbol{A}} = \boldsymbol{J}(\boldsymbol{x}) + \frac{1}{2\mu_0}\frac{\delta(\boldsymbol{\nabla} \times \boldsymbol{A})^2}{\delta \boldsymbol{A}} = 0 \tag{3.50}$$

A bit of index playing provides

$$\frac{1}{2\mu_0}\delta(\boldsymbol{\nabla} \times \boldsymbol{A})^2 = \frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B} \cdot \delta \boldsymbol{A} \tag{3.51}$$

and the second GL reads

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{3.52}$$

The Meissner-Ochsenfeld Effect

From the rotational on the second GL and the consideration of uniformity for the order parameter,

$$\nabla^2 \boldsymbol{B} = \frac{1}{\lambda^2}\boldsymbol{B} \ , \ \lambda = (\frac{m^*}{e^{*2} n_s})^{1/2} \tag{3.53}$$

When $n_s = 0$ (absence of condensate) any solution to the Laplace equation holds. When the condensate is present, $n_s \neq 0$, the decaying behavior $e^{-z/\lambda}$ produces the (practical) elimination of the induction within a finite shell from the material boundary. λ is coined the 'London penetration depth'.

The Critical Field and the Stability of Solutions

The normal state is defined by the absence of Bose-Einstein condensation, $\psi = 0$, for which follows $\boldsymbol{B} = \mu_0 H$ due to the current relation and the boundary condition. In the Meissner state, $B = 0$, and the uniform order-parameter minimizing the free energy is $\psi = \psi_0 = \sqrt{\frac{\alpha_0(T-T_c)}{b}}$. It is convenient to transform the Landau free energy to the Gibbs free energy such that we have an explicit function on the control parameter $\boldsymbol{H}$. As these solutions are uniform, for either one of them

$$g[\psi, \boldsymbol{H}] \equiv \frac{G[\psi, \boldsymbol{H}]}{V} = \alpha_0(T - T_c)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{B^2}{2\mu_0} - \boldsymbol{B} \cdot \boldsymbol{H} \tag{3.54}$$

It follows

$$g_{normal}[\psi, \boldsymbol{H}] = -\frac{\mu_0}{2}\boldsymbol{H}^2 \ , \ g_{superc.}[\psi, \boldsymbol{H}] = -\frac{\alpha_0^2(T - T_c)^2}{2b} \tag{3.55}$$

Therefore there is a ‘critical’ field above which the normal state is prevalent relatively to the Meissner

$$H > H_c = \sqrt{\frac{\alpha_0^2(T - T_c)^2}{\mu_0 b}} \tag{3.56}$$

since it imply $g_{\text{supercond}} > g_{\text{normal}}$. When $H = H_c$ both the normal and the Meissner states are degenerate. It suggests the existence of a state of matter in between the normal and superconducting states. The effect of spatial variation in between these two limiting cases is to be considered. In general, we may seek the stability of general solution with respect to the Meissner phase by comparing their energies at the critical temperature. The solutions are stable relatively to the normal state when at the critical field H_c, $\Delta g[H_c] = g_{\text{sol.}} - g_{\text{normal}}|_{H=H_c} < 0$. The normal state energy at H_c is $g_{H_c} = -\frac{\mu_0}{2}H_c^2(-\frac{1}{8\pi}H_c^2$ in cgs unities). In this way,

$$\Delta g[H_c] = \frac{\hbar^2}{2m}|(\boldsymbol{\nabla} - \frac{ie^*\boldsymbol{A}}{\hbar})\psi|^2 + \alpha_0(T - T_c)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{B^2}{2\mu_0} - \boldsymbol{B} \cdot \boldsymbol{H}_c + \frac{\mu_0}{2}H_c^2 \tag{3.57}$$

In fact, each non-uniform solution defines a critical field by setting the last expression equal to zero. We are concerned, though, about comparing the energy of solutions with respect to the critical field of the uniform phase.

3.5 DOMAIN-WALL: SOLUTION AND STABILITY

If we restrict our attention to the domain wall solutions (1d solutions), the space is isotropic in two dimensions such that the energy of a domain-wall slice is

$$\sigma = \frac{\int \Delta g[H_c] d^3x}{A} = \int dx \left\{ \frac{\hbar^2}{2m}|(\boldsymbol{\nabla} - \frac{ie^*}{\hbar}\boldsymbol{A})\psi|^2 + \alpha_0(T - T_c)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{(B - B_c)^2}{2\mu_0} \right\} \tag{3.58}$$

Here we have defined $H_c = B_c/\mu_0$. The thermodynamic stability of the solution relies on the sign of the surface tension. By letting the incidence of the magnetic field to be in the z direction and the magnetic flux to be homogeneous in the normal phase, inside the superconductor we expect $\boldsymbol{B} = (0, 0, B(x))$. It suffices to choose a gauge $\boldsymbol{A} = (0, A(x), 0)$, from which $A'(x) = B(x)$. Due to the isotropy along the direction $y - z$ directions, $\psi(\boldsymbol{x}) = \psi(x)$, $\nabla\psi \cdot \boldsymbol{A} = 0$, yielding

$$\sigma = \int dx \Big\{ \frac{\hbar^2}{2m^*}(\psi'^2 + \frac{e^{*2}A^2}{\hbar^2}\psi^2) + \alpha_0(T - T_c)\psi^2 + \frac{b}{2}\psi^4 + \frac{(B - B_c)^2}{2\mu_0} \Big\} \tag{3.59}$$

It is convenient to use dimensionless variables

$$\tilde{x} = \frac{x}{\lambda} \text{ , } \tilde{\psi} = \frac{\psi}{\psi_0} \text{ , } \tilde{A} = \frac{A}{B_c\lambda} \text{ , } \tilde{B} = \frac{B}{B_c} = \frac{\mu_0 B}{H_c} \text{ , } \bar{\sigma} = \frac{\mu_0}{B_c^2\lambda}\sigma \tag{3.60}$$

, such that (omitting the bar)

$$\sigma = \int dx \{ \frac{\psi'^2}{\kappa^2} + (\frac{1}{2}A^2 - 1)\psi^2 + \frac{1}{2}\psi^4 + \frac{1}{2}(A' - 1)^2 \} \tag{3.61}$$

In this scaled unities, the minimum with respect to ψ and A is found to be

$$-\frac{\psi''}{\kappa^2} + \frac{1}{2}A^2\psi + (\psi^2 - 1)\psi = 0 \tag{3.62}$$

$$A\psi^2 - A'' = 0 \tag{3.63}$$

As we wish to evaluate the surface tension for solutions of these equations it is convenient to rewrite the surface tension in terms of the second derivative,

$$\sigma = \frac{B_c^2\lambda}{2\mu_0} \int dx \{ -2\frac{\psi\psi''}{\kappa^2} + A^2\psi^2 + (\psi^4 - 2\psi^2) + (A' - 1)^2 \} \tag{3.64}$$

Therefore, due to the first GL (3.62),

$$\sigma = \frac{B_c^2\lambda}{2\mu_0} \int dx [\underbrace{(B(x) - 1)^2}_{\text{Mag. field energy}} - \overbrace{\psi(x)^4}^{\text{condensate energy}}] \tag{3.65}$$

In this way, we were able to decouple the energy contribution of the magnetic and condensate field when treating domain walls. The normalized condition Meissner-Ochsenfeld is $B = 0$, $\psi = 1$. In the normal phase, $\psi = 0$ and $B = 1$. At the critical temperature, either of these uniform solutions has the same energy, as we would expect from the prior section. From this, a much stronger conclusion can be drawn. At $\kappa = 1/\sqrt{2}$, known as the Bogomol'nyi point, the relation $B = 1 - |\psi|^2$ is valid, as we prove (chap. 6). Hence, at the Bogomol'nyi point (BP), any domain-wall solution is degenerate. In general, any solution is degenerate at the BP (chap. 6). When κ is modified from this equilibrium point, the relation between the two characteristic lengths change, and either the magnetic or the condensate contributes the most, causing the surface tension to acquire a sign ruling the stability of the solution.

3.6 FINAL REMARKS

Within this chapter, we have covered the relationship between the symmetry-breaking proposal due to N.Bogoliubov and the appearance of coherent states by R.Glauber. We have also considered the physics of a macroscopic wave function in the view of the phenomenological Ginzburg-Landau theory. Now that we have relied on some of the microscopic results to justify the phenomenological macroscopic Ginzburg-Landau theory, we will fully enter the realm of the microscopic theory. Considerations due to the Fhrölich, BCS, and Bogoliubov will be considered in the chapter to follow.

4 MICROSCOPIC THEORY OF SUPERCONDUCTIVITY

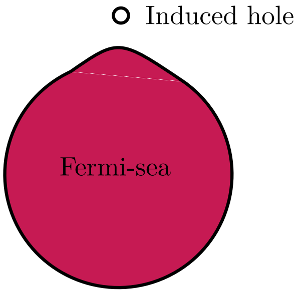
4.1 PROLOGUE

R. A. Ogg Jr. was the first to propose that electrons could act in pairs as a result of material lattice vibrations (JR, 1946). The isotope effect observed in superconductors provided evidence for this. Fröhlich recognized this and proposed his model (FRÖHLICH, 1954), which justified the appearance of an electron-electron attractive interaction mediated by lattice vibration. This was a necessary component in the discovery of the pairing stability (Cooper, 1956)(COOPER, 1956). The Fröhlich model is used as the starting point for a Bardeen-Pines-like (BP) Hamiltonian electron in this chapter, which is the precursor to the Bardeen-Cooper-Schrieffer model (BCS) (BARDEEN; COOPER; SCHRIEFFER, 1957). We demonstrate the formal similarities between the BP and BCS theories. We formulate the Bogoliubov theory (BOGOLIUBOV, 1947) using Landau-Fermi liquid theory reasoning. Then, we specialize to the case of uniformly anomalous averages (uniform gap), recovering the mean-field BCS theory with a cutoff for interaction energy (Heaviside modelling). Additionally, we illustrate (i) the Dyson series for the uniform-gap theory diagrammatically and (ii) demonstrate the theory's early success in describing superconductivity thermodynamics. At the conclusion of the chapter, we present the Hubbard-Stratonovitch (HUBBARD, 1959) method transformation, which provides an alternative interpretation of the results obtained in the preceding section via path integrals. The purpose of this chapter is to (i) introduce the initial predictions of superconductivity using a microscopic theory, and (ii) to introduce the mathematical framework for developing the Ginzburg-Landau and recently developed Extended Ginzburg-Landau Theory, both of which have as their object the non-uniform gap in increasing accuracy orders.

4.2 THE FRÖHLICH HYPOTHESIS

Consider how an electron distorts its environment, resulting in the presence of a 'positive' environment that can be thought of as an induced hole. We will consider the effect of this hole interacting with the lattice by distorting the Fermi-size sea's but not sufficiently to remove an electron from the Fermi-sea.

Figure 5 – A sketch of the distortion of the Fermi-surface by a hole induced by the passage of a fast electron.



Source: The author

In general, the fermi-energy is locally related to the density of electrons $\rho(\boldsymbol{x})$ [1],

$$\varepsilon_F(\boldsymbol{x}) = \frac{1}{2m}(3\pi^2\rho(\boldsymbol{x}))^{2/3} \tag{4.1}$$

Supposing no particle modification occur within the shell, the sole effect being the distortion of the volume in near-zero temperature

$$\delta\varepsilon_F(\boldsymbol{x}) = -\frac{2}{3}\varepsilon_F\frac{\delta V}{V}(\boldsymbol{x}) \tag{4.2}$$

The above is the direct-space representation of the reciprocal-space disturbance of the Fermi-energy ($\boldsymbol{x}$ lies in the vicinity of each of the atoms uttermost shells). The total change of the energy in the direct-space is provided by the number of electrons between $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{dx}$ times the energy modification per electron in between this interval,

$$\delta E = -\frac{2}{3}\varepsilon_F\sum_{\sigma\boldsymbol{x}} c^\dagger_\sigma(\boldsymbol{x})c_\sigma(\boldsymbol{x})\frac{\delta V}{V}(\boldsymbol{x}) \tag{4.3}$$

We define the displacement field $\boldsymbol{\Phi}$ as in elasticity theory (Appendix C.8 for details),

$$\delta V(\boldsymbol{x}) \equiv \boldsymbol{\Phi}\cdot\boldsymbol{\Delta S}\ ,\ \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} = \lim_{V\to 0}\frac{\delta V}{V} \tag{4.4}$$

$\Phi(\boldsymbol{x})$ measures the displacement of Φ each point $\boldsymbol{x}$ of the material under deformation. From (4.3) and (4.4),

$$H_I = -\frac{2}{3}\varepsilon_F\int d^3\boldsymbol{x}c^\dagger_\sigma(\boldsymbol{x})c_\sigma(\boldsymbol{x})\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\exp[i\boldsymbol{q}\cdot\boldsymbol{x}] \tag{4.5}$$

In connection, the displacement of longitudinal phonon modes is provided

$$\boldsymbol{\Phi}(\boldsymbol{x}) = -i\sum_{\boldsymbol{q}}\hat{\boldsymbol{x}}_{\boldsymbol{q}}\Phi_{\boldsymbol{q}}\exp[i\boldsymbol{q}\cdot\boldsymbol{x}]\ , \tag{4.6}$$

$$\Phi_{\boldsymbol{q}} = \Delta\boldsymbol{x}_{\omega_{\boldsymbol{q}}}\left[b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}}\right]\ ,\ \Delta\boldsymbol{x}_{\omega_{\boldsymbol{q}}} = \sqrt{\frac{\hbar}{2M_{\text{lattice}}\omega_{\boldsymbol{q}}}} \tag{4.7}$$

---

[1] $\rho = \frac{N}{V} = \frac{1}{V}\sum_{(x,y,z)}\frac{\Delta k_x}{(2\pi/L_x)}\frac{\Delta k_y}{(2\pi/L_y)}\frac{\Delta k_z}{(2\pi/L_z)}\Theta(\varepsilon_F - \varepsilon_{\boldsymbol{k}}) = \frac{1}{\pi^3}\int dk4\pi k^2\theta(\varepsilon_{k_F} - \varepsilon_{\boldsymbol{k}}) = \frac{k_F^3}{3\pi^2}.$

a wave-like displacement expression for a set of coupled harmonic oscillators for different crystal modes. The factor $-i$ is introduced as to assure the hermiticity of $\nabla \cdot \Phi$, and thus, of the Hamiltonian (the minus sign selects $\nabla \cdot \boldsymbol{\Phi} > 0$, $\delta V > 0$ as standard, due to the tendency of the hole of pulling the Fermi-surface). In the reciprocal space,

$$c_\sigma(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma} \exp[i\boldsymbol{k} \cdot \boldsymbol{x}] \tag{4.8}$$

Thus, provided (4.6),

$$H_{\mathsf{I}} = \sum_{\boldsymbol{q}} g_{\boldsymbol{q}} c^\dagger_{\boldsymbol{k}'\sigma} c_{\boldsymbol{k}\sigma} [b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}}] \frac{1}{V} \int d^3x \exp[i(\boldsymbol{q} + \boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{x}] \tag{4.9}$$

$$H_{\mathsf{I}} = \sum_{\boldsymbol{q}} g_{\boldsymbol{q}} c_{\boldsymbol{k}+\boldsymbol{q}\sigma} c_{\boldsymbol{k}\sigma} [b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}}] \text{ , } g_{\boldsymbol{q}} = -\frac{2}{3} \varepsilon_F q \Delta \boldsymbol{x}_{\omega_{\boldsymbol{q}}} \tag{4.10}$$

In this way, the Fröhlich model is written

$$H = \underbrace{\sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left[b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} + \frac{1}{2} \right]}_{\text{phonon}} + \underbrace{\sum_{\boldsymbol{k}} \omega_{\boldsymbol{q}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}}}_{\text{fermion}} + \underbrace{\sum_{\sigma \boldsymbol{k} \boldsymbol{q}} g_{\boldsymbol{q}} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma} c_{\boldsymbol{k}\sigma} (b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}})}_{\text{interaction}} \tag{4.11}$$

The above Hamiltonian carries remarkable consequences. The modification of the partition function modification for the system is

$$Z = Z_0 \left\langle \mathcal{T} \exp\left[- \int_0^\beta V_I(\tau) d\tau \right] \right\rangle_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 ... d\tau_n \left\langle \mathcal{T} V_I(\tau_1) V_I(\tau_2) ... V_I(\tau_n) \right\rangle \tag{4.12}$$

with

$$\left\langle (...) \right\rangle_0 = \frac{\mathrm{Tr}[\exp[-\beta H_0](...)]}{\mathrm{Tr}[\exp[-\beta H_0]]} \tag{4.13}$$

where (...) refers to any physical measurable. By considering the average on the Fermionic $\otimes$ Bosonic space, the first term of the expansion cancels ($\langle b_k \rangle = 0$). As for the second, it is the first non-null term. Non-zero terms mixes an equal number of the creation and annihilation to phonons,

$$\begin{aligned} \frac{Z}{Z_0} = 1 + \frac{1}{2} \int_0^\beta d\tau d\tau_0 \sum_{\sigma\sigma'; \boldsymbol{q}\boldsymbol{q}'\boldsymbol{k}\boldsymbol{k}'} g_{q'} g_q \Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}'+\boldsymbol{q}'\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c_{\boldsymbol{k},\sigma}(\tau_0) \\ \Big[b_{\boldsymbol{q}'}(\tau) b^\dagger_{-\boldsymbol{q}}(\tau_0) + b^\dagger_{-\boldsymbol{q}'}(\tau) b_{\boldsymbol{q}}(\tau_0) \Big] \Big\rangle_0 \end{aligned} \tag{4.14}$$

As the bosonic and fermionic operators are in independent Hilbert spaces, the time-ordering of the product equals the time-ordering over each contraction set. Only $\boldsymbol{q}' = -\boldsymbol{q}$ is selected.

such that

$$\frac{Z}{Z_0} = 1 - \frac{1}{2} \sum_{\sigma\sigma'\boldsymbol{q}\boldsymbol{k}\boldsymbol{k}'} \int_0^\beta d\tau d\tau_0 g_{\boldsymbol{q}}^2 \Big\langle \mathcal{T}\Big[b_{-\boldsymbol{q}}(\tau) b^\dagger_{-\boldsymbol{q}}(\tau_0) + b^\dagger_{\boldsymbol{q}}(\tau) b_{\boldsymbol{q}}(\tau_0) \Big] \Big\rangle_0 \\ \Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c_{\boldsymbol{k}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) \Big\rangle_0 \tag{4.15}$$

By applying the time-ordering (Appendix A.2) of fermions[2]

$$\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) c_{\boldsymbol{k}'\sigma'}(\tau) \Big\rangle_0 = -\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) \Big\rangle_0 \tag{4.17}$$

and defining the dimensionless quantity

$$\Big\langle \mathcal{T} \frac{\Phi_{\boldsymbol{q}'}(\tau)\Phi^\dagger_{\boldsymbol{q}}(\tau_0)}{\Delta^2_{\omega_{\boldsymbol{q}}}} \Big\rangle_0 = \overbrace{\Big\langle \mathcal{T}\Big[b_{-\boldsymbol{q}}(\tau) b^\dagger_{-\boldsymbol{q}}(\tau_0) + b^\dagger_{\boldsymbol{q}}(\tau) b_{\boldsymbol{q}}(\tau_0) \Big] \Big\rangle_0}^{\equiv \mathcal{D}_{\text{ph.}\boldsymbol{q}}(\tau-\tau_0)} \delta_{\boldsymbol{q}',-\boldsymbol{q}} \tag{4.18}$$

with $\mathcal{D}_{\text{ph.}\boldsymbol{q}}$ frequently referred as the dimensionless elastic (or phonon) propagator. Explicitly, the contraction in the index for the exchange of momenta yields,

$$\frac{Z}{Z_0} = 1 + \frac{1}{2} \sum_{\sigma\sigma'\boldsymbol{q}\boldsymbol{k}\boldsymbol{k}'} \int_0^\beta d\tau d\tau_0 g_{\boldsymbol{q}}^2 \mathcal{D}_{\text{ph.}\boldsymbol{q}}(\tau-\tau_0) \Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) \Big\rangle_0 \tag{4.19}$$

We have ignored higher-order $2n$-legged diagrams with $n > 2$ in favor of the four-legged diagram, but the same reasoning applies. By contracting the momentum $\boldsymbol{q}$, and the four-operator average through Wick's theorem for finite temperature (for a reference, section (8.4) of (COLEMAN, 2015)).

$$\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) \Big\rangle_0 = \underbrace{-\mathcal{G}^{(0)}_{\boldsymbol{k}}(0)\mathcal{G}^{(0)}_{\boldsymbol{k}'}(0)\delta_{\boldsymbol{q}0}}_{\text{Hartree term}} + \\ \underbrace{\delta_{\sigma\sigma'}\delta_{\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{q}}\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau_0-\tau)\mathcal{G}^{(0)}_{\boldsymbol{k}'}(\tau-\tau_0)}_{\text{Fock term}} \tag{4.20}$$

It is convenient to consider the variables $\tau - \tau_0$ and $\tau + \tau_0$ [3]. Carrying out the sum in the spin index and shifting the momentum sign due to the arbitrariness of the sum over the momenta,

$$\frac{Z}{Z_0} = 1 - \frac{\beta}{2} \int_0^\beta d\tau \sum_{\boldsymbol{k}\boldsymbol{k}'} \Big[(2S+1)^2 \mathcal{G}^{(0)}_{\boldsymbol{k}}(0) V_{\text{eff}\boldsymbol{0}}(\tau) \mathcal{G}^{(0)}_{\boldsymbol{k}'}(0) - \\ (2S+1)\mathcal{G}^{(0)}_{\boldsymbol{k}}(-\tau) V_{\text{eff}(\boldsymbol{k}'-\boldsymbol{k})}(\tau) \mathcal{G}^{(0)}_{\boldsymbol{k}'}(\tau) \Big] \tag{4.21}$$

---

[2]

$$\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c_{\boldsymbol{k},\sigma}(\tau_0) \Big\rangle = -\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c_{\boldsymbol{k}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) \Big\rangle_0 = \\ \Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) c_{\boldsymbol{k}'\sigma'}(\tau) \Big\rangle_0 = -\Big\langle \mathcal{T} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\sigma}(\tau_0) c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\sigma'}(\tau) c_{\boldsymbol{k}'\sigma'}(\tau) c_{\boldsymbol{k}\sigma}(\tau_0) \Big\rangle_0 \tag{4.16}$$

.

[3] The integrand content is a function of $\tau - \tau_0$. The integration on the second variable becomes trivial and equal to β - consider, for instance, the case $\tau > \tau_0$.

with

$$V_{\mathsf{eff}.\boldsymbol{q}}(\tau) \equiv g_{\boldsymbol{q}}^2 \mathcal{D}_{\mathsf{ph}.\boldsymbol{q}}(\tau) \tag{4.22}$$

Its convenient to evaluate directly the phonon propagator

$$\mathcal{D}_{\mathsf{ph}.\boldsymbol{q}}(\tau) = \underbrace{\theta(\tau)(1+n_{\boldsymbol{q}}(1))\exp[\omega_{\boldsymbol{q}}\tau]}_{(i\nu-\omega_q)^{-1}} - \underbrace{\theta(-\tau)(1+n_{\boldsymbol{q}}(1))\exp[-\omega_{\boldsymbol{q}}\tau]}_{(i\nu+\omega_{\boldsymbol{q}})^{-1}} \ , \tag{4.23}$$

where we identify it is written in terms of the frequencies of forward and backwards propagating bosons by considering the zero-temperature limit scenario (Appendix A.6). Within the Fhrölich model, the mechanism responsible for mediating the interaction between the electrons is the phonon. The phonon acquires a momentum equal to the difference of the momenta between the involved electrons, $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$. From both of these facts, we conclude that it is the movement of the phonon going backwards and forwards in between electrons which causes them to exchange momenta and implying the appearance of an effective interaction.

$$\mathcal{D}^{(0)}_{\mathsf{ph}.\boldsymbol{q}}(i\nu) = \frac{2\omega_{\boldsymbol{q}}}{(i\nu)^2 - \omega_{\boldsymbol{q}}^2} \tag{4.24}$$

We recall the Fourier series in Matsubara frequencies

$$\mathcal{D}_{\mathsf{ph}.\boldsymbol{q}}(\tau) = \frac{1}{\beta}\sum_{\nu} \mathcal{D}_{\mathsf{ph}.\boldsymbol{q}}(i\nu)\exp[-i\nu\tau] \ , \ \nu \in \{ \text{ even } \} \tag{4.25}$$

$$\mathcal{G}^{(0)}_{\boldsymbol{k}}(\tau) = \frac{1}{\beta}\sum_{\omega} \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega)\exp[-i\omega\tau] \ , \ \omega \in \{ \text{ odd } \} \tag{4.26}$$

Both the Fock and Hartree terms are present; we examine them separately.

The Fock Interaction

From the Fourier series (4.25,4.26) into (4.21) for the Fock term,

$$\frac{Z}{Z_0} = 1 + \frac{1}{\beta^2}\sum_{\boldsymbol{k}i\omega}\int_0^\beta d\tau \sum_{\boldsymbol{k}'i\omega' i\nu} \mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega)[g^2_{\boldsymbol{k}'-\boldsymbol{k}}\mathcal{D}_{\mathsf{ph}(\boldsymbol{k}'-\boldsymbol{k})}(i\nu)]$$

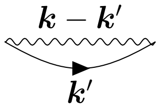
$$\mathcal{G}^{(0)}_{\boldsymbol{k}'}(i\omega')\exp[-i(\omega'-\omega+\nu)\tau] \tag{4.27}$$

The integral selects $\omega' = \omega - \nu$ (the minus sign below is due to the exponential being odd),

$$\frac{Z}{Z_0} = 1 - T\sum_{\boldsymbol{k}i\omega}\Big\{\sum_{\boldsymbol{k}'i\nu}\underbrace{\frac{1}{i\omega-\omega_{\boldsymbol{k}}}}_{\mathcal{G}^{(0)}_{\boldsymbol{k}}(i\omega)}\underbrace{\left[g^2_{\boldsymbol{k}'-\boldsymbol{k}}\frac{2\omega_{\boldsymbol{k}'-\boldsymbol{k}}}{(i\nu)^2-\omega^2_{\boldsymbol{k}'-\boldsymbol{k}}}\right]}_{V_{\mathsf{eff}.(\boldsymbol{k}'-\boldsymbol{k})}(i\nu)}\underbrace{\frac{1}{i\omega-i\nu-\omega_{\boldsymbol{k}'}}}_{\mathcal{G}^{(0)}_{\boldsymbol{k}'}(i\omega-i\nu)}\Big\} \tag{4.28}$$

We identify the self energy as a combination of the phonon propagator and the electron propagator (Appendix C.3), corresponding to the diagrammatic representation in chap. 2.

$$\Sigma_{\text{Fock},\boldsymbol{k}}(i\omega) = T\sum_{\boldsymbol{k}',i\nu} g^2_{\boldsymbol{k}-\boldsymbol{k}'}\frac{2\omega_{\boldsymbol{k}-\boldsymbol{k}'}}{[(i\nu)^2-\omega^2_{\boldsymbol{k}-\boldsymbol{k}'}]}\frac{1}{[i\omega-i\nu-\omega_{\boldsymbol{k}'}]} = \tag{4.29}$$



It is convenient to explicitly separate the forward and backward contribution of the phonon propagator. To carry out the sum we apply the contour integral method to obtain.

$$-T\sum_{i\nu} = \oint_{\mathsf{C}}\frac{dz}{2\pi i}n_z(1)\Sigma_{\text{Fock},\boldsymbol{k}}(z),\ \text{C is counterclockwise oriented} \tag{4.30}$$

By analytically continuing the result to the complex plane ($i\omega \to z$), while separating the forward and backwards propagating bosons,

$$\Sigma_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{k}'} g^2_{\boldsymbol{k}-\boldsymbol{k}'}\left[\frac{1+n_{\boldsymbol{k}-\boldsymbol{k}'}(1)-n_{\boldsymbol{k}'}(-1)}{z-(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}-\boldsymbol{k}'})}+\frac{n_{\boldsymbol{k}-\boldsymbol{k}'}(1)+n_{\boldsymbol{k}'}(-1)}{z-(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}'-\boldsymbol{k}})}\right] \tag{4.31}$$

In the zero-temperature limit, $n(1) \to 0,\ z \to \omega \in \mathrm{Re}$ (Appendix A.6). Even though there is interaction, we may consider the zero-interaction limit, in which quasi-holes can only be found below the Fermi surface, and quasi-electrons above the Fermi-surface. With this consideration, the first term designates a quasi-particle contribution, as it only survives $k' > k_F$; the second term designates a quasi-hole contribution - only non-zero to $k' < k_F$. Therefore, $-\omega_{\boldsymbol{k}'} > 0$ in the second term and we identify positive and negative contributions modifying the energy of the propagating electron while it interacts with the forward and back-propagating boson.

$$\Sigma_{\boldsymbol{k}}(\omega) = -\sum_{|\boldsymbol{k}'|>k_F} g^2_{\boldsymbol{k}-\boldsymbol{k}'}\left[\frac{1}{(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}-\boldsymbol{k}'})-\omega}\right]+\sum_{|\boldsymbol{k}'|<k_F} g^2_{\boldsymbol{k}-\boldsymbol{k}'}\left[\frac{1}{\omega+(|\omega_{\boldsymbol{k}'}|+\omega_{\boldsymbol{k}-\boldsymbol{k}'})}\right] \tag{4.32}$$

The first term contains the interaction between the forward-propagating boson and the quasi-particle, contributing for minimizing the energy. On the second, we have the interaction of the backward-propagating boson with the quasi-hole (or backward-propagating electron), contributing to the increase of the energy. Notice that to find the renormalized energy correction to a higher-order we may solve for ω^* the relation

$$\Sigma_{\boldsymbol{k}}(\omega^*) = \omega^* - \omega \tag{4.33}$$

As the dependence on ω is solely on the $\boldsymbol{k}$; solving is to obtain the renormalized energy $\omega^*_{\boldsymbol{k}}$.

The Hartree Interaction

From a mathematical standpoint, the Hartree interaction is simpler to analyze. The Fock contribution to self-energy is both momentum and frequency independent. It is equivalent to i) a collision with no exchange of momentum and ii) an instantaneous interaction. One could think of self-energy as causing a particle with a given momentum to transform into a particle-hole pair with zero lifetime 'before' reverting to its particle state. This is similar to the bubble portion of chapter's two third diagram, except that nothing emerges from the bubble at a different time. This behavior is denoted by a diagram displayed in chap. 2, which takes on the mathematical form in the Fröhlich model.

$$\Sigma_{\text{Hartree}} = \quad = \sum_{\boldsymbol{k}} \mathcal{G}^{(0)}_{\boldsymbol{k}}(0) V_{\text{eff}\mathbf{0}}(0) = (2S+1)^2 \sum_{\boldsymbol{k}} \frac{1}{\omega_{\boldsymbol{k}}} \lim_{\boldsymbol{q}\to 0}[-\frac{2g^2_{\boldsymbol{q}}}{\omega_{\boldsymbol{q}}}] \tag{4.34}$$

It is usual to denote $g \equiv \lim_{\boldsymbol{q}\to 0} \frac{2g^2_{\boldsymbol{q}}}{\omega_{\boldsymbol{q}}}$, and, in most materials, $\lambda \equiv N(0)g < 1$, λ is the 'coupling constant', with $N(0)$, as before, the density of states at the Fermi-level.

Comments on the Fröhlich Model

From the Fröhlich self-energy, it is clear that the effective interaction is smaller away from the Fermi-energy. In regions away from the Fermi-surface, the lifetime of excited particles is much smaller than any adiabatic switch on of interaction, and the theory of Landau would fail; however, this is the limit where, in the Fröhlich model, the contribution due to the perturbation is less relevant - in agreement with Landau considerations. When $\nu = 0$, the effective interaction is negative, and as this is the term contributing the most, the total effective interaction is negative. We conclude that the phonon mediating the interaction between both electrons causes them to attract each other. By attracting one another, the question arises as to what happens next; is the paired state stable? Indeed, the electron pairing is stabilized (COOPER, 1956). Other hypotheses for the mediation of electron-electron pairings have permeated the literature most recently, such as magnetism, etc. We will not focus on the detailed microscopic mechanisms of particular systems, but rather on the general consequences of the BCS model.

4.3 FRÖHLICH MODEL, BARDEEN-PINES, BCS AND LANDAU THEORIES

In low-energy physics, it is common to keep only the $i\nu = 0$ contribution, corresponding to the largest term contributing to the self-energy.

$$\max\{V_{\text{eff}\boldsymbol{q}}(i\nu)\} = V_{\text{eff}\boldsymbol{q}}(0) = -\frac{2g^2_{\boldsymbol{q}}}{\omega_{\boldsymbol{q}}} < 0 \tag{4.35}$$

From the Fhrölich model, in this limit, we have, effectively, to first-order of the expansion, a theory of the kind

$$H_{\text{I,BP}} = \sum_{\sigma\sigma';\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} V_{\text{eff}\boldsymbol{q}} c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q}\sigma'} c_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}\sigma} \text{ with } V_{\text{eff}\boldsymbol{q}} = -\frac{2g^2_{\boldsymbol{q}}}{\omega_{\boldsymbol{q}}} \ , \tag{4.36}$$

With a displacement of the kind $\boldsymbol{k}' \rightarrow \boldsymbol{k}' + \boldsymbol{q}$, the above becomes

$$H_{\text{I,BP}} = \sum_{\sigma\sigma';\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} V_{\text{eff}\boldsymbol{q}} c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma} c^{\dagger}_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}'+\boldsymbol{q}\sigma'} c_{\boldsymbol{k}\sigma} \ , \tag{4.37}$$

which is precisely the Bardeen-Pines Hamiltonian (BP). We notice that an average of the BP hamiltonian over the Fermi-Landau vacuum at zero-temperature yields a theory within the scope of the Landau model

$$\langle H_{\text{BP}} \rangle = \sum_{\boldsymbol{k}\sigma} \omega_{\boldsymbol{k}\sigma} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}\boldsymbol{k}'} [V_{\boldsymbol{0}} - \frac{1}{2} V_{\boldsymbol{k}-\boldsymbol{k}'} \delta_{\sigma\sigma'}] n_{\boldsymbol{k}'\sigma'}(-1) n_{\boldsymbol{k}\sigma}(-1) \tag{4.38}$$

(see Appendix A.6 for the meaning of $n(\zeta = -1)$). The eq. (4.37) is the interaction part of the Bardeen-Pines Hamiltonian (ignoring the particular form of the interaction). This Hamiltonian model is a predecessor of the Bardeen-Cooper-Schrieffer Hamiltonian in considering low-energy physics. Let us consider the displacement $\boldsymbol{q} \rightarrow [\boldsymbol{q} - (\boldsymbol{k} + \boldsymbol{k}')]$,

$$H_{\text{I,BCS}} = \sum V_{\boldsymbol{q}-(\boldsymbol{k}+\boldsymbol{k}')} c^{\dagger}_{-\boldsymbol{k}'\sigma} c^{\dagger}_{\boldsymbol{k}'\sigma'} c_{-\boldsymbol{k}\sigma'} c_{\boldsymbol{k}\sigma} \tag{4.39}$$

But provided the low-energy consideration of momentum conservation, $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$, and the center of mass of the electrons to be fixed, $\boldsymbol{k} \sim -\boldsymbol{k}'$, the approach $V_{-2\boldsymbol{k}'} \sim V_{2\boldsymbol{k}} \sim V_{\boldsymbol{k}-\boldsymbol{k}'}$ and the BCS modelling,

$$H_{\text{BCS}} = \sum_{\boldsymbol{k}\sigma} \omega_{\boldsymbol{k}\sigma} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum V_{\boldsymbol{k}-\boldsymbol{k}'} c^{\dagger}_{-\boldsymbol{k}'\sigma} c^{\dagger}_{\boldsymbol{k}'\sigma'} c_{-\boldsymbol{k}\sigma'} c_{\boldsymbol{k}\sigma} \tag{4.40}$$

The average on the Fermi vacuum yields a limiting case of the prior result, as we see

$$\langle H_{\text{I,BCS}} \rangle_{\text{FLV}} = \sum_{\boldsymbol{k},\sigma\neq\sigma'} V_{\boldsymbol{k}} n_{\boldsymbol{k}\sigma}(-1) n_{-\boldsymbol{k}\sigma'}(-1) \tag{4.41}$$

This is a version of the Fermi-Landau theory where the interaction due to which the "exchange" of identical particles is ignored. It might be also be seen as the result of neglecting the exchange interaction in the Bardeen-Pines (BP) model, and considering $V_0 \rightarrow V_{\boldsymbol{k}-\boldsymbol{k}'}$. The first historical choice for modelling the interaction has been the s-wave interaction

$$V_{\boldsymbol{k}} = -g[1 - \Theta(\omega_{\boldsymbol{k}} - \omega_D)] \tag{4.42}$$

with Θ denoting the Heaviside function. As we are about to see, the mean-field Heaviside modelling due to BCS is reproduced by the mean-field Bogoliubov formulation with uniform anomalous averages. In the next section we begin by providing the fundamental connection between the Fermi-Landau theory and the Bogoliubov theory of superconductivity.

4.4 THE BOGOLIUBOV THEORY OF SUPERCONDUCTIVITY

Bogoliubov considered the interaction part of the Hamiltonian to look like the Landau energy in the position space,

$$H_I = \sum_\sigma \int \boldsymbol{dxdx'} V_{\boldsymbol{x}-\boldsymbol{x}'} n_\sigma(\boldsymbol{x}) n_{\sigma'}(\boldsymbol{x}') = \int \boldsymbol{dxdx'} V_{\boldsymbol{x}-\boldsymbol{x}'} \sum_\sigma c^\dagger_\sigma(\boldsymbol{x}) c_\sigma(\boldsymbol{x}) \sum_{\sigma'} c^\dagger_{\sigma'}(\boldsymbol{x}') c_{\sigma'}(\boldsymbol{x}') \tag{4.43}$$

Then, we may normal-order [4] it. If $\sigma \neq \sigma'$, we simply commute the operators without residue, while if $\sigma = \sigma'$ we have to take into account the energy due to $\boldsymbol{x} = \boldsymbol{x}'$.

$$\begin{aligned} H_I &= \sum_\sigma \int \boldsymbol{dxdx'} V_{\boldsymbol{x}-\boldsymbol{x}'} n_\sigma(\boldsymbol{x}) n_{\sigma'}(\boldsymbol{x}') = \\ &= \int \boldsymbol{dxdx'} \sum_{\{\sigma,\sigma'\},\sigma\neq\sigma'} V_{\boldsymbol{x}-\boldsymbol{x}'} c^\dagger_\sigma(\boldsymbol{x}) c^\dagger_{\sigma'}(\boldsymbol{x}) c_{\sigma'}(\boldsymbol{x}') c_\sigma(\boldsymbol{x}') + \sum_\sigma \int \boldsymbol{dxdx'} V_{\boldsymbol{0}} c^\dagger_\sigma(\boldsymbol{x}) c_\sigma(\boldsymbol{x}) \end{aligned}$$

Neglecting the last term whose sole effect is to uniformly displace the energy per particle, the kinetic term,

$$H_I = \sum_\sigma \int \boldsymbol{dxdx'} n_\sigma(\boldsymbol{x}) n_{\sigma'}(\boldsymbol{x}') = \int \boldsymbol{dxdx'} V_{\boldsymbol{x}-\boldsymbol{x}'} \sum_\sigma c^\dagger_\sigma(\boldsymbol{x}) c_\sigma(\boldsymbol{x}) \sum_{\sigma'} c^\dagger_{\sigma'}(\boldsymbol{x}') c_{\sigma'}(\boldsymbol{x}') \tag{4.44}$$

Denoting $\boldsymbol{y} = (\boldsymbol{x}, \sigma)$ and $\int \boldsymbol{dy} \equiv \sum_\sigma \boldsymbol{dx}$, we apply the mean-field approximation (Appendix C.4),

$$\begin{aligned} H_I = \int \boldsymbol{dydy'} V_{\boldsymbol{y}',\boldsymbol{y}} c^\dagger(\boldsymbol{y}) c^\dagger(\boldsymbol{y}') c(\boldsymbol{y}') c(\boldsymbol{y}) = \int \boldsymbol{dydy'} V_{\boldsymbol{y}-\boldsymbol{y}'} \Big[\langle c^\dagger(\boldsymbol{y}) c^\dagger(\boldsymbol{y}') \rangle c(\boldsymbol{y}') c(\boldsymbol{y}) + \\ c^\dagger(\boldsymbol{y}) c^\dagger(\boldsymbol{y}') \langle c(\boldsymbol{y}') c(\boldsymbol{y}) \rangle - \langle c^\dagger(\boldsymbol{y}) c^\dagger(\boldsymbol{y}') \rangle \langle c(\boldsymbol{y}') c(\boldsymbol{y}) \rangle \Big] \end{aligned} \tag{4.45}$$

The assumption due to Bogoliubov is

$$V_{\boldsymbol{x}-\boldsymbol{x}'} = -g\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{4.46}$$

which implies in the $\boldsymbol{k}$ space that $V_{\boldsymbol{k}}$ is uniform, which already provides a brief indication of the connection to the energy-cutoff modelling in (4.42). The Hamiltonian is rewritten as

$$H = \sum_\sigma \int \boldsymbol{dx} c^\dagger_\sigma(\boldsymbol{x}) T_{\boldsymbol{x}} c_\sigma(\boldsymbol{x}) + \int \boldsymbol{dx} [c^\dagger_\uparrow(\boldsymbol{x}) \Delta(\boldsymbol{x}) c^\dagger_\downarrow(\boldsymbol{x}) + c_\downarrow(\boldsymbol{x}) \Delta^*(\boldsymbol{x}) c_\uparrow(\boldsymbol{x})] + \frac{|\Delta(\boldsymbol{x})|^2}{g} \tag{4.47}$$

$$\boldsymbol{T_x} \equiv -\frac{\hbar^2}{2m}(\boldsymbol{\nabla} - i\frac{\boldsymbol{A}}{\Phi_0})^2 - \varepsilon_F \ , \ \Delta(\boldsymbol{x}) \equiv -g\langle c_\downarrow(\boldsymbol{x}) c_\uparrow(\boldsymbol{x}) \rangle \tag{4.48}$$

---

[4] The process of moving the annihilation operators to the right (priority of its application on the vector space).

We may rewrite the kinetic term as

$$\int \boldsymbol{dx}[c^\dagger_\uparrow(\boldsymbol{x})\boldsymbol{T_x}c_\uparrow(\boldsymbol{x}) + c^\dagger_\downarrow(\boldsymbol{x})\boldsymbol{T_x}c_\downarrow(\boldsymbol{x})] = [c^\dagger_\uparrow(\boldsymbol{x})\boldsymbol{T_x}c_\uparrow(\boldsymbol{x}) - c_\downarrow(\boldsymbol{x})\boldsymbol{T}^*_{\boldsymbol{x}}c^\dagger_\downarrow(\boldsymbol{x})] \ , \tag{4.49}$$

provided an integration by parts on the second term and assuming the vanishing of the surface term. More will be stated on this condition in the chapter to follow. By defining the 'Nambu-spinor' vector

$$\Psi(\boldsymbol{x},\tau) \equiv \begin{pmatrix} c_\uparrow(\boldsymbol{x},\tau) \\ c^\dagger_\downarrow(\boldsymbol{x},\tau) \end{pmatrix} \ , \tag{4.50}$$

the Hamiltonian may be represented as

$$H = \int \boldsymbol{dx}\Psi^\dagger(\boldsymbol{x})\varepsilon(\boldsymbol{x})\Psi(\boldsymbol{x}) = \boldsymbol{\Psi}^\dagger\boldsymbol{\varepsilon}\boldsymbol{\Psi} \tag{4.51}$$

$$\varepsilon(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{T_x} & \Delta(\boldsymbol{x}) \\ \Delta^*(\boldsymbol{x}) & -\boldsymbol{T}^*_{\boldsymbol{x}} \ . \end{pmatrix} \tag{4.52}$$

$\boldsymbol{\Psi}$ is easily checked to obey the fermionic commutation relations in the matrix space,

$$\{\boldsymbol{\Psi}^\dagger, \boldsymbol{\Psi}\} = \boldsymbol{1} \tag{4.53}$$

The Go'rkov-Nambu equations in the basis-independent form (Appendix C.2) is summarized in

$$(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})\boldsymbol{\mathcal{G}} = -\boldsymbol{1} \ , \tag{4.54}$$

$$\boldsymbol{\mathcal{G}} = -\langle \mathcal{T}\boldsymbol{\Psi}\boldsymbol{\Psi}^\dagger \rangle \ , \tag{4.55}$$

providing each matrix component explicitly and contracting in the position and time basis,

$$\begin{aligned}\boldsymbol{\mathcal{G}}(\boldsymbol{x},\tau,\boldsymbol{x}',\tau') \equiv& \begin{pmatrix} \mathcal{G}(\boldsymbol{x}\tau;\boldsymbol{x}'\tau') & \mathcal{F}(\boldsymbol{x}\tau;\boldsymbol{x}'\tau') \\ \tilde{\mathcal{F}}(\boldsymbol{x}\tau;\boldsymbol{x}'\tau') & \tilde{\mathcal{G}}(\boldsymbol{x},\tau;\boldsymbol{x}'\tau') \end{pmatrix} = \\ =& - \begin{pmatrix} \langle \mathcal{T}c_\uparrow(\boldsymbol{x},\tau)c^\dagger_\uparrow(\boldsymbol{x}'\tau')\rangle & \langle \mathcal{T}c_\uparrow(\boldsymbol{x}\tau)c_\downarrow(\boldsymbol{x}'\tau')\rangle \\ \langle \mathcal{T}c^\dagger_\downarrow(\boldsymbol{x}\tau)c^\dagger_\uparrow(\boldsymbol{x}'\tau')\rangle & \langle \mathcal{T}c^\dagger_\downarrow(\boldsymbol{x}\tau)c_\downarrow(\boldsymbol{x}'\tau')\rangle \end{pmatrix}\end{aligned} \tag{4.56}$$

By comparing (4.48) to the above defined correlations $\mathcal{F}$ and $\tilde{\mathcal{F}}$, we have the known coupling properties,

$$\lim_{\tau-\tau'\to0^+,\boldsymbol{x}'\to\boldsymbol{x}} \mathcal{F}(\boldsymbol{x}\tau;\boldsymbol{x}'\tau') = -\langle c_\uparrow(\boldsymbol{x})c_\downarrow(\boldsymbol{x})\rangle = -\frac{\Delta}{g} \tag{4.57}$$

$$\lim_{\tau-\tau'\to0^+,\boldsymbol{x}'\to\boldsymbol{x}} \tilde{\mathcal{F}}(\boldsymbol{x}\tau;\boldsymbol{x}'\tau') = -\langle c^\dagger_\downarrow(\boldsymbol{x})c^\dagger_\uparrow(\boldsymbol{x})\rangle = -\frac{\Delta^*}{g} \tag{4.58}$$

. The dynamics for the matrix is expressed in the coined Gor'kov-Nambu representation,

$$(\partial_\tau + \varepsilon_{\boldsymbol{x}})\boldsymbol{\mathcal{G}}(\boldsymbol{x},\tau;\boldsymbol{x}',\tau') = -\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau')\hat{1}$$

Considering a representation in momentum, equivalent to a Fourier transform, we have

$$(\boldsymbol{\partial}_\tau + \boldsymbol{\varepsilon}_{\boldsymbol{k}})\boldsymbol{\mathcal{G}}_{\boldsymbol{k},\boldsymbol{k}'}(\tau,\tau') = -\mathbf{1}\delta(\tau-\tau') \tag{4.59}$$

However the Fourier transform presents the extra-information that $\boldsymbol{\mathcal{G}}$ is diagonal in its momentum index provided the system is invariant by translation $\boldsymbol{\mathcal{G}}(\boldsymbol{x},\tau;\boldsymbol{x}'\tau') = \boldsymbol{\mathcal{G}}(\boldsymbol{x}-\boldsymbol{x}';\tau,\tau')$. By transforming to the frequency domain, we identify

$$\boldsymbol{\mathcal{G}}_{\boldsymbol{k}} \equiv (i\omega - \boldsymbol{\varepsilon}_{\boldsymbol{k}})^{-1} \tag{4.60}$$

We will proceed to derive the above results to the stricter BCS mean-field theory. This route will be of use in the chapter to follow.

4.5 MEAN-FIELD BCS THEORY

The Particle-hole View

We consider the BCS hamiltonian with the model consideration

$$V_{\boldsymbol{q}} = \begin{cases} -g \text{ if } |\omega_{\boldsymbol{k}}| < \omega_D \\ 0 \text{ otherwise} \end{cases} \tag{4.61}$$

By applying the mean-field approximation to the BCS Hamiltonian (4.39) with (4.42), it results in

$$\begin{aligned} H^{\text{Mean}}_{\text{I,BCS}} &= \sum_{|\omega_{\boldsymbol{k}}|<\omega_D \sigma} \omega_{\boldsymbol{k}\sigma} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{|\varepsilon_{\boldsymbol{k}}|<\omega_D} \left[c_{-\boldsymbol{k}\downarrow}\Delta^* c_{\boldsymbol{k}\uparrow} + c^\dagger_{\boldsymbol{k}\uparrow}\Delta c^\dagger_{-\boldsymbol{k}\downarrow}\right] + \frac{|\Delta|^2}{g} \\ \text{with } \Delta^* &= -\frac{g}{V}\sum_{|\varepsilon_{\boldsymbol{k}}|<\omega_D} \langle c^\dagger_{-\boldsymbol{k}\downarrow} c^\dagger_{\boldsymbol{k}\uparrow}\rangle \end{aligned} \tag{4.62}$$

(with V the volume), in which we consider all of the kinetic contributions such that $|\omega_{\boldsymbol{k}}| > \omega_D$ to be irrelevant in contributing to the dynamics. These might be removed through a displacement of the energy from the true particle vacuum to the Fermi vacuum. This is a version of (4.47) in which Δ is uniform as one might notice by expanding it in the momenta space. The nontrivial part of the proof is to show the equivalence of the Δ^* definition in both versions. If Δ^* is uniform, it must be such that $\Delta^* = \frac{\int \Delta^*(\boldsymbol{x}) d\boldsymbol{x}}{V}$. Indeed, this and (4.62) provides the above gap dependence in the reciprocal space (Appendix C.7(a) for a proof).

To write this expression in the particle-hole picture explicitly, we need to define the hole operators

$$h^\dagger_{\boldsymbol{k}} \equiv c_{-\boldsymbol{k}} \ , \ h_{\boldsymbol{k}} \equiv c^\dagger_{-\boldsymbol{k}} \ , \tag{4.63}$$

since to create or remove a negative particle moving with momentum $-\boldsymbol{k}$ corresponds, respectively, to add or remove a positive particle with momentum $\boldsymbol{k}$. The hole operators obey the same algebra of the quasi-particle operators.

We, therefore, may rewrite the BCS mean-field hamiltonian in the particle-hole view

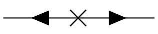
$$H = \sum_{|\varepsilon_{\boldsymbol{k}}|<\omega_D} \omega_{\boldsymbol{k}} (c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow} - h^\dagger_{\boldsymbol{k}\uparrow} h_{\boldsymbol{k}\uparrow}) + \sum_{|\omega_{\boldsymbol{k}}|<\omega_D} [c^\dagger_{\boldsymbol{k}\uparrow} \Delta h_{\boldsymbol{k}\downarrow} + h^\dagger_{\boldsymbol{k}\downarrow} \Delta^* c_{\boldsymbol{k}\uparrow}] + \frac{|\Delta|^2}{g} \tag{4.64}$$

which is equivalent to the Anderson impurity model for local magnetic moment in dilute alloy (ANDERSON, 1961), with the s-d mixing replaced for the spin mixing.

From this we see that there are two ways for the hole and electron scattering to the first order of the Gell-Man expansion - creating a hole and destroying an electron, or the converse, creating an electron and destroying a hole. These virtual processes are coined 'Andreev scattering', hence represented diagrammatically

$$h^\dagger_{\boldsymbol{k}\uparrow} \Delta^* c_{\boldsymbol{k}\uparrow} = c_{-\boldsymbol{k}\downarrow} \Delta^* c_{\boldsymbol{k}\uparrow} \equiv$$

$$c^\dagger_{\boldsymbol{k}\uparrow} \Delta h_{\boldsymbol{k}\downarrow} = c^\dagger_{\boldsymbol{k}\uparrow} \Delta c^\dagger_{-\boldsymbol{k}\downarrow} \equiv$$



A convenient representation to the particle-hole excitation is the Nambu-spinor in the momenta space, the reciprocal-space version of (4.50),

$$\Psi_{\boldsymbol{k}} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ h_{\boldsymbol{k}\downarrow} \end{pmatrix} \ , \ \Psi^\dagger_{\boldsymbol{k}} = \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & h^\dagger_{\boldsymbol{k}\downarrow} \end{pmatrix} \tag{4.65}$$

for the Hamiltonian may be rewritten as

$$H' \equiv H - \frac{|\Delta|^2}{g} = \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & h^\dagger_{\boldsymbol{k}\downarrow} \end{pmatrix} \begin{pmatrix} \omega_{\boldsymbol{k}} & \Delta \\ \Delta^* & -\omega_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c^{\uparrow}_{\boldsymbol{k}} \\ h_{\boldsymbol{k}\downarrow} \end{pmatrix} = \Psi^\dagger_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \ . \tag{4.66}$$

The Zeeman-like form of the Anderson model

Following similar footsteps to those of J.R Schrieffer and P.A.Wolf (SCHRIEFFER; WOLFF, 1966), one can prove that the exemplar of the impurity Anderson model (4.64) may be rewritten in the Zeeman-like form, as in chap. 14 of the introductory ref. (COLEMAN, 2015)

$$H' = -\sum_{\boldsymbol{k}} \boldsymbol{B_k} \cdot \hat{\boldsymbol{\sigma}}_{\boldsymbol{k}} \text{ with } \hat{\boldsymbol{\sigma}}_{\boldsymbol{k}} = \Psi^{\dagger}_{\boldsymbol{k}} \boldsymbol{\sigma} \Psi_{\boldsymbol{k}} \tag{4.67}$$

The operator $\hat{\boldsymbol{\sigma}}_{\boldsymbol{k}}$ is often refereed as isospin. $\boldsymbol{\sigma}$ comprises the Pauli matrices as components.

$$\boldsymbol{\sigma} = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \tag{4.68}$$

To prove this interesting result, it is a matter of clever algebraic procedure. By defining

$$\boldsymbol{B_k} = (-\operatorname{Re}[\Delta], \operatorname{Im}[\Delta], -\omega_{\boldsymbol{k}}) \tag{4.69}$$

It is immediate that

$$-\boldsymbol{B_k} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \omega_{\boldsymbol{k}} & \Delta \\ \Delta^* & -\omega_{\boldsymbol{k}} \end{pmatrix} \tag{4.70}$$

we might rearrange (4.67) in the form,

$$H' = -\sum_{\boldsymbol{k}} \Psi^{\dagger}_{\boldsymbol{k}} (\boldsymbol{B_k} \cdot \boldsymbol{\sigma}) \Psi_{\boldsymbol{k}} \tag{4.71}$$

A useful vector representation to $\boldsymbol{B_k}$ is provided in polar and azihmuthal angles

$$-\boldsymbol{B_k} = |\boldsymbol{B_k}|(\sin\theta_{\boldsymbol{k}} \cos\phi_{\boldsymbol{k}}, \sin\theta_{\boldsymbol{k}} \sin\phi_{\boldsymbol{k}}, \cos\theta_{\boldsymbol{k}}) \tag{4.72}$$

$$\theta_{\boldsymbol{k}} = \cos^{-1}(\omega_{\boldsymbol{k}}/\sqrt{\Delta^2 + \omega^2_{\boldsymbol{k}}}) \text{ , } \phi_{\boldsymbol{k}} = -\tan^{-1}(\operatorname{Im}[\Delta]/\operatorname{Re}[\Delta]) \tag{4.73}$$

Hence, the Hamiltonian is rewritten accordingly

$$H' = \sum_{\boldsymbol{k}} |\boldsymbol{B_k}| \Psi^{\dagger}_{\boldsymbol{k}} \underbrace{\begin{pmatrix} \cos\theta_{\boldsymbol{k}} & \sin\theta_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \\ \sin\theta_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} & -\cos\theta_{\boldsymbol{k}} \end{pmatrix}}_{\equiv \mathcal{P}_{\boldsymbol{k}}} \Psi^{\dagger}_{\boldsymbol{k}} \tag{4.74}$$

The Hamiltonian mixes particles and holes explicitly. We wish to find the quasi-particles of the system such that the Hamiltonian becomes effectively non-interacting concerning these. This is done by computing the eigenstates, as pointed out in chap. 1.

The eigenvalues of the matrix are ± 1, for which it follows the corresponding eigenstates

$$\begin{pmatrix} \cos\frac{\theta_{\boldsymbol{k}}}{2} e^{-i\frac{\phi_{\boldsymbol{k}}}{2}} \\ \\ \sin\frac{\theta_{\boldsymbol{k}}}{2} e^{i\frac{\phi_{\boldsymbol{k}}}{2}} \end{pmatrix} ; \begin{pmatrix} -\sin\frac{\theta_{\boldsymbol{k}}}{2} e^{-i\frac{\phi_{\boldsymbol{k}}}{2}} \\ \\ \cos\frac{\theta_{\boldsymbol{k}}}{2} e^{i\frac{\phi_{\boldsymbol{k}}}{2}} \end{pmatrix} \tag{4.75}$$

The unitary matrix produces the coordinate transformation to the mixed particle-hole eigenstate.

$$U = \begin{pmatrix} u_{\boldsymbol{k}} & -v^*_{\boldsymbol{k}} \\ \\ v_{\boldsymbol{k}} & u^*_{\boldsymbol{k}} \end{pmatrix} \text{ with } u_{\boldsymbol{k}} = \cos\frac{\theta_{\boldsymbol{k}}}{2} \exp\left[-i\frac{\phi_{\boldsymbol{k}}}{2}\right] \text{ , } v_{\boldsymbol{k}} = \sin\frac{\theta_{\boldsymbol{k}}}{2} \exp\left[i\frac{\phi_{\boldsymbol{k}}}{2}\right] \tag{4.76}$$

Hence,

$$H' = \sum_{\boldsymbol{k}} |\boldsymbol{B}_{\boldsymbol{k}}| \Psi^\dagger_{\boldsymbol{k}} \mathcal{P}_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} = \sum_{\boldsymbol{k}} |\boldsymbol{B}_{\boldsymbol{k}}| (\Psi^\dagger_{\boldsymbol{k}} U)(U^\dagger \mathcal{P}_{\boldsymbol{k}} U)(U^\dagger \Psi_{\boldsymbol{k}}) \text{ , } |\boldsymbol{B}_{\boldsymbol{k}}| = \sqrt{\Delta^2 + \omega^2_{\boldsymbol{k}}} \tag{4.77}$$

Therefore, we identify the quasi-particle operator associated with the eigenstates of the Hamiltonian.

$$H = \sum_{\boldsymbol{k}} |\boldsymbol{B}_{\boldsymbol{k}}| \Psi^\dagger_{\boldsymbol{k}\text{quasi}} \sigma_3 \Psi_{\boldsymbol{k}\text{quasi}} + \frac{|\Delta|^2}{g} \text{ , } \Psi_{\boldsymbol{k}\text{quasi}} = U^\dagger \Psi_{\boldsymbol{k}} \tag{4.78}$$

or explicitly,

$$H = \sum_{\boldsymbol{k}} |\boldsymbol{B}_{\boldsymbol{k}}| \left[c^\dagger_{\boldsymbol{k}\uparrow\text{quasi}} c_{\boldsymbol{k}\uparrow\text{quasi}} - h^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} h_{\boldsymbol{k}\downarrow\text{quasi}} \right] + \frac{|\Delta|^2}{g} \tag{4.79}$$

The above hamiltonian is explicitly not symmetric in its spin index. A way to write it symmetrically is to consider the division of the Brillouin zone in symmetric halves [5],

$$\begin{aligned} H' = & \sum_{\boldsymbol{k}, k_z > 0} |\boldsymbol{B}_{\boldsymbol{k}}| \left[c^\dagger_{\boldsymbol{k}\uparrow\text{quasi}} c_{\boldsymbol{k}\uparrow\text{quasi}} - h^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} h_{\boldsymbol{k}\downarrow\text{quasi}} \right] + \\ & \sum_{\boldsymbol{k}, k_z > 0} |\boldsymbol{B}_{\boldsymbol{k}}| \left[c^\dagger_{-\boldsymbol{k}\uparrow\text{quasi}} c_{-\boldsymbol{k}\uparrow\text{quasi}} - h^\dagger_{-\boldsymbol{k}\downarrow\text{quasi}} h_{-\boldsymbol{k}\downarrow\text{quasi}} \right] \end{aligned} \tag{4.80}$$

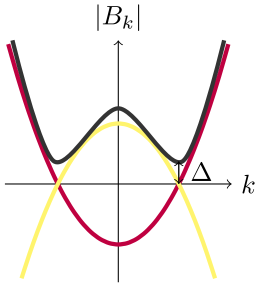
where we account for the $k_z > 0$ in the first term and $k_z < 0$ in the second term. The explicit sum over $k_z < 0$ is not present as we have done the next step directly, by reversing the sign of the mute variable on which the sum is made. Hence, we identify

$$\begin{aligned} H' = & \sum_{\boldsymbol{k}, k_z > 0} |\boldsymbol{B}_{\boldsymbol{k}}| \left[c^\dagger_{\boldsymbol{k}\uparrow\text{quasi}} c_{\boldsymbol{k}\uparrow\text{quasi}} - h^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} h_{\boldsymbol{k}\downarrow\text{quasi}} \right] \\ & + \sum_{\boldsymbol{k}, k_z > 0} |\boldsymbol{B}_{\boldsymbol{k}}| \left[h_{\boldsymbol{k}\uparrow,\text{quasi}} h^\dagger_{\boldsymbol{k}\uparrow\text{quasi}} - c_{\boldsymbol{k}\downarrow,\text{quasi}} c^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} \right] \end{aligned} \tag{4.81}$$

---

[5] This procedure in the $\boldsymbol{k}$ space (in the reverse order, i.e, the passage from a spin-index symmetric to an asymmetric representation) is analogous to the passage from (4.45) to (4.47), in the position space.

Figure 6 – The free-particle (stable above k_F, unstable below¸F), the free-hole spectrum (stable below k_F, unstable above k_F) and the gaped particle-hole (stable) spectra. The positive spectrum due to the quasi-state mixing particle and holes, is predicted due to the inclusion of interaction, in the microscopic theory.



Source: The author, based on (COLEMAN, 2015)

Or equivalently,

$$H = \sum_{\boldsymbol{k};\sigma k_z>0} |\boldsymbol{B_k}|[n_{\boldsymbol{k}\sigma\text{quasi}} - \bar{n}_{\boldsymbol{k}\sigma\text{quasi}}] + \frac{|\Delta|^2}{g} \tag{4.82}$$

with the bar present in the hole-occupancy. The Hamiltonian does not mix the quasi-particle and quasi-hole states. Therefore, we have an explicit one-to-one correspondence between the problem of the interacting system in which particles and holes are mixed to a non-interacting problem of quasi-particles and quasi-holes which do not interact. At $k = k_F$, $|\boldsymbol{B_k}| = \Delta$, featuring the gap of the energy in between the hole and particle energy spectrum. The energy minimization is favored by the presence of the quasi-particle state. We remember the quasi-particle is made up of a mixture of holes and particles of the original system.

The mixture is explicitly provided by means of the unitary transformation

$$\begin{aligned} c^\dagger_{\boldsymbol{k}\uparrow\text{quasi}} &= c^\dagger_{\boldsymbol{k}\uparrow} u_{\boldsymbol{k}} + h^\dagger_{\boldsymbol{k}\downarrow} v_{\boldsymbol{k}} \\ h^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} &= h^\dagger_{\boldsymbol{k}\downarrow} u^*_{\boldsymbol{k}} - c^\dagger_{\boldsymbol{k}\uparrow} v^*_{\boldsymbol{k}} \end{aligned} \tag{4.83}$$

If one wishes to apply the symmetric form (4.82) of the Hamiltonian (4.81), the conjugate of these relations and the proper identification is of use, explicitly,

$$\begin{aligned} h_{\boldsymbol{k}\uparrow\text{quasi}} &= h_{\boldsymbol{k}\uparrow} u^*_{\boldsymbol{k}} + c^\dagger_{\boldsymbol{k}\downarrow} v^*_{\boldsymbol{k}} \\ c^\dagger_{\boldsymbol{k}\downarrow\text{quasi}} &= c^\dagger_{\boldsymbol{k}\downarrow} u_{\boldsymbol{k}} - h^\dagger_{\boldsymbol{k}\uparrow} v_{\boldsymbol{k}} \end{aligned} \tag{4.84}$$

As one can easily verify, the commutation rules continue to apply to quasi-operators. This fact enables laddering across the spectrum. In superconductivity, the quasi-particle excitations are dubbed 'Bogolons.'

BCS Propagator and the Feynman Dictionary in Superconductivity

Now we make a correspondence of the BCS theory with energy cutoff with the derivations in the Nambu-Gor'kov representation in the momentum space. We know that the quasi-particles $\Psi^\dagger_{\boldsymbol{k}\sigma}\,|\text{Fermi vacuum}\rangle$ are eigenstates of the problem, therefore, only the diagonal momentum survives

$$\mathcal{G}_{\alpha\beta\boldsymbol{k}}(\tau,\tau') = -\delta_{\boldsymbol{k}\boldsymbol{k}'}\langle\mathcal{T}\Psi_{\boldsymbol{k}\alpha}(\tau)\Psi^\dagger_{\boldsymbol{k}'\beta}(\tau')\rangle \tag{4.85}$$

Explicitly the diagonal (momenta) terms of the Greens's function operator are represented in the isospin space,

$$\boldsymbol{\mathcal{G}}(\boldsymbol{k},\tau-\tau') = -\begin{pmatrix}\langle\mathcal{T}c_{\boldsymbol{k}\uparrow}(\tau)c^\dagger_{\boldsymbol{k}\uparrow}(\tau')\rangle & \langle\mathcal{T}c_{\boldsymbol{k}\uparrow}(\tau)h^\dagger_{\boldsymbol{k}\downarrow}(\tau')\rangle \\ \langle\mathcal{T}h_{\boldsymbol{k}\downarrow}(\tau)c^\dagger_{\boldsymbol{k}\uparrow}(\tau')\rangle & \langle\mathcal{T}h_{\boldsymbol{k}\downarrow}(\tau)h^\dagger_{\boldsymbol{k}\downarrow}(\tau')\rangle\end{pmatrix} \tag{4.86}$$

This matrix is precisely the Fourier transform of the position representation (for a detailed proof, see Appendix C.7(b)). We notice that the coupling relation in the momentum space becomes $\lim_{\tau\to\tau'}\tilde{\mathcal{F}}_{\boldsymbol{k}}(\tau-\tau') = -g\Delta^*$. In the frequency domain,

$$\boldsymbol{\mathcal{G}}(\boldsymbol{k},i\omega) = \begin{pmatrix}\mathcal{G}_\omega(\boldsymbol{k}) & \mathcal{F}_\omega(\boldsymbol{k}) \\ \tilde{\mathcal{F}}_\omega(\boldsymbol{k}) & \tilde{\mathcal{G}}_\omega(\boldsymbol{k})\end{pmatrix} \tag{4.87}$$

Now we remind the operator result

$$(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon_k})\boldsymbol{\mathcal{G}_k} = -\mathbf{1} \tag{4.88}$$

in the time-independent index. From which,

$$\boldsymbol{\mathcal{G}_k}(\boldsymbol{k},\tau-\tau') = -(\boldsymbol{\partial_\tau} - \boldsymbol{B_k}\cdot\boldsymbol{\sigma})^{-1} \tag{4.89}$$

In the frequency domain, $\partial_\tau \to -i\omega$, thus,

$$\boldsymbol{\mathcal{G}_k} \equiv \frac{1}{i\omega + \boldsymbol{B_k}\cdot\boldsymbol{\sigma}} = \frac{i\omega - \boldsymbol{B_k}\cdot\boldsymbol{\sigma}}{(i\omega + \boldsymbol{B_k}\cdot\boldsymbol{\sigma})(i\omega - \boldsymbol{B_k}\cdot\boldsymbol{\sigma})} = \frac{i\omega - \boldsymbol{B_k}\cdot\boldsymbol{\sigma}}{(i\omega)^2 - E_{\boldsymbol{k}}^2\mathbf{1}} \tag{4.90}$$

where, in the denominator, the proof that $(\boldsymbol{B_k}\cdot\boldsymbol{\sigma})^2 = \mathbf{1}|\boldsymbol{B_k}|^2$ is of use. It is trivial since $\boldsymbol{\sigma} \equiv (\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3), \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j = \delta_{ij}\mathbf{1}$. Its convenient to define $E_{\boldsymbol{k}} \equiv |\boldsymbol{B_k}| = \sqrt{\varepsilon_{\boldsymbol{k}}^2 + \Delta^2}$ ($\omega_{\boldsymbol{k}} \equiv \varepsilon_{\boldsymbol{k}}$, not to be confused with the operator $\boldsymbol{\varepsilon_k}$, containing the gap dependence). Therefore,

$$\boldsymbol{\mathcal{G}}(\boldsymbol{k},i\omega) = \frac{1}{(i\omega)^2 - |\boldsymbol{B_k}|^2}\begin{pmatrix}i\omega + \varepsilon_{\boldsymbol{k}} & \Delta \\ \Delta^* & i\omega - \varepsilon_{\boldsymbol{k}}\end{pmatrix} \tag{4.91}$$

The bare propagator $\boldsymbol{\mathcal{G}}^{(0)}$ consists of the absence of the Andreev scattering, i.e, null condensation.

$$\boldsymbol{\mathcal{G}}_{\mathbf{0}}(\boldsymbol{k}, i\omega) = \begin{pmatrix} \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}}} & 0 \\ 0 & \frac{1}{i\omega + \varepsilon_{\boldsymbol{k}}} \end{pmatrix} \tag{4.92}$$

The scattering matrix is

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \tag{4.93}$$

We may prove that the dressed propagator is provided algebraically by the Dyson series,

$$\boldsymbol{\mathcal{G}}_\omega(\boldsymbol{k}) = \boldsymbol{\mathcal{G}}^0_\omega(\boldsymbol{k}) + \boldsymbol{\mathcal{G}}^0_\omega(\boldsymbol{k})\boldsymbol{\Delta}\boldsymbol{\mathcal{G}_\omega}(\boldsymbol{k}) \ , \tag{4.94}$$

by applying successively the matrix operator (4.87),

$$\tilde{\mathcal{F}}_\omega = \tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega + \tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega\Delta\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega + \tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega\Delta\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega\Delta\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega + \ldots. \tag{4.95}$$

Therefore,

$$\tilde{\mathcal{F}}_\omega = \tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega \sum_n (\Delta\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega)^n = \frac{\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega}{1 - \Delta\tilde{\mathcal{G}_\omega}^{(0)}\Delta^*\mathcal{G}^{(0)}_\omega} \tag{4.96}$$

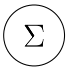
Indeed,

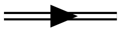
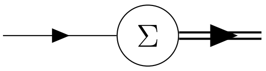
$$\tilde{\mathcal{F}}_\omega(\boldsymbol{k}) = \frac{\Delta^*}{[(i\omega)^2 - \varepsilon^2_{\boldsymbol{k}}]}\frac{1}{\left[1 - \frac{|\Delta|^2}{(i\omega)^2 - \varepsilon^2_{\boldsymbol{k}}}\right]} = \frac{\Delta^*}{(i\omega)^2 - E^2_{\boldsymbol{k}}} \tag{4.97}$$

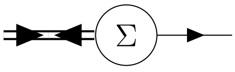
which agrees with (4.87) and (4.91). The consistency is readily checked for the other components. In the elegant and concise language of Feynman diagrams,

$\mathcal{G}^{(0)}(k) \equiv$  $= \dfrac{1}{i\omega - \varepsilon_{\boldsymbol{k}}}$

$\bar{\mathcal{G}}^{(0)}(k) \equiv$  $= \dfrac{1}{i\omega + \varepsilon_{\boldsymbol{k}}}$

 =  $= \Delta\dfrac{1}{i\omega + \varepsilon}\Delta^*$

$\mathcal{G}(k) \equiv$  =  + 

$\tilde{F}(k) \equiv$  =  + 

4.6 THE HUBBARD-STRATONOVITCH TRANSFORMATION

The objective of this section is to introduce some of the methods in path integrals. Here we present an alternative way to understand the formulation of the mean-field theory in the context of BCS.

$$H' = \sum_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \tag{4.98}$$

Defining

$$A = \sum_{\boldsymbol{k}\in\Lambda(k_F)} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} \tag{4.99}$$

with the sum carried out in the vicinity $\Lambda(k_F)$ of the Fermi-surface. Under the path integral formalism (appendices C.5 and C.6),

$$\begin{aligned} Z &= \mathrm{Tr}\left[\exp\left[-\beta \sum_{\boldsymbol{k}\sigma} c^{\dagger}_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma} - A^{\dagger} A\right]\right] \\ &= \int \mathcal{D}[\boldsymbol{c}^{\dagger} \cdot \boldsymbol{c}] \exp\left[-\int d\tau \sum_{\boldsymbol{k}} c^{*}_{k}(\partial_{\tau} + \varepsilon_{\boldsymbol{k}}) c_{k} - g A^{*}(c^{*}_{\uparrow}, c^{*}_{\downarrow}) A(c_{\uparrow}, c_{\downarrow})\right] \end{aligned} \tag{4.100}$$

The Hubbard-Stratonovitch approach includes a noise effect to the Hamiltonian by making it to interact with another system of particles,

$$Z \to Z' = Z \times Z_{\gamma} \tag{4.101}$$

with

$$Z_{\gamma} = \int \mathcal{D}[\gamma^{*}, \gamma] \exp\left[-\frac{1}{g}\int_{0}^{\beta} d\tau \gamma^{*}\gamma\right] \text{ , i.e , } H' \to H' - \frac{1}{g}\gamma^{\dagger}\gamma \tag{4.102}$$

The correlation function associated to the operator newly introduced operator presents a white noise

$$\langle \gamma_{\alpha}(\tau) \gamma^{\dagger}_{\beta}(\tau') \rangle = g \delta_{\alpha\beta} \delta(\tau' - \tau) \tag{4.103}$$

With the introduction of this second set of free particles we have two uncoupled systems

$$\begin{aligned} Z' &= \mathrm{Tr}\left[\exp\left[-\beta \sum_{\boldsymbol{k}\sigma} \hat{c}^{\dagger}_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}\sigma} - g \hat{A}^{\dagger} \hat{A} + \frac{1}{g} \hat{\gamma}^{\dagger} \gamma\right]\right] \\ &= \int \mathcal{D}[\boldsymbol{c}^{\dagger} \cdot \boldsymbol{c}] d\gamma d\gamma^{*} \exp\left[-\int d\tau \sum_{\boldsymbol{k}} c^{*}_{k}(\partial_{\tau} + \varepsilon_{\boldsymbol{k}}) c_{k} - g A^{*} A - \frac{1}{g}\gamma^{*}\gamma\right] \end{aligned} \tag{4.104}$$

Defining the variable $\Delta(\tau) = \gamma(\tau) - gA(\tau)$ coupling both the novel field and the Fermionic field, we may rewrite the integral

$$Z' = \underbrace{\int \mathcal{D}[\Delta^*, \Delta] \exp\left[-\int d\tau \frac{|\Delta|^2}{g}\right]}_{\text{Fluctuating field measuring the coupling}} \times$$
$$\underbrace{\int \mathcal{D}[\boldsymbol{c}^\dagger \cdot \boldsymbol{c}] \exp\left[-\int d\tau [\sum_{\boldsymbol{k}\sigma} c^*_{\boldsymbol{k}\sigma} \partial_\tau c_{\boldsymbol{k}\sigma} + c^*_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}} c_{-\boldsymbol{k},\downarrow} c_{\boldsymbol{k}\uparrow} \Delta + c^*_{\boldsymbol{k}\uparrow}, c^*_{\boldsymbol{k}\downarrow} \Delta^*]\right]}_{\text{Electrons moving in a fixed external field}}$$
(4.105)

In the same footsteps of the chapter on the BCS mean-field theory, it is possible to rewrite it in the bilinear form provided in the Nambu representation,

$$Z' = \int \mathcal{D}[\Delta^*, \Delta] \exp\left[-\int d\tau \frac{|\Delta|^2}{g}\right] \int \mathcal{D}[\Psi^\dagger \cdot \Psi] \exp\left[-\int d\tau \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} (\partial_\tau - \boldsymbol{B}_{\boldsymbol{k}}(\Delta, \Delta^*) \cdot \boldsymbol{\sigma}) \Psi_{\boldsymbol{k}}\right] \tag{4.106}$$

By incorporating the time as an index (Appendix C.2) in the operators, we simply identify the second integral as a fermionic Gaussian integral. Hence,

$$Z' = \int \mathcal{D}[\Delta^*, \Delta] \exp\left[-\int d\tau \frac{|\Delta|^2}{g} - \ln[\prod_{\boldsymbol{k}} \det(\boldsymbol{\partial_\tau} - \boldsymbol{B}_{\boldsymbol{k}}(\Delta, \Delta^*) \cdot \boldsymbol{\sigma})]\right] \tag{4.107}$$

We rewrite the above as

$$Z' = \int \mathcal{D}[\Delta^*, \Delta] \exp[-S] \tag{4.108}$$

$$S[\Delta^*, \Delta] = \int_0^\beta d\tau \frac{|\Delta|^2}{g} + \sum_{\boldsymbol{k}} \mathrm{Tr}[\ln(\boldsymbol{\partial_\tau} - \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma})] \tag{4.109}$$

Since $\ln\det[\mathcal{O}] = \mathrm{Tr}[\ln \mathcal{O}]$ and, thus, $\ln\det[\prod_{\boldsymbol{k}} \mathcal{O}_{\boldsymbol{k}}] = \sum_{\boldsymbol{k}} \ln\det[\mathcal{O}_{\boldsymbol{k}}] = \sum_{\boldsymbol{k}} \mathrm{Tr}[\ln \mathcal{O}_{\boldsymbol{k}}]$ hold. If the fluctuating field has a peak around a given field, $\Delta \equiv \delta(\Delta - \Delta_0)$, the above reduces to the BCS mean-field action for the uniform gap.

The Free Energy for the Uniform-Gap Theory

We consider an expansion of the quasi-state in the frequency domain

$$\Psi_{\boldsymbol{k}} = \frac{1}{\sqrt{\beta}} \sum_{\omega} \Psi_{\boldsymbol{k}\omega} e^{-i\omega\tau} \tag{4.110}$$

The sum over frequencies makes needless to include the time as an index for the operators. The matrix elements are only featured by the particle type

$$S = \sum_{\boldsymbol{k},\omega} \Psi^*_{\boldsymbol{k}\omega}(-i\omega - \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma})\Psi_{\boldsymbol{k}\omega} + \int_0^\beta d\tau \frac{1}{g}|\Delta|^2 \tag{4.111}$$

$$Z = \int \mathcal{D}[\Psi \cdot \Psi^\dagger] \exp[-S] \tag{4.112}$$

Therefore, in the thermodynamic limit,

$$Z_{\mathsf{BCS}} = \prod_n \det[-i\omega - \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}] \exp\left[-\frac{\beta}{g}|\Delta|^2\right] , \tag{4.113}$$

as

$$\det[-i\omega - \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}] = \det \begin{pmatrix} -i\omega + \varepsilon_{\boldsymbol{k}} & \Delta \\ \Delta^* & i\omega - \varepsilon_{\boldsymbol{k}} \end{pmatrix} = \omega^2 + \varepsilon^2_{\boldsymbol{k}} + |\Delta|^2 \tag{4.114}$$

The free energy is provided

$$F = -\frac{1}{\beta}\log Z = -\beta \sum_{\boldsymbol{k}\omega} \ln[\omega^2 + \varepsilon^2_{\boldsymbol{k}} + |\Delta|^2] + \frac{|\Delta|^2}{g} \tag{4.115}$$

This is the free energy for the mean-field 'Heaviside' model (uniform gap).

4.7 THE SUCCESS OF THE MICROSCOPIC THEORY

The minimization of the Free energy provides the gap equation ($k_B = 1$)

$$\frac{1}{g} = T \sum_{\boldsymbol{k}\omega} \frac{1}{\omega^2 + E^2_{\boldsymbol{k}}}. \tag{4.116}$$

This can be integrated out through the contour integral technique and the residue theorem in the complex plane. For $f(z)$ the Fermi distribution

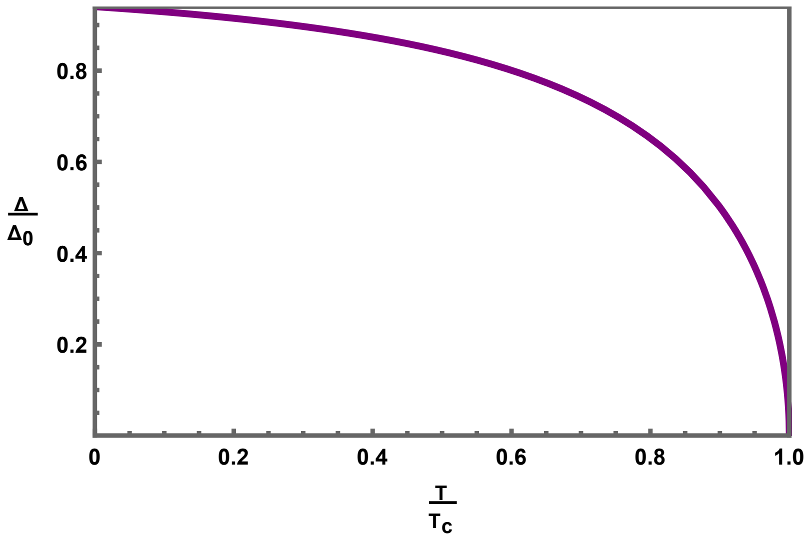
$$T \sum_\omega \frac{1}{\omega^2 + E^2_{\boldsymbol{k}}} = -\oint \frac{dz}{2\pi i} f(z) \frac{1}{z^2 - E^2_{\boldsymbol{k}}} = \frac{\tanh\left(\frac{E_{\boldsymbol{k}}}{2T}\right)}{2E_{\boldsymbol{k}}} \tag{4.117}$$

For all of the sums over $\boldsymbol{k}$ we have omitted that $\varepsilon_{\boldsymbol{k}} < \omega_D$. Therefore, the BCS s-wave gap equation is rewritten

$$\frac{1}{gN(0)} = \int_0^{\omega_D} d\varepsilon \frac{\tanh\left(\frac{1}{2T}\sqrt{\varepsilon^2 + \Delta^2}\right)}{\sqrt{\varepsilon^2 + \Delta^2}} \tag{4.118}$$

A common technique is to consider $\omega_D \to \infty$, as the integrand saturates rapidly. The critical temperature is such that $\Delta(T_c) = 0^+$, measuring the starting-point of the condensation. It is convenient to define $\Delta_0 \equiv \Delta(T = 0)$. The ratio Δ/Δ_0 is then, parametrized in the ratio T/T_c obeying a curve coinciding with reported experimental measurements (CHEN et al., 2008).

Figure 7 – The uniform gap evolution in deviations from the critical temperature as predicted by the microscopic theory.



Source: The author

4.8 FINAL REMARKS

The emergence of a space-varying gap results in a continuous change in the optimal isospin direction $\boldsymbol{\sigma}$, analogous to the case of Néel walls in magnetism. In magnetism, the wall is formed by actual spins that are used to determine the direction of magnetization. In superconductivity, the analogous isospin is defined by angles that define the ratio of particle-hole presence in the quasi-state. We will connect the microscopic theory to the Ginzburg-Landau theory and to the more accurate theories away from the critical point, Extended Ginzburg-Landau Theory, in the chapter to follow. In this chapter we investigated the existence of a uniform-order parameter. In the following chapter, we will see the possibility for it to vary. A uniform order-parameter can not produce complexity. If we are to study complexity phenomena, the introduction of spatial variations - derivatives - must play a central role in the modelling. And these plays a relevant role if we are further away from the critical temperature, as we are about to understand.

5 EXTENDED GINZBURG-LANDAU THEORY

5.1 PROLOGUE

In 1950, Ginzburg-Landau proposed the generalization of the phenomenological second-order Landau-theory of phase transition to encompass a complex-order and the accountability of the magnetic field (LANDAU; GINZBURG, 1950). Such description is treated in detail in the chapter on phase transitions of this dissertation. Seven years later, Bardeen, Cooper, and Schrieffer (BCS) proposed a theory valid on the microscopic scale for the superconductivity phenomena. An alternative representation of the same theory is provided by Bogoliubov in his seminar papers (BOGOLIUBOV, 1958). For such theory, many representations were given, such as the Gor'kov-Nambu representation, which when carefully treated links the microscopic theory to the phenomenological Ginzburg-Landau theory (GOR'KOV, 1959). The Ginzburg-Landau theory has its unique and pioneering merits, however, it fails to describe the phenomena of phase transition further away from the critical temperature. As we pointed out earlier in the chapter on phase transition, Landau asked himself how should the theory behave in the vicinity of the critical temperature. An extension of such phenomenological thought is less accurate if we wish to account for 'distances' further away from the locus of phase transition. In the context of superconductivity, many phenomenological expansions with different motivations were provided over the years (to mention a few (TEWORDT, 1963; WERTHAMER, 1963; TAKANAKA; KUBOYA, 1995; ICHIOKA et al., 1996; ICHIOKA et al., 1996; ICHIOKA; HASEGAWA; MACHIDA, 1999; ADACHI; IKEDA, 2003; HOUZET; BUZDIN, 2001)). These phenomenological expansions were based on the expansion of the self-consistent gap equation accounting for the inclusion of higher powers of the order parameter and its spatial gradients. The question of which terms should be considered in these expansions is a fundamental problem whose relevance is only considered with a systematic treatment in the first decade of this century (KOGAN; SCHMALIAN, 2011; SHANENKO et al., 2011).

The truncation criteria problem - what terms to consider for a given theory accuracy - is present at a fundamental level, in any expansion in physics. A procedure named partial-summation is performed, in which many of the diagrams are neglected on the basis of phenomenological justifications to favorable 'physical diagrams'. Plenty of efforts have been driven in the direction of establishing the correct criteria for summing the diagrams such as in the RPA approach proposed by D.Bohm and coauthors (BOHM; PINES, 1951; PINES; BOHM, 1952).

The partial summation approach is not quite the complete theory in treating finite-temperature physics. This chapter, in a sense, might be viewed as the modification of the partial summation approach such that it takes into account a different collection of diagrams for each order of accuracy in deviations from the critical temperature.

The problem of the truncation criteria, in the context of superconductivity, is solved by identifying the scaling of the physical quantities of interest in the vicinity of the critical temperature. (KOGAN; SCHMALIAN, 2011; SHANENKO et al., 2011; VAGOV; SHANENKO A, 2012). This creates a set of hierarchical equations for each order of accuracy one is interested in.

By choosing dimensionless quantities to represent the physical variables in the GL theory, it follows the scaling

$$\Delta = \tau^{1/2}\bar{\Delta}\ ,\ \boldsymbol{r} = \tau^{-1/2}\bar{\boldsymbol{r}}\ ,\ \boldsymbol{\nabla} = \tau^{1/2}\bar{\boldsymbol{\nabla}}\ ,\ \boldsymbol{A} = \tau^{1/2}\bar{\boldsymbol{A}}\ ,\ \boldsymbol{B} = \tau\bar{\boldsymbol{B}} \tag{5.1}$$

with $\tau = 1 - T/T_c$. An effortless way to confirm the scaling is consistent is by noticing the GL equation contains a second order derivative of the gap ($\sim \tau^{3/2}$), the third power of the gap ($\sim \tau^{3/2}$), and a liner gap which is multiplied by a coefficient proportional to τ, yielding the same order of accuracy. As for the vector potential, the second-GL equation is of use for immediate verification.

Therefore, if we wish to expand our theory to a higher-order of accuracy in τ, we perform an expansion of the BCS theory via the Gor'kov-Nambu representation and keep track of the τ terms. In order to find a set of self-consistent equation for each order of accuracy in τ, we simply apply the usual asymptotic expansion, in the barred quantities

$$\bar{f}_s - \bar{f}_{n,B=0} = \bar{f}_0 + \tau\bar{f}_1 + ... \tag{5.2}$$

$$\bar{\Delta} = \bar{\Delta}_0 + \tau\bar{\Delta}_1 + ... \tag{5.3}$$

$$\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}_0 + \tau\bar{\boldsymbol{A}}_1 + ... \tag{5.4}$$

$$\bar{\boldsymbol{B}} = \bar{\boldsymbol{B}}_0 + \tau\bar{\boldsymbol{B}}_1 + ... \tag{5.5}$$

around the critical point. Often we will not treat the bar coordinates explicitly, but just keep in mind to which order of tau the unbarred argument corresponds.

Though such a solution is quite simple, one may say, the simplest ideas are quite often difficult to be found, as more than half a century separates the GL from the EGL despite the renewed interests in understanding the behavior of material away from the critical point. This comes with some profound consequences, such as the appearance of the intertype domain regime in superconductivity (VAGOV et al., 2016).

In this chapter, we construct both the Ginzburg-Landau (GL) and the Extended Ginzburg-Landau theory (EGL) departing from the Bogoliubov Hamiltonian. We apply the same truncation and accuracy criterion of the authors and collaborators A.Vagov and A.Shanenko. We initially center our attention on the case of zero magnetic field, where there is no coupling of the magnetic field with neat charges, as in ref. (VAGOV; SHANENKO A, 2012). In the zero magnetic field, we have a superfluidity theory, which we name the Extended Landau theory (EL). Distinguishing from the advisors and collaborators A.Vagov and A. Shanenko, we investigate an alternative route which seems to reduce remarkably the calculation efforts towards EGL and to any intended higher-order theory. In fact, the problem becomes identical to a connecting-dot-like exercise together with a dictionary. The ingredient we need is the existence of a smooth varying order parameter (Appendix D.1). In a first reading, one may jump the section concerning the scheme for developing theories beyond the order of the Extended Landau theory, as we do not wish the reader to deviate much from the main purpose of this chapter: to present the EGL theory.

Though the BCS is a complete theory, its implementation in capturing spatial variations imposes technical difficulties. On the other hand, due to their simplicity, the GL equations have been studied numerically over the years, since 1996 (GROPP et al., 1996), due to their importance in describing superconductivity for technological purposes. Many other works have been carried out around this topic, such as (GEIM et al., 1997; SCHWEIGERT; PEETERS; DEO, 1998; MEL'NIKOV et al., 2002), and the doctorate thesis (DARKO, 2018). As pointed out by S.Darko (DARKO, 2018), '(...) the Ginzburg-Landau theory is only valid in the vicinity of the critical temperature. Coming up with a new theoretical solution that can overcome this restriction remains paramount for many technological applications'. In this way, we would like to produce a theory that is neither unpractical as the BCS full theory, nor restrict to the very vicinity of the phase transition. This is the idea provoking the formulation of the Extended Ginzburg-Landau theory.

5.2 ALTERNATIVE ROUTE TO THE EXTENDED LANDAU THEORY

For a generic interaction, the interaction part of the Bogoliubov Hamiltonian reads

$$H_I = \int d\boldsymbol{x} d\boldsymbol{x}' \Phi(\boldsymbol{z}-\boldsymbol{z}')\Big[c_{\downarrow}(\boldsymbol{z}')\Delta^*(\boldsymbol{z},\boldsymbol{z}')c_{\uparrow}(\boldsymbol{z}) + \text{H.C}\Big] + \frac{|\Delta(\boldsymbol{z},\boldsymbol{z}')|^2}{g} \tag{5.6}$$

In general the Bogoliubov centers the discussion over the simplified hard interaction $\Phi(\boldsymbol{z}-\boldsymbol{z}') = -g\delta(\boldsymbol{z}-\boldsymbol{z}')$. It is convenient to write the expression in terms of

$$H_I = \int d\boldsymbol{z}\Big[c_\downarrow(\boldsymbol{z})\Delta^*(\boldsymbol{z})c_\uparrow(\boldsymbol{z}) + \text{H.C}\Big] + \frac{|\Delta(\boldsymbol{z})|^2}{g}\ ,\ \Delta^*(\boldsymbol{z}) = -g\langle c^\dagger_\uparrow(\boldsymbol{z})c^\dagger_\downarrow(\boldsymbol{z})\rangle \tag{5.7}$$

We consider the expansion around a point where such expansion is possible (any analytic region, away from the exact locus of vortices),

$$\Delta^*(\boldsymbol{z}) = \sum_\alpha \frac{1}{\alpha!}\frac{\partial^{n_\alpha}\Delta^*(\boldsymbol{z})}{\partial \boldsymbol{z}^{n_\alpha}}|_{\boldsymbol{z}=\boldsymbol{x}}(z_1-x_1)^{\alpha_1}(z_2-x_2)^{\alpha_2}(z_3-x_3)^{\alpha_3} \tag{5.8}$$

with $\alpha! = \alpha_1!\alpha_2!\alpha_3!$, $n_\alpha = \sum_j \alpha_j$, and $\frac{\partial^{n_\alpha}\Delta(\boldsymbol{z})}{\partial \boldsymbol{z}^{n_\alpha}}$ a short-hand notation for the known derivative form. We reckon, yet, each derivative produces a contribution of order $\tau^{1/2}$ in the accuracy - keeping the same accuracy of it in the lowest-level order of the theory (GL).

$$H_I = \sum_\alpha \frac{1}{\alpha}\frac{\partial^n\Delta^*(\boldsymbol{x})}{\partial \boldsymbol{x}^n}\int d\boldsymbol{z}c_\downarrow(\boldsymbol{z})(z_1-x_1)^{\alpha_1}(z_2-x_2)^{\alpha_2}(z_3-x_3)^{\alpha_3}c_\uparrow(\boldsymbol{z}) + \text{H.C} \tag{5.9}$$

As the integration is an unbounded domain, we may consider the displacement $\boldsymbol{z}\to\boldsymbol{z}+\boldsymbol{x}$ without changing the (unbounded) integration limits,

$$H_I = \sum_\alpha \frac{1}{\alpha!}\frac{\partial^{n_\alpha}\Delta(\boldsymbol{x})}{\partial \boldsymbol{x}^{n_\alpha}}\int d\boldsymbol{z}c_\downarrow(\boldsymbol{z}+\boldsymbol{x})z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}c_\uparrow(\boldsymbol{z}+\boldsymbol{x}) + \text{H.C}\ . \tag{5.10}$$

In the momenta space,

$$H_I = \sum_\alpha \frac{1}{\alpha!}\frac{\partial^{n_\alpha}\Delta(\boldsymbol{x})}{\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}}\sum_{\boldsymbol{k}'\boldsymbol{k}} e^{i(\boldsymbol{k}'+\boldsymbol{k})\cdot\boldsymbol{x}}c_{\boldsymbol{k}'\downarrow}\Bigg[(\frac{\partial_{\boldsymbol{k}_1}}{i})^{\alpha_1}(\frac{\partial_{\boldsymbol{k}_2}}{i})^{\alpha_2}(\frac{\partial_{\boldsymbol{k}_3}}{i})^{\alpha_3}\int\frac{d\boldsymbol{z}}{V}e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{z}}\Bigg]c_{\boldsymbol{k}\uparrow} + \text{H.C} \tag{5.11}$$

We may consider the momenta to vary continuously near the boundary, so that we are able to eliminate the surface term in the next step. By explicitly integrating in the momenta space, the surface term vanishes once the boundary condition $c_{\downarrow-\boldsymbol{k}}c_{\uparrow\boldsymbol{k}} = 0$ (also, its derivatives) and hermitian conjugate are met. This accounts for the absence of Cooper pairs in the boundary of the analytic region of interest. We are safe unless we meet vortices! As we are to see in the chapter to follow, many of the solutions have large regions where vortices do not appear. Following these ideas (for a in depth proof, check the Appendix D.5),

$$H_I = \sum_{\boldsymbol{k}} h^\dagger_{\boldsymbol{k}\downarrow}\Bigg\{\sum_\alpha \frac{(-1)^{n_\alpha}}{\alpha!}\frac{\partial^{n_\alpha}\Delta^*(\boldsymbol{x})}{\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}}(\frac{\partial_{\boldsymbol{k}_1}}{i})^{\alpha_1}(\frac{\partial_{\boldsymbol{k}_2}}{i})^{\alpha_2}(\frac{\partial_{\boldsymbol{k}_3}}{i})^{\alpha_3}\Bigg\}[c_{\boldsymbol{k}\uparrow}] + \text{H.C} \tag{5.12}$$

By including the kinetic term as in the fourth chapter, in the particle-hole view, the full Hamiltonian reads

$$H_{\text{BCS}} = \sum_{\boldsymbol{k}}\varepsilon_{\boldsymbol{k}}(c^\dagger_{\boldsymbol{k}\uparrow}c_{\boldsymbol{k}\uparrow} - h^\dagger_{\boldsymbol{k}\uparrow}h_{\boldsymbol{k}\uparrow}) + \sum_{\boldsymbol{k}\sigma}(c^\dagger_{\boldsymbol{k}\uparrow}\boldsymbol{\Delta}[h_{\boldsymbol{k}\downarrow}] + h^\dagger_{\boldsymbol{k}\downarrow}\boldsymbol{\Delta}^\dagger[c_{\boldsymbol{k}\uparrow}])\ , \tag{5.13}$$

with the difference that Δ and Δ^* in the interaction matrix, are now promoted to operators, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^\dagger$, respectively. Hence, we infer directly that the action is quite similar to that in chapter four, following the same footsteps,

$$S = \int \mathcal{D}[c^*, c] \exp\Big[-\int d\tau \sum_{\boldsymbol{k}} \boldsymbol{\Psi}^\dagger_{\boldsymbol{k}}(\tau) \cdot (\partial_\tau - \boldsymbol{B_k} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\Psi}_{\boldsymbol{k}}(\tau) \Big] \tag{5.14}$$

The interaction part of the Hamiltonian, $\boldsymbol{\varepsilon_k} = -\boldsymbol{B_k} \cdot \boldsymbol{\sigma}$ might be seen as an operator. As in chap.4, on the microscopic theory, $\mathcal{G}_{\boldsymbol{k}} = -(\partial_\tau + \boldsymbol{\varepsilon_k})^{-1}$,

$$\boldsymbol{\mathcal{G}_k}(\boldsymbol{k}, \tau - \tau') = -(\boldsymbol{\partial_\tau} - \boldsymbol{B_k} \cdot \boldsymbol{\sigma})^{-1} \tag{5.15}$$

The expansion is exactly as the Dyson series in chap. 4, except that scattering matrix Σ is replaced for an operator on the $\boldsymbol{k}$ space acting on the closest neighbor. The sum over α is omitted by the presence of repeated index. For the bare propagator it is that part of the Green's function matrix not converting electron into holes or vice-versa ($\Delta = 0$). Explicitly,

$$\boldsymbol{\mathcal{G}^{(0)}}(\boldsymbol{k}, i\omega) = \begin{pmatrix} \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}}} & 0 \\ 0 & \frac{1}{i\omega + \varepsilon_{\boldsymbol{k}}} \end{pmatrix} \tag{5.16}$$

Thus,

$$\boldsymbol{\Delta_k} = \sum_\alpha \frac{(-1)^{n_\alpha}}{\alpha!} \begin{pmatrix} 0 & \frac{\partial^{n_\alpha} \Delta(\boldsymbol{x})}{\partial \boldsymbol{x}^{n_\alpha}} (\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_2} (\frac{\partial_{\boldsymbol{k}_3}}{-i})^{\alpha_3} \\ \frac{\partial^{n_\alpha} \Delta^*(\boldsymbol{x})}{\partial \boldsymbol{x}^{n_\alpha}} (\frac{\partial_{\boldsymbol{k}_1}}{i})^{\alpha_1} (\frac{\partial_{\boldsymbol{k}_2}}{i})^{\alpha_2} (\frac{\partial_{\boldsymbol{k}_3}}{i})^{\alpha_3} & 0 \end{pmatrix} \tag{5.17}$$

We, on purpose, preserved the ordering suitable for the operator substitution in (4.96). We replace the uniform Δ for an operator acting on its closest neighbor.

$$\tilde{\mathcal{F}}_\omega(\boldsymbol{k}) = \sum_\alpha \frac{(\partial_1^{\alpha_1} \partial_2^{\alpha_1} \partial_3^{\alpha_3} \Delta^*)}{\alpha! (-1)^{n_\alpha}} \frac{1}{i\omega + \varepsilon_{\boldsymbol{k}}} \Big[(\frac{\partial_{\boldsymbol{k}_1}}{-i})^{\alpha_1} (\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_2} (\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_3} \Big] \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}}} \sum_{n=0} (\mathcal{X})^n \ , \tag{5.18}$$

$$\mathcal{X} = \sum_\gamma \frac{(\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \Delta)}{\gamma! (-1)^{n_\gamma}} \Big[(\frac{\partial_{k_1}}{i})^{\gamma_1} (\frac{\partial_{k_2}}{i})^{\gamma_3} (\frac{\partial_{k_2}}{i})^{\gamma_3} \Big] \frac{1}{i\omega + \varepsilon_{\boldsymbol{k}}} \tag{5.19}$$

$$\sum_\beta \frac{(\partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} \Delta^*)}{\beta! (-1)^{n_\beta}} \Big[(\frac{\partial_{k_1}}{-i})^{\beta_1} (\frac{\partial_{k_2}}{-i})^{\beta_2} (\frac{\partial_{k_3}}{-i})^{\beta_3} \Big] \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}}} \ , \tag{5.20}$$

Finally, (Appendix D.4)

$$\frac{\Delta^*(\boldsymbol{x})}{g} = -\frac{1}{\beta} \sum_\omega \int \tilde{\mathcal{F}}_\omega(\boldsymbol{k}) d\boldsymbol{k} = -\frac{N(0)}{\beta} \sum_\omega \int \tilde{\mathcal{F}}_\omega(\varepsilon) d\varepsilon \tag{5.21}$$

Such that $(k_B = 1)$,

$$\frac{\Delta^*(\boldsymbol{x})}{g} = -T \sum_\omega \int \boldsymbol{dk} \sum_{\boldsymbol{\alpha}} \frac{(\partial_1^{\alpha_1} \partial_2^{\alpha_1} \partial_3^{\alpha_3} \Delta^*)}{\alpha! (-1)^{n_\alpha}} \frac{1}{i\omega + \varepsilon_{\boldsymbol{k}}} \Big[(\frac{\partial_{\boldsymbol{k}_1}}{-i})^{\alpha_1} (\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_2} (\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_3} \Big] \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}}} \sum_{n=0} (\mathcal{X})^n \tag{5.22}$$

with

$$\mathcal{X} = \sum_{\boldsymbol{\gamma\beta}} \frac{(\partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3}\Delta)}{\gamma!(-1)^{n_\gamma}} \frac{(\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}\Delta^*)}{\beta!(-1)^{n_\beta}} \left[(\frac{\partial_{k_1}}{i})^{\gamma_1}(\frac{\partial_{k_2}}{i})^{\gamma_3}(\frac{\partial_{k_2}}{i})^{\gamma_3}\right]\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}$$
$$\left[(\frac{\partial_{k_1}}{-i})^{\beta_1}(\frac{\partial_{k_2}}{-i})^{\beta_2}(\frac{\partial_{k_3}}{-i})^{\beta_3}\right]\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}} \tag{5.23}$$

I.e,

$$\frac{\Delta^*(\boldsymbol{x})}{g} = -T\sum_{\omega}\sum_{n}\sum_{\boldsymbol{\alpha\beta\gamma}} \frac{(\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3})\Delta}{\alpha!(-1)^{n_\alpha}}\left[\frac{(\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3})\Delta^*}{\beta!(-1)^{n_\beta}}\frac{(\partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3})\Delta}{\gamma!(-1)^{n_\gamma}}\right]^n$$
$$\int d\boldsymbol{k}\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}\left[(\frac{\partial_{\boldsymbol{k}_1}}{-i})^{\alpha_1}(\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_2}(\frac{\partial_{\boldsymbol{k}_2}}{-i})^{\alpha_3}\right]\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}}\Bigg\{\left[(\frac{\partial_{k_1}}{i})^{\gamma_1}(\frac{\partial_{k_2}}{i})^{\gamma_3}(\frac{\partial_{k_2}}{i})^{\gamma_3}\right]\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}$$
$$\left[(\frac{\partial_{k_1}}{-i})^{\beta_1}(\frac{\partial_{k_2}}{-i})^{\beta_2}(\frac{\partial_{k_3}}{-i})^{\beta_3}\right]\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}}\Bigg\}^n \tag{5.24}$$

The overall sign for each set of α, β and γ (not provenient from the realization of the product of imaginary terms) is provided through $(-1)^{n_\alpha+n(n_\beta+n_\gamma)+1}$.

Selection Rules and Simplifications

The final expression seems a bit cumbersome and has a lot of indices. The reader less used to index manipulation might prefer to simply modify the order parameter as follows

$$\Delta \to \underbrace{\Delta}_{O(\tau^{1/2})} - \sum_j \underbrace{\partial_j\Delta(\frac{\partial_{k_j}}{i})}_{O(\tau^{1/2})} + \sum_j \underbrace{\frac{1}{2}\partial_j^2\Delta(\frac{\partial_{k_j}}{i})^2}_{O(\tau^{3/2})} - \sum_{j\neq l}\underbrace{\frac{\partial_j^2\partial_l\Delta}{2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})}_{O(\tau^{3/2})} -$$
$$\sum_{j\neq l\neq m}\underbrace{\partial_j\partial_l\partial_m\Delta(\frac{\partial_{k_j}}{i})(\frac{\partial_{k_l}}{i})(\frac{\partial_{k_m}}{i})}_{O(\tau^4)} + \sum_{j\neq l}\underbrace{\frac{\partial_j^2\partial_l^2\Delta}{2!2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})^2}_{O(\tau^{5/2})} + \text{Higher order terms} \tag{5.25}$$

Implying, thus,

$$\tilde{\mathcal{F}}_\omega = \tilde{\mathcal{G}}_\omega^{(0)}\{\Delta^* - \sum_j \partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_{jl}\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 - \sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i}) -$$
$$\sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i}) + \sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\mathcal{G}_\omega^{(0)}\sum_n\Bigg[\{\Delta - \sum_j\partial_j\Delta(\frac{\partial_{k_j}}{i}) +$$
$$\sum_j\frac{1}{2}\partial_j^2\Delta(\frac{\partial_{k_j}}{i})^2 - \sum_{jl}\frac{\partial_j^2\partial_l\Delta}{2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i}) - \sum_{jlm}\partial_j\partial_l\partial_m\Delta(\frac{\partial_{k_j}}{i})(\frac{\partial_{k_l}}{i})(\frac{\partial_{k_m}}{i}) +$$
$$\sum_{jl}\frac{\partial_j^2\partial_l^2\Delta}{2!2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})^2\}\tilde{\mathcal{G}}_\omega^{(0)}\{\Delta^* - \sum_j\partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_j\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 -$$
$$\sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i}) - \sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i}) + \sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\mathcal{G}_\omega^{(0)}\Bigg]^n \tag{5.26}$$

The self-consistent gap equation is, therefore,

$$\begin{aligned}\frac{\Delta^*}{g} = & -T\sum_{\omega}\int \boldsymbol{dk}\frac{1}{i\omega+\varepsilon}\{\Delta^* - \sum_j \partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_i \frac{1}{2}\partial_i^2\Delta^*(\frac{\partial_{k_i}}{-i})^2 - \sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i}) - \\ & \sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i}) + \sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\frac{1}{i\omega-\varepsilon}\sum_n\Bigg[\{\Delta - \sum_j \partial_j\Delta(\frac{\partial_{k_j}}{i}) + \\ & \sum_j \frac{1}{2}\partial_j^2\Delta(\frac{\partial_{k_j}}{i})^2 - \sum_{jl}\frac{\partial_j^2\partial_l\Delta}{2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i}) - \sum_{jlm}\partial_j\partial_l\partial_m\Delta(\frac{\partial_{k_j}}{i})(\frac{\partial_{k_l}}{i})(\frac{\partial_{k_m}}{i}) + \\ & \sum_{jl}\frac{\partial_j^2\partial_l^2\Delta}{2!2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})^2\}\frac{1}{i\omega+\varepsilon}\{\Delta^* - \sum_j \partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_j \frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 - \\ & \sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i}) - \sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i}) + \sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\frac{1}{i\omega-\varepsilon}\Bigg]^n\end{aligned} \tag{5.27}$$

The advantage of such representation might be dubious at first glance, yet, the reader might be convinced of the contrary by applying two simple rules, the first one being simply the truncation order in τ,

- The order of the product terms is not greater than the maximum order in the expansion of Δ.
- There are trivial null terms - these are such that the number of times the derivative with respect to a given coordinate is odd.

We notice that for each spatial derivative of a given coordinate direction there is a derivative with respect to the respective momentum direction. Next, since odd momentum derivatives of either the electron or hole greens function are odd, and odd terms do cancel when integrated in a symmetric interval, the selection rule holds.

$$\partial_{\boldsymbol{k}}^{\alpha}\{\mathcal{G}^{(0)}, \tilde{\mathcal{G}}^{(0)}\} = \text{odd} \text{ , provided } \varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m} - \mu \tag{5.28}$$

with $\boldsymbol{k}$ measured from the Fermi-Level. This can be checked in practice with any symbolic software. By following these rules, many of the terms are trivially disregarded at a first glance. Also, accounting for the number of terms providing the same contribution becomes a connecting-dot-like exercise. The following section will show the usefulness of this in practice.

THE LANDAU-THEORY

The only way to capture terms of order $3/2$ is by considering the zeroth-order expansion $n = 0$. The second most relevant term is the subsequent $n = 1$ term, which gives origin to the Extended Landau theory ($\tau^{5/2}$). It is important to notice that $n = 1$ also contains a single term of the order $\tau^{3/2}$- that without derivatives. For $n = 0$, the odd terms in the derivative are null by applying the selection rule. Therefore,

$$\frac{\Delta^*}{g} = -T\int \boldsymbol{dk}\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}\{\Delta^* - \overbrace{\sum_j \partial_j \Delta^* (\frac{\partial_{k_j}}{-i})}^{\text{odd parity}} + \sum_j \frac{1}{2}(\frac{\partial_j}{-i})^2 \Delta^* (\partial_{k_j})^2 - \sum_j \frac{1}{2}(\frac{\partial_j}{-i})^2(\frac{\partial_l}{-i})\Delta^*(\partial_{k_j})^2 + \\ \overbrace{\sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i})}^{\text{odd parity}} + \overbrace{\sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}}^{O(\tau^{5/2})}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}} \tag{5.29}$$

Consider, for instance, the term with the second-order derivative (for $n = 0$). It follows, in the energy measure,

$$a_2 = -N(0)T\sum_\omega \int_{-\infty}^{\infty} d\varepsilon \frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_i}}{-i})^2\Big[\frac{1}{i\omega-\varepsilon}\Big]|_{\frac{\hbar^2 k_i^2}{2m}=\frac{\varepsilon+\mu_F}{3}} = N(0)T\sum_\omega \frac{\mu_F\pi}{3|\omega|^3} \tag{5.30}$$

Provided isotropy, and changing to the energy measure. Provided the Matsubara frequency $\omega = (2n+1)\pi T$,

$$a_2 = \mathcal{K}(1 + 2\tau + \mathcal{O}(\tau^2)) \text{ , } \mathcal{K} = N(0)\frac{7\zeta(3)}{48\pi^2T_c^2}\hbar^2 v_F^2 \tag{5.31}$$

The calculation of the remaining integrals ($n = 0$) leads to the remaining microscopic coefficients of the GL theory, b, and a_1 (Appendix D.2).

$$b = N(0)\frac{7\zeta(3)}{8\pi^2T_c^2} \text{ , } a_1 = \frac{1}{g} - a\Big(\tau + \frac{\tau^2}{2} + \mathcal{O}(\tau^3)\Big) \text{ , } a = -N(0) \tag{5.32}$$

THE EXTENDED LANDAU THEORY

For the Extended Landau theory we stop the expansion in the next-to-leading order (of the GL theory), $\tau^{5/2}$. We rewrite eq. (5.27) and explicitly exclude terms due to the combination of the selection rule and the truncation criterion.

$$
\begin{aligned}
&\frac{\Delta^*}{g} = -T\sum_{\omega}\int \boldsymbol{dk}\frac{1}{i\omega+\varepsilon}\{\Delta^* - \overbrace{\sum_j \partial_j\Delta^*(\frac{\partial_{k_j}}{-i})}^{\text{Possible at } n=1} + \sum_j \frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 - \cancel{\sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})} - \\
&\cancel{\sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})} + \overbrace{\cancel{\sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2}\}}^{\text{unless } n=0}\ \frac{1}{i\omega-\varepsilon}\sum_n\Bigg[\{\Delta - \sum_j\partial_j\Delta(\frac{\partial_{k_j}}{i}) + \\
&\sum_i\frac{1}{2}\partial_j^2\Delta(\frac{\partial_{k_j}}{i})^2 - \cancel{\sum_{jl}\frac{\partial_j^2\partial_l\Delta}{2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})} - \cancel{\sum_{jlm}\partial_j\partial_l\partial_m\Delta(\frac{\partial_{k_j}}{i})(\frac{\partial_{k_l}}{i})(\frac{\partial_{k_m}}{i})} + \\
&\cancel{\sum_{jl}\frac{\partial_j^2\partial_l^2\Delta}{2!2!}(\frac{\partial_{k_j}}{i})^2(\frac{\partial_{k_l}}{i})^2}\}\frac{1}{i\omega+\varepsilon}\{\Delta^* - \sum_j\partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_j\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 - \\
&\cancel{\sum_{jl}\frac{\partial_j^2\partial_l\Delta^*}{2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})} - \cancel{\sum_{jlm}\partial_j\partial_l\partial_m\Delta^*(\frac{\partial_{k_j}}{-i})(\frac{\partial_{k_l}}{-i})(\frac{\partial_{k_m}}{-i})} + \cancel{\sum_{jl}\frac{\partial_j^2\partial_l^2\Delta^*}{2!2!}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2}\}\frac{1}{i\omega-\varepsilon}\Bigg]^n
\end{aligned}
\tag{5.33}
$$

We remark that for $n = 0$ the single derivative term is not present due to the second rule, however, it might be present for $n = 1$ in combination with other terms if passing the selection criteria. The exclusion of these terms is justified by a connecting-dot exercise as promised. This term passes the selection criteria when $n = 0$. When $n = 1$, it is not possible to connect the terms with $\partial_j^2\partial_l^2\Delta$ or its conjugate to any other term and produce an accuracy smaller or equal to the order of the theory, as the smallest term it could possibly multiply is of the order $\tau^{1/2}$. The other excluded terms are those with $\partial_j^2\partial_l\Delta$ and $\partial_j\partial_l\partial_m\Delta$ or its conjugate. They could be present for $n = 0$ if not for the second rule. For $n = 1$ a combination of the first and second selection rules is required; it survives the application of the first rule only when attached to Δ, but not the second.

By applying the selection rule and the constraint on the truncation order,

$$
\begin{aligned}
&\frac{1}{g}\Delta^* = -T\sum_{\omega}\int \boldsymbol{dk}\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}\{\Delta^* + \sum_j\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2 + \sum_{jl}\frac{\partial_j^2\partial_l^2}{2!2!}\Delta^*(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\}\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}} + \\
&-T\sum_{\omega}\int \boldsymbol{dk}\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}\{\Delta^* - \sum_j\partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_j\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2\}\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}}\{\Delta - \sum_j\partial_j\Delta(\frac{\partial_{k_j}}{i}) + \\
&\sum_j\frac{1}{2}\partial_j^2\Delta(\frac{\partial_{k_j}}{i})^2\}\frac{1}{i\omega+\varepsilon_{\boldsymbol{k}}}\{\Delta^* - \sum_j\partial_j\Delta^*(\frac{\partial_{k_j}}{-i}) + \sum_j\frac{1}{2}\partial_j^2\Delta^*(\frac{\partial_{k_j}}{-i})^2\}\frac{1}{i\omega-\varepsilon_{\boldsymbol{k}}}
\end{aligned}
\tag{5.34}
$$

We collect the remaining terms passing the selection rules by looking for every possible multiplication in (5.34) - with the sum on indices made implicit. By connecting dots and applying once more the selection rule to exclude derivatives present in an odd number, the extra contributing terms are easily identified. It follows ($k_B = 1$): for $n = 0$,

$$\sum_j \frac{\partial_j^4}{4!}\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\Big[\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})^4\frac{1}{i\omega-\varepsilon}\Big] \tag{5.35}$$

$$\frac{\partial_j^2\partial_l^2}{2!2!}\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\Big[\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})^2(\frac{\partial_{k_l}}{-i})^2\frac{1}{i\omega-\varepsilon}\Big]\ , \tag{5.36}$$

for $n = 1$,

$$\Delta^*\Delta\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\right] \tag{5.37}$$

$$(\partial_j\Delta^*)(\partial_j\Delta)\Delta^*[-N(0)T]\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})(\frac{1}{i\omega-\varepsilon})(\frac{\partial_{k_j}}{i})(\frac{1}{i\omega+\varepsilon})\frac{1}{i\omega-\varepsilon}\right] \tag{5.38}$$

$$(\partial_j\Delta^*)\Delta(\partial_j\Delta^*)(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})(\frac{1}{i\omega-\varepsilon})\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})(\frac{1}{i\omega-\varepsilon})\right] \tag{5.39}$$

$$\Delta^*(\partial_j\Delta)(\partial_j\Delta^*)(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}(\frac{\partial_{k_j}}{i})(\frac{1}{i\omega+\varepsilon})(\frac{\partial_{k_j}}{-i})(\frac{1}{i\omega-\varepsilon})\right] \tag{5.40}$$

$$(\frac{1}{2}\partial_j^2\Delta^*)\Delta\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})^2(\frac{1}{i\omega-\varepsilon})\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\right] \tag{5.41}$$

$$\Delta^*(\frac{1}{2}\partial_j^2\Delta)\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}(\frac{\partial_{k_j}}{i})^2(\frac{1}{i\omega+\varepsilon})\frac{1}{i\omega-\varepsilon}\right] \tag{5.42}$$

$$\Delta^*\Delta(\frac{1}{2}\partial_j^2\Delta^*)(-T)\sum_\omega \int \boldsymbol{dk}\left[\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\frac{1}{i\omega+\varepsilon}(\frac{\partial_{k_j}}{-i})^2(\frac{1}{i\omega-\varepsilon})\right]\ , \tag{5.43}$$

for $n = 2$,

$$\Delta^*\Delta\Delta^*\Delta\Delta^*(-T)\sum_\omega \int \boldsymbol{dk}\Big[\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\frac{1}{i\omega+\varepsilon}\frac{1}{i\omega-\varepsilon}\Big]\ . \tag{5.44}$$

Any other contribution of five (or more) multiplying terms would overcome the order of the EGL theory.

Dictionary

Below we describe a dictionary. We infer on a set of simple rules producing the integral factor multiplying the corresponding terms depending on Δ, Δ^* and its derivatives. It is, we 'translate' the differential terms to the shape of the integral (with respect to $\boldsymbol{k}$) yielding the respective coefficient.

- The first factor in the integrand is always $T(-1)^{n_\alpha+n(n_\beta+n_\gamma)+1}\frac{1}{i\omega+\varepsilon}$;

- To each Δ^* alone we associate $\frac{1}{i\omega-\varepsilon}$. To each Δ alone we associate a factor of $\frac{1}{i\omega+\varepsilon}$;
- To $\partial_j^{n_j}\Delta$ we associate $(\frac{\partial_{k_j}}{i})^{n_j}(\frac{1}{i\omega+\varepsilon})$; To each $\partial_j^{n_j}\Delta^*$ we associate $(\frac{\partial_{k_j}}{-i})^{n_j}(\frac{1}{i\omega-\varepsilon})$.

These rules are of general nature to any order n of the expansion. If we want rules for the energy measure, we apply the extra procedures.

- The multiplication for $N(0)$ on top of the first item in the momenta space;
- If a single quadratic term component of momenta appears, $\frac{k_j^2}{2m} = \frac{\varepsilon-\mu}{3}$ (isotropic case) and the integration is carried out in the energy measure;
- If two quadratic terms appear, one for each component, the spherical components are to be integrated over, and, provided the radial component, $k^2 = k_j^2 + k_l^2 = \frac{2}{3}(\varepsilon+\mu)$ $(j \neq l)$, the integration is carried out in the energy measure;
- If tree-different momenta appears, the spherical components are to be integrated over, and, provided the radial component, $k^2 = k_j^2 + k_l^2 + k_m^2 = \varepsilon + \mu$ $(j \neq l \neq m)$, the integration is carried out in the energy measure.

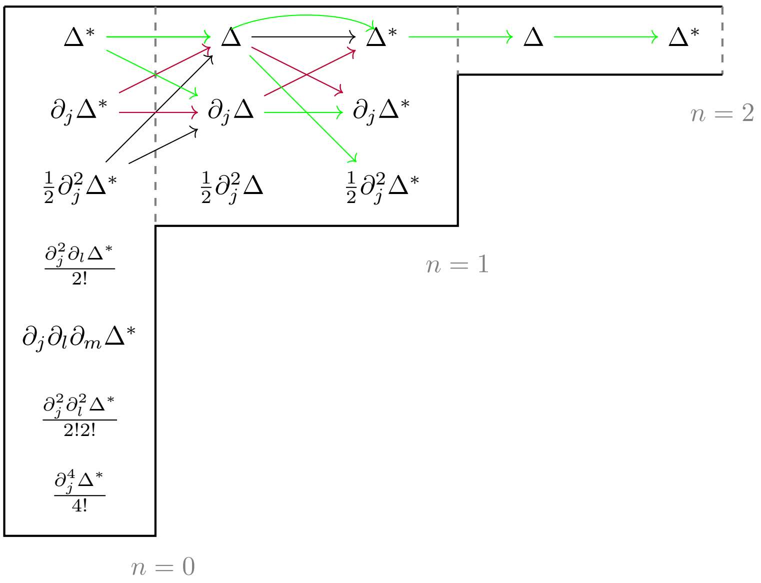
The case of item two quadratic terms appears only one time in the Extended-theory, while the case of three quadratic terms does not appear neither in the Extended Landau theory (EL) nor in the leading order of the Extended Landau theory (EL$^{(2)}$).

Connecting-Dot Exercise, a Map to the Road

We may map our problem to the following connecting-dot diagram. The cases $n = 0$, $n = 1$ and $n = 2$ contribute separately with a different number of terms, respectively 1, 3 and 5. We omit the sum over indices in the diagram for aesthetic reasons.

The dashed line denotes the ‘stopping point’ (in a sense to be understood) for the accounting of terms of the powers of n. Before the $n = 0$ stopping point, we add a single column, and after each ’stopping point’ we add a layer of two columns. The terms contributing to the power n are those consisting of the multiplication of column elements -each column contributes with a single term in the multiplication- up to the stopping-point relative to n. The multiplications are represented by arrows and an arrow path is a chain of arrows collecting the multiplications before a stopping point. A path relative to a given n-power stopping point finishes its trajectory in the last element before the corresponding dashed line is reached. If

Figure 8 – Diagram to the Extended Landau Theory (EL), O($\leq \tau^{5/2}$) .



Source: The author

the arrow path is relevant, the term it corresponds to must obey the selection rules. We infer, on these assumptions, that, for $n = 0$, there is no terms to connect. For paths relative to $n = 1$, there should be contributions with a three-term multiplication. For paths relative to $n = 2$, there should be contributions with a five-term multiplication, and, in general, $2n + 1$ terms for each power considered. The diagram portrayed is built such that a first filtering is made on terms not respecting the truncation criterion. We see how this is done in practice.

Before the $n = 0$ stopping point, we list the terms in Δ and its derivatives which obey the truncation criterion, but not the selection rule on the parity. Next, O($\tau^{\text{Order of the theory}-1}$) is the maximum order allowed for the element of the columns added in between the stopping points $n = 0$ and $n = 1$. This is since before the final point of a given $n = 1$ path is reached, it will at least have collected two $(2n)$ terms of the minimum order of $\tau^{1/2}$ each. The same idea can be applied to the next added layer. Before the final point of a given $n = 2$ is reached, it has at least collected four $(2n)$ terms of the minimum order of $\tau^{1/2}$ each, yielding O($\tau^{\text{Order of the theory}-2}$) as the maximum order allowed for the element of the columns added in between the stopping points of $n = 1$ and $n = 2$. In general, the maximum order of the added columns in between the stopping points $n - 1$ and n is O($\tau^{\text{Order of theory}-n}$) for $n > 0$. Indeed, this corresponds to our diagram arrangement. In between the $n = 0$ and $n = 1$ stopping points in EL (O($\tau^{5/2}$)), the maximum order is that due to the second-order derivative on the gap, $\tau^{3/2}$. In between the $n = 1$ and $n = 2$ stopping points, the maximum order is $\tau^{1/2}$, corresponding to Δ, or Δ^*.

We would like to understand some of the drawn arrows - and those not drawn (for the $n = 0$ power) - in the context of EL. In the first stopping point, $n = 0$, we just apply the second selection rule, filtering terms such as $\partial_j^2 \partial_l \Delta^*$, and $\partial_j \partial_l \partial_m \Delta^*$, which do not contribute to this power, but could be of importance in connecting to other terms related to the contribution of higher powers. The first selection rule is attended in connecting $\frac{1}{2}\partial_j^2 \Delta^*$ to $\partial_j \Delta$. But in order for the second selection rule to hold, we need to connect it to $\partial_j \Delta^*$ - however the EGL order is surpassed. Therefore, this is not a valid arrow path. We may represent the non-validity of this arrow path by not making this to reach the element of the column before the 'stopping point'. There is also no validity for terms starting with $\partial_l^2 \partial_j \Delta^*$ from $n = 0$. This path is not drawn in the diagram, but $\partial_l^2 \partial_j \Delta^* \Delta \partial_j \Delta^*$ is the only possibility encompassing the correct parity for $n = 1$. Though it violates the truncation criterion. We did not write this arrow, but if it were the case we would not make it reach the final column before the $n = 1$ stopping point.

The integrals in the range (5.38) to (5.40) are equal, as well, the integrals from (5.41) to (5.43) are equal, reducing the number of integrals to be computed to two extra terms for $n = 0$ and six extra terms for $n = 1$. By following the graphics, this is equivalent to accounting for the number of valid arrow paths.

Integrals Evaluation

By defining the integral coefficients as I_i with i varying from 1 to 8, the first and second integral coefficient they relate as $I_1 = I_2/3$, causing

$$\sum_{jl} \frac{1}{2!2!} \partial_j^2 \partial_l^2 \Delta^* I_1 + \sum_j \frac{1}{4!} \partial_j^4 \Delta^* I_2 = \frac{1}{8} (\nabla^2 (\nabla^2 \Delta^*)) I_2 \tag{5.45}$$

The first integral is one of the cases where distinct k appears, case (ii) as refereed in the dictionary section on the energy measure. We make a table summarizing the resulting integrals.

$$I_1 = \frac{I_2}{3} \tag{5.46}$$

$$I_2 = N(0) \hbar^4 v_F^4 \frac{W_5^4}{3} \tag{5.47}$$

$$I_4 = I_6 = -N(0) \frac{5}{36} \hbar^2 v_F^2 W_5^4 \tag{5.48}$$

$$I_5 = 3 I_4 \tag{5.49}$$

$$I_7 = I_9 = -N(0) \frac{5}{9} \hbar^2 v_F^2 W_5^4 \tag{5.50}$$

$$I_8 = \frac{I_7}{2} \tag{5.51}$$

with $W_5^4 = 93\zeta(5)/160\pi^4 T_c^4$ with ζ the usual Zeta-Riemann function. We remember that, attached to the integrals I_7 to I_9, there is the multiplication factor of $\frac{1}{2}$. This done, we write down the complete microscopic theory.

$$
\begin{aligned}
0 = (a_1 - \tfrac{1}{g})\tau^{1/2}\Delta^* + a_2\tau^{3/2}\nabla^2\Delta^* + a_3\tau^{5/2}\nabla^2(\nabla^2\Delta^*) - b_1\tau^{3/2}|\Delta|^2\Delta^* - \\
b_2\tau^{5/2}\Big[2\Delta^*|\nabla\Delta|^2 + 3\Delta(\nabla\Delta^*)^2 + (\Delta^*)^2\nabla^2\Delta + 4|\Delta|^2\nabla^2\Delta^*\Big] + c_1\tau^{5/2}|\Delta|^4\Delta
\end{aligned}
\tag{5.52}
$$

With the microscopic coefficients

$$a_1 = \frac{1}{g} - a\Big(\tau + \frac{\tau^2}{2} + \mathcal{O}(\tau^3)\Big) \ , \ a = -N(0) \tag{5.53}$$

$$b_1 = b(1 + 2\tau + \mathcal{O}(\tau^2)) \ , \ b = N(0)\frac{7\zeta(3)}{8\pi^2 T_c^2} \tag{5.54}$$

$$c_1 = c(1 + \mathcal{O}(\tau)) \ , \ c = N(0)\frac{93\zeta(5)}{128\pi^4 T_c^4} \tag{5.55}$$

$$a_2 = \mathcal{K}(1 + 2\tau + \mathcal{O}(\tau^2)) \ , \ \mathcal{K} = \frac{b}{6}\hbar^2 v_F^2 \tag{5.56}$$

$$a_3 = \mathcal{Q}(1 + \mathcal{O}(\tau)), \mathcal{Q} = \frac{c}{30}\hbar^4 v_F^4 \tag{5.57}$$

$$b_2 = \mathcal{L}(1 + \mathcal{O}(\tau)), \mathcal{L} = \frac{c}{9}\hbar^2 v_F^2 \tag{5.58}$$

For the computation of integrals in the level of GL equations, one is referred to Appendix D.2. The same procedure holds for EL/EGL integrals. Computing the integrals becomes a very straightforward activity employing a symbolic software.

BEYOND THE EXTENDED LANDAU THEORY

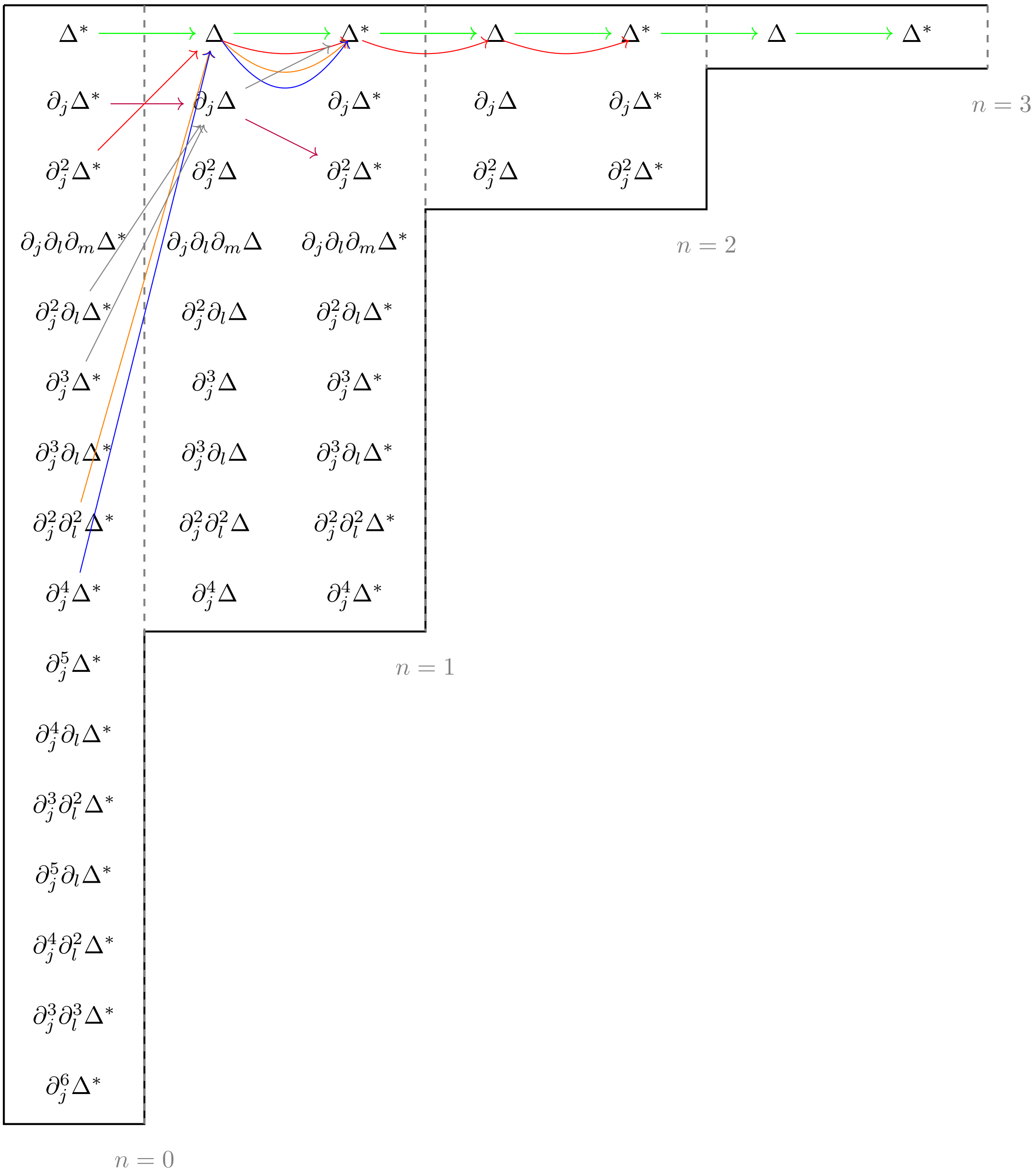
The theory of the next-to-leading EL order, which we coin E$^{(2)}$L theory, for simplicity of reference, is an expansion in the order of $\tau^{7/2}$ in terms of τ. In the first column we start with the maximum order of derivative for reaching the maximum accuracy of the theory, in case of E$^{(2)}$L, six. In between the stopping points $n = 0$ and $n = 1$, the maximum order in the first layer (of two columns) is $5/2$, in between $n = 1$ and $n = 2$, the maximum order in the second layer is $3/2$, and in the last layer $1/2$. It follows the diagram.

where we omitted the numerical factors $(1/\alpha!)$ preceding the derivatives for simplicity. As before, the odd terms cancels out for the power $n = 0$. For the power $n = 1$, the only 'kind' [1] of odd derivative terms which contribute are the paths in gray, which must be partnered as

---

[1] There are terms deriving from these which appear by exchanging the order of the terms, and conjugating

Figure 9 – Diagram of the theory of the next-to-next-leading order (E$^{(2)}$L), O($\leq \tau^{7/2}$).



Source: The author

above not to violate the selection rule. The reader may verify that this is indeed the case by trying to connect them to other terms without violating the selection rules.

We remember that, in connecting the terms, we are actually summing over indices. For instance, the product $\partial_j\Delta^*\partial_j\Delta\Delta^*$ denotes $\sum_{jl}\partial_j\Delta^*\partial_j\Delta\frac{\partial_l^2\Delta^*}{2}\times$ Integral$_{ij}$, where the integral is provided in the dictionary, and transformed to an index-independent form in the energy-measure, for either $l=j$ or $l\neq j$, each of the cases corresponding to the specific rules in the section 'dictionary'. Its remarkable that no term of the form $\partial_i\partial_j\partial_l\Delta^*$ is connected via a path contributing to the theory. This is equivalent to say that it does not appear a quadratic term in the three components (as we claim in the dictionary section, on the rules for the energy-measure in the EGL theory).

As we are to see in chap. 6, an extension of the Extended theory would provide a better understanding of the $\kappa-T$ phase diagram, which up to the EGL order (and a κ expansion) describes only linear curves separating the multiple domains. The extension of the EGL theory could allow for the appearance of boundaries between phases which otherwise would not make contact (Fig.15, chap.6). Higher-order theories could answer qualitative questions concerning the superconducting behaviour, their importance being not merely a quantitative sophistication. Apart from this commentary we will not go into much detail about the higher-order theory except in providing a diagrammatic straightforward tool to extended theories. It allows one not to be lost in an infinite amount of calculations; now one has a graphical map and a dictionary. The integrals can be made via symbolic software such as *Mathematica*, as the time consumption for such calculus is virtually null.

Procedure to Any Order of the Superfluid Theory

We propose a simple schedule to expand the theory to any order.

- Draw the diagram scope - now you should be able to write it for any order of the theory.
- Draw the arrows within the diagram scope, connecting the terms obeying the selection rules.
- Apply the dictionary so that the integral coefficient is known.

---

it, for instance,$\Delta^*\partial_j^2\partial_l\Delta\partial_j\Delta^*$ is another possible arrow of the same 'kind'. We did not draw these sibling lines, as we have also not drawn them in the EL diagram.

- Integrate it with symbolic software such as *Mathematica* and perform the integral and discrete sum. They save the time of unnecessary work.

SUPERCONDUCTIVITY: NON-ZERO MAGNETIC-FIELD EFFECTS AND THE EXTENDED GINZBURG-LANDAU THEORY

Now we turn our attention to charged superfluids, 'superconductors'. The most direct approach to provide some of the important contributions departing from the previous results is by considering the formalism in the position space. We will follow this path even though up to this point we have worked out mostly in the momenta space. The Gor'kov modification in the Green's function provides

$$\mathcal{G}^{(0)}_{\text{Gor'kov}}(\boldsymbol{r},\boldsymbol{r'}) = \exp\left[\frac{ei}{\hbar c}\int_{\boldsymbol{r}}^{\boldsymbol{r'}} \boldsymbol{A}(\boldsymbol{q})d\boldsymbol{q}\right] \tag{5.59}$$

$$\tilde{\mathcal{G}}^{(0)}_{\text{Gor'kov}}(\boldsymbol{r},\boldsymbol{r'}) = \exp\left[-\frac{ei}{\hbar c}\int_{\boldsymbol{r}}^{\boldsymbol{r'}} \boldsymbol{A}(\boldsymbol{q})d\boldsymbol{q}\right]\tilde{\mathcal{G}}^{(0)}_{\boldsymbol{B}=0}(\boldsymbol{x},\boldsymbol{y}) \tag{5.60}$$

where the path is a straight line. In fact, the only deviation provided by the classical movement in the field $\boldsymbol{B}$

$$\ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \times \boldsymbol{\Omega} \ ; \ \boldsymbol{\Omega} = \frac{2e}{\hbar c}\boldsymbol{B} \tag{5.61}$$

In the theory written for the space, we have convolutions of the kind

$$\begin{aligned}\mathcal{F}_\omega(\boldsymbol{r},\boldsymbol{r'}) &= \int d^3\boldsymbol{y}\mathcal{G}^{(0)}_\omega(\boldsymbol{r},\boldsymbol{y})\Delta(\boldsymbol{y})\tilde{\mathcal{G}_\omega}^{(0)}(\boldsymbol{y},\boldsymbol{r'})\\ \mathcal{F}_\omega(\boldsymbol{r},\boldsymbol{r'}) &= \mathcal{G}^{(0)}_\omega(\boldsymbol{r},\boldsymbol{r'}) + \int d^3\boldsymbol{y}\tilde{\mathcal{G}}^{(0)}_\omega(\boldsymbol{r},\boldsymbol{y})\Delta^*(\boldsymbol{y})\mathcal{F}_\omega(\boldsymbol{y},\boldsymbol{r'})\end{aligned} \tag{5.62}$$

See, for instance, ref. (VAGOV; SHANENKO A, 2012). Therefore, by applying the equations self-consistently, each term in the exponent appears with exchanged integration limits on the right and left-hand sides. As both $\mathcal{G}$ and $\tilde{\mathcal{G}}$ appear in pairs, it is convenient to provide the corrections entirely in the order parameter while keeping the original Green's function unmodified,

$$\Delta(\boldsymbol{r}) \to \Delta(\boldsymbol{r},\boldsymbol{r'})\exp\left[-\frac{2ei}{\hbar c}\int_{\boldsymbol{r}}^{\boldsymbol{r'}} \boldsymbol{A}(\boldsymbol{q})d\boldsymbol{q}\right], \tag{5.63}$$

which defines an auxiliary two-point order-parameter. As the phase factor is present on the left and on the right of the self-consistent expansion, it follows that taking the limit $\boldsymbol{r} \to \boldsymbol{r'}$ in the final result is equivalent to cancelling the remaining exponents. Term by term, the extended theory term-by-term is rewritten with the correct magnetic dependence,

$$\begin{aligned}
&a_1\tau^{1/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}\Delta(\boldsymbol{r},\boldsymbol{r'}) = a_1\tau^{1/2}\Delta(\boldsymbol{r})\\
&a_2\tau^{3/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}\nabla^2\Delta(\boldsymbol{r},\boldsymbol{r'}) = a_2\tau^{3/2}\boldsymbol{\mathcal{D}}^2\Delta(\boldsymbol{r})\\
&a_3\tau^{5/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}\nabla^2(\nabla^2\Delta(\boldsymbol{r},\boldsymbol{r'})) = a_3\tau^{5/2}\{\boldsymbol{\mathcal{D}}^4 - \frac{4ei}{3\hbar c}\boldsymbol{\nabla}\times\boldsymbol{B}\cdot\boldsymbol{\mathcal{D}} + \frac{4e^2}{\hbar^2c^2}\boldsymbol{B}^2\}\Delta\\
&b_1\tau^{3/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}|\Delta(\boldsymbol{r},\boldsymbol{r'})|^2\Delta(\boldsymbol{r},\boldsymbol{r'}) = b_1\tau^{3/2}|\Delta(\boldsymbol{r},\boldsymbol{r'})|^2\Delta(\boldsymbol{r},\boldsymbol{r'})\\
&b_2\tau^{5/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}\Big[2\Delta(\boldsymbol{r},\boldsymbol{r'})|\nabla\Delta(\boldsymbol{r},\boldsymbol{r'})|^2 + 3\Delta^*(\boldsymbol{x},\boldsymbol{y})(\nabla\Delta(\boldsymbol{r},\boldsymbol{r'}))^2 + \Delta(\boldsymbol{r},\boldsymbol{r'})^2\nabla^2(\Delta^*(\boldsymbol{x},\boldsymbol{y})) +\\
&\quad + 4|\Delta(\boldsymbol{r},\boldsymbol{r'})|^2\nabla^2\Delta(\boldsymbol{r},\boldsymbol{r'})\Big] = b_2\tau^{5/2}\Big[2\Delta|\boldsymbol{\mathcal{D}}\Delta|^2 + 3\Delta^*(\boldsymbol{\mathcal{D}}\Delta)^2 + \Delta^2(\boldsymbol{\mathcal{D}}^2\Delta)^* + 4|\Delta|^2\boldsymbol{\mathcal{D}}^2\Delta\Big]\\
&c_1\tau^{5/2}\lim_{\boldsymbol{r'}\to\boldsymbol{r}}|\Delta(\boldsymbol{r},\boldsymbol{r'})|^2\Delta(\boldsymbol{r},\boldsymbol{r'}) = c_1\tau^{5/2}|\Delta(\boldsymbol{r})|^4\Delta(\boldsymbol{r})\ , \qquad (5.64)
\end{aligned}$$

where we have factored out each contribution of the same order in deviations from the critical temperature. The only contribution different from those provided above with the classical correction in a $\tau^{5/2}$ theory is

$$a_4\tau^{5/2}\boldsymbol{B}^2\lim_{\boldsymbol{r'}\to\boldsymbol{r}}\Delta(\boldsymbol{r},\boldsymbol{r'}) = -a_4\tau^{5/2}\boldsymbol{B}^2\Delta(\boldsymbol{r}) \qquad (5.65)$$

with $a_4 = \frac{b\hbar^2e^2}{36m^2c^2}$ a new coefficient adding to those derived in the absence of the magnetic field. If one is concerned to where this comes from, it comes from accounting for considering the classical evolution of a system and truncating up to the order of the theory. This modifies the Gor'kov greens function,

$$\mathcal{G}^0_\omega(\boldsymbol{r},\boldsymbol{r'}) = \mathcal{G}^{(0)}_{\text{Gor'kov}}(\boldsymbol{r},\boldsymbol{r'})\Big\{1 + \frac{e^2\tau^2B^2}{24m^2c^2}[\partial^2_\omega + \frac{i}{\hbar}m(\boldsymbol{r}-\boldsymbol{r'})^2\partial_\omega] + O(\tau^{5/2})\Big\} \qquad (5.66)$$

As this term produces a correction of order τ^2, an inspection shows that it only produces modifications in the terms accounting for the GL order, the accountability of other terms trespassing the EGL order, $\tau^{5/2}$. The reader interested in the Pierls phase correction up to the proper order is encouraged to read ref. (VAGOV et al., 2016). From the above,

$$\begin{aligned}
&0 = (a_1 - \frac{1}{g})\tau^{1/2}\Delta + a_2\tau^{3/2}\boldsymbol{\mathcal{D}}^2\Delta + a_3\tau^{5/2}\Big[\boldsymbol{\mathcal{D}}^4\Delta - \frac{4ei}{3\hbar c}\nabla\times\boldsymbol{B}\cdot\boldsymbol{\mathcal{D}}\Delta + \frac{4e^2}{\hbar^2c^2}\boldsymbol{B}^2\Delta\Big] -\\
&a_4\tau^{5/2}\boldsymbol{B}^2|\Delta|^2 - b_1\tau^{3/2}|\Delta|^2\Delta - b_2\tau^{5/2}\Big[2\Delta|\boldsymbol{\mathcal{D}}\Delta|^2 + 3\Delta^*(\boldsymbol{\mathcal{D}}\Delta)^2 + \Delta^2\boldsymbol{\mathcal{D}}^{*2}\Delta^* + 4|\Delta|^2\boldsymbol{\mathcal{D}}^2\Delta\Big] +\\
&c_1\tau^{5/2}|\Delta|^4\Delta \qquad (5.67)
\end{aligned}$$

If one wishes to consider a higher-order theory for superconductors the classical contribution provides more terms to be aware of. We will not proceed in this fashion for a superconductor, in the same manner. It was previously done by setting up rules for the n-order theory of a chargeless superfluid. We believe there is a set of simple rules holding when the magnetic field is included, but the exploration of this fact is beyond the scope of this dissertation.

The Energy Functional

We seek a functional such that:

$$\frac{\delta f}{\delta \Delta^*} = (\tau^{-2} \cdot \tau^{1/2})\Big\{(a_1 - \frac{1}{g})\tau^{1/2}\Delta + a_2\tau^{3/2}\boldsymbol{\mathcal{D}}^2\Delta + a_3\tau^{5/2}\Big[\boldsymbol{\mathcal{D}}^4\Delta - \frac{4ei}{3\hbar c}\nabla \times \boldsymbol{B} \cdot \boldsymbol{\mathcal{D}}\Delta + \frac{4e^2}{\hbar^2 c^2}\boldsymbol{B}^2\Delta\Big] - a_4\tau^{5/2}\boldsymbol{B}^2|\Delta|^2 - b_1\tau^{3/2}|\Delta|^2\Delta - b_2\tau^{5/2}\Big[2\Delta|\boldsymbol{\mathcal{D}}\Delta|^2 + 3\Delta^*(\boldsymbol{\mathcal{D}}\Delta)^2 + \Delta^2\boldsymbol{\mathcal{D}}^{*2}\Delta^* + 4|\Delta|^2\boldsymbol{\mathcal{D}}^2\Delta\Big] + c_1\tau^{5/2}|\Delta|^4\Delta\Big\} \tag{5.68}$$

$$\frac{\delta f}{\delta \Delta} = (\frac{\delta f}{\delta \Delta^*})^* \tag{5.69}$$

in a symmetric form with respect to the the order-parameter and its conjugate. The prefactor multiplication has its origim in: i) $f_s = \mathcal{F}/L^n$, with n the dimension of the space. Since $L \sim \tau$, in two dimensions, $f \to f/\tau^2$, ii) The factor $1/2$ appears since in fact the variation is taken w.r.t to Δ^* is in fact w.r.t $\Delta^*\tau^{1/2}$ in dimensionless unities. Hence, we have the mathematical problem of finding the functional f by knowing its derivatives. The integration of any function has an arbitrary constant. This constant is set by the magnetic field which shall contribute in cgs unities with $(\frac{B^2}{2}\frac{1}{4\pi})$. In scaled unities,

$$f_s = f_{n,B=0} + \frac{\boldsymbol{B}^2}{8\pi} + \frac{1}{\tau}(g^{-1} - a_1)|\Delta|^2 + a_2|\boldsymbol{\mathcal{D}}\Delta|^2 - \tau a_3(|\boldsymbol{\mathcal{D}}^2\Delta|^2 + \frac{1}{3}\nabla \times \boldsymbol{i}_{\mathsf{G.L}} + \frac{4e^2}{\hbar^2 c^2}\boldsymbol{B}^2|\Delta|^2) + \\ + \tau a_4\boldsymbol{B}^2|\Delta|^2 + \frac{b_1}{2}|\Delta|^4 - \tau\frac{b_2}{2}\Big[8|\Delta|^2|\boldsymbol{\mathcal{D}}\Delta|^2 + \Delta^{*2}(\boldsymbol{\mathcal{D}}\Delta)^2 + \Delta^2(\boldsymbol{\mathcal{D}}^*\Delta^*)^2\Big] - \tau\frac{c_1}{3}|\Delta|^6 \tag{5.70}$$

with the current of the Ginzburg-Landau functional

$$\boldsymbol{i}_{\mathsf{GL}} = \frac{2ei}{\hbar c}(\Delta\boldsymbol{\mathcal{D}}^*\Delta^* - \Delta^*\boldsymbol{\mathcal{D}}\Delta) \tag{5.71}$$

One may confirm that this indeed produces the result. An exercise of thinking backward also leads to this functional, though we will not care about this discussion in detail, since it is long but not very challenging.

THE EXTENDED-GINZBURG-LANDAU EQUATIONS

With the inclusion of the magnetic field, we seek the modified equations of motion directly from the functional. For instance, we did not have an equation for the magnetic field, which would limit further important analysis in superconductivity. To capture each order of correct accuracy, we seek solutions in powers of the τ contribution.

$$f_s - f_{n,B=0} = f_0 + \tau f_1 + ... \tag{5.72}$$

$$\Delta = \Delta_0 + \tau\Delta_1 + ... \tag{5.73}$$

$$\boldsymbol{A} = \boldsymbol{A}_0 + \tau\boldsymbol{A}_1 + ... \tag{5.74}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \tau\boldsymbol{B}_1 + ... \tag{5.75}$$

By denoting $\boldsymbol{\mathcal{D}}_0 \equiv \nabla - \frac{2ei}{\hbar c}A_0$,

$$f_0 = \frac{\boldsymbol{B}_0^2}{8\pi} + a|\Delta_0|^2 + \frac{b}{2}|\Delta_0|^4 + \mathcal{K}|\boldsymbol{\mathcal{D}_0}\Delta_0|^2 \tag{5.76}$$

Denoting

$$f_1 = f_1^{(0)} + f_1^{(1)} \tag{5.77}$$

$$\begin{aligned} f_1^{(0)} = {} & \frac{a}{2}|\Delta_0|^2 + 2\mathcal{K}|\boldsymbol{\mathcal{D}_0}\Delta_0|^2 - \mathcal{Q}\Big(|\boldsymbol{\mathcal{D}}_0^2\Delta_0|^2 + \frac{1}{3}(\boldsymbol{\nabla}\times\boldsymbol{B}_0)\cdot\boldsymbol{i}_0 + \frac{4e^2}{\hbar^2c^2}\boldsymbol{B}_0^2|\Delta_0|^2\Big) + \\ & \frac{b}{36}\frac{e^2\hbar^2}{36m^2c^2}\boldsymbol{B}_0^2|\Delta_0|^2 + b|\Delta_0|^4 - \frac{\mathcal{L}}{2}\Big[8|\Delta_0|^2|\boldsymbol{\mathcal{D}}_0\Delta_0|^2 + \Delta_0^{*2}(\boldsymbol{\mathcal{D}_0}\Delta_0)^2 + \Delta_0^2(\boldsymbol{\mathcal{D}_0^*}\Delta_0^*)^2\Big] - \frac{c}{3}|\Delta|^6 \end{aligned} \tag{5.78}$$

$$f_1^{(1)} = \frac{\boldsymbol{B}_0\cdot\boldsymbol{B}_1}{4\pi} + (a + b|\Delta_0|^2)(\Delta_0^*\Delta_1 + \Delta_0\Delta_1^*) + \mathcal{K}\Big[\boldsymbol{\mathcal{D}}_0\Delta_0\cdot\mathcal{D}_0^*\Delta_1^* + \text{c.c} - \boldsymbol{A}_1\cdot\boldsymbol{i}_0\Big] \tag{5.79}$$

with

$$\boldsymbol{i}_0 = \frac{2ei}{\hbar c}(\Delta_0\boldsymbol{\mathcal{D}}_0^*\Delta_0^* - \Delta_0^*\boldsymbol{\mathcal{D}}_0\Delta_0) \tag{5.80}$$

For each order in τ we have an equation. In the zeroth-order

$$a\Delta_0 + b|\Delta_0|^2\Delta_0 - \mathcal{K}\boldsymbol{\mathcal{D}}_0^2\Delta_0 = 0 \tag{5.81}$$

$$\nabla\times\boldsymbol{B}_0 = (\frac{4\pi}{c})\mathcal{K}ci_0 \tag{5.82}$$

the Ginzburg-Landau equations are presented. We are interested in the higher-order contributions,

$$a\Delta_1 + b(2|\Delta_0|^2\Delta_1 + \Delta_0^2\Delta_1^*) - \mathcal{K}\boldsymbol{\mathcal{D}}_0^2\Delta_1 = F \tag{5.83}$$

$$\nabla \times \boldsymbol{B}_1 = \frac{4\pi}{c}\boldsymbol{j}_1 \tag{5.84}$$

with

$$\begin{aligned} F =& -\frac{a}{2}\Delta_0 + 2\mathcal{K}\boldsymbol{\mathcal{D}}_0^2\Delta_0 + \mathcal{Q}\Big[\boldsymbol{\mathcal{D}}_0^2(\boldsymbol{\mathcal{D}}_0^2\Delta_0) - \frac{4ei}{3\hbar c}\boldsymbol{\nabla} \times \boldsymbol{B} \cdot \boldsymbol{D}_0\Delta_0 + \frac{4e^2}{\hbar^2 c^2}\boldsymbol{B}_0^2\Delta_0\Big] \\ & - \frac{b}{36}\frac{e^2\hbar^2}{36m^2c^2}\boldsymbol{B}_0^2\Delta_0 - 2b|\Delta_0|^2 - \boldsymbol{\mathcal{L}}\Big[2\Delta_0|\boldsymbol{\mathcal{D}}_0\Delta_0|^2 + 3\Delta_0^*(\boldsymbol{\mathcal{D}}_0\Delta_0)^2 + \Delta_0^2(\boldsymbol{D}_0^2\Delta_0)^* + \\ & 4|\Delta_0|^2\boldsymbol{\mathcal{D}}_0^2\Delta_0\Big] + c|\Delta_0|^4\Delta_0 - \frac{2ei}{\hbar c}\mathcal{K}\{\boldsymbol{A}_1 \cdot \boldsymbol{D}_0\}\Delta_0 \end{aligned} \tag{5.85}$$

Provided

$$\boldsymbol{j}_1 = \mathcal{K}ci_1 + \boldsymbol{J} \tag{5.86}$$

$$\boldsymbol{i}_1 = \frac{2ei}{\hbar c}(\Delta_0\boldsymbol{\mathcal{D}}_0^*\Delta_1^* + \Delta_1\boldsymbol{\mathcal{D}}_0^*\Delta_0^* - \Delta_1^*\boldsymbol{\mathcal{D}}_0\Delta_0) \tag{5.87}$$

$$\begin{aligned} \boldsymbol{J} =& c\Big\{(2\mathcal{K} - 3\mathcal{L}|\Delta_0|^2)\boldsymbol{i}_0 + \mathcal{Q}\boldsymbol{\mathcal{D}}_0\boldsymbol{i}_0 + \frac{\mathcal{Q}}{3}\nabla \times \nabla\boldsymbol{i}_0 + \mathcal{Q}\frac{8e^2}{\hbar^2c^2}\Big[\boldsymbol{\nabla} \times (\boldsymbol{B}_0|\Delta_0|^2) - \\ & \frac{1}{3}|\Delta_0|^2(\boldsymbol{\nabla} \times \boldsymbol{B}_0) - \frac{b}{18}\frac{e^2\hbar^2}{m^2c^2}\boldsymbol{\nabla} \times (\boldsymbol{B}_0|\Delta_0|^2)\Big]\Big\} \end{aligned} \tag{5.88}$$

and,

$$\{\boldsymbol{A}_1 \cdot \boldsymbol{\mathcal{D}}_0\} = \frac{1}{2}(\mathcal{A}_1\boldsymbol{\mathcal{D}}_0 + \boldsymbol{\mathcal{D}}_0\mathcal{A}_1) \tag{5.89}$$

In the calculus of the above equations of motion we used the generalization of the GL boundary condition

$$\boldsymbol{D}_\perp\Delta|_{\partial V} = 0\ , \tag{5.90}$$

since it reduces to GL plus higher order contributions

$$\boldsymbol{D}_{0,\perp}\Delta_0|_{\partial V} = 0 \text{ GL} \tag{5.91}$$

$$\boldsymbol{D}_{0,\perp}\Delta_1 - \frac{2ei}{\hbar c}\boldsymbol{A}_{1\perp}\Delta_0|_{\partial V} = 0 \text{ from } \mathcal{O}(\tau) \tag{5.92}$$

EGL THEORY COMPARED TO GL AND BCS

Concerning the uniform gap of EGL and BCS compared (see Fig. 10) theory, it responds quite well up to $\tau \leq 0.8$. At $T/T_c = 0.2$ a qualitative difference appears as the EGL curve tends to decrease . About the critical field (see Fig. 11), it responds reasonably to the BCS theory for $\tau \leq 0.7$ ($T/T_c \geq 0.3$). In particular, at lower temperatures the curve representing EGL only differs from BCS for an amount varying from 10% to 20% while the relative difference to GL is greater than 73%.

Gap Dependence on the Temperature

The uniform solution considering the zero order equation (5.81) and first-order (5.83) of the self-consistent gap expansion are

$$\Delta_0 = -\sqrt{\frac{a}{b}} \tag{5.93}$$

$$\frac{\Delta_1}{\Delta_0}|_{\text{bulk}} = -\frac{3}{4} - \frac{ac}{2b^2} = -\tau\frac{3}{4}(1 - \frac{31\zeta(5)}{49\zeta^2(3)}) \tag{5.94}$$

Returning to the unscaled representation,

$$\frac{\Delta}{\Delta_{\text{BCS}}(0)} = e^{\gamma}\sqrt{\frac{8}{7\zeta(3)}}\tau^{1/2}\Big[1 - \frac{3}{4}\tau(1 - \frac{31\zeta(5)}{49\zeta^2(3)})\Big] \tag{5.95}$$

the zero-temperature gap.

Critical Field

From the definition of the normal-Meissner critical field, and defining the Gibbs free energy relative to the Meissner state, $g \equiv \Delta g|_{H=H_c}$,

$$g = g_0 + \tau g_1 = -\frac{H_c^2}{8\pi} \tag{5.96}$$

Therefore,

$$H_c = H_{c0} + H_{c1}\tau \tag{5.97}$$

defines the coefficients of zero order and the first-order correction. We identify this terms as

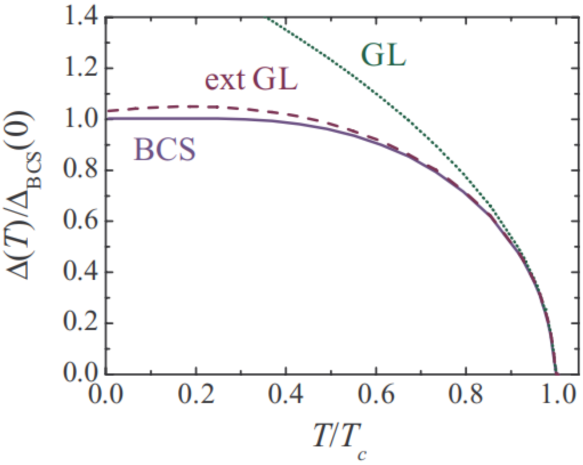
$$H_{c_0} = \sqrt{\frac{4\pi a^2}{b}}\ ,\ H_{c_1} = -H_{c_0}(\frac{1}{2} + \frac{ac}{3b^2}) \tag{5.98}$$

$$\frac{H_c}{H_{c0}} = 1 - \frac{\tau}{2}\Big(1 - \frac{31\zeta(5)}{49\zeta^2(3)}\Big) + \mathcal{O}(\tau^2) = 1 - 0.273\tau + \mathcal{O}(\tau^2) \tag{5.99}$$

In the unscaled variable,

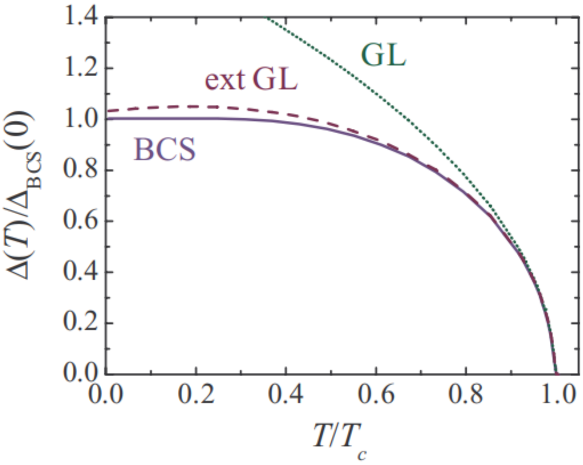
$$\frac{H_c(T)}{H_{c,\text{BCS}}(0)} = e^{\gamma}\sqrt{\frac{8}{7\zeta(3)}}\tau\Big[1 - \frac{\tau}{2}(1 - \frac{31\zeta(5)}{49\zeta^2(3)})\Big] \text{ with } H_{c,\text{BCS}}(0) = [4\pi N(0)]^{1/2}\frac{\pi T_c}{e^{\gamma}} \tag{5.100}$$

Figure 10 – Comparison of uniform-gap solution between Ginzburg-Landau,Extended-Ginzburg Landau and the full microscopic model. A higher-order theory would approach the full microscopic model further away from the critical temperature.



Source: From the original, (VAGOV; SHANENKO A, 2012)

Figure 11 – Comparison of the critical field between Ginzburg-Landau,Extended-Ginzburg Landau and the full microscopic model.



Source: From the original, (VAGOV; SHANENKO A, 2012)

5.3 1D BOGOMOL'NYI EQUATION

We consider the scaling,

$$\bar{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\lambda} \ , \ \bar{\boldsymbol{A}} = \frac{\boldsymbol{A}}{\lambda H_{c0}} \ , \ \bar{\boldsymbol{B}} = \frac{\boldsymbol{B}}{H_{c_0}} \ , \ \bar{\Delta} = \frac{\Delta}{\Delta_0} \ , \ \bar{f} = \frac{4\pi}{H_{c_0}^2} f \ , \ \bar{\sigma} = \frac{4\pi}{\lambda H_{c_0}^2}\sigma \tag{5.101}$$

With this scaling, the zero-order expansion of the theory reads

$$f_0 = \frac{B^2}{2} - \Delta_0^2 + \frac{\Delta_0^4}{2} - \kappa^2 \Delta_0'^2 \ , \tag{5.102}$$

since $\xi^2 = -\frac{K}{a}$. Defining the energy relative to Meissner as $\Delta g|_{H=H_c} \equiv g$.

$$g_0 = -\frac{\Delta_0 \Delta_0''}{\kappa^2} + (\frac{A_0^2}{2} - 1)\Delta_0^2 + \frac{\Delta_0^4}{2} + \frac{1}{2}(A_0' - 1)^2 \tag{5.103}$$

From this we may derive the 1d Bogomol'nyi relations. The stationary GL equations reads

$$-\frac{1}{\kappa^2}\Delta_0'' - \Delta_0 + \Delta_0^3 + \frac{1}{2}A_0^2\Delta_0 = 0 \tag{5.104}$$

$$A_0'' = A_0\Delta_0 \tag{5.105}$$

but these may be rewritten in a first order system for $\kappa = \kappa_0 \equiv \frac{1}{\sqrt{2}}$. This is said to be the Bogomol'nyi point. In the GL formalism, it is in this point that superconductivity exchanges type. We rewrite the first GL equation as

$$(\partial_x - \frac{\kappa}{\sqrt{2}}A_0)(\partial_x + \frac{\kappa}{\sqrt{2}}A_0)\Delta_0 + \kappa^2(\Delta_0^3 - \Delta_0) + \frac{\kappa}{\sqrt{2}}A_0'\Delta_0 = 0 \tag{5.106}$$

We consider solutions to above with the Sarma ansatz, i.e, such that

$$(\partial_x + \frac{\kappa}{\sqrt{2}}A_0)\Delta_0 = 0 \tag{5.107}$$

$$A_0' = (1 - \Delta_0^2)\sqrt{2}\kappa \tag{5.108}$$

These clearly satisfy the first GL differential equation. But to satisfy the second, the 'compatibility condition', as often called, imply $(\sqrt{2}\kappa(1 - \Delta_0)^2)' = A_0\Delta_0^2$ and from the first Bogomol'nyi, $\kappa = \kappa_0 \equiv \frac{1}{\sqrt{2}}$. Therefore, this is the locus where the first-order system is valid, and it is named 'Bogomol'nyi point' or, saving words 'B-point'. Summarizing, at $\kappa = \kappa_0$ the GL equations can be rewritten as a set of first-order differential equations

$$\Delta_0' + \frac{1}{2}A_0\Delta_0 = 0 \tag{5.109}$$

$$A_0' = (1 - \Delta_0^2) \tag{5.110}$$

Further, at the Bogomol'nyi point, it is quite trivial to prove that $g_0 = 0$. More importantly than a mathematical generic point, it has the physical importance of being the point where the solutions change stability. In the subsequent section, we will see that the extended theory will produce instead a straight line which is temperature-dependent dividing two domains. And in the next chapter provide that in between the types I and II, other superconductivity phenomena are appearing in a region of a $\kappa - (T/T_c)$ phase diagram coined intertype domain.

ON THE STABILITY OF SUPERCONDUCTIVITY IN EGL

As done previously, we wish to find the condition $g(\Delta) < 0$ for a given solution $\Delta(x)$. The EL$^{(2)}$ theory reads

$$g_1 = g_0 + \tau g_1 \tag{5.111}$$

It is convenient to denote $g_1 = g_1^{(0)} + g_1^{(1)}$ such that

$$\begin{aligned} g_1^{(0)} =& -\frac{\Delta_0^2}{2} + 2\frac{\Delta_0'^2}{\kappa^2} + A_0^2\Delta_0^2 + \tilde{\mathcal{Q}}\Big[(\frac{\Delta_0''}{\kappa^2} - \frac{A_0^2}{2}\Delta_0)^2 + \frac{1}{3\kappa^2}A_0''A_0\Delta_0^2 + \frac{(A_0')^2}{2\kappa^2}\Delta_0^2\Big] + \\ & + \frac{(A_0')^2\Delta_0^2}{48(k_F\lambda)^2} + \Delta_0^4 + \tilde{\mathcal{L}}\Big[\frac{5}{\kappa^2}(\Delta_0')^2\Delta_0^2 + \frac{3}{2}A_0^2\Delta_0^4\Big] + \tilde{c}\Delta_0^6 \end{aligned} \tag{5.112}$$

$$g_1^{(1)} = 2\Big[(\Delta_0^2 - 1)\Delta_0\Delta_1 + \frac{A_0^2}{2}\Delta_0\Delta_1 + \frac{1}{\kappa^2}\Delta_0'\Delta_1'\Big] - A_1 i_0 + (A_0' - 1)\Big(A_1' + \frac{1}{2} + \tilde{c}\Big) \tag{5.113}$$

with the definitions

$$\tilde{c} \equiv \frac{ac}{3b^2} \ , \ \tilde{\mathcal{Q}} \equiv \frac{\mathcal{Q}a}{\mathcal{K}^2} \ , \ \tilde{\mathcal{L}} \equiv \frac{\mathcal{L}a}{b\mathcal{K}} \tag{5.114}$$

We point out that the coefficients above are dimensionless and can be computed from the parameters of the microscopic theory,

$$\tilde{c} = -0.227 \ , \ \tilde{\mathcal{Q}} \equiv -0.454 \ , \ \tilde{\mathcal{L}} = -0.817 \tag{5.115}$$

For conventional superconductivity, $(k_F\lambda)$ is a very large term and is often neglected. For this dissertation we do not go beyond this treatment. However, we emphasize that this term is suggested to be important for the microscopic scale of superconductivity, for which fluctuations become relevant (VAGOV et al., 2016).

To solve for the entry solution Δ, minimizing the free energy for Δ_1 and A_1 yields these as functions of A_0 and Δ_0, as common in a self-consistent theory. For any integration by parts we may use the following properties for the boundary condition

$$\begin{pmatrix} \Delta_1(-\infty) = -\frac{3}{4}(1 - \frac{31\zeta(5)}{49\zeta^2(3)}) \\ \\ \Delta_1(\infty) = 0 \end{pmatrix} \text{ and } \begin{pmatrix} A_1'(-\infty) = 0 \\ \\ A_1'(\infty) = -\frac{1}{2} - \tilde{c} \end{pmatrix} \tag{5.116}$$

and

$$\begin{pmatrix} \Delta_0(-\infty) = 1 \\ \\ \Delta_0(\infty) = 0 \end{pmatrix} \text{ and } \begin{pmatrix} A_0'(-\infty) = 0 \\ \\ A_0'(\infty) = 1 \end{pmatrix} \tag{5.117}$$

plus the bulk condition

$$\Delta'_{0,1}(\pm\infty) = 0 \text{ and } A_{0,1}(\pm\infty) = 0 \tag{5.118}$$

As a result of minimizing the free-energy of zero-order in τ (in unscaled variables, of order $\tau^{3/2}$) we obtain the GL equations

$$-\frac{\Delta''_0}{\kappa^2} - \Delta_0 + \Delta_0^3 + \frac{A_0^2}{2}\Delta_0 = 0 \tag{5.119}$$

$$A''_0 = A_0\Delta_0^2 \tag{5.120}$$

which may also be used to express the Gibbs energy g_1 in terms of the solutions to the usual GL equation.

$$g_1^{(1)} = (\frac{1}{2} + \tilde{c})(A'_0 - 1) \tag{5.121}$$

With these, we wish to find the value of κ such that the surface energy becomes null. This sets the thermodynamic criteria for the emergence of superconductivity, as before. The surface energy is

$$\sigma(\kappa) = \underbrace{\int dx g_0(\Delta_0(x,\kappa), A_0(x,\kappa), \kappa)}_{\equiv\sigma_0(\kappa)} + \tau \underbrace{\int dx g_1(\Delta_0(x,\kappa), A_0(x,\kappa), A'_0(x,\kappa), \kappa)}_{\equiv\sigma_1(\kappa)} = 0 \tag{5.122}$$

since from the Gibbs free energy we see that varying κ the equation of motion for Δ_0 and A_0 changes, i.e, $\Delta_0 = \Delta_0(x,\kappa)$, $A_0 = A_0(x,\kappa)$. We have seen that when $\kappa = \kappa_0 \equiv \frac{1}{\sqrt{2}}$, $g_0(\Delta_0, A'_0, \kappa_0) = 0$. The deviations from κ_0 are, therefore, analytic in τ and obey

$$\kappa = \kappa_0 + \kappa_1\tau \tag{5.123}$$

in a self-consistent first-order correction. We wish to understand the behavior of the surface energy in the vicinity of the Bogomol'nyi point ($\kappa = \kappa_0$).

$$\begin{aligned}\sigma(\kappa) = \sigma_1(\kappa_0) + \Big[\frac{\partial\sigma_0(\kappa_0)}{\partial\kappa_0} + \cancel{\frac{\delta\sigma_0(\kappa_0)}{\delta\Delta_0}}\frac{\partial\Delta_0(\kappa_0)}{\partial\kappa_0} + \cancel{\frac{\delta\sigma_0(\kappa_0)}{\delta A_0}}\frac{\partial A_0(\kappa_0)}{\partial\kappa_0} + \\ \cancel{\frac{\delta\sigma_0(\kappa_0)}{\delta A'_0}}\frac{\partial A'_0(\kappa_0)}{\partial\kappa_0}\Big](\kappa_1\tau) = 0\end{aligned} \tag{5.124}$$

neglecting the next-order contribution in τ. Since $\sigma_0(\kappa_0) = 0$, only the term $\sigma_1(\kappa_0)\tau$ survives. In the second term, the only contribution (keeping the accuracy) is due to $\sigma \to \sigma_0$. $\sigma_0(\kappa_0) = 0$ clearly does not require the condition of null κ-derivative; changing the argument κ_0 while keeping the solutions at the Bogomol'nyi point shifts the energy. Any variation with respect to Δ_0, A_0, A_0' along a path in which $\kappa = \kappa_0$ will only change the solutions to other solutions yet at the Bogomol'nyi point, the resulting integral being null. Hence,

$$\kappa_1 = -\sigma_1(\kappa_0)\Big(\frac{\partial\sigma_0(\kappa_0)}{\partial\kappa_0}\Big)^{-1} \tag{5.125}$$

Some strong simplifications are made by the use of the 1d Bogomol'nyi equations. Immediately, from the second-Bogomol'nyi equation, it is possible to eliminate the A_0' dependence in $\sigma_1(\kappa_0)$.

$$\sigma_1(\kappa_0) = \int dx g_1(\Delta_0, A_0, \kappa_0) \tag{5.126}$$

From GL at the B-point plus boundary conditions, it is possible to infer (Appendix D.3)

$$\begin{aligned}\frac{\partial\sigma_0(\kappa_0)}{\partial\kappa_0} &= -\sqrt{2}\mathcal{I}\\ \sigma_1 &= \mathcal{I}\Big(1-\tilde{c}+2\tilde{\mathcal{Q}}\Big) + \mathcal{J}\Big(2\tilde{\mathcal{L}}-\tilde{c}-\frac{5}{3}\tilde{\mathcal{Q}}\Big)\end{aligned} \tag{5.127}$$

with

$$\mathcal{I} = \int dx\Delta_0^2(1-\Delta_0^2) \tag{5.128}$$

$$\mathcal{J} = \int dx\Delta_0^4(1-\Delta_0^2) \tag{5.129}$$

with Δ_0 solution of the GL equations at the Bogomol'nyi-point. Therefore,

$$\sigma(\kappa) = -\sqrt{2}\mathcal{I}\delta\kappa + \tau\Big[(1-\bar{c}+2\tilde{\mathcal{Q}})\mathcal{I} + (2\tilde{\mathcal{L}}-\tilde{c}-\frac{5}{3}\tilde{\mathcal{Q}})\mathcal{J}\Big] + \mathcal{O}(\tau^2) \tag{5.130}$$

in which we denote $\delta\kappa = \kappa - \kappa_0 = \kappa_1\tau$.

By numerically solving the GL equation for a flat normal-superconducting domain wall with the aforementioned boundary condition, we find

$$\kappa = \frac{1}{\sqrt{2}}(1 - 0.027\tau + \mathcal{O}(\tau^2)) \tag{5.131}$$

such that the frontier between type I and type II becomes temperature-dependent. We will further see the richness of the EL theory in providing results beyond the usual Type I/Type II dichotomous understanding shortly. We have computed κ_1 for a specific domain wall solution. Other solutions, however, open up an 'intertype' region, in between the Types I/Type II, where a different behavior is found - we will shortly examine it in the next chapter. Though we have derived this criterion of stability for 1d solutions, the same criteria are kept for 2d solutions (VAGOV et al., 2016).

5.4 FINAL REMARKS

The reader was presented to the Extended Landau Theory and its magnetic field-dependent counter-part, the Extended Ginzburg-Landau theory. These were reached by a self-consistent asymptotic expansion in deviations from the critical temperature. We have provided means to evaluate the stability of solutions within the $\kappa - T/T_c$ diagram. Such facts will be explored with exotic analytic solutions in the next chapter. By accounting for the proper terms in the expansion, one is expected to foresee that we are adding or removing terms in the diagrammatic partial summation method. Indeed, this is actually what we are doing for each of the intended orders of accuracy in deviation from the critical temperature. Therefore, this procedure selects precisely the correct diagrams for a self-consistent theory without relying on unsystematic or strong phenomenological assumptions. Finally, we have set the limit of trust for the Extended-Ginzburg-Landau theory and provided a way to analytically account for the stability of solutions.

Lastly, along the way, we have produced a systematic graphical scheme for the Extended Landau theory and its next-to-leading order theory through a simple set of graphics and a dictionary. These are intended tools for computing any generic order of the theory. More is required, however, if we are to systematically investigate a reasonable way of treating higher-orders expansions of the Extended-Ginzburg-Landau theory. Systematic and graphic inclusion of the magnetic field has not yet been proposed by the time this manuscript has been written.

6 SELF-DUAL SOLUTIONS AND THE EMERGENCE OF COMPLEXITY VIA COOPERATION

PROLOGUE

It is now widely accepted that spontaneous structures form as a result of length-scale competition, (SEUL; ANDELMAN, 1995). For the second question, while adaptation is frequently mentioned in biology, frustration caused by competing interactions has also been suggested as a possible origin of numerous nearly degenerate patterns with a variety of arrangements, (WOLF; KATSNELSON; KOONIN, 2018). In any case, the critical word in this scenario of emergence is 'competition'. We present a qualitatively distinct framework in which the keyword is 'cooperation'.

In the 2003 Nobel Prize, Alexei Abrikosov received 1/3 of the award for his contribution to the theory of superconductivity, for which it has been provided the succinct justification

> *When certain substances are cooled to extremely low temperatures, they become superconductors, conducting electrical current entirely without resistance. With one type of superconductivity, the magnetic field is forced away from the conductor, but with another type of superconductivity, the magnetic field is admitted into the conductor. The different types of superconductivity cannot be described with the same theory. At the end of the 1950s, Alexei Abrikosov formulated a theory for the second type of superconductor. He introduced a mathematical function that described vortices whereby an external magnetic field can intrude into the conductor* - Nobel Foundation (NobelPrize.org. Nobel Prize Outreach AB 2022,)

This work provides alternative novel solutions (in the place of vortices) which we prove to be stable in the superconductivity domain. This can be done provided

$$\mathcal{G} = -\sqrt{2}\mathcal{I}\delta\kappa + \tau\Big[(1 - \bar{c} + 2\tilde{\mathcal{Q}})\mathcal{I} + (2\tilde{\mathcal{L}} - \tilde{c} - \frac{5}{3}\tilde{\mathcal{Q}})\mathcal{J}\Big] \tag{6.1}$$

with

$$\mathcal{I} = \int \boldsymbol{dx}|\Psi|^2(1 - |\Psi|^2) \tag{6.2}$$

$$\mathcal{J} = \int \boldsymbol{dx}|\Psi|^4(1 - |\Psi|^2) \tag{6.3}$$

with Ψ solution of GL at the Bogomol'nyi-point (VAGOV et al., 2016). It is the equivalent of the 1d result (chap.5) generalized to 2d. We spare the effort of reproducing the results for 2d - which is analogous to the 1d procedure - in the benefit of exploring other topics.

The mathematical functions provided adds to the vortex solution, also describing the superconducting behavior in a region coined 'intertype domain' emerging in between the standard types I and II. The functions are obtained by nontrivial semi-analytic solutions for the Bogomol'nyi equation. We have obtained these by employing the methods described in this chapter. Our solutions were labelled 'bubble' (or 'droplet'), 'donut' and 'stripe' (or filament'). The bubble consists of an isolated superconducting domain separated from a normal boundary region. The donuts, on the other hand, alternates normal-superconducting-normal domains, and it has as a limiting case, the suppression of the most external domain, presenting the feature of a vortex (single-quantum or multi-quanta). This suggests the donuts is a combination of the droplet solution with a vortex placed in the center. Another possible solution is the stripe, which is the 1d solution presenting subsequently the increase and decrease of the order-parameter. After displaying the solutions, we indicate their stability in a $\kappa - T/T_c$ phase-diagram. Next, we numerically confirm their presence as a possible superconducting phase by running usual simulations with the Time-Dependent-Ginzburg-Landau formulation (GROPP et al., 1996) and applying the solutions as ansatz to the time-dependent GL (TDGL) in the vicinity of the Bogomol'nyi point (B-point, $\kappa = \frac{1}{\sqrt{2}}$). For the numerical achievements due to the TDGL method, I am very much indebted to W.Córdoba. As a result of the combination of the semi-analytic results in this chapter and the numerical TDGL results, we were able to reproduce theoretically the multitude of phases of matter reported in the experimental results reported more than five decades ago (1969,U. Krägeloh in single-band materials (KRAGELOH, 1969)). Furthermore, this is the starting point for casting doubt on the fundamentals for complexity emergence.

Figure 12 – Flux lines experimentally obtained by U.Krägeloh, in the late sixties.

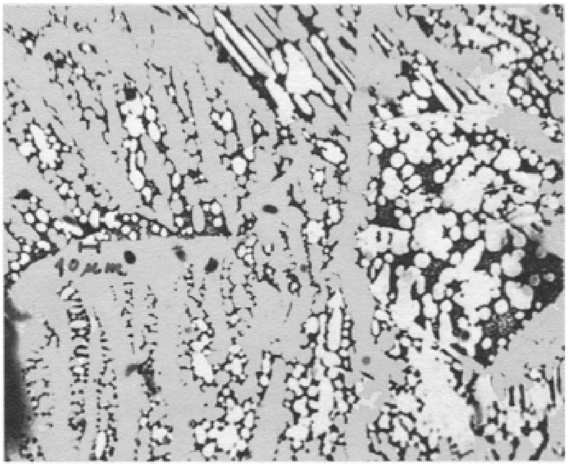


Fig. 1. Intermediate state structures observed in the optical microscope in a disk of Pb–1.89 wt% Tl in a field of 329 Oe at 1.25°K

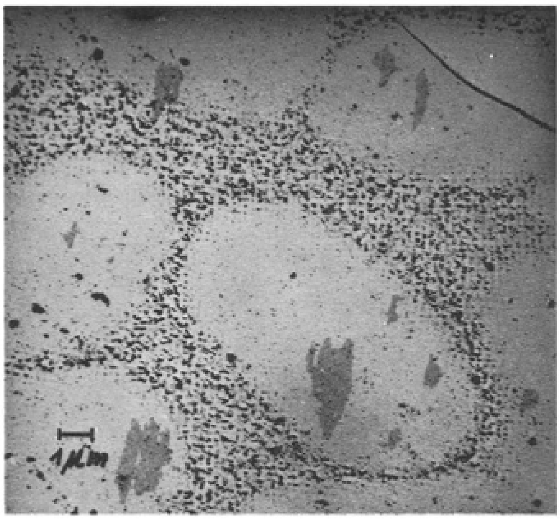


Fig. 2. Same specimen as fig. 1, electron micrograph showing coesistence of the Meissner phase with a phase consisting of flux lines arranged in a square lattice.

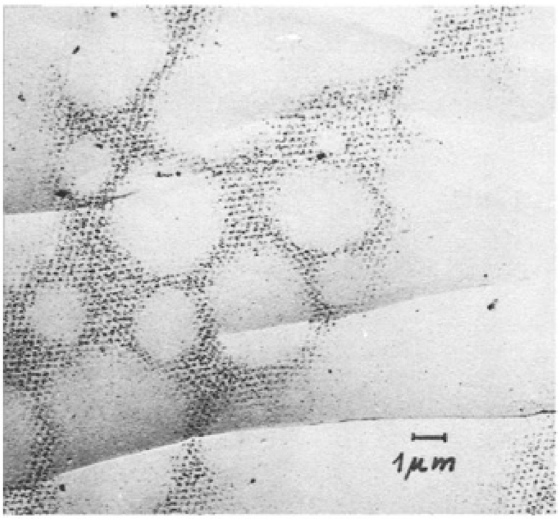


Fig. 3. Electron micrograph showing a triangular lattice ($H = 343$ Oe at 1.25°K).

Source: (KRAGELOH, 1969), permission granted by Elsevier.

6.1 SELF-DUAL BOGOMOL'NYI SOLUTIONS: BUBBLE, STRIPE AND DONUT

Consider the self-dual Bogomol'nyi equations

$$(\partial_y + i\partial_x)\Psi = (A_x - iA_y)\Psi \tag{6.4}$$

$$\mathcal{B} = 1 - |\Psi|^2 \tag{6.5}$$

and the following ansatz

$$\Psi_N = \frac{(x-iy)^N}{\rho^N}\Phi_N(\rho) = \Phi_N(\rho)\exp[-iN\theta] \tag{6.6}$$

$$\vec{A} = (0, A(\rho), 0) \tag{6.7}$$

with $\vec{A}$ the vector potential in cylindrical coordinates.

The first ansatz simply states we are looking for topological defects with N as winding number. If existent, these solutions have polar symmetry for a $\hat{z}$ directed magnetic field due to the second condition (6.7). A further consistent assumption for solutions is the link between the winding number and the order of the polynomial expansion around the origin,

$$\partial_\rho^N \Phi_N(0) = \mathcal{C} \tag{6.8}$$

since it ensures $\Phi_N \propto \rho^N$, $\Psi_N \propto (\frac{1}{z})^N$ as it should be for a N-quanta vortex solution.

Writing the curl of the vector potential in cylindrical coordinates,

$$\mathcal{B} = A' + \frac{A}{\rho} \tag{6.9}$$

From (6.6) and (6.7) into the first Bogomol'nyi equation (6.4),

$$\Phi_N' - N\frac{\Phi_N}{\rho} = -A\Phi_N \tag{6.10}$$

It is possible to obtain an expression for Φ alone if going to the second-order GL equation. For $N = 1$, this second-order equation is

$$-\Phi_1'' - \frac{\Phi_1'}{\rho} + \frac{\Phi_1'^2}{\Phi_1} - \Phi_1 + \Phi_1^3 = 0 \tag{6.11}$$

$$\partial_\rho \Phi_1 = \mathcal{C} \tag{6.12}$$

which is equivalent to the modified Liouville equation

$$\nabla^2 \ln \Phi_1 = \Phi_1^2 - 1 \tag{6.13}$$

In general, it is also possible to prove that the same Liouville equation is valid for an arbitrary winding number choice,

$$\nabla^2 \ln \Phi_N = \Phi_N^2 - 1 \tag{6.14}$$

with the N factor cancelling out.

In general, for practical purposes, it is usual to rewrite the Liouville equation for $\phi_N = \Phi_N^{\frac{1}{N}}$ as

$$\nabla^2 \ln \phi_N = \frac{1}{N}(\phi_N^{2N} - 1) \tag{6.15}$$

$$\partial_\rho \phi_N = \mathcal{C} \tag{6.16}$$

which is suitable for numerical purposes as to avoid the N-th order derivative as the boundary condition.

From Bogomol'nyi to the modified Liouville equation

At the Bogomolnyi point, we wish to prove the solutions obey the ‘modified’ Liouville equation

$$\nabla^2 \ln \Psi = |\Psi|^2 - 1 \tag{6.17}$$

differing by the adding constant -1 from the Liouville equation ((LIOUVILLE, 1853), (CROWDY, 1997)). Though a constant addition is a problem of trivial solution when the linear solution is known, it poses a major technical difficult in solving the modified non-linear equation, even when one has knowledge of the original non-linear equation.

For a proof, we operate $(\partial_y - i\partial_x)$ onto the first Bogomol'nyi equation, $(\partial_y^2 + \partial_x^2)\Psi = \Psi(\partial_y - i\partial_x)(A_x - iA_y) + (A_x - iA_y)(\partial_y - i\partial_x)\Psi$ and express $(A_x - iA_y)$ as a function of Ψ alone, according to the Bogomolnyi first relation. Applying the Coulomb gauge condition and the second Bogomol'nyi equation, the representation

$$\frac{\nabla^2\Psi}{\Psi} - \frac{(\nabla\Psi)^2}{\Psi^2} = |\Psi|^2 - 1 \tag{6.18}$$

is obtained by excluding $\Psi = 0$. The above is equivalent to the modified Liouville, proving the claim (taking care to exclude the $\Psi = 0$ points).

We compare this result with that of an isolated solution, for which

$$\nabla^2 \ln |\Psi| = |\Psi|^2 - 1 \tag{6.19}$$

noticing the distinction on the left side.

Another synthetic proof is provided in the characteristic coordinate representation, by rewriting the first-Bogomol'nyi as

$$\partial_z \ln \Psi = -(iA_x + A_y) \ , \tag{6.20}$$

$z = x+iy$ (excluding $\Psi = 0$). By applying the conjugate derivative and the second Bogomol'nyi equation within the Coulomb Gauge retrieves the modified Liouville equation.

Note on Singularities

With this we notice we have missed to include singularities in the first section on isolated solutions. Notice that by considering $\Psi = \frac{\bar{z}^n}{|z|^{2n}}|\Phi_N|$ into 6.17, the results of the previous equations will be the same apart from the singularity at the origin, as $\nabla^2 \ln z \propto \delta(z)$. For the isolated solutions the origin point can be punctured out for the integration purpose.

Bubble, Donut and Vortex Solution

The representation to bubble, donuts and vortex is similar. It is obtained for different choices of sign and initial condition to

$$\phi'_N = \pm \frac{1}{r} \phi_N \gamma_N \tag{6.21}$$

$$\gamma'_N = \pm \frac{r}{N} (\phi_N^{2N} - 1) \tag{6.22}$$

for the initial value problem $\lim_{r_0 \to 0} \phi_N(r_0) = \phi_{N,0}$ and $\lim_{r_0 \to 0} \gamma_N(r_0) = \gamma_{N,0}$ (remember $\phi_N = \Phi_N^{1/N}$ is the solution). The first-order Bogomol'nyi system solves the second-order Liouville equation.

For both donuts and vortex, $\phi_{N,0} \to 0$; for bubbles $\phi_{N,0} \to 1$. In the positive sign choice ($+$ in both (6.21) and (6.22)), each value to N produces the appropriate asymptotic behavior to the N-th quanta solution, either donuts or vortex. When $\gamma_{0,N}$ trespass a critical value, the vortex solution with vorticity N is obtained, below that value, only donuts are present. Bubbles are obtained with the negative choice for the sign ($-$ in both (6.21) and (6.22)). N controls the asymptotic profile for bubbles and is not related to a quantized flux - there is no bubble vorticity (Appendix E.2). The sole contribution of N is in modifying the long-range profile of the condensate decay.

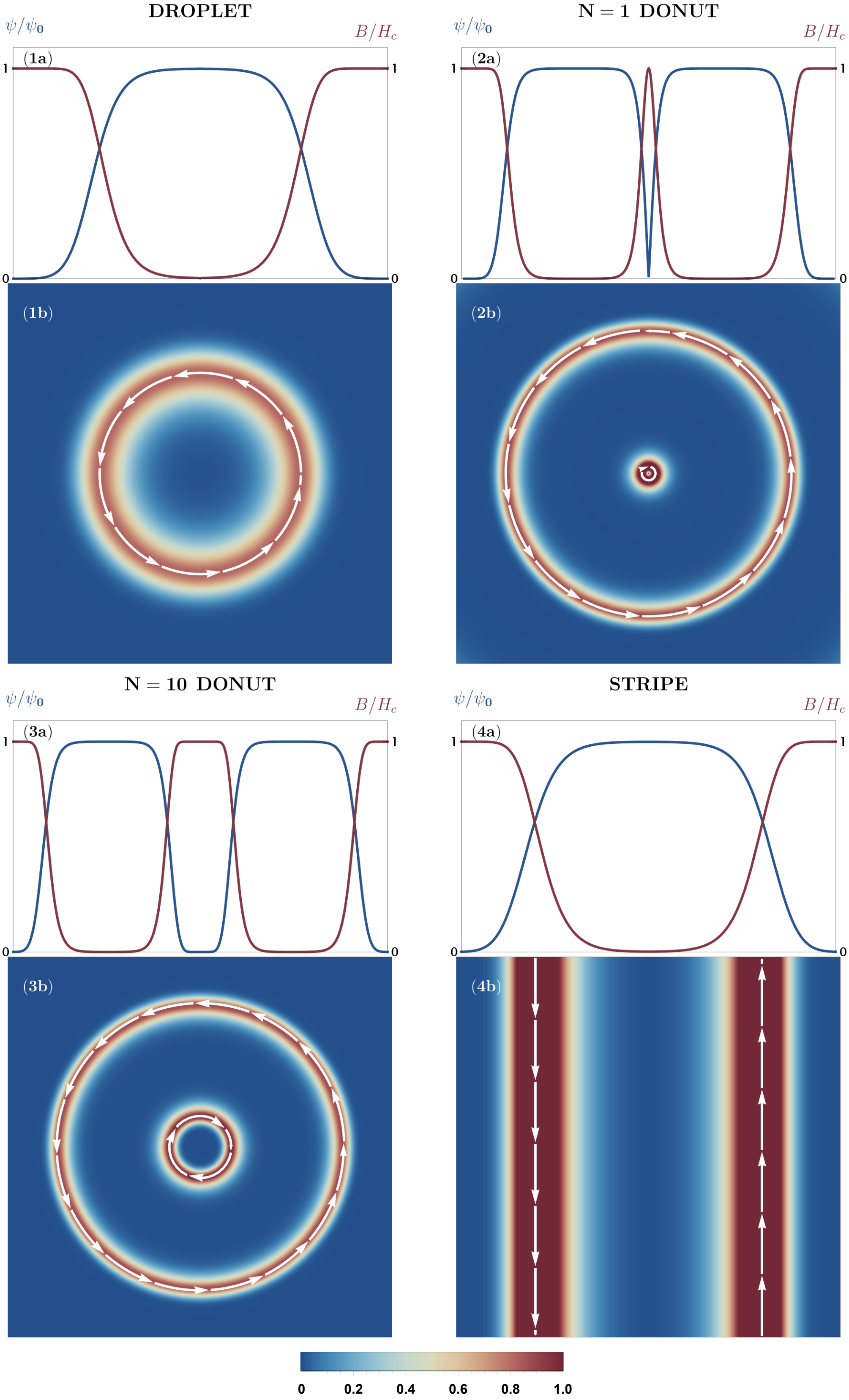
Stripes (or Filaments)

In 1d no phase change is expected, therefore, $\Psi = |\Psi|$. The stripe solution is represented in the first-order differential form

$$\Psi' = \pm\Psi\sqrt{\Psi^2 - 2\ln[c|\Psi|]} \tag{6.23}$$

by gluing each branch at the locus of maximum amplitude. Both branches of (6.23) are consistent with the 1d modified Liouville. The critical point is $\Psi_c = W^{\frac{1}{2}}[\frac{1}{c^2}]$ and $\Psi_c = 0$ (at infinity), for W the Lambert-W function. Expanding the solution at the locus of the critical point x_c, the approximate parabolic behaviour $\Psi(x) = \Psi_c - (1 - \Psi_c^2)\Psi_c(x - x_c)^2$ is verified. For stripes, we choose the initial amplitude value of δ_1 to be small (for instance, $\delta_1 = 10^{-2}\Psi_c$) for integration purposes. The equation (6.23) is autonomous, hence for a set of choices to δ_1, there is a set of translations of the same solution.

Figure 13 – The order parameter and its respective induction along a 1d cut; the condensate and current (arrows) in the 2d graphics. In donuts solution, two counter-propagating currents appear, as if it were a bidimensional realization of the stripe structure. Contrary to stripes where the inner and outward currents are the same in magnitude, though, the external donuts ‘ring’ carries the greater current.

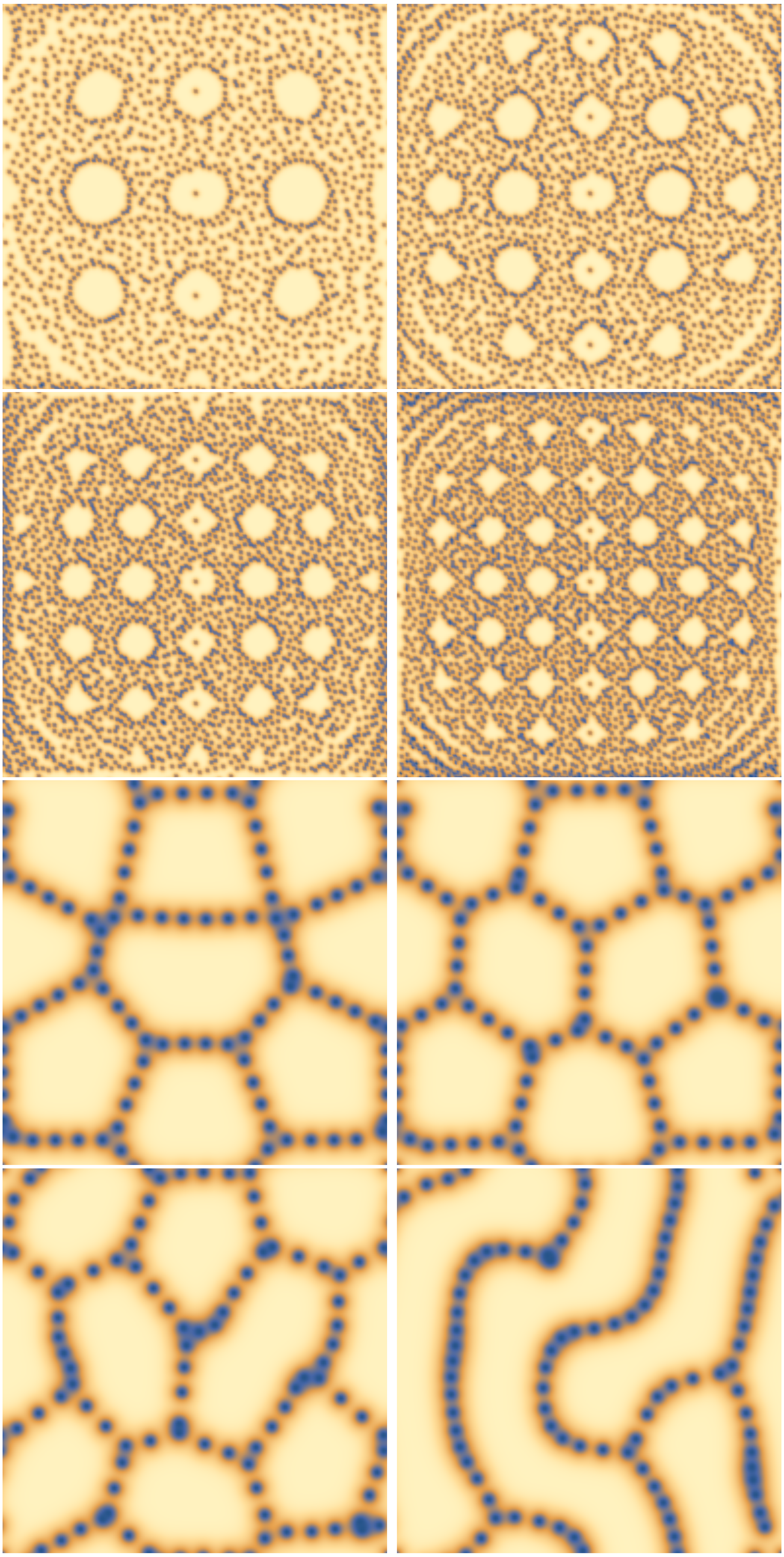


Source: The author

TDGL NUMERICS

In collaboration with W.Córdoba, by applying the Time-Dependent Ginzburg-Landau equation to the evolution of novel-like solutions ansatz, a plethora of configurations emerged in the vicinity of $\kappa = 1/\sqrt{2}$. Needless to say the similarities between the above and the superconducting island observed in Krägeloh experiments. Beyond this, the filaments with low curvature in the first Krägeloh figure are also present, now surrounding the bulk containing a periodic structure of islands with (donuts) and without (bubble) a vortex inside. The presence of vortices in some of the islands is not easily detected by following the images of the Krägeloh experiment.

Figure 14 – Condensate reproduction via TDGL.

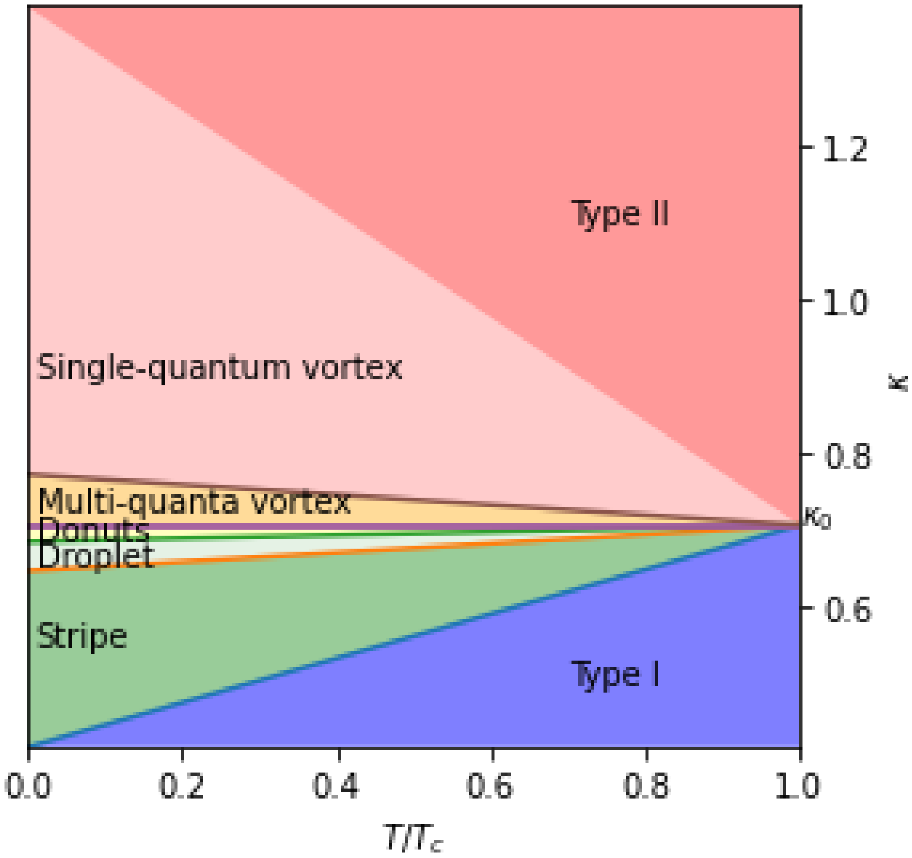


Source: W.Córdoba

ON THE STABILITY OF THE NOVEL SOLUTIONS IN THE INTERTYPE DOMAIN

The intertype domain consists of the exotic configurations which are opened up in between the standard superconducting types I and II. EGL produces the first linear corrections for the dependence of the critical κ as a function of the temperature. Obtaining the next-leading-order expansion to EGL would lead to better results in regions with $\tau \geq 0.3$ (see chap.5), allowing the curves to bend in the phase diagram. The criteria for the stability classification of each region below is the starting point of the stability relative of the Meissner state.

Figure 15 – Superconducting phase diagram of the Extended Ginzburg-Landau theory. The next-to-leading order theory could allow for the appearance of novel phase boundaries; beyond the one-to-one boundary contacts within the intertype diagram.

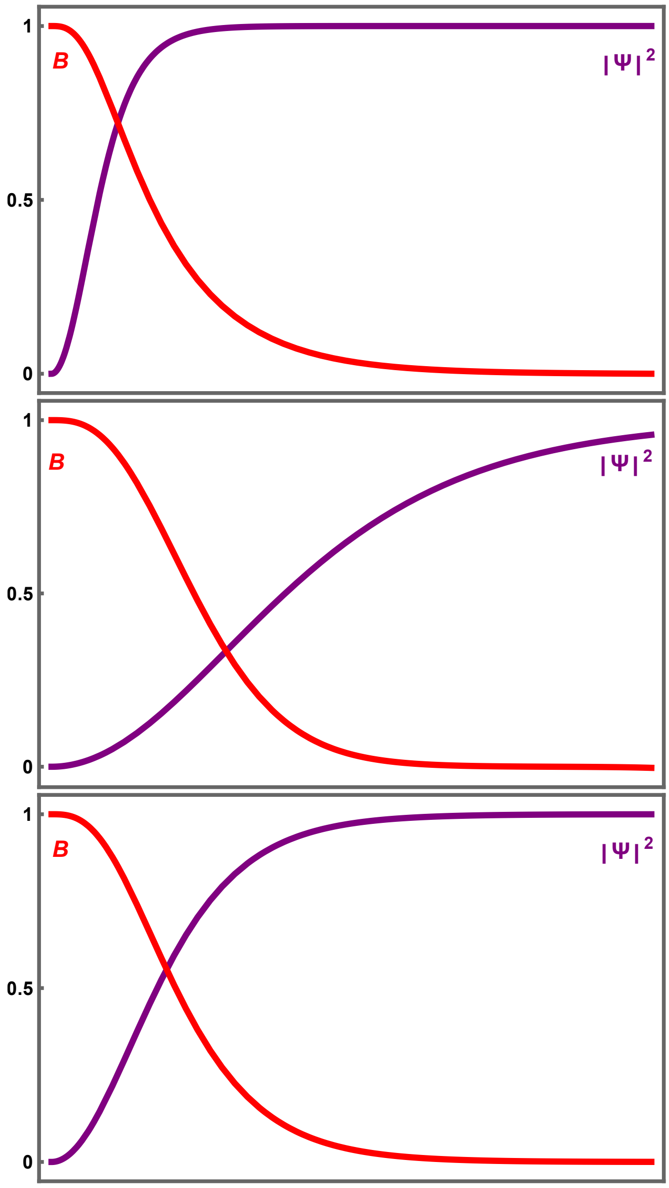


Source: The author

6.2 SELF-DUALITY AS MECHANISM TO THE EMERGENCE OF COMPLEXITY

Away from the Bogomol'nyi point, a competition between the condensate density ($|\Psi|^2$) and magnetic field (B) length-scales is held. To see this, observe the figures below, either deep in type II or type I, respectively. The fields represented in the images are normalized.

Figure 16 – Pictures of Type II ($\kappa > 1/\sqrt{2}$), Type I ($\kappa < 1/\sqrt{2}$), and the Bogomol'nyi point ($\kappa = 1/\sqrt{2}$), in this order.



Source: The author

At the Bogomol'nyi point, the competition between the two length-scales dictating the formation of patterns vanishes, both $|\Psi|^2$ and B presents precisely the same scale, as may be inferred from the second Bogomo'lnyi equation (6.5). In this way, both the magnetic field and the order-parameter act cooperatively to the formation of patterns, being present in precisely the same amount. As we proved, in chap.3, at the B-point, the second Bogomol'nyi equation imply zero free-energy relative to the Meissner-state. Therefore, any solution at the B-point is degenerate. This is also the case in the EGL theory, since i) the Bogomol'nyi point of the extended theory becomes the pair $(\delta\kappa = 0, \tau = 0)$ ii) (6.1) holds.

For the point of the phase diagram, at the critical temperature and parameter κ fixed at $\kappa = \frac{1}{\sqrt{2}}$, every solution is degenerate. It implies the possibility of a plethora of solutions, such as the solutions introduced in this chapter, which yields the emergence of complexity. Despite the absolutely richness of this mechanism in generating complexity, the literature has always attributed the formation of complexity to competition among order-parameters. In this mechanism, however, we present a phenomenology infinitely degenerate, able to produce complexity. Yet, it is based on the two physical quantities, $|\Psi|^2$ and B in cooperation, acting as a single entity. We notice that the first and second Bogomol'nyi relations can be decoupled into the modified-Liouville (6.17), and it together with the explicit link between the condensate density and the induction field (6.5) recovers the original Bogomol'nyi equations. Thus, determining the order-parameter becomes independent from the induction field, and once it is determined, the induction is a polynomial function of the order-parameter. It is remarkable that it is not necessary to specify boundary conditions for the induction field, for this is constrained by the second-Bogomol'nyi. As a consequence, the two order-parameter theory becomes effectively a theory at a single order-parameter at the Bogomol'nyi point. The phenomenology at the B-point defies the fundamental and dominant view of complexity emergence as a result of competition between length-scales.

7 CONCLUSIONS AND PERSPECTIVES

Though superconductivity has been studied for nearly a century, recent years have demonstrated that much can be learned from ‘material-independent’ theories of superconductivity. Recently, several properties beyond the usual type I and II superconductivity have been described using the simplest possible microscopic theory and the generic s-wave interaction. The Extended-Ginzburg-Landau theory contributes to our understanding of superconductivity beyond the critical temperature. Previously, only phenomenological theories would germinate in areas in desperate need of accurate treatment.

As a first technical contribution worth noting, we developed an efficient method for obtaining extended-Landau theories in any order (in the language of this manuscript, E$^{(n)}$L). This is accomplished by developing a diagrammatic scheme and a set of simple rules for identifying the pertinent terms. However, we should keep in mind that at the time this manuscript was written, we were unable to incorporate vector-field coupling in a systematic manner into the expansion. We have not been able to produce their Ginzburg counterpart (E$^{(n)}$GL) in the language presented throughout the text until the date of this manuscript submission. While a schedule for an extended theory may be of little interest in the absence of magnetic coupling in superconductivity, we may raise a fundamental question. If we take the Landau’s concept of universality of phase transitions seriously in describing phase transitions near the critical temperature, a natural question is whether the universal form of the equations is maintained further away from the transition temperature. While addressing this question is undoubtedly beyond the scope of this dissertation, the answer may have profound implications for our understanding of phase-transition processes in nature.

As a second technical contribution, we established semi-analytic self-dual solutions permitted within the intertype domain’s zoo of solutions. The terms bubble, stripe, and donut were used to describe these. The thermodynamic stability of these solutions relative to the Meissner-Ochsenfel state has been established, and the presented intertype diagram has represented this relative stability. With the assistance of W.Córdoba, we calculated the numerical time evolution of droplets, stripes, and donuts *ansatz* at $\kappa = 1/\sqrt{2}$. As a result, we were able to extract numerical results containing at least three phases of matter simultaneously in a single image. We have not seen these solutions reproduced in the literature, but they appear to be representative of Krägeloh’s pioneering experiments in single-band materials. This result, when compared to other model mechanisms that drive patterns, implies a paradigm shift in our understanding of complexity emergence. We have provided an explicit example in which the keyword is the 'cooperation' between length-scales, in contrast to the numerous predecessor models that attribute complexity to the 'competition' between these.

As future research many directions may be devised departing from this writing, among which we begin to itemize.

> i) We have analysed the stability of the solutions along the intertype in terms of the Meissner-Ochsenfel state, which is frequently used as a 'absolute' stability criterion. We may gain a better understanding of how the class of stripes, bubbles, and donuts solutions shape the intertype in understanding their relative stability. This is not a difficult task to accomplish, but due to the large number of parameters and the time commitment required for this writing, it has been postponed.
>
> ii) Improved control of semi-analytic solutions through numerical exploration of the associated phase. This is not necessary for understanding the associate current in the intertype domain, nor for its stability, but it would be critical for improving numerical control in time-dependent evolution;
>
> iii) Investigate the phase diagram of multi-band materials;
>
> iv) Develop an alternative scheme expansion in diagrams/rules for Extended-Ginzburg-Landau theories (incorporate the induction).
>
> v) Seek a deeper-level understanding of the relation of the self-dual mechanism to other models leading to complexity;

Although the dissertation's primary focus has been on fundamental understanding of the superconducting state and complexity, numerous practical applications may result from a numerical program that incorporates geometry and other ingredients, such as magnetism. To my knowledge, the numerical components displayed in the literature have been limited to simulations near the critical temperature. As a result, some relevant aspects of the phenomenology may have been overlooked, particularly those further away from the critical temperature.

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APPENDIX A – FOR THE CHAPTER 2

A.1 ADIABATICITY

Consider the slow turning on of an interaction as

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + e^{t/t_A}\boldsymbol{H}_I \tag{1}$$

in an otherwise unperturbed system. By slow, we mean $t_A \to \infty$. At $t = -\infty$, $\boldsymbol{H} = \boldsymbol{H}_0$, and when $t = 0$, the interaction is fully turned on. The order of the energy coupled to this time scale is $\epsilon = 1/\tau_A$ in plank-energy unities.

A.2 TIME-ORDERING

Let $\{t_1, t_2..., t_N\}$ define an unordered set of times. Given the ordered set $\{t_{P_1}, t_{P_2}, ..., t_{P_N}\}$ such that $t_{P_1} > t_{P_2} > ...t_{P_N}$. We define the time-ordering of the operators $\mathcal{O}_i$ by

$$\mathcal{T}\{\prod_{i=1} \boldsymbol{\mathcal{O}}_i(t_i)\} = \zeta^P \prod_{i=1} \boldsymbol{\mathcal{O}}(t_{P_i}) \tag{2}$$

in which P is the number of permutations necessary to bring the unordered set to the ordered set.

$$\zeta = \begin{cases} 1 \text{ , for bosonic operators} \\ -1 \text{ , for fermionic operators} \end{cases} \tag{3}$$

As bosonic and fermionic operators acts in distinct spaces, commuting, the time ordering to the mixture of bosonic and fermionic operators equals the product of the respective time-orderings.

A.3 INTERACTION PICTURE REMINDER

The interaction picture is the easier representation to a perturbed Hamiltonian of the form

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_I \tag{4}$$

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The interaction picture is the easier representation to a perturbed Hamiltonian of the form

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_I \tag{4}$$

In this representation, we propose a slow evolution of the form by taking back part of the effect of the unperturbed dynamics

$$\begin{aligned} &|\Psi_I\rangle = e^{i\boldsymbol{H}_0 t}|\Psi_S\rangle \text{ with } |\Psi_S\rangle = e^{-i\boldsymbol{H}t}|\Psi_H\rangle \ , \ |\Psi_I\rangle = e^{i\boldsymbol{H}_0 t}e^{-i\boldsymbol{H}t}|\Psi_H\rangle \\ &|\Psi_H\rangle = e^{i\boldsymbol{H}t}e^{-i\boldsymbol{H}_0 t}|\Psi_I\rangle \end{aligned} \tag{5}$$

Where we have supposed $\boldsymbol{H}_I$ to be time-independent, such that $\boldsymbol{H}$ is time-independent. Since the average shall be the same in any representation,

$$\begin{aligned} &\langle\Psi_S|\,\boldsymbol{O}_S\,|\Psi_S\rangle = \langle\Psi_H|\,e^{i\boldsymbol{H}t}\boldsymbol{O}_S e^{-i\boldsymbol{H}t}\,|\Psi_H\rangle = \\ &\langle\Psi_I|\,e^{i\boldsymbol{H}_0 t}e^{-i\boldsymbol{H}t}e^{i\boldsymbol{H}t}\boldsymbol{O}_S e^{-i\boldsymbol{H}t}e^{i\boldsymbol{H}t}e^{-i\boldsymbol{H}_0 t}\,|\Psi_I\rangle\,\langle\Psi_I|\,e^{i\boldsymbol{H}_0 t}\boldsymbol{O}_S e^{-i\boldsymbol{H}_0 t}\,|\Psi_I\rangle \end{aligned} \tag{6}$$

Therefore, in the interaction picture,

$$|\Psi_I(t)\rangle = \boldsymbol{U}(t)\,|\Psi_I(0)\rangle = e^{i\boldsymbol{H}_0 t}e^{-i\boldsymbol{H}t}\,|\Psi_I(0)\rangle \tag{7}$$

$$\boldsymbol{O}_I = e^{i\boldsymbol{H}_0 t}\boldsymbol{O}_S e^{-i\boldsymbol{H}_0 t} \tag{8}$$

where it is convenient to define the unitary operator providing the wave-function dynamics in the interaction picture. It is useful to define the 'S' evolution matrix as the transformation leading to an evolution from the state $|\Psi(t')\rangle$ to $|\Psi(t)\rangle$ whatever the time ordering. From

$$\begin{aligned} &|\Psi_I\rangle(t) = \boldsymbol{U}(t)\,|\Psi_I(0)\rangle \ ; \boldsymbol{U}^\dagger(t')\,|\Psi_I\rangle(t') = |\Psi_I(0)\rangle \\ &|\Psi_I(t)\rangle \equiv \boldsymbol{S}(t,t')\,|\Psi_I(t')\rangle = \boldsymbol{U}(t)\boldsymbol{U}^\dagger(t')\,|\Psi_I(t')\rangle \end{aligned} \tag{9}$$

Finally, since

$$i\frac{\partial \boldsymbol{U}}{\partial t} = e^{i\boldsymbol{H}_0 t}(\boldsymbol{H}-\boldsymbol{H}_0)e^{-i\boldsymbol{H}t} = \boldsymbol{H}_I(t)\boldsymbol{U}(t) \tag{10}$$

Where $\boldsymbol{H}_I(t)$ is the $\boldsymbol{H}_I$ operator evolution in the interaction picture. At a first view, the immediate evaluation of $\boldsymbol{S}(t,t')$ is difficult to be made in the representation $\boldsymbol{S}(t,t') = \boldsymbol{U}(t)\boldsymbol{U}^\dagger(t')$. However, by studying its dynamics, a very simple result follows

$$i\frac{\partial \boldsymbol{S}(t,t')}{\partial t} = \boldsymbol{H}_I(t)\boldsymbol{S}(t,t') \tag{11}$$

For which the solution is

$$\boldsymbol{S}(t,t') = \mathcal{T}[e^{-i\int_{t'}^{t}\boldsymbol{H}_I(t^*)dt^*}] \tag{12}$$

This is said to be the matrix element 'S' connecting the states at different times.

A.4 GELL-MANN-LOW THEOREM

From the idea of adiabaticity introduced by Murray Gell-Man and Francis Low, it follows a convenient theorem of strong consequences, coined with their names. The theorem produces a one-to-one correspondence between (i) and (ii) with (i) the correlation/green's function of observables of an adiabatically perturbed Hamiltonian in the Heisenberg picture, (ii) the dynamics of the non-interacting evolution of the same observables (interaction picture). The link is provided via the 'S' matrix in the one-to-one relation

$$\langle\Psi_H|\,\mathcal{T}\prod_i \boldsymbol{\mathcal{O}}_H^{(i)}(t_i)\,|\Psi_H\rangle = \langle\Psi_I(t)|\,\mathcal{T}\{\boldsymbol{S}(t,-\infty)\prod_i \boldsymbol{\mathcal{O}}_I^{(i)}(t_i)\}\,|\Psi_I(-\infty)\rangle \tag{13}$$

As $|\Psi_H\rangle = |\Psi_I(-\infty)\rangle$ and $\boldsymbol{\mathcal{O}}_H(t) = \boldsymbol{U}^\dagger(t)\boldsymbol{\mathcal{O}}_I(t)\boldsymbol{U}(t)$,

$$\begin{aligned}\langle\Psi_H|\,\mathcal{T}\prod_i \boldsymbol{\mathcal{O}}_H^{(i)}(t_i)\,|\Psi_H\rangle = \langle\Psi_I(-\infty)|\,\mathcal{T}\{\,&\underbrace{\boldsymbol{U}^\dagger(t_1)}_{\boldsymbol{S}^\dagger(t_1,-\infty)}\,\boldsymbol{\mathcal{O}}_I(t_1)\underbrace{\boldsymbol{U}(t_1)\boldsymbol{U}^\dagger(t_2)}_{\boldsymbol{S}(t_1,t_2)}...\\ \underbrace{\boldsymbol{U}(t_{n-1})\boldsymbol{U}^\dagger(t_n)}_{\boldsymbol{S}(t_{n-1},t_n)}\boldsymbol{\mathcal{O}}_I(t_n)\,\underbrace{\boldsymbol{U}(t_n)}_{\boldsymbol{S}(t_n,-\infty)}\,\}\,|\Psi_I(-\infty)\rangle\ \ ,&\end{aligned} \tag{14}$$

with no a priori time-ordering for the times t_i. Hence, supposing $t_1 > t_i$ for every i,

$$\begin{aligned}\langle\Psi_H|\,\mathcal{T}\prod_i \boldsymbol{\mathcal{O}}_H^{(i)}(t_i)\,|\Psi_H\rangle = \langle\Psi_I(-\infty)|\,&\underbrace{\boldsymbol{U}^\dagger(t_1)}_{S^\dagger(t_1,-\infty)}\,\mathcal{T}\{\mathcal{O}_I(t_1)\underbrace{\boldsymbol{U}(t_1)\boldsymbol{U}^\dagger(t_2)}_{\boldsymbol{S}(t_1,t_2)}...\\ &\underbrace{\boldsymbol{U}(t_{n-1})\boldsymbol{U}^\dagger(t_n)}_{\boldsymbol{S}(t_{n-1},t_n)}\boldsymbol{\mathcal{O}}_I(t_n)\,\underbrace{\boldsymbol{U}(t_n)}_{\boldsymbol{S}(t_n,-\infty)}\,\}\,|\Psi_I(-\infty)\rangle\end{aligned} \tag{15}$$

and since

$$\langle\Psi_I(-\infty)|\,\boldsymbol{S}^\dagger(t_1,-\infty) = (\boldsymbol{S}(t_1,-\infty)\,|\Psi_I(-\infty)\rangle)^\dagger = \langle\Psi_I(t_1)| = \langle\Psi_I(t)|\,\boldsymbol{S}(t,t_1). \tag{16}$$

Next, we may use the flexibility that the time-ordering operator provides to mix the arguments at our convenience. It does not matter the mixing, since after the time-ordering, just a single ordering survives. We also might move back the operator with higher time dependence inside the time-ordering, without loss,

$$\langle\Psi_I(t)|\,\mathcal{T}\{\boldsymbol{S}(t,t_1)\boldsymbol{S}(t_1,t_2)...\boldsymbol{S}(t_{n-1},t_n)\boldsymbol{S}(t_n,-\infty)\prod_i \boldsymbol{\mathcal{O}}_I(t_i)\}\,|\Psi_I(-\infty)\rangle\ \ . \tag{17}$$

Choosing $t_1 > t_i$ for every $1 < i < n$ is not a limitation, since any other choice produces invariably the same result by immediate inspection.

A.5 FINITE-TEMPERATURE PHYSICS

We are indebted to the physicist Matsubara (1955) for the smart observation that the partition function is related to the time-evolution operator in the imaginary time.

$$Z = \mathrm{Tr}[\exp[-\beta \boldsymbol{H}]] = \mathrm{Tr}[\boldsymbol{U}(-i\hbar\beta)] \ , \tag{18}$$

with $\boldsymbol{U}(t) = \exp\Big[-\frac{i}{\hbar}\boldsymbol{H}t\Big]$ the Schrodinger evolution operator. Also, from statistical mechanics,

$$\langle \boldsymbol{\mathcal{O}} \rangle = \frac{\mathrm{Tr}[\exp[-\beta \boldsymbol{H}]\boldsymbol{\mathcal{O}}]}{\mathrm{Tr}[\exp[-\beta \boldsymbol{H}]]} = \frac{\mathrm{Tr}[\boldsymbol{U}(-i\hbar\beta)\boldsymbol{\mathcal{O}}]}{\mathrm{Tr}[\boldsymbol{U}(-i\hbar\beta)]} \ . \tag{19}$$

Such result is the analytic continuation of the Gell-man Low theorem, with the difference that an average on several eigenstates in the place of a single state. We live on the surface, exploring the important results needed for carrying out the standard calculations in many-body physics. This said the physics of finite temperature concerns the transformation of time such that $it \to \tau$ and its consequences. An important identity based on the reasoning above is

$$\frac{Z}{Z_0} = \Big\langle \mathcal{T} \exp\Big[-\int_0^\beta d\tau \boldsymbol{H}_I(\tau)\Big]\Big\rangle_0 = \frac{\mathrm{Tr}\Big[\exp[-\beta \boldsymbol{H}_0]\mathcal{T}\exp\Big[-\int_0^\beta d\tau \boldsymbol{H}_I(\tau)\Big]\Big]}{\mathrm{Tr}[\exp[-\beta \boldsymbol{H}_0]]} \tag{20}$$

A.6 IMAGINARY-TIME GREEN'S FUNCTION

The definition of the Green's function or propagator is

$$\mathcal{G}_{\alpha\alpha'}(\tau-\tau') = -\langle \mathcal{T}\Psi_\alpha(\tau)\Psi^\dagger_{\alpha'}(\tau')\rangle = -\frac{\mathrm{Tr}\Big[e^{-\beta H}\mathcal{T}\Psi_\alpha(\tau)\Psi^\dagger_{\alpha'}(\tau')\Big]}{\mathrm{Tr}[e^{-\beta H}]} \ . \tag{21}$$

We will compute the Green's function for a free-system of bosons or fermions (propator for non-interacting fermions or bosons). For a free system of either bosons or fermions,

$$H = \sum_\alpha \omega_\alpha \underbrace{\Psi^\dagger_\alpha \Psi_\alpha}_{n_\alpha} \tag{22}$$

with $\omega_\alpha = \epsilon_\alpha$ ($\hbar = 1$). Concerning bosons, $\alpha \in \{0,1,...,\infty\}$, for fermions, $\alpha \in \{0,1\}$. Thus,

$$n_\alpha = \frac{\mathrm{Tr}\Big[\exp\Big[-\sum_\beta \omega_\beta n_\beta\Big]n_\alpha\Big]}{\mathrm{Tr}\Big[\exp\Big[-\sum_\beta \omega_\beta n_\beta\Big]\Big]} = \frac{\partial \log Z}{\partial \omega_\alpha} = \begin{cases} \Big[\exp[\beta\omega_\alpha]-1\Big]^{-1} & \text{boson} \\ \Big[\exp[\beta\omega_\alpha]+1\Big]^{-1} & \text{fermion} \end{cases} \tag{23}$$

The Heisenberg equation of motion leads to a particular time-evolution of the operators in the Schrodinger picture

$$\Psi_\alpha(\tau) = e^{-\omega_\alpha \tau}\Psi_\alpha(0) \tag{24}$$

$$\Psi^\dagger_\alpha(\tau) = e^{\omega_\alpha \tau}\Psi^\dagger_\alpha(0) \tag{25}$$

With the operator in the Schrodinger picture as $\Psi_\alpha(0) \equiv \Psi_\alpha$, we write the time-ordering

$$\mathcal{G}^{(0)}_{\alpha\alpha'}(\tau - \tau') = -\exp[-\omega_\alpha(\tau-\tau')][\theta(\tau-\tau')\langle\Psi_\alpha\Psi^\dagger_{\alpha'}\rangle + \zeta\theta(\tau'-\tau)\langle\Psi^\dagger_{\alpha'}\Psi_\alpha\rangle] \tag{26}$$

with $\zeta = 1$ for boson and $\zeta = -1$ for fermion.

The off-diagonal average on the operator product $\Psi^\dagger_{\alpha'}\Psi_\alpha$ vanishes for 'good' quantum numbers α (conserved quantities such as momentum in a translational invariant system). Hence,

$$\langle\Psi^\dagger_\alpha\Psi_{\alpha'}\rangle = n_\alpha\delta_{\alpha\alpha'} \tag{27}$$

$$\langle\Psi_\alpha\Psi^\dagger_{\alpha'}\rangle = \delta_{\alpha\alpha'} \pm \langle\Psi^\dagger_{\alpha'}\Psi_\alpha\rangle = \delta_{\alpha'\alpha}(1+\zeta n_\alpha) \tag{28}$$

Then

$$\mathcal{G}^{(0)}_{\alpha'\alpha}(\tau - \tau') = -\delta_{\alpha\alpha'}\exp[-\omega_\alpha(\tau-\tau')][\theta(\tau-\tau')(1+\zeta n_\alpha) + \zeta\theta(\tau'-\tau)n_\alpha] \tag{29}$$

Defining the RHS term in $\mathcal{G}_{\alpha\alpha'} = \delta_{\alpha\alpha'}\mathcal{G}_\alpha(\tau-\tau')$, we set $\tau' = 0$, to obtain

$$\mathcal{G}^{(0)}_\alpha(\tau) = -e^{-\omega_\alpha\tau}[\theta(\tau)(1+\zeta n_\alpha) + \zeta\theta(-\tau)n_\alpha] \tag{30}$$

with

$$n_\alpha(\zeta) \equiv \frac{1}{e^{\beta\omega_\alpha} - \zeta} \tag{31}$$

in which $n_\alpha(1)$ designate bosons and $n_\alpha(-1)$ designates fermions. As a mnemonic tool, the argument relates to the symmetric and anti-symmetric commutation properties of bosonic and fermionic particles, respectively.

It might be verified the property of the bosonic/fermionic Green's function,

$$\mathcal{G}_{\alpha\alpha'}(\tau+\beta) = \zeta\mathcal{G}_{\alpha\alpha'}(\tau) \tag{32}$$

such that each negative time is related to a positive time. Such property is valid also for any non-free propagator, reason why we omitted the superscript index.

This allows for a even/odd Fourier expansions over the ‘Matsubara frequencies’,

$$\mathcal{G}_{\alpha\alpha'}(\tau) = \frac{1}{\beta}\sum_{n} \mathcal{G}_{\alpha\alpha'}(i\omega_n)\exp[-i\omega_n\tau] \tag{33}$$

$$\omega_n = \frac{\pi n}{\beta} \text{ with } n \in \{\text{odd - fermion; even - boson}\} \tag{34}$$

We may obtain, therefore, the frequency modes of the Green’s function provided via

$$\mathcal{G}_{\alpha\alpha'}(i\omega_n) = \int_{-\beta}^{\beta} d\tau \mathcal{G}_{\alpha\alpha'}(\tau)\exp[i\omega_n\tau] \tag{35}$$

For a well behave function we get the wrong result by a factor of two. What makes this representation valid is the presence of the step-function selecting the positive or negative times.

If $\tau > 0$,

$$\begin{aligned} G^{(0)}_{\alpha\alpha'}(i\omega_n) \equiv \mathcal{G}_{\alpha\alpha'}(i\omega_n)|_{\tau<0} = -\int_0^{\beta} d\tau e^{i\omega_n\tau - \omega_\alpha\tau}\theta(\tau)(1+\zeta n_\alpha) = \\ = \frac{1}{i\omega_n - \omega_\alpha} \equiv \relbar\!\relbar\!\blacktriangleright\!\relbar\!\relbar \end{aligned} \tag{36}$$

If $\tau < 0$,

$$\begin{aligned} \tilde{G}^{(0)}_{\alpha\alpha'}(i\omega_n) \equiv \mathcal{G}_{\alpha\alpha'}(i\omega_n)|_{\tau<0} = -\int_{-\beta}^{0} d\tau e^{i\omega_n\tau - \omega_\alpha\tau}\theta(-\tau)\zeta n_\alpha = \\ = \frac{1}{i\omega_n + \omega_\alpha} \equiv \relbar\!\relbar\!\blacktriangleleft\!\relbar\!\relbar \end{aligned} \tag{37}$$

Hence, $G(i\omega_n)$ denotes the free-particle propagator and $\tilde{G}(i\omega_n)$ denotes the free-hole propagator. Notice how their energy are related by a negative sign one to another, in agreement with the hole concept.

We may always consider the zero temperature limit. Mathematically this can be reached by an analytic continuation of the imaginary axis $i\omega$. If we let $i\omega \to z = i\omega + \omega_0$, the effects of the pure zero temperature are those such that $\omega \to 0$ and $\omega_0 \neq 0$, and for values of $\omega \neq 0$ and $\omega_0 \neq 0$ we approach an intermediate scenario, partly as if at zero temperature and partly as if at higher temperatures.

In Feynman’s/Stueckelberg interpretation, the first propagator appearing in (30) moves forwards in time and the second denotes a virtual process for which the particle moves backwards in time with the same energy of the first propagating particle. The first term denotes a ‘real’ process, whose contribution is progressively greater up to the point where the temperature reaches zero. Feynman provided this as an alternative theory to the particle-hole theory. In the equivalence between the theories, particles moving backwards in time are equivalent to holes (or positrons, in high-energy physics) moving forward in time.

APPENDIX B – FOR THE CHAPTER 3

B.1 JUSTIFICATION OF NONZERO ANOMALOUS AVERAGES

To understand the expectation of nonzero ‘anomalous’ averages ($\langle cc\rangle$,$\langle c^\dagger c^\dagger\rangle$) the ‘weak correlation principle’ is of use. This principle is known to hold from integrable models,

$$\langle \boldsymbol{\mathcal{O}}_1(\boldsymbol{x}_1)\boldsymbol{\mathcal{O}}_2(\boldsymbol{x}_2)\rangle \rightarrow \langle \boldsymbol{\mathcal{O}}_1(\boldsymbol{x}_1)\rangle\langle \boldsymbol{\mathcal{O}}_2(\boldsymbol{x}_2)\rangle \text{ when } |\boldsymbol{r}_1-\boldsymbol{r}_2|\rightarrow\infty \tag{1}$$

or, most generally,

$$\langle \boldsymbol{\mathcal{O}}_1(\boldsymbol{x}_1)...\boldsymbol{\mathcal{O}}_n(\boldsymbol{x}_n)\boldsymbol{\mathcal{O}}'_{1'}(\boldsymbol{x}'_1)...\boldsymbol{\mathcal{O}}'_{n'}(\boldsymbol{x}'_n)\rangle \rightarrow \langle \boldsymbol{\mathcal{O}}_1(\boldsymbol{x}_1)...\boldsymbol{\mathcal{O}}_n(\boldsymbol{x}_n)\rangle\langle \boldsymbol{\mathcal{O}}'_1(\boldsymbol{x}'_1)...\boldsymbol{\mathcal{O}}'_n(\boldsymbol{x}'_n)\rangle \tag{2}$$
$$\text{when } |\boldsymbol{x}_1+...+\boldsymbol{x}_n|-|\boldsymbol{x}'_1+...+\boldsymbol{x}'_n|\rightarrow\infty$$

In particular, for a two-field average,

$$\langle c^\dagger(\boldsymbol{x}_1)c^\dagger(\boldsymbol{x}_2)c(\boldsymbol{x}'_1)c(\boldsymbol{x}'_2)\rangle \rightarrow \langle c^\dagger(\boldsymbol{x}_1)c^\dagger(\boldsymbol{x}_2)\rangle\langle c(\boldsymbol{x}'_2)c(\boldsymbol{x}'_1)\rangle \text{ when } |\boldsymbol{R}-\boldsymbol{R}'|\rightarrow\infty \tag{3}$$

with $\boldsymbol{R}$ denoting the center of mass. Hence, the Bogoliubov interaction term would vanish. However, also the limit $x_1 \rightarrow x_2$ and $x'_1 \rightarrow x'_2$ would yield zero, and as a consequence, the average of the Fermi-Landau liquid Hamiltonian -from which the Bogoliubov theory departs- would be null.

B.2 GRASSMAN ALGEBRA

If the numbers c and c^* are Grassman numbers, they obey anti-commutation relations amongst themselves and fermionic operators. The aim of this new algebra is to smoothly change from all the results derived for the commuting bosonic operators to the anti-commuting operators. As a consequence, $c^2 = 0$ and $c^{*2} = 0$, implying that any function on Grasmman numbers is truncated.

$$f(\bar{c},c) = a_0 + c^*a_1 + a_1^*c + a_{12}c^*c \ , \tag{4}$$

a_1 and a_{12} are the coefficients of the expansion. Also,

$$|c\rangle = (1+\hat{c}^\dagger c)\,|0\rangle = |0\rangle + |1\rangle\, c \ . \tag{5}$$

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In particular, for a two-field average,

$$\langle c^\dagger(\boldsymbol{x}_1)c^\dagger(\boldsymbol{x}_2)c(\boldsymbol{x}'_1)c(\boldsymbol{x}'_2)\rangle \rightarrow \langle c^\dagger(\boldsymbol{x}_1)c^\dagger(\boldsymbol{x}_2)\rangle\langle c(\boldsymbol{x}'_2)c(\boldsymbol{x}'_1)\rangle \text{ when } |\boldsymbol{R} - \boldsymbol{R}'| \rightarrow \infty \tag{3}$$

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a_1 and a_{12} are the coefficients of the expansion. Also,

$$|c\rangle = (1 + \hat{c}^\dagger c)\,|0\rangle = |0\rangle + |1\rangle\, c \ . \tag{5}$$

In the Grassman calculus,

$$\partial_c f = -a_1^* - a_{12} c^* \ , \tag{6}$$

$$\partial_{c^*} a = a_1 + a_{12} c \tag{7}$$

Finally, the integration is equivalent to the differentiation.

$$\int dc \equiv \partial_c \ . \tag{8}$$

APPENDIX C – FOR THE CHAPTER 4

C.1 GREEN'S FUNCTION OF BILINEAR HAMILTONIANS

We consider the evolution of the Green's function operator for the fermionic bilinear Hamiltonian

$$H = \boldsymbol{\Psi}^\dagger \boldsymbol{\varepsilon} \boldsymbol{\Psi} = \sum_{\alpha\beta} \Psi^\dagger_\alpha \varepsilon_{\alpha\beta} \Psi_\beta, \tag{1}$$

$$\{\boldsymbol{\Psi}^\dagger, \boldsymbol{\Psi}\} = \mathbf{1} \text{ or } \{\Psi_\alpha, \Psi_\beta\} = \delta_{\alpha\beta} \tag{2}$$

where α and β are each a set of indices and the sum may be continuous or discrete.

$$\begin{aligned} \mathcal{G}_{\alpha\beta}(\tau_1 - \tau_2) = -\langle \mathcal{T} \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle = &- \langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \theta(\tau_1 - \tau_2) \\ &+ \langle \Psi_\beta(\tau_2)^\dagger \Psi_\alpha(\tau_1) \rangle \theta(\tau_2 - \tau_1) \end{aligned} \tag{3}$$

Hence,

$$\begin{aligned} \partial_{\tau_1} \mathcal{G}_{\alpha\beta}(\tau_1 - \tau_2) = -\langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) - \langle \Psi^\dagger_\beta(\tau_2) \Psi_\alpha(\tau_1) \rangle \delta(\tau_2 - \tau_1) = \\ = -\langle \mathcal{T} \partial_{\tau_1} \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \end{aligned} \tag{4}$$

We investigate the following term whose non-zero contribution is provided by the limit of the first term for the Dirac-delta,

$$\begin{aligned} \langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) = \lim_{\tau_1 \to \tau_2} \frac{\mathrm{Tr}\Big[e^{-\beta H} e^{H\tau_1} \Psi_\alpha e^{-H\tau_1} e^{-H\tau_2} \Psi^\dagger_\beta e^{H\tau_2}\Big]}{\mathrm{Tr}[e^{-\beta H}]} \delta(\tau_1 - \tau_2) = \\ = \langle \Psi_\alpha \Psi^\dagger_\beta \rangle \delta(\tau_1 - \tau_2) \end{aligned} \tag{5}$$

Hence, the two initial terms are rewritten

$$\begin{aligned} -\langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) - \langle \Psi^\dagger_\beta(\tau_2) \Psi_\alpha(\tau_1) \rangle \delta(\tau_1 - \tau_2) &= -\langle \{\Psi_\alpha, \Psi_\beta\} \rangle \delta(\tau_1 - \tau_2) \\ &= \delta_{\alpha\beta} \delta(\tau_1 - \tau_2) \end{aligned} \tag{6}$$

As for the final term,

$$\langle \mathcal{T} \partial_{\tau_1} \Psi_\alpha(\tau_1) \Psi_\beta(\tau_2) \rangle = -\langle \mathcal{T} [H, \Psi_\alpha(\tau_1)] \Psi_\beta(\tau_2) \rangle = \varepsilon_{\alpha\gamma} \langle \mathcal{T} \Psi_\gamma(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle = \varepsilon_{\alpha\gamma} \mathcal{G}_{\gamma\beta}(t_1 - t_2) \tag{7}$$

C.1 GREEN'S FUNCTION OF BILINEAR HAMILTONIANS

We consider the evolution of the Green's function operator for the fermionic bilinear Hamiltonian

$$H = \mathbf{\Psi}^\dagger \boldsymbol{\varepsilon} \mathbf{\Psi} = \sum_{\alpha\beta} \Psi^\dagger_\alpha \varepsilon_{\alpha\beta} \Psi_\beta, \tag{1}$$

$$\{\mathbf{\Psi}^\dagger, \mathbf{\Psi}\} = \mathbf{1} \text{ or } \{\Psi_\alpha, \Psi_\beta\} = \delta_{\alpha\beta} \tag{2}$$

where α and β are each a set of indices and the sum may be continuous or discrete.

$$\begin{aligned} \mathcal{G}_{\alpha\beta}(\tau_1 - \tau_2) = -\langle \mathcal{T} \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle = &- \langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \theta(\tau_1 - \tau_2) \\ &+ \langle \Psi_\beta(\tau_2)^\dagger \Psi_\alpha(\tau_1) \rangle \theta(\tau_2 - \tau_1) \end{aligned} \tag{3}$$

Hence,

$$\begin{aligned} \partial_{\tau_1} \mathcal{G}_{\alpha\beta}(\tau_1 - \tau_2) &= -\langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) - \langle \Psi^\dagger_\beta(\tau_2) \Psi_\alpha(\tau_1) \rangle \delta(\tau_2 - \tau_1) = \\ &= -\langle \mathcal{T} \partial_{\tau_1} \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \end{aligned} \tag{4}$$

We investigate the following term whose non-zero contribution is provided by the limit of the first term for the Dirac-delta,

$$\begin{aligned} \langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) &= \lim_{\tau_1 \to \tau_2} \frac{\mathrm{Tr}\Big[e^{-\beta H} e^{H\tau_1} \Psi_\alpha e^{-H\tau_1} e^{-H\tau_2} \Psi^\dagger_\beta e^{H\tau_2} \Big]}{\mathrm{Tr}[e^{-\beta H}]} \delta(\tau_1 - \tau_2) = \\ &= \langle \Psi_\alpha \Psi^\dagger_\beta \rangle \delta(\tau_1 - \tau_2) \end{aligned} \tag{5}$$

Hence, the two initial terms are rewritten

$$\begin{aligned} -\langle \Psi_\alpha(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle \delta(\tau_1 - \tau_2) - \langle \Psi^\dagger_\beta(\tau_2) \Psi_\alpha(\tau_1) \rangle \delta(\tau_1 - \tau_2) &= -\langle \{ \Psi_\alpha, \Psi_\beta \} \rangle \delta(\tau_1 - \tau_2) \\ &= \delta_{\alpha\beta} \delta(\tau_1 - \tau_2) \end{aligned} \tag{6}$$

As for the final term,

$$\langle \mathcal{T} \partial_{\tau_1} \Psi_\alpha(\tau_1) \Psi_\beta(\tau_2) \rangle = -\langle \mathcal{T} [H, \Psi_\alpha(\tau_1)] \Psi_\beta(\tau_2) \rangle = \varepsilon_{\alpha\gamma} \langle \mathcal{T} \Psi_\gamma(\tau_1) \Psi^\dagger_\beta(\tau_2) \rangle = \varepsilon_{\alpha\gamma} \mathcal{G}_{\gamma\beta}(t_1 - t_2) \tag{7}$$

Therefore,

$$(\partial_{\tau_1} + \varepsilon_{\alpha\gamma})\mathcal{G}_{\alpha\beta}(\tau_1 - \tau_2) = -\delta_{\alpha\beta}\delta(\tau_1 - \tau_2) \tag{8}$$

In matrix form,

$$(\mathbf{1}\partial_\tau + \boldsymbol{\varepsilon}) \cdot \boldsymbol{\mathcal{G}}(\tau - \tau') = -\mathbf{1}\delta(\tau - \tau') \tag{9}$$

If we consider the time as an index, this is rewritten formally as

$$(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon}) \cdot \boldsymbol{\mathcal{G}} = -\mathbf{1} \tag{10}$$

I.e,

$$\boldsymbol{\mathcal{G}} = -(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})^{-1} \tag{11}$$

The proof of the result for bosons follows analogous footsteps, with the final result unaltered.

The bold-typing justify the absence of contraction. The bold typing is often only absent when the contraction is fully carried out with respect to the indices, discrete or continuous.

$$\langle\tau|\, \boldsymbol{\mathcal{G}}\, |\tau'\rangle = \boldsymbol{\mathcal{G}}(\tau - \tau') \tag{12}$$

$$\langle\tau|\, \boldsymbol{\partial_\tau}\, |\tau'\rangle = \mathbf{1}\delta(\tau - \tau')\partial_\tau \tag{13}$$

$$\langle\alpha|\, \boldsymbol{\mathcal{G}}\, |\beta\rangle = \boldsymbol{\mathcal{G}}_{\alpha\beta} \tag{14}$$

$$\langle\alpha|\, \boldsymbol{\partial_\tau}\, |\beta\rangle = \delta_{\alpha\beta}\boldsymbol{\partial_\tau} \tag{15}$$

$$\langle\tau, \alpha|\, \boldsymbol{\mathcal{G}}\, |\tau', \beta\rangle = \mathcal{G}_{\alpha\beta}(\tau - \tau') \tag{16}$$

$$\langle\tau, \alpha|\, \boldsymbol{\partial_\tau}\, |\tau', \beta\rangle = \delta(\tau - \tau')\delta_{\alpha\beta}\partial_\tau \tag{17}$$

C.2 PROOF OF RELATION OF SELF-ENERGY AND PARTITION FUNCTION

Since for fermions

$$\boldsymbol{\mathcal{G}}(\tau - \tau') = -(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})^{-1} \tag{18}$$

and from the fermionic path integral (the interested reader is refereed to appendix C.5)

$$\log[Z] = \log\det\Big[(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})^{-1}\Big] = \operatorname{Tr}\log[\boldsymbol{-\mathcal{G}}^{-1}] \tag{19}$$

Therefore,

$$\log[\frac{Z}{Z_0}] = \mathrm{Tr}\Big[\log\Big(\frac{\boldsymbol{\mathcal{G}}^{(0)}}{\boldsymbol{\mathcal{G}}}\Big)\Big] = \mathrm{Tr}\Big[\log\Big(\frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{\Sigma}\boldsymbol{\mathcal{G}}}\Big)\Big] \tag{20}$$

where in the last step we applied the Dyson series

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}^{(0)} + \boldsymbol{\mathcal{G}}^{(0)}\boldsymbol{\Sigma}\boldsymbol{\mathcal{G}} \tag{21}$$

If we truncate the result to the first-order of the expansion in the self-energy,

$$\log[\frac{Z}{Z_0}] \sim \mathrm{Tr}\Big[\log\Big(\mathbf{1} - \boldsymbol{\Sigma}\boldsymbol{\mathcal{G}}^{(0)}\Big)\Big] + \mathcal{O}(\boldsymbol{\Sigma}^2) = -\,\mathrm{Tr}\Big[\boldsymbol{\Sigma}\boldsymbol{\mathcal{G}}^{(0)}\Big] + \mathcal{O}(\boldsymbol{\Sigma}^2) \tag{22}$$

Often $\boldsymbol{\Sigma}$ is left not contracted with respect to time or frequency - the contraction of the trace is only on the momenta space. By comparing it with the Hartree-Fock expansion,

$$\log[\frac{Z}{Z_0}] = -T\sum_{\boldsymbol{k}}\Big\{\sum_{\boldsymbol{k'}}\mathcal{G}_{\boldsymbol{k'}}[(2S+1)^2V(0) - (2S+1)V_{\mathrm{eff}}(\boldsymbol{k'}-\boldsymbol{k})]\Big\}\mathcal{G}^{(0)}_{\boldsymbol{k}} \tag{23}$$

From which we identify

$$\boldsymbol{\Sigma}_{\boldsymbol{k}} = -T\sum_{\boldsymbol{k'}}\mathcal{G}_{\boldsymbol{k'}}[-(2S+1)^2V_{\mathbf{0}} + (2S+1)V_{\mathrm{eff}\boldsymbol{k'}-\boldsymbol{k}}] \tag{24}$$

By contracting it with respect to the frequency,

$$\Sigma_{\boldsymbol{k}}(i\omega) = -T\sum_{i\nu}\sum_{\boldsymbol{k'}}\mathcal{G}_{\boldsymbol{k'}}(i\omega)[-(2S+1)^2V_{\mathbf{0}}(i\nu)\delta_{i\omega,0}\delta_{i\nu,0} + (2S+1)V_{\mathrm{eff}(\boldsymbol{k'}-\boldsymbol{k})}(i\nu)] \tag{25}$$

C.3 MEAN-FIELD THEORY

The mean-field-theory of concerns the killing of quadratic or higher-order terms in the deviation from the mean- field. The Fermi-liquid theory is a mean-field of the four-point interaction. Hence,

$$\boldsymbol{\mathcal{O}}_i\boldsymbol{\mathcal{O}}_j \to \boldsymbol{\mathcal{O}}'_i\boldsymbol{\mathcal{O}}'_j = \boldsymbol{\mathcal{O}}_i\boldsymbol{\mathcal{O}}_j + \Delta_{ij} + O(\Delta_{ij}^2)\ ,\ \Delta_{ij} = \langle\boldsymbol{\mathcal{O}}_i\boldsymbol{\mathcal{O}}_j\rangle - \boldsymbol{\mathcal{O}}_i\boldsymbol{\mathcal{O}}_j \tag{26}$$

It follows

$$\boldsymbol{\mathcal{O}}_1\boldsymbol{\mathcal{O}}_2\boldsymbol{\mathcal{O}}_3\boldsymbol{\mathcal{O}}_4 \to \boldsymbol{\mathcal{O}}_1\boldsymbol{\mathcal{O}}_2\Delta_{34} + \boldsymbol{\mathcal{O}}_3\boldsymbol{\mathcal{O}}_4\Delta_{12} + \langle\boldsymbol{\mathcal{O}}_1\boldsymbol{\mathcal{O}}_2\rangle\langle\boldsymbol{\mathcal{O}}_3\boldsymbol{\mathcal{O}}_4\rangle \tag{27}$$

C.4 COHERENT STATE AND IDENTITIES

Coherent states are eigenstates of the creation/annihilation operators. The coherent states are not orthogonal. Despite this, they obey simple properties

$$\langle C_1|C_2\rangle = e^{C_1^* C_2} \tag{28}$$

$$\langle C_1| : \mathcal{O}[C^\dagger, C] : |C_2\rangle = \mathcal{O}[C_1^*, C_2] e^{C_1^* C_2} \tag{29}$$

The coherent states are an 'overcomplete' basis. For either bosonic (denoted $|b\rangle$) or fermionic (denoted $|c\rangle$) coherent states, adding up to unity obeys different measurements.

$$\hat{1} = \int \frac{db^* db}{2\pi i} e^{-b^* b} |b\rangle \langle b^*| \text{, bosonic identity} \tag{30}$$

$$\hat{1} = \int dc^* dc e^{-c^* c} |c\rangle \langle c^*| \text{, fermionic identity} \tag{31}$$

From this, it is possible to infer the property

$$\mathrm{Tr}[\mathcal{O}] = \int \frac{db^* db}{2\pi i} e^{-b^* b} \langle b^*| \mathcal{O} |b\rangle \text{, bosonic identity} \tag{32}$$

$$\mathrm{Tr}[\mathcal{O}] = \int dc^* dc e^{-c^* c} \langle -c^*| \mathcal{O} |c\rangle \text{, fermionic identity} \tag{33}$$

C.5 GAUSSIAN INTEGRALS

Bosonic integral

The bosonic Gaussian integral of interest is of the form

$$Z \equiv \int \prod_j db_j^* db_j \exp\Big[-\boldsymbol{b}^\dagger \boldsymbol{\epsilon} \boldsymbol{b}\Big] = \frac{1}{\det \boldsymbol{\epsilon}} \tag{34}$$

and the generalization to the source case provided via

$$Z_s \equiv \int \prod_j db_j^* db_j \exp\Big[-\boldsymbol{b}^\dagger \boldsymbol{\epsilon} \boldsymbol{b} - \boldsymbol{j}^\dagger \cdot \boldsymbol{b} - \boldsymbol{b}^\dagger \cdot \boldsymbol{j}\Big] = \frac{1}{\det \boldsymbol{\epsilon}} \exp\Big[\boldsymbol{j}^\dagger \boldsymbol{\epsilon} \boldsymbol{j}\Big] \tag{35}$$

Considering 34, the introduction of the current term is obtained through the shift

$$\boldsymbol{b} \to \boldsymbol{b} - \boldsymbol{\epsilon}^{-1} \boldsymbol{j} \tag{36}$$

Since,

$$\int \mathcal{D}[\boldsymbol{b}^{*T}, \boldsymbol{b}] \exp\Big[-\boldsymbol{b}^{*T} \boldsymbol{\epsilon} \boldsymbol{b} + \boldsymbol{b}^{*T} \boldsymbol{j} + \boldsymbol{j}^{*T} \boldsymbol{b} - \boldsymbol{j}^{*T} \boldsymbol{\epsilon}^{-1} \boldsymbol{j}\Big] = \frac{1}{\det \boldsymbol{\epsilon}} \tag{37}$$

The measure in the integral is not changed. Therefore,

$$\exp\Big[-\boldsymbol{j}^{*T}\boldsymbol{\epsilon}^{-1}\boldsymbol{j}\Big]\int\mathcal{D}[\boldsymbol{b}^{*T},\boldsymbol{b}]\exp\Big[-\boldsymbol{b}^{*T}\boldsymbol{\epsilon}\boldsymbol{b}+\boldsymbol{b}^{*T}\boldsymbol{j}+\boldsymbol{j}^{*T}\boldsymbol{b}\Big]=\frac{1}{\det\boldsymbol{\epsilon}} \tag{38}$$

Finally,

$$Z_s=\int\mathcal{D}[\boldsymbol{b}^{*T},\boldsymbol{b}]\exp\Big[-\boldsymbol{b}^{*T}\boldsymbol{\epsilon}\boldsymbol{b}+\boldsymbol{b}^{*T}\boldsymbol{j}+\boldsymbol{j}^{*T}\boldsymbol{b}\Big]=\frac{\exp\Big[\boldsymbol{j}^{*T}\boldsymbol{\epsilon}^{-1}\boldsymbol{j}\Big]}{\det\boldsymbol{\epsilon}} \tag{39}$$

From 11, it follows the identity

$$\begin{aligned}Z_s=\int\mathcal{D}[\boldsymbol{b}^{\dagger}\cdot\boldsymbol{b}]\exp\Bigg[-\int_0^{\beta}d\tau[\boldsymbol{b}^{*T}(\boldsymbol{\partial_\tau}+\boldsymbol{\varepsilon})\boldsymbol{b}-j^*(\tau)b(\tau)-b^*(\tau)j(\tau)]\Bigg]=\\=\frac{\exp\Big[-\int_0^{\beta}d\tau d\tau' j^*(\tau)G(\tau-\tau')j(\tau')\Big]}{\det[\boldsymbol{\partial_\tau}+\boldsymbol{\varepsilon}]}\end{aligned} \tag{40}$$

The usefulness of this expression is that it allows for obtaining all of correlation in arbitrary order. It is also an effectively easier path to the proof of the Wick's theorem.

Proof

We wish to prove the relevant Gaussian identity 34.

$$\boldsymbol{b}=\boldsymbol{U}\boldsymbol{d} \tag{41}$$

$$\boldsymbol{b}^{\dagger}=\boldsymbol{d}^{\dagger}\boldsymbol{U}^{\dagger} \tag{42}$$

in which the bilinear form is diagonal,

$$\boldsymbol{b}^{\dagger}\boldsymbol{\epsilon}\boldsymbol{b}=\boldsymbol{d}^{\dagger}\boldsymbol{D}\boldsymbol{d}=\sum_{\alpha}d^*_{\alpha}D_{\alpha}d_{\alpha},\ \boldsymbol{D}\equiv\boldsymbol{U}^{\dagger}\boldsymbol{\epsilon}\boldsymbol{U} \tag{43}$$

The change in the integral measure is the identity, due to the Jacobian of the transformation 41,42,

$$\det\begin{pmatrix}\boldsymbol{U}^{\dagger} & 0\\ 0 & \boldsymbol{U}\end{pmatrix}=\mathbf{1} \tag{44}$$

Therefore, in the novel basis

$$Z=\int\prod_{\alpha}\frac{dd^*_{\alpha}dd_{\alpha}}{2\pi i}\exp[-D_{\alpha}d^*_{\alpha}d_{\alpha}] \tag{45}$$

Changing from the characteristic coordinates (d_α, d_α^*) to the canonical geometric coordinates, the Jacobian may be readily checked to be $2i$, thus,

$$\begin{aligned} Z &= \int \prod_\alpha \frac{dd_{\alpha,x} dd_{\alpha,y}}{\pi} \exp\Big[-D_\alpha (d_{\alpha,x}^2 + d_{\alpha,y}^2)\Big] = \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^\infty \prod_{dr_\alpha} dr_\alpha \exp\Big[-D_\alpha r_\alpha^2\Big] \\ &= \prod_\alpha \frac{1}{D_\alpha} = \frac{1}{\det[\boldsymbol{D}]} \end{aligned} \tag{46}$$

The result is proved due to the determinant invariance by basis modification.

Fermionic Gaussian integral

The relevant fermionic Gaussian integral is of the form

$$Z \equiv \int \prod_j dc_j^* dc_j \exp\Big[-\boldsymbol{c}^\dagger \boldsymbol{\epsilon} \boldsymbol{c}\Big] = \det \boldsymbol{\epsilon} \tag{47}$$

Considering this to hold, the generalization to the source case is provided via

$$Z_s \equiv \int \prod_j dc_j^* dc_j \exp\Big[-\boldsymbol{c}^\dagger \boldsymbol{\epsilon} \boldsymbol{c} - \boldsymbol{j}^\dagger \cdot \boldsymbol{c} - \boldsymbol{c}^\dagger \cdot \boldsymbol{j}\Big] = \det \boldsymbol{\epsilon} \exp\Big[\boldsymbol{j}^\dagger \boldsymbol{\epsilon} \boldsymbol{j}\Big] \tag{48}$$

by a transformation analogue to 36. Assuming 11 to hold,

$$\begin{aligned} Z_s &= \int \mathcal{D}[\boldsymbol{c}^\dagger \cdot \boldsymbol{c}] \exp\left[-\int_0^\beta d\tau [\boldsymbol{c}^{*T} (\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon}) \boldsymbol{c} - j^*(\tau) c(\tau) - c^*(\tau) j(\tau)]\right] = \\ &= \exp\left[-\int_0^\beta d\tau d\tau' j^*(\tau) G(\tau - \tau') j(\tau')\right] \det[\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon}] \end{aligned} \tag{49}$$

Proof

We wish to prove the basic Gaussian integral to Fermions 47

$$Z \equiv \int \boldsymbol{dc}^\dagger \cdot \boldsymbol{dc} \exp\Big[-\boldsymbol{c}^\dagger \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{c}\Big] = \det[\boldsymbol{\epsilon}] \tag{50}$$

We diagonalize the matrix with a unitary transformation, as in the derivation of the bosonic path integral.

$$\boldsymbol{d} = \boldsymbol{U} \boldsymbol{c} \tag{51}$$

$$\boldsymbol{d}^\dagger = \boldsymbol{c}^\dagger \boldsymbol{U}^\dagger \tag{52}$$

with $\boldsymbol{U}$ consisting of bosonic numbers ('normal' numbers). There is no change in the integral measurement, as

$$\int \boldsymbol{dc}^{\dagger} \cdot \boldsymbol{dc} = \partial_{\boldsymbol{c}^{\dagger}}\partial_{\boldsymbol{c}} = \partial_{\boldsymbol{d}^{\dagger}}\partial_{\boldsymbol{d}}\underbrace{[\frac{\partial \boldsymbol{d}^{\dagger}}{\partial \boldsymbol{c}} + \frac{\partial \boldsymbol{d}^{\dagger}}{\partial \boldsymbol{c}^{\dagger}}] \cdot [\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{c}} + \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{c}^{\dagger}}]}_{\boldsymbol{U}^{\dagger}\boldsymbol{U}} = \partial_{\boldsymbol{d}'}\partial_{\boldsymbol{d}} = \int \boldsymbol{dd}^{\dagger} \cdot \boldsymbol{dd} \tag{53}$$

Therefore,

$$Z = \int \prod_{\alpha} dd_{\alpha}^{*} dd_{\alpha} \exp[-D_{\alpha} d_{\alpha}^{*} d_{\alpha}] \tag{54}$$

Applying the rules of the Grassman algebra (Appendix B.2),

$$Z = \prod_{\alpha} \partial_{dd_{\alpha}^{*}} \partial_{dd_{\alpha}} (1 - D_{\alpha} d_{\alpha}^{*} d_{\alpha}) = \prod_{\alpha} D_{\alpha} = \det[\boldsymbol{D}] \tag{55}$$

which proves the claim.

C.6 FUNDAMENTALS OF PATH INTEGRALS IN FINITE-TEMPERATURE

Feynman formulation of operator quantum mechanics states

$$\begin{aligned} \langle f| \exp\left[-i\frac{Ht}{\hbar}\right] |i\rangle &= \sum_{\text{paths}} \exp[iS_{\text{path},i\to j}/\hbar] \\ S_{\text{path}} &= \int_0^t dt'(pq' - H[p,q]) \end{aligned} \tag{56}$$

The path integral provided by Feynman connects the classical functional with quantum mechanical amplitudes. In the limit of $\hbar \to 0$, it reduces to the Hamilton principle. By considering the above at $t' \to \tau(t') = \frac{it'}{\hbar}$,

$$\begin{aligned} \langle f| \exp[-H\tau(t)] |i\rangle &= \sum_{\text{paths}} \exp[-S_{\text{path},i\to j}] \\ S_{\text{path}} &= \int_0^{\tau(t)} d\tau(pq'\frac{dt'}{d\tau} - H[p,q]) \end{aligned} \tag{57}$$

Since the partition function is provided through

$$Z = \text{Tr}[\exp[-H\beta]] = \sum_{\alpha} \langle\alpha| \exp[-H\beta] |\alpha\rangle \tag{58}$$

The Feynman techniques are extended to be applicable to statistical mechanics (identifying $\tau(t) = \beta$). The initial and final states are identical, as the time evolution of bosonic states is periodic (Matsubara bosonic frequency). In this way, $|\alpha(0)\rangle = |\alpha(\beta)\rangle$, and it follows

$$\begin{aligned} Z &= \text{Tr}[\exp[-\beta H]] = \sum_{\alpha} \sum_{\text{per.paths},\alpha\to\alpha} \exp[-S] = \sum_{\text{per. paths}} \exp[-S] \\ S &= \int_0^{\beta} d\tau(-\frac{i}{\hbar}p\partial_{\tau}q + H[p,q]) \end{aligned} \tag{59}$$

Bosons

From the identity for the bosonic trace 32,

$$Z = \mathrm{Tr}\Big[e^{-\beta H}\Big] = \int \frac{db_0^* db_0}{2\pi i} e^{-b_0^* b_0} \langle b_0^*| e^{-\beta H} |b_0\rangle \tag{60}$$

By computing the amplitude due to any set of $n-1$ intermediate coherent states in between the initial $|b_0\rangle$ and final states $|b_n\rangle = |b_0\rangle$ within n time slices $\Delta\tau_j = \tau_{j+1} - \tau_j$ ($\sum_{j=0}^{n-1} \Delta\tau_j = \beta$),

$$Z = \mathrm{Tr}\Big[e^{-\beta H}\Big] = \int \frac{db_0^* db_0}{2\pi i} e^{-b_0^* b_0} \langle b_0| e^{-\Delta\tau_{n-1} H} \hat{1}_{N-1} ... \hat{1}_j e^{-\Delta\tau_j H} \hat{1}_{j-1} ... \hat{1}_1 e^{-\Delta\tau_0 H} |b_0\rangle \tag{61}$$

With

$$\hat{1}_j = \int \frac{db_j^* db_j}{2\pi i} e^{-b_j^* b_j} |b_j\rangle \langle b_j| \tag{62}$$

Therefore,

$$Z = \int \prod_{j=0}^{N-1} \frac{db_j^* db_j}{2\pi i} \exp\Big[-b_j^* b_j\Big] \langle b_{j+1}| e^{-\Delta\tau_j H[\hat{b}^\dagger, \hat{b}]} |b_j\rangle \tag{63}$$

We would like to apply the identity 29, however it does only works if the operator is normal ordered version. However, at first order in $\Delta\tau_i$, the operator is immediately normal-ordered. Therefore,

$$\langle b_{j+1}| e^{-\Delta\tau_j H[\hat{b}^\dagger, \hat{b}]} |b_j\rangle = \exp\Big[b_{j+1}^* b_j - \Delta\tau H[b_{j+1}^*, b_j]\Big] \tag{64}$$

$$Z = \int \prod_{j=0}^{N-1} \frac{db_j^* db_j}{2\pi i} \exp\Big[b_j^*(b_{j+1} - b_j) + \Delta\tau_j H[b_{j+1}^*, b_j]\Big] + O[n(\Delta\tau)^2] \tag{65}$$

A useful shorthand definition is often employed

$$\mathcal{D}_N[\boldsymbol{b}^\dagger \cdot \boldsymbol{b}] = \prod_{j=0}^{N-1} \frac{db_j^* db_j}{2\pi i} = \frac{\boldsymbol{db}^\dagger \cdot \boldsymbol{db}}{2\pi i} \tag{66}$$

We have defined $\boldsymbol{b}$ as a column matrix whose entries are b evaluated at distinct discrete times. We identify

$$Z = \int \mathcal{D}_N[\boldsymbol{b}^\dagger \cdot \boldsymbol{b}] \exp[-S_N]$$

$$S_N = \sum_{j=0}^{N-1} \Delta\tau (b_j^* \frac{(b_{j+1} - b_j)}{\Delta\tau_j} + H[b_{j+1}^*, b_j]) \tag{67}$$

In the continuous limit, when the number of time-slices tends to infinity, defining $\lim_{n\to\infty}\mathcal{D}_n[\boldsymbol{b}^\dagger\cdot\boldsymbol{b}]\equiv\mathcal{D}[\boldsymbol{b}^\dagger\cdot\boldsymbol{b}]$,

$$Z=\int\mathcal{D}[\boldsymbol{b}^\dagger\cdot\boldsymbol{b}]\exp[-S] \tag{68}$$

$$S=\int_0^\beta d\tau(b^*\partial_\tau b+H[b^*,b]) \tag{69}$$

The term $O(n(\Delta\tau)^2)\to 0$ when $n\to\infty$, since $\Delta\tau\sim\frac{1}{n}$.

Consider the bilinear hamiltonian with bosonic operators

$$H=H[\hat{b}^\dagger_\alpha,\hat{b}_\beta] \tag{70}$$

with

$$H=\sum_{\alpha\beta}\hat{b}^\dagger_\alpha\varepsilon_{\alpha\beta}\hat{b}_\beta\equiv\hat{\boldsymbol{b}}^\dagger\cdot\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{b}} \tag{71}$$

Applying the same procedure with

$$\hat{1}_j=\int\prod_\alpha\frac{db^*_{j\alpha}db_{j\alpha}}{2\pi i}e^{-b^*_{j\alpha}b_{j\alpha}}\prod_\beta\frac{db^*_{j\beta}db_{j\beta}}{2\pi i}e^{-b^*_{j\beta}b_{j\beta}}\left|b_{j\alpha},b_{j\beta}\right\rangle\left\langle b_{j\alpha},b_{j\beta}\right| \tag{72}$$

We must evaluate

$$\left\langle b_{j+1,\alpha},b_{j+1,\beta}\right|e^{-\Delta\tau_jH[\hat{b}^\dagger_\alpha,\hat{b}_\beta]}\left|b_{j,\alpha},b_{j,\beta}\right\rangle=\exp\Big[b^*_{j+1,\alpha}b_{j,\alpha}+b_{j+1,\beta}b_{j,\beta}-\Delta\tau_jb^*_{j+1,\alpha}\varepsilon_{\alpha\beta}b_{j,\beta}\Big] \tag{73}$$

Therefore,

$$\begin{aligned}Z=\int\prod_{j=0}^{N-1}\prod_{\alpha\beta}&\frac{db^*_{j\alpha}db_{j\alpha}}{2\pi i}\frac{db^*_{j\beta}db_{j\beta}}{2\pi i}\\ &\exp\left[-\Delta\tau_j\{b^*_j\frac{(b_{j+1,\beta}-b_{j,\beta})}{\Delta_{\tau_j}}+b^*_{j,\beta}\frac{(b_{j+1,\beta}-b_{j,\beta})}{\Delta\tau_j}+b^*_{j+1,\alpha}\varepsilon_{\alpha,\beta}b_{j,\beta}\}\right]\end{aligned} \tag{74}$$

Equivalently,

$$Z=\int\mathcal{D}[\boldsymbol{b}^\dagger,\boldsymbol{b}]\exp\left[-\{\sum_{j,\alpha}\Delta\tau_jb^*_{j,\alpha}\partial_{\tau_j}b_{j,\alpha}+\sum_{j,\beta}\Delta\tau_jb^*_{j,\beta}\partial_{\tau_j}b_{j,\beta}+\sum_{j,\alpha,\beta}\Delta\tau_jb^*_{\alpha,j}\varepsilon_{\alpha\beta}b_{j,\beta}\}\right] \tag{75}$$

This might be rewritten to include sums in time indices,

$$Z=\int\mathcal{D}[\boldsymbol{b}^{*T},\boldsymbol{b}]\exp\left[-\sum_{jj',\alpha\beta}\delta_{jj'}\Delta\tau_jb^*_{\alpha,j}\epsilon_{\alpha\beta;jj'}b_{\beta,j'}\right] \tag{76}$$

In the continuous limit, since $\delta_{jj'}=\Delta\tau_{j'}\delta(\tau_{j'}-\tau_j)$, assuming the variables as continuous,

$$Z=\int\mathcal{D}[\boldsymbol{b}^\dagger\cdot\boldsymbol{b}]\exp\Big[-\int d\tau'\int d\tau b_\alpha(\tau)\epsilon_{\alpha\beta}(\tau-\tau')\delta(\tau'-\tau)b_\beta(\tau')\Big] \tag{77}$$

$$\epsilon_{\alpha\beta}(\tau,\tau')=(\partial_\tau+\varepsilon_{\alpha\beta})\delta(\tau-\tau') \tag{78}$$

We notice that in the general basis consisting of both particle index and time,

$$\int\int d\tau' d\tau b_\alpha(\tau)\epsilon(\tau-\tau')\delta(\tau'-\tau)b_\beta(\tau') = \boldsymbol{b}^\dagger \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{b} \tag{79}$$

In which we identify the identity to hold

$$\boldsymbol{\epsilon} = (\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon}) = -\boldsymbol{\mathcal{G}}^{-1} \tag{80}$$

the last equality resulting from 11. This justifies the bosonic Gaussian integral of relevance to be that mentioned in C.6.

Time domain

We consider the multidimensional Gaussian identity

$$\int \mathcal{D}[\boldsymbol{b}^\dagger \cdot \boldsymbol{b}] \exp\Big[-\boldsymbol{b}^\dagger \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{b}\Big] = \frac{1}{\det \boldsymbol{\epsilon}} \tag{81}$$

with $\boldsymbol{\varepsilon}$ a bilinear form.

Thence, we may rewrite 77,

$$Z = \int \mathcal{D}[\boldsymbol{b}^\dagger \cdot \boldsymbol{b}] \exp\Big[-\boldsymbol{b}^{*T}(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})\boldsymbol{b}\Big] = \frac{1}{\det[\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon}]} = -\det\Big[\boldsymbol{\mathcal{G}}^{-1}\Big] \tag{82}$$

Evaluation on the frequency domain

Contracting the discrete and continuous indices in the bilinear form,

$$Z = \int \mathcal{D}[\boldsymbol{b}^\dagger \cdot \boldsymbol{b}] \exp\left[-\int d\tau \begin{pmatrix} b_\alpha^* & b_\beta^* \end{pmatrix} \left\{ \begin{pmatrix} \partial_\tau & 0 \\ 0 & \partial_\tau \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \right\} \begin{pmatrix} b_\alpha \\ b_\beta \end{pmatrix}\right] \tag{83}$$

It is often convenient to expand in frequency modes.

$$b_\gamma = \frac{1}{\beta^{1/2}} \sum_n b_\gamma(i\nu_n) e^{-i\omega_n\tau} \text{ with } \omega_n = 2\pi n/\beta \tag{84}$$

Then,

$$S = \sum_{\gamma_1\gamma_2 ln} b_{\gamma_1}^*(i\omega_l) b_{\gamma_2}(i\omega_n) \frac{1}{\beta} \int_0^\beta d\tau e^{i\omega_l\tau}(\partial_\tau + \varepsilon_{\gamma_1\gamma_2}) e^{-i\omega_n\tau} \tag{85}$$

Since

$$\frac{1}{\beta}\int_0^\beta e^{i(\omega_l - \omega_n)\tau} = \delta_{nl} \tag{86}$$

Then,

$$S = \sum_{\gamma_1\gamma_2 n} b^*_{\gamma_1}(i\omega_n)(-i\omega_n\delta_{\gamma_1\gamma_2} + \varepsilon_{\gamma_1\gamma_2})b_{\gamma_2}(i\omega_n) \tag{87}$$

This might be seen as

$$S = \sum_{\gamma_1\gamma_2 n} b^*_{\gamma_1}(i\omega_n)(-i\omega_n\delta_{\gamma_1\gamma_2} + \varepsilon_{\gamma_1\gamma_2})b_{\gamma_2}(i\omega_n) = \sum_n \boldsymbol{b}^{*T}_n \cdot \boldsymbol{\epsilon}_n \cdot \boldsymbol{b}_n \tag{88}$$

where we define the vector $\boldsymbol{b}$ for which the entries are sequenced by the particle index alone in this case. Then,

$$Z = \int \prod_n \mathcal{D}[\boldsymbol{b}^{*T}_n, \boldsymbol{b}_n] \exp\Big[-\boldsymbol{b}^{*T}_n \boldsymbol{\epsilon}_n \boldsymbol{b}_n\Big] = \prod_n \frac{1}{\det \boldsymbol{\epsilon}_n} \tag{89}$$

Also,

$$Z = \prod_n \frac{1}{\det[-i\omega_n \mathbf{1} + \boldsymbol{\varepsilon}]} = \prod_{n,j} \frac{1}{\det[-i\omega_n + \varepsilon_j]} \tag{90}$$

The free energy is expressed as

$$F = -\beta \ln Z = \beta \ln\left(\prod_n \det[-i\omega_n \mathbf{1} + \boldsymbol{\varepsilon}]\right) = \beta \sum_n \mathrm{Tr}[\ln(\boldsymbol{\varepsilon} - i\omega_n)] = \beta \sum_{nj} \ln(\varepsilon_j - i\omega_n) \tag{91}$$

since

$$\det[\boldsymbol{\mathcal{O}}] = \mathrm{Tr}[\ln[\boldsymbol{\mathcal{O}}]] \tag{92}$$

From our study of the free bosonic Green's function in the frequency domain.

$$\mathcal{G}_{\alpha\alpha'}(\omega_n) = \frac{\delta_{\alpha\alpha'}}{i\omega_n - \varepsilon_\alpha} \tag{93}$$

By hiding the contraction in the particle-index basis,

$$\boldsymbol{\mathcal{G}}(\omega_n) = \frac{1}{i\omega_n - \boldsymbol{\varepsilon}} \tag{94}$$

By promoting this to a more general operator on both frequency and particle-index,

$$\boldsymbol{\mathcal{G}} = \frac{1}{i\boldsymbol{\Omega} - \boldsymbol{\varepsilon}} \tag{95}$$

$$\langle n'| \boldsymbol{\Omega} |n\rangle = \omega_n \delta_{nn'} \tag{96}$$

Therefore,

$$F = \beta \ln\Big(\det\Big[-\boldsymbol{\mathcal{G}}^{-1}\Big]\Big) = \beta\, \mathrm{Tr} \ln[-\boldsymbol{\mathcal{G}}^{-1}] \tag{97}$$

By comparing to 80, the correspondence between the time-dependent and the frequency-dependent operators is provided.

$$i\boldsymbol{\Omega} \longleftrightarrow \boldsymbol{\partial_\tau} \tag{98}$$

Fermions

The steps in developing the fermionic path integral are about the same of the bosonic case. For simplicity, yet as a representative illustration, we consider the Hamiltonian comprising a single fermion,

$$H = \varepsilon c^{\dagger} c \tag{99}$$

Applying the identity for the trace 33,

$$Z = \mathrm{Tr}[\exp[-\beta H]] = -\int dc_N^* dc_0 \exp[c_N^* c_0] \langle c_n | \exp[-\beta H] | c_0 \rangle \tag{100}$$

To take a faster route we divide the exponential in n identical time-slices,

$$\exp[-\beta H] = (\exp[-\Delta\tau H])^N \ , \ \beta = n\Delta\tau \tag{101}$$

As before, in between time-slices we introduce the completeness relation

$$\int dc_j^* dc_j \left| c_j \middle\rangle\middle\langle c_j^* \right| \exp\Big[-c_j^* c_j\Big] = 1 \tag{102}$$

Hence,

$$Z = -\int dc_n^* dc_0 \exp[c_n^* c_0] \prod_{j=1}^{n-1} dc_j^* dc_j \exp\Big[-c_j^* c_j\Big] \prod_{j=1}^{n} \left\langle c_j^* \right| \exp[-H\Delta\tau] \left| c_{j-1} \right\rangle \tag{103}$$

By using the antiperiodic boundary condition to fermions,

$$c(\tau + \beta) = -c(\tau) \ , \ c^*(\tau + \beta) = -\bar{c}(\tau) \ , \tag{104}$$

it follows the identification $c_n = -c_0$, from which the last may be rewritten as the product of n terms,

$$Z = \int \prod_{j=1}^{n} dc_j^* dc_j \exp\Big[-c_j^* c_j\Big] \langle c_j | \exp[-H\Delta\tau] | c_{j-1} \rangle \tag{105}$$

Hence, from 29, at first-order in $\Delta\tau$, the operator is identical to its normal-ordered form.

$$\left\langle c_j^* \right| \exp[-H\Delta\tau] \left| c_{j-1} \right\rangle = \exp\Big[c_j^* c_{j-1}^*\Big] \exp\Big[-H[c_j^* c_{j-1}]\Delta\tau\Big] + \mathcal{O}(\Delta\tau^2) \tag{106}$$

Hence,

$$Z_n = \int \prod_{j=1}^{n} dc_j^* dc_j \exp[-S] \tag{107}$$

$$S = \sum_{j=1}^{n} [c_j^* \frac{(c_j - c_{j-1})}{\Delta\tau} + \varepsilon c_j^* c_{j-1}] \Delta\tau \tag{108}$$

The continuous limit is obtained in the limit $n \to \infty$, and considering $\boldsymbol{c}$ to be a vector in which the components correspond each to an specific time,

$$Z = \int \boldsymbol{dc}^* \cdot \boldsymbol{dc} \exp[-S] \tag{109}$$

$$S = \int_0^\beta d\tau [c^* \epsilon c] = \boldsymbol{c}^\dagger \boldsymbol{\epsilon c} \tag{110}$$

$$\boldsymbol{\epsilon} = -(\boldsymbol{\partial_\tau} + \boldsymbol{\varepsilon})^{-1} = \boldsymbol{\mathcal{G}}^{-1} \tag{111}$$

The last equality results from the comparison to the general result (11) in C.2.

C.7 SOME PROOFS INVOLVING FOURIER SERIES

(a)

We consider the Kronecker delta-relation,

$$\delta_{\boldsymbol{k},\boldsymbol{k}'} \equiv \frac{1}{V} \int \boldsymbol{dx} \exp[i(\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{x}] \tag{112}$$

and the the Fourier expansion,

$$c_{\boldsymbol{x}} = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} c_{\boldsymbol{k}} \exp[i\boldsymbol{k} \cdot \boldsymbol{x}] \tag{113}$$

Proceeding, we want to find uniform Δ, i.e, such that $\Delta = \frac{\int \Delta(\boldsymbol{x}) \boldsymbol{dx}}{V}$, with V the volume, hold. Considering the expansion in the momenta,

$$\Delta = -g \sum_{\boldsymbol{k}\boldsymbol{k}'} \int \boldsymbol{dx} \langle c_{\downarrow \boldsymbol{k}} c_{\uparrow \boldsymbol{k}'} \rangle \frac{1}{V^2} \exp[i(\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{x}] \tag{114}$$

Integrating in $\boldsymbol{x}$ and making continuum the space of momenta

$$\Delta = -g \int V \boldsymbol{dk} V \boldsymbol{dk'} \int \boldsymbol{dx} \langle c_{\downarrow \boldsymbol{k}} c_{\uparrow \boldsymbol{k}'} \rangle \frac{1}{V} \delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{0}} \tag{115}$$

Finally, since $\lim_{V\to\infty} V \delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{0}} = \delta(\boldsymbol{k} + \boldsymbol{k}')$, integrating in $\boldsymbol{k}'$ and returning to the discrete space,

$$\Delta = -g \int \boldsymbol{dk} \langle c_{\downarrow \boldsymbol{k}} c_{\uparrow -\boldsymbol{k}} \rangle = -\frac{g}{V} \sum_{\boldsymbol{k}} \langle c_{\downarrow \boldsymbol{k}} c_{\uparrow -\boldsymbol{k}} \rangle \tag{116}$$

(b)

Another proof we carry out is to consider

$$\boldsymbol{\mathcal{G}}(\boldsymbol{x},\tau,\boldsymbol{x}',\tau') = -\mathcal{T}\langle\Psi_{\boldsymbol{x}}(\tau)\Psi^{\dagger}_{\boldsymbol{x}'}(\tau')\rangle = -\sum_{\boldsymbol{k}\boldsymbol{k}'} \overbrace{\langle\Psi_{\boldsymbol{k}}(\tau)\Psi^{\dagger}_{\boldsymbol{k}'}(\tau')\rangle}^{\propto\delta_{\boldsymbol{k},\boldsymbol{k}'}} \frac{\exp[i\boldsymbol{k}'\boldsymbol{x}'-\boldsymbol{k}\boldsymbol{x}]}{V} =$$

$$= -\sum_{\boldsymbol{k}}\langle\Psi_{\boldsymbol{k}}(\tau)\Psi^{\dagger}_{\boldsymbol{k}}(\tau')\rangle\frac{\exp[i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')]}{V} = -\int \boldsymbol{dk}\langle\Psi_{\boldsymbol{k}}(\tau)\Psi^{\dagger}_{\boldsymbol{k}}(\tau')\rangle\exp[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')]$$

where we considered that the operators defining the Green's function are quasi-operators (they commute with the Hamiltonian).

C.8 ON THE ELASTIC DEFORMATION THEORY.

Consider

$$\Delta V(\boldsymbol{x}) = \boldsymbol{\Phi}(\boldsymbol{x})\cdot\boldsymbol{\Delta S}\ , \tag{117}$$

with $\boldsymbol{\Delta S}$ understood as a small oriented area. This defines the elastic displacement $\boldsymbol{\Phi}$. The only component of the displacement relevant for changing the volume is that carried out parallel (or anti-parallel) to the vector of the oriented surface. It follows

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \lim_{\boldsymbol{h}\to 0}\frac{V(\boldsymbol{x}+\boldsymbol{h})-V(\boldsymbol{x})}{\Delta S} \tag{118}$$

Next, consider a box with a small volume V with the surfaces at $\boldsymbol{x}_{||1}$ and $\boldsymbol{x}_{||2}$, with parallel vectors to the displacement, both with areas ΔS. $\boldsymbol{x}_{\perp 1}$ and $\boldsymbol{x}_{\perp 2}$ are irrelevant for the expansion process.

$$\boldsymbol{\Phi}(\boldsymbol{x}_1) = \lim_{\boldsymbol{h}\to 0}\frac{V(\boldsymbol{x}_1+\boldsymbol{h})-V(\boldsymbol{x}_1)}{\Delta S} \tag{119}$$

$$\boldsymbol{\Phi}(\boldsymbol{x}_2) = -\lim_{\boldsymbol{h}\to 0}\frac{V(\boldsymbol{x}_2+\boldsymbol{h})-V(\boldsymbol{x}_2)}{\Delta S} = \tag{120}$$

The sign difference holds as the normal vector at one of the surfaces points in an opposite direction with respect to the displacement vector, assumed to change continuously. By choosing the system of coordinates such that the displacement is carried out along the x-axis,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\Phi}(\boldsymbol{x})}{\partial x} = \frac{\boldsymbol{\Phi}(\boldsymbol{x}+h_x\hat{\boldsymbol{x}})-\boldsymbol{\Phi}(\boldsymbol{x})}{h_x} \tag{121}$$

Taken special care on the sign,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} &= \lim_{h_x\to 0}\frac{\partial\boldsymbol{\Phi}(\boldsymbol{x})}{\partial x} = \lim_{h_x\to 0}\frac{1}{h_x\Delta S}[(V(\boldsymbol{x}+2h_x\hat{\boldsymbol{x}}) - V(\boldsymbol{x}+h_x\hat{\boldsymbol{x}})) - (-1)[V(\boldsymbol{x}+h_x\hat{\boldsymbol{x}}) - V(\boldsymbol{x})] \\
\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} &= \lim_{h_x\to 0}\frac{\partial\boldsymbol{\Phi}(\boldsymbol{x})}{\partial x} = \lim_{h_x\to 0}\frac{1}{h_x\Delta S}[V(\boldsymbol{x}+2h_x\hat{\boldsymbol{x}}) - V(\boldsymbol{x})] = \lim_{V\to 0}\frac{1}{V}[\Delta V(\boldsymbol{x}) + \cancel{h_x\partial_x V(\boldsymbol{x})}]\ ,
\end{aligned} \tag{122}$$

where the higher-order term in h_x is neglected.

APPENDIX D – FOR THE CHAPTER 5

D.1 ON THE ANALYTICITY OF THE ORDER PARAMETER

When the magnetic field is turned on, it penetrates the material introducing vortex solutions containing singularity. As we are about to see in chapter 6, many solutions are mostly smooth and well behaved, only possibly having its boundary made up of vortices. This allows for the consideration of an order parameter that can be analytically continued once it does not trespass a vortex singularity.

D.2 ON THE EVALUATION OF INTEGRALS

Considering the terms linked to $\nabla^2\Delta^*$ in the GL level,

$$\begin{aligned} a_2 &= -N(0)T\frac{\hbar^2}{2m}\lim_{\omega_D\to\infty}\sum_{\omega}\int_{-\omega_D/2}^{\omega_D/2}\frac{\frac{4}{3}\varepsilon_F(\omega^4-\varepsilon^4)}{(\omega^2+\varepsilon^2)^4} \\ &= N(0)T\frac{\hbar^2}{2m}\sum_{\omega}\frac{\pi\mu_F}{3|\omega|^3} \end{aligned} \tag{1}$$

We may remove the odd parity terms in ω and ε in instantly; these vanish in a symmetric interval. The sum is on the fermionic Matsubara frequencies, $\omega = (2n+1)\pi T$,

$$a_2 = N(0)T\frac{\hbar^2}{2m}\frac{\varepsilon_F}{3\pi^2T^3}\underbrace{\sum_{n}\frac{1}{|2n+1|^3}}_{\frac{7\zeta(3)}{4}} = N(0)\frac{\hbar^2}{2m}\frac{7\varepsilon_F\zeta(3)}{12\pi^2T^2} = \frac{1}{6}N(0)\frac{7\zeta(3)}{8\pi^2T^2}(\frac{\hbar^2}{m})^2\hbar^2k_F^2 \tag{2}$$

As

$$\frac{1}{T^2} = \frac{1}{T_c^2}(\frac{T_c}{T})^2 = \frac{1}{T_c^2}(\frac{1}{1-\tau})^2 = \frac{1}{T_c^2}(1+2\tau+O(\tau^{\geq 2}))\ ,\ m = \frac{p_F}{v_F} = \frac{\hbar k_F}{v_F}\ ,\ W_3^2 \equiv \frac{7\zeta(3)}{8\pi^2T_c^2}\ , \tag{3}$$

it follows

$$a_2 = \frac{1}{6}N(0)W_3^2\hbar^2v_F^2(1+2\tau) \tag{4}$$

which agrees with the GL theory written in terms of the microscopic parameters in different derivations. As for the terms linked to Δ^* on the right-hand-side, it is the only term which requires some extra care for the convergence. We have to consider a small parameter for the convergence to hold,

$$a_1 = \sum_{\omega}N(0)T\int_{-\omega_D/2}^{\omega_D/2}d\varepsilon\frac{1}{\omega^2+\varepsilon^2} = N(0)T\int\frac{1}{2\varepsilon}\sum_{\omega}\exp[i\omega\eta]\Big(\frac{1}{\varepsilon+i\omega}+\frac{1}{\varepsilon-i\omega}\Big) \tag{5}$$

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$$a_1 = \sum_\omega N(0)T\int_{-\omega_D/2}^{\omega_D/2}d\varepsilon\frac{1}{\omega^2+\varepsilon^2} = N(0)T\int\frac{1}{2\varepsilon}\sum_\omega\exp[i\omega\eta]\Big(\frac{1}{\varepsilon+i\omega}+\frac{1}{\varepsilon-i\omega}\Big) \tag{5}$$

By applying the contour method established in chapter two we reach

$$a_1 = N(0)\int_{-\omega_D/2}^{\omega_D/2}\frac{d\varepsilon}{2\varepsilon}\tanh\left(\frac{\beta\varepsilon}{2}\right) = N(0)\int_0^{\omega_D/2} d\varepsilon\frac{1}{\varepsilon}\tanh\left(\frac{\beta\varepsilon}{2}\right) = N(0)\ln\left(\frac{2e^{\gamma}\omega_D}{\pi T}\right) \ , \quad (6)$$

γ is the Euler number. The linear coefficient in Δ^* must go to zero (GL theory),

$$(a_1 - \frac{1}{g})|_{T\to T_c} \to 0 \ . \quad (7)$$

The product $gN(0) \equiv \lambda$ is dimensionless, thus,

$$T_c = \frac{e^{\gamma}}{\pi}2\omega_D \exp\left[-\frac{1}{\lambda}\right] \quad (8)$$

With this, the dependence on Δ^* for a temperature different than zero may be recast with the linear term written in terms of an expansion in terms of τ,

$$\begin{aligned}\Delta^* &= [\lambda\log[1-\tau]^{-1}\Delta^* + 1]\Delta^* + \text{Other GL terms}\\ &\sim [\lambda(\tau + \frac{1}{2}\tau^2 + ...) + 1]\Delta^* + \text{Other GL terms}\end{aligned} \quad (9)$$

The term with Δ^* alone cancels; as a result, there is not a dependence of the form $\tau^{1/2}$ in the GL equation - it captures only the order $\tau^{3/2}$. The other integrals/sums follow the spirit of the a_2 calculation and are solved by employing a symbolic software.

D.3 ON THE STABILITY OF 1D SOLUTIONS

First we prove the way of rewriting g_1 is consistent.

$$g_1^{(1)} = 2\Big[(\Delta_0^2 - 1)\Delta_0\Delta_1 + \frac{A_0^2}{2}\Delta_0\Delta_1 + \frac{1}{\kappa^2}\Delta_0'\Delta_1'\Big] - A_1 i_0 + (A_0' - 1)\Big(A_1' + \frac{1}{2} + \tilde{c}\Big) \quad (10)$$

Due to an integration by parts and the subsequent use of the first GL equation,

$$\sigma_1^{(1)} = \frac{1}{\kappa^2}\Big[\Delta_0'\Delta_1\Big]|_{-\infty}^{\infty} + \int\Big[-A_1 i_0 + (A_0' - 1)(A_1' + \frac{1}{2} + \tilde{c})\Big] \quad (11)$$

According to the second GL equation

$$i_0 = -A_0'' \quad (12)$$

An integration by parts yields

$$\sigma_1^{(1)} = \frac{1}{\kappa^2}\Big[\Delta_0'\Delta_1\Big]|_{-\infty}^{\infty} + \Big[A_0'A_1\Big]|_{-\infty}^{\infty} + \int(A_0' - 1)(\frac{1}{2} + \tilde{c})$$

We identify the derivative of the GL free functional

$$\frac{\partial\sigma_0}{\partial\kappa_0}=\frac{2}{\kappa_0^3}\int dx\Delta_0\Delta_0''=4\sqrt{2}\int dx\Big[(\Delta_0\Delta_0')'-\Delta_0'^2\Big]=4\sqrt{2}\Big[\Delta_0\Delta_0'\Big]|_{-\infty}^{\infty}-4\sqrt{2}\int\Delta_0'^2dx$$

and apply the first and second Bogomol'nyi equations

$$\int\Delta_0'^2dx=\int\Delta_0'(-\frac{1}{2}A_0\Delta_0)=-\int dx\frac{1}{4}(\Delta_0^2)'A_0=-\frac{1}{4}\Big[\Delta_0^2A_0\Big]_{-\infty}^{\infty}+\frac{1}{4}\int dx\Delta_0^2(1-\Delta_0^2)$$

Thus,

$$\frac{\partial\sigma_0}{\partial\kappa_0}=4\sqrt{2}[\Delta_0\Delta_0'+\frac{1}{4}\Delta_0^2A_0]|_{-\infty}^{\infty}-\sqrt{2}\mathcal{I}\tag{13}$$

As for the contribution

$$\sigma_1(\kappa_0)=\int dx\Big[g_1^{(0)}(\kappa_0)+g_1^{(1)}(\kappa_0)\Big]\tag{14}$$

we divide the relevant integrals as those attached to the dimensionless coefficients (and its absence)

$$I=\int dx\Big[-\frac{\Delta_0^2}{2}+2\frac{\Delta_0'}{\kappa_0^2}+A_0^2\Delta_0^2+\Delta_0^4+\frac{1}{2}(A_0'-1)\Big]\tag{15}$$

$$\frac{II}{\tilde{\mathcal{Q}}}=\int dx\Big[\Big(\frac{\Delta_0''}{\kappa_0^2}-\frac{A_0^2}{2}\Delta_0\Big)+\frac{1}{3\kappa_0^2}A_0''A_0\Delta_0^2+\frac{A_0'^2}{2\kappa_0^2}\Delta_0^2\Big]\tag{16}$$

$$\frac{III}{\tilde{c}}=\int dx\Big[(A_0'-1)+\Delta_0^6\Big]\tag{17}$$

$$\frac{IV}{\tilde{\mathcal{L}}}=\int dx\Big[\frac{5}{\kappa_0^2}(\Delta_0')^2\Delta_0^2+\frac{3}{2}A_0^2\Delta_0^4\Big]\tag{18}$$

For the first, we use the second Bogomol'nyi relation

$$I=\int dx\Big[-\frac{\Delta_0^2}{2}+\frac{2}{\kappa_0^2}(\Delta_0'\Delta_0)'-\underbrace{\frac{2}{\kappa_0^2}\Delta_0''\Delta_0}_{2\Delta_0^2(1-\Delta_0^2)-A_0^2\Delta_0})+A_0^2\Delta_0^2+\Delta_0^4-\frac{\Delta_0^2}{2}\Big]\tag{19}$$

where the first GL equation has been employed in the underbrace. Hence,

$$I=\Big[\frac{2}{\kappa_0^2}\Delta_0'\Delta_0\Big]|_{-\infty}^{\infty}+\int dx\Delta_0^2(1-\Delta_0^2)\tag{20}$$

As for the second integral, the first term in parenthesis is reduced immediately due to the second GL equation

$$\frac{II}{\tilde{\mathcal{Q}}}=\int dx\Delta_0^2(\Delta_0^2-1)^2+\frac{II_{\mathrm{aux}}}{\tilde{\mathcal{Q}}}\tag{21}$$

such that the second integral accounts for the remaining terms.

$$\begin{aligned}
\frac{II_{\rm aux}}{\tilde{\mathcal{Q}}} &= \frac{2}{3}\int \Delta_0^2[(A_0'A_0)' - A_0'^2]dx + \int dx\Delta_0^2(A_0')^2 dx = \int dx\frac{2}{3}\Delta_0^2(A_0'A_0)' + \frac{1}{3}\int dx A_0'^2\Delta_0^2 \\
&= \int dx\frac{2}{3}\Delta_0^2((1-\Delta_0^2)A_0)' + \frac{1}{3}\int dx\Delta_0^2(1-\Delta_0^2)^2 \\
&= \int dx\Delta_0^2(1-\Delta_0^2)^2 + \int dx\frac{2}{3}\Delta_0^2 A_0(-2\Delta_0'\Delta_0) \qquad (22)
\end{aligned}$$

Hence, as a result of identifying

$$\int dx\Delta_0^2(\Delta_0^2 - 1)^2 = \mathcal{I} - \mathcal{J}\ , \qquad (23)$$

it follows

$$\begin{aligned}
\frac{II}{\tilde{\mathcal{Q}}} &= 2(\mathcal{I}-\mathcal{J}) - \frac{1}{3}\int dx(\Delta_0^4)'A_0 = 2(\mathcal{I}-\mathcal{J}) - \frac{1}{3}\Big[\Delta_0^4 A_0\Big]|_{-\infty}^{\infty} + \frac{1}{3}\mathcal{J} \\
\frac{II}{\tilde{\mathcal{Q}}} &= 2\mathcal{I} - \frac{5}{3}\mathcal{J} - \frac{1}{3}\Big[\Delta_0^4 A_0\Big]|_{-\infty}^{\infty} \qquad (24)
\end{aligned}$$

The third is quite easy to identify, from the second Bogomol'nyi relation as

$$\frac{III}{\tilde{c}} = \int dx\Delta_0^2(\Delta_0^4 - 1) = \mathcal{I} + \mathcal{J} \qquad (25)$$

As for the final identity, we use the first-Bogomol'nyi equation read from the left to the right in the first term and from the right to the right in the second term. In the end, we also apply the second Bogomol'nyi equation.

$$\begin{aligned}
\frac{IV}{\tilde{\mathcal{L}}} &= \int dx\Big[\frac{5}{\kappa_0^2}(\Delta_0')^2 + \frac{3}{2}A_0^2\Delta_0^2\Big]\Delta_0^2 = \int dx\Big[\frac{5}{\kappa_0^2}\Delta_0'(-\frac{1}{2}A_0\Delta_0) + \frac{3}{2}A_0\Delta_0(-2\Delta_0')\Big]\Delta_0^2 \\
\frac{IV}{\tilde{\mathcal{L}}} &= -2\int dx A_0(\Delta_0^4)' = -2\Big[A_0\Delta_0^4\Big]|_{-\infty}^{\infty} + 2\int dx A_0'\Delta_0^4 \\
\frac{IV}{\tilde{\mathcal{L}}} &= -2\Big[A_0\Delta_0^4\Big]|_{-\infty}^{\infty} + 2\mathcal{J} \qquad (26)
\end{aligned}$$

Applying the bulk properties, the surface terms vanish.

D4. PROOF OF IDENTITY

We wish to prove that

$$\frac{\Delta^*(x)}{g} = -\frac{N(0)}{\beta}\sum_\omega \int \tilde{\mathcal{F}}_\omega(\varepsilon)d\varepsilon\ . \qquad (27)$$

For a proof,

$$\tilde{\mathcal{F}}(\boldsymbol{x},\tau,\boldsymbol{x}',\tau') = \frac{1}{\beta}\sum_\omega \tilde{\mathcal{F}}_\omega(\boldsymbol{x},\boldsymbol{x}')e^{-i\omega(\tau-\tau')} \qquad (28)$$

$$-\frac{\Delta^*}{g} = \lim_{\tau-\tau'\to 0^+,\boldsymbol{x}\to\boldsymbol{x}'} \tilde{\mathcal{F}}(\boldsymbol{x},\tau,\boldsymbol{x}',\tau') = \frac{1}{\beta}\sum_\omega \lim_{\boldsymbol{x}\to\boldsymbol{x}'}\int \boldsymbol{dk}\tilde{\mathcal{F}}_\omega(\boldsymbol{k})\exp[i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')] \qquad (29)$$

Finally, in the vicinity of the Fermi-surface, the density of states is $N(0)$, therefore,

$$\frac{\Delta^*}{g} = -\frac{1}{\beta}\sum_{\omega}\lim_{\boldsymbol{x}\to\boldsymbol{x'}}\int \boldsymbol{dk}\tilde{\mathcal{F}}_{\omega}(\boldsymbol{k}) = -\frac{N(0)}{\beta}\sum_{\omega}\int d\varepsilon\tilde{\mathcal{F}}_{\omega}(\varepsilon) \tag{30}$$

D5. PROOF OF INTERACTION TERM FORM

Applying the Kronecker-delta relation (Appendix C.7),

$$H_I = \sum_{\alpha}\frac{1}{\alpha!}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\Delta(\boldsymbol{x})\int V\boldsymbol{dk}V\boldsymbol{dk'}e^{i(\boldsymbol{k'}+\boldsymbol{k})\cdot\boldsymbol{x}}c_{\boldsymbol{k'}\downarrow}\left[\prod_{j=1}^{3}(\frac{\partial_{k_j}}{i})^{\alpha_j}\right]\delta_{\boldsymbol{k}+\boldsymbol{k'},\boldsymbol{0}}c_{\boldsymbol{k}\uparrow} + \text{H.C} \tag{31}$$

where the derivative operator acts on the Kronecker-delta alone. Since $\lim_{V\to\infty}V\delta(\boldsymbol{k},\boldsymbol{k'}) = \delta(\boldsymbol{k}-\boldsymbol{k'})$, and that the derivative acts on the terms dependent on $\boldsymbol{k}$ alone (the derivative is partial),

$$H_I = \sum_{\alpha}\frac{1}{\alpha!}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\Delta(\boldsymbol{x})\int V\boldsymbol{dk}\boldsymbol{dk'}\left[\prod_{j=1}^{3}(\frac{\partial_{k_j}}{i})^{\alpha_j}\right][c_{\boldsymbol{k'}\downarrow}\delta(\boldsymbol{k}+\boldsymbol{k'})]e^{i(\boldsymbol{k'}+\boldsymbol{k})\cdot\boldsymbol{x}}c_{\boldsymbol{k}\uparrow} + \text{H.C} \tag{32}$$

By integrating by parts, we eliminate the many surfaces terms appearing, and we are left with

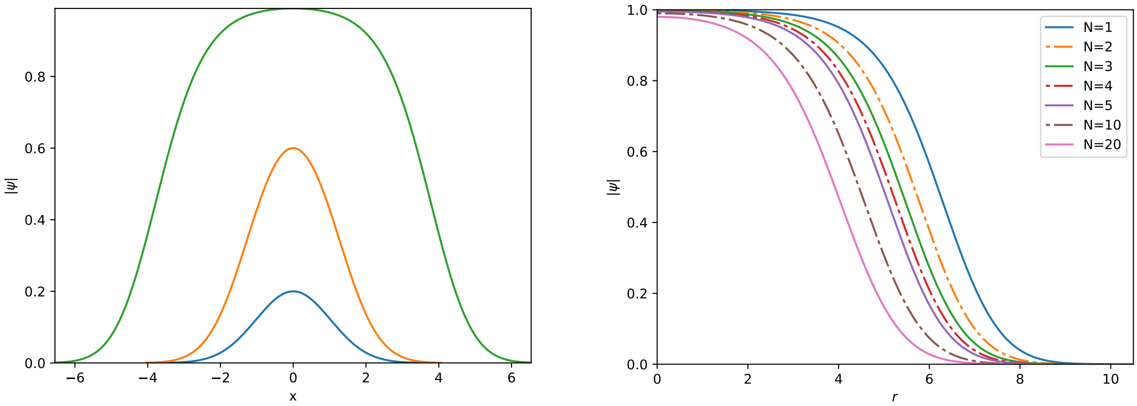
$$H_I = \sum_{\alpha}\frac{1}{\alpha!}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\Delta^*(\boldsymbol{x})\int V\boldsymbol{dk}\boldsymbol{dk'}c_{\boldsymbol{k'}\downarrow}\delta(\boldsymbol{k}+\boldsymbol{k'})\left[(-1)^{n_\alpha}\prod_{j=1}^{3}(\frac{\partial_{k_j}}{i})^{\alpha_j}\right][e^{i(\boldsymbol{k'}+\boldsymbol{k})\cdot\boldsymbol{x}}c_{\boldsymbol{k}\uparrow}] \\ +\text{H.C}\ , \tag{33}$$

where we recall, each integration by parts produces a sign, and there is $n_\alpha = \sum_j \alpha_j$ integration by parts. The surface terms, contains terms responsible for the destruction of Cooper-pairs - the complex conjugate contains the terms responsible for the creation. Once we request these to vanish further away from the bulk, we have a reliable expression. Integrating in the $\boldsymbol{k'}$ variable. We return to the discretized expression, $\boldsymbol{k'} = -\boldsymbol{k}$, the dependence on V being naturally removed.

$$H_I = \sum_{\alpha}\frac{1}{\alpha!}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\Delta^*(\boldsymbol{x})\sum_{\boldsymbol{k}}c_{-\boldsymbol{k}\downarrow}^{\dagger}\left[(-1)^{n_\alpha}\prod_{j=1}^{3}(\frac{\partial_{k_j}}{i})^{\alpha_j}\right][c_{\boldsymbol{k}\uparrow}] + \text{H.C}\ . \tag{34}$$

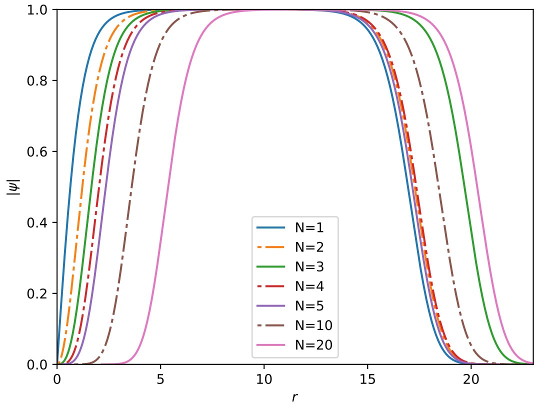
APPENDIX E – FOR THE CHAPTER 6

E.1 EXAMPLE SOLUTIONS



(a) Stripe

(b) Bubble



(c) Donut

Source: The author

E.2 CURRENT OF SOLUTIONS

The current behave as

$$\vec{i} = \frac{1}{\mathcal{K}} \nabla \times \vec{B} = (-\partial_y B, \partial_x B, 0) \tag{1}$$

For the bidimensional defects the second Bogomolnyi yields

$$\vec{i} = 2\mathcal{K}\phi_N(\rho)\phi'_N(\rho)(-\sin\theta, \cos\theta, 0) \tag{2}$$

In case of vortex, $\phi'_N(\rho) > 0$, for bubbles, $\phi'_N(\rho) < 0$, the current running in different direction. For donuts there is a critical point where ϕ'_N changes sign from a positive

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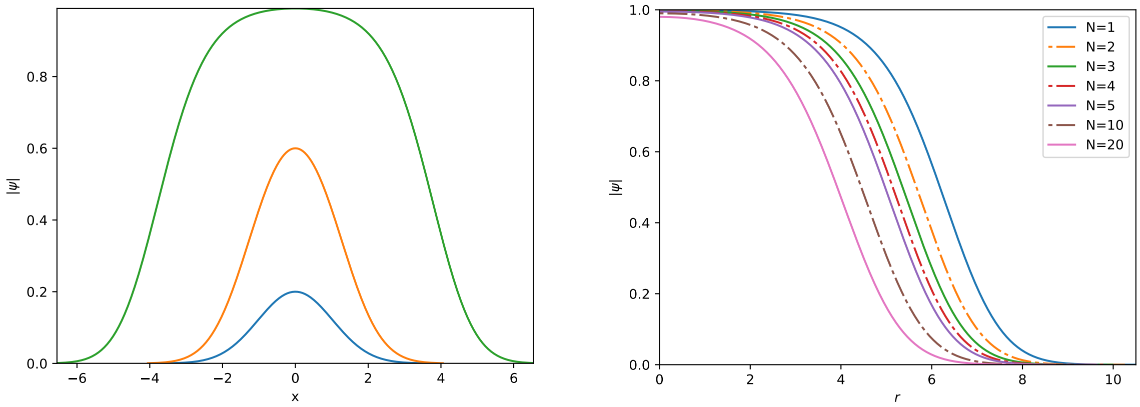
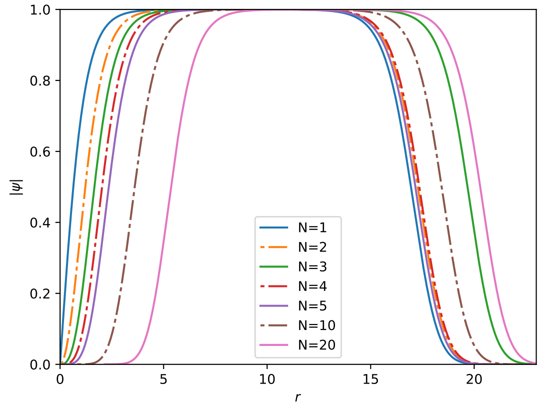


(a) Stripe (b) Bubble



(c) Donut

Source: The author

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The current behave as

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For the bidimensional defects the second Bogomolnyi yields

$$\vec{i} = 2\mathcal{K} \phi_N(\rho) \phi'_N(\rho)(-\sin\theta, \cos\theta, 0) \tag{2}$$

In case of vortex, $\phi'_N(\rho) > 0$, for bubbles, $\phi'_N(\rho) < 0$, the current running in different direction. For donuts there is a critical point where ϕ'_N changes sign from a positive

to negative inclination, implying the current to be like vortex in the core, zero at some distance and change orientation (behave like bubble) from this distance on.
For filament solutions, $\partial_y B = 0$, and due to each of the branches and the second Bogomol'nyi relation,

$$\vec{B} = (0, \pm 2\Psi^2\sqrt{\Psi^2 - 2\ln[c|\Psi|]}, 0)$$

which displays the behaviour of the current to reverse in direction and points along the stripe orientation. It is interesting to notice that the sense of the current change differently if we choose either the filament to be on either the x-axis of y-axis. For the y-axis choice,

$$\vec{B} = (\mp 2\Psi^2\sqrt{\Psi^2 - 2\ln[c|\Psi|]}, 0, 0)$$

. One may think of stripes as the superposition of an infinity array of vortex or bubbles depending on the orientation of the current.

E.2 ABSCENCE OF BUBBLE VORTICITY

We provide a proof by contradiction. We consider the following ansatz

$$\Psi = (\frac{z}{|z|^2})^N|\Psi| \tag{3}$$

including the vorticity is valid for bubbles.

The modified Liouville introduces the Dirac deltas. Consider the general Bogomolnyi equation in the form of the modified Liouville equation,

$$\partial_x^2 \ln|\Psi| + \partial_y^2 \ln|\Psi| = |\Psi|^2 - 1 - 2\pi N\delta(x)\delta(y) \tag{4}$$

As usual it is assumed the continuity of Ψ in both x and y, but the derivative is allowed to jump. For instance, integrating the modified Liouville equation around $\boldsymbol{x} = 0$,

$$\frac{|\Psi|_x(0^+, y) - |\Psi|_x(0^-, y)}{\Psi(0, y)} = -2\pi N\delta(y) \tag{5}$$

Proceeding to integrate this equation in between a symmetric y interval, $y \in (-\epsilon, \epsilon)$,

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dy[|\Psi|_x(0^+, y) - |\Psi|_x(0^-, y)] = -2\pi N\Psi(0,0) \neq 0 \text{ for a bubble} \tag{6}$$

However, for a bubble, due to the symmetry of the solution, $|\Psi|_x(0^+, y) = |\Psi|_x(0^-, y)$, causing one side of the equation to differ from the other.