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CANONICAL QUANTIZATION OF GENERAL RELATIVITY WITH APPLICATION TO THE SCHWARZSCHILD BLACK HOLE

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ABSTRACT

This dissertation aims to discuss the canonical quantization of general relativity and apply it to the Schwarzschild black hole, and thus it can be divided into two main parts. To implement this quantization process, it is essential to obtain a Hamiltonian for the gravitational field, and a variational formulation of general relativity becomes necessary. With the Hamiltonian in hand, we can define the mass of spacetime and, with it, the gravitational ADM action. The system constraints are obtained by using the Dirac–Bergmann algorithm, and the quantization then proceeds in the usual way, changing the nature of the canonical variables by promoting them to operators. In the quantum theory, such constraints become conditions on the state vector of the system, whose wave function satisfies the Wheeler–DeWitt equation. In the case of the Schwarzschild black hole, we only have the degree of freedom given by its mass. Therefore, we are dealing with an effectively one-dimensional system whose wave function is a linear combination of confluent hypergeometric functions and whose mass spectrum derives from the appropriate boundary condition. In this scenario, the transition between states is responsible for the emission of Hawking radiation, and the temperature of the black hole is obtained through the Stefan–Boltzmann law.

Keywords: Hamiltonian formalism; canonical quantization; general relativity; Schwarzschild black hole.

RESUMO

Esta dissertação tem por objetivo discutir a quantização canônica da relatividade geral e aplica-la ao buraco negro de Schwarzschild, podendo assim ser dividida em duas partes principais. Para implementarmos esse processo de quantização, é essencial obtermos um Hamiltoniano para o campo gravitacional e uma formulação variacional da relatividade geral se faz necessária. Com o Hamiltoniano em mãos, somos capazes de definir a massa do espaço-tempo e com isso a ação gravitacional ADM. Os vínculos do sistema são obtidos usando-se o algoritmo de Dirac-Bergmann e a quantização então procede de maneira usual, mudando-se a natureza das variáveis canônicas ao promovê-las a operadores. Na teoria quântica, tais vínculos se tornam condições sobre o vetor de estado do sistema, cuja função de onda satisfaz a equação de Wheeler–DeWitt. No caso do buraco negro de Schwarzschild temos apenas o grau de liberdade dado pela sua massa. Sendo assim, estamos lidando com um sistema efetivamente unidimensional cuja função de onda é uma combinação linear de funções hipergeométricas confluentes e cujo espectro de massa decorre da condição de contorno apropriada. Nesse cenário, a transição entre estados é responsável pela emissão de radiação Hawking e a temperatura do buraco negro é obtida através da lei de Stefan-Boltzmann.

Palavras-chaves: Formalismo Hamiltoniano; quantização canônica; relatividade geral; buraco negro de Schwarzschild.

CONTENTS

1 INTRODUCTION

As living beings that belong to a specific range on the scale spectrum of physical quantities, all our senses have been shaped throughout our evolutionary history to capture and process phenomena within this range of magnitudes, thus subjugating our perceptions and intuitions about the world we live in. Therefore, in this context, it is natural for the limits of applicability of the theories we developed to be contained in this range of scales since we first attempt to understand the phenomena with which we have direct contact, the phenomena we experience in our daily lives. Quantum mechanics (QM) and general relativity (GR) arise as two results of our endeavor to expand the limits of applicability of our theories across the scale spectrum. However, despite the universal character of both theories, QM and GR, they find relevance in regions of the spectrum that are quite distant from each other. How wonderful it would be if there existed objects whose full description requires the merge of the methods and tools of quantum mechanics with those of general relativity.

In October 1914, Albert Einstein presented to the Prussian Academy of Sciences his paper entitled *The Formal Foundations of the General Theory of Relativity*. However, at this stage, the field equations of the theory lacked general covariance and did not reproduce the correct value of $43''$ per century for the perihelion advance of Mercury's orbit. On November 25, 1915, Einstein finally came across the correct value for the perihelion advance in a paper entitled *Explanation of the Perihelion Motion of Mercury from General Relativity Theory*. In this paper, Einstein obtained the perihelion advance formula from an equation of motion derived from approximate field equations for the gravitational field of a point mass. Surprisingly, on December 22, less than one month after his paper, Einstein received a letter sent by the German physicist and astronomer Karl Schwarzschild, where he presented an exact solution to the problem of a point mass. Schwarzschild published his solution on February 3, 1916, under the name *On the Gravitational Field of a Point Mass, according to Einstein's Theory*, and died of pemphigus on 11 May of that same year. The uniqueness of the Schwarzschild metric as the exact vacuum solution to the Einstein field equations for a static, spherically symmetric, asymptotically flat spacetime was proved after 51 years by Werner Israel (ISRAEL, 1967). We know that divergences in general indicate points where the theory no longer works. Schwarzschild's work revealed the possibility of the existence of singularities, bringing up objects that lie outside the limits of general relativity.

Stephen Hawking's studies of the behavior of quantum fields near the black hole's event horizon led him to publish an article (HAWKING, 1974) in March 1974 communicating that black holes can create and emit particles, radiating as if they were hot bodies with a temperature proportional to its inverse mass. This thermal radiation gradually reduces the mass of the black hole until it finally disappears, leaving us with the question of what happens with all information that fell into the hole during its lifetime. In August 1975, Hawking published a paper (HAWKING, 1975) detailing this evaporation mechanism. Hawking's work shed some light on the quantum aspects related to black holes, showing that we can not fully understand these objects without a proper quantum formulation of general relativity.

Chapter 2 develops the Hamiltonian formalism for general relativity, and we closely follow Ref. (POISSON, 2004). We start by introducing the notation used throughout the dissertation by means of a quick discussion of some basic concepts in GR, followed by a derivation of the Gauss–Codazzi equations. Once we have set the stage, we are free to talk about the variational principle formulation of GR. At the level of the Lagrangian, the Hamilton variational principle yields the standard form of the Einstein field equations (EFEs). Our interest in passing to the Hamiltonian formulation is not in getting the Hamiltonian version of the field equations but in the Hamiltonian itself, which formally defines what we might mean by the total energy of spacetime. Our approach consists of thought of an arbitrary region $\mathscr{V}$ of spacetime as the final picture of the time evolution of an arbitrary spacelike hypersurface Σ. Thus we decompose the spacetime into a foliation where each leaf is a hypersurface representing the gravitational field configuration at that specific time. After we dismembered $\mathscr{V}$ and its boundary, $\partial\mathscr{V}$, we adapt the gravitational action to this decomposition and then define the field velocity and its canonical conjugate momentum in order to introduce the gravitational Hamiltonian. For vacuum solutions of the EFEs, the Gauss–Codazzi equations manage to vanish the bulk part of the Hamiltonian, which becomes a pure boundary term. This Hamiltonian defines the energy of spacetime when we push the boundary $\partial\mathscr{V}$ all way to an asymptotic flat infinity and specify the asymptotic values of the lapse function and the shift vector.

In chapter 3, we use the Dirac–Bergmann algorithm to extract all the constraints of the theory. The so-called primary constraints arise from the definition of the momentum variables, thus being consequences merely of the form of the Lagrangian. The consistency conditions represent the requirement that these constraints be constant over time: the time derivative of the constraints must be zero. By using the Poisson brackets to express time evolution, these conditions lead to more constraints, called secondary constraints. The consistency conditions of these new constraints are identically satisfied, and the algorithm ends here, totaling eight constraints for each point in space. To proceed with the canonical quantization is simple: we map all the canonical variables into Hermitian operators and take the constraints to be conditioned on the state kets of spacetime. We obtain a quantum mechanical equation for general relativity, the Wheeler–DeWitt equation, by choosing the configuration space representation for these state kets. This equation is supposed to be the dynamical equation of quantum spacetime, but there is no time variable. In addition, this equation suffers from a massive operator ordering problem, yielding divergences.

Chapter 4 is devoted to applying the previous discussion to the Schwarzschild black hole (SBH). Here we start from a spherically symmetric spacelike hypersurface Σ and then build up a spherical spacetime $\mathcal{M}$ utilizing a time parameterization. The evolution of Σ is dictated by the ADM action, from which the constraints follow. Due to the spherical symmetry, there are four constraints instead of eight, and through a judicious canonical transformation, followed by elimination of the constraints, we bring the action to the reduced form, where the reduced Hamiltonian is simply the Schwarzschild mass, which is the only degree of freedom of $\mathcal{M}$. Before proceeding with the quantization, we do one more canonical transformation and use the Schwarzschild wormhole throat and its conjugate momentum as our canonical variables. The quantization is done straightforwardly by promoting the canonical variables to Hermitian operators. The Wheeler–DeWitt equation is the equation of the one-dimensional harmonic oscillator with position-dependent angular frequency. Its general solution is a linear combination of confluent hypergeometric functions with an overall Gaussian exponential factor. We restrict ourselves to black holes whose masses are much larger than Planck's mass, which allows us to explore the asymptotic behavior of the confluent hypergeometric functions and obtain a semi-classical approximation for the wave function of the SBH. There are two boundary conditions to this wave function, and from one of them, we get the hole's mass spectrum. This spectrum and the assumption of spontaneous emission and the Stefan–Boltzmann law constitute the basis for the quantized SBH thermodynamics described here. The idea is not to follow Hawking's steps to get thermodynamics out of the black hole but to see how far the canonical quantization can take us. We trace an independent path here, and we refer to the Hawking temperature and the Bekenstein–Hawking entropy to visualize our results as corrections to it.

2 HAMILTONIAN FORMULATION OF GENERAL RELATIVITY

2.1 PRELIMINARIES

2.1.1 Some basic concepts in general relativity

Consider an arbitrary region $\mathscr{V}$ of spacetime, a four- dimensional manifold $\mathcal{M}$ that represents the structure of our universe, and its boundary $\partial\mathscr{V}$, a three- dimensional submanifold. We will refer to $\partial\mathscr{V}$ as a hypersurface. To be able to locate points in $\mathscr{V}$, we install an atlas $\{(V_i, x_i^\alpha),\, i \in \mathbb{N}\}$, where V_i is a neighborhood around a point $P \in \mathscr{V}$ and x_i^α are the coordinates of this point (Greek indices for quantities defined in the spacetime bulk), while on the hypersurface $\partial\mathscr{V}$ we place an atlas $\{(U_j, y_j^a),\, j \in \mathbb{N}\}$ (Lower-case Latin indices for quantities defined only on the hypersurface). For simplicity, we will refer to these atlases as the spacetime coordinates x^α and the intrinsic coordinates y^a of the hypersurface, respectively. The coordinate system x^α must overlap y^a. Therefore, there are points in spacetime that can be represented by both coordinate systems, x^α and y^a, which means that there must exist a reversible transformation $x^\alpha = x^\alpha(y^a)$.

We let $g_{\alpha\beta}$ denote the Lorentzian metric in $\mathscr{V}$ with signature $(-,+,+,+)$. The metric evaluated at a certain point only gives us access to an infinitesimal distance around this point. This distance is known as the interval and its square $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ is usually referred to as the metric since $g_{\alpha\beta}$ can be read directly from it. On the left-hand side, we see an invariant line element $\mathrm{d}s$, a proper distance about which all observers must agree. At the same time, on the right-hand side, there is an explicit coordinate system x^α, which is completely arbitrary. The metric $g_{\alpha\beta}$ is then what translates arbitrariness of coordinate choice to invariant distances. If we want to measure larger distances, we have to integrate the interval along the straightest possible curve $\gamma(\lambda)$, where $\lambda \in (a,b)$ is a real parameter, connecting the desired two points, A and B for instance, which means that we must know the metric at all points of the curve since the shape of the coordinate grid might change from one point to the other:

$$\Delta s = \int_A^B \mathrm{d}s = \int_a^b \sqrt{g_{\alpha\beta}v^\alpha v^\beta}\,\mathrm{d}\lambda, \tag{2.1}$$

where $v^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$ is the α-th component of the tangent vector field $v = \mathrm{d}/\mathrm{d}\lambda$ to the curve in the vector basis $\{e_\alpha \coloneqq \partial/\partial x^\alpha\}$.

To know how the coordinate grid changes in each direction, we decompose the covariant derivative of the basis vectors e_α:

$$\nabla_{e_\beta} e_\alpha = \Gamma^\mu{}_{\alpha\beta} e_\mu, \tag{2.2}$$

where the components $\Gamma^\mu{}_{\alpha\beta}$ are called the connection components. Since the covariant derivative operator obeys the product rule and its action on a function is simply the function's usual derivative, we have

$$\nabla_{e_\beta} v = \nabla_{e_\beta}(v^\alpha e_\alpha) = \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\mu\beta} v^\mu\right) e_\alpha = \nabla_{e_\beta} v^\alpha e_\alpha, \tag{2.3}$$

where $\nabla_{e_\beta} v^\alpha := \partial_\beta v^\alpha + \Gamma^\alpha{}_{\mu\beta} v^\mu$ merely denotes the α-th component of $\nabla_{e_\beta} v$ and should not be confused with the covariant derivative of the component v^α, which is a function, of the vector v. We have made such a definition because we intend to represent tensors through their components without having to write down the basis vectors all the time.

The connection is then the quantity (non-tensorial) that contains the information about how the grid changes. If we require the metric to be covariantly constant, $\nabla_\gamma g_{\alpha\beta} = 0$ (the inner product between two vectors remains constant under their parallel transport along any curve), and the connection to be symmetric, $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ (no antisymmetric torsion tensor), then $\Gamma^\mu{}_{\alpha\beta}$ can be completely determined by the metric through the following expression:

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right), \tag{2.4}$$

in which case $\Gamma^\mu{}_{\alpha\beta}$ is called the Christoffel symbol.

If a given vector field u does not change along a curve γ with tangent vector field v, then we write $\nabla_v u = 0$ and say that u is parallel transported along γ. If v itself is parallel transported along γ, then the curve γ is called a geodesic and we have $\nabla_v v = v^\alpha \nabla_\alpha v = 0$, the geodesic equation. Objects are represented by worldlines in spacetime. A worldline is a curve parameterized by the object's proper time τ, which is the time that ticks on the clock attached to the object's reference frame. If there is no net force acting upon the object, then the worldline is a curve without acceleration, or in other words, a geodesic. Therefore, it is the global behavior of geodesics that indicates the presence of curvature, also known as gravity, in spacetime.

The concept of curvature can be precisely defined using the parallel transport of our basis vectors e_α from a point P to a point Q. Firstly, we choose a basis vector $e_\beta(P)$ and two coordinates to transport the vector, say x^μ and x^ν. Transporting $e_\beta(P)$ first along x^μ then x^ν and secondly along x^ν then x^μ, yields two images of $e_\beta(P)$, namely $e^T_\beta(Q)_{\nu\mu}$ and

$e^T_\beta(Q)_{\mu\nu}$, respectively. We then compute the difference between these two vectors, $R_{\beta\mu\nu} \coloneqq e^T_\beta(Q)_{\mu\nu} - e^T_\beta(Q)_{\nu\mu}$:

$$R_{\beta\mu\nu} = [\nabla_\mu, \nabla_\nu]e_\beta = (\partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\gamma{}_{\beta\nu}\Gamma^\alpha{}_{\gamma\mu} - \Gamma^\gamma{}_{\beta\mu}\Gamma^\alpha{}_{\gamma\nu})e_\alpha. \tag{2.5}$$

The 256 components $R^\alpha{}_{\beta\mu\nu}$ of all 64 curvature vectors $R_{\beta\mu\nu}$ form the Riemann tensor:

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\gamma\mu}\Gamma^\gamma{}_{\beta\nu} - \Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\beta\mu}. \tag{2.6}$$

Although the Riemann tensor fully describes the spacetime curvature, even if we take into account its symmetries, its large number of components makes it a heavy object to handle in terms of practical calculations. To describe curvature in a way more convenient for calculations, we introduce two new quantities:

$$R_{\alpha\beta} \coloneqq R^\nu{}_{\alpha\nu\beta} = g^{\mu\nu}R_{\mu\alpha\nu\beta} \tag{2.7}$$

called the Ricci tensor, and its trace

$$R \coloneqq R^\alpha_{\ \alpha} = g^{\alpha\beta}R_{\alpha\beta} \tag{2.8}$$

called the Ricci scalar.

We conclude this subsection by turning our attention to $\partial\mathscr{V}$. The hypersurface $\partial\mathscr{V}$ can be localized in spacetime by a restriction on the spacetime coordinates: $\Phi(x^\alpha) = 0$. Hence, to move away from $\partial\mathscr{V}$ perpendicularly, one must follow the gradient of Φ, since the value of Φ changes only in the direction perpendicular to $\partial\mathscr{V}$. We can thus define a unit vector field n^α normal to $\partial\mathscr{V}$ as follows:

$$n_\alpha \coloneqq \frac{\varepsilon \partial_\alpha \Phi}{|g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi|^{1/2}}, \tag{2.9}$$

where $\varepsilon \coloneqq n_\alpha n^\alpha = \pm 1$, with $\varepsilon = +1$ where $\partial\mathscr{V}$ is timelike and $\varepsilon = -1$ where $\partial\mathscr{V}$ is spacelike. We follow the convention that n^α points in the direction of increasing Φ: $n^\alpha\partial_\alpha\Phi > 0$. Notice that (2.9) can not be used to define a unit normal where $\partial\mathscr{V}$ is null, because n^α is orthogonal to itself in this case: $n_\alpha n^\alpha = 0$ (being both normal and tangent to $\partial\mathscr{V}$). Due to its particularities, we will avoid the case of null hypersurface by saying that $\partial\mathscr{V}$ is almost-nowhere null, which means that the regions where $\partial\mathscr{V}$ is null are too small compared to the regions where $\partial\mathscr{V}$ is spacelike or timelike.

Using the transformation $x^\alpha = x^\alpha(y^a)$, we can define the vector field

$$e^\alpha_a \coloneqq \frac{\partial x^\alpha}{\partial y^a} \tag{2.10}$$

tangent to $\partial\mathscr{V}$: $n_\alpha e^\alpha_a = 0$. The vectors n^α and e^α_a form a vector basis for objects on $\partial\mathscr{V}$ embedded in $\mathscr{V}$. The projection of $g_{\alpha\beta}$ onto $\partial\mathscr{V}$ defines the induced metric on the hypersurface:

$$h_{ab} \coloneqq g_{\alpha\beta} e^\alpha_a e^\beta_b. \tag{2.11}$$

The induced metric is a three-tensor: it is invariant with respect to spacetime coordinate transformations, but behaves as a tensor under hypersurface coordinate transformations. With this, we can decompose the metric $g_{\alpha\beta}$ into two mutually orthogonal parts, one normal to $\partial\mathscr{V}$ and the other tangent to $\partial\mathscr{V}$. This decomposition is called completeness relation and for the inverse metric we have:

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta}, \tag{2.12}$$

which defines $h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$.

2.1.2 Intrinsic and extrinsic geometries

In this subsection, we attempt to characterize the embedding of the hypersurface in spacetime by constructing three-quantities that live on the hypersurface. We start this task by defining the intrinsic covariant derivative of a three-vector field obtained from the projection onto the hypersurface of a vector field $V^\alpha = V^a e^\alpha_a$ purely tangent to the hypersurface, $V^\alpha n_\alpha = 0$. This will give rise to a three-connection from which we build the Riemann three-tensor.

We define the intrinsic covariant derivative of $V_a = V_\alpha e^\alpha_a$ to be the projection of $\nabla_\beta V_\alpha$ onto the hypersurface:

$$D_b V_a \coloneqq e^\alpha_a e^\beta_b \nabla_\beta V_\alpha. \tag{2.13}$$

Working out the right-hand side:

$$\begin{aligned} e^\alpha_a e^\beta_b \nabla_\beta V_\alpha &= e^\beta_b \nabla_\beta (e^\alpha_a V_\alpha) - V_\alpha e^\beta_b \nabla_\beta e^\alpha_a \\ &= e^\beta_b \partial_\beta V_a - (V^c e^\alpha_c) e^\beta_b \nabla_\beta e_{a\alpha} \\ &= \frac{\partial x^\beta}{\partial y^b} \frac{\partial V_a}{\partial x^\beta} - (e^\alpha_c e^\beta_b \nabla_\beta e_{a\alpha}) V^c \\ &= \partial_b V_a - \Gamma_{cab} V^c, \end{aligned} \tag{2.14}$$

where we define the three-connection as $\Gamma_{cab} \coloneqq e^\alpha_c e^\beta_b \nabla_\beta e_{a\alpha}$.
Therefore, definition (2.13) results in the usual expression for covariant differentiation (things still keep working the same in one less dimension):

$$D_b V_a = \partial_b V_a - \Gamma^c{}_{ab} V_c, \tag{2.15}$$

where $\Gamma^c{}_{ab}$ is compatible with h_{ab}: $D_c h_{ab} := e^\alpha_a e^\beta_b e^\gamma_c \nabla_\gamma h_{\alpha\beta} = 0$, which one can easily demonstrate using the completeness relation (2.12). Hence, the symmetry requirement enables this connection to be completely determined by h_{ab}:

$$\Gamma^c{}_{ab} = \frac{1}{2} h^{cd} (\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab}). \tag{2.16}$$

With this, the Riemann three-tensor, also known as intrinsic curvature of the hypersurface $\partial\mathscr{V}$, is given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \Gamma^a{}_{de} \Gamma^e{}_{bc}. \tag{2.17}$$

We observe that $D_b V_a = e^\alpha_a e^\beta_b \nabla_\beta V_\alpha$ can be thought of as the tangent component of the vector $e^\beta_b \nabla_\beta V^\alpha$, which leads us to ask whether this vector also has a normal component to the hypersurface. To answer this question we re-write $e^\beta_b \nabla_\beta V^\alpha$ in terms of the metric and use the completeness relation to split the vector into a normal part and a tangent part:

$$\begin{aligned} e^\beta_b \nabla_\beta V^\alpha &= g^\alpha{}_\mu e^\beta_b \nabla_\beta V^\mu \\ &= (\varepsilon n^\alpha n_\mu + h^{ac} e^\alpha_a e_{c\mu}) e^\beta_b \nabla_\beta V^\mu \\ &= \varepsilon (n_\mu e^\beta_b \nabla_\beta V^\mu) n^\alpha + h^{ac} (e^\mu_c e^\beta_b \nabla_\beta V_\mu) e^\alpha_a \\ &= \varepsilon [e^\beta_b \nabla_\beta (n_\mu V^\mu) - V^\mu e^\beta_b \nabla_\beta n_\mu] n^\alpha + h^{ac} (D_b V_c) e^\alpha_a \\ &= -\varepsilon V^a (e^\mu_a e^\beta_b \nabla_\beta n_\mu) n^\alpha + e^\alpha_a D_b V^a, \end{aligned} \tag{2.18}$$

where in the fourth line we have used the fact that $V^\mu = V^a e^\mu_a$ is orthogonal to n^μ and that $D_b h^{ac} = 0$. To the term within parentheses in the normal component, we give a name and a new look:

$$K_{ab} := e^\alpha_a e^\beta_b \nabla_\beta n_\alpha, \tag{2.19}$$

called the extrinsic curvature of the hypersurface $\partial\mathscr{V}$. The extrinsic curvature is a symmetric three-tensor, $K_{ab} = K_{ba}$, and its trace is $K := h^{ab} K_{ab} = \nabla_\alpha n^\alpha$. The following equation summarizes the answer to our question:

$$e^\beta_b \nabla_\beta V^\alpha = e^\alpha_a D_b V^a - \varepsilon V^a K_{ab} n^\alpha. \tag{2.20}$$

We now want to investigate how the intrinsic and extrinsic curvatures of $\partial\mathscr{V}$ are related to the spacetime curvature. Our first move is to replace V^α with e^α_a. We notice that $e^\alpha_a = V^c e^\alpha_c$ implies in $V^c = \delta^c{}_a$, then (2.20) becomes:

$$\begin{aligned} e^\beta_b \nabla_\beta e^\alpha_a &= (\partial_b \delta^c{}_a + \Gamma^c{}_{db} \delta^d{}_a) e^\alpha_c - \varepsilon \delta^c{}_a K_{cb} n^\alpha \\ &= \Gamma^c{}_{ab} e^\alpha_c - \varepsilon K_{ab} n^\alpha. \end{aligned} \tag{2.21}$$

We plan to take a covariant derivative in the above equation to obtain a second covariant derivative of e_a^α, so we can bring the spacetime Riemann tensor into our calculation using its definition as the action of the commutator $[\nabla_\beta, \nabla_\gamma]$ upon e_a^α. Acting with $e_c^\gamma \nabla_\gamma$ on both sides of (2.21) gives us the following identity:

$$e_c^\gamma \nabla_\gamma (e_b^\beta \nabla_\beta e_a^\alpha) = e_c^\gamma \nabla_\gamma ({\Gamma^d}_{ab} e_d^\alpha - \varepsilon K_{ab} n^\alpha). \tag{2.22}$$

We first expand the left-hand side:

$$\begin{aligned} e_c^\gamma \nabla_\gamma (e_b^\beta \nabla_\beta e_a^\alpha) &= e_c^\gamma e_b^\beta \nabla_\gamma \nabla_\beta e_a^\alpha + (e_c^\gamma \nabla_\gamma e_b^\beta) \nabla_\beta e_a^\alpha \\ &= e_c^\gamma e_b^\beta \nabla_\gamma \nabla_\beta e_a^\alpha + ({\Gamma^d}_{bc} e_d^\beta - \varepsilon K_{bc} n^\beta) \nabla_\beta e_a^\alpha \\ &= e_c^\gamma e_b^\beta \nabla_\gamma \nabla_\beta e_a^\alpha + {\Gamma^d}_{bc} ({\Gamma^e}_{ad} e_e^\alpha - \varepsilon K_{ad} n^\alpha) - \varepsilon K_{bc} n^\beta \nabla_\beta e_a^\alpha, \end{aligned} \tag{2.23}$$

where we have used (2.21) twice.

Next, we work on the right-hand side:

$$\begin{aligned} e_c^\gamma \nabla_\gamma ({\Gamma^d}_{ab} e_d^\alpha - \varepsilon K_{ab} n^\alpha) &= e_d^\alpha (e_c^\gamma \nabla_\gamma {\Gamma^d}_{ab}) + {\Gamma^d}_{ab} (e_c^\gamma \nabla_\gamma e_d^\alpha) + \\ &\quad - \varepsilon n^\alpha (e_c^\gamma \nabla_\gamma K_{ab}) - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha \\ &= e_d^\alpha \partial_c {\Gamma^d}_{ab} + {\Gamma^d}_{ab} ({\Gamma^e}_{dc} e_e^\alpha - \varepsilon K_{dc} n^\alpha) + \\ &\quad - \varepsilon n^\alpha \partial_c K_{ab} - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha. \end{aligned} \tag{2.24}$$

In the following development, after we equate (2.23) and (2.24) and solve for $e_c^\gamma e_b^\beta \nabla_\gamma \nabla_\beta e_a^\alpha$, we collect the terms with e_e^α and n^α, gathering them in two sets within parentheses, aiming to introduce the intrinsic curvature in the first set and the covariant derivative of the extrinsic curvature in the second one:

$$\begin{aligned} e_c^\gamma e_b^\beta \nabla_\gamma \nabla_\beta e_a^\alpha &= e_d^\alpha \partial_c {\Gamma^d}_{ab} + {\Gamma^d}_{ab} ({\Gamma^e}_{cd} e_e^\alpha - \varepsilon K_{cd} n^\alpha) - {\Gamma^d}_{bc} ({\Gamma^e}_{ad} e_e^\alpha - \varepsilon K_{ad} n^\alpha) + \\ &\quad - \varepsilon n^\alpha \partial_c K_{ab} + \varepsilon K_{bc} n^\beta \nabla_\beta e_a^\alpha - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha \\ &= e_d^\alpha \partial_c {\Gamma^d}_{ab} + ({\Gamma^d}_{ab} {\Gamma^e}_{cd} - {\Gamma^d}_{bc} {\Gamma^e}_{ad}) e_e^\alpha - \varepsilon ({\Gamma^d}_{ab} K_{cd} - {\Gamma^d}_{bc} K_{ad}) n^\alpha + \\ &\quad - \varepsilon n^\alpha \partial_c K_{ab} + \varepsilon K_{bc} n^\beta \nabla_\beta e_a^\alpha - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha \\ &= (\partial_c {\Gamma^e}_{ab} + {\Gamma^e}_{cd} {\Gamma^d}_{ab} - {\Gamma^e}_{ad} {\Gamma^d}_{bc}) e_e^\alpha - \varepsilon (\partial_c K_{ab} - {\Gamma^d}_{bc} K_{ad} + {\Gamma^d}_{ab} K_{cd}) n^\alpha + \\ &\quad + \varepsilon K_{bc} n^\beta \nabla_\beta e_a^\alpha - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha \\ &= ({R^e}_{acb} + \partial_b {\Gamma^e}_{ac} + {\Gamma^e}_{bd} {\Gamma^d}_{ac}) e_e^\alpha - \varepsilon (D_c K_{ab} + {\Gamma^d}_{ac} K_{db} + {\Gamma^d}_{ab} K_{cd}) n^\alpha + \\ &\quad + \varepsilon K_{bc} n^\beta \nabla_\beta e_a^\alpha - \varepsilon K_{ab} e_c^\gamma \nabla_\gamma n^\alpha. \end{aligned} \tag{2.25}$$

A similar expression for $e^\beta_b e^\gamma_c \nabla_\beta \nabla_\gamma e^\alpha_e$ is given by

$$e^\beta_b e^\gamma_c \nabla_\beta \nabla_\gamma e^\alpha_e = (R^e{}_{abc} + \partial_c \Gamma^e{}_{ab} + \Gamma^e{}_{cd}\Gamma^d{}_{ab}) e^\alpha_e - \varepsilon (D_b K_{ac} + \Gamma^d{}_{ab} K_{dc} + \Gamma^d{}_{ac} K_{bd}) n^\alpha + \\ + \varepsilon K_{cb} n^\gamma \nabla_\gamma e^\alpha_a - \varepsilon K_{ac} e^\beta_b \nabla_\beta n^\alpha. \quad (2.26)$$

We now subtract (2.25) from (2.26) and use $R^\alpha{}_{\mu\beta\gamma} e^\mu_a = [\nabla_\beta, \nabla_\gamma] e^\alpha_a$ to obtain

$$R^\mu{}_{\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c = R^e{}_{abc} e^\mu_e - \varepsilon (D_b K_{ac} - D_c K_{ab}) n^\mu + \varepsilon K_{ab} e^\gamma_c \nabla_\gamma n^\mu - \varepsilon K_{ac} e^\beta_b \nabla_\beta n^\mu, \quad (2.27)$$

where we exchanged the indices α and μ.

Multiplying this equation by $e_{d\mu}$, we obtain $R^\mu{}_{\alpha\beta\gamma}$ fully evaluated on $\partial\mathscr{V}$, where it can be directly related with the curvatures of the hypersurface:

$$\begin{aligned} R_{\mu\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c e^\mu_d &= (g_{\mu\nu} e^\mu_d e^\nu_c) R^e{}_{abc} + \varepsilon K_{ab} (e^\mu_d e^\gamma_c \nabla_\gamma n_\mu) - \varepsilon K_{ac} (e^\mu_d e^\beta_b \nabla_\beta n_\mu) \\ &= R_{dabc} + \varepsilon (K_{ab} K_{cd} - K_{ac} K_{bd}), \end{aligned} \quad (2.28)$$

where we have used the definitions of the induced metric and the extrinsic curvature. This is the curvature relationship that we were looking for. Notice that the term between parentheses has the same symmetries of R_{dabc}. If we multiply (2.27) by n_μ instead, we find

$$R_{\mu\alpha\beta\gamma} n^\mu e^\alpha_a e^\beta_b e^\gamma_c = D_c K_{ab} - D_b K_{ac}. \quad (2.29)$$

Equations (2.28) and (2.29) are known as the general form of the Gauss-Codazzi equations. They show that some projections of the spacetime curvature tensor along e^α_a and n^α can be written in terms of the curvatures of the hypersurface. There are also zero projections:

$$R_{\mu\alpha\nu\beta} n^\mu n^\alpha n^\nu n^\beta = 0 \quad \text{and} \quad R_{\mu\alpha\nu\beta} n^\mu e^\alpha_a n^\nu n^\beta = 0.$$

The Gauss-Codazzi equations can be written in terms of the Einstein tensor,

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}. \quad (2.30)$$

The advantage of writing this way is that using Einstein's field equations we can relate the matter content in the region $\mathscr{V}$ to the intrinsic and extrinsic curvatures of its boundary. We start from the definitions of the Ricci tensor,

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= (\varepsilon n^\mu n^\nu + h^{mn} e^\mu_m e^\nu_n) R_{\mu\alpha\nu\beta} \\ &= \varepsilon R_{\mu\alpha\nu\beta} n^\mu n^\nu + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\nu_n, \end{aligned} \quad (2.31)$$

and the Ricci scalar,

$$\begin{aligned}R &= g^{\alpha\beta}R_{\alpha\beta} \\ &= (\varepsilon n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b)(\varepsilon R_{\mu\alpha\nu\beta}n^\mu n^\nu + h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\nu_n) \\ &= R_{\mu\alpha\nu\beta}n^\mu n^\alpha n^\nu n^\beta + \varepsilon h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m n^\alpha e^\nu_n n^\beta + \\ &\quad + \varepsilon h^{ab}R_{\mu\alpha\nu\beta}n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\alpha_a e^\nu_n e^\beta_b. \end{aligned} \tag{2.32}$$

The first term in (2.32) is zero, while the second and third are the same, which one can see by doing the exchanges $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$ in the indices and knowing that $R_{\alpha\mu\beta\nu} = R_{\mu\alpha\nu\beta}$. Hence, we are left with

$$R = 2\varepsilon h^{ab}R_{\mu\alpha\nu\beta}n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\alpha_a e^\nu_n e^\beta_b. \tag{2.33}$$

Substituting the expressions (2.31) and (2.33) into (2.30) results in

$$\begin{aligned}G_{\alpha\beta} &= \varepsilon R_{\mu\alpha\nu\beta}n^\mu n^\nu + h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\nu_n + \\ &\quad - \frac{1}{2}(2\varepsilon h^{ab}R_{\mu\alpha\nu\beta}n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\alpha_a e^\nu_n e^\beta_b)g_{\alpha\beta}. \end{aligned} \tag{2.34}$$

Projecting twice along n^α and using (2.28):

$$\begin{aligned}-2\varepsilon G_{\alpha\beta}n^\alpha n^\beta &= -2\varepsilon h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m n^\alpha e^\nu_n n^\beta + 2\varepsilon h^{ab}R_{\mu\alpha\nu\beta}n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\alpha_a e^\nu_n e^\beta_b \\ &= h^{ab}h^{mn}[R_{manb} + \varepsilon(K_{an}K_{bm} - K_{ab}K_{nm})] \\ &= {}^3R + \varepsilon(K^{bm}K_{bm} - K^2), \end{aligned} \tag{2.35}$$

where ${}^3R \coloneqq h^{ab}R^n{}_{anb}$ represents the three-Ricci scalar.
Multiplying by $e^\alpha_a n^\beta$ and using (2.29):

$$\begin{aligned}G_{\alpha\beta}e^\alpha_a n^\beta &= h^{mn}R_{\mu\alpha\nu\beta}e^\mu_m e^\alpha_a e^\nu_n n^\beta \\ &= h^{mn}R_{\beta\nu\alpha\mu}n^\beta e^\nu_n e^\alpha_a e^\mu_m \\ &= h^{mn}(D_m K_{na} - D_a K_{nm}) \\ &= D_m K^m{}_a - D_a K. \end{aligned} \tag{2.36}$$

Equations (2.35) and (2.36) are known as the contracted form of the Gauss-Codazzi equations and shows us the close relation between those components of the Einstein tensor and the curvatures of the hypersurface.

Before we leave this subsection, we need to derive one last expression that will come in handy later. Observe that in the first term on the right-hand side of (2.33) we have

$R_{\mu\alpha\nu\beta}n^\mu e^\alpha_a n^\nu e^\beta_b$, which is the missing component of the spacetime curvature tensor. We will work on this term and end up with a more convenient expression for the spacetime Ricci scalar. As usual, using the completeness relation, we have:

$$\begin{aligned}2\varepsilon(h^{ab}e^\alpha_a e^\beta_b)R_{\mu\alpha\nu\beta}n^\mu n^\nu &= 2\varepsilon(g^{\alpha\beta} - \varepsilon n^\alpha n^\beta)R_{\mu\alpha\nu\beta}n^\mu n^\nu \\ &= 2\varepsilon g^{\alpha\beta}R_{\mu\alpha\nu\beta}n^\mu n^\nu \\ &= 2\varepsilon R_{\mu\nu}n^\mu n^\nu, \end{aligned} \tag{2.37}$$

with

$$\begin{aligned}R_{\mu\nu}n^\mu n^\nu &= n^\nu(R^\alpha{}_{\mu\alpha\nu}n^\mu) \\ &= n^\nu(\nabla_\alpha\nabla_\nu n^\alpha - \nabla_\nu\nabla_\alpha n^\alpha) \\ &= \nabla_\alpha(n^\nu\nabla_\nu n^\alpha) - \nabla_\alpha n^\nu\nabla_\nu n^\alpha - \nabla_\nu(n^\nu\nabla_\alpha n^\alpha) + \nabla_\nu n^\nu\nabla_\alpha n^\alpha \\ &= \nabla_\alpha(n^\nu\nabla_\nu n^\alpha - n^\alpha\nabla_\nu n^\nu) - \nabla_\alpha n^\nu\nabla_\nu n^\alpha + K^2, \end{aligned} \tag{2.38}$$

where we recognized $K = \nabla_\alpha n^\alpha$, the trace of the extrinsic curvature. The second term in the above equation can be written as

$$\begin{aligned}\nabla_\alpha n^\beta\nabla_\beta n^\alpha &= g^{\beta\mu}g^{\alpha\nu}\nabla_\alpha n_\beta\nabla_\mu n_\nu \\ &= (\varepsilon n^\beta n^\mu + h^{\beta\mu})(\varepsilon n^\alpha n^\nu + h^{\alpha\nu})\nabla_\alpha n_\beta\nabla_\mu n_\nu \\ &= (\varepsilon n^\beta n^\mu + h^{\beta\mu})h^{\alpha\nu}\nabla_\alpha n_\beta\nabla_\mu n_\nu \\ &= (h^{bm}e^\beta_b e^\mu_m)(h^{an}e^\alpha_a e^\nu_n)\nabla_\alpha n_\beta\nabla_\mu n_\nu \\ &= h^{bm}h^{an}(e^\beta_b e^\alpha_a\nabla_\alpha n_\beta)(e^\nu_n e^\mu_m\nabla_\mu n_\nu) \\ &= h^{bm}h^{an}K_{ba}K_{nm} \\ &= K^{ab}K_{ab}. \end{aligned} \tag{2.39}$$

In the second and third lines, we have used $n^\alpha\nabla_\beta n_\alpha = \frac{1}{2}\nabla_\beta(n^\alpha n_\alpha) = 0$. Finally, we gather these results in a new expression for the spacetime Ricci scalar:

$$R = {}^3R - \varepsilon(K^{ab}K_{ab} - K^2) + 2\varepsilon\nabla_\alpha(n^\beta\nabla_\beta n^\alpha - n^\alpha\nabla_\beta n^\beta). \tag{2.40}$$

2.1.3 Lagrangian formulation of general relativity

We close this section with a brief discussion about the Lagrangian formalism, which application in general relativity yields the Einstein field equations. Consider a scalar field $\phi(x^\alpha)$

defined in $\mathscr{V}$. Suppose ϕ held fixed on the boundary $\partial\mathscr{V}$. The necessary information to determine the dynamic of this field is contained in the Lagrangian density, a scalar function $\mathscr{L}(\phi, \partial_\alpha\phi)$ from which we build the action functional $S[\phi]$:

$$S[\phi] = \int_{\mathscr{V}} \mathscr{L}(\phi, \partial_\alpha\phi)\sqrt{-g}\,\mathrm{d}^4x, \tag{2.41}$$

where $g = \det(g_{\alpha\beta}) < 0$ is the determinant of the metric in $\mathscr{V}$.

We state that among all possible time evolutions for the field configuration, the action functional should be an extremum only for the evolution that the field configuration will truly perform. This statement is the so-called Hamilton's variational principle. Considering arbitrary variations $\delta\phi(x^\alpha)$ around the true evolution, we calculate, in first-order in $\delta\phi$, the variation of the action. The condition $\delta\phi|_{\partial\mathscr{V}} = 0$ annihilate an eventual hypersurface integral. Demanding $\delta S = 0$, we obtain the Euler-Lagrange equation:

$$\frac{\partial\mathscr{L}}{\partial\phi} - \nabla_\alpha\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha\phi)} = 0. \tag{2.42}$$

Once we are given a Lagrangian density, the solution of this equation, together with a boundary condition, uniquely determines the field dynamics.

Following the example of scalar field, we are led to choose for the gravitational field $g_{\alpha\beta}$ a Lagrangian density $\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta})$. However, the first derivatives of the metric are not tensorial quantities, so one can not build a scalar by combining them. Therefore, we allow the inclusion of the next order derivatives: $\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \partial_\mu\partial_\nu g_{\alpha\beta})$. The simplest scalar quantity built from the metric up to its second derivatives and that contains the necessary information to describe the spacetime geometry is the Ricci scalar. But the variation of a Lagrangian density that depends on the field's second derivatives gives rise to an additional hypersurface integral that does not vanish by the condition of a fixed field on the boundary $\partial\mathscr{V}$. To circumvent this problem without imposing one more condition on the field, which would reduce the scope of our formalism, we add a boundary term to the gravitational action, which variation must cancel out this additional hypersurface integral. Thus, the gravitational action S_G is given by two terms:

$$S_G[g] = S_H[g] + S_B[g], \tag{2.43}$$

where

$$S_H[g] = \frac{1}{2\kappa}\int_{\mathscr{V}} R\sqrt{-g}\,\mathrm{d}^4x \tag{2.44}$$

is known as the Hilbert term ($\kappa = 8\pi G$, where G is Newton's gravitational constant) and

$$S_B[g] = \frac{1}{\kappa} \oint_{\partial \mathscr{V}} \varepsilon K \sqrt{|h|}\, \mathrm{d}^3 y \tag{2.45}$$

is the boundary term, with $h = \det(h_{ab})$ representing the determinant of the metric on $\partial \mathscr{V}$.

To derive a field equation for the gravitational field, we must now apply Hamilton's principle to the total action

$$S = S_G[g] + S_M[\phi, g], \tag{2.46}$$

where we include the matter action

$$S_M[\phi, g] = \int_{\mathscr{V}} \mathscr{L}_M \left(\phi, \partial_\alpha \phi, g^{\alpha\beta}\right) \sqrt{-g}\, \mathrm{d}^4 x \tag{2.47}$$

to account for the possible presence of matter, represented by the scalar field ϕ, in $\mathscr{V}$. After we calculate the variations of all these terms ((POISSON, 2004), sec. 4.1), S_H, S_B, and S_M, we impose $\delta S = 0$ and arrive at the Einstein field equations:

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{2.48}$$

where

$$T_{\alpha\beta} := \mathscr{L}_M g_{\alpha\beta} - 2 \frac{\partial \mathscr{L}_M}{\partial g^{\alpha\beta}}, \tag{2.49}$$

is the energy-momentum tensor.

2.2 ADM FORMALISM

2.2.1 3+1 decomposition

In our discussion above, we have introduced the idea of the time evolution of a field configuration. One way to think about this idea is through an analogy, imagining the field as a book, where each sheet represents the field at a given time. The i-th sheet corresponds to a field configuration $\phi(t_i, \mathbf{x})$. Thus, the time evolution has its meaning translated into the act of reading the book: the history of the field is told as we turn the pages, stacking more and more sheets. The time evolution perspective promotes the timelike coordinate to a prominent position by designating it as responsible for bringing about the change of state of the system, which is dissonant with the covariant way in which we have written the Einstein field equations, with timelike and spacelike coordinates treated on equal footing.

To clarify this point, consider the simple case of a system with only one degree of freedom, q, and Lagrangian $L(q, \dot{q})$ homogeneous of first degree in $\dot{q}$: $L(q, \lambda\dot{q}) = \lambda L(q, \dot{q})$. According to Euler's homogeneous function theorem:

$$L = \dot{q}\frac{\partial L}{\partial \dot{q}}. \tag{2.50}$$

But this is equivalent to the statement that the Hamiltonian of the system is zero:

$$p\dot{q} - L = 0, \tag{2.51}$$

where $p = \partial L/\partial\dot{q}$ is the momentum conjugate to q. Now consider a change of independent time variable, from t to τ. Using the homogeneity of the Lagrangian:

$$L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}\right) = L\left(q, \frac{\mathrm{d}q/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}\right) = \frac{\mathrm{d}\tau}{\mathrm{d}t} L\left(q, \frac{\mathrm{d}q}{\mathrm{d}\tau}\right). \tag{2.52}$$

The action then becomes

$$\int L(q, \dot{q})\,\mathrm{d}t = \int L\left(q, \frac{\mathrm{d}q}{\mathrm{d}\tau}\right)\mathrm{d}\tau. \tag{2.53}$$

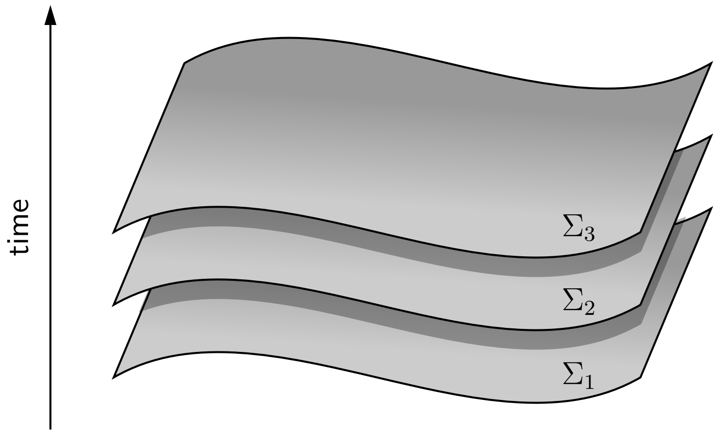
Therefore, for a dynamical system with vanishing Hamiltonian, we can change the time parameterization, $t \to \tau$, and define a new Lagrangian

$$\widetilde{L} = \frac{\mathrm{d}t}{\mathrm{d}\tau} L, \tag{2.54}$$

such that the action stays invariant. As we will see further, the gravitational field is an example of such a system with a vanishing Hamiltonian, and although we select a specific Lorentz frame (time parameterization) in formulating the Hamiltonian of the field, we can switch frames insofar we have an invariant variational principle.

We can incorporate the time evolution perspective directly into our description of spacetime geometry. That is the idea behind the $3+1$ decomposition, which consists of foliate the spacetime with a family of arbitrary, non-intersecting, spacelike hypersurfaces $\{\Sigma_\Phi\}$, where the index $\Phi \in \mathbb{R}$ selects a member of the family.

Figure 1 – Foliation of spacetime by spacelike hypersurfaces.

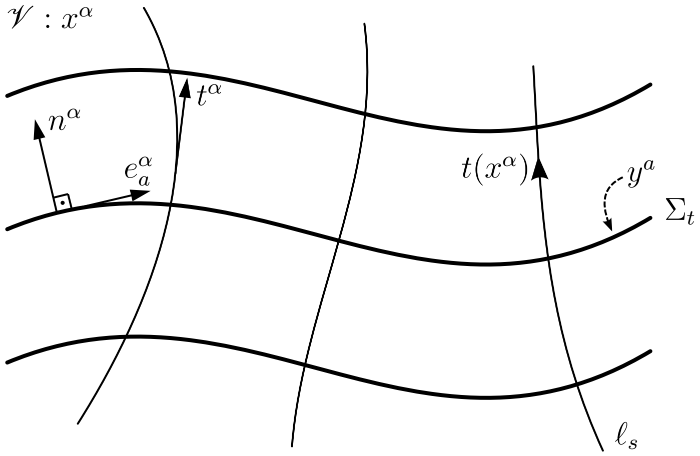


Source: the author (2021).

The specification of a particular hypersurface of the foliation can be made by a restriction on the spacetime coordinates x^α if we introduce a single-valued scalar field $\Phi(x^\alpha)$, usually called the time function, such that $\Phi = constant$ defines the hypersurface $\Sigma_{constant}$. We require the unit normal to the hypersurfaces, $n_\alpha \propto \partial_\alpha \Phi$, to be a future-directed timelike vector field, $n_\alpha n^\alpha = -1$. Therefore, the convention $n^\alpha \partial_\alpha \Phi > 0$ implies that Φ must increase monotonically with the timelike coordinate in $\mathscr{V}$.

Now that we have $\mathscr{V}$ foliated, we search for an alternative coordinate system that better suits this construction. On each hypersurface Σ_Φ we have the freedom of setting up a coordinate system $(y^a)_\Phi$ independently of the coordinate systems that we lay down on the other hypersurfaces of the foliation. In $\mathscr{V}$ we define a congruence of curves $\{\ell_s\}$, where the index $s \in \mathbb{R}$ selects a member of the congruence, all of them parameterized by Φ and with tangent vector field t^α intersecting the hypersurfaces. We do not require that these curves intersect the hypersurfaces orthogonally, which means that t^α does not need to be parallel to n^α, nor that they are geodesics, in which case t^α should satisfy the geodesic equation. By construction, the parameter Φ of all curves crossing a given hypersurface Σ_Φ has the same value. We now link the initially unrelated intrinsic coordinates $(y^a)_\Phi$ by imposing the coordinates y^a of all points pierced by a given curve ℓ_s to be the same. With this picture of the congruence carrying through spacetime a grid of intrinsic coordinates placed on a hypersurface, we have the alternative coordinate system that we were looking for, which is (t, y^a) (from now on, we will use $t(x^\alpha)$ instead of $\Phi(x^\alpha)$).

Figure 2 – Construction of the coordinate system (t, y^a).



Source: the author (2021).

The time function $t(x^\alpha)$ determines the way we slice $\mathscr{V}$, so the arbitrariness of the foliation is a consequence of the arbitrariness of t. A displacement along a curve ℓ_s can be written as $\mathrm{d}x^\alpha = \mathrm{d}t\, t^\alpha$. But a change in $t(x^\alpha)$ is given by $\mathrm{d}t = \partial_\alpha t\, \mathrm{d}x^\alpha$. Hence, we obtain

$$t^\alpha \partial_\alpha t = 1. \tag{2.55}$$

The original and the alternative coordinate systems are related by some well behaved parametric relations $x^\alpha(t, y^a)$. Once the $3+1$ decomposition is made, we must be able to express the spacetime metric in terms of the new coordinate system. The unit normal to the hypersurfaces is

$$n_\alpha = -N\partial_\alpha t, \tag{2.56}$$

where the normalization function N is called the lapse. In coordinates (t, y^a) the lapse function is $N = (-g^{\alpha\beta}\partial_\alpha t \partial_\beta t)^{-1/2} = (-g^{tt})^{-1/2}$. The vector fields t^α and e^α_a tangent to the congruence and the hypersurfaces, respectively, are "naturally" given by

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} \quad \text{and} \quad e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t, \tag{2.57}$$

with e^α_a orthogonal to n^α: $n_\alpha e^\alpha_a = 0$. The statement that we can interchange the order of second partial derivatives,

$$\frac{\partial}{\partial t} e^\alpha_a = \frac{\partial}{\partial y^a} t^\alpha, \tag{2.58}$$

can be cast into the form of $t^\beta \nabla_\beta e^\alpha_a = e^\beta_a \nabla_\beta t^\alpha$, which implies that t^α and e^α_a are Lie transported along each other:

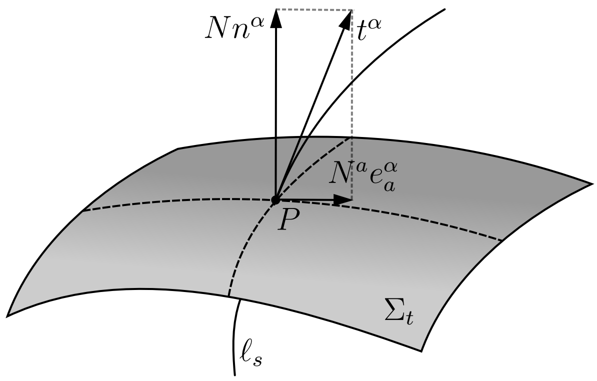
$$\mathcal{L}_t e^\alpha_a = 0. \tag{2.59}$$

Using n^α and e^α_a as basis vectors, we can decompose t^α as follows:

$$t^\alpha = Nn^\alpha + N^a e^\alpha_a, \tag{2.60}$$

where the three-vector N^a is called the shift. Using (2.55) and (2.56) it is easy to see that the normal component of t^α indeed must agree with the lapse function.

Figure 3 – Decomposition of the flow vector t^α in the basis $\{n^\alpha, e^\alpha_a\}$.



Source: the author (2021).

With this, a displacement in $\mathscr{V}$ is written as

$$\begin{aligned} \mathrm{d}x^\alpha &= t^\alpha \mathrm{d}t + e^\alpha_a \mathrm{d}y^a \\ &= (Nn^\alpha + N^a e^\alpha_a)\mathrm{d}t + e^\alpha_a \mathrm{d}y^a \\ &= (N\mathrm{d}t)n^\alpha + (N^a \mathrm{d}t + \mathrm{d}y^a)e^\alpha_a. \end{aligned} \tag{2.61}$$

Therefore, the spacetime metric expressed in coordinates (t, y^a) is:

$$\begin{aligned} \mathrm{d}s^2_{\mathsf{ADM}} &= g_{\alpha\beta}\left[(N\mathrm{d}t)n^\alpha + (N^a\mathrm{d}t + \mathrm{d}y^a)e^\alpha_a\right]\left[(N\mathrm{d}t)n^\beta + (N^b\mathrm{d}t + \mathrm{d}y^b)e^\beta_b\right] \\ &= -N^2\mathrm{d}t^2 + h_{ab}(\mathrm{d}y^a + N^a\mathrm{d}t)(\mathrm{d}y^b + N^b\mathrm{d}t) && (2.62a) \\ &= -(N^2 - N_a N^a)\mathrm{d}t^2 + 2N_a \mathrm{d}y^a \mathrm{d}t + h_{ab}\mathrm{d}y^a \mathrm{d}y^b, && (2.62b) \end{aligned}$$

where $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ is the induced metric on the hypersurfaces. In matrix notation:

$$\begin{pmatrix} g_{tt} & g_{ta} \\ g_{at} & g_{ab} \end{pmatrix} = \begin{pmatrix} -(N^2 - N_a N^a) & N_a \\ N_a & h_{ab} \end{pmatrix}, \tag{2.63}$$

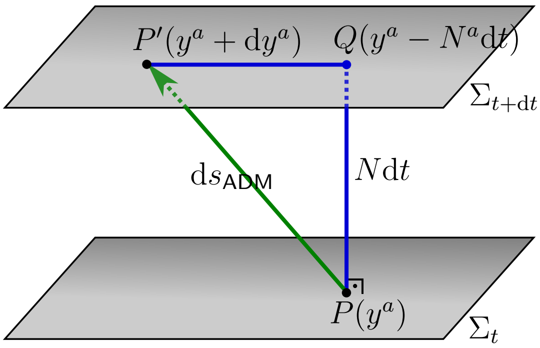
$$\begin{pmatrix} g^{tt} & g^{ta} \\ g^{at} & g^{ab} \end{pmatrix} = \frac{1}{N^2}\begin{pmatrix} -1 & N^a \\ N^a & N^2 h^{ab} - N^a N^b \end{pmatrix}, \tag{2.64}$$

from where one can check the following relation:

$$\sqrt{-g} = N\sqrt{h}. \tag{2.65}$$

Eq. (2.62a) makes clear the roles of the lapse function and the shift vector: N translates an elapsed time $\mathrm{d}t$ to a proper time, while N^a tells us how much the initial coordinates have shifted in the spatial directions.

Figure 4 – Lapse and shift visualization.



Source: the author (2021).

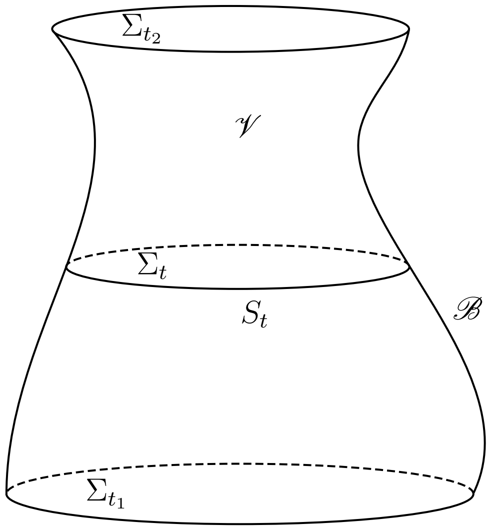
Furthermore, (2.63) explicitly shows that the information contained in the spacetime metric is preserved on the hypersurfaces Σ_t and the flow ℓ: on the right-hand side we have the six independent components $h_{ab}(t, \mathbf{y})$, the lapse function $N(t, \mathbf{y})$, and the three components $N^a(t, \mathbf{y})$ of the shift vector, matching the $10 \cdot \infty^3$ degrees of freedom represented by the ten independent components $g_{\alpha\beta}(t, \mathbf{y})$ for every point of space on the left-hand side. However, N and N^a are redundant degrees of freedom since they reflect the arbitrariness of the time function $t(x^\alpha)$ and of the congruence ℓ: $N = (-g^{tt})^{-1/2}$ and $N^\alpha := N^a e^\alpha_a = t^\alpha - N n^\alpha$, which can be thought of as a measure of the deviation from the orthogonality. Therefore, the choices of N and N^a cannot affect the physical content of the theory. Thus, we are left with $6 \cdot \infty^3$ physical degrees of freedom, and we expect to come across the "four" constraints responsible for this reduction.

2.2.1.1 Foliation of the boundary

Now that we have the fundamental results of the $3+1$ decomposition, equations (2.60), (2.62b), and (2.65), we need to provide more details about the foliation of the boundary $\partial\mathscr{V}$ before we get busy adapting the gravitational action to the Hamiltonian formalism. We established that the region $\mathscr{V} = \bigcup_{t=t_1}^{t_2} \Sigma_t$ is foliated by spacelike hypersurfaces Σ_t with coordinates y^a

and described by equations of the form $t(x^\alpha) = constant$, or by parametric relations $x^\alpha(y^a)$. It has a unit normal $n_\alpha \propto \partial_\alpha t$ and tangent vectors $e^\alpha_a = \partial x^\alpha/\partial y^a$, so that $n_\alpha e^\alpha_a = 0$. Its embedding in spacetime is characterized by the induced metric $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ and the extrinsic curvature $K_{ab} = e^\alpha_a e^\beta_b \nabla_\beta n_\alpha$. The completeness relation is $g^{\alpha\beta} = -n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b$. These hypersurfaces are bounded by closed two-surfaces S_t, such that $\mathscr{V}$ is bounded by $\partial\mathscr{V} = \Sigma_{t_1} \bigcup \mathscr{B} \bigcup \Sigma_{t_2}$, where $\mathscr{B} = \bigcup\limits_{t=t_1}^{t_2} S_t$ is the timelike boundary.

Figure 5 – $\mathscr{V}$ is foliated by spacelike hypersurfaces Σ_t, which are bounded by closed two-surfaces S_t, which in turn foliates the timelike boundary $\mathscr{B}$ of $\mathscr{V}$.



Source: the author (2021).

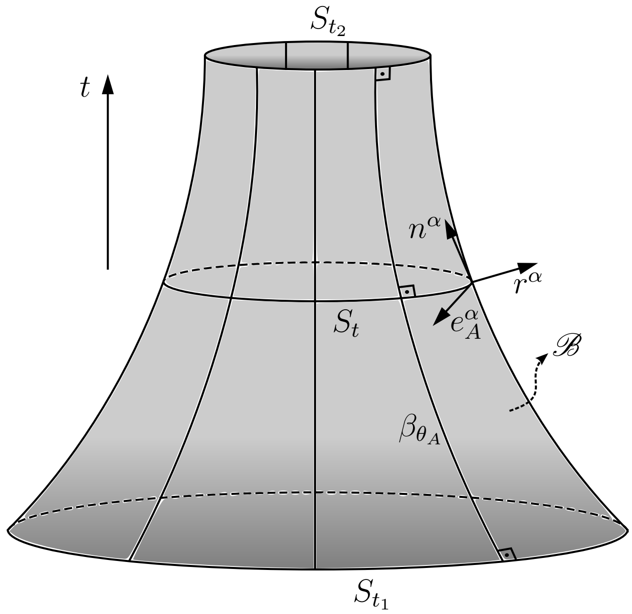
On the surfaces S_t we define coordinates $\theta^A = (\theta^1, \theta^2)$ and as embedded in Σ_t they are described by $\Theta(y^a) = constant$ or $y^a(\theta^A)$. Their spacelike unit normal and tangent vectors are $r_a \propto \partial_a \Theta$ and $e^a_A = \partial y^a \big/ \partial \theta^A$, with $r_a e^a_A = 0$. The induced metric is given by $\sigma_{AB} = h_{ab} e^a_A e^b_B$ and the extrinsic curvature by $k_{AB} = e^a_A e^b_B D_b r_a$. The three-dimensional completeness relation is $h^{ab} = r^a r^b + \sigma^{AB} e^a_A e^b_B$. As embedded in spacetime, on the other hand, the restriction must be of the form $\Psi(x^\alpha) = constant$ and the parametric relations are $x^\alpha(\theta^A) = x^\alpha(y^a(\theta^A))$. We then associate to r^a and e^a_A the four-vectors r^α and e^α_A defined by $r^\alpha := r^a e^\alpha_a$, such that $r^\alpha r_\alpha = 1$ and $r^\alpha n_\alpha = 0$, and

$$e^\alpha_A := \frac{\partial x^\alpha}{\partial \theta^A} = e^\alpha_a e^a_A,$$

which satisfies $r_\alpha e^\alpha_A = 0$. The induced metric can be expressed as $\sigma_{AB} = (g_{\alpha\beta} e^\alpha_a e^\beta_b) e^a_A e^b_B = g_{\alpha\beta} e^\alpha_A e^\beta_B$, and using definition (2.13) for the three-covariant derivative, $D_b r_a = e^\alpha_a e^\beta_b \nabla_\beta r_\alpha$, the extrinsic curvature becomes $k_{AB} = (e^\alpha_a e^\beta_b \nabla_\beta r_\alpha) e^a_A e^b_B = e^\alpha_A e^\beta_B \nabla_\beta r_\alpha$. The spacetime completeness relation on S_t can be written as $g^{\alpha\beta} = -n^\alpha n^\beta + r^\alpha r^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$.

The union of all surfaces S_t is represented by $\mathscr{B}$. In $\mathscr{B}$ we define a congruence of curves $\{\beta_s\}$ parameterized by t and with tangent vector n^α intersecting the surfaces S_t orthogonally. We use this congruence to relate the coordinates θ^A through all surfaces S_t in the same way we did before: θ^A is constant along a curve. We choose $z^i = (t, \theta^A)$ to be coordinates on $\mathscr{B}$, which will be defined by $\Omega(x^\alpha) = constant$ or $x^\alpha(z^i)$. Its unit normal is $r_\alpha \propto \partial_\alpha \Omega$ and its tangent vectors we denote by $e_i^\alpha = \partial x^\alpha / \partial z^i$. The induced metric on $\mathscr{B}$ is given by $\gamma_{ij} = g_{\alpha\beta} e_i^\alpha e_j^\beta$ and we let $\mathcal{K}_{ij} = e_i^\alpha e_j^\beta \nabla_\beta r_\alpha$ be its extrinsic curvature. The completeness relation on $\mathscr{B}$ is simply $g^{\alpha\beta} = r^\alpha r^\beta + \gamma^{ij} e_i^\alpha e_j^\beta$.

Figure 6 – New congruence of curves β orthogonal to S_t within $\mathscr{B}$ along which θ^A is constant.



Source: the author (2021).

Due to the orthogonality of this new congruence β, there will be no shift vector associated with it. Moreover, since the surfaces S_t are the boundaries of the hypersurfaces Σ_t, we will still be talking about the same lapse function N here. Obviously, equations $n_\alpha = -N\partial_\alpha t$ and $n^\alpha n_\alpha = -1$ are still valid, such that, as the unit tangent vector to the congruence β, n^α may be expressed as

$$n^\alpha = \frac{1}{N}\left(\frac{\partial x^\alpha}{\partial t}\right)_{\theta^A}. \tag{2.66}$$

With this, a displacement on $\mathscr{B}$ is given by:

$$\begin{aligned} \mathrm{d}x^\alpha &= \left(\frac{\partial x^\alpha}{\partial t}\right)\mathrm{d}t + \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)\mathrm{d}\theta^A \\ &= N n^\alpha \mathrm{d}t + e_A^\alpha \mathrm{d}\theta^A. \end{aligned} \tag{2.67}$$

Therefore, the metric on $\mathscr{B}$ expressed in coordinates (t, θ^A) is:

$$\begin{aligned} \mathrm{d}s^2_{\mathscr{B}} &= g_{\alpha\beta}(Nn^\alpha \mathrm{d}t + e^\alpha_A \mathrm{d}\theta^A)(Nn^\beta \mathrm{d}t + e^\beta_B \mathrm{d}\theta^B) \\ &= -N^2 \mathrm{d}t^2 + \sigma_{AB} \mathrm{d}\theta^A \mathrm{d}\theta^B. \end{aligned} \tag{2.68}$$

From this expression for $\gamma_{ij}\mathrm{d}z^i\mathrm{d}z^j$, it is straightforward to obtain $\sqrt{-\gamma} = N\sqrt{\sigma}$, where $\gamma = \det(\gamma_{ij})$ and $\sigma = \det(\sigma_{AB})$.

2.2.1.2 Scalar field in curved spacetime

As we saw in the previous section, the field equation (2.42) for a scalar field $\phi(x^\alpha)$ with Lagrangian density $\mathscr{L}(\phi, \partial_\alpha\phi)$ was derived from the action functional (2.41) via Hamilton's principle. In the new coordinates (t, y^a), the derivatives $\partial_\alpha\phi$ are $\partial_t\phi := \dot{\phi}$, for the time derivative, and $\partial_a\phi := e^\alpha_a\partial_\alpha\phi$, for the spatial derivatives. The canonical momentum conjugate to ϕ is defined by

$$p := \frac{\partial}{\partial\dot{\phi}}\left(\mathscr{L}\sqrt{-g}\right). \tag{2.69}$$

Assuming that one can solve this equation for $\dot{\phi}$, which gives $\dot{\phi}(\phi, \partial_a\phi, p)$, we define the Hamiltonian density by

$$\mathscr{H}(\phi, \partial_a\phi, p) := p\,\dot{\phi} - \mathscr{L}(\phi, \dot{\phi}, \partial_a\phi)\sqrt{-g}. \tag{2.70}$$

The Hamiltonian functional $H[\phi, p](t)$ is obtained by integrating $\mathscr{H}$ over Σ_t and is a function of time t. The action functional (2.41) then becomes

$$S[\phi, p] = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} (p\,\dot{\phi} - \mathscr{H})\ \mathrm{d}^3y\,. \tag{2.71}$$

We could get Hamilton's equations of motion by varying this action with respect to ϕ and p. However, the equations of motion can be concisely written in the Poisson brackets (PBs) formalism:

$$\dot{\phi} = \left\{\phi, \mathscr{H}\right\}, \tag{2.72a}$$

$$\dot{p} = \left\{p, \mathscr{H}\right\}. \tag{2.72b}$$

Using the Poisson brackets, we can shortly express time evolution. Let $\mathscr{P}$ denote the $(2\cdot\infty^3)$–dimensional phase space formed by all pairs $(\phi(t, \mathbf{y}), p(t, \mathbf{y}))$, and $C^\infty(\mathscr{P})$ denote the space of all smoothly differentiable functionals $U[\phi, p]$ of the phase space, also known as dynamical variables:

$$\begin{aligned} U: \quad & \mathscr{P} \longrightarrow C^\infty(\mathscr{P}) \\ & (\phi, p) \mapsto U[\phi, p]. \end{aligned}$$

Thus, $\{\ ,\ \}$ is a structure that generates dynamical variables. Just take two arbitrary dynamical variables, stick into the PB and get a third one:

$$\{\ ,\ \}: \quad C^\infty(\mathscr{P}) \times C^\infty(\mathscr{P}) \longrightarrow C^\infty(\mathscr{P}).$$

This map obeys some properties:

$$\Big\{U, V\Big\} = -\Big\{V, U\Big\} \quad \text{(antisymmmetry)}, \tag{2.73a}$$

$$\Big\{U, V + \lambda W\Big\} = \Big\{U, V\Big\} + \lambda \Big\{U, W\Big\} \quad \text{(bilinearity)}, \tag{2.73b}$$

$$\Big\{U, VW\Big\} = V\Big\{U, W\Big\} + \Big\{U, V\Big\}W \quad \text{(product rule)}, \tag{2.73c}$$

$$\Big\{U, \Big\{V, W\Big\}\Big\} + \Big\{V, \Big\{W, U\Big\}\Big\} + \Big\{W, \Big\{U, V\Big\}\Big\} = 0 \quad \text{(Jacobi identity)}. \tag{2.73d}$$

The equal-time PB of two canonical variables is a fundamental PB:

$$\Big\{\phi(t, \mathbf{y}), \phi(t, \mathbf{y}')\Big\} = 0, \tag{2.74a}$$

$$\Big\{p(t, \mathbf{y}), p(t, \mathbf{y}')\Big\} = 0, \tag{2.74b}$$

$$\Big\{\phi(t, \mathbf{y}), p(t, \mathbf{y}')\Big\} = \delta(\mathbf{y} - \mathbf{y}'). \tag{2.74c}$$

The definition of the PB of two functionals U and V with respect to the canonical variables ϕ and p is ((LEMOS, 2018), sec. 11.4):

$$\Big\{U(t, \mathbf{y}), V(t, \mathbf{y}')\Big\} = \int_{\Sigma_t} \left(\frac{\delta U(t, \mathbf{y})}{\delta \phi(t, \mathbf{y}'')} \frac{\delta V(t, \mathbf{y}')}{\delta p(t, \mathbf{y}'')} - \frac{\delta U(t, \mathbf{y})}{\delta p(t, \mathbf{y}'')} \frac{\delta V(t, \mathbf{y}')}{\delta \phi(t, \mathbf{y}'')} \right) \mathrm{d}^3 y''. \tag{2.75}$$

This is the prescription to generate dynamical variables using PBs.

2.2.2 Gravitational field

We are now in position to introduce the $3+1$ decomposition to the gravitational action. By substituting (2.44) and (2.45) into (2.43), we have

$$2\kappa S_G = \int_{\mathscr{V}} R\sqrt{-g}\, \mathrm{d}^4 x + 2 \oint_{\partial \mathscr{V}} \varepsilon K \sqrt{|h|}\, \mathrm{d}^3 y. \tag{2.76}$$

We intend to express everything in the equation above in terms of foliated quantities. Since the boundary $\partial\mathscr{V}$ is composed by three parts, the overall boundary integral breaks down into three integrals referring to the two spacelike ($\varepsilon = -1$) boundaries Σ_{t_1} and Σ_{t_2}, and the timelike ($\varepsilon = +1$) boundary $\mathscr{B}$. Moreover, because the unit normal to $\partial\mathscr{V}$ must point outward, the integration over Σ_{t_1} carries an extra minus sign since the unit normal to Σ_{t_1} is future-directed and therefore points inward:

$$2\oint_{\partial\mathscr{V}} \varepsilon K\sqrt{|h|}\,\mathrm{d}^3y = 2\int_{\Sigma_{t_1}} K\sqrt{h}\,\mathrm{d}^3y - 2\int_{\Sigma_{t_2}} K\sqrt{h}\,\mathrm{d}^3y + 2\int_{\mathscr{B}} \mathcal{K}\sqrt{-\gamma}\,\mathrm{d}^3z. \tag{2.77}$$

Making use of expression (2.40),

$$R = {}^3R + K^{ab}K_{ab} - K^2 - 2\nabla_\alpha(n^\beta\nabla_\beta n^\alpha - n^\alpha\nabla_\beta n^\beta), \tag{2.78}$$

and writing $\sqrt{-g}\,\mathrm{d}^4x = N\sqrt{h}\,\mathrm{d}t\,\mathrm{d}^3y$, the volume integral in (2.76) becomes:

$$\begin{aligned}\int_{\mathscr{V}} R\sqrt{-g}\,\mathrm{d}^4x = &\int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t}({}^3R + K^{ab}K_{ab} - K^2)N\sqrt{h}\,\mathrm{d}^3y + \\ &- 2\oint_{\partial\mathscr{V}}(n^\beta\nabla_\beta n^\alpha - n^\alpha\nabla_\beta n^\beta)\mathrm{d}\Sigma_\alpha.\end{aligned} \tag{2.79}$$

Evaluating this last boundary integral on Σ_{t_1}, where $\mathrm{d}\Sigma_\alpha = n_\alpha\sqrt{h}\,\mathrm{d}^3y$ (extra minus sign already included):

$$\begin{aligned}-2\int_{\Sigma_{t_1}}(n^\beta\nabla_\beta n^\alpha - n^\alpha\nabla_\beta n^\beta)\mathrm{d}\Sigma_\alpha &= -2\int_{\Sigma_{t_1}}(n_\alpha n^\beta\nabla_\beta n^\alpha + \nabla_\beta n^\beta)\sqrt{h}\,\mathrm{d}^3y \\ &= -2\int_{\Sigma_{t_1}}\left[\frac{1}{2}n^\beta\nabla_\beta(n_\alpha n^\alpha) + \nabla_\beta n^\beta\right]\sqrt{h}\,\mathrm{d}^3y \\ &= -2\int_{\Sigma_{t_1}} K\sqrt{h}\,\mathrm{d}^3y.\end{aligned} \tag{2.80}$$

The integration over Σ_{t_2} gives us the same result but with a plus sign. The contribution from $\mathscr{B}$, on which $\mathrm{d}\Sigma_\alpha = r_\alpha\sqrt{-\gamma}\,\mathrm{d}^3z$, is:

$$\begin{aligned}-2\int_{\mathscr{B}}(n^\beta\nabla_\beta n^\alpha - n^\alpha\nabla_\beta n^\beta)\mathrm{d}\Sigma_\alpha &= -2\int_{\mathscr{B}} r_\alpha n^\beta\nabla_\beta n^\alpha\sqrt{-\gamma}\,\mathrm{d}^3z \\ &= -2\int_{\mathscr{B}}[n^\beta\nabla_\beta(r_\alpha n^\alpha) - n^\alpha n^\beta\nabla_\beta r_\alpha]\sqrt{-\gamma}\,\mathrm{d}^3z \\ &= 2\int_{\mathscr{B}} n^\alpha n^\beta\nabla_\beta r_\alpha\sqrt{-\gamma}\,\mathrm{d}^3z.\end{aligned} \tag{2.81}$$

When we put together these results, we see that the integrals over the spacelike boundaries Σ_{t_1} and Σ_{t_2} cancel out so that we are left with

$$2\kappa S_G = \int_{t_1}^{t_2}\mathrm{d}t\int_{\Sigma_t}({}^3R + K^{ab}K_{ab} - K^2)N\sqrt{h}\,\mathrm{d}^3y + 2\int_{\mathscr{B}}(\mathcal{K} + n^\alpha n^\beta\nabla_\beta r_\alpha)\sqrt{-\gamma}\,\mathrm{d}^3z. \tag{2.82}$$

In Eq. (2.82) we have the gravitational action subjected to the foliation of $\mathscr{V}$. We can simplify this expression if we rewrite the integrand of the integral over $\mathscr{B}$ using the discussion about the foliation of $\mathscr{B}$ that we've made previously. For the trace $\mathcal{K}$ we have:

$$\begin{aligned}\mathcal{K} &= \gamma^{ij}\mathcal{K}_{ij} \\ &= (\gamma^{ij}e^{\alpha}_{i}e^{\beta}_{j})\nabla_{\beta}r_{\alpha} \\ &= (g^{\alpha\beta} - r^{\alpha}r^{\beta})\nabla_{\beta}r_{\alpha}.\end{aligned} \tag{2.83}$$

Hence,

$$\begin{aligned}\mathcal{K} + n^{\alpha}n^{\beta}\nabla_{\beta}r_{\alpha} &= (g^{\alpha\beta} - r^{\alpha}r^{\beta} + n^{\alpha}n^{\beta})\nabla_{\beta}r_{\alpha} \\ &= (\sigma^{AB}e^{\alpha}_{A}e^{\beta}_{B})\nabla_{\beta}r_{\alpha} \\ &= \sigma^{AB}k_{AB} \\ &= k.\end{aligned} \tag{2.84}$$

Finally, using $\sqrt{-\gamma}\,\mathrm{d}^3z = N\sqrt{\sigma}\,\mathrm{d}t\,\mathrm{d}^2\theta$, we end up with an expression that displays the familiar way of writing the action S_G as the time integral of a Lagrangian L_G:

$$2\kappa S_G = \int_{t_1}^{t_2}\mathrm{d}t\left[\int_{\Sigma_t}({}^3R + K^{ab}K_{ab} - K^2)N\sqrt{h}\,\mathrm{d}^3y + 2\oint_{S_t}kN\sqrt{\sigma}\,\mathrm{d}^2\theta\right]. \tag{2.85}$$

We notice that the bulk part $\mathscr{L}_{\Sigma_t}$ is written as extrinsic geometry minus intrinsic geometry, reproducing the classic form of a quadratic term $K^{ab}K_{ab} - K^2$, playing the role of kinetic energy, minus $-{}^3R$, as the potential energy. Eq. (2.85) is manifestly invariant under general spatial coordinate transformations.

2.2.2.1 The gravitational Hamiltonian

The $3+1$ decomposition establish the metric field h_{ab} as our dynamical coordinates. The rate of change of a field configuration h_{ab} along the congruence ℓ is given by the associated velocity $\dot{h}_{ab}$ defined by the Lie derivative of h_{ab} along the vector field t^{α} tangent to this congruence:

$$\begin{aligned}\dot{h}_{ab} &\coloneqq \mathcal{L}_t h_{ab} \\ &= \mathcal{L}_t(g_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}) \\ &= (\mathcal{L}_t g_{\alpha\beta})e^{\alpha}_{a}e^{\beta}_{b} \\ &= (\nabla_{\beta}t_{\alpha} + \nabla_{\alpha}t_{\beta})e^{\alpha}_{a}e^{\beta}_{b},\end{aligned} \tag{2.86}$$

where property (2.59) was used in the second line. Recalling that $t^\alpha = Nn^\alpha + N^\alpha$, we have:

$$\begin{aligned} e^\alpha_a e^\beta_b \nabla_\beta t_\alpha &= e^\alpha_a e^\beta_b (n_\alpha \nabla_\beta N + N \nabla_\beta n_\alpha + \nabla_\beta N_\alpha) \\ &= N(e^\alpha_a e^\beta_b \nabla_\beta n_\alpha) + e^\alpha_a e^\beta_b \nabla_\beta N_\alpha \\ &= NK_{ab} + D_b N_a. \end{aligned} \tag{2.87}$$

Therefore,

$$\dot{h}_{ab} = 2NK_{ab} + D_a N_b + D_b N_a. \tag{2.88}$$

The canonical momentum conjugate to h_{ab} is defined by:

$$\Pi^{ab} := \frac{\partial \mathscr{L}_G}{\partial \dot{h}_{ab}}. \tag{2.89}$$

But we see that only the bulk part of the gravitational Lagrangian density in (2.85) depends on $\dot{h}_{ab}$ via the relation between the extrinsic curvature K_{ab} and the velocity established in (2.88). Hence, we may write

$$2\kappa \Pi^{ab} = \frac{\partial K_{mn}}{\partial \dot{h}_{ab}} \frac{\partial}{\partial K_{mn}} (2\kappa \mathscr{L}_{\Sigma_t}), \tag{2.90}$$

where

$$2\kappa \mathscr{L}_{\Sigma_t} = [{}^3R + (h^{ac}h^{bd} - h^{ab}h^{cd})K_{ab}K_{cd}]N\sqrt{h}. \tag{2.91}$$

Using (2.88) we can then proceed with the calculation of Π^{ab}:

$$\begin{aligned} 2\kappa \Pi^{ab} &= \frac{1}{2N} \delta^a{}_m \delta^b{}_n \frac{\partial}{\partial K_{mn}} \left\{ [{}^3R + (h^{ac}h^{bd} - h^{ab}h^{cd})K_{ab}K_{cd}]N\sqrt{h} \right\} \\ &= \frac{1}{2N} \frac{\partial}{\partial K_{ab}} \left\{ [{}^3R + (h^{ac}h^{bd} - h^{ab}h^{cd})K_{ab}K_{cd}]N\sqrt{h} \right\} \\ &= \frac{1}{2}\sqrt{h}\,(h^{ac}h^{bd} - h^{ab}h^{cd}) \frac{\partial}{\partial K_{ab}} (K_{ab}K_{cd}) \\ &= \frac{1}{2}\sqrt{h}\,(h^{ac}h^{bd} - h^{ab}h^{cd})(K_{cd} + \delta^a{}_c \delta^b{}_d K_{ab}) \\ &= \sqrt{h}\,(K^{ab} - Kh^{ab}). \end{aligned} \tag{2.92}$$

The gravitational Hamiltonian density is defined by

$$\mathscr{H}_G := \Pi^{ab}\dot{h}_{ab} - \mathscr{L}_G. \tag{2.93}$$

Using (2.88), (2.91) and (2.92), its bulk part is

$$\begin{aligned}
2\kappa\mathscr{H}_{\Sigma_t} &= \sqrt{h}\,(K^{ab}-Kh^{ab})(2NK_{ab}+2D_{(b}N_{a)}) - N\sqrt{h}\,({}^3R+K^{ab}K_{ab}-K^2)\\
&= N\sqrt{h}\,(2K^{ab}K_{ab}-2K^2-{}^3R-K^{ab}K_{ab}+K^2)+\sqrt{h}\,(K^{ab}-Kh^{ab})2D_bN_a\\
&= N\sqrt{h}\,(K^{ab}K_{ab}-K^2-{}^3R)+\\
&\quad+2\sqrt{h}\,D_b[(K^{ab}-Kh^{ab})N_a]-2\sqrt{h}\,N_aD_b(K^{ab}-Kh^{ab}). \qquad (2.94)
\end{aligned}$$

In the first line we have used the fact that both K^{ab} and h^{ab} are symmetric. To obtain the full gravitational Hamiltonian we just need to integrate $\mathscr{H}_{\Sigma_t}$ over Σ_t and subtract the boundary term in (2.85):

$$\begin{aligned}
2\kappa H_G = &\int_{\Sigma_t} 2\kappa\mathscr{H}_{\Sigma_t}\,\mathrm{d}^3y - 2\oint_{S_t} kN\sqrt{\sigma}\,\mathrm{d}^2\theta\\
= &\int_{\Sigma_t}(K^{ab}K_{ab}-K^2-{}^3R)N\sqrt{h}\,\mathrm{d}^3y+\\
&+2\oint_{S_t}(K^{ab}-Kh^{ab})N_a\,\mathrm{d}S_b-2\int_{\Sigma_t}N_aD_b(K^{ab}-Kh^{ab})\sqrt{h}\,\mathrm{d}^3y+\\
&-2\oint_{S_t}kN\sqrt{\sigma}\,\mathrm{d}^2\theta, \qquad (2.95)
\end{aligned}$$

where $\mathrm{d}S_b = r_b\sqrt{\sigma}\,\mathrm{d}^2\theta$. Gathering these terms we finally arrive at

$$\begin{aligned}
2\kappa H_G = &\int_{\Sigma_t}\Big[(K^{ab}K_{ab}-K^2-{}^3R)N-2N_aD_b(K^{ab}-Kh^{ab})\Big]\sqrt{h}\,\mathrm{d}^3y+\\
&-2\oint_{S_t}\Big[kN-(K^{ab}-Kh^{ab})N_ar_b\Big]\sqrt{\sigma}\,\mathrm{d}^2\theta. \qquad (2.96)
\end{aligned}$$

Here, K_{ab} must be seen as a function of h_{ab} and Π^{ab}: $K_{ab}=K_{ab}(h_{ab},\Pi^{ab})$. This dependence can be written explicitly using (2.92), from which $\sqrt{h}\,K=-\kappa\Pi$, where $\Pi \coloneqq h_{ab}\Pi^{ab}$. This allows us to write

$$\sqrt{h}\,K^{ab} = 2\kappa\left(\Pi^{ab}-\frac{1}{2}\Pi h^{ab}\right). \qquad (2.97)$$

Eq. (2.96) is just the gravitational part of the total Hamiltonian associated with the total action (2.46). But we do not need to worry about subjecting the matter action to the $3+1$ decomposition because we are only interested in vacuum solutions to the Einstein field equations. From the Gauss-Codazzi equations (2.35) and (2.36), such solutions must satisfy the following constraints:

$${}^3R+K^2-K^{ab}K_{ab}=0, \qquad (2.98a)$$

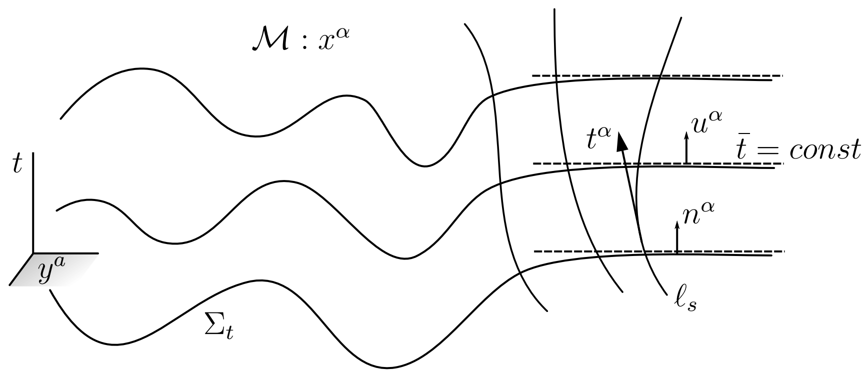
$$D_b(K^{ab}-Kh^{ab})=0. \qquad (2.98b)$$

Therefore, the gravitational Hamiltonian for a vacuum solution is a pure boundary term:

$$H_G^{(0)} = -\frac{1}{\kappa}\oint_{S_t}\left[kN - (K^{ab} - Kh^{ab})N_a r_b\right]\sqrt{\sigma}\,\mathrm{d}^2\theta. \tag{2.99}$$

We must now think about what this quantity $H_G^{(0)}$ is supposed to mean physically. We see that, in addition to h_{ab} and Π^{ab}, $H_G^{(0)}$ also depends on the embedding of the surface S_t in Σ_t, given the presence of $k = \sigma^{AB}k_{AB}$ in (2.99). But more interesting is to note the dependence on the foliation, since that $H_G^{(0)}$ is evaluated where S_t is located, hence depends on how we choose to slice $\mathscr{V}$, and the dependence on the flow lines, which are specified by N and N^a. Each choice that we make for the lapse and shift will produce a different value for the Hamiltonian and perhaps with a different meaning. This undetermined meaning of the Hamiltonian is the price we pay for all arbitrariness that we've kept in constructing the theory. We should now ask ourselves how to reinsert physical meaning into our theory by making meaningful choices for N and N^a. The first thing we must do is to go far away from the gravitational source and consider surfaces S_t in the asymptotic regions of spacetime. From now on, we take $\mathscr{V}$ to be the whole spacetime $\mathcal{M}$, which must be asymptotically flat. In these distant regions we can place a Minkowski frame $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ and restrict the hypersurfaces Σ_t such that it smoothly approach flat hypersurfaces $\bar{t} = constant$. It is worth emphasizing that this is just an assertion about the asymptotic portions of Σ_t, which are still arbitrary in the spacetime bulk.

Figure 7 – Asymptotic behaviour of the foliation in asymptotically flat spacetime.



Source: the author (2021).

The arbitrary coordinates y^a and x^α are asymptotically related to the Minkowski coordinates: $y^a(\bar{x}, \bar{y}, \bar{z})$ and $x^\alpha(\bar{t}, \bar{x}, \bar{y}, \bar{z})$. The unit normal n^α to Σ_t must coincide with the unit normal $u^\alpha := \partial x^\alpha \big/ \partial \bar{t}$ to $\bar{t} = constant$. We then have the following asymptotic expression for the flow vector:

$$t^\alpha \to N\left(\frac{\partial x^\alpha}{\partial \bar{t}}\right)_{\bar{x},\bar{y},\bar{z}} + N^a\left(\frac{\partial x^\alpha}{\partial y^a}\right)_{\bar{t}}. \tag{2.100}$$

The choice $N = 1$ and $N^a = 0$ implies in $t^\alpha \to \partial x^\alpha \big/ \partial \bar{t}$, which generates an asymptotic time translation. The Hamiltonian (2.99) subjected to this particular choice defines the ADM mass:

$$M := -\lim_{S_t\to\infty} \frac{1}{\kappa} \oint_{S_t} k\sqrt{\sigma}\,\mathrm{d}^2\theta. \tag{2.101}$$

With this definition, we succeed in establishing a formal connection between the total energy of spacetime and time translations. Notice that because the metric field h_{ab} is asymptotically flat, $h_{ab} \to \delta_{ab}$, and $N^a = 0$, the extrinsic curvature K_{ab} must asymptotically vanish, so does the momentum Π^{ab}. Flat spacetime is not dynamical. Eq. (2.101) allows us to write the gravitational action (2.85) as

$$S_G = \int \mathrm{d}t \int_{\Sigma_t} \frac{1}{2\kappa}({}^3R + K^{ab}K_{ab} - K^2)N\sqrt{h}\,\mathrm{d}^3y - \int M(t)\,\mathrm{d}t. \tag{2.102}$$

Another fruitful choice is $N = 0$ and $N^a = \partial y^a/\partial\bar{x}$, which generates an asymptotic spatial translation along $\bar{x}$, and the definition

$$P_{\bar{x}} := \lim_{S_t\to\infty} \frac{1}{\kappa} \oint_{S_t} (K^{ab} - Kh^{ab})N_a r_b\sqrt{\sigma}\,\mathrm{d}^2\theta \tag{2.103}$$

establish a formal connection between the total linear momentum of spacetime and spatial translations, both along $\bar{x}$. For the total angular momentum of spacetime associated with asymptotic rotations, we make the choice $N = 0$ and $N^a = \partial y^a/\partial\bar{\varphi} := \varphi^a$, where φ is the angle around some rotation axis in the asymptotic region, which gives

$$-J := \lim_{S_t\to\infty} \frac{1}{\kappa} \oint_{S_t} (K^{ab} - Kh^{ab})\varphi_a r_b\sqrt{\sigma}\,\mathrm{d}^2\theta. \tag{2.104}$$

3 CANONICAL QUANTIZATION OF GENERAL RELATIVITY

3.1 CONSTRAINED HAMILTONIAN SYSTEM

We have already anticipated that there must exist at least "four" constraints in the theory. The fact that the ADM action (2.102) is independent of $\dot{N}$ and $\dot{N}_a$ is expressed by the following constraints:

$$P_N := \frac{\partial \mathscr{L}_G}{\partial \dot{N}} = 0, \tag{3.1a}$$

$$P^a := \frac{\partial \mathscr{L}_G}{\partial \dot{N}_a} = 0. \tag{3.1b}$$

These two independent relations, $P_N(N, N_a, h_{ab}, \dot{h}_{ab}) = 0$ and $P^a(N, N_a, h_{ab}, \dot{h}_{ab}) = 0$, are called primary constraints and imply that $\dot{N}$ and $\dot{N}_a$ are arbitrary and cannot be re-expressed in terms of momenta.

Because of these constraints, the Hamiltonian is not uniquely determined since we could work with

$$2\kappa\widetilde{\mathscr{H}_G} := 2\kappa\mathscr{H}_G + P_N\dot{N} + P^a\dot{N}_a \tag{3.2}$$

and our theory should not be able to distinguish between $\mathscr{H}_G$ and $\widetilde{\mathscr{H}_G}$. However, this seems not to be the case when we compute the PB of $U \in C^\infty(\mathscr{P})$, where $\mathscr{P}$ now denotes the $(20 \cdot \infty^3)$–dimensional phase space built with $(N, N_a, h_{ab}, P_N, P^a, \Pi^{ab})$, with $\widetilde{\mathscr{H}_G}$:

$$\begin{aligned} 2\kappa\dot{U} &= \Big\{U, 2\kappa\widetilde{\mathscr{H}_G}\Big\} \\ &= \Big\{U, 2\kappa\mathscr{H}_G + P_N\dot{N} + P^a\dot{N}_a\Big\} \\ &= 2\kappa\Big\{U, \mathscr{H}_G\Big\} + P_N\Big\{U, \dot{N}\Big\} + \Big\{U, P_N\Big\}\dot{N} + P^a\Big\{U, \dot{N}_a\Big\} + \Big\{U, P^a\Big\}\dot{N}_a, \end{aligned} \tag{3.3}$$

where we have used properties (2.73). Following Dirac (DIRAC, 1964), we only make use of the constraint equations (3.1) after we have worked out the relevant Poisson brackets, and introduce the weak equality symbol $\approx$ to remind us of this rule:

$$P_N \approx 0, \tag{3.4a}$$

$$P^a \approx 0. \tag{3.4b}$$

The equation of motion of the generic dynamical variable U then becomes:

$$2\kappa\dot{U} \approx 2\kappa\Big\{U, \mathscr{H}_G\Big\} + \dot{N}\Big\{U, P_N\Big\} + \dot{N}_a\Big\{U, P^a\Big\}. \tag{3.5}$$

The presence of $\dot{N}$ and $\dot{N}_a$ above seems to represent an issue in our theory because the time evolution of the system is not uniquely determined by an initial state $(N, N_a, h_{ab}, P_N, P^a, \Pi^{ab})(t_0)$ but also by the specification of these arbitrary coefficients, such that two different choices of them could lead an initial state to two different states at later times $t > t_0$. To examine the consequences of this equation of motion, firstly note that (3.4) must hold for all time, which means $\dot{P} \approx 0$ and $\dot{P}^a \approx 0$. If we take U to be P_N and P^a, we thus obtain the two consistency conditions:

$$\dot{P}_N \approx \Big\{P_N, \mathscr{H}_G\Big\} \approx 0, \tag{3.6a}$$

$$\dot{P}^a \approx \Big\{P^a, \mathscr{H}_G\Big\} \approx 0, \tag{3.6b}$$

where we used the fundamental Poisson bracket $\Big\{P_N(t,\mathbf{y}), P^a(t,\mathbf{y}')\Big\} = 0$. It is worth at this point to write down the non-zero fundamental PBs:

$$\Big\{N(t,\mathbf{y}), P_N(t,\mathbf{y}')\Big\} = \delta(\mathbf{y}-\mathbf{y}'), \tag{3.7a}$$

$$\Big\{N_a(t,\mathbf{y}), P^b(t,\mathbf{y}')\Big\} = \delta_a{}^b\, \delta(\mathbf{y}-\mathbf{y}'), \tag{3.7b}$$

$$\Big\{h_{ab}(t,\mathbf{y}), \Pi^{cd}(t,\mathbf{y}')\Big\} = \delta_a{}^{(c}\delta_b{}^{d)}\, \delta(\mathbf{y}-\mathbf{y}'). \tag{3.7c}$$

The symmetrization on the r.h.s. of (3.7c) is necessary because the l.h.s. is symmetric on ab and cd. The PB of two dynamical variables, U and V, with respect to the canonical variables N, N_a, h_{ab}, P_N, P^a, Π^{ab} is:

$$\begin{aligned}\Big\{U(t,\mathbf{y}), V(t,\mathbf{y}')\Big\} = \int_{\Sigma_t}\Bigg[&\left(\frac{\delta U(t,\mathbf{y})}{\delta N(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta P_N(t,\mathbf{y}'')} - \frac{\delta U(t,\mathbf{y})}{\delta P_N(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta N(t,\mathbf{y}'')}\right) + \\ &+\left(\frac{\delta U(t,\mathbf{y})}{\delta N_a(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta P^a(t,\mathbf{y}'')} - \frac{\delta U(t,\mathbf{y})}{\delta P^a(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta N_a(t,\mathbf{y}'')}\right) + \\ &+\left(\frac{\delta U(t,\mathbf{y})}{\delta h_{ab}(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta \Pi^{ab}(t,\mathbf{y}'')} - \frac{\delta U(t,\mathbf{y})}{\delta \Pi^{ab}(t,\mathbf{y}'')}\frac{\delta V(t,\mathbf{y}')}{\delta h_{ab}(t,\mathbf{y}'')}\right)\Bigg]\mathrm{d}^3y''. \end{aligned} \tag{3.8}$$

To work out the consistency conditions (3.6), we first must write (2.96),

$$2\kappa\,(H_G - M) = \int_{\Sigma_t}[N\sqrt{h}\,(K^{ab}K_{ab} - K^2 - {}^3R) - 2N_a\sqrt{h}\,D_b(K^{ab} - Kh^{ab})]\,\mathrm{d}^3y, \tag{3.9}$$

in terms of the canonical variables only. Using (2.97) and $\sqrt{h}\,K = -\kappa\Pi$:

$$\begin{aligned}\sqrt{h}\,(K^{ab}K_{ab} - K^2 - {}^3R) &= \frac{2\kappa^2}{\sqrt{h}}(2\Pi^{ab}\Pi_{ab} - \Pi^2) - \sqrt{h}\,{}^3R \\ &= \frac{2\kappa^2}{\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd})\Pi^{ab}\Pi^{cd} - \sqrt{h}\,{}^3R. \end{aligned} \tag{3.10}$$

Using (2.92):

$$
\begin{aligned}
-2\sqrt{h}\, D_b(K^{ab} - Kh^{ab}) &= -4\kappa\sqrt{h}\, D_b\left(\frac{\Pi^{ab}}{\sqrt{h}}\right) \\
&= -4\kappa D_b\Pi^{ab} + 4\kappa\left(\frac{\Pi^{ab}}{\sqrt{h}}\right) D_b\sqrt{h} \\
&= -4\kappa D_b\Pi^{ab} - 4\kappa\left(\frac{\Pi^{ab}}{\sqrt{h}}\right)\frac{1}{2}\sqrt{h}\, h_{cd}D_b h^{cd} \\
&= -4\kappa D_b\Pi^{ab}. \qquad (3.11)
\end{aligned}
$$

By defining the super-Hamiltonian

$$\chi \coloneqq G_{abcd}\Pi^{ab}\Pi^{cd} - \sqrt{h}\,{}^3R, \qquad (3.12)$$

where

$$G_{abcd} \coloneqq \frac{2\kappa^2}{\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}), \qquad (3.13)$$

and the supermomentum

$$\chi^a \coloneqq -4\kappa D_b\Pi^{ab}, \qquad (3.14)$$

we can write (3.9) in the following compacted way:

$$2\kappa\,(H_G - M) = \int_{\Sigma_t}(N\chi + N_a\chi^a)\,\mathrm{d}^3y. \qquad (3.15)$$

With this form of the Hamiltonian, we compute the PBs in (3.6):

$$
\begin{aligned}
\left\{P_N(t,\mathbf{y}), \mathscr{H}_G(t,\mathbf{y}')\right\} &= -\int_{\Sigma_t}\frac{\delta P_N(t,\mathbf{y})}{\delta P_N(t,\mathbf{y}'')}\,\frac{\delta\mathscr{H}_G(t,\mathbf{y}')}{\delta N(t,\mathbf{y}'')}\,\mathrm{d}^3y'' \\
&= -\int_{\Sigma_t}\delta(\mathbf{y}-\mathbf{y}'')\,\frac{\delta\mathscr{H}_G(t,\mathbf{y}')}{\delta N(t,\mathbf{y}'')}\,\mathrm{d}^3y'' \\
&= -\frac{\delta\mathscr{H}_G(t,\mathbf{y}')}{\delta N(t,\mathbf{y})} \\
&= -\frac{1}{2\kappa}\chi(t,\mathbf{y})\,\delta(\mathbf{y}-\mathbf{y}'). \qquad (3.16)
\end{aligned}
$$

Analogously,

$$\left\{P^a(t,\mathbf{y}), \mathscr{H}_G(t,\mathbf{y}')\right\} = -\frac{1}{2\kappa}\chi^a(t,\mathbf{y})\,\delta(\mathbf{y}-\mathbf{y}'). \qquad (3.17)$$

Therefore, the consistency conditions give rise to the following secondary constraints:

$$\chi \approx 0, \qquad (3.18a)$$

$$\chi^a \approx 0. \qquad (3.18b)$$

As stated by DeWitt (DEWITT, 1967), Eq. (3.18a) is known as the Hamiltonian constraint in virtue of the classical appearance of $\chi = \sqrt{h}\,(K^{ab}K_{ab} - K^2 - {}^3R)$ as the sum of kinetic and potential energies. The fact that this constraint must hold for all time shows a balance between the extrinsic and intrinsic curvatures on each hypersurface Σ_t. The quantity G_{abcd} defined in (3.13) tells us how to contract the momentum. We call it the DeWitt metric, and it is easy to see that it has the following symmetries:

$$G_{abcd} = \begin{cases} G_{bacd}, & (3.19a) \\ G_{abdc}, & (3.19b) \\ G_{cdab}. & (3.19c) \end{cases}$$

To explore the consistency conditions related to the secondary constraints, we need to calculate some PBs first. The PBs of χ and χ^a with P_N and P^a must vanish since χ and χ^a depend only on h_{ab} and Π^{ab}. The PBs involving χ and χ^a are listed bellow (DEWITT, 1967):

$$\Big\{\chi_a(t,\mathbf{y}), \chi_b(t,\mathbf{y}')\Big\} = \chi_b(t,\mathbf{y})\,\partial_a\delta(\mathbf{y}-\mathbf{y}') + \chi_a(t,\mathbf{y}')\,\partial_b\delta(\mathbf{y}-\mathbf{y}'), \tag{3.20a}$$

$$\Big\{\chi_a(t,\mathbf{y}), \chi(t,\mathbf{y}')\Big\} = \chi(t,\mathbf{y})\,\partial_a\delta(\mathbf{y}-\mathbf{y}'), \tag{3.20b}$$

$$\Big\{\chi(t,\mathbf{y}), \chi(t,\mathbf{y}')\Big\} = 2\chi^a(t,\mathbf{y})\,\partial_a\delta(\mathbf{y}-\mathbf{y}') + \partial_a\chi^a(t,\mathbf{y})\,\delta(\mathbf{y}-\mathbf{y}'). \tag{3.20c}$$

Using the total Hamiltonian

$$2\kappa\left(\widetilde{H}_G - M\right) = \int_{\Sigma_t} (P_N\dot{N} + P^a\dot{N}_a + N\chi + N_a\chi^a)\,\mathrm{d}^3y, \tag{3.21}$$

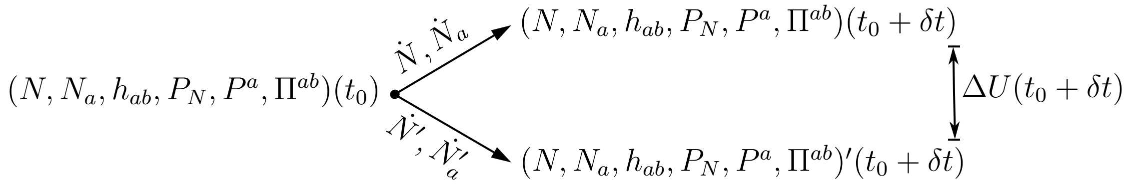
we have:

$$\begin{aligned} 2\kappa\dot{\chi} &= \Big\{\chi(t,\mathbf{y}), 2\kappa\widetilde{\mathscr{H}_G}(t,\mathbf{y}')\Big\} \\ &\approx N(t,\mathbf{y}')\Big\{\chi(t,\mathbf{y}), \chi(t,\mathbf{y}')\Big\} + N^a(t,\mathbf{y}')\Big\{\chi(t,\mathbf{y}), \chi_a(t,\mathbf{y}')\Big\}. \end{aligned} \tag{3.22}$$

But the Poisson brackets (3.20) are linear combinations of χ and χ_a, and thereby the equation above weakly vanishes. The same conclusion applies to $\dot{\chi}^a$. Therefore, the consistency conditions related to the secondary constraints do not give rise to more secondary constraints, and we end up with $2\cdot\infty^3$ physical degrees of freedom.

Since $\mathscr{H}_G$ is zero, we are free to change the time scale of Eq. (3.5). The form of our equation of motion thus does not change under time reparameterization. Before we move on to the quantization procedure, we must unravel the issue we pointed out in (3.5). We will show that the difference between two final states corresponds to the increment of an infinitesimal canonical transformation. Thus, these final states are related by a canonical transformation and thereby represent the same physical state.

Figure 8 – Time evolution of an initial state.



Source: the author (2022).

In first order in δt:

$$\begin{aligned} U(t_0 + \delta t) &= U(t_0) + \dot{U}\,\delta t \\ &= U(t_0) + \Big\{U, \widetilde{\mathscr{H}_G}\Big\}\delta t \\ &\approx U(t_0) + \left(\Big\{U, \mathscr{H}_G\Big\} + \frac{\dot{N}}{2\kappa}\Big\{U, P_N\Big\} + \frac{\dot{N}_a}{2\kappa}\Big\{U, P^a\Big\}\right)\delta t. \end{aligned} \tag{3.23}$$

A different choice N', N'_a yields a difference

$$\begin{aligned} \Delta U(t_0 + \delta t) &= \delta t\frac{(\dot{N}' - \dot{N})}{2\kappa}\Big\{U, P_N\Big\} + \delta t\frac{(\dot{N}'_a - \dot{N}_a)}{2\kappa}\Big\{U, P^a\Big\} \\ &:= \epsilon\Big\{U, P_N\Big\} + \epsilon_a\Big\{U, P^a\Big\}, \end{aligned} \tag{3.24}$$

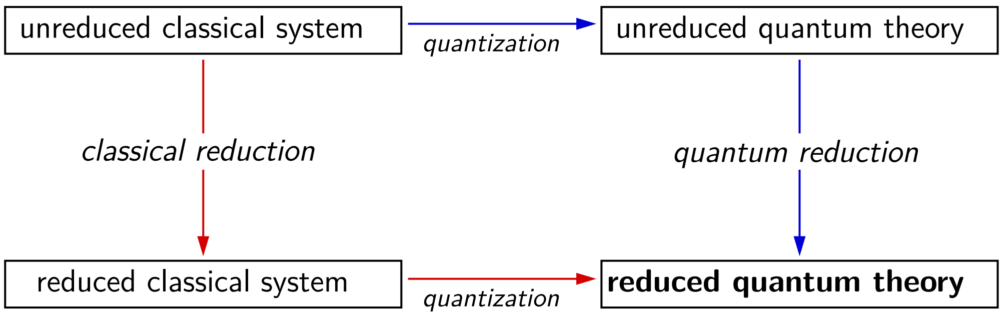
where ϵ and ϵ_a are small arbitrary numbers. We conclude that the primary constraints of our theory are generating functions of infinitesimal canonical transformations. This means that the change of canonical variables $U' = U + \Delta U$, where ΔU is given by (3.24), does not affect the physical state of the system.

3.2 THE CANONICAL QUANTIZATION

To communicate with phenomena at the level of quantum scales, we need to use a different mathematical language than we use for phenomena on classical scales. Quantum mechanics postulates that at a time t_0, the state of a physical system is completely specified by a ket $|\Psi(t_0, \mathbf{x})\rangle$ belonging to a vector space $\mathscr{S}$. Therefore, all the information in the physical degrees of freedom of the system must be contained in $|\Psi(t_0, \mathbf{x})\rangle$. However, the state ket alone only gives us the probability density of finding the system around a certain point of space at time t_0. So we need tools with which we can extract more information about the system. Suppose our system is some water in a glass, and we want to know the temperature of this system. We just act with a thermometer on the water and read its temperature on a scale of real numbers selected by the mercury level. The thermometer is meant to give only the temperature of the water, nothing else. In quantum mechanics, Hermitian operators play the role of the thermometer, acting on $|\Psi(t_0, \mathbf{x})\rangle$ and giving specific information about the system.

When it comes to the quantization of a constrained Hamiltonian system, we face the question of whether we should quantize first the unreduced classical system (gauge + physical degrees of freedom) and then apply some quantum reduction to obtain a genuine quantum theory, or reduce first and then quantize the classical reduced system.

Figure 9 – Two ways of quantizing constrained systems.



Source: reproduction of the diagram in (LOLL, 1990).

We follow the Dirac quantization procedure (blue path), which consists of mapping all the canonical variables into Hermitian operators. Under this quantization map Q from the phase space $\mathscr{P}$ to the space of Hermitian operators H on the state space $\mathscr{S}$, the Poisson brackets are taken into commutators:

$$Q: \quad \mathscr{P} \longrightarrow H(\mathscr{S})$$
$$\{\ ,\ \} \mapsto \frac{1}{i\hbar}[\ ,\ ].$$

The fundamental PBs (3.7) then become:

$$\left[\hat{N}(t, \mathbf{y}), \hat{P}_N(t, \mathbf{y}')\right] = i\hbar\, \delta(\mathbf{y} - \mathbf{y}'), \tag{3.25a}$$
$$\left[\hat{N}_a(t, \mathbf{y}), \hat{P}^b(t, \mathbf{y}')\right] = i\hbar\, \delta_a{}^b\, \delta(\mathbf{y} - \mathbf{y}'), \tag{3.25b}$$
$$\left[\hat{h}_{ab}(t, \mathbf{y}), \hat{\Pi}^{cd}(t, \mathbf{y}')\right] = i\hbar\, \delta_a{}^{(c}\delta_b{}^{d)}\, \delta(\mathbf{y} - \mathbf{y}'). \tag{3.25c}$$

We use the constraints of the classical theory to filter the state kets relevant to our quantum theory. These constraints become conditions on $|\Psi\rangle$:

$$\hat{P}_N |\Psi\rangle = 0, \tag{3.26a}$$

$$\hat{P}^a |\Psi\rangle = 0, \tag{3.26b}$$

$$\hat{\chi} |\Psi\rangle = 0, \tag{3.26c}$$

$$\hat{\chi}_a |\Psi\rangle = 0. \tag{3.26d}$$

All kets that satisfy these eight constraint conditions are said to belong to the physical subspace $\mathscr{S}_{\text{phys}}$ of the state space $\mathscr{S}$. We demand (3.26) to be the only conditions on the state kets. Take (3.26c) for example and multiply it by $\hat{\chi}_a$:

$$\hat{\chi}_a \hat{\chi} |\Psi\rangle = 0. \tag{3.27}$$

On the other hand, if we multiply (3.26d) by $\hat{\chi}$, we get

$$\hat{\chi} \hat{\chi}_a |\Psi\rangle = 0. \tag{3.28}$$

If we subtract these two equations, we obtain

$$[\hat{\chi}_a, \hat{\chi}] |\Psi\rangle = 0. \tag{3.29}$$

For this result to be just a consequence of the constraint conditions, and not a new condition that $|\Psi\rangle$ must satisfy, the commutator $[\hat{\chi}_a, \hat{\chi}]$ must be some linear combination of the constraint operators. The same holds for the remaining commutators, $[\hat{\chi}_a, \hat{\chi}_b]$ and $[\hat{\chi}, \hat{\chi}']$. This conclusion is in agreement with (3.20), where we have linear combinations of χ and χ^a.

To obtain a wave function for $\mathcal{M}$, we need to specify a representation for the state kets. In the configuration space representation, we use $|g_{\alpha\beta}\rangle = |N, N_a, h_{ab}\rangle$ as base kets for $\mathscr{S}$. All the information in the degrees of freedom of $\mathcal{M}$ is contained in these base kets, and to extract it, we must act with the appropriate operator:

$$\hat{N} |N, N_a, h_{ab}\rangle = N |N, N_a, h_{ab}\rangle, \tag{3.30a}$$

$$\hat{N}_a |N, N_a, h_{ab}\rangle = N_a |N, N_a, h_{ab}\rangle, \tag{3.30b}$$

$$\hat{h}_{ab} |N, N_a, h_{ab}\rangle = h_{ab} |N, N_a, h_{ab}\rangle. \tag{3.30c}$$

From the fundamental commutation relations, we know that the observables $\hat{N}$, $\hat{N}_a$, $\hat{h}_{ab}$ are mutually compatible, so they were expected to have simultaneous eigenkets. In this representation, $|\Psi\rangle$ becomes a functional of the configuration space:

$$\langle N, N_a, h_{ab} | \Psi\rangle = \Psi[N, N_a, h_{ab}]. \tag{3.31}$$

If $\{|g_{\alpha\beta}\rangle\}$ is a basis, then it defines $10 \cdot \infty^3$ linear independent directions in $\mathscr{S}$, along which any $|\Psi\rangle \in \mathscr{S}$ can be projected down. Therefore, the sum of all projection operators $|g_{\alpha\beta}\rangle\langle g_{\alpha\beta}|$ must be the unit operator, otherwise there would be directions along which $|\Psi\rangle$ could not be projected. This means that our set of kets would not span the entire state space, so it would not be a basis. We then have

$$\sum_{\alpha\leq\beta}\int_{\Sigma_t} \mathrm{d}^3y\,\sqrt{h}\;|g_{\alpha\beta}(t,\mathbf{y})\rangle\langle g_{\alpha\beta}(t,\mathbf{y})| = \hat{1}, \tag{3.32}$$

where we must restrict the indices of the sum because the metric is symmetric, so $|g_{\alpha\beta}\rangle\langle g_{\alpha\beta}| = |g_{\beta\alpha}\rangle\langle g_{\beta\alpha}|$, and we do not end up with $6\cdot\infty^3$ directions been counted twice. This closure relation allows us to write

$$|\Psi\rangle = \sum_{\alpha\leq\beta}\int_{\Sigma_t} \mathrm{d}^3y\,\sqrt{h}\;|g_{\alpha\beta}(t,\mathbf{y})\rangle\;\Psi[g_{\alpha\beta}(t,\mathbf{y})]. \tag{3.33}$$

If we interpret the modulus square of the wave functional, $|\Psi[g_{\alpha\beta}(x)]|^2$, as the probability density of $\mathcal{M}$ to have a local metric field $g_{\alpha\beta}(t,\mathbf{y})$ around a point $\mathbf{y}\in\Sigma_t$, then Ψ must be square-integrable:

$$\sum_{\alpha\leq\beta}\int_{\Sigma_t} |\Psi[g_{\alpha\beta}(t,\mathbf{y})]|^2\sqrt{h}\;\mathrm{d}^3y < \infty. \tag{3.34}$$

We now need to define an inner product in $\mathscr{S}$ under which the state kets can be normalized and the observables are Hermitian. Using (3.32):

$$\begin{aligned}\langle\Psi_2|\Psi_1\rangle &= \sum_{\alpha\leq\beta}\int_{\Sigma_t} \mathrm{d}^3y\,\sqrt{h}\;\langle\Psi_2|g_{\alpha\beta}(t,\mathbf{y})\rangle\,\langle g_{\alpha\beta}(t,\mathbf{y})|\Psi_1\rangle \\ &= \sum_{\alpha\leq\beta}\int_{\Sigma_t} \mathrm{d}^3y\,\sqrt{h}\,\Psi_2^*[g_{\alpha\beta}(t,\mathbf{y})]\Psi_1[g_{\alpha\beta}(t,\mathbf{y})]. \end{aligned}\tag{3.35}$$

This equation gives the probability amplitude for state $|\Psi_1\rangle$ to be found in state $|\Psi_2\rangle$. In particular, we must have $\langle\Psi|\Psi\rangle = 1$, and thereby our quantum states are normalized. The state space $\mathscr{S}$ equipped with the inner product (3.35) is a Hilbert space $\mathcal{H}$.

In the configuration space representation, the momenta become functional derivative operators:

$$\langle N, N_a, h_{ab}|\hat{P}_N|\Psi\rangle = -i\hbar\frac{\delta}{\delta N}\Psi[N, N_a, h_{ab}], \tag{3.36a}$$

$$\langle N, N_a, h_{ab}|\hat{P}^a|\Psi\rangle = -i\hbar\frac{\delta}{\delta N_a}\Psi[N, N_a, h_{ab}], \tag{3.36b}$$

$$\langle N, N_a, h_{ab}|\hat{\Pi}^{ab}|\Psi\rangle = -i\hbar\frac{\delta}{\delta h_{ab}}\Psi[N, N_a, h_{ab}]. \tag{3.36c}$$

Eqs. (3.26a) and (3.26b) then tell us that Ψ depends only on h_{ab}. The physical quantum states thus belong to the physical Hilbert space $\mathcal{H}_{\text{phys}}$, spanned by $\{|h_{ab}\rangle\}$.

We introduce dynamics in $\mathcal{H}_{\text{phys}}$ through the Hamiltonian constraint, which is the "normal component" of the gravitational Hamiltonian. In the "metric representation", Eqs. (3.26c) and (3.26d), with a simple choice of factor ordering, take the form

$$\left(-\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \sqrt{h}\,{}^3R\right)\Psi[h_{ab}] = 0, \tag{3.37a}$$

$$4i\hbar\kappa D_c h_{ab}\frac{\delta}{\delta h_{bc}}\Psi[h_{ab}] = 0. \tag{3.37b}$$

Taken together, these equations are the quantum mechanical version of the Einstein field equations. Eq. (3.37a) is the so-called Wheeler–DeWitt equation. This second order functional derivative equation is based on a wave function over three-geometries and imply the possibility of superposition of different spacetimes.

4 CANONICAL QUANTIZATION OF SCHWARZSCHILD BLACK HOLE

4.1 THE SCHWARZSCHILD BLACK HOLE

In this chapter, we apply the theory we have developed so far to the Schwarzschild spacetime. From the no-hair theorem (ISRAEL, 1967), we know in advance about the uniqueness of the Schwarzschild metric:

$$\mathrm{d}s^2_{\mathsf{SCH}} = -F(R)\mathrm{d}T^2 + F^{-1}(R)\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2, \tag{4.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on the two-sphere S^2 and $x^\alpha = (T, R, \theta, \phi)$ are the Schwarzschild spacetime coordinates. The function $F(R)$ has the explicit form

$$F(R) = 1 - \frac{2GM}{R}, \tag{4.2}$$

where $M \geq 0$ is the constant Schwarzschild mass parameter. The T independence of the metric coefficients entails $\partial/\partial T$ to be a Killing vector field of the metric (4.1). For this reason we call T the Killing time. On the other hand, the radial curvature coordinate $R \geq 0$ is invariantly defined by the requirement that $4\pi R^2$ be the area of the surface $T = const$, $R = const$.

To explore the causal structure of this spacetime, let us look at the behavior of the light cones. Evaluating (4.1) along radial null geodesics, we find

$$\frac{\mathrm{d}T}{\mathrm{d}R} = \pm F^{-1}, \tag{4.3}$$

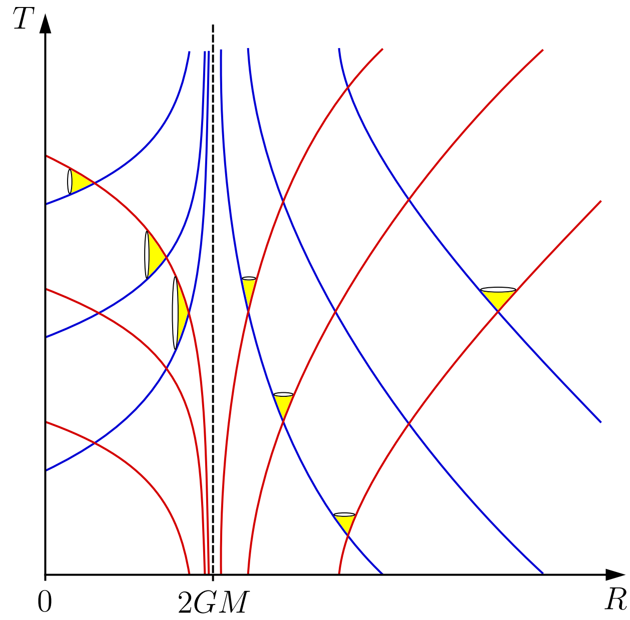
which measures the slope of the light cones. As one approaches the black hole from far away, the slope goes from ± 1 at $R \gg 2GM$ to $\pm\infty$ near $R = 2GM$. In the Schwarzschild coordinates, the light cones then close up when we move from the asymptotic regions of spacetime and get close to the surface $R = 2GM$. This behavior of the light cones gives the impression that a free-falling observer would never cross the surface $R = 2GM$.

We know that the Schwarzschild coordinates are ill-behaved on this surface, since $g_{TT} \to 0$ and $g_{RR} \to \infty$ as $R \to 2GM$. Besides, the metric signature swap from $(-,+,+,+)$ above the surface to $(+,-,+,+)$ beneath it, setting R as the timelike coordinate. However, the curvature invariant

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48G^2M^2}{R^6} \tag{4.4}$$

shows that $R = 2GM$ is just a coordinate singularity. The physical singularity is located at $R = 0$. What the Schwarzschild coordinates are trying to tell us is that the black hole event horizon is located at $R = 2GM$. Below the horizon, everything is destined to fall downwards with no hope of ever coming out.

Figure 10 – The Schwarzschild coordinates can not be extended into (or out of) the black hole, such that we need two patches to cover both regions, $R > 2GM$ and $R < 2GM$.



Source: the author (2022).

To clarify the problem of the asymptotic approach of a free-falling observer, as seen by a stationary (at rest with respect to the hole) observer far away outside the black hole, consider the latter sitting at constant spatial coordinates (R_0, θ_0, ϕ_0), and the former's radial trajectory starting at this very position. Let R and τ denote the radius and the proper time of the falling observer. To calculate the coordinate speed $v = \mathrm{d}R/\mathrm{d}T$ of the falling observer, we start evaluating the Schwarzschild metric, $\mathrm{d}s^2_{\mathsf{SCH}} = -\mathrm{d}\tau^2$, along her radial trajectory. Using (4.1), we have

$$\mathrm{d}\tau^2 = F\mathrm{d}T^2 - F^{-1}\mathrm{d}R^2, \tag{4.5}$$

from which we get

$$v = \pm F\sqrt{1 - F^{-1}\left(\frac{\mathrm{d}\tau}{\mathrm{d}T}\right)^2}, \tag{4.6}$$

where we must take the negative sign. Since the metric is independent of T, there is a timelike Killing vector $K^\alpha = (1, 0, 0, 0)$, and the energy per unit mass of the falling observer

$$E = F\frac{\mathrm{d}T}{\mathrm{d}\tau} \tag{4.7}$$

is a conserved quantity along the trajectory (see (CARROLL, 2019), Eq. 5.61). By substituting (4.7) into (4.6), we obtain

$$v = -F\sqrt{1 - \frac{F}{E^2}}. \tag{4.8}$$

Using the initial condition $v(R_0) = 0$, we set $E^2 = F_0 := F(R_0)$, so the coordinate speed of the falling observer is given by

$$v(R) = -F\sqrt{1 - \frac{F}{F_0}}, \tag{4.9}$$

which indeed goes to zero at $R = 2GM$.

Similarly, to calculate the proper speed $u = \mathrm{d}R/\mathrm{d}\tau$, we use (4.5) and (4.7), with $E^2 = F_0$, and conclude that

$$u(R) = -\sqrt{F_0 - F}, \tag{4.10}$$

which goes to $-\sqrt{F_0} \neq 0$ at $R = 2GM$. The greater the value of the initial radius R_0, the greater the proper speed $|u|$ at the event horizon. In particular, for $R_0 \to \infty$, $u(2GM)$ approaches the speed of light.

Therefore, although the stationary observer sees her image fade and gradually slow down until freezing on the horizon, the falling observer faces no trouble in crossing the horizon in a finite amount of her proper time, given by:

$$\begin{aligned}\Delta\tau &= \frac{1}{\sqrt{2GM}} \int_{2GM}^{R_0} \frac{\mathrm{d}R}{\sqrt{\dfrac{1}{R} - \dfrac{1}{R_0}}} \\ &= \frac{1}{\sqrt{2GM}} \left[R_0^{3/2} \arctan\left(R_0^{1/2} \sqrt{\frac{1}{2GM} - \frac{1}{R_0}} \right) + 2GMR_0 \sqrt{\frac{1}{2GM} - \frac{1}{R_0}} \right].\end{aligned} \tag{4.11}$$

We now look for a coordinate system well-behaved everywhere. A coordinate patch that covers the entire manifold, except the physical singularity, is given by the Kruskal coordinates $(\widetilde{T}, \widetilde{R}, \theta, \phi)$ ((MISNER; THORNE; WHEELER, 1973), sec. 31), where

$$\textbf{(I)}\begin{cases} \widetilde{T}(T,R) = \left(\dfrac{R}{2GM} - 1\right)^{1/2} \mathrm{e}^{R/4GM} \sinh\left(\dfrac{T}{4GM}\right), & \text{(4.12a)} \\ \widetilde{R}(T,R) = \left(\dfrac{R}{2GM} - 1\right)^{1/2} \mathrm{e}^{R/4GM} \cosh\left(\dfrac{T}{4GM}\right), & \text{(4.12b)} \end{cases}$$

for the outer region $R > 2GM$, and

$$\textbf{(II)}\begin{cases} \widetilde{T}(T,R) = \left(1 - \dfrac{R}{2GM}\right)^{1/2} \mathrm{e}^{R/4GM} \cosh\left(\dfrac{T}{4GM}\right), & \text{(4.13a)} \\ \widetilde{R}(T,R) = \left(1 - \dfrac{R}{2GM}\right)^{1/2} \mathrm{e}^{R/4GM} \sinh\left(\dfrac{T}{4GM}\right), & \text{(4.13b)} \end{cases}$$

for the inner region $0 < R < 2GM$. The Schwarzschild metric in terms of the Kruskal coordinates reads

$$\mathrm{d}s^2_{\mathsf{SCH}} = -\frac{32G^3M^3}{R}\,\mathrm{e}^{-R/2GM}\,(\mathrm{d}\widetilde{T}^2 - \mathrm{d}\widetilde{R}^2) + R^2\mathrm{d}\Omega^2, \tag{4.14}$$

where R must be understood as a function of $\widetilde{T}$ and $\widetilde{R}$, implicitly defined by

$$\widetilde{R}^2 - \widetilde{T}^2 = \left(\frac{R}{2GM} - 1\right)\mathrm{e}^{R/2GM}. \tag{4.15}$$

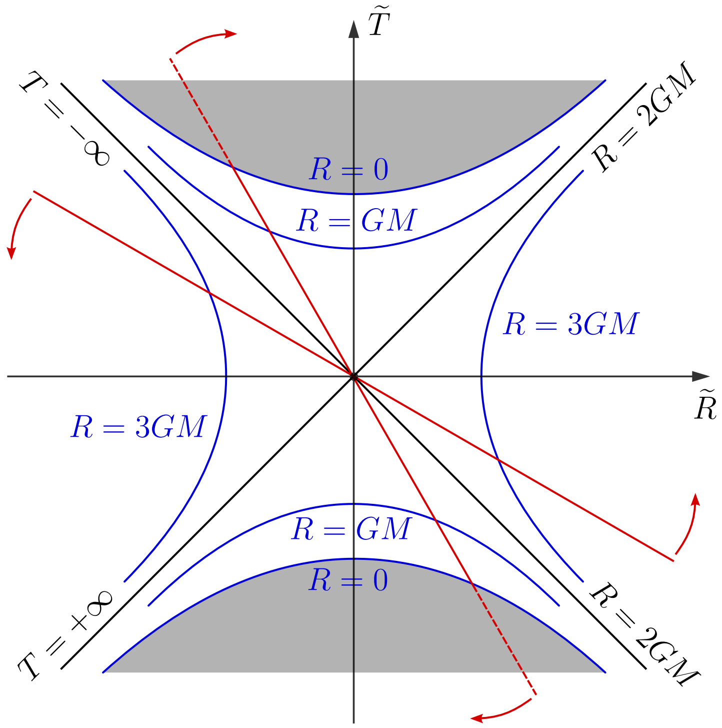
This metric is manifestly regular at $R = 2GM$, and evaluating it along radial null geodesics, we see that the light cones on the Kruskal diagram ($\widetilde{R} - \widetilde{T}$ plane) are oriented at 45 degrees everywhere. On a Kruskal diagram, the surfaces $R = constant$, both outside and inside the black hole, are defined by (4.15). Hence, they will be represented by lateral hyperbolas outside the black hole, while inside, we get vertical hyperbolas. At $R = 2GM$, these hyperbolas degenerate into their asymptotes $\widetilde{T} = \pm\widetilde{R}$. On the other hand, the surfaces of simultaneity $T = constant$ are straight lines through the origin:

$$\widetilde{T} = \tanh\left(\frac{T}{4GM}\right)\widetilde{R}, \tag{4.16a}$$

$$\widetilde{R} = \tanh\left(\frac{T}{4GM}\right)\widetilde{T}, \tag{4.16b}$$

outside and inside the black hole, respectively. We observe that $T \to \pm\infty$ implies in $\widetilde{T} \to \pm\widetilde{R}$, such that the surfaces $R = 2GM$ and $T = \pm\infty$ coincide. The Kruskal coordinates gave us two copies of each surface $R = constant$, thus creating two copies of the universe, and ended up revealing that the region of the manifold not covered by the Schwarzschild coordinates is much larger than we thought.

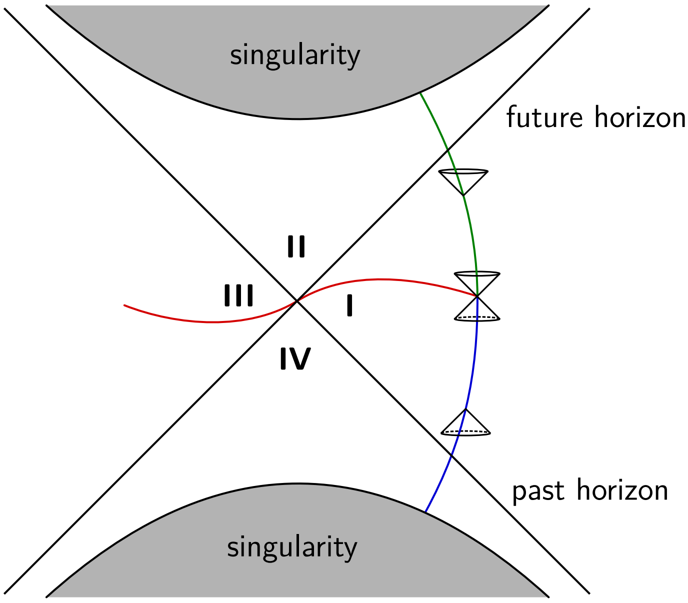
Figure 11 – Kruskal diagram of Schwarzschild spacetime. The red lines represent the surfaces $T = constant$.



Source: the author (2022).

With the two angular coordinates suppressed, each point of this diagram represents a spherical surface. The origin, in particular, is called the bifurcation two-sphere, and this whole manifold that was revealed to us is called the maximally extended Schwarzschild spacetime, being divided into four regions. Region **I** is where we live safely outside the black hole. By following a future-directed timelike worldline, which lies within the light cones, one falls into region **II**, the black hole region, while following a past-directed timelike worldline, one reaches region **IV**, usually called the white hole region. It is in this sense that we refer to the surface $R = 2GM$ that separates regions **I** and **II** as a future horizon, while the surface $R = 2GM$ that separates regions **I** and **IV** is the past horizon. Spacelike geodesics, which lie outside the light cones, would lead us to region **III**, which represents an asymptotically flat universe identical, but distinct, to that of region **I**.

Figure 12 – Regions of the maximally extended Schwarzschild spacetime.



Source: the author (2022).

Of course that this mirror universe where time seems to pass in reverse only exists if we are talking about eternal black holes. For a real black hole, which is the outcome of the gravitational collapse of a sufficiently massive star, the diagram must be cut off at a timelike boundary representing the surface of the collapsing star, so regions **III** and **IV** would not exist.

4.2 GEOMETRODYNAMICS OF SBH

4.2.1 Hamiltonian formalism for a spherically symmetric spacetime

The following geometrodynamical approach stems from Kuchař (KUCHAR, 1994) and consists in evolving, according to the ADM action (2.102), the geometry of a spherically symmetric spacelike hypersurface Σ, and then reconcile our solution with Schwarzschild's by requiring them to be locally isomorphic. On Σ, we place coordinates $y^a = (r, \theta, \phi)$ adapted to its symmetry. In this coordinates, the metric h_{ab} on Σ is completely characterized by two functions $\Lambda(r)$ and $R(r)$ of the radial label r:

$$\mathrm{d}s^2_\Sigma = \Lambda^2(r)\mathrm{d}r^2 + R^2(r)\mathrm{d}\Omega^2. \tag{4.17}$$

In matrix notation:

$$(h_{ab}) = \begin{pmatrix} \Lambda^2 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\theta \end{pmatrix}. \tag{4.18}$$

We take $\Lambda(r)$ to be positive, and since Σ is asymptotically flat, we have $\Lambda(|r| = \infty) = 1$ and $R(|r| = \infty) = |r|$, such that (4.17) becomes the flat spherical polar metric. Let us proceed with the calculations of the intrinsic geometry of Σ. The non-zero spatial Christoffel symbols ${}^3\Gamma^c{}_{ab}$ are displayed bellow:

$$\begin{aligned}
&{}^3\Gamma^r{}_{rr} = \frac{\Lambda'}{\Lambda}, && {}^3\Gamma^\theta{}_{r\theta} = \frac{R'}{R}, && {}^3\Gamma^\phi{}_{r\phi} = \frac{R'}{R}, \\
&{}^3\Gamma^r{}_{\theta\theta} = -\left(\frac{R}{\Lambda}\right)^2 \frac{R'}{R}, && {}^3\Gamma^\theta{}_{\phi\phi} = \sin\theta\cos\theta, && {}^3\Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}, \\
&{}^3\Gamma^r{}_{\phi\phi} = {}^3\Gamma^r{}_{\theta\theta}\sin^2\theta,
\end{aligned}$$

where the primes $'$ denote differentiation with respect to r. These Christoffel symbols yield a diagonal spatial Ricci tensor ${}^3R_{ab}$:

$$\begin{aligned}
{}^3R_{rr} &= -2\frac{R''}{R} + 2\frac{\Lambda' R'}{\Lambda R}, \\
{}^3R_{\theta\theta} &= -\left(\frac{R}{\Lambda}\right)^2 \frac{R''}{R} - \left(\frac{R'}{\Lambda}\right)^2 + \left(\frac{R}{\Lambda}\right)^2 \frac{R'\Lambda'}{R\Lambda} + 1, \\
{}^3R_{\phi\phi} &= {}^3R_{\theta\theta}\sin^2\theta.
\end{aligned}$$

By contracting ${}^3R_{ab}$ with the inverse metric h^{ab}, we obtain the spatial Ricci scalar:

$${}^3R = -4\frac{R''}{\Lambda^2 R} + 4\frac{\Lambda' R'}{\Lambda^3 R} - 2\frac{R'^2}{\Lambda^2 R^2} + \frac{2}{R^2}. \tag{4.19}$$

We now make a $3+1$ composition and generate the spherically symmetric spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$ by labeling Σ with a time parameter $t \in \mathbb{R}$ and stacking the many hypersurfaces Σ_t into a foliation. Consequently, the metric coefficients Λ and R become also functions of t. We then associate with this nesting of three-dimensional spheres a lapse function $N(t, r)$ and a shift vector $N^a(t, r)$. Due to the spherical symmetry, the shift vector has only the radial component: $N^a = (N^r, 0, 0)$.

Let us now calculate the extrinsic curvature of each hypersurface Σ_t. From (2.88), we have:

$$K_{ab} = \frac{1}{2N}(\dot{h}_{ab} - \partial_a N_b - \partial_b N_a + 2\,{}^3\Gamma^c{}_{ab} N_c). \tag{4.20}$$

We found that K_{ab} is diagonal, with

$$K_{rr} = \frac{\Lambda}{N}(\dot{\Lambda} - (\Lambda N^r)'),$$
$$K_{\theta\theta} = \frac{R}{N}(\dot{R} - R'N^r),$$
$$K_{\phi\phi} = K_{\theta\theta}\sin^2\theta.$$

The "kinetic energy" of the hypersurfaces is

$$\begin{aligned} K^{ab}K_{ab} - K^2 = &\left[\frac{1}{N^2\Lambda^2}(\dot{\Lambda} - (\Lambda N^r)')^2 + \frac{2}{N^2R^2}(\dot{R} - R'N^r)^2\right] + \\ &- \left[\frac{1}{N^2\Lambda^2}(\dot{\Lambda} - (\Lambda N^r)')^2 + \frac{4}{N^2R^2}(\dot{R} - R'N^r)^2 + \right. \\ &\left. + \frac{4}{N^2\Lambda R}(\dot{\Lambda} - (\Lambda N^r)')(\dot{R} - R'N^r)\right] \\ = &- \frac{4}{N^2R^2}\left[\frac{1}{2}(\dot{R} - R'N^r)^2 + \frac{R}{\Lambda}(\dot{\Lambda} - (\Lambda N^r)')(\dot{R} - R'N^r)\right]. \end{aligned} \tag{4.21}$$

The metric $g_{\alpha\beta}$ of this $3+1$ composed spacetime is given by the ADM metric (2.62b):

$$\mathrm{d}s^2_{\mathsf{ADM}} = -[N^2 - (\Lambda N^r)^2]\mathrm{d}t^2 + 2\Lambda^2 N^r \mathrm{d}t\mathrm{d}r + \Lambda^2\mathrm{d}r^2 + R^2\mathrm{d}\Omega^2. \tag{4.22}$$

In matrix notation:

$$(g_{\alpha\beta}) = \begin{pmatrix} -[N^2 - (\Lambda N^r)^2] & \Lambda^2 N^r & 0 & 0 \\ \Lambda^2 N^r & \Lambda^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix}, \tag{4.23}$$

from which

$$\sqrt{-g} = N\Lambda R^2 \sin\theta = N\sqrt{h}. \tag{4.24}$$

Gathering (4.19), (4.21), and (4.24) into the ADM action (2.102), we have:

$$\begin{aligned} \mathscr{L}_{\Sigma_t} &= \frac{1}{2\kappa}({}^3R + K^{ab}K_{ab} - K^2)N\sqrt{h} \\ &= \frac{2}{\kappa}\left\{N\left(-\frac{RR''}{\Lambda} + \frac{\Lambda'RR'}{\Lambda^2} - \frac{R'^2}{2\Lambda} + \frac{\Lambda}{2}\right) + \right. \\ &\quad \left. - \frac{1}{N}\left[\frac{\Lambda}{2}(\dot{R} - R'N^r)^2 + R(\dot{\Lambda} - (\Lambda N^r)')(\dot{R} - R'N^r)\right]\right\}\sin\theta. \end{aligned} \tag{4.25}$$

Therefore, the bulk action $S_\Sigma[\Lambda, R; N, N^r]$ is given by:

$$S_\Sigma = \int \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}r\, \frac{8\pi}{\kappa}\Bigg\{ N\left(-\frac{RR''}{\Lambda} + \frac{\Lambda' R R'}{\Lambda^2} - \frac{R'^2}{2\Lambda} + \frac{\Lambda}{2}\right) + \\ - \frac{1}{N}\left[\frac{\Lambda}{2}(\dot{R} - R'N^r)^2 + R(\dot{\Lambda} - (\Lambda N^r)')(\dot{R} - R'N^r)\right]\Bigg\}, \tag{4.26}$$

where the solid angle 4π came from the integration over θ and ϕ, and we let the parameter r run from $-\infty$ to $+\infty$ since we intend to cover all the extended Schwarzschild spacetime.

To pass from the Lagrangian to the Hamiltonian, we need to calculate all the conjugate momenta. The angular independence in the integrand of (4.26) reflects the spherical symmetry of $\mathcal{M}$. Such as $\mathscr{L}_{\Sigma_t}$, this integrand is also a linear Lagrangian density, and the dynamics of Λ and R will follow from it. We shall denote it by $\mathscr{L}$. As we already know, N and N^r are not dynamical:

$$P_N = \frac{\partial \mathscr{L}}{\partial \dot{N}} = 0 \quad \text{and} \quad P_r = \frac{\partial \mathscr{L}}{\partial \dot{N}^r} = 0. \tag{4.27}$$

On the other hand, we have:

$$P_\Lambda = \frac{\partial \mathscr{L}}{\partial \dot{\Lambda}} = -\frac{m_P^2 R}{N}(\dot{R} - R'N^r), \tag{4.28a}$$

$$P_R = \frac{\partial \mathscr{L}}{\partial \dot{R}} = -\frac{m_P^2}{N}[\Lambda(\dot{R} - R'N^r) + R(\dot{\Lambda} - (\Lambda N^r)')], \tag{4.28b}$$

where $m_P = G^{-1/2}$ is the Planck mass ($\hbar = 1$). We can solve these equations for the velocities $\dot{R}$ and $\dot{\Lambda}$:

$$\dot{R} = -\frac{N}{m_P^2 R} P_\Lambda + R'N^r, \tag{4.29a}$$

$$\dot{\Lambda} = -\frac{N}{m_P^2 R^2}(RP_R - \Lambda P_\Lambda) + (\Lambda N^r)'. \tag{4.29b}$$

If we substitute (4.29) into (4.26), we obtain $\mathscr{L}$ in terms of the canonical variables Λ, R, P_Λ, P_R:

$$\mathscr{L} = \frac{N\Lambda}{2m_P^2 R^2} P_\Lambda^2 - \frac{N}{m_P^2 R} P_\Lambda P_R + m_P^2 N\left(-\frac{RR''}{\Lambda} + \frac{\Lambda' R R'}{\Lambda^2} - \frac{R'^2}{2\Lambda} + \frac{\Lambda}{2}\right). \tag{4.30}$$

The Hamiltonian density $\mathscr{H}$ is then given by:

$$\begin{aligned}\mathscr{H} &:= P_\Lambda \dot{\Lambda} + P_R \dot{R} - \mathscr{L} \\ &= \frac{N\Lambda}{2m_P^2 R^2} P_\Lambda^2 - \frac{N}{m_P^2 R} P_\Lambda P_R - m_P^2 N\left(-\frac{RR''}{\Lambda} + \frac{\Lambda' R R'}{\Lambda^2} - \frac{R'^2}{2\Lambda} + \frac{\Lambda}{2}\right) + \\ &\quad + N^r(P_R R' - \Lambda P_\Lambda') + (N^r \Lambda P_\Lambda)'. \end{aligned} \tag{4.31}$$

Not only the shift vector N^r but also the momenta P_Λ and P_R are asymptotically zero, so the term $(N^r \Lambda P_\Lambda)'$ does not contribute to the action,

$$\int \mathrm{d}t \, (N^r \Lambda P_\Lambda)\Big|_{r=-\infty}^{r=+\infty} = 0, \tag{4.32}$$

and we can remove it from (4.31):

$$\mathscr{H} = N\chi + N^r \chi_r, \tag{4.33}$$

where

$$\chi \coloneqq \frac{\Lambda}{2m_P^2 R^2} P_\Lambda^2 - \frac{1}{m_P^2 R} P_\Lambda P_R - m_P^2 \left(-\frac{RR''}{\Lambda} + \frac{\Lambda' R R'}{\Lambda^2} - \frac{R'^2}{2\Lambda} + \frac{\Lambda}{2} \right) \tag{4.34}$$

is the super-Hamiltonian, and

$$\chi_r \coloneqq P_R R' - \Lambda P_\Lambda' \tag{4.35}$$

is the radial component of the supermomentum.

The primary constraints (4.27) together with the Hamilton's equations of motion,

$$\dot{P}_N = -\frac{\partial \mathscr{H}}{\partial N} = -\chi, \tag{4.36a}$$

$$\dot{P}_r = -\frac{\partial \mathscr{H}}{\partial N^r} = -\chi_r, \tag{4.36b}$$

give the secondary constraints:

$$\chi = 0 \quad \text{and} \quad \chi_r = 0. \tag{4.37}$$

Using the Lagrange multipliers N and N^r, the action S_Σ can be cast into canonical form:

$$S_\Sigma[\Lambda, R, P_\Lambda, P_R; N, N^r] = \int \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}r \, (P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\chi - N^r \chi_r). \tag{4.38}$$

4.2.2 Schwarzschild spacetime

We now wonder if the boundary action

$$S_{\partial\Sigma} = \int M(t) \, \mathrm{d}t \tag{4.39}$$

is also a functional of the canonical variables. For this to be true, the ADM mass must be a dynamical variable that satisfies $M' = 0$, which guarantees M to depend solely on time t. Since the ADM metric (4.22) must be locally isometric to the Schwarzschild metric (4.1), then the ADM mass of $\mathcal{M}$ is the Schwarzschild mass. Substituting $T = T(t,r)$ and $R = R(t,r)$ into (4.1), we get:

$$\begin{aligned} \mathrm{d}s^2_{\mathsf{SCH}} = & -(F\dot{T}^2 - F^{-1}\dot{R}^2)\mathrm{d}t^2 + 2(-F\dot{T}T' + F^{-1}\dot{R}R')\mathrm{d}t\mathrm{d}r + \\ & + (-FT'^2 + F^{-1}R'^2)\mathrm{d}r^2 + R^2\mathrm{d}\Omega^2. \end{aligned} \tag{4.40}$$

Comparing these metric coefficients with those of (4.22), we see that

$$\Lambda^2 = -FT'^2 + F^{-1}R'^2, \tag{4.41a}$$

$$\Lambda^2 N^r = -F\dot{T}T' + F^{-1}\dot{R}R', \tag{4.41b}$$

$$N^2 - (\Lambda N^r)^2 = F\dot{T}^2 - F^{-1}\dot{R}^2. \tag{4.41c}$$

We can use the first two equations to solve the third one for N. After a few algebraic steps, one finds

$$N = \frac{\dot{T}R' - T'\dot{R}}{\Lambda}. \tag{4.42}$$

By using this result together with

$$N^r = \frac{-F\dot{T}T' + F^{-1}\dot{R}R'}{\Lambda^2} \tag{4.43}$$

and the first equation (4.41a) in Eq. (4.28a), the time derivatives $\dot{T}$ and $\dot{R}$ drop out, and we obtain the relation

$$-FT' = \frac{\Lambda}{m_P^2 R} P_\Lambda. \tag{4.44}$$

The substitution of this relation into (4.41a) gives F in terms of the canonical variables:

$$F = \left(\frac{R'}{\Lambda}\right)^2 - \left(\frac{P_\Lambda}{m_P^2 R}\right)^2. \tag{4.45}$$

Lastly, bringing Eq. (4.2), we show that the Schwarzschild mass can indeed be read from the canonical data:

$$M = \frac{P_\Lambda^2}{2m_P^2 R} - \frac{m_P^2 R R'^2}{2\Lambda^2} + \frac{m_P^2 R}{2}. \tag{4.46}$$

If we differentiate this expression with respect to r and then add zero,

$$\frac{R'}{m_P^2 \Lambda R} P_\Lambda P_R - \frac{R'}{m_P^2 \Lambda R} P_\Lambda P_R,$$

by recalling definitions (4.34) and (4.35), we find that M' is a linear combination of the secondary constraints:

$$M' = -\frac{1}{\Lambda}\left(R'\chi + \frac{P_\Lambda}{m_P^2 R}\chi_r\right). \tag{4.47}$$

A remarkable outcome of this algebraic procedure of constructing M from the canonical data is that M and $P_M := -T'$, given by (4.44), form a pair of canonically conjugate variables:

$$\begin{aligned}\left\{M(t,r), P_M(t,r')\right\} &= \int \left(\frac{\delta M(t,r)}{\delta \Lambda(t,r'')} \frac{\delta P_M(t,r')}{\delta P_\Lambda(t,r'')} - \frac{\delta M(t,r)}{\delta P_\Lambda(t,r'')} \frac{\delta P_M(t,r')}{\delta \Lambda(t,r'')} \right) \mathrm{d}r'' \\ &= \left(\frac{R'}{\Lambda}\right)^2 F^{-1} \left[1 + 2F^{-1} \left(\frac{P_\Lambda}{m_P^2 R}\right)^2\right] \delta(r-r') + \\ &\quad - \left(\frac{P_\Lambda}{m_P^2 R}\right)^2 F^{-1} \left[1 + 2F^{-1} \left(\frac{R'}{\Lambda}\right)^2\right] \delta(r-r') \\ &= \delta(r-r'). \end{aligned} \tag{4.48}$$

We may want to replace Λ, P_Λ with M, P_M, given that with this new pair of canonical variables the boundary action $S_{\partial\Sigma}$ is a way simpler. With the help of (4.45), the full expression for P_M is

$$P_M = \frac{\Lambda P_\Lambda}{m_P^2 R} \left[\left(\frac{R'}{\Lambda}\right)^2 - \left(\frac{P_\Lambda}{m_P^2 R}\right)^2 \right]^{-1}. \tag{4.49}$$

This expression is the negative of $\mathrm{d}T/\mathrm{d}r$, and if we integrate it, we obtain the difference of the Killing times $T(r_1)$ and $T(r_2)$ between any two points, r_1 and r_2, of the hypersurface Σ_t. As we can see, neither (4.46) nor (4.49) contains P_R, so M and P_M have vanishing PBs with R. However, their PBs with P_R do not vanish. To implement the desired transformation of canonical variables, we need to modify the troublesome momentum P_R in such a way that the new momentum, $P_\mathcal{R}$, will have vanishing PBs with M and P_M, but still remains conjugate to

$$\mathcal{R} := R. \tag{4.50}$$

If we add to P_R a dynamical variable Θ that does not depend on P_R, then we may have a transformation that fulfills the above requirements:

$$P_\mathcal{R} = P_R + \Theta[\Lambda, R, P_\Lambda]. \tag{4.51}$$

To determine Θ, we use the fact that 1-forms, $\omega = p_i \mathrm{d}x^i$, are well-defined objects, invariant under changes of coordinates:

$$P_R \mathrm{d}R + P_\Lambda \mathrm{d}\Lambda = P_\mathcal{R} \mathrm{d}\mathcal{R} + P_M \mathrm{d}M. \tag{4.52}$$

We know that, because of (4.32), the action (4.38) cannot tell the difference between $P_R R' - \Lambda P_\Lambda'$ and $P_R R' + P_\Lambda \Lambda'$, so we can write this identity as

$$P_R R' - \Lambda P_\Lambda' = P_\mathcal{R} \mathcal{R}' + P_M M' \tag{4.53}$$

and take

$$\chi_r = P_{\mathcal{R}}\mathcal{R}' + P_M M' \tag{4.54}$$

to be the supermomentum in the new canonical variables.

We now get Θ just by comparing (4.53) with (4.51):

$$\Theta = -\frac{\Lambda P_\Lambda'}{R'} - \frac{P_M M'}{R'}. \tag{4.55}$$

Therefore, to find the expression for the new momentum is a matter of substituting (4.44) and (4.47) and using (4.45). We end up with a linear combination of the constraints:

$$P_{\mathcal{R}} = F^{-1}\left(\frac{P_\Lambda}{m_P^2 R}\chi + \frac{R'}{\Lambda^2}\chi_r\right). \tag{4.56}$$

This result completes our new canonical chart on phase space. The transition

$$\Lambda, R, P_\Lambda, P_R \mapsto M, \mathcal{R}, P_M, P_{\mathcal{R}} \tag{4.57}$$

is a canonical transformation (see (KUCHAR, 1994) for the proof) easily reversible:

$$\Lambda = (F^{-1}R'^2 - FT'^2)^{1/2} = (\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)^{1/2}, \tag{4.58a}$$

$$P_\Lambda = -\frac{m_P^2 R}{\Lambda}FT' = \frac{m_P^2\mathcal{R}}{(\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)^{1/2}}\mathcal{F}P_M, \tag{4.58b}$$

$$R = \mathcal{R}, \tag{4.58c}$$

$$P_R = P_{\mathcal{R}} - \Theta = P_{\mathcal{R}} + \frac{m_P^2}{\mathcal{R}'}\left[(\mathcal{R}\mathcal{F}P_M)' - \frac{\mathcal{R}\mathcal{F}P_M}{2}\frac{(\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)'}{\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2} + \frac{P_M M'}{m_P^2}\right], \tag{4.58d}$$

where $\mathcal{F} \coloneqq F(\mathcal{R})$.

The super-Hamiltonian in the new canonical variables is obtained from (4.47) by using (4.58a) and (4.58b):

$$-R'\chi = \Lambda M' + P_\Lambda\frac{\chi_r}{m_P^2 R} = \frac{(\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)M' + \mathcal{F}P_M\chi_r}{(\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)^{1/2}}. \tag{4.59}$$

By adding $\mathcal{F}P_M P_{\mathcal{R}}\mathcal{R}' - \mathcal{F}P_M P_{\mathcal{R}}\mathcal{R}'$ to the numerator and recognizing expression (4.54), we get

$$\chi = -\frac{\mathcal{F}^{-1}M'\mathcal{R}' + \mathcal{F}P_M P_{\mathcal{R}}}{(\mathcal{F}^{-1}\mathcal{R}'^2 - \mathcal{F}P_M^2)^{1/2}}. \tag{4.60}$$

We can therefore express the constraints in terms of the new canonical variables using (4.54) and (4.60), or equivalently (4.47) and (4.56):

$$M' = 0 \quad \text{and} \quad P_{\mathcal{R}} = 0. \tag{4.61}$$

When we apply these constraints to the gravitational action $S_G = S_\Sigma + S_{\partial\Sigma}$,

$$S_G[M, \mathcal{R}, P_M, P_\mathcal{R}; N, N^r] = \int \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}r\, (P_M \dot{M} + P_\mathcal{R} \dot{\mathcal{R}} - N\chi - N^r \chi_r) - \int M\, \mathrm{d}t, \quad (4.62)$$

we are left with the only physical degree of freedom of the Schwarzschild spacetime:

$$S_G[M, \mathcal{P}_M] = \int (\mathcal{P}_M \dot{M} - H)\, \mathrm{d}t, \quad (4.63)$$

where

$$H(M) \coloneqq M \quad (4.64)$$

is the reduced Hamiltonian, and the homogeneity of M allowed us to define the new momentum

$$\mathcal{P}_M(t) \coloneqq \int_{-\infty}^{\infty} P_M\, \mathrm{d}r = T(r = -\infty) - T(r = \infty), \quad (4.65)$$

which carries the information about the evolution of the asymptotic ends of the hypersurfaces Σ_t in spacetime, and takes the values $-\infty < \mathcal{P}_M < \infty$.

The extended Schwarzschild spacetime has two spacelike infinities, and the reader may have bothered that apparently, we have not taken into account the contribution of the left-hand side infinity to the boundary action (4.39). Firstly, stationary observers placed at the spatial infinities of regions **I** and **III** (refer back to Fig. 12) should agree about the mass of the hole: $M_\pm = M$, where the minus sign refers to the left-hand side infinity and the plus sign to the right-hand side infinity. Secondly, the lapse function is the rate of change of their proper time $\tau_\pm$ with respect to the label time t in the direction normal to the hypersurfaces Σ_t (remember that $N^r_\pm = 0$): $N_\pm = \pm\dot{\tau}_\pm$. The boundary action then becomes

$$S_\Sigma = \int M(\dot{\tau}_+ - \dot{\tau}_-)\, \mathrm{d}t. \quad (4.66)$$

However, this time parameterization at infinities is arbitrary, and we can set $\dot{\tau}_+ - \dot{\tau}_- = 1$. In particular, the choice $\dot{\tau}_- = 0$ and $\dot{\tau}_+ = 1$ freezes the evolution of the hypersurfaces at the left-hand side infinity, while their evolution at the right-hand side infinity proceeds at unit rate.

The equations of motion for the reduced canonical variables follow from the variation of the reduced action (4.63) and show that M is a constant of motion:

$$\dot{M} = 0, \quad (4.67a)$$

$$\dot{\mathcal{P}}_M = \dot{\tau}_- - \dot{\tau}_+ = -1. \quad (4.67b)$$

So far we assumed that we could extend the hypersurfaces Σ_t through the singularities in **II** and **IV**, which requires the existence of a wormhole linking regions **I** and **III** at their horizons. Following Ref. (JALALZADEH; SILVA; MONIZ, 2021), we map the black hole solution into a wormhole solution by using the canonical transformation $M, \mathcal{P}_M \mapsto x, p$ introduced by Louko and Mäkelä (LOUKO; MÄKELÄ, 1996), where $x \geq 0$ is the wormhole throat and $-\infty < p < \infty$ its conjugate momentum:

$$|\mathcal{P}_M(x)| = \int_x^{2M/m_P^2} \frac{\mathrm{d}x'}{\sqrt{\frac{2M}{m_P^2 x'} - 1}}, \tag{4.68a}$$

$$M(x,p) = \frac{p^2}{2m_P^2 x} + \frac{m_P^2 x}{2}. \tag{4.68b}$$

To satisfy $\dot{M} = 0$, x and p must evolve according to

$$\dot{x} = \frac{p}{m_P^2 x}, \tag{4.69a}$$

$$\dot{p} = \frac{p^2}{2m_P^2 x^2} - \frac{m_P^2}{2}. \tag{4.69b}$$

If we parameterize x and t with a dimensionless conformal time η,

$$\mathrm{d}t = x\,\mathrm{d}\eta, \tag{4.70}$$

these equations of motion become:

$$\frac{\mathrm{d}x}{\mathrm{d}\eta} = \frac{p}{m_P^2}, \tag{4.71a}$$

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}\eta} &= \frac{p^2}{2m_P^2 x} - \frac{m_P^2 x}{2} \\ &= M - m_P^2 x.\end{aligned} \tag{4.71b}$$

By differentiating (4.71a) with respect to η and then using (4.71b), we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\eta^2} + x = \frac{M}{m_P^2}. \tag{4.72}$$

Therefore, we can write $x(\eta)$ and $t(\eta)$ as

$$x(\eta) = \frac{M}{m_P^2}(1 + \cos\eta), \tag{4.73a}$$

$$t(\eta) = \frac{M}{m_P^2}(\eta + \sin\eta). \tag{4.73b}$$

We see that x is periodic and reaches its turning values, 0 and $2M/m_P^2$, at integer multiples of π, $x(\eta = n\pi)$, which correspond to

$$t(n\pi) = \frac{n\pi M}{m_P^2} = \frac{n}{8T_\mathsf{H}},\ n \in \mathbb{Z} \tag{4.74}$$

where T_H is, remarkably, the Hawking temperature of the SBH.

4.3 QUANTIZED SBH

4.3.1 The mass spectrum

Following the canonical quantization procedure, the reduced Hamiltonian becomes an operator $\hat{H}(\hat{x}, \hat{p})$ acting on a state ket $|\Psi\rangle$. Eq. (4.64) then becomes an eigenvalue equation:

$$\hat{H}(\hat{x}, \hat{p})\,|\Psi\rangle = M\,|\Psi\rangle\,. \tag{4.75}$$

The dependence of the Hamiltonian on $\hat{x}$ and $\hat{p}$ comes from (4.68b). By writing this dependence explicitly, we get

$$\left(\frac{\hat{p}^2}{2m_P^2} + \frac{m_P^2}{2}\hat{x}^2\right)|\Psi\rangle = M\hat{x}\,|\Psi\rangle\,. \tag{4.76}$$

In the coordinate representation,

$$\hat{x} \to x, \quad \hat{p} \to -i\frac{\mathrm{d}}{\mathrm{d}x}, \quad |\Psi\rangle \to \Psi(x), \tag{4.77}$$

Eq. (4.76) becomes

$$-\frac{1}{2m_P^2}\frac{\mathrm{d}^2\Psi(x)}{\mathrm{d}x^2} + \frac{m_P^2}{2}\left(x - \frac{M}{m_P^2}\right)^2\Psi(x) = \frac{M^2}{2m_P^2}\Psi(x). \tag{4.78}$$

We can simplify this equation with the following definitions:

$$z \coloneqq \sqrt{2}\,m_P\left(x - \frac{M}{m_P^2}\right), \tag{4.79a}$$

$$\widetilde{\Psi}(z) \coloneqq \Psi(x(z)), \tag{4.79b}$$

$$\nu \coloneqq \frac{M^2}{2m_P^2} - \frac{1}{2}. \tag{4.79c}$$

In terms of this new dimensionless variable $z \geq z_0 \coloneqq -\sqrt{2}\,M/m_P$, Eq. (4.78) reads

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{z^2}{4}\right)\widetilde{\Psi}(z) = \left(\nu + \frac{1}{2}\right)\widetilde{\Psi}(z), \tag{4.80}$$

which is the equation of the one-dimensional harmonic oscillator with position-dependent angular frequency:

$$\widetilde{\Psi}'' + \omega^2\widetilde{\Psi} = 0, \tag{4.81}$$

where

$$\omega(z) \coloneqq \sqrt{\nu + \frac{1}{2} - \frac{z^2}{4}}. \tag{4.82}$$

This wave function must be square-integrable,

$$\int_{z_0}^{\infty} \left|\widetilde{\Psi}(z)\right|^2 \mathrm{d}z < \infty, \tag{4.83}$$

and we require that it must vanish at infinity:

$$\widetilde{\Psi}(z \to \infty) = 0. \tag{4.84}$$

To ensure this boundary condition, we may consider an explicitly Gaussian exponential factor in the two independent solutions of (4.80):

$$\widetilde{\Psi}_1(z) = f(z)\, \mathrm{e}^{-z^2/4}, \tag{4.85a}$$

$$\widetilde{\Psi}_2(z) = z g(z)\, \mathrm{e}^{-z^2/4}, \tag{4.85b}$$

where $f(z)$ and $g(z)$ satisfy the respective equations upon substitution in (4.80). The substitution of the first solution $\widetilde{\Psi}_1(z)$ yields

$$f'' - zf' + \nu f = 0. \tag{4.86}$$

By using the definitions $\xi := z^2/2$ and $\tilde{f}(z) := f(z(\xi))$ to rewrite the above equation, we obtain

$$\xi \tilde{f}'' + \left(\frac{1}{2} - \xi\right) \tilde{f}' + \frac{\nu}{2} \tilde{f} = 0, \tag{4.87}$$

which is a confluent hypergeometric equation. Therefore, its solution is a confluent hypergeometric function:

$$\tilde{f}(\xi) = {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \xi\right). \tag{4.88}$$

By substituting (4.85b) into (4.80), we get

$$g'' + \left(\frac{2}{z} - z\right) g' + (\nu - 1) g = 0. \tag{4.89}$$

In terms of ξ and $\tilde{g}(\xi) := g(z(\xi))$:

$$\xi \tilde{g}'' + \left(\frac{3}{2} - \xi\right) \tilde{g}' - \frac{(1-\nu)}{2} \tilde{g} = 0, \tag{4.90}$$

which solution is

$$\tilde{g}(\xi) = {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \xi\right). \tag{4.91}$$

The general solution of (4.80) is a linear combination of its two independent solutions:

$$\widetilde{\Psi}(z) = \mathrm{e}^{-z^2/4}\left[A\, {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + Bz\, {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right], \tag{4.92}$$

where A and B are constants. Before we apply the boundary condition (4.84), we need to know about the asymptotic behaviour of the confluent hypergeometric functions in the above general solution. From Abramowitz and Stegun (ABRAMOWITZ; STEGUN, 1964), since $\xi > 0$ and $\xi \to \infty$ along the positive real axis, we have the following:

$$ {}_1F_1\,(a;c;\xi \to \infty) \to \frac{\Gamma(c)}{\Gamma(a)}\,\mathrm{e}^{\xi}\,\xi^{a-c}. \tag{4.93}$$

Therefore:

$$ {}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2} \to \infty\right) \to \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{-\nu}{2}\right)}\,\mathrm{e}^{z^2/2}\,z^{-(\nu+1)}\,2^{(\nu+1)/2}, \tag{4.94a}$$

$$ {}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2} \to \infty\right) \to \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}\,\mathrm{e}^{z^2/2}\,z^{-(\nu+2)}\,2^{(\nu+2)/2}. \tag{4.94b}$$

If we substitute these relations in (4.92), we obtain an exponential factor $\mathrm{e}^{z^2/4}$ multiplying remaining terms within the square brackets. Hence, $\Psi(z \to \infty)$ vanishes only if the square brackets vanish. This fixes the relative values of A and B:

$$ \frac{A}{B} = -\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}\frac{\Gamma\left(\frac{-\nu}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{2^{(\nu+2)/2}}{2^{(\nu+1)/2}} = -\frac{\Gamma\left(\frac{-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}\frac{1}{\sqrt{2}}. \tag{4.95}$$

With this result, the wave function becomes

$$ \widetilde{\Psi}(z) = N\,\mathrm{e}^{-z^2/4}\left[\frac{\Gamma\left(\frac{-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}\,{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right) - \sqrt{2}\,z\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right], \tag{4.96}$$

where $N \coloneqq -B/\sqrt{2}$ is a normalization constant. This wave function is normalized by using the following inner product:

$$ \left\langle \widetilde{\Psi}_2 \middle| \widetilde{\Psi}_1 \right\rangle = \int_{z_0}^{\infty} \mathrm{d}z\,\widetilde{\Psi}_2^*(z)\widetilde{\Psi}_1(z). \tag{4.97}$$

A second boundary condition arises when we require the reduced Hamiltonian operator $\hat{H}$ to be Hermitian. Note that the operator, $\hat{\mathcal{H}}$, acting on $\widetilde{\Psi}(z)$ on the left-hand side of the eigenvalue equation (4.80) is proportional to $\hat{H}$:

$$ \hat{\mathcal{H}} = \frac{M}{2m_P^2}\hat{H}. \tag{4.98}$$

Its adjoint operator $\hat{\mathcal{H}}^\dagger$ satisfies

$$ \left\langle \widetilde{\Psi}_2 \middle| \hat{\mathcal{H}}\widetilde{\Psi}_1 \right\rangle = \left\langle \hat{\mathcal{H}}^\dagger\widetilde{\Psi}_2 \middle| \widetilde{\Psi}_1 \right\rangle, \tag{4.99}$$

and the validity of the self-adjoint condition $\hat{\mathcal{H}} = \hat{\mathcal{H}}^\dagger$ holds if a certain boundary term vanishes. Let us show this explicitly:

$$\begin{aligned}\left\langle \widetilde{\Psi}_2 \middle| \hat{\mathcal{H}} \widetilde{\Psi}_1 \right\rangle &= \int_{z_0}^{\infty} \mathrm{d}z\, \widetilde{\Psi}_2^* \left(-\frac{\mathrm{d}^2 \widetilde{\Psi}_1}{\mathrm{d}z^2} + \frac{z^2}{4} \widetilde{\Psi}_1 \right) \\ &= \int_{z_0}^{\infty} \mathrm{d}z \left[-\frac{\mathrm{d}(\widetilde{\Psi}_2^* \widetilde{\Psi}_1')}{\mathrm{d}z} + \frac{\mathrm{d}\widetilde{\Psi}_2^*}{\mathrm{d}z} \widetilde{\Psi}_1' + \frac{z^2}{4} \widetilde{\Psi}_2^* \widetilde{\Psi}_1 \right] \\ &= \int_{z_0}^{\infty} \mathrm{d}z \left[-\frac{\mathrm{d}(\widetilde{\Psi}_2^* \widetilde{\Psi}_1')}{\mathrm{d}z} + \frac{\mathrm{d}(\widetilde{\Psi}_2^{*\prime} \widetilde{\Psi}_1)}{\mathrm{d}z} + \left(-\frac{\mathrm{d}^2 \widetilde{\Psi}_2^*}{\mathrm{d}z^2} + \frac{z^2}{4} \widetilde{\Psi}_2^* \right) \widetilde{\Psi}_1 \right] \\ &= \left\langle \hat{\mathcal{H}} \widetilde{\Psi}_2 \middle| \widetilde{\Psi}_1 \right\rangle + (\widetilde{\Psi}_2^{*\prime} \widetilde{\Psi}_1 - \widetilde{\Psi}_2^* \widetilde{\Psi}_1') \Big|_{z_0}^{\infty}. \end{aligned} \tag{4.100}$$

Since $\widetilde{\Psi}(z \to \infty) = 0$, we must have

$$\frac{\widetilde{\Psi}_1(z_0)}{\widetilde{\Psi}_1'(z_0)} = \frac{\widetilde{\Psi}_2^*(z_0)}{\widetilde{\Psi}_2^{*\prime}(z_0)}. \tag{4.101}$$

But the wave function is not a complex function, and $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ are arbitrary, so these quotients must be constant. In terms of $\Psi(x)$, we have

$$\frac{\Psi(0)}{\Psi'(0)} = \gamma, \tag{4.102}$$

where the prime symbol denotes derivative with respect to x, and the constant γ has dimension of length. Therefore, the demand for a Hermitian Hamiltonian leads to a new fundamental constant of the theory unless we take the wave function to be zero at the singularity:

$$\Psi(0) = 0. \tag{4.103}$$

If we subject (4.96) to this new boundary condition,

$$\widetilde{\Psi} \left(z_0 = -\sqrt{2} \frac{M}{m_P} \right) = 0, \tag{4.104}$$

we get:

$$\frac{{}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{M^2}{m_P^2} \right)}{\Gamma\left(\frac{1-\nu}{2} \right)} + 2\frac{M}{m_P} \frac{{}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{M^2}{m_P^2} \right)}{\Gamma\left(\frac{-\nu}{2} \right)} = 0. \tag{4.105}$$

We are interested in the black hole states whose mass M is much larger than Planck's mass m_P. Putting into perspective, stellar black holes, which are the final state of the gravitational collapse of extremely massive stars, typically has a mass between about 5 and 10 solar masses, $M_\odot = 1,988 \times 10^{30}$ kg, while supermassive black holes, which can be found in the center of almost every large galaxy, can have masses ranging from millions to billions solar masses. Comparing with $m_P = 2,176 \times 10^{-8}$ kg, we can safely say that $M/m_P \gg 1$ ($\nu \gg 1$) for this masses regime. Therefore, we can use (4.93) to obtain a semi-classical approximation:

$$ {}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};2\nu+1\right) \approx \frac{\sqrt{\pi}}{\Gamma\left(\frac{-\nu}{2}\right)}\,\mathrm{e}^{2\nu+1}\,(2\nu+1)^{-(\nu+1)/2}, \tag{4.106a} $$

$$ {}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};2\nu+1\right) \approx \frac{\sqrt{\pi}/2}{\Gamma\left(\frac{1-\nu}{2}\right)}\,\mathrm{e}^{2\nu+1}\,(2\nu+1)^{-(\nu+2)/2}. \tag{4.106b} $$

Eq. (4.105) then becomes

$$ \begin{aligned} 0 &= \frac{2\sqrt{\pi}\,\mathrm{e}^{2\nu+1}}{(2\nu+1)^{\frac{\nu+1}{2}}\,\Gamma\left(\frac{-\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)} \\ &= \frac{\mathrm{e}^{2\nu+1}}{(2\nu+1)^{\frac{\nu+1}{2}}\,2^{\nu}\,\Gamma(-\nu)}, \end{aligned} \tag{4.107} $$

where we have used the Legendre duplication formula for the gamma function:

$$ \Gamma(c)\,\Gamma\left(c+\frac{1}{2}\right) = 2^{1-2c}\sqrt{\pi}\,\Gamma(2c), \quad \text{with } c \in \mathbb{C}. \tag{4.108} $$

Thereby, to attend the boundary condition (4.104), $\Gamma(-\nu)$ must diverge, hence ν must be a positive integer. From (4.79c),

$$ n = \frac{1}{2}\left(\frac{M^2}{m_P^2} - 1\right) \in \mathbb{N}, \tag{4.109} $$

we then obtain the mass spectrum of the Schwarzschild black hole in terms of m_P, which can be thought of as the elementary mass of a black hole with Schwarzschild radius equals twice the reduced Compton wavelength:

$$ M(n) = m_P\sqrt{2n+1},\ n \gg 1. \tag{4.110} $$

The black hole's surface area is proportional to M^2, since that it is given by $A = 4\pi R_{\mathsf{SCH}}^2$, where R_{SCH} is the Schwarzschild radius. The area spectrum of the quantized SBH then follows from (4.110):

$$ A(n) = 16\pi(2n+1)\ell_P^2, \tag{4.111} $$

where $\ell_P = 1/m_P$ is the Planck length.

4.3.2 The thermodynamics

As suggested by Mukhanov (MUKHANOV, 1986) and Li Xiang (XIANG, 2004), the existence of a quantum number n and mass levels M_n make the quantized SBH analogous to an atomic system. If a black hole in level $n+1$ is allowed to interact with the vacuum, then it may decay into the closest lower level n, and a photon with minimal frequency ω_0 will be emitted similarly to an atomic transition. This frequency is determined as follows:

$$\begin{aligned}\omega_0 &= m_P\sqrt{2n+3} - m_P\sqrt{2n+1} \\ &= m_P\sqrt{2n}\left(\sqrt{1+\frac{3}{2n}} - \sqrt{1+\frac{1}{2n}}\right).\end{aligned} \tag{4.112}$$

Since $n \gg 1$, we can use $(1 \pm \epsilon)^a \approx 1 \pm a\epsilon$ to get $\omega_0 = m_P/\sqrt{2n}$, which shows that the mass spectrum is not equally spaced, but becomes broader as n decreases, such that low level transitions emit more energetic photons. If we now use (4.109), we end up with

$$\omega_0 \approx \frac{m_P^2}{M}\left[1 - \frac{1}{2}\left(\frac{m_P}{M}\right)^2\right]. \tag{4.113}$$

The finite lifetime of the black hole in level $n+1$ is its characteristic time τ_n defined as

$$\tau_n^{-1} := \frac{\dot{M}}{\omega_0}, \tag{4.114}$$

where $\dot{M}$ denotes the mass loss rate of the transition. By using (4.113), we conclude that

$$\tau_n^{-1} \approx \frac{M\dot{M}}{m_P^2}\left[1 - \frac{1}{2}\left(\frac{m_P}{M}\right)^2\right]. \tag{4.115}$$

The width of level n is proportional to the difference $M_{n+1} - M_n$:

$$W_n = \alpha(M_{n+1} - M_n) = \alpha\,\omega_0, \tag{4.116}$$

where $\alpha \ll 1$ is a dimensionless constant.

If we assume that the lifetime τ_n of the black hole becomes shorter as n increases, then in the semi-classical limit, it is expected intense low-energy emission. But to measure the energy of a state with good precision, the state must be observed for many cycles, which becomes hard to accomplish if the state has a short τ_n. This suggests an uncertainty relation between τ_n and W_n:

$$\tau_n W_n \approx 1. \tag{4.117}$$

This uncertainty causes a blurring in the energy of excited states: each time an excited state decays into the nearest level, the emitted energy may be slightly different, but the average energy of the emitted photons corresponds to the theoretical value ω_0. Using (4.114) and (4.115), we can solve (4.117) for $\dot{M}$ and find

$$\begin{aligned} \dot{M} &= \alpha\,\omega_0^2 \\ &\approx \alpha\,\frac{m_P^4}{M^2}\left[1+\left(\frac{m_P}{M}\right)^2\right], \end{aligned} \tag{4.118}$$

which exhibits a divergent behaviour of the radiated power $\dot{M}$ in late stages of the black hole evaporation, $M \to 0$. But this divergence is an extrapolation of our current formalism since such small mass scales lies outside of its scope.

Due to its discrete mass spectrum, the quantized SBH absorbs and emits radiation only with frequencies corresponding to the distance between two levels. Thus, the quantized SBH does not radiate as a black body, and to define its temperature from the Stefan–Boltzmann law,

$$\dot{M} = \varepsilon\sigma A T^4, \tag{4.119}$$

where $\varepsilon \in (0,1)$ is the emissivity of the radiating body and $\sigma = \pi^2/60$ is the Stefan-Boltzmann constant ($k_B = 1$), one must determine the gray body factor ε. It was by using the Stefan–Boltzmann law to calculate the emission rate of the SBH emitting as a grey body that Giddings (GIDDINGS, 2016) concluded that the effective emitting area should be larger than the horizon area. The region from which the emission originates he called the black hole's "quantum atmosphere", and its radius is about 2.6 times the Schwarzschild radius of the black hole. However, for our purposes, the black hole lies at the semi-classical limit, and its spectrum is very narrow, such that the emitted radiation is almost thermal. In the black body approximation ($\varepsilon \approx 1$), if we use (4.118) and $A = 16\pi M^2/m_P^4$, Eq. (4.119) becomes

$$\alpha\,\frac{m_P^4}{M^2}\left[1+\left(\frac{m_P}{M}\right)^2\right] = \frac{4\pi^3}{15}\frac{M^2}{m_P^4}T^4, \tag{4.120}$$

which can be solved for the black hole temperature:

$$T \approx \left(\frac{15\alpha}{4\pi^3}\right)^{1/4}\frac{m_P^2}{M}\left[1+\frac{1}{4}\left(\frac{m_P}{M}\right)^2\right]. \tag{4.121}$$

The black hole entropy can be expressed as

$$
\begin{aligned}
S &= \int \frac{\mathrm{d}M}{T} \\
&\approx \left(\frac{4\pi^3}{15\alpha}\right)^{1/4} \frac{1}{m_P^2} \int M \left[1 - \frac{1}{4}\left(\frac{m_P}{M}\right)^2\right] \mathrm{d}M \\
&= \left(\frac{4\pi^3}{15\alpha}\right)^{1/4} \left(\frac{M^2}{2m_P^2} - \frac{1}{4}\ln M\right) + const. \qquad (4.122)
\end{aligned}
$$

We fix the value of α by requiring the leading term of the entropy to be the Bekenstein-Hawking entropy $S_{\mathsf{BH}} = 4\pi M^2/m_P^2$:

$$
\alpha = \frac{1}{15360\pi} \Rightarrow \left(\frac{4\pi^3}{15\alpha}\right)^{1/4} = 8\pi. \qquad (4.123)
$$

Finally, we arrive at a negative logarithmic correction to the Bekenstein-Hawking entropy:

$$
S = S_{\mathsf{BH}} - 2\pi \ln M + const. \qquad (4.124)
$$

Another important thermodynamic quantity is heat capacity. According to Eq. (4.121), the black hole gets hotter as it loses mass, thus having a negative heat capacity:

$$
C = \left(\frac{\partial T}{\partial M}\right)^{-1} \approx -\frac{8\pi}{m_P^2 M^2}\left[1 - \frac{3}{4}\left(\frac{m_P}{M}\right)^2\right] < 0. \qquad (4.125)
$$

Unlike the heat capacity for the Hawking temperature $T_{\mathsf{H}} = m_P^2/8\pi M$ that vanishes at $M = 0$, the heat capacity of the quantized SBH becomes vanishing when the mass approaches the non-zero Planck scale $2m_P/\sqrt{3}$, which can be regarded as the new ground state of the black hole. The negativity of the heat capacity prevents the evaporating black hole from reaching the thermal equilibrium with its surrounding.

We end our analysis by obtaining the lifetime of the quantized SBH, which can be made by integrating (4.118):

$$
\begin{aligned}
\Delta t &= -\frac{1}{\alpha\, m_P^4} \int_M^{2m_P/\sqrt{3}} M'^2 \left[1 + \left(\frac{m_P}{M'}\right)^2\right]^{-1} \mathrm{d}M' \\
&\approx \frac{1}{\alpha\, m_P^4} \int_{2m_P/\sqrt{3}}^{M} (M'^2 - m_P^2)\, \mathrm{d}M' \\
&= \frac{M^3}{3\alpha\, m_P^4}\left[1 - 3\left(\frac{m_P}{M}\right)^2\right] + \frac{10}{9\sqrt{3}\,\alpha\, m_P}, \qquad (4.126)
\end{aligned}
$$

which gives a lifetime of the order of 10^{74} seconds to a solar mass black hole.

5 CONCLUSION

Chapter 2 of this dissertation was devoted to constructing a Hamiltonian for the gravitational field. To this end, we decomposed the spacetime into arbitrary spacelike hypersurfaces Σ_t and related them by introducing the lapse function N and shift vector N^a, which, together with the induced metric h_{ab} on the hypersurfaces, amounted to the form the canonical configuration variables of the theory. When evaluated on vacuum solutions to the Einstein field equations, the meaning of this Hamiltonian turned out to be a boundary term sensitive to the asymptotic behavior of N and N^a. By means of a suitable choice of these, the ADM definition of the spacetime energy was attained. In chapter 3, we showed that there are eight constraints relating to the canonical variables and that the gravitational Hamiltonian consequently vanishes, which results in equations of motion with no absolute time variable. Then, we attempted to give a quantum mechanical treatment to this Hamiltonian constrained system by promoting the classical constraints to operators that annihilate the quantum state of spacetime. Following the standard procedures of quantum mechanics, we ended up with the Wheeler–DeWitt equation.

Once the general discussion was concluded, we proceeded in chapter 4 with the geometrodynamics of the maximally extended Schwarzschild spacetime. We showed that the constant Schwarzschild mass is the only degree of freedom of this spacetime. The theory was no longer a field theory but a theory of finite degrees of freedom. The quantization of this one-dimensional classical system was then addressed within ordinary quantum mechanics. This quantization led us to a mass spectrum for the SBH, and the thermodynamics we have obtained qualitatively reproduces the features of the black hole evaporation.

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