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# Bifurcations of Two Symmetric Families of Dziobek Configurations

Recife

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Trabalho apresentado ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal de Pernambuco como requisito parcial para a obtenção do grau de Doutora em Matemática.

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**Advisor:** Prof. Dr. Eduardo Shirlippe Goes Leandro.

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**MICHELLE GONZAGA DOS SANTOS**

# BIFURCATIONS OF TWO SYMMETRIC FAMILIES OF DZIOBEK CONFIGURATIONS

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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## RESUMO

Neste trabalho, investigamos bifurcações de configurações de Dziobek nos problemas de quatro e cinco corpos, considerando o expoente da função potencial de cada sistema negativo e menor do que menos um. O objetivo deste estudo é encontrar novas configurações centrais. Inicialmente estudamos as bifurcações de uma configuração triangular com corpos de massas unitárias em seus vértices e um corpo de massa arbitrária no centro. Utilizando o método de Redução de Liapunov-Schmidt e o Teorema da Ramificação Equivariante, encontramos três famílias de configurações centrais que bifurcam da configuração triangular centrada degenerada. No caso Newtoniano, realizamos uma análise completa das soluções e também encontramos três famílias de configurações centrais assim como em (MEYER; SCHMIDT, 1987). Em seguida, investigamos as bifurcações de uma configuração de Dziobek do problema de cinco corpos no espaço. Mais precisamente, consideramos uma configuração tetraedral com corpos de massas unitárias nos vértices e centrada num corpo de massa arbitrária. Primeiramente, analisamos o que ocorre numa vizinhança da configuração degenerada variando igualmente três das massas dos vértices. Em seguida, variamos igualmente duas das massas dos vértices. Utilizamos o método de Redução de Liapunov-Schmidt, a equivariância das equações que descrevem o problema e expansão de Taylor para obter novas configurações centrais. No primeiro caso, encontramos quatro novas famílias simétricas que surgem da configuração degenerada e no segundo, encontramos três novas famílias simétricas.

**Palavras-chaves**: Problema de $N$ Corpos. Configurações Centrais Simétricas. Configurações de Dziobek. Bifurcações.

## ABSTRACT

In this work, we investigate bifurcations of Dziobek configurations in four and five-body problems, considering the exponent of the potential function of each system to be negative and less than minus one. The aim of this study is to find new central configurations. Initially, we investigate the bifurcations of the equilateral triangular configuration with bodies of unit mass at its vertices and a body with a mass of arbitrary value at its center. Using the Liapunov-Schmidt reduction method and the Equivariant Branching Theorem, we find three families of central configurations that bifurcate from the degenerate centered triangular configuration. In the Newtonian case, we performed a complete analysis of the solutions found and also found three families of central configurations with the same behavior as well as in (MEYER; SCHMIDT, 1987). Next, we study the bifurcations of a Dziobek configuration of the five-body problem in space. More precisely, we consider the regular tetrahedral configuration with bodies of unit mass at the vertices and a body of arbitrary mass at the center. Firstly, we analyze what happens in a neighborhood of the degenerate configuration by varying three of the vertex masses in the same fashion. Next, we vary two of the vertex masses equally. We use the Liapunov-Schmidt reduction method, the equivariance of the equations that describe the problem and Taylor's formula to obtain new central configurations. In the first case, we found four new symmetrical families that arise from the degenerate configuration and, in the second, we found three new symmetrical families.

**Keywords**: $N$-Body Problem. Symmetrical Central Configurations. Dziobek Configurations. Bifurcations.

## LIST OF FIGURES

## CONTENTS

# 1 INTRODUCTION

The $N$-body problem is a mathematical problem that has existed for more than three centuries and continues to challenge mathematicians in various fields. In essence, this problem is the study of the dynamics of $N$ massive bodies which are attracted to each other by the gravitational force that each body exerts on the others. So far, our understanding of the general solution to this problem is limited. For two bodies, the problem was completely solved by Isaac Newton $[1687]$ . The only explicitly known solutions are so-called homographic solutions. These solutions are such that the bodies start in a special configuration, called central configuration. The main property of central configurations is that the gravitational acceleration vector produced on each mass by all the others must point towards the center of mass and be a multiple of the position vector of this body relative to the center of mass. These configurations govern the behavior of solutions near collisions. Finding homographic solutions to the $N$-body problem seems to be the main motivation for studying central configurations, but other reasons have motivated the study of such configurations, as stressed in some references to this subject (ALBOUY, 2003), (ALBOUY; LLIBRE, 2002) and (MOECKEL, 2014).

Central configurations are defined by a complicated system of algebraic equations, which does not make it easy to find homographic solutions to the $N$-body problem. Nevertheless, once we find a solution to the equations of central configurations and perform a rotation, translation or dilation of this solution, we have a new solution for this system. For this reason, configurations obtained in this way are said to belong to the same equivalence (similarity) class.

An important question about the central configuration equations is: for given $N$ positive masses, is the number (equivalence classes) of solutions finite? This problem is sometimes called the finiteness problem, and its answer in the affirmative is known as the Chazy-Wintner-Smale conjecture. There are few definitive answers to this question. There are three classes of central configurations for any ordering of three bodies with positive masses on a straight line (EULER, 1764) and two classes in which the bodies are at the vertices of an equilateral triangle (LAGRANGE, 1772). These are all the possible three-body central configurations. (MOULTON, 1910) generalized Euler's result proving that for each ordering of the $N$ bodies along a straight line, there is exactly one collinear central configuration. A generalization of Lagrange's result is such that, for $N$ arbitrary masses, the only associated $(N-1)$-dimensional configuration is the regular simplex (SAARI, 1980).

The classification of central configurations can be explored through the dimension of the configuration. The dimension of a configuration is understood as the dimension of the smallest affine space containing the positions of the point masses at the configuration. The $(N-1)$-dimensional configurations are classified as we have already mentioned, but the $(N-2)$-dimensional configurations are not yet well understood. These configurations are often called Dziobek configurations and stand out, among other reasons, because they allow for a simpler formulation of the equations of central configuration.

For four bodies in the plane with equal masses, (ALBOUY, 1996) proved that there exists an upper bound for the number of classes of Dziobek configurations. A lower bound can be found in (SIMO, 1978). More generally, for any four positive masses, (HAMPTON; MOECKEL, 2006) and (ALBOUY; KALOSHIN, 2012) showed that the number of equivalence classes of central configurations is finite. For a particular family of $d$-dimensional symmetric configurations of $d+2$ bodies of point masses, (LEANDRO, 2003) showed the finiteness and studied the bifurcations of these configurations, providing the exact number of central configurations when $d = 2, 3$. There are many other enumerative studies for particular cases. For five bodies with any positive masses, except for an explicitly determined set of zero measure in the space of masses, (ALBOUY; KALOSHIN, 2012) showed that the number of classes of central configurations is finite (generic finiteness).

For six bodies or more, even generic finiteness is an open problem. However, recently some contributions have been made, see (DIAS; PAN, 2020) and (CHEN; CHANG, 2023).

The equations for central configurations suggest that we interpret a central configuration as a critical point of the potential function of the $N$-body system under the condition that the moment of inertia is constant. A central configuration is called degenerate if it is not isolated, already considering its equivalence class modulo rotations, translations and dilations. Bifurcation theory has proved to be an important way of obtaining new central configurations that shoot out from a degenerate configuration. For this purpose, the masses of the bodies are typically considered to be the bifurcation parameters.

The use of bifurcation methods in Celestial Mechanics was probably pioneered by (PALMORE, 1973) considering a configuration consisting of three bodies with unitary masses at the vertices of an equilateral triangle centered at a fourth body with arbitrary mass. He also considered a configuration consisting of four bodies of unitary masses at the vertices of a square centered at a fifth body of arbitrary mass. In both cases, the author showed that there exists a unique positive value of the central mass for which these configurations are degenerate. Considering the central mass as the bifurcation parameter in each case, Palmore found a new central configuration family bifurcating from the degenerate configuration. (MEYER; SCHMIDT, 1987), repeated the bifurcation analysis of these problems using mutual distances as coordinates and showed the existence of new central configurations. Namely, the configurations that bifurcate from the centered equilateral triangle are pseudo- centered isosceles triangles. For five bodies, the configurations that bifurcate from the centered square form either a pseudo-centered kite or an isosceles trapezoid.

In the case of a rhombus configuration with unitary masses at the vertices and centered at a mass of arbitrary value, there is a negative value for the central mass which makes this configuration degenerate. It was shown in (ROBERTS, 1999) that there is a continuum of central configurations bifurcating from this one. This is an important result that shows that considering positive masses is a necessary condition for finiteness.

Applying the same analytical techniques as in (MEYER; SCHMIDT, 1987), (SCHMIDT, 1988) studied the bifurcations of a regular tetrahedral configuration with unitary masses at the vertices and a mass of arbitrary value at the barycenter. It was shown that there is a single positive value for the central mass which makes the configuration degenerate. Considering the central mass as the bifurcation parameter, the author proved the existence of four new families of symmetric tetrahedrons. This work was continued in (SANTOS, 2004). The author showed, after reformulating the problem using Dziobek equations and bifurcation theory with symmetry, that there are in fact at least seven families of central configurations close to the centered regular tetrahedron. The three new configurations that were found have a so-called planar symmetry.

(MEYER; SCHMIDT, 1988) extended the method to find the value of the central mass that makes the central configuration formed by any centered regular polygon with equal masses at the vertices degenerate. They used the so-called Palmore coordinates. In addition, with the help of the software Macsyma and Polypak, they carried out a complete bifurcation analysis for problems of five to thirteen bodies where the central mass is the bifurcation parameter.

In a numerical study, (GLASS, 1997) examined the bifurcations of central configurations in the planar four-body problem under the condition of equal masses and using the exponent of the system's potential as the bifurcation parameter. In the same way, (GLASS, 2000) discussed the many bifurcations that occur in the central configuration equation with the number of bodies between five and eight. Some asymmetrical configurations were found for the six-body problem.

Assuming again the centered tetrahedral configuration with equal masses at its vertices and the central mass as the bifurcation parameter, (ALVAREZ-RAMÍREZ; CORBERA; LLIBRE, 2016) analyzed numerically this family with the central mass varying from zero to one. However, no central configurations were found other than those by Schmidt and Santos.

Considering homogeneous potentials with negative exponents, (SANTOS et al., 2017) proved the existence of new Dziobek configurations arising from the centered equilateral triangular configuration and the centered regular tetrahedron with unitary masses at their vertices and an arbitrary mass at the barycenter. In both cases, the mass at one of the vertices of the configuration was taken as the bifurcation parameter, with the mass at the barycenter assuming the specific degenerate value and the remaining masses equal to one. For the four-bodies problem, there are two symmetrical configurations and two non-symmetrical configurations bifurcating from the degenerate centered equilateral triangle. For the five-bodies problem, there are five symmetrical configurations which were found bifurcating from the degenerate centered regular tetrahedron, two with so-called axial symmetry and three with so-called planar symmetry.

The present thesis is divided into three parts and was mostly inspired by (SANTOS et al., 2017) and (SANTOS, 2004). Our main results derive from the study of the bifurcations of the centered tetrahedral configuration. We will continue to treat the masses of the vertices as bifurcation parameters, but the degenerate configuration will be approached differently than in preceding works.

The second chapter provides a foundation for the technical concepts that will be used to study bifurcations of central configurations in the following chapters. It also reviews relevant concepts about bifurcations (GOLUBITSKY; STEWART; SCHAEFFER, 1985) and (GOLUBITSKY; SCHAEFFER, 1988) and central configurations (ALBOUY, 2003).

In the third chapter, we analyze the bifurcations of the centered triangular configuration. We consider the exponent of the potential function of the system to be less than minus one. The central mass that makes the configuration degenerate is determined as a function of this exponent. We consider the central mass as the bifurcation parameter, and apply the Liapunov-Schimidt reduction method to the equation that describes the problem. In the simplified equations, we apply a theorem, known as Equivariant Branching Theorem, which guarantees the existence and uniqueness of solutions for particular bifurcation problems. We find three families of central configurations that bifurcate from the centered equilateral triangle. In the

Newtonian case, a more detailed analysis of the three families revealed that the configurations that bifurcate from the centered equilateral triangle are pseudo-centered isosceles triangles, as in (MEYER; SCHMIDT, 1987).

Finally in chapter four, we investigate the bifurcations of the centered regular tetrahedron with equal masses at the vertices and a body of arbitrary mass at the barycenter. Keeping in mind that this is a bifurcation problem with five different parameters, we study two particular cases of this general problem. The first problem consists of varying three vertex masses equally, while the second consists of varying two of the vertex masses equally. We use the same technique to solve both problems. We start by applying the Liapunov-Schmidt reduction method to simplify the equations that describe these problems. Next, we make use of the equivariance of the reduced equations to find some solutions. To look for more solutions, we use analyticity to obtain an interesting factorization of the Taylor expansion of the functions involved. The implicit function theorem is an important tool in this process. For the first problem, we found four central configuration families emerging from the degenerate configuration with the same type of symmetry as the solutions presented in (SANTOS et al., 2017). In the second problem, we found three new central configuration families bifurcating from the degenerate configuration. These configurations are symmetrical with respect to a plane containing one edge of the tetrahedron and the midpoint of the opposite edge. To display the solutions analytically, we use Taylor series.

We used the computer algebra system Maple to perform the extensive calculations in this thesis.

# 2 PRELIMINARIES

## 2.1 MOTIVATION

Consider $N$ particles with masses $m_1, ..., m_N \in \mathbb{R}^+$ located at the positions $q_1, ..., q_N$ at time $t$, with each $q_i$ in $\mathbb{R}^d$. Determining the positions and velocities of these particles at each instant is the classical problem of Celestial Mechanics known as the $N$-body problem, whose mathematical formulation is given by the system of equations of motion

$$\ddot{q}_i = \sum_{j \neq i}^{N} m_j ||q_i - q_j||^{2a}(q_j - q_i), \;\; i = 1, ..., N, \tag{2.1}$$

when $a = -3/2$. We define the potential function of the system by

$$U = \frac{1}{2a+2} \sum_{i<j}^{N} m_i m_j \parallel q_i - q_j \parallel^{2a+2}, \; \forall \, a \neq -1. \tag{2.2}$$

Finding solutions for the system of equations (2.1) is a challenging task and has motivated many research works covering different areas of Mathematics. In this sense the study of central configurations plays an important role, one major reason being that these configurations are initial conditions for a special family of solutions of the $N$-body problem called *homographic solutions* (see chapter $2$ in (LLIBRE; MOECKEL; SIMO, 2015) and (MOECKEL, 2014)).

**Definition 2.1.** A configuration $q = (q_1, ..., q_n) \in \mathbb{R}^{dn}$ is *central* if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\sum_{j \neq i}^{N} m_j \parallel q_i - q_j \parallel^{2a} (q_i - q_j) = \lambda(q_i - q_G), \;\; \forall \, i = 1, ..., N, \tag{2.3}$$

where $q_G = \frac{1}{M} \sum_{i=1}^{N} m_i q_i$ is center of mass when $M = \sum_{i=1}^{N} m_i \neq 0$.

**Remark 2.2.** It is important to emphasize that the value of the exponent $a$ may play a fundamental role in the dynamics of the system (see (ALBOUY, 2003)), but if the interest is to study central configurations, which is a problem from statics, its value seems to be not so relevant. In this work, we usually consider $a < -1$ and, when it is convenient, we will concentrate on the Newtonian case, $a = -3/2$.

**Remark 2.3.** The dimension of a configuration is understood as the dimension of the smallest affine subspace of $\mathbb{R}^d$ containing the points $q_1, ..., q_N$, or simply the dimension of the space generated by $q_1 - q_i, ..., q_N - q_i$, which does not depend on the choice of the point $q_i$.

**Proposition 2.4.** *For any value of $a \neq 0$, a configuration of $N$ non-zero masses of dimension exactly $N-1$ is* central *if and only if it is the regular simplex.*

**Remark 2.5.** The value of the exponent $a = 0$ implies every configuration is central.

Additional fundamental information can be found in (ALBOUY, 2003).

**Lemma 2.6.** *Let $q = (q_1, ..., q_N)$ be a configuration. If the dimension of $q$ is exactly $N-2$, then there exists a unique non-zero $N-$tuple $X = (x_1, ..., x_N) \in \mathbb{R}^N$, up to a common factor, such that*

$$\sum_{i=1}^{N} x_i = 0 \quad \textit{and} \quad \sum_{i=1}^{N} x_i q_i = 0. \tag{2.4}$$

*Proof.* The dimension of a configuration is $N-2$, if and only if, for each $1 \leq i \leq N$, there is $k \in \{1, ...i-1, i+1, ..., N\}$ such that the vectors $\{q_j - q_i\}_{j \neq k,i}$ are linearly independent. So, there exists $\alpha_j \in \mathbb{R}$ such that

$$\begin{aligned} q_k - q_i = \sum_{j \neq i,k}^{N} \alpha_j (q_j - q_i) &= \alpha_1 (q_1 - q_i) + ... + \alpha_{i-1}(q_{i-1} - q_i) + \alpha_{i+1}(q_{i+1} - q_i) + ... \\ &+ \alpha_{k-1}(q_{k-1} - q_i) + \alpha_{k+1}(q_{i+1} - q_i) + ... + \alpha_N (q_N - q_i). \end{aligned}$$

Grouping common terms, we have

$$\begin{aligned} &\alpha_1 q_1 + ... + \alpha_{i-1} q_{i-1} + \alpha_{i+1} q_{i+1} + ... \alpha_{k-1} q_{k-1} + \alpha_{k+1} q_{k+1} + ... \alpha_N q_N \\ &+ (1 - \alpha_1 - ... - \alpha_{k-1} - \alpha_{k+1} - ... - \alpha_N) q_i - q_k = 0. \end{aligned}$$

Hence, taking $x_i = 1 - \alpha_1 - ... - \alpha_{k-1} - \alpha_{k+1} - ... - \alpha_N$, $x_k = -1$ and $x_j = \alpha_j$, $j \neq i, k$, the existence follows. Now, suppose that there are $X$ and $Y \in \mathbb{R}^N$ satisfying (2.4). Since the dimension of $q$ is $N-2$, we have

$$x_k (q_k - q_i) = - \sum_{j \neq k}^{N} x_j (q_j - q_i), \tag{2.5}$$

$$y_k (q_k - q_i) = - \sum_{j \neq k}^{N} y_j (q_j - q_i). \tag{2.6}$$

By multiplying (2.5) by $-y_k$, (2.6) by $x_k$ and adding the resulting expressions, we get

$$\sum_{j \neq k} (y_k x_j - y_j x_k)(q_j - q_i) = 0.$$

By linear independence, it follows that $y_k x_j - y_j x_k = 0$, so $x_j = \mu y_j$, for all $j \neq k, i$, where $\mu = \dfrac{x_k}{y_k}$, since for some $k$ we have that $y_k \neq 0$. □

**Definition 2.7.** A *Dziobek configuration* is a configuration $q = (q_1, ..., q_N)$, with non-zero masses, such that there exists a non-zero $X \in \mathbb{R}^N$ satisfying (2.4) and, for some pair $(\xi, \eta) \in \mathbb{R}^2$, we have that

$$s_{ij}^a = \xi + \eta \frac{x_i}{m_i} \frac{x_j}{m_j} \quad \forall\, i \neq j, \tag{2.7}$$

where $s_{ij} = \| q_i - q_j \|^2$ .

**Lemma 2.8.** *A Dziobek configuration is a central configuration with* $\xi = \dfrac{\lambda}{M}$.

**Lemma 2.9.** *A central configuration with non-zero masses and* $M \neq 0$ *of dimension exactly* $N - 2$ *is a Dziobek configuration.*

A proof of both lemmas is found in (ALBOUY, 2003) as well as that $\eta < 0$ for any Dziobek configuration, with $a < 0$ and $m_i > 0$, for all $i$.

**Remark 2.10.** Setting $t_i = \sum_{j \neq i} s_{ij} x_j$, equations (2.4) are equivalent to

$$\sum_{i=1}^{N} x_i = 0 \ \text{ and } t_i = t_j, \ \ \forall\, i \neq j.$$

See (SANTOS, 2004).The variables $x_i$ can be interpreted as barycentric coordinates and are sometimes called barycentric weights (ALBOUY, 2003). For more details about these coordinates, see (BERGER, 2009) and (COXETER, 1969).

The central configuration equations can be written equivalently as:

$$\nabla_i U(q) - \frac{\lambda}{2} \nabla_i I(q) = 0, \ \ i = 1, ..., N,$$

where $U$ is the potential function given by (2.2) and $I$ is the moment of inertia of the system

$$I(q) = \frac{1}{2} \sum_{i=1}^{N} m_i \| q_i \|^2. \tag{2.8}$$

The constant $\lambda$ can be interpreted as a Lagrange multiplier. Thus, central configurations are critical points of $U(q)$ under the condition that $I(q) = I_0$, where $I_0$ is a constant.

Bifurcation Theory allows us to find new central configurations close to a *degenerate central configuration*.

**Remark 2.11.** We must keep in mind that a central configuration is understood as being degenerate if it is not isolated as we vary the masses, already considering its equivalence class modulo rotations, translations, and dilations.

## 2.2 SOME BIFURCATION THEORY

In this section, we will describe some techniques, concepts, and results that will help us to study central configurations in some degenerate cases, as well to analyze the behavior of the bifurcation branches arising from a degenerate central configuration.

We begin by studying the structure of the bifurcations of an equilibrium solution of a system of ODEs

$$\dot{x} = \Phi(x, \epsilon),$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a smooth map. Assuming that $x = x_0$ is an equilibrium solution, i.e. $\Phi(x_0, \epsilon) = 0$ for all $\epsilon \in \mathbb{R}^k$, we consider a bifurcation problem as a problem of finding solutions to the equation

$$\Phi(x, \epsilon) = 0 \tag{2.9}$$

in the neighborhood of a degenerate solution $(x_0, \epsilon_0)$ in which $\epsilon$ is the bifurcation parameter. These solutions are usually of the form $x(\epsilon)$ but can also be of the form $\epsilon(x)$.

**Definition 2.12.** A degenerate solution of (2.9) is a solution such that $\Phi(x_0, \epsilon_0) = 0$ and $D_x\Phi(x_0, \epsilon_0) = 0$. We call *bifurcation diagram* the set $S = \{(x, \epsilon) \in \mathbb{R}^n \times \mathbb{R}^k \,|\, \Phi(x, \epsilon) = 0$ near $(x_0, \epsilon_0)\}$. If $n(\epsilon)$ denotes the number of solutions $x(\epsilon)$ of $\Phi(x, \epsilon) = 0$, a *bifurcation point* is a point $(x_0, \epsilon_0) \in S$ at which $n(\epsilon)$ changes when we vary the parameter in a neighborhood of $\epsilon_0$.

The vanishing of the determinant of $D_x\Phi(x_0, \epsilon_0)$ is a necessary condition for the point $(x_0, \epsilon_0)$ to be a bifurcation point. As a matter of fact, if rank$(D_x\Phi)(x_0, \epsilon_0)$ is maximum, then the implicit function theorem ensures that equation $(2.9)$ may be solved uniquely for $x$ in terms of $\epsilon$ in a neighborhood of $\epsilon_0$. Therefore, in a neighborhood of $(x_0, \epsilon_0)$ the number of solutions is always the same. So, by definition, $(x_0, \epsilon_0)$ can not be a bifurcation point (for more details, see (GOLUBITSKY; SCHAEFFER, 1988) ).

**Example 2.13.** Consider the ODE

$$\dot{x} = x^2 + \epsilon,$$

where $x, \epsilon \in \mathbb{R}$ and $\Phi(x, \epsilon) = x^2 + \epsilon$. The equilibrium points are $x = \pm\sqrt{-\epsilon}$. If $\epsilon < 0$, then there are two real equilibrium points, but if $\epsilon = 0$, then there is only one. It is clear that $\dfrac{d\Phi}{dx}(0, 0) = 0$, so the point $(0, 0)$ is the bifurcation point, and in a neighborhood of zero the equilibrium points appear and disappear. This type of bifurcation is known in the literature as a *saddle node bifurcation.*

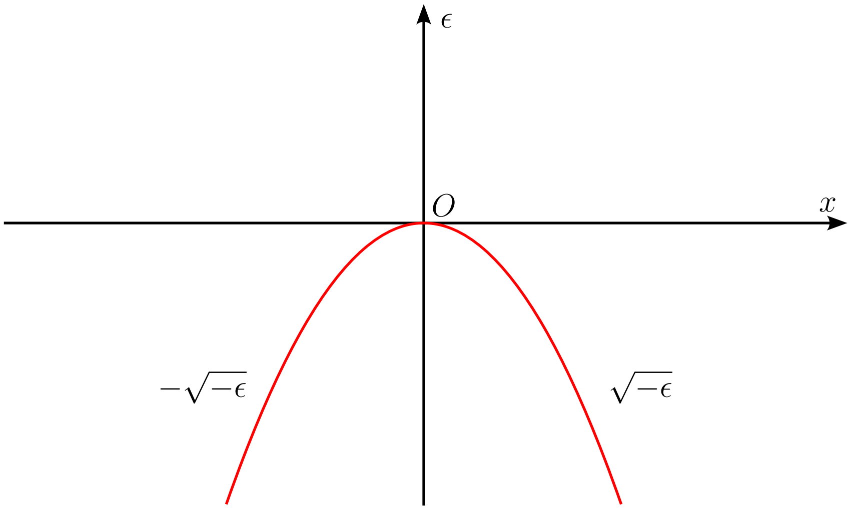


**Figure** 1. Saddle node bifurcation at $(0,0)$. The bifurcation branches are defined for $\epsilon < 0$.

**Example 2.14.** Consider the ODE

$$\dot{x} = x^3 - \epsilon\, x,$$

where $x, \epsilon \in \mathbb{R}$ and $\Phi(x,\epsilon) = x^3 - \epsilon\, x$. The equilibrium points are $x = 0$ and $x = \pm\sqrt{\epsilon}$. Similar to the previous example, the point $(0,0)$ is the only bifurcation point. The number of solutions of $\Phi = 0$ jumps from one to three when $\epsilon$ crosses zero and the equilibrium points that appear are symmetrical with respect to the $\epsilon-$axis. In this case, the bifurcation is known in the literature as a *pitchfork bifurcation*

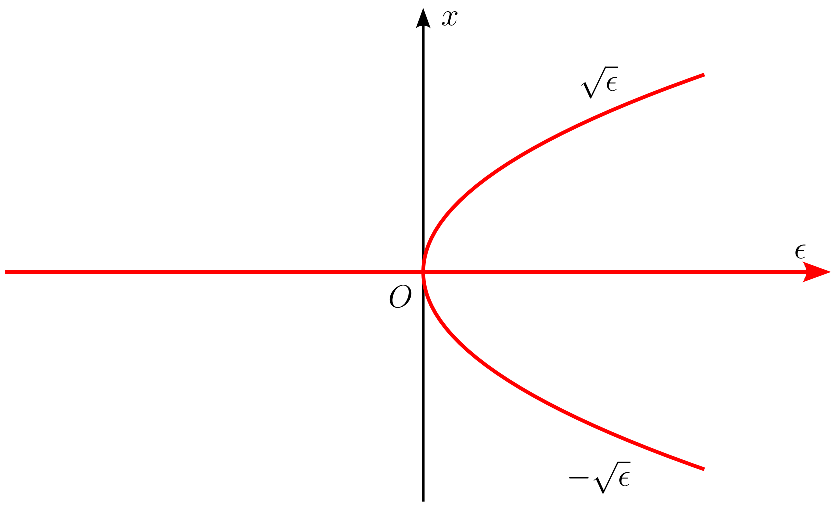


**Figure** 2. Pitchfork Bifurcation at $(0,0)$. The bifurcation branches are defined for $\epsilon > 0$.

For more examples and basic concepts, see (STROGATZ, 2000), (GOLUBITSKY; SCHAEFFER, 1988) and (HALE; KOÇAK, 2012).

**Remark 2.15.** This thesis is concerned only with finding bifurcations that arise from degenerate configurations, but not in classifying them.

### 2.2.1 Liapunov-Schmidt Reduction Method

The Liapunov-Schmidt reduction can help to solve, or at least simplify, degenerate problems. This method allows us to apply the implicit function theorem in situations where it apparently cannot be used. More precisely, we can apply it to minimally degenerate problems. The purpose of this section is to present the Liapunov-Schimidt reduction method for finite dimensional problems with or without symmetry. (for further details see (GOLUBITSKY; SCHAEFFER, 1988)).

Let us start with the Liapunov-Schmidt reduction method without symmetry. We consider a bifurcation problem as (2.9), where $\Phi(0,0) = 0$, rank$(D_x\Phi(0,0)) = n - r$, and let $L = D_x\Phi(0,0)$. By choosing subspaces complementary to the kernel and the image of $L$, it is possible to obtain a direct sum decompositions of the domain and the condomain of $\Phi$

$$\mathbb{R}^n = \ker(L) \oplus M, \tag{2.10}$$

$$\mathbb{R}^n = N \oplus \text{Im}(L). \tag{2.11}$$

**Lemma 2.16.** *Let P be the projection of $\mathbb{R}^n$ onto Im$(L)$ and $I - P$ the projection of $\mathbb{R}^n$ onto $N$. If $u \in \mathbb{R}^n$, then $u = 0$ if and only if $P(u) = 0$ and $(I - P)(u) = 0$.*

*Proof.*
($\Rightarrow$) It is clear since $P$ and $(I - P)$ are linear maps.
($\Leftarrow$) If $u \in \ker(P) = N$ and $u \in \ker(I - P) = \text{Im}(L)$, since $N \cap \text{Im}(L) = \{0\}$, it follows that $u = 0$. □

By lemma 2.16, equation (2.9) is equivalent to the pair of equations

$$(P \circ \Phi)(x, \epsilon) \quad = 0, \tag{2.12}$$

$$[(I - P) \circ \Phi](x, \epsilon) \quad = 0. \tag{2.13}$$

Dividing the problem into this pair of equations becomes interesting because it allows us to solve equation (2.12) for $(n - r)$ variables by applying the implicit function theorem to the function $(P \circ \Phi)(x, \epsilon)$. Actually, according to decomposition (2.10), any vetor $x \in \mathbb{R}^n$ can be written as $x = v + w$, where $v \in \ker(L)$ and $w \in M$. So, defining a map

$$\begin{aligned} F : \ker(L) \times M \times \mathbb{R}^k &\longrightarrow \mathrm{Im}(L) \\ (v, w, \epsilon) &\longmapsto F(v, w, \epsilon) = (P \circ \Phi)(v + w, \epsilon), \end{aligned}$$

we have that $F(0, 0, 0) = 0$ and

$$D_w F(v, w, \epsilon) = (P \circ D_w \Phi)(v + w, \epsilon).$$

At $(0, 0, 0)$, it follows that $D_w F(0, 0, 0) = L\big|_M$ is an invertible linear map. Hence, by the implicit function theorem, there exists a neighborhood $V \subset \ker(L) \times \mathbb{R}^k$ such that for all $(v, \epsilon) \in V$ there is a unique smooth function $w = W(v, \epsilon) : V \longrightarrow M$, where $W(0, 0) = 0$ and $F(v, W(v, \epsilon), \epsilon) = 0$.

Now, we substitute $W$ into equation (2.13) to obtain the map

$$\begin{aligned} \phi : \ker(L) \times \mathbb{R}^k &\longrightarrow N \\ (v, \epsilon) &\longmapsto \phi(v, \epsilon) = [(I - P) \circ \Phi](v + W(v, \epsilon), \epsilon). \end{aligned}$$

Thus, for all $(v, \epsilon)$ near $(0, 0)$, the zeros of $\phi(v, \epsilon)$ determine the zeros of $\Phi(x, \epsilon)$.

After replacing $W$ in equation (2.12), if we choose explicit bases for $\ker(L)$ and $N$, namely $\{v_1, ..., v_r\}$ and $\{v_1^*, ..., v_r^*\}$, respectively, we can define a map

$$\begin{aligned} g : \mathbb{R}^r \times \mathbb{R}^k &\longrightarrow \mathbb{R}^r \\ (y, \epsilon) &\longmapsto (g_1, ..., g_r), \end{aligned}$$

by $g_i = \langle v_i^*, \phi\left(\sum_{i=1}^r y_i v_i, \epsilon\right)\rangle$, when $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^r$, and $y = (y_1, ..., y_r)$ are the coordinates of the vector $v \in \ker(L)$ with respect to the chosen basis. So, $g(y, \epsilon) = 0$ if and only if $\phi\left(\sum_{i=1}^r y_i v_i, \epsilon\right) = 0$. Both of these equations are called *reduced equations.*

In particular, we can choose $M = \ker(L)^\perp$ and $N = \mathrm{Im}(L)^\perp$.

**Remark 2.17.** The function $g$ can be seen as a representation of $\phi$ in specific coordinates. Indeed, since $\phi\left(\sum_{j=1}^r y_j v_j, \epsilon\right) \in N$, we have $\phi(\sum_{j=1}^r y_j v_j, \epsilon) = \sum_{i=1}^r \alpha_i v_i^*$ and taking the inner product on both sides with $v_i^*$, and assuming that the basis of $N$ is an orthogonal basis, we obtain

$$\alpha_i(y, \epsilon) = \frac{\langle v_i^*, \phi\left(\sum_{i=1}^r y_i v_i, \epsilon\right)\rangle}{\| v_i^* \|^2}, \ i = 1, ..., r.$$

So, by definition of $\phi(v, \epsilon)$, if $N = \text{Im}(L)^{\perp}$, we have

$$\alpha_i(y, \epsilon) = \frac{\langle v_i^*, \Phi\left(\sum_{i=1}^{r} y_i v_i + W(\sum_{i=1}^{r} y_i v_i, \epsilon), \epsilon\right)\rangle}{\| v_i^* \|^2}, \; i = 1, ..., r, \tag{2.14}$$

for $v_i^* \in \text{Im}(L)^{\perp}$ whereas $(P \circ \Phi)\left(\sum_{i=1}^{r} y_i v_i + W\left(\sum_{i=1}^{r} y_i v_i, \epsilon\right), \epsilon\right) \in \text{Im}(L)$. Therefore, we can define $g_i := \alpha_i$.

### 2.2.2 Groups in Bifurcation Theory

In this section, we consider the Liapunov-Schmidt reduction method with symmetry. We treat the case when the operator $L$ in the bifurcation problem $(2.9)$ commutes with a group of symmetries. Usually, the symmetries present in the problem furnish a simplification for the equations after the Liapunov-Schmidt reduction. The reduced equation inherits the symmetry of the complete equation (the best reference, in my opinion, is chapter XVII of (GOLUBITSKY; SCHAEFFER, 1988)).

**Definition 2.18.** Let $\Gamma$ be a group and $\mathbb{V}$ a nonzero finite-dimensional vector space. A *representation* of $\Gamma$ on $\mathbb{V}$ is a homomorphism $\rho : \Gamma \longrightarrow GL(\mathbb{V})$. In other words, $\Gamma$ *acts linearly* on $\mathbb{V}$ if to each element $\gamma \in \Gamma$ we can associate an isomorphism $\rho_\gamma \in GL(\mathbb{V})$ defined by

$$\begin{aligned} \rho_\gamma : \mathbb{V} &\longrightarrow \mathbb{V} \\ v &\longmapsto \rho_\gamma(v), \end{aligned}$$

such that

$$\rho_{\gamma_1 \cdot \gamma_2}(v) = \left(\rho_{\gamma_1} \circ \rho_{\gamma_2}\right)(v), \quad \forall \; \gamma_1, \gamma_2 \in \Gamma.$$

**Definition 2.19.** A representation $\rho$ of a group $\Gamma$ on $\mathbb{V}$ is *absolutely irreducible* if the only linear operators on $\mathbb{V}$ that commute with each $\rho_\gamma$ are scalar multiples of the identity.

**Definition 2.20.** A map $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ *commutes* with the representation $\rho$ of a group of symmetries $\Gamma$ if

$$\Phi(\rho_\gamma(v)) = \rho_\gamma(\Phi(v)), \quad \forall \, \gamma \in \Gamma, v \in \mathbb{V}.$$

In this case we say that $\Phi$ is $\Gamma-$ *equivariant.*

**Remark 2.21.** If we want the representation $\rho$ act on the domain of $\Phi$ in the same way that it acts on the codomain, we must have that $\mathbb{V} \subseteq \mathbb{W}$. Otherwise, it is necessary to define a representation different from $\rho$ for act on $\mathbb{V}$.

In the context of the Liapunov-Schmidt reduction, definition 2.20 makes sense if we restrict the choice of the complements of the kernel and the image of the derivative map to $\Gamma$-invariant ( or, simply, invariant) subspaces M and N, respectively. The next two lemmas will allow us to construct such subspaces.

**Lemma 2.22.** *Consider $\Gamma$ a finite group, $\Phi : \mathbb{V} \longrightarrow \mathbb{W}$ a $\Gamma-$equivariant map, where $\mathbb{V}$ and $\mathbb{W}$ are finite-dimensional vector subspaces, and $L = D\Phi(0)$. Then,*
*(a) $L$ commutes with $\Gamma$.*
*(b) $\ker(L)$ is $\Gamma$-invariant subspace of $\mathbb{V}$.*
*(c) Im($L$) is $\Gamma$-invariant subspace of $\mathbb{W}$.*

*Proof.*
(a) Let $\rho$ be a representation of $\Gamma$. By hypothesis, $\Phi(\rho_\gamma(v)) = \rho_\gamma(\Phi(v))$, for all $\gamma \in \Gamma$. So, by the chain rule

$$D\Phi(\rho_\gamma(v))D\rho_\gamma(v)\Big|_{v=0} = D\rho_\gamma(v)D\Phi(\rho_\gamma(v))\Big|_{v=0}.$$

It follows that $L \circ \rho_\gamma = \rho_\gamma \circ L$, for all $\gamma \in \Gamma$ and $v \in \mathbb{V}$.

(b) Pick $v \in \ker(L)$. In accordance with (a), we have

$$L(\rho_\gamma(v)) = \rho_\gamma(L(v)) = \rho_\gamma(0) = 0, \ \ \forall \, \gamma \in \Gamma.$$

Thus, $\rho_\gamma(v) \in \ker(L)$. Item (c) is proved similarly to item (b). □

**Lemma 2.23.** *If $\mathbb{V}$ is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, which is preserved by the action of a group $\Gamma$, i.e.,*

$$\langle \rho_\gamma(v_1), \rho_\gamma(v_2) \rangle = \langle v_1, v_2 \rangle, \ \ \forall \, v_1, v_2 \in \mathbb{V} \text{ and } \gamma \in \Gamma, \quad (\bigstar)$$

*then the orthogonal complements of $\ker(L)$ and Im($L$) are $\Gamma$-invariant.*

*Proof.* Assume $u \in \ker(L)^\perp$ and $v \in \ker(L)$. We want to show that $\rho_\gamma(u) \in \ker(L)^\perp$. Since $\ker(L)$ is $\Gamma$-invariant, by the previous lemma, we have $\rho_\gamma^{-1}(v) \in \ker(L)$ and $\langle u, \rho_\gamma^{-1}(v) \rangle = 0$. Hence,

$$\langle \rho_\gamma(u), v \rangle = \langle \rho_\gamma(u), \rho_\gamma(\rho_\gamma^{-1}(v)) \rangle = \langle u, \rho_\gamma^{-1}(v) \rangle = 0,$$

where the second equality follows from the hypothesis ($\bigstar$). Since $u$ was arbitrarily chosen, we proved the first claim. Similarly, we show that Im$(L)^\perp$ is $\Gamma$-invariant. □

In order to show that the reduced map is equivariant, we need the next lemma. Let us assume that the parameters are not affected by the action of $\Gamma$.

**Lemma 2.24.** *Consider $\Gamma$ a finite group. If $W$ is an implicit solution of equation (2.12), then*

$$W(\rho_\gamma(v), \epsilon) = \rho_\gamma(W(v, \epsilon)),$$

*for all $\gamma \in \Gamma$ and $v \in \ker(L)$ near the origin.*

*Proof.* Firstly, we prove that $P$ commutes with the action of $\Gamma$ (recall that $P : N \oplus \text{Im}(L) \longrightarrow \text{Im}(L)$ is the projection on the image and its kernel is $N$). If $u \in N$, $w \in \text{Im}(L)$ and $x = u+w$, then

$$\rho_\gamma(P(x)) = \rho_\gamma(w) = P(\rho_\gamma(w)) = P(\rho_\gamma(u)) + P(\rho_\gamma(w)) = P(\rho_\gamma(u) + \rho_\gamma(w)) = P(\rho_\gamma(x)),$$

where it was used that the projection is the identity on $\text{Im}(L)$, as well as the linearity of $P$ and $\rho_\gamma$ for all $\gamma \in \Gamma$. In the same way, we prove that $(I - P)$ commutes with $\rho$. Now, we verify that $\rho_{\gamma^{-1}}(W(\rho_\gamma(v), \epsilon))$ is also a solution of (2.12) for all $v \in \ker(L)$ near the origin. Indeed,

$$\begin{aligned}(P \circ \Phi)(v + \rho_{\gamma^{-1}}(W(\rho_\gamma(v), \epsilon)), \epsilon) &= (P \circ \Phi)(\rho_{\gamma^{-1}}(\rho_\gamma(v) + W(\rho_\gamma(v), \epsilon)), \epsilon) \\ &= \rho_{\gamma^{-1}}((P \circ \Phi)(\rho_\gamma(v) + W(\rho_\gamma(v), \epsilon), \epsilon)) \\ &= 0,\end{aligned}$$

provided $v$ is near the origin, so $\rho_\gamma(v) \in \ker(L)$ is also near the origin. So, $\rho_{\gamma^{-1}}(W(\rho_\gamma(v), \epsilon))$ is a solution of equation (2.12) with $\rho_{\gamma^{-1}}(W(\rho_\gamma(0), 0)) = \rho_{\gamma^{-1}}(W(0, 0)) = 0$. Therefore, by the uniqueness of the implicit solution, the statement follows. □

**Proposition 2.25.** *In the situation given in the previous section, if $M$ and $N$ are $\Gamma$-invariant subspaces, then the reduced map $\phi : \ker(L) \times \mathbb{R}^k \longrightarrow N$ commutes with $\rho_\gamma$ for all $\gamma \in \Gamma$.*

*Proof.* Indeed,

$$\begin{aligned}\phi(\rho_\gamma(v), \epsilon) &= (I - P)(\Phi(\rho_\gamma(v) + W(\rho_\gamma(v), \epsilon), \epsilon)), \\ &= (I - P)(\Phi(\rho_\gamma(v + W(v, \epsilon)), \epsilon)) \quad \text{(by lemma 2.24 and linearity of } \rho_\gamma\text{)}, \\ &= \rho_\gamma(\phi(v, \epsilon)), \ \forall\, \gamma \in \Gamma \qquad \text{(by the equivariance of } \Phi \text{ and } I - P\text{)}.\end{aligned}$$

□

In coordinates, let $\beta = \{v_1, ..., v_n\}$ and $\beta^* = \{v_1^*, ..., v_n^*\}$ be bases of $\ker(L)$ and $\text{Im}(L)^\perp$, respectively. Let $\rho_\gamma^{(1)}$ and $\rho_\gamma^{(2)}$ be the actions of $\gamma$ on $\ker(L)$ and $\text{Im}(L)^\perp$, respectively, so that

$$\rho_\gamma^{(1)}(v_j) = \sum_{i=1}^{n} a_{ij}(\gamma) v_i, \tag{2.15}$$

$$\rho_\gamma^{(2)}(v_j^*) = \sum_{i=1}^{n} b_{ij}(\gamma) v_i^*. \tag{2.16}$$

The actions are determined by the matrices $A(\gamma)$ and $B(\gamma)$, called *action matrices* in relation to the respective ordered bases. The actions $\rho^{(1)}_\gamma$ and $\rho^{(2)}_\gamma$ are isomorphic if there exists a linear isomorphism $\tau : \ker(L) \longrightarrow \mathsf{Im}(L)^\perp$ such that

$$\tau \circ \rho^{(1)}_\gamma = \rho^{(2)}_\gamma \circ \tau, \quad \forall\, \gamma \in \Gamma.$$

As a consequence, there is an inverse matrix $S$ such that

$$B(\gamma) = S^{-1}A(\gamma)S, \;\; \forall\, \gamma \in \Gamma.$$

Hence, by changing the basis of $\mathsf{Im}(L)^\perp$, we can have the same action on $\ker(L)$ and $\mathsf{Im}(L)^\perp$ (for a more general discussion in the context of group representation theory see (SERRE, 1977)).

With respect to the bases $\beta$ and $\beta^*$, the reduced equation satisfies

$$g(A(\gamma)y, \epsilon) = A(\gamma)g(y, \epsilon). \tag{2.17}$$

As a matter of fact, from (2.15) and (2.16), the $i$-th coordinate of the vector on the right-hand side of equation 2.17 is

$$\sum_{j=1}^{r} a_{ij}(\gamma) \left\langle v_j^*, \phi\left(\sum_{i=1}^{r} y_i v_i, \epsilon\right)\right\rangle. \tag{2.18}$$

On the other hand,

$$g_i(A(\gamma)y, \epsilon) = \left\langle v_i^*, \phi\left(\sum_{i=1}^{r}\left(\sum_{j}^{r} a_{ij}(\gamma)y_j\right) v_i, \epsilon\right)\right\rangle, \tag{2.19}$$

but, for each $v \in \ker(L)$, we have $\rho_\gamma(v) = \sum_{i=1}^{r}\left(\sum_{j=1}^{r} a_{ij}(\gamma)y_j\right) v_i$. So,

$$\begin{aligned} g_i(A(\gamma)y, \epsilon) &= \langle v_i^*, \phi(\rho_\gamma(v), \epsilon)\rangle = \langle v_i^*, (\rho_\gamma \circ \phi)(v, \epsilon)\rangle = \langle \rho_{\gamma^{-1}}(v_i^*), \phi(v, \epsilon)\rangle, \\ &= \sum_{j=i}^{r} a_{ji}(\gamma^{-1})\langle v_j^*, \phi(v, \epsilon)\rangle, \end{aligned} \tag{2.20}$$

since, as long as $\rho_\gamma$ is orthogonal, it follows that $A(\gamma^{-1}) = [A(\gamma)]^t$. Hence the last expression in (2.20) is equal to (2.18) and equation (2.17) is verified.

**Lemma 2.26.** *In the context discussed previously, the derivatives of the functions $W$ and $g$ at $(0, 0)$ are zero.*

*Proof.* Differentiating both sides of equality (2.12) with respect to $v \in \ker(L)$ at (0,0), we obtain, for $u \in \ker(L)$,

$$\begin{aligned} PD\Phi(0 + W(0, 0), 0)(\mathsf{I} + D_vW(0, 0))u = 0, \\ PL(u + D_vW(0, 0)u) = 0. \end{aligned}$$

Since $L$ is an invertible operator and $P$ acts as the identity when restricted to M, it follows that $D_vW(0,0) = 0$. Recall $g_i = \langle v_i^*, \phi(\sum_{i=1}^r y_iv_i, \epsilon)\rangle$. Differentiating $g_i$ with respect to $y_j$, we obtain

$$D_{y_j}g_i(0,0) = \langle v_i^*, (I-P)DG(0+W(0,0),0)(v_j + D_vW(0,0)v_j)\rangle = 0,$$

for all $i, j = 1, ...r$. □

### 2.2.3 Restriction to Fixed-Point Subspaces

According to (GOLUBITSKY; STEWART; SCHAEFFER, 1985), although the Liapunov-Schmidt reduction is useful for simplifying bifurcation problems with symmetry, often, after applying this method, we are still left with a problem that is difficult to solve. Restricting the equations to the subspace of fixed points of a certain subgroup of the symmetry group of the problem allows us to reduce the number of equations to be solved. To be more precise, the number of equations left to solve after applying the method is not greater than the dimension of $\ker(L)$, since the fixed point subspace is a subspace of $\ker(L)$. So we may effectively reduce the number of equations to be solved and the chances of obtaining solutions are higher. When we make such a restriction we are also restricting our search to solutions with a certain type of symmetry.

Let $\Gamma$ be the symmetry group of (2.9) and $y$ a solution of the reduced equation (after applying Liapunov-Schmidt reduction). This solution has the symmetry of some subgroup $\Sigma$ of $\Gamma$ if it satisfies $\rho_\sigma(y) = y$, for all $\sigma \in \Sigma$; besides, it is in

$$\text{Fix}(\Sigma) = \{y \in \ker(L) \,|\, \rho_\sigma(y) = y, \, \forall\, \sigma \in \Sigma\}.$$

This set is called *fixed point subspace* for $\Sigma$. In this context, we have the following lemma.

**Lemma 2.27.** *Let $g : \mathbb{V} \times \mathbb{R} \longrightarrow \mathbb{V}$ be a $\Gamma-$equivariant map. The fixed point subspaces, for each $\Sigma \leq \Gamma$ are invariant by $g$.*

*Proof.* Assume $y \in$ Fix($\Sigma$), for some subgroup $\Sigma$ of $\Gamma$. Thus,

$$\rho_\sigma(g(y,\epsilon)) = g(\rho_\sigma(y),\epsilon) = g(y,\epsilon), \;\; \forall\, \sigma \; \in \Sigma.$$

Hence, $g(y,\epsilon) \in$ Fix($\Sigma$). □

Due to the last lemma, when we want to find solutions of the reduced equation with some specific symmetry, it is sufficient to solve the restricted system of equations

$$g\big|_{\mathsf{Fix}(\Sigma)\times\mathbb{R}} = 0. \tag{2.21}$$

A question asked in (GOLUBITSKY; SCHAEFFER, 1988) is: when does making this restriction help us find a solution to the bifurcation problem? In other words, which isotropy subgroup $\Sigma$ guarantees that there is a solution with this type of symmetry? A partial answer occurs for subgroups such that

$$\dim \mathsf{Fix}(\Sigma) = 1. \tag{2.22}$$

The following result ensures the existence of symmetrical solutions.

**Theorem 2.28.** ***(Equivariant Branching Theorem)*** *Suppose that the action of a finite group $\Gamma$ on a vector space $\mathbb{V}$ is absolutely irreducible and $g : \mathbb{V}\times\Lambda \longrightarrow \mathbb{V}$ defines a smooth and $\Gamma$-equivariant bifurcation problem (consequently $Dg(0,\epsilon) = c(\epsilon)Id$). Assume that $0 \in \Lambda \subset \mathbb{R}$ with $c(0) = 0$ and $c'(0) \neq 0$. Moreover, let $\Sigma \leq \Gamma$ be an isotropy subgroup which satisfies (2.22). Then there exists a unique branch $(y, \epsilon(y))$ of solutions to $g(y,\epsilon) = 0$ near the trivial solution with symmetry $\Sigma$.*

Theorem 2.28 is due to (VANDERBAUWHEDE, 1980) and (CICOGNA, 1981). In (GOLUBITSKY; SCHAEFFER, 1988) there is a slightly more general statement of this theorem and a similar theorem when $\dim$ Fix$(\Sigma) = 2$. This is particularly true when analyzing Hopf bifurcations.

**Remark 2.29.** Although this theorem is an important tool for ensuring the existence of certain symmetric solutions that bifurcate from the degenerate solution, it does not guarantee that other solutions do not also bifurcate.

In (SANTOS, 2004), theorem 2.28 was used to prove the existence of some central configurations which arise from the centered regular tetrahedron. In the next chapter, we will apply it to find central configurations bifurcating from the centered equilateral triangle.

# 3 BIFURCATIONS OF THE CENTERED EQUILATERAL TRIANGULAR CONFIGURATION

The goal of this chapter is to study central configurations of the four-body problem in the plane, specifically the bifurcations that occur from a centered triangular configuration with unit masses at its vertices and a mass of arbitrary value at the center. Our approach is based on (SANTOS, 2004). Initially, we were motivated to explore the symmetry of the problem so that we can make a more general study and find new solutions. We assume that the exponent of the potential function of the system is negative and can take any value less than $-1$. We find three families of central configurations bifurcating from the centered equilateral triangle. So far, we have been able to fully analyze the Newtonian case, and the solutions found behave in the same way as the solutions in (MEYER; SCHMIDT, 1987), i.e. configurations in the shape of isosceles triangles.

## 3.1 PROBLEM OVERVIEW

Let $(q_1, q_2, q_3, q_4)$ be a planar configuration with three unitary masses located at the vertices of an equilateral triangle and one variable mass at its barycenter. More precisely, we consider $m_1 = m_2 = m_3 = 1$, $m_4 = m$ and the squares of the mutual distances $s_{ij} = \| q_i - q_j \|^2$ between the bodies given by $s_{12} = s_{13} = s_{23} = 3$ and $s_{14} = s_{24} = s_{34} = 1$. It is already known that this configuration is central for all values of the central mass. This configuration has dimension $2 = 4 - 2$, so, by lemma 2.9, it is a Dziobek configuration and must satisfy the system of equations

$$\begin{aligned} \sum_{i=1}^{4} x_i &= 0, \\ t_i &= t_j, \\ s_{ij}^a - \frac{\lambda}{M} &= -\frac{x_i x_j}{m_i m_j}, \end{aligned} \tag{3.1}$$

$1 \le i < j \le 4$, where $t_i = \sum_{j \ne i} s_{ij} x_j$, $M = \sum_{i=1}^{4} m_i$ and $\lambda = m + 3^{1+a}$. Indeed, if $m_i = m_j$ and $s_{ik} = s_{jk}$, for some $k$, then $x_i = x_j$. Thus, $x_1 = x_2 = x_3$ and $x_4 = -3x_1$, and substituting these in the system (3.1), the last equations are reduced to just two,

$$\begin{cases} \dfrac{\lambda}{3+m} - 3^a &= x_1^2, \\ \dfrac{\lambda}{3+m} - 1 &= -\dfrac{3x_1^2}{m}. \end{cases}$$

Multiplying the first equation by $3$, the second by $m$, and adding them together, we obtain the expression for $\lambda$. Consequently, we have $(x_1, x_2, x_3, x_4) = (k, k, k, -3k)$, where $k = k(m, a) = \sqrt{\dfrac{m(1-3^a)}{3+m}}$ represents the centered triangular family of central configurations.

Let us assume that $M$ is positive and $a < -1$.

Since $x_4 = -x_1 - x_2 - x_3$ and $s_{ij} = \left(\dfrac{\lambda}{M} - \dfrac{x_i}{m_i}\dfrac{x_j}{m_j}\right)^{1/a}$, system (3.1) can be represented by equation

$$F(X, m) = 0, \tag{3.2}$$

where $X = (x_1, x_2, x_3)$ and $F : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3$ is a smooth function given by $F_i = t_i - t_4$, $i = 1, 2, 3$. Explicitly,

$$\begin{aligned}
F_1(X, m) =& x_2\left(\frac{\lambda}{M} - x_1x_2\right)^{1/a} + x_3\left(\frac{\lambda}{M} - x_1x_3\right)^{1/a} \\
&- (x_1 + x_2 + x_3)\left(\frac{\lambda}{M} + \frac{x_1(x_1 + x_2 + x_3)}{m}\right)^{1/a} - t_4, \\
F_2(X, m) =& x_1\left(\frac{\lambda}{M} - x_1x_2\right)^{1/a} + x_3\left(\frac{\lambda}{M} - x_2x_3\right)^{1/a} \\
&- (x_1 + x_2 + x_3)\left(\frac{\lambda}{M} + \frac{x_2(x_1 + x_2 + x_3)}{m}\right)^{1/a} - t_4, \\
F_3(X, m) =& x_1\left(\frac{\lambda}{M} - x_1x_3\right)^{1/a} + x_2\left(\frac{\lambda}{M} - x_2x_3\right)^{1/a} \\
&- (x_1 + x_2 + x_3)\left(\frac{\lambda}{M} + \frac{x_3(x_1 + x_2 + x_3)}{m}\right)^{1/a} - t_4,
\end{aligned}$$

and

$$\begin{aligned}
t_4 =& x_1\left(\frac{\lambda}{M} + \frac{x_1(x_1 + x_2 + x_3)}{m}\right)^{1/a} + x_2\left(\frac{\lambda}{M} + \frac{x_2(x_1 + x_2 + x_3)}{m}\right)^{1/a} \\
&+ x_3\left(\frac{\lambda}{M} + \frac{x_3(x_1 + x_2 + x_3)}{m}\right)^{1/a}.
\end{aligned}$$

We can immediately verify that $F$ is $S_3$-equivariant, i.e.,

$$F(\rho_\gamma(X), m) = \rho_\gamma F(X, m), \tag{3.3}$$

where $\rho$ is the representation of $S_3$, the permutation group of the three symbols $1, 2, 3$, on $\mathbb{R}^3$ given by $\rho_\gamma(x_1, x_2, x_3) = (x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)})$, for all $\gamma \in S_3$.

**Remark 3.1.** The matrices of the representation $\rho$ with respect to the canonical basis of $\mathbb{R}^3$ are

$$\rho_{(12)} : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \rho_{(13)} : \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \rho_{(23)} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\rho_{(123)}: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ \rho_{(132)}: \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $F$ is $S_3$-equivariant and $X^0(m) = (k(m), k(m), k(m)) \in \text{Fix}(S_3)$ for all $m$, we can apply the chain rule in (3.3) and obtain

$$D_X F(X^0(m), m)\rho_\gamma = \rho_\gamma D_X F(X^0(m), m),$$

i,e., $D_X F(X^0(m), m)$ is also $S_3$-equivariant. Hence, it follows from equivariance that

$$D_X F(X^0(m), m) = \begin{bmatrix} \eta_1 & \eta_2 & \eta_2 \\ \eta_2 & \eta_1 & \eta_2 \\ \eta_2 & \eta_2 & \eta_1 \end{bmatrix}, \tag{3.4}$$

where

$$\eta_1 = \frac{\partial F_1}{\partial x_1}\Big|_{(X^0(m),m)} = -2 - \frac{2k^2}{a}\left(3^{1-a} + \frac{9}{m}\right) \text{ and } \eta_2 = \frac{\partial F_1}{\partial x_2}\Big|_{(X^0(m),m)} = 1 - \frac{k^2}{a}\left(3^{1-a} + \frac{9}{m}\right).$$

Thus, the determinant of (3.4) is

$$\det(D_X F(X^0, m)) = (\eta_1 + 2\eta_2)(\eta_1 - \eta_2)^2.$$

The term $\eta_1 + 2\eta_2 = -\frac{4k^2}{a}\left(\frac{9}{m} + 3^{1+a}\right)$ is positive whenever $a < -1$ and $m > 0$. On the other hand, we have that

$$\eta_1 - \eta_2 = -3 - \frac{3k^2}{a}\left(\frac{3}{m} + 3^{-a}\right) = 0 \iff m = m^* := 3\left(\frac{3^a - a - 1}{3^{-a} + a - 1}\right).$$

**Remark 3.2.** When $a = -3/2$, we have $m^* = \frac{81 + 64\sqrt{3}}{249}$ which agrees with the value found in (PALMORE, 1973) and (MEYER; SCHMIDT, 1987).

**Proposition 3.3.** For $a < -1$, the value of $m^*$ is in the interval $(0, 1)$.

*Proof.* See (SANTOS et al., 2017). □

Equation (3.2) describes a bifurcation problem with four parameters. For each exponent $a$, the pair $(X(m^*), m^*)$ represents a degenerate centered equilateral triangular configuration, which is a candidate for a bifurcation point. Actually the degenerate mass vector is the quadruple $(m_1, m_2, m_3, m_4) = (1, 1, 1, m^*)$. We will study what happens with the number of solutions of equation (3.2) in a neighborhood of the point $(1,1,1,m^*)$. In other words, we want to approach this point on the straight line generated by $(0,0,0,\epsilon)$. As in (SANTOS, 2004), we will make use of the Equivariant Branch Theorem to guarantee the existence and uniqueness of symmetric solutions to this problem.

## 3.2 BIFURCATION ANALYSIS

Consider $m_1 = m_2 = m_3 = 1$ and $m_4 = m^* + \epsilon$. Replacing in (3.1), we obtain a bifurcation problem

$$F(X,\epsilon) = 0, \tag{3.5}$$

with three variables $X = (x_1, x_2, x_3)$ and a parameter $\epsilon$. Let us represent the centered triangular configuration by $\bar{X}(\epsilon) = (k(\epsilon), k(\epsilon), k(\epsilon))$ which is non-degenerate for all $\epsilon \neq 0$, where $k(\epsilon) = \sqrt{\dfrac{(m^*+\epsilon)(1-3^a)}{3+m^*+\epsilon}}$.

The derivative of $F$ at $(\bar{X}(0), 0)$ is

$$D_X F(\bar{X}(0),0) = \begin{bmatrix} \eta_1 & \eta_1 & \eta_1 \\ \eta_1 & \eta_1 & \eta_1 \\ \eta_1 & \eta_1 & \eta_1 \end{bmatrix}. \tag{3.6}$$

Let us apply Liapunov-Schmidt reduction with symmetry. If we define $L = D_X F(\bar{X}(0), 0)$, it follows that

$$\ker(L) = \left\{ (v_1, v_2, v_3) \in \mathbb{R}^3 \,\middle|\, \sum_{i=1}^{3} v_i = 0 \right\} \text{ and } \mathrm{Im}(L) = \{(s,s,s) \mid s \in \mathbb{R}\}.$$

These subspaces are $\rho$-invariant. It is convenient to pick

$$\beta_1 = \left\{ u_1 = \left( -\frac{\sqrt{6}}{6}, \frac{2\sqrt{6}}{6}, -\frac{\sqrt{6}}{6} \right), u_2 = \left( -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right) \right\}$$

and

$$\beta_2 = \left\{ u_3 = \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \right\},$$

as bases for the kernel and the image of $L$, respectively. Moreover, we decompose the domain and codomain of $L$ as

$$\mathbb{R}^3 = \ker(L) \oplus \mathrm{Im}(L)$$

and perform the change of variables

$$(x_1, x_2, x_3) = \sum_{i=1}^{3} y_i u_i. \tag{3.7}$$

Explicitly,

$$x_1 = -\frac{\sqrt{6}}{6}y_1 - \frac{\sqrt{2}}{2}y_2 + \frac{\sqrt{3}}{3}y_3,\ x_2 = \frac{\sqrt{6}}{3}y_1 + \frac{\sqrt{3}}{3}y_3,\ x_3 = -\frac{\sqrt{6}}{6}y_1 + \frac{\sqrt{2}}{2}y_2 + \frac{\sqrt{3}}{3}y_3.$$

**Remark 3.4.** Let $A$ be the change of basis matrix from $\beta = \beta_1 \cup \beta_2$ to the canonical basis of $\mathbb{R}^3$. Since $A$ is orthogonal, we can obtain the matrices of the representation $\rho$ with respect to basis $\beta$ by computing the matrices $[\rho_\gamma]_\beta = A^t[\rho_\gamma]_c A$, for all $\gamma \in S_3$. Thus,

$$\rho_{(12)} : \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \rho_{(13)} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \rho_{(23)} : \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\rho_{(123)} : \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \rho_{(132)} : \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting the change of variable (3.7) in (3.5), we have a new equation

$$G(Y, \epsilon) = 0. \tag{3.8}$$

**Lemma 3.5.** The function $G(Y, \epsilon)$ is $S_3-$equivariant.

*Proof.* Since $F(X, \epsilon)$ and $X(Y)$ are $S_3$-equivariant, it follows that

$$G(\rho_\gamma(Y), \epsilon) = F(X(\rho_\gamma(Y)), \epsilon) = F(\rho_\gamma(X(Y)), \epsilon) = \rho_\gamma F(X(Y), \epsilon) = \rho_\gamma G(Y, \epsilon).$$

□

The solution of equation (3.8) which corresponds to the centered triangular configuration is given by

$$\overline{Y}(\epsilon) = A^t \bar{X}(\epsilon) = (0, 0, \sqrt{3}k(\epsilon)).$$

Due to the decomposition of $\mathbb{R}^3$, we can define a projection onto Im$(L)$, denoted by $P : \mathbb{R}^3 \longrightarrow$ Im$(L)$ such that $\ker(L) = \ker(P)$, and another projection onto $\ker(L)$ given by $(I - P) : \mathbb{R}^3 \longrightarrow \ker(L)$. Keeping this in mind, equation (3.8) is equivalent to system

$$(P \circ G)(Y, \epsilon) = \frac{\sqrt{3}}{3}\langle (1, 1, 1), G(Y, \epsilon)\rangle\, u_3 = 0, \tag{3.9}$$

$$((I - P) \circ G)(Y, \epsilon) = \frac{\sqrt{6}}{6}\langle (-1, 2, -1), G(Y, \epsilon)\rangle\, u_1 + \frac{\sqrt{2}}{2}\langle (-1, 0, 1), G(Y, \epsilon)\rangle\, u_2 = 0. \tag{3.10}$$

Let us verify that equation (3.9) may be solved for $y_3$. We must check the hypotheses of the implicit function theorem for the function $\Psi(Y,\epsilon) = \dfrac{\sqrt{3}}{3}\left(\sum_{i=1}^{3} G_i(Y,\epsilon)\right)$. We have that $\dfrac{\partial \Psi}{\partial y_i}(\overline{Y}(0),0) = 0,\ i = 1,2$, whereas $\dfrac{\partial \Psi}{\partial y_3}(\overline{Y}(0),0) = -\dfrac{4k^2}{a}\left(3^{1-a} + \dfrac{9}{m^*}\right) > 0$, for all $a < -1$. Thus, there exist $U \subset \mathbb{R}^2 \times \mathbb{R}$ and $V \subset \mathbb{R}$ neighborhoods of $(0,0,0)$ and $\sqrt{3}k$, respectively, such that for each $(y_1, y_2, \epsilon) \in U$ there is a unique smooth function $y_3 = W : U \longrightarrow V$, such that $W(0,0,0) = \sqrt{3}k$ and $\Psi(y_1, y_2, W(y_1, y_2, \epsilon), \epsilon) = 0$ (we denoted $k = k(0)$).

**Lemma 3.6.** The function $\Psi$ is $S_3$-invariant.

*Proof.* Indeed,

$$\begin{aligned}
\Psi(\rho_\gamma(Y),\epsilon) &= \frac{\sqrt{3}}{3}\left(\sum_{i=1}^{3} G_i(\rho_\gamma(Y),\epsilon)\right) \\
&= \frac{\sqrt{3}}{3}\left(\sum_{i=1}^{3} G_{\gamma(i)}(Y,\epsilon)\right) \quad \text{(by the equivariance of } G\text{)} \\
&= \frac{\sqrt{3}}{3}\left(\sum_{j=1}^{3} G_j(Y,\epsilon)\right) \\
&= \Psi(Y,\epsilon) \quad (\text{if } j = \gamma(i),\ \ \forall \gamma \in S_3).
\end{aligned}$$

□

**Remark 3.7.** We define the representation of $S_3$ on $\ker(L)$ as $\tilde{\rho}$ and check that the function $W$ is $S_3$-invariant. In fact, for $(y_1, y_2)$ near the origin, it follows that

$$\Psi(y_1, y_2, W(\tilde{\rho}_\gamma(y_1,y_2),\epsilon),\epsilon) = \Psi(\tilde{\rho}_\gamma(y_1,y_2), W(\tilde{\rho}_\gamma(y_1,y_2),\epsilon),\epsilon) = 0,$$

by invariance of $\Psi$ for $(y_1, y_2)$ near the origin. Hence, $W(\tilde{\rho}_\gamma(y_1,y_2),\epsilon)$ is a solution of $\Psi = 0$. Provided the implicit solution is unique, we have that $W(\tilde{\rho}_\gamma(y_1,y_2),\epsilon) = W(y_1,y_2,\epsilon)$.

Substituting $W$ in equation (3.10), we define

$$g(y_1, y_2, \epsilon) = 0, \tag{3.11}$$

where $g : U \longrightarrow \mathbb{R}^2$ is an analytic function given by

$$g_i(y_1, y_2, \epsilon) = \langle u_i, G(y_1, y_2, W(y_1, y_2, \epsilon), \epsilon)\rangle,\ i = 1, 2,$$

which becomes the bifurcation problem to be solved, that is, $g_1 = 0$ and $g_2 = 0$.

To find solutions to equation (3.11) we will apply the Equivariant Branching Theorem to the function $g$. The following lemma states that $g$ verifies the hypotheses of this theorem.

**Lemma 3.8.** *The map $g$, which we have previously defined, satisfies:*

*a)* $g(0,0,0) = 0$.

*b)* $g$ *is* $S_3$*-equivariant.*

*c) The action of* $S_3$ *on* $\ker(L)$ *is absolutely irreducible.*

*d)* $Dg(0,0,\epsilon) = c(\epsilon)I_{2\times 2}$, *and* $Dg(0,0,0) = 0$ *with* $\left.\dfrac{dc(\epsilon)}{d\epsilon}\right|_{\epsilon=0} \neq 0.$

*Proof.*

(a) Since $G(0,0,\sqrt{3}k,0) = 0$, the statement follows directly from definition of $g$.

b) Representing the matrices of the action of $S_3$ on $\ker(L)$ with respect to basis $\beta_1$ by

$$\tilde{\rho}_\gamma = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and with respect to canonical basis by $\rho_\gamma$, as defined before for each $\gamma \in S_3$, we have by (2.20) that

$$\begin{aligned} g_i(\tilde{\rho_\gamma}(\tilde{y}),\epsilon) &= \langle u_i, G(\tilde{\rho_\gamma}(\tilde{y}), W(\tilde{\rho_\gamma}(\tilde{y}),\epsilon),\epsilon)\rangle = \langle u_i, G(\tilde{\rho_\gamma}(\tilde{y}), W(\tilde{y},\epsilon),\epsilon)\rangle, \\ &= \langle u_i, G(\rho_\gamma(\tilde{y}, W(\tilde{y},\epsilon)),\epsilon)\rangle = \langle u_i, \rho_\gamma G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle, \\ &= \langle \rho_\gamma^t(u_i), G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle, \ \ i = 1,2, \end{aligned}$$

where $\tilde{y} = (y_1, y_2)$. Hence,

$$\begin{aligned} g(\tilde{\rho_\gamma}(\tilde{y}),\epsilon) &= (\langle \rho_\gamma^t(u_1), G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle, \langle \rho_\gamma^t(u_2), G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle), \\ &= (\langle a_{11}u_1 + a_{12}u_2, G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle, \langle a_{21}u_1 + a_{22}u_2, G(\tilde{y}, W(\tilde{y},\epsilon),\epsilon)\rangle), \\ &= (a_{11}g_1 + a_{12}g_2, a_{21}g_1 + a_{22}g_2), \\ &= \tilde{\rho}_\gamma(g(\tilde{y},\epsilon)), \end{aligned}$$

The second equality follows from the form of the matrix representation of the $\rho$ in the remarks 3.1 and 3.4, noting for example, that $\rho^t_{(124)} = \rho_{(142)}$.

c) By definition of absolutely irreducible, taking a $2\times 2$ matrix representing a linear transformation of $\mathbb{R}^2$, and making it commute with $\tilde{\rho}_\gamma$, for all $\gamma \in S_3$, $\tilde{\rho}_\gamma$ as defined in item (b), we verify that this matrix must be a multiple of the identity.

d) Since $g$ is an $S_3$-equivariant map, we have that $D_{\tilde{y}}g(0,\epsilon)$ is a linear map which is also $S_3$-equivariant. So, by item (c), the derivative of $g$ is a multiple of the identity which depends on $\epsilon$. Let us check its expression. Recall equation (3.6). We have that

$$
\begin{aligned}
D_Y G(\overline{Y}(\epsilon),\epsilon) &= D_X F(\bar{X}(\epsilon),\epsilon) D_Y X(\overline{Y}(\epsilon)) I_{3\times 3} \\
&= \begin{bmatrix}
\frac{\sqrt{6}}{6}(\eta_2-\eta_1) & \frac{\sqrt{2}}{2}(\eta_2-\eta_1) & \frac{\sqrt{3}}{3}(\eta_1+2\eta_2) \\
\frac{2\sqrt{6}}{6}(\eta_1-\eta_2) & 0 & \frac{\sqrt{3}}{3}(\eta_1+2\eta_2) \\
\frac{\sqrt{6}}{6}(\eta_2-\eta_1) & \frac{\sqrt{2}}{2}(\eta_1-\eta_2) & \frac{\sqrt{3}}{3}(\eta_1+2\eta_2)
\end{bmatrix}.
\end{aligned}
$$

Since $\dfrac{\partial g_i}{\partial y_j}(0,\epsilon) = \langle u_i, D_{\tilde{y}} G(0,\epsilon) e_j\rangle$, we have that

$$
\begin{aligned}
\frac{\partial g_1}{\partial y_1}(0,\epsilon) &= \frac{1}{6}\left\langle(-1,2,-1),(\eta_2-\eta_1, 2(\eta_1-\eta_2), \eta_2-\eta_1))\right\rangle = \eta_1-\eta_2, \\
\frac{\partial g_1}{\partial y_2}(0,\epsilon) &= \frac{\sqrt{3}}{6}\left\langle(-1,2,-1),(\eta_2-\eta_1, 0, \eta_1-\eta_2))\right\rangle = 0, \\
\frac{\partial g_2}{\partial y_1}(0,\epsilon) &= \frac{\sqrt{3}}{6}\left\langle(-1,0,1),(\eta_2-\eta_1, 2(\eta_1-\eta_2), \eta_2-\eta_1))\right\rangle = 0, \\
\frac{\partial g_2}{\partial y_2}(0,\epsilon) &= \frac{1}{2}\left\langle(-1,0,1),(\eta_2-\eta_1, 0, \eta_1-\eta_2))\right\rangle = \eta_1-\eta_2.
\end{aligned}
$$

Hence,

$$
D_{\tilde{y}} g(0,\epsilon) = (\eta_1-\eta_2)(\epsilon) I_{2\times 2},
$$

where $(\eta_1-\eta_2)(\epsilon) = c(\epsilon)$, so

$$
\begin{aligned}
c(\epsilon) = &-\left[\frac{3^{1+a}+3^a(m^*+\epsilon)-\epsilon}{m^*+3+\epsilon}\right]^{\frac{1}{a}} - \left[\frac{9\,(1-3^a)}{a\,(3+m^*+\epsilon)}\right]\left(\frac{m^*+3}{m^*+3+\epsilon}\right)^{\frac{1-a}{a}} \\
&- \left[\frac{(m^*+\epsilon)\,(1-3^a)}{(3+m^*+\epsilon)a}\right]\left[\frac{3^{1+a}+3^a(m^*+\epsilon)-\epsilon}{m^*+3+\epsilon}\right]^{\frac{1-a}{a}}.
\end{aligned}
$$

When $\epsilon = 0$, we have that $D_{\tilde{y}} g(0,0,0) = 0$. Furthermore, differentiating $c(\epsilon)$ with respect to $\epsilon$, we get

$$
c'(0) = \frac{1}{3}\,\frac{[3^a(1-a)-1]\,[3(3^a-1)a^2 + 3^{a+1}\,(1-3^a)\,a - 2\cdot 3^{a+1} + 3^{2a+1} + 3]}{3^a a^2\,(3^a-1)^3}.
$$

All that is left to show is that $c'(0) \neq 0$ for all $a < -1$. Since the denominator is never $0$ for $a < -1$, we can see this fact directly from its numerator, which is already decomposed in two factors, none of which vanish for $a < -1$.

The first factor

$$
p(a) := 3^a(1-a) - 1,
$$

can be readily shown to vanish if, and only if,

$$
p_1(a) = 1 - a - 3^{-a}
$$

vanishes. This last expression has derivative:

$$p_1'(a) = -1 + 3^{-a}\ln(3)$$

which we can estimate from below term by term

$$p_1'(a) > -1 + 3 \cdot 1 = 2,$$

so $p_1$ is monotonically increasing on the interval $(-\infty, -1)$, hence it attains its maximum value at $-1$, $p_1(-1) = -1$, which in particular is less than zero. It follows that $p_1$ never vanishes on this interval, and so does not the original $p$.

As for the second term

$$q(a) = 3(3^a - 1)a^2 + 3^{a+1}(1 - 3^a)\,a - 2 \cdot 3^{a+1} + 3^{2a+1} + 3.$$

Notice we can manipulate the last few terms

$$\begin{aligned} -2 \cdot 3^a + 3^{2a} + 1 &= (-3^a - 3^a) + 3^a 3^a + 1 = -3^a + (-3^a + 3^a 3^a) + 1 \\ &= -3^a + 3^a(-1 + 3^a) + 1 = 3^a(-1 + 3^a) + (1 - 3^a), \end{aligned}$$

by substituting these terms into q(a), we obtain the following short factorization:

$$q(a) = 3(1 - 3^a)(-a^2 + 3^a a - 3^a + 1)$$

the factor $3(1 - 3^a)$ is positive for $a < -1$, while the first three terms of the other factor can be seen to be negative, with the first term being the main reason behind this entire expression being negative since $-a^2 + 1 < 0$ and $3^a a$ and $-3^a$ are also less than zero for $a < -1$.

Therefore there are no roots of the numerator in the interval for which we are considering. We conclude that $c'(0) \neq 0$ for all $a < -1$.

The figure (3) illustrates the behavior of $c'(0)$ as well as its derivative with respect to $a$. □

Now, we need to determine all isotropy subgroups whose fixed point subspaces are one-dimensional. Actually, three subgroups which satisfy this condition, namely

$$\Sigma_1 = \{\mathsf{id}, (12)\},\ \Sigma_2 = \{\mathsf{id}, (13)\},\ \text{and } \Sigma_3 = \{\mathsf{id}, (23)\}.$$

Let us denote the corresponding subspaces of fixed points by

$$\mathsf{Fix}(\Sigma_1) = \left\{\left(y, -\sqrt{3}y\right)\right\},\ \mathsf{Fix}(\Sigma_2) = \{(y, 0)\},\ \text{and } \mathsf{Fix}(\Sigma_3) = \left\{\left(y, \sqrt{3}y\right)\right\}.$$

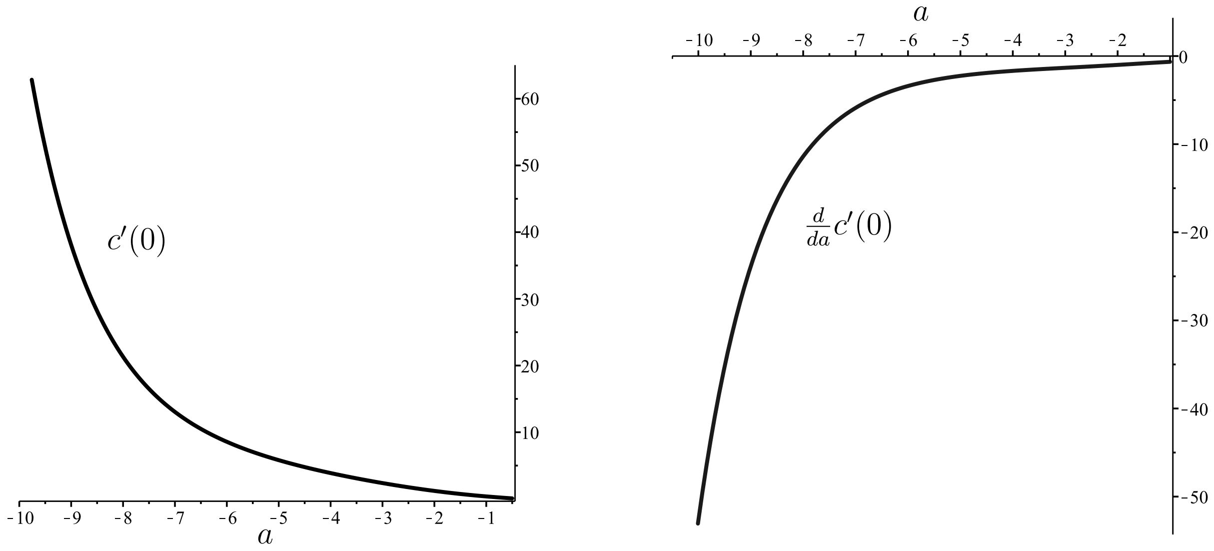


**Figure** 3**:** The figure on the left shows how the function $c'(0)$ behaves when $a$ decreases, while the figure on the right shows how the derivative of $c'(0)$ with respect to $a$ behaves.

Provided all hypotheses of the Equivariant Branching Theorem are satisfied, for each isotropy subgroup $\Sigma_i$, there is only one solution branch of the bifurcation problem (3.11). Thus, we have proved the main result of this chapter, see the theorem below.

**Theorem 3.9.** *Let $q(1,1,1,m)$ be a central configuration of four bodies in the plane forming a centered equilateral triangle. Denote by $m^* > 0$ the value of the central mass, which depends on the exponent $a < -1$, such that the configuration is degenerate. For values of $m$ close to $m^*$, there are only three symmetrical families of central configurations which bifurcate from the degenerate solution $q(1,1,1,m^*)$.*

## 3.3 BIFURCATION BRANCHES

In this section, we present analytical expressions for the solutions obtained in the previous section. Using the analyticity of functions $G_i$, $i = 1,2,3$, we can determine the Taylor series of the implicit solution $W$ of equation (3.9) and express $y_1$ and $y_2$ as analytical functions of $\epsilon$ . Firstly, let us make the translation of the variable $y_3$ to $y_3 + \sqrt{3}k(\epsilon)$ in the function $G_i$, $i = 1,2,3,$ before calculating the terms of the Taylor expansion at $(0,0,0,0)$. Thus,

$$\Psi(y_1, y_2, y_3, \epsilon) = b_3 y_3 + b_\epsilon \epsilon + \frac{1}{2}(b_{11}y_1^2 + b_{22}y_2^2 + b_{33}y_3^2 + b_{\epsilon\epsilon}\epsilon^2) + b_{3\epsilon}y_3\epsilon + O(3).$$

Recall that $G_i$ is the $i$-th coordinate function of $G$ and $\Psi = \sum_{i=1}^{3} G_i$.

Since W is a solution of $\Psi = 0$, let us substitute the generic Taylor series of $W$ into the above expansion of $\Psi$ and compare each term with the null series. We obtain

$$W(y_1, y_2, \epsilon) = -\frac{1}{b_3}\left\{b_\epsilon \epsilon + \frac{b_{11}}{2}y_1^2 + \frac{b_{22}}{2}y_2^2 + \left[b_{\epsilon\epsilon} + b_{3\epsilon}\left(\frac{b_\epsilon}{b_3}\right) - b_{33}\left(\frac{b_\epsilon}{b_3}\right)^2\right]\epsilon^2 + O(3)\right\},$$

where

$$b_3 = -\frac{4k^2}{a}\left(3^{1-a} + \frac{9}{m^*}\right) > 0,$$

$$b_\epsilon = \frac{6\left[3^a(1-a) - 1\right]\left[(3^a-1)^2 + a(1-3^a)\right]}{a(3^a-1)^3\sqrt{3^{a+1}\left(\frac{1+a-3^a}{3^a-1}\right)}} > 0,$$

$$b_{11} = \frac{2k}{3}\left[\frac{\left(3^{a+1}m^{*2} - 3^{3a+3} - 3m^{*2}\right)(a-1) - 3^{2a+2}(am^* + 3)}{a^2 m^*(3+m^*)3^{2a}}\right],$$

$$b_{22} = b_{11},$$

$$b_{33} = \frac{2\sqrt{3}k}{3}\left\{\frac{(3^a-1)^2\left[12m^{*^2}(a-1) - (8m^{*^2} - 12)3^{a+1}a + 3^{2a+1}(a+1) - 4\cdot 3^{2a+2}m^*a\right]}{3^{2a}a^2(3+m^*)m^*}\right\}$$

$$b_{\epsilon\epsilon} = -\frac{\sqrt{3}}{2}\left\{\frac{(3^a-1)^2\left[12m^{*^2}(a-1) - (8m^{*^2} - 12)3^{a+1}a + 3^{2a+1}(a+1) - 4\cdot 3^{2a+2}m^*a\right]}{3^{2a}a^2(3+m^*)^4k^3}\right\}$$

$$b_{3\epsilon} = \frac{2\left[2(3^a-1)a^2 + (3 - 3^a(2+3^a))a + 2(1 - 3^a(2+3^a))\right](1 + 3^a a - 3^a)}{3^a a^2(3^a-1)^3}$$

Substituting the expansion of W into the expansions of the functions $g_i$, $i = 1, 2$, and combining common terms, we get

$$g_1(y_1, y_2, \epsilon) = \frac{1}{2}(\tau_{11}y_1^2 + \tau_{22}y_2^2) + (\tau_{13}w_\epsilon + \tau_{14})y_1\epsilon + O(3), \tag{3.12}$$

$$g_2(y_1, y_2, \epsilon) = \eta_{12}y_1y_2 + (\eta_{23}w_\epsilon + \eta_{24})y_2\epsilon + O(3), \tag{3.13}$$

where

$$\tau_{11} = \frac{\sqrt{2}\zeta\left[3^{2a+1}(a^3 + 5a + 2) - 3^{a+1}(6a^2 + 8a + 4) + 3^{2a+2}(a^2 + a) + (15a^2 + 18a) + 6\right]}{3^a a^2(3^a - a - 1)(1 - 3^a)}$$

$$\tau_{13} = \frac{2\zeta\left[3^{2a+1}(2a^2 - a - a^3) + 3^{a+1}(2a - 3a^2) + 3(a^3 + a^2 - a)\right]}{3^{a+1}a^2(3^a - a - 1)(3^a - 1)},$$

$$\tau_{14} = \frac{(1 + 3^a(a-1))\left[3^{2a+1}(1-a) + 3^{a+1}(a^2 + a - 2) - 3a^2 + 3\right]}{3^{a+1}a^2(3^a-1)^3},$$

$$w_\epsilon = -\frac{b_\epsilon}{b_3},\ \zeta = \sqrt{\frac{3^{a+1}(1 + a - 3^a)}{3^a - 1}},\ \tau_{22} = \eta_{12} = -\tau_{11},\ \eta_{23} = \tau_{13},\ \eta_{24} = \tau_{14}.$$

**Remark 3.10.** The terms of the Taylor series that were displayed are nonzero in a neighborhood of $a = -3/2$. However, no analysis has been performed to determine whether this property holds for arbitrary values of $a$.

Since Fix($\Sigma_1$)= $[(1, -\sqrt{3})]$, by the equivariance of $g$, we have that $g_1 = -\sqrt{3}g_2$. Indeed, substituting $y_1 = \mu$ and $y_2 = -\sqrt{3}\mu$ in expression (3.13) we have

$$\begin{aligned} g_2(\mu, -\sqrt{3}\mu, \epsilon) &= -\sqrt{3}\eta_{12}\mu^2 - \sqrt{3}(\eta_{23}w_\epsilon + \eta_{24})\mu\epsilon + O(2) \\ &= \sqrt{3}\tau_{11}\mu - \sqrt{3}(\tau_{13}w_\epsilon + \tau_{14})\mu\epsilon + O(2) \\ &= -\sqrt{3}g_1(\mu, -\sqrt{3}\mu, \epsilon). \end{aligned}$$

We obtain similar expressions for the other fixed point subspace $\Sigma_2$ and $\Sigma_3$.

The solutions guaranteed by the Equivariant Branching Theorem are written as functions of the variable $(\mu, \epsilon(\mu))$ (for example, setting $y_1 = \mu$). Let

$$\epsilon(\mu) = \delta\mu + O(\mu^2),$$

where $\delta$ is nonzero. By restricting $g = (g_1, g_2)$ to Fix($\Sigma_1$)= $[(1, -\sqrt{3})]$, we restricted ourselves to solving the equation $g_1 = 0$, where

$$g_1(\mu, -\sqrt{3}\mu, \epsilon(\mu)) = [-\tau_{11} + \delta(\tau_{13}\omega_\epsilon + \tau_{14})]\mu^2 + O(\epsilon^3).$$

Since $g_1(\mu, -\sqrt{3}\mu, \epsilon(\mu)) = 0$, the quadratic term gives us

$$\delta = \frac{\tau_{11}}{\tau_{13}\omega_\epsilon + \tau_{14}}. \tag{3.14}$$

Therefore, undoing the translation made in the variable $y_3$, the solutions of (3.8) are

$$\text{Families (I),(II):} \begin{cases} y_1 &= \mu, \\ y_2 &= \pm\sqrt{3}\mu, \\ y_3 &= \sqrt{3}k(\mu) + w_\epsilon\delta\mu + O(\mu^2), \\ \epsilon &= \delta\mu + O(\mu^2). \end{cases}$$

$$\text{Family (III):} \begin{cases} y_1 &= \mu, \\ y_2 &= 0, \\ y_3 &= \sqrt{3}k(\mu) + w_\epsilon\bar{\delta}\mu + O(\mu^2), \\ \epsilon &= \bar{\delta}\mu + O(\mu^2), \quad \bar{\delta} = -\dfrac{1}{2\delta}. \end{cases}$$

Returning to the variables $x_i$, we obtain

$$\text{Family (I):}\begin{cases} x_1 &= k(\delta\mu) + \left[\dfrac{\sqrt{6}}{3} + \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_2 &= k(\delta\mu) + \left[\dfrac{\sqrt{6}}{3} + \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_3 &= k(\delta\mu) - \left[\dfrac{2\sqrt{6}}{3} - \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_4 &= -3k(\delta\mu) - \sqrt{3}w_\epsilon\delta\mu + O(\mu^2). \end{cases}$$

$$\text{Family (II):}\begin{cases} x_1 &= k(\delta\mu) - \left[\dfrac{2\sqrt{6}}{3} - \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_2 &= k(\delta\mu) + \left[\dfrac{\sqrt{6}}{3} + \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_3 &= k(\delta\mu) + \left[\dfrac{\sqrt{6}}{3} + \dfrac{\sqrt{3}}{3}w_\epsilon\delta\right]\mu + O(\mu^2), \\ x_4 &= -3k(\delta\mu) - \sqrt{3}w_\epsilon\delta\mu + O(\mu^2). \end{cases}$$

$$\text{Family (III):}\begin{cases} x_1 &= k(\bar{\delta}\mu) - \left[\dfrac{\sqrt{6}}{6} - \dfrac{\sqrt{3}}{3}w_\epsilon\bar{\delta}\right]\mu + O(\mu^2), \\ x_2 &= k(\bar{\delta}\mu) + \left[\dfrac{\sqrt{6}}{3} + \dfrac{\sqrt{3}}{3}w_\epsilon\bar{\delta}\right]\mu + O(\mu^2), \\ x_3 &= k(\bar{\delta}\mu) - \left[\dfrac{\sqrt{6}}{6} - \dfrac{\sqrt{3}}{3}w_\epsilon\bar{\delta}\right]\mu + O(\mu^2), \\ x_4 &= -3k(\bar{\delta}\mu) - \sqrt{3}w_\epsilon\bar{\delta}\mu + O(\mu^2). \end{cases}$$

The behavior of the solutions can be analyzed by looking at the growth of the mutual distances. Let

$$s_{ij}(\mu) = s_{ij}(0) + v_{ij}\mu + O(\mu^2),$$

where $v_{ij} = \left.\dfrac{ds_{ij}(\mu)}{d\mu}\right|_{\mu=0}$. Recalling that $s_{ij} = \left(\dfrac{\lambda}{M} - \dfrac{x_i x_j}{m_i m_j}\right)^{1/a}$ and assuming $a = -3/2$, we calculate the values of $v_{ij}$ for the family I. Namely,

$$\begin{cases} v_{12} &= \left(\dfrac{216}{1909} - \dfrac{40130}{17181}\sqrt{3}\right)\sqrt{\dfrac{2(27+29\sqrt{3})}{39}} \approx -7.8, \\ v_{13} &= v_{23} = \left(\dfrac{216}{1909} - \dfrac{91673}{17181}\sqrt{3}\right)\sqrt{\dfrac{2(27+29\sqrt{3})}{39}} \approx -18.7, \\ v_{14} &= v_{24} = \left(\dfrac{2313}{1909} - \dfrac{122300}{51543}\sqrt{3}\right)\sqrt{\dfrac{2(27+29\sqrt{3})}{39}} \approx -5.8, \\ v_{34} &= \left(\dfrac{21124}{51543}\sqrt{3} - \dfrac{4410}{1909}\right)\sqrt{\dfrac{2(27+29\sqrt{3})}{39}} \approx -3.2. \end{cases}$$

Therefore, this analysis shows that the central configuration that bifurcates from the centered equilateral triangle is an isosceles triangle. Due to the relationship between $\mu$ and $\epsilon$, considering $\delta = -\left(\frac{3632612}{475341} + \frac{212892}{158447}\sqrt{3}\right)\sqrt{\frac{27+29\sqrt{3}}{78}} < 0$, when $\mu < 0$ we have $\epsilon > 0$. Then, the mass $m^*$ moves along the axis of symmetry and approaches the mass $m_3$. On the other hand, when $\mu > 0$ we have $\epsilon < 0$ and the opposite motion of $m^*$ happens.

Similarly, we analyze the other two families. Due to the symmetry, we have an isosceles triangle with the mass $m^*$ moving along the axis of symmetry passing through the mass $m_1$ and $m_2$.

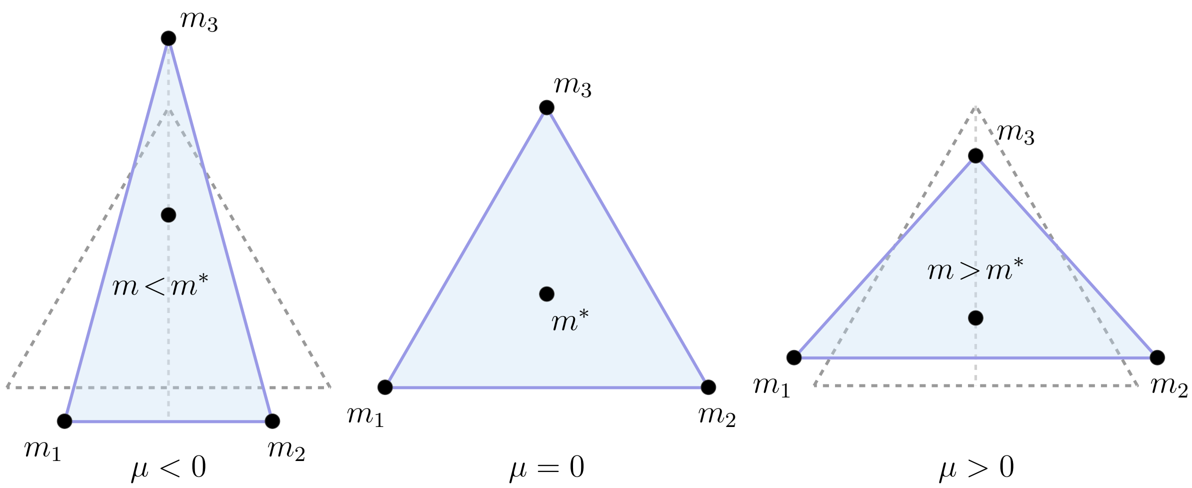


**Figure** 5. Bifurcation of the centered equilateral triangle representing family I. The central mass $m^*$ moves along the axis of symmetry. When $\epsilon > 0$, $m^*$ approaches $m_3$. Conversely, when $\epsilon < 0$, $m^*$ moves away from $m_3$.

# 4 BIFURCATIONS OF THE CENTERED REGULAR TETRAHEDRAL CONFIGURATION

## 4.1 INTRODUCTION

Consider a configuration $q = (q_1, q_2, q_3, q_4, q_5)$ of massive points in $\mathbb{R}^3$ forming a centered regular tetrahedron with masses $m_1 = m_2 = m_3 = m_5 = 1$ at the vertices and $m_4 = m$ at the barycenter. For the sake of comparison, we denotation and ordering of masses are the same as in (SANTOS et al., 2017). The squares of the mutual distances $s_{ij} = \| q_i - q_j \|^2$ between the bodies are $s_{12} = s_{13} = s_{15} = s_{23} = s_{25} = s_{35} = \nu$ and $s_{14} = s_{24} = s_{34} = s_{45} = 1$, where we set $\nu = \frac{8}{3}$ as the value of the edge of the tetrahedron. As we saw in chapter 2, this configuration of five bodies in space is a Dziobek configuration and must satisfy the system of equations

$$\begin{aligned} \sum_{i=1}^{5} x_i &= 0, \\ \sum_{j\neq i} s_{ij}x_j &= \sum_{i\neq j} s_{ij}x_i, \\ s_{ij}^a - \frac{\lambda}{M} &= -\frac{x_i x_j}{m_i m_j}, \end{aligned} \tag{4.1}$$

for all $1 \leq i < j \leq 5$, where $\lambda$ must be equal to $m + 4\nu^a$. Indeed, if $m_i = m_j$ and $s_{ik} = s_{jk}$, for some $k$, then $x_i = x_j$. Substituting these in system (4.1), the last equations are reduced to just two:

$$\begin{cases} \dfrac{\lambda}{4+m} - \nu^a &= x_1^2, \\ \dfrac{\lambda}{4+m} - 1 &= -\dfrac{4x_1^2}{m}. \end{cases}$$

Multiplying the first equation by $4$, the second by $m$, and adding them together, we obtain the expression for $\lambda$.

As on chapter $3$, we assume $M > 0$ and $a < -1$.

Note that the first equation in (4.1) gives us $x_4 = -\left(\sum_{i=1}^{3} x_i + x_5\right)$, and the variables

$$x_1^0 = x_2^0 = x_3^0 = x_5^0 = k \text{ and } x_4^0 = -4k,$$

where $k = \pm\sqrt{\dfrac{m(1-\nu^a)}{4+m}}$ determine the centered tetrahedron. We call it the *trivial solution* and denote it by $X^0$. We choose the positive sign of $k$ for convenience. If we put $t_i = \sum_{j\neq i} s_{ij}x_j$, with $s_{ij} = \left(\dfrac{\lambda}{M} - \dfrac{x_i x_j}{m_i m_j}\right)^{1/a}$, from the third equation in (4.1) we have a system of four equations given by $t_i - t_4 = 0,\ i = 1,2,3,5$. We denote the new system by

$$F(X, m) = 0, \tag{4.2}$$

where $X = (x_1, x_2, x_3, x_5)$ and $F = (F_1, F_2, F_3, F_5) : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4$ is a smooth function such that $F(X^0(m), m) = 0$, for any given value of $m$.

The symmetry group of (4.2) is $S_4$, the permutation group of the four symbols $1, 2, 3, 5$, i.e., the function $F$ commutes with the action of $S_4$ on $\mathbb{R}^4$ defined by the representation $\rho$ given by $\rho_\sigma(x_1, x_2, x_3, x_5) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(5)}), \forall\, \sigma \in S_4$.

**Proposition 4.1.** The mapping $F$ is $S_4-$equivariant, i.e.,

$$F(\rho_\sigma \cdot X, m) = \rho_\sigma \cdot F(X, m), \quad \forall\ \sigma \in S_4.$$

*Proof.* By defining the function $\psi\left(\mathcal{X}\right) = \left(\dfrac{\lambda}{4+m} - \mathcal{X}\right)^{1/a}$, we have that

$$F_k(X, m) = \sum_{\substack{i=1 \\ i \neq k,4}}^{5} x_i \left(\psi\left(x_i x_k\right) - \psi\left(\frac{x_i x_4}{m}\right)\right) + (x_4 - x_k)\psi\left(\frac{x_4 x_k}{m}\right).$$

Hence,

$$\begin{aligned} F_k(\rho_\sigma(X), m) &= \sum_{\substack{i=1 \\ i \neq k,4}}^{5} x_{\sigma(i)} \left(\psi\left(x_{\sigma(i)} x_{\sigma(k)}\right) - \psi\left(\frac{x_{\sigma(i)} x_4}{m}\right)\right) + (x_4 - x_{\sigma(k)})\psi\left(\frac{x_4 x_{\sigma(k)}}{m}\right), \\ &= \sum_{\substack{j=1 \\ j \neq \sigma(k),4}}^{5} x_j \left(\psi\left(x_j x_{\sigma(k)}\right) - \psi\left(\frac{x_j x_4}{m}\right)\right) + (x_4 - x_{\sigma(k)})\psi\left(\frac{x_4 x_{\sigma(k)}}{m}\right), \\ &= F_{\sigma(k)}(X, m), \quad k = 1,2,3,5 \text{ and } \sigma \in S_4, \end{aligned}$$

where the second equality follows from $j = \sigma(i)$. □

Consequently, the derivative of $F$ at the point $(X^0(m), m)$ is a linear transformation which commutes with $\rho_\sigma$ for all $\sigma \in S_4$, since $X^0 \in$ Fix$(S_4)$. We have that

$$D_X F(X^0, m) = \begin{bmatrix} b & c & c & c \\ c & b & c & c \\ c & c & b & c \\ c & c & c & b \end{bmatrix}, \tag{4.3}$$

where

$$b = \frac{\partial F_1}{\partial x_1}\Big|_{(X^0,m)} = -2 - \frac{k^2}{a}\left(3\nu^{1-a} + \frac{28}{m}\right), \quad \text{and}$$
$$c = \frac{\partial F_1}{\partial x_2}\Big|_{(X^0,m)} = (\nu - 2) - \frac{k^2}{a}\left(\nu^{1-a} + \frac{12}{m}\right).$$

So, the determinant of $D_X F(X^0, m)$ is

$$|D_X F(X^0, m)| = (b + 3c)(b - c)^3.$$

The term

$$b + 3c = -\frac{m(1-\nu^a)}{a(4+m)}\left(\frac{64}{m} + 6\nu^{1-a}\right)$$

is positive for $a < -1$ and $m > 0$, whereas

$$b - c = -\nu - \frac{m(1-\nu^a)}{a(4+m)}\left(\frac{16}{m} + 2\nu^{1-a}\right),$$

is zero if and only if $m$ is equal to

$$m^* = 2\left(\frac{3\nu^a - 2a - 3}{2\nu^{-a} + a - 2}\right).$$

When $a = -\frac{3}{2}$, the Newtonian case, we have the value $m^* = \dfrac{10368 + 1701\sqrt{6}}{54952}$ found by Schmidt in (SCHMIDT, 1988).

**Proposition 4.2.** The value of $m^*$ is positive for all $a < -1$.

*Proof.* See (SANTOS et al., 2017). □

Equation (4.2) describes a bifurcation problem where the central mass $m$ plays the role of the bifurcation parameter. For each exponent $a$ the pair $(X^0, m^*)$ represents the degenerate centered tetrahedral configuration, where the degenerate mass vector is the quintuple $(m_1, m_2, m_3, m_4, m_5) = (1, 1, 1, m^*, 1)$. Therefore, $(X^0, m^*)$ is a candidate for a bifurcation point. In the most general sense, we have a bifurcation problem where the five masses are the bifurcation parameters. However, bifurcation problems with more than one parameter can be technically very challenging and we shall restrict ourselves to studying particular cases. The fundamental idea is to study what happens with the number of solutions of equation (4.2) in a neighborhood of the point $(1, 1, 1, m^*, 1)$.

For the first problem studied in this chapter, we will consider the mass vector $(1+\epsilon_1, 1+\epsilon_2, 1+\epsilon_3, m^*+\epsilon_4, 1+\epsilon_5)$ with $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$ and $\epsilon_4 = \epsilon_5 = 0$. While in the second problem, we will consider $\epsilon_1 = \epsilon_2 = \epsilon$ and $\epsilon_3 = \epsilon_4 = \epsilon_5 = 0$. It is worth emphasizing that the problems which we propose to study are not equivalent to the problem addressed in (SANTOS et al., 2017), where the authors considered $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$ and $\epsilon_5 = \epsilon$.

## 4.2 BIFURCATION PROBLEM WITH THREE EQUALLY VARYING MASSES

Let us consider the bifurcation problem for the centered tetrahedron with three vertex masses as bifurcation parameters. This problem consists in studying the solutions of the system $(4.1)$ in a neighborhood of the degenerate configuration, i.e., we keep fixed the masses $m_5 = 1$ and $m_4 = m^*$ and we approach the remaining three unit masses in the same way, $m_1 = m_2 = m_3 = 1 + \epsilon$. We substitute these mass values in $(4.1)$ to obtain the equation

$$F(X, \epsilon) = 0, \tag{4.4}$$

where $X = (x_1, x_2, x_3, x_5)$, $\epsilon$ is the bifurcation parameter and $F = (F_1, F_2, F_3, F_4) : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4$ is a smooth function such that

$$F_1(X,\epsilon) = \sum_{\substack{i=2 \\ i\neq 4}}^{5} x_i \left( \frac{\lambda}{M} - \frac{x_1 x_i}{(1+\epsilon)m_i} \right)^{1/a} - \left( \sum_{i=1}^{3} x_i + x_5 \right) \left( \frac{\lambda}{M} + \frac{x_1(\sum_{i=1}^{3} x_i + x_5)}{(1+\epsilon)m^*} \right)^{1/a} - t_5$$

$$F_2(X,\epsilon) = \sum_{\substack{i=1 \\ i\neq 2,4}}^{5} x_i \left( \frac{\lambda}{M} - \frac{x_2 x_i}{(1+\epsilon)m_i} \right)^{1/a} - \left( \sum_{i=1}^{3} x_i + x_5 \right) \left( \frac{\lambda}{M} + \frac{x_2(\sum_{i=1}^{3} x_i + x_5)}{(1+\epsilon)m^*} \right)^{1/a} - t_5$$

$$F_3(X,\epsilon) = \sum_{\substack{i=1 \\ i\neq 3,4}}^{5} x_i \left( \frac{\lambda}{M} - \frac{x_3 x_i}{(1+\epsilon)m_i} \right)^{1/a} - \left( \sum_{i=1}^{3} x_i + x_5 \right) \left( \frac{\lambda}{M} + \frac{x_3(\sum_{i=1}^{3} x_i + x_5)}{(1+\epsilon)m^*} \right)^{1/a} - t_5$$

$$F_4(X,\epsilon) = \sum_{\substack{i=1 \\ i\neq 4}}^{5} x_i \Big( \frac{\lambda}{M} + \frac{(\sum_{i=1}^{3} x_i + x_5) x_i}{m_i m^*} \Big)^{1/a} - t_5,$$

and

$$t_5 = \sum_{i=1}^{3} x_i \left( \frac{\lambda}{M} - \frac{x_i x_5}{(1+\epsilon)} \right)^{1/a} - \left( \sum_{i=1}^{3} x_i + x_5 \right) \left( \frac{\lambda}{M} + \frac{x_5(\sum_{i=1}^{3} x_i + x_5)}{m^*} \right)^{1/a},$$

for all $a < -1$ fixed. We already know that when $\epsilon = 0$ the degenerate configuration is represented by the point $(X^0, 0) = ((k(m^*), k(m^*), k(m^*), k(m^*)), 0)$; for this reason $F(X^0, 0) = 0$. Furthermore, $F$ is $S_3$-equivariant, that is, the symmetry group of (4.4) is $\Sigma = \{\sigma \in S_4 \,|\, \sigma(4) = 4\} \cong S_3$. For $\sigma \in \Sigma$, we set $\rho_\sigma(x_1, x_2, x_3, x_5) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_5)$.

Hence,

$$\begin{aligned}F_k(\rho_\sigma\cdot X,\epsilon)&=\sum_{\substack{i=1\\ i\neq 4,k}}^{5}x_{\sigma(i)}\left(\frac{\lambda}{M}-\frac{x_{\sigma(k)}x_{\sigma(i)}}{m_{\sigma(k)}m_{\sigma(i)}}\right)^{1/a}\\&\quad-\left(\sum_{i=1}^{3}x_{\sigma(i)}+x_5\right)\left(\frac{\lambda}{M}+\frac{x_{\sigma(k)}(\sum_{i=1}^{3}x_{\sigma(i)}+x_5)}{m_{\sigma(k)}m}\right)^{1/a}\\&=\sum_{\substack{i=1\\ \sigma(i)\neq 4,k}}^{5}x_i\left(\frac{\lambda}{M}-\frac{x_{\sigma(k)}x_i}{m_{\sigma(k)}m_i}\right)^{1/a}\\&\quad-\left(\sum_{i=1}^{3}x_i+x_5\right)\left(\frac{\lambda}{M}+\frac{x_{\sigma(k)}(\sum_{i=1}^{3}x_i+x_5)}{m_{\sigma(k)}m}\right)^{1/a}\\&=F_{\sigma(k)}(X,\epsilon),\quad k=1,2,3,\end{aligned}$$

and it is immediate to check that $t_5$ and $F_4$ have $\rho_\sigma-$invariant expressions.

**Remark 4.3.** The matrices of the representation $\rho$ with respect to the canonical basis of $\mathbb{R}^4$ are as follows

$$\rho_{(12)}:\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad\rho_{(13)}:\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\quad\rho_{(23)}:\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix},$$

$$\rho_{(123)}:\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\quad\rho_{(132)}:\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}.$$

Consequently, by the chain rule, since $\rho_\sigma(X^0)=X^0$, the derivative of $F$ at the point $(X^0,0)$ is also $S_3-$equivariant. We have that

$$D_XF(X^0,0)=\begin{bmatrix}b&c&c&d\\c&b&c&d\\c&c&b&d\\f&f&f&e\end{bmatrix},$$

where

$$\begin{aligned}
b &= \frac{\partial F_1}{\partial x_1}\Big|_{(X^0,0)} = -\nu - \frac{2k^2}{a}\Big(\nu^{1-a} + \frac{8}{m}\Big) = -\frac{\partial F_1}{\partial x_5}\Big|_{(X^0,0)} = d,\\
c &= \frac{\partial F_1}{\partial x_2}\Big|_{(X^0,0)} = 0,\\
f &= \frac{\partial F_4}{\partial x_1}\Big|_{(X^0,0)} = 2 - \nu + \frac{k^2}{a}\Big(\nu^{1-a} + \frac{12}{m}\Big),\\
e &= \frac{\partial F_4}{\partial x_5}\Big|_{(X^0,0)} = 2 + \frac{k^2}{a}\Big(3\nu^{1-a} + \frac{28}{m}\Big) = f - b.
\end{aligned}$$

Hence,

$$D_X F(X^0, 0) = \begin{bmatrix} b & 0 & 0 & -b \\ 0 & b & 0 & -b \\ 0 & 0 & b & -b \\ f & f & f & f-b \end{bmatrix}.$$

So, the determinant

$$|D_X \Phi(X^0, 0)| = b^3(4e - b).$$

The term $4e - b = \frac{2k^2}{a}\left(3\nu^{1-a} + \frac{32}{m}\right)$ is negative for all $a < -1$ and $m$ positive. On the other hand, $b = 0$ if only if $m = m^*$ (as defined earlier). Therefore, since $m = m^*$, we have

$$D_X F(X^0, 0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f & f & f & f \end{bmatrix},$$

where $f = -\frac{4(1-\nu^a)(1-2a)}{a(4-3\nu^a)} \neq 0$ for $a < -1$, when we substitute $m^*$ and $k(m^*)$ in its previously defined expression.

Let $L = D_X F(X^0, 0)$ and apply the Liapunov-Schmidt reduction process. From the structure of $L$, it is immediate to obtain its kernel and image

$$\ker(L) = \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \,\Big|\, \sum_{i=1}^{4} v_i = 0\right\} \text{ and } \mathrm{Im}(L) = \{(0,0,0,v) \mid v \in \mathbb{R}\},$$

and to check that both are $\rho$-invariant subspaces. We make the decomposition $\mathbb{R}^4 = \ker(L) \oplus \mathrm{Im}(L)$, choose convenient bases for the kernel and the image, respectively

$$\begin{aligned}
\beta_1 &= \{u_1 = (1,0,0,-1), u_2 = (0,1,0,-1), u_3 = (0,0,1,-1)\},\\
\beta_2 &= \{u_4 = (0,0,0,-1)\},
\end{aligned}$$

and perform the change of variables

$$(x_1, x_2, x_3, x_5) = \sum_{i=1}^{4} y_i u_i.$$

Explicitly,

$$x_1 = y_1, \;\; x_2 = y_2, \;\; x_3 = y_3, \;\; x_5 = -\sum_{i=1}^{4} y_i,$$

since $\beta = \beta_1 \cup \beta_2$ forms a basis for $\mathbb{R}^4$. We keep in mind that $x_4 = -\left(\sum\limits_{i=1}^{3} x_i + x_5\right)$.

**Remark 4.4.** The matrices of the representation of $S_3$ with respect to the basis $\beta$ and the canonical basis are the same.

By changing variables, we obtain a new function and the problem is described by the equation

$$G(Y, \epsilon) = 0, \tag{4.5}$$

where

$$\begin{aligned}
G_1(Y,\epsilon) = & \, y_2 \left[\frac{\lambda}{M} - \frac{y_1 y_2}{(1+\epsilon)^2}\right]^{1/a} + y_3 \left[\frac{\lambda}{M} - \frac{y_1 y_3}{(1+\epsilon)^2}\right]^{1/a} + y_4 \left[\frac{\lambda}{M} - \frac{y_1 y_4}{(1+\epsilon) m^*}\right]^{1/a} \\
& - y_1 \left[\frac{\lambda}{M} + \frac{y_1 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_2 \left[\frac{\lambda}{M} + \frac{y_2 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} \\
& - y_3 \left[\frac{\lambda}{M} + \frac{y_3 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_4 \left[\frac{\lambda}{M} + \frac{y_4 (\sum_{i=1}^4 y_i)}{m^*}\right]^{1/a} \\
& - \left(\sum_{i=1}^{4} y_i\right) \left[\frac{\lambda}{M} + \frac{y_1 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a}, \\
G_2(Y,\epsilon) = & \, y_1 \left[\frac{\lambda}{M} - \frac{y_1 y_2}{(1+\epsilon)^2}\right]^{1/a} + y_3 \left[\frac{\lambda}{M} - \frac{y_2 y_3}{(1+\epsilon)^2}\right]^{1/a} + y_4 \left[\frac{\lambda}{M} - \frac{y_2 y_4}{(1+\epsilon) m^*}\right]^{1/a} \\
& - y_1 \left[\frac{\lambda}{M} + \frac{y_1 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_2 \left[\frac{\lambda}{M} + \frac{y_2 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} \\
& - y_3 \left[\frac{\lambda}{M} + \frac{y_3 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_4 \left[\frac{\lambda}{M} + \frac{y_4 (\sum_{i=1}^4 y_i)}{m^*}\right]^{1/a} \\
& - \left(\sum_{i=1}^{4} y_i\right) \left[\frac{\lambda}{M} + \frac{y_2 (\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a},
\end{aligned}$$

$$
\begin{aligned}
G_3(Y,\epsilon) = & y_1\left[\frac{\lambda}{M}-\frac{y_1y_3}{(1+\epsilon)^2}\right]^{1/a} + y_2\left[\frac{\lambda}{M}-\frac{y_2y_3}{(1+\epsilon)^2}\right]^{1/a} + y_4\left[\frac{\lambda}{M}-\frac{y_3y_4}{(1+\epsilon)m^*}\right]^{1/a}\\
& - y_1\left[\frac{\lambda}{M}+\frac{y_1(\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_2\left[\frac{\lambda}{M}+\frac{y_2(\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a}\\
& - y_3\left[\frac{\lambda}{M}+\frac{y_3(\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a} - y_4\left[\frac{\lambda}{M}+\frac{y_4(\sum_{i=1}^4 y_i)}{m^*}\right]^{1/a}\\
& - \left(\sum_{i=1}^4 y_i\right)\left[\frac{\lambda}{M}+\frac{y_3(\sum_{i=1}^4 y_i)}{(1+\epsilon)}\right]^{1/a},\\
G_4(Y,\epsilon) = & y_1\left[\frac{\lambda}{M}-\frac{y_1y_4}{(1+\epsilon)m^*}\right]^{1/a} + y_2\left[\frac{\lambda}{M}-\frac{y_2y_4}{(1+\epsilon)m^*}\right]^{1/a} + y_3\left[\frac{\lambda}{M}-\frac{y_3y_4}{(1+\epsilon)m^*}\right]^{1/a}\\
& - y_1\left[\frac{\lambda}{M}+\frac{y_1\left(\sum_{i=1}^4 y_i\right)}{(1+\epsilon)}\right]^{1/a} - y_2\left[\frac{\lambda}{M}+\frac{y_2\left(\sum_{i=1}^4 y_i\right)}{(1+\epsilon)}\right]^{1/a}\\
& - y_3\left[\frac{\lambda}{M}+\frac{y_3\left(\sum_{i=1}^4 y_i\right)}{(1+\epsilon)}\right]^{1/a} - y_4\left[\frac{\lambda}{M}+\frac{y_4(\sum_{i=1}^4 y_i)}{m^*}\right]^{1/a}\\
& - \left(\sum_{i=1}^4 y_i\right)\left[\frac{\lambda}{M}+\frac{y_4(\sum_{i=1}^4 y_i)}{m^*}\right]^{1/a},
\end{aligned}
$$

are the coordinates of $G(Y,\epsilon)$ with respect to canonical basis and $\lambda = m^* + 4\nu^a$, $M = 4 + m^* + 3\epsilon$ and $k = \sqrt{\frac{m^*}{4+m^*}(1-\nu^a)}$ are fixed for each $a < -1$.

**Remark 4.5.** $G(Y,\epsilon)$ is $S_3-$equivariant.

Let us write the degenerate solution in the new coordinates by $(Y^0, 0) = (k, k, k, -4k, 0)$ and note that $D_Y G(X^0, 0) = D_X F(X^0, 0) D_Y X(Y^0) D_Y Y(Y^0)$. Thus

$$
D_Y G(X^0,0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f & f & f & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \end{bmatrix}.
$$

In order to solve equation $(4.5)$, we make use of the equivariance of G and the implicit function theorem with the same technique as in (SANTOS et al., 2017).

According to the decomposition of $\mathbb{R}^4 = \ker(L) \oplus \mathrm{Im}(L)$, equation (4.5) is equivalent to the system

$$
(P \circ G)(Y,\epsilon) = \langle u_4, G_4(Y,\epsilon)\rangle u_4 = 0, \tag{4.6}
$$

$$
((I-P) \circ G)(Y,\epsilon) = \sum_{i=1}^{3} \langle u_i, G(Y,\epsilon)\rangle u_i = 0, \tag{4.7}
$$

where $P : \mathbb{R}^4 \longrightarrow \mathrm{Im}(L)$ is the canonical projection on $\mathrm{Im}(L)$ with $\ker(P) = \ker(L)$, and $I - P$ is the complementary projection.

We verify that equation (4.6) may be solved for $y_4$. Firstly, observe that any vector $v \in \mathbb{R}^4$ may be decomposed in the form $v = v_1 + v_2$, where $v_1 \in \ker(L)$ and $v_2 \in \mathrm{Im}(L)$. Therefore, it is sufficient to solve the equation $G_4 = 0$, since $G_4$ is the coordinate of $(P \circ G)(Y, \epsilon)$ with respect to $\beta_2$.

We check that $G_4(Y^0, 0) = 0$ and $\dfrac{\partial G_4}{\partial y_i}(Y^0, 0) = 0$, $i = 1, 2, 3$, but,

$$\frac{\partial G_4}{\partial y_4}(Y^0, 0) = \frac{4(1-\nu^a)(1-2a)}{a(4-3\nu^a)} < 0,$$

for all $a < -1$. Hence, by the implicit function theorem, there exists an open set $U \times V \subset \mathbb{R}^3 \times \mathbb{R}$ containing $(\widetilde{Y}^0, 0)$, where $\widetilde{Y}^0 = (k(m^*), k(m^*), k(m^*))$, such that there is a unique analytic function $y_4 = W : U \times V \longrightarrow \mathbb{R}$ with $W(\widetilde{Y}^0, 0) = -4k(m^*)$, and for all $(y_1, y_2, y_3) \in U$ and $\epsilon \in V$ we have $G_4(y_1, y_2, y_3, W(y_1, y_2, y_3, \epsilon), \epsilon) = 0$. Analogously to lemma 2.24, we can show that the function $W$ is $S_3-$invariant.

Now, substituting $W(y_1, y_2, y_3, \epsilon)$ into equations (4.7) yields an $S_3$-equivariant system of three equations, three variables and one parameter, namely

$$\widetilde{G}_i(y_1, y_2, y_3, \epsilon) = 0, \quad i = 1, 2, 3. \tag{4.8}$$

**Lemma 4.6.** *The system (4.8) does not admit an implicit differentiable solution defined around $\epsilon = 0$.*

*Proof.* Indeed, if we had a solution $\widetilde{Y}(\epsilon) = (y_1(\epsilon), y_2(\epsilon), y_3(\epsilon))$ of (4.8), then

$$D_\epsilon \widetilde{G}\left(\widetilde{Y}^0, 0\right) = D_{\widetilde{Y}} \widetilde{G}\left(\widetilde{Y}^0, 0\right) \frac{d\widetilde{Y}}{d\epsilon}(0) + \frac{\partial \widetilde{G}}{\partial \epsilon}\left(\widetilde{Y}^0, 0\right) = 0. \tag{4.9}$$

But, $D_{\widetilde{Y}} \widetilde{G}\left(\widetilde{Y}^0, 0\right) = 0$, whereas

$$\frac{\partial G_i}{\partial \epsilon}(\widetilde{Y}^0, 0) = \frac{2k^3}{a}\left(\nu^{1-a} + \frac{8}{m^*}\right) < 0, \ i = 1, 2, 3,$$

for $a < -1$. So, the claim is proved. □

Let us consider the parameter $\epsilon$ as an additional variable of the problem, and solve one of the equations $\widetilde{G}_i = 0$, $i = 1, 2, 3$, for $\epsilon$ in terms of $(y_1, y_2, y_3)$. More precisely, we consider $i = 3$ and since we already know that $G_3(\widetilde{Y}^0, 0) = 0$ and $\dfrac{\partial G_3}{\partial \epsilon} \neq 0$, by the implicit function theorem, there exist neighborhoods $V_1(\widetilde{Y}^0) \subset \mathbb{R}^3$ and $V_2(0) \subset \mathbb{R}$ such that, for all $(y_1, y_2, y_3) \in V_1(\widetilde{Y^0}) \cap U$ there is $\epsilon(y_1, y_2, y_3) \in V_2(0)$, such that

$$\widetilde{G}_3(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = 0,$$

with $\epsilon(\widetilde{Y}^0) = 0$.

Due to the $S_3$-equivariance of system (4.8), the remaining two equations can also be solved if we restrict ourselves to subspaces of fixed points. In fact, since

$$\rho_{(123)} : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \rho_{(132)} : \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

the fixed points $(y_1, y_2, y_3)$ by $\rho_{(123)}$ and $\rho_{(132)}$ are $\widetilde{Y} = (\delta, \delta, \delta)$ (setting $y_1 = y_2 = y_3 = \delta$). Therefore,

$$\widetilde{G}_1(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = \widetilde{G}_2(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = \widetilde{G}_3(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)),$$

for all $(y_1, y_2, y_3) \in V_1(\widetilde{Y}^0) \cap U \cap \text{Fix}(\Sigma_{(123)})$ or $(y_1, y_2, y_3) \in V_1(\widetilde{Y}^0) \cap U \cap \text{Fix}(\Sigma_{(132)})$. Thus, we have that $y_1 = y_2 = y_3$ is a solution of system (4.8). On the other hand, the points $(y_1, y_2, y_3)$ fixed by

$$\rho_{(12)} : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \rho_{(13)} : \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \rho_{(23)} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

are $\widetilde{Y} = (\delta, \delta, y_3)$ (setting $y_1 = y_2 = \delta$), $\widetilde{Y} = (\delta, y_2, \delta)$ (setting $y_1 = y_3 = \delta$) and $\widetilde{Y} = (y_1, \delta, \delta)$ (setting $y_2 = y_3 = \delta$). Therefore,

$$\widetilde{G}_1(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = \widetilde{G}_3(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)),$$
$$\widetilde{G}_2(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = \widetilde{G}_3(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)),$$

for all $(y_1, y_2, y_3) \in V_1(\widetilde{Y}^0) \cap U \cap \text{Fix}(\Sigma_{(13)})$ and $(y_1, y_2, y_3) \in V_1(\widetilde{Y}^0) \cap U \cap \text{Fix}(\Sigma_{(23)})$. Thus, we have that $y_1 = y_3$ and $y_2 = y_3$ are solutions of system (4.8). To show that $y_1 = y_2$ is also a solution of system (4.8), we just have to solve the equation $\widetilde{G}_1(y_1, y_1, y_3, \epsilon(y_1, y_1, y_3)) = 0$.

Without imposing any restrictions and setting

$$H_1(y_1, y_2, y_3) := \widetilde{G}_1(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)), \tag{4.10}$$

$$H_2(y_1, y_2, y_3) := \widetilde{G}_2(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)), \tag{4.11}$$

we can solve the system $H_1 = 0$ and $H_2 = 0$ completely.

The following theorem, found in (SANTOS et al., 2017), allows us to rewrite the functions $H_i$ in an interesting way.

**Theorem 4.7.** *If $H(x, y, z)$ is an analytic function defined in a neighborhood of the origin $(0, 0, 0)$, such that $H(0, 0, 0) = 0$ and $H(x, x, z) = 0$, then*

$$H(x, y, z) = (x - y)h(x, y, z),$$

*where $h(x, y, z)$ is an analytic function in a neighborhood of $(0, 0, 0)$.*

*Proof.* We first define an auxiliary function $\hat{H}(\xi, \eta, z) = H(\xi+\eta, \xi-\eta, z)$ for all $(\xi, \eta, z)$ near the origin $(0, 0, 0)$. The function $\hat{H}$ is an analytic function because $H$ is. Its Taylor expansion at $(0, 0, 0)$ has the form

$$\begin{aligned}\hat{H}(\xi, \eta, z) = {} & \frac{\partial \hat{H}}{\partial \xi}(0)\xi + \frac{\partial \hat{H}}{\partial \eta}(0)\eta + \frac{\partial \hat{H}}{\partial z}(0)z + \frac{\partial^2 \hat{H}}{\partial \xi \partial \eta}(0)\xi\eta + \frac{\partial^2 \hat{H}}{\partial \xi \partial z}(0)\xi z + \frac{\partial^2 \hat{H}}{\partial \eta \partial z}(0)\eta z \\ & + \frac{1}{2}\left(\frac{\partial^2 \hat{H}}{\partial \xi^2}(0)\xi^2 + \frac{\partial^2 \hat{H}}{\partial \eta^2}(0)\eta^2 + \frac{\partial^2 \hat{H}}{\partial z^2}(0)z^2\right) + O(3).\end{aligned}$$

By hypothesis, it follows that $\hat{H}(\xi, 0, z) = H(\xi, \xi, z) = 0$. So,

$$\hat{H}(\xi, 0, z) = \frac{\partial \hat{H}}{\partial \xi}(0)\xi + \frac{\partial \hat{H}}{\partial z}(0)z + \frac{\partial^2 \hat{H}}{\partial \xi \partial z}(0)\xi z + \frac{1}{2}\left(\frac{\partial^2 \hat{H}}{\partial \xi^2}(0)\xi^2 + \frac{\partial^2 \hat{H}}{\partial z^2}(0)z^2\right) + O(3)$$

is equal to the null series, which implies that

$$\frac{\partial^i \hat{H}}{\partial \xi^i}(0) = 0, \ \frac{\partial^i \hat{H}}{\partial z^i}(0) = 0, \ \text{and} \ \frac{\partial^i \hat{H}}{\partial \xi^j \partial z^k}(0) = 0, \ \forall\, i, j, k = 1, 2, 3, \dots .$$

Thus,

$$\hat{H}(\xi, \eta, z) = \eta\left(\frac{\partial \hat{H}}{\partial \eta}(0) + \frac{1}{2}\frac{\partial^2 \hat{H}}{\partial \eta^2}(0)\eta + \frac{\partial^2 \hat{H}}{\partial \xi \partial \eta}(0)\xi + \frac{\partial^2 \hat{H}}{\partial \eta \partial z}(0)z + O(2)\right). \tag{4.12}$$

for all $(\xi, \eta, z)$ near of the origin. Setting $\eta = x - y$ and $\xi = x + y$, the result follows. □

Since functions $H_i$, $i = 1, 2$, previously defined, satisfy theorem 4.7 ($H_1(\tau_1, y_2, \tau_1) = 0$ and $H_2(y_1, \tau_2, \tau_2) = 0$), so we must have

$$H_1(y_1, y_2, y_3) = (y_1 - y_3)h_1(y_1, y_2, y_3) = 0, \tag{4.13}$$

$$H_2(y_1, y_2, y_3) = (y_2 - y_3)h_2(y_1, y_2, y_3) = 0, \tag{4.14}$$

where the analytic functions $h_i's$ are defined in $V_1(O) \cap U$ (by translating the point $\widetilde{Y}^0$ to the origin).

Next, in order to analyze the bifurcations arising from the centered regular tetrahedral configuration, we will use the analyticity of the function $G$ to obtain analytical expressions for the implicit solutions $W$ and $\epsilon$ found so far, and the analytical expressions for the functions $h_1$ and $h_2$ in equations (4.13) and (4.14).

### 4.2.1 Analytic Expressions

Let us write Taylor expansions for the functions $G_i$ around the origin. For this, we make a translation of the variables $y_i = y_i + k,\ i = 1, 2, 3$, and $y_4 = y_4 - 4k$. Note that the function $G_1$ is $\rho_{(23)}$-invariant. Thus, the expression of the Taylor series up to second order terms has the form

$$\begin{aligned} G_1(Y, \epsilon) =& b_5\epsilon + b_{11}y_1^2 + b_{22}(y_2^2 + y_3^2) + b_{44}y_4^2 + b_{55}\epsilon^2 + b_{12}(y_2 + y_3)y_1 + b_{23}y_2y_3 + b_{14}y_1y_4 \\ &+ b_{24}(y_2 + y_3)y_4 + b_{15}y_1\epsilon + b_{25}(y_2 + y_3)\epsilon + b_{45}y_4\epsilon + O(3). \end{aligned}$$

Moreover, due to the equivariance of $G$,

$$\begin{aligned} G_2(y_1, y_2, y_3, y_4, \epsilon) &= G_1(y_2, y_1, y_3, y_4, \epsilon), \\ G_3(y_1, y_2, y_3, y_4, \epsilon) &= G_1(y_3, y_2, y_1, y_4, \epsilon), \end{aligned}$$

thus we get the Taylor expansions of $G_2$ and $G_3$.

On the other hand, the function $G_4$ is $S_3$-invariant. Thus, its Taylor series can be written as

$$\begin{aligned} G_4(Y, \epsilon) =& -fy_4 + c_5\epsilon + c_{11}(y_1^2 + y_2^2 + y_3^2) + c_{44}y_4^2 + c_{55}\epsilon^2 + c_{12}(y_1y_2 + y_1y_3 + y_2y_3) \\ &+ c_{14}(y_1 + y_2 + y_3)y_4 + c_{15}(y_1 + y_2 + y_3)\epsilon + c_{45}y_4\epsilon + O(3). \end{aligned}$$

The terms in the expansion of $G$ at $(0, 0, 0, 0, 0)$ which are the most important for the analysis, will be made explicit, and the remaining ones will be omitted. The expressions of the coefficients are

$$
\begin{aligned}
b_5 &= \frac{\partial G_1}{\partial \epsilon} = \frac{2k^3}{a}\left(\nu^{1-a} + \frac{8}{m^*}\right),\\
b_{22} &= \frac{\partial^2 G_1}{\partial y_2^2} = -\frac{6}{a}k\nu^{1-a} + \frac{64k^3}{a^2 m^{*2}}(1-a),\\
c_5 &= \frac{\partial G_4}{\partial \epsilon} = -\frac{12k}{a}\left[\frac{2(m^*+4\nu^a)}{(4+m^*)^2} + \frac{k^2}{m^*}\right] - \frac{3k}{a}\nu^{1-a}\left[k^2 - \frac{3(m^*+4\nu^a)}{(4+m^*)^2}\right],\\
c_{11} &= \frac{\partial^2 G_4}{\partial y_1^2} = \frac{2k}{a}\left(\frac{8}{m^*} - \nu^{1-a}\right) + \frac{2k^3}{a^2}\left(\frac{48}{m^{*2}} - \nu^{1-2a}\right)(1-a),\\
c_{12} &= \frac{\partial^2 G_4}{\partial y_1 \partial y_2} = \frac{4k}{a}\left(\frac{2}{m^*} - \nu^{1-a}\right) + \frac{k^3}{a^2}\left(\frac{80}{m^{*2}} - \nu^{1-2a}\right)(1-a),\\
c_{15} &= \frac{\partial^2 G_4}{\partial y_1 \partial \epsilon} = \frac{k}{a}\left(\nu^{1-a} - \frac{8}{m^*}\right) + \frac{2k^4}{a^2}\left(\frac{8}{m^{*2}} - \nu^{1-2a}\right)(1-a),\\
c_{44} &= \frac{\partial^2 G_4}{\partial y_4^2} = \frac{k^3}{a^2}\left(\frac{128}{m^{*2}} - 3\nu^{1-2a}\right)(1-a) + \frac{30k}{am^*},\\
c_{45} &= \frac{\partial^2 G_4}{\partial y_4 \partial \epsilon} = \frac{3k^4}{a^2}\left(\frac{4}{m^{*2}} - \nu^{1-2a}\right)(1-a) + \frac{3k^2}{a}\left(\frac{1}{m^*} + \nu^{1-a}\right),\\
c_{55} &= \frac{\partial^2 G_4}{\partial \epsilon^2} = \frac{3k^5}{a^2}\left(\frac{16}{m^{*2}} - \nu^{1-2a}\right)(1-a) + \frac{6k^3}{a}\left(\frac{4}{m^*} + \nu^{1-a}\right),\\
b_{11} &= \frac{\partial^2 G_1}{\partial y_1^2} = 0, \quad b_{12} = \frac{1}{2}\frac{\partial^2 G_1}{\partial y_1 \partial y_2} = b_{22}, \quad c_{14} = \frac{\partial^2 G_4}{\partial y_1 \partial y_4} = c_{11}.
\end{aligned}
$$

The analytic expression of the implicit solution $W$ can be obtained by substituting its generic Taylor series into $G_4$ and setting each term equal to zero. Moreover, since $W$ is $S_3$-invariant, we have

$$
\begin{aligned}
W(y_1,y_2,y_3,\epsilon) =& \frac{1}{f}\Big(c_5\epsilon + c_{11}(y_1^2+y_2^2+y_3^2) + \left(c_{44}\frac{c_5^2}{f^2} + c_{55} + c_{45}\frac{c_5}{f}\right)\epsilon^2\\
&+ c_{12}(y_1y_2 + y_1y_3 + y_2y_3) + \left(c_{14}\frac{c_5}{f} + c_{15}\right)(y_1+y_2+y_3)\epsilon + O(3)\Big),
\end{aligned}
$$

Recall that $f = \dfrac{4(1-\nu^a)(1-2a)}{a(4-3\nu^a)}$. Substituting $y_4 = W$ into the system $G_i = 0, i = 1,2,3,$ and setting

$$
\widetilde{G}_i(y_1,y_2,y_3,\epsilon) = G_i(y_1,y_2,y_3,W(y_1,y_2,y_3,\epsilon),\epsilon),\ \ i = 1,2,3,
$$

we get

$$
\begin{aligned}
\widetilde{G}_1(y_1,y_2,y_3,\epsilon) =& b_5\epsilon + b_{22}\left[(y_2^2+y_3^2)+2(y_1y_2+y_1y_3+y_2y_3)\right] + \left(\frac{b_{14}}{f}c_5+b_{15}\right)y_1\epsilon \\
&+\left(\frac{b_{44}}{f^2}c_5^2+\frac{b_{45}}{f}c_5+b_{55}\right)\epsilon^2+\left(\frac{b_{24}}{f}+b_{25}\right)(y_2+y_3)\epsilon+O(3),\\
\widetilde{G}_2(y_1,y_2,y_3,\epsilon) =& b_5\epsilon + b_{22}\left[(y_1^2+y_3^2)+2(y_1y_2+y_2y_3+y_1y_3)\right] + \left(\frac{b_{14}}{f}c_5+b_{15}\right)y_2\epsilon \\
&+\left(\frac{b_{44}}{f^2}c_5^2+\frac{b_{45}}{f}c_5+b_{55}\right)\epsilon^2+\left(\frac{b_{24}}{f}+b_{25}\right)(y_1+y_3)\epsilon+O(3),\\
\widetilde{G}_3(y_1,y_2,y_3,\epsilon) =& b_5\epsilon + b_{22}\left[(y_2^2+y_1^2)+2(y_3y_2+y_1y_3+y_2y_1)\right] + \left(\frac{b_{14}}{f}c_5+b_{15}\right)y_3\epsilon \\
&+\left(\frac{b_{44}}{f^2}c_5^2+\frac{b_{45}}{f}c_5+b_{55}\right)\epsilon^2+\left(\frac{b_{24}}{f}+b_{25}\right)(y_2+y_1)\epsilon+O(3).
\end{aligned}
$$

Now, the series for $\epsilon(y_1,y_2,y_3)$ can be obtained by substituting its generic Taylor series into $G_3$ and setting each term equal to zero. It follows that

$$
\begin{aligned}
\epsilon(y_1,y_2,y_3) =& -\frac{b_{22}}{b_5}(y_1+y_2)(2y_3+y_1+y_2)+d_1(y_1^3+y_2^3)+d_2y_3^3+d_3y_1y_2y_3 \\
&+d_4(y_1^2y_2+y_2^2y_1)+d_5(y_1^2y_3+y_2^2y_3)+d_6(y_3^2y_1+y_3^2y_2)+O(4).
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\partial^3\epsilon}{\partial y_1^3} = -\frac{1}{b_5}\left[b_{24}\frac{c_{11}}{f}+b_{222}-\frac{b_{22}}{b_5}\left(c_5\frac{b_{24}}{f}+b_{25}\right)\right],\\
d_2 &= \frac{\partial^3\epsilon}{\partial y_3^3} = -\frac{1}{b_5}\left[b_{14}\frac{c_{11}}{f}+b_{111}\right],\\
d_3 &= \frac{\partial^3\epsilon}{\partial y_1\partial y_2\partial y_3} = -\frac{1}{b_5}\left[b_{123}+\frac{c_{12}}{f}(b_{14}+2b_{24})-\frac{b_{12}}{b_5}\left(\frac{c_5}{f}(b_{14}+2b_{24})+b_{15}\right)\right],\\
d_4 &= \frac{\partial^3\epsilon}{\partial y_1^2\partial y_2} = -\frac{1}{b_5}\left[b_{223}-\frac{b_{12}}{b_5}\left(b_{25}+\frac{b_{24}}{f}(c_5+c_{12})\right)\right],\\
d_5 &= \frac{\partial^3\epsilon}{\partial y_1^2\partial y_3} = -\frac{1}{b_5}\left[b_{112}+\frac{1}{f}(b_{14}c_{11}+b_{24}c_{12})-\frac{b_{22}}{b_5}\left(b_{15}+\frac{b_{14}}{f}\right)-\frac{b_{12}}{b_5}\left(b_{25}+c_5\frac{b_{24}}{f}\right)\right],\\
d_6 &= \frac{\partial^3\epsilon}{\partial y_3^2\partial y_1} = -\frac{1}{b_5}\left[b_{112}+\frac{1}{f}(b_{14}c_{12}+b_{24}c_{11})-\frac{b_{12}}{b_5}\left(b_{15}+c_5\frac{b_{14}}{f}\right)\right].
\end{aligned}
$$

**Remark 4.8.** To express the Taylor series for $\epsilon$ up to order-three, it was necessary to calculate the order three terms of the Taylor series of $G_3$. However, these terms are omitted because their expressions were too long.

Finally, we define

$$
H_i(y_1,y_2,y_3)=\widetilde{G}_i(y_1,y_2,y_3,\epsilon(y_1,y_2,y_3)),
$$

by replacing the series of $\epsilon$ into series of the $\widetilde{G}_i,\ i = 1, 2$. Hence,

$$H_1(y_1, y_2, y_3) = -b_{22}(y_3 + y_1)(y_1 - y_3) + O(3), \tag{4.15}$$

$$H_2(y_1, y_2, y_3) = -b_{22}(y_3 + y_2)(y_2 - y_3) + O(3), \tag{4.16}$$

Recall that $b_{22} = -\frac{6}{a}k\nu^{1-a} + \frac{64k^3}{a^2 m^{*2}}(1-a)$.

Since the Taylor series of a function is unique, we compare (4.13) with (4.15), and (4.14) with (4.16). We conclude that the analytic functions $h_i$ are given by

$$h_1(y_1, y_2, y_3) = -b_{22}(y_1 + y_3) + O(2), \tag{4.17}$$

$$h_2(y_1, y_2, y_3) = -b_{22}(y_2 + y_3) + O(2). \tag{4.18}$$

The factorization guaranteed by theorem 4.7 allows us to look for solutions of the system of equations $(4.13)$ and (4.14) using the zeros of the functions $h_i,\ i = 1, 2$. As a matter of fact, firstly we observe that

$$\frac{\partial h_1}{\partial y_1} = -b_{22} \neq 0 \ \text{ and } \ h_1(0,0,0) = 0.$$

By the implicit function theorem, there is an analytic function $\tau_1 : \widetilde{V_1}(0) \subseteq \mathbb{R}^2 \longrightarrow \widetilde{V_2}(0) \subseteq \mathbb{R}$ such that

$$h_1(\tau_1(y_2, y_3), y_2, y_3) = 0 \text{ and } \tau_1(0,0) = 0.$$

More precisely, we solve $h_1 = 0$ for the variable $y_1 = \tau_1(y_2, y_3)$. Due to the expansion of $h_1$, we conclude that

$$\tau_1(y_2, y_3) = -y_3 + O(2).$$

Similarly, there exists a analytic function $\tau_2 : \widetilde{U}_1(0) \subseteq \mathbb{R}^2 \longrightarrow \widetilde{U}_2(0) \subseteq \mathbb{R}$ such that

$$h_2(y_1, \tau_2(y_1, y_3), y_3) = 0, \quad \tau_2(0,0) = 0.$$

Due to the expansion of $h_2$, we conclude that

$$\tau_2(y_1, y_3) = -y_3 + O(2).$$

Therefore, we implicitly obtain all possible solutions for the system of equations (4.13) and (4.14), i.e., by combining the solutions of (4.13) with the solutions of $(4.14)$, we determine four bifurcation branches which form the solution set of the bifurcation problem (4.5), namely

(I) $y_1 = y_3$ and $y_2 = y_3$,

(II) $y_1 = -y_3 + O(2)$ and $y_2 = -y_3 + O(2)$,

(III) $y_2 = y_3$ and $y_1 = -y_3 + O(2)$,

(IV) $y_1 = y_3$ and $y_2 = -y_3 + O(2)$.

In order, to obtain family (IV), we substitute the solution $y_1 = y_3$ of (4.13) into equation $(4.14)$, which produces $H_2(y_3, y_2, y_3) = (y_2 - y_3)h_2(y_3, y_2, y_3)$. Assuming that $y_2 \neq y_3$, $H_2 = 0$ if $y_2 = \tau_2(y_3) = -y_3 + O(2)$. Family (III) was obtained by substituting the solution $y_2 = y_3$ of (4.14) into equation $(4.13)$. So, we have $H_1(y_1, y_3, y_3) = (y_1 - y_3)h_1(y_1, y_3, y_3)$ and assuming that $y_1 \neq y_3$, $H_1 = 0$ if $y_2 = \tau_1(y_3) = -y_3 + O(2)$. Finally, family (II) is the combination of the zeros of the functions $h_i$.

Our approach coincides with the one in (SANTOS et al., 2017). Although the problems are similar, mainly because the symmetry in both problems is the same, in each case one can see subtle changes in the solutions found. Since the problems have the same symmetry, we could expect that the solutions found behave similarly.

In the next section we will analyze in more detail two of the four families of solutions found above and we will compare them with the solutions found in (SANTOS et al., 2017).

### 4.2.2 The Behavior of Bifurcation Branches

In the last section, the variables and the bifurcation parameter were determined in terms of the variable $y_3$ (using the functions $\tau_1$ and $\tau_2$). In order to describe the families of central configurations corresponding to the solutions of equation (4.5), let us set $y_3 = \delta$ as the bifurcation parameter of the problem and study the behavior of solutions when we approach the degenerate configuration (by letting $\delta \to 0$). We should keep in mind that it is necessary to undo the translation of variables made to study the Taylor series expansions. We will analyze the four central configurations obtained from the system of equations (4.5) starting with family (I):

$$\text{Family (I):} \begin{cases} y_1(\delta) &= k + \delta, \\ y_2(\delta) &= k + \delta, \\ y_3(\delta) &= k + \delta, \\ y_4(\delta) &= -4k + \left[\dfrac{3b_5(c_{11} + c_{12}) - 8b_{22}c_5}{fb_5}\right]\delta^2 + O(3), \\ \epsilon(\delta) &= -\left(\dfrac{8b_{22}}{b_5}\right)\delta^2 + O(3). \end{cases}$$

Since $b_{22} > 0$ and $b_5 < 0$ for $a \in (-\infty, -1)$, the derivative $\dfrac{d^2}{d\delta^2}(\epsilon(0))$ is positive, which indicates that $\epsilon(\delta)$ has a minimum at $\delta = 0$. Furthermore, we consider $\delta$ near the origin, so the quadratic term controls the behavior of the function. For this reason, we can state that

$\epsilon(\delta) > 0$ and for each value of $\epsilon$, we have two values for $\delta$, $\delta_1 = \delta$ and $\delta_2 = -\delta$, such that $\epsilon(\delta_1) = \epsilon(\delta_2)$ and $Y_1(\delta_1) \neq Y_2(\delta_2)$. Thus, for the masses $m_4 = m^*$ and $m_5 = 1$ fixed and $m_i \in (1, 1+\epsilon(\delta))$, $i = 1, 2, 3$, we have two branches of solution arising from the centered tetrahedron. Moreover, these branches only exist for positive values of the parameter $\epsilon$.

**Remark 4.9.** When the bifurcation parameter is the mass of one of the vertices of the tetrahedron (SANTOS et al., 2017), the branches of bifurcations with this same symmetry exist for negative values of the parameter $\epsilon$.

Returning to the variables $x_i$, we find that

$$\begin{cases} x_1(\delta) &= x_2(\delta) = x_3(\delta) = k + \delta, \\ x_4(\delta) &= -4k + \left[\dfrac{3b_5(c_{11}+c_{12}) - 8b_{22}c_5}{fb_5}\right]\delta^2 + O(3), \\ x_5(\delta) &= k - 3\delta - \left[\dfrac{3b_5(c_{11}+c_{12}) - 8b_{22}c_5}{fb_5}\right]\delta^2 + O(3). \end{cases}$$

Finally, to establish the behavior of the solutions, we use the expressions for the squares of the distances

$$s_{ij}(\delta) = \left[\frac{\lambda}{4 + m^* + 3\epsilon(\delta)} - \frac{x_i(\delta)x_j(\delta)}{m_i m_j}\right]^{1/a}, \ 1 \le i < j < 5. \tag{4.19}$$

These expressions give information about the growth of the distances between the bodies when we change the parameter.

We write the Taylor series of $s_{ij}$ around $\delta = 0$ as follows

$$s_{ij}(\delta) = s_{ij}^0 + v_{ij}\delta + \alpha_{ij}\delta^2 + O(\delta^3), \tag{4.20}$$

where $v_{ij} = \left.\dfrac{ds_{ij}}{d\delta}\right|_{\delta=0}$ and $\alpha_{ij} = \frac{1}{2}\left.\dfrac{d^2 s_{ij}}{d\delta^2}\right|_{\delta=0}$. The linear terms are

$$\begin{aligned} v_{12} = v_{13} = v_{23} &= -\frac{2\nu^{1-a}}{a}k > 0, \\ v_{15} = v_{25} = v_{35} &= \frac{2\nu^{1-a}}{a}k < 0, \\ v_{14} = v_{24} = v_{34} &= \frac{4k}{am^*} < 0, \\ v_{45} &= -\frac{12}{am^*}k > 0, \quad \text{for } a \in (-\infty, -1), \end{aligned}$$

and, the quadratic terms are

$$\alpha_{12} = \left(\frac{4k^2}{a^2}\nu^{1-2a}\right)(1-a) + \frac{\nu^{1-a}}{a}\left\{-2 + 48\left[\frac{b_{22}}{b_5}\frac{(m^* + 4\nu^a)}{(4+m^*)^2}\right] - 32k^2\frac{b_{22}}{b_5}\right\},$$

$$\begin{aligned} \alpha_{15} = &\left(\frac{4k^2}{a^2}\nu^{1-2a}\right)(1-a) \\ &+ \frac{2\nu^{1-a}}{a}\left\{3 + 24\left[\frac{b_{22}}{b_5}\frac{(m^* + 4\rho^a)}{(4+m^*)^2}\right] + k\left[\frac{3b_5(c_{11}+c_{12}) - 8b_{22}c_5}{fb_5}\right] - 8k^2\frac{b_{22}}{b_5}\right\}, \end{aligned}$$

$$\alpha_{14} = \frac{16}{a^2(m^*)^2}k^2(1-a)$$
$$+\frac{1}{a}\left\{48\left[\frac{b_{22}(m^*+4\nu^a)}{b_5(4+m^*)^2}\right] - 2k\left[\frac{3b_5(c_{11}+c_{12})-8b_{22}c_5}{fb_5m^*}\right] + \left(\frac{64k^2}{m^*}\right)\frac{b_{22}}{b_5}\right\},$$
$$\alpha_{45} = \left(\frac{144k^2}{a^2(m^*)^2}\right)(1-a) + \frac{1}{a}\left\{48\left[\frac{b_{22}(m^*+4\nu^a)}{b_5(4+m^*)^2}\right] - 10k\left[\frac{8b_{22}c_5+3b_5(c_{11}+c_{12})}{fb_5m^*}\right]\right\}.$$

Calculating the terms of the $s_{ij}$ expansion found in (SANTOS et al., 2017) and comparing them with ours, we see that the linear terms agree, but the quadratic terms do not ( the terms of the $s_{ij}$ of (SANTOS et al., 2017) are displayed in Appendix A). Therefore, in the linear approximation, the solutions have the same behavior, but they must diverge for higher order terms.

A natural question is whether it is possible to obtain a reparametrization of one of the solutions capable of obtaining a solution of the other problem, i.e., for different values of the mass vector, is it possible to obtain the same solution curve? This is an unlikely fact and would imply the existence of a configuration of the type called *perverse* in (CHENCINER, 2003).

In order to answer this question we assume that such an analytic reparametrization exists and let

$$\delta(t) = \zeta + \beta t + \gamma t^2 + O(t^3), \quad \zeta, \beta, \gamma \in \mathbb{R},$$

where $t$ is the parameter used in (SANTOS et al., 2017). Substituting in (4.20) it is possible to determine the expressions for possible coefficients. In fact,

$$s_{12}(\delta(t)) = \left(\frac{8}{3} + v_{12}\zeta + \alpha_{12}\zeta^2\right) + \beta\left(v_{12} + 2\zeta\alpha_{12}\right)t + \left(v_{12}\gamma + \alpha_{12}(\beta^2 + 2\zeta\gamma)\right)t^2 + O(3)$$

and comparing each term with the terms in the expression $s_{12}$ in (SANTOS et al., 2017), we have

$$\zeta = 0 \text{ or } \zeta = -\frac{v_{12}}{\alpha_{12}},$$
$$\beta = 1 \text{ or } \beta = -1,$$
$$\gamma = \pm\frac{1}{k}\left[16k^2 - \frac{32(m^*+4\nu^a)}{(4+m^*)^2}\right]\frac{b_{22}}{b_5}.$$

In the same way for

$$s_{15}(\delta(t)) = \left(1 + v_{15}\zeta + \alpha_{15}\zeta^2\right) + \beta\left(v_{15} + 2\zeta\alpha_{15}\right)t + \left(v_{15}\gamma + \alpha_{15}(\beta^2 + 2\zeta\gamma)\right)t^2 + O(3).$$

To agree with the coefficients found for $s_{12}$, we get that $\zeta = 0$ and consequently $\beta = 1$. However, in this case we obtain

$$\gamma = \left\{16k^2 + \frac{8k}{f}\left[\left(\frac{12k\nu^{1-a}}{a} - 32\right)\left(\frac{m^*+4\nu^a}{(4+m^*)^2}\right) - \frac{32k^3}{am^*}\right] - 32\frac{(m^*+4\nu^a)}{(4+m^*)^2}\right\}\frac{b_{22}}{b_5}.$$

Unfortunately, it is not possible yet to give a complete analysis when $a < -1$. For the particular case when $a = -3/2$, we can see that we would need different coefficients for $\delta(t)$. This leads us to conclude that, for $a = -3/2$, there is no class $C^2$ reparametrization and, from this analysis, the solutions seem to behave differently for orders higher than two.

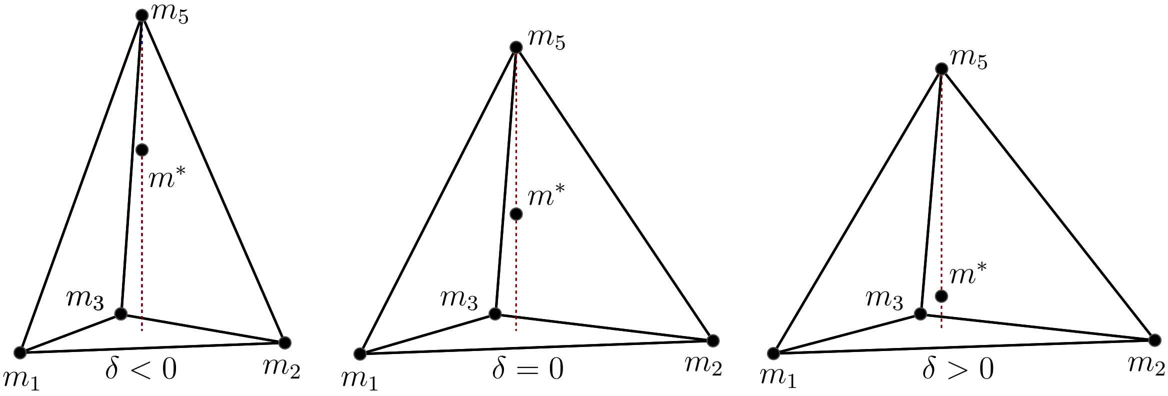


**Figure 6** : Bifurcations of the centered regular tetrahedron when $\epsilon > 0$ (corresponding to family (I)). The axis of symmetry passes through the barycenter of the tetrahedron and $m^*$ moves along this axis as $\epsilon$ varies. Notice that the base of the tetrahedron remains an equilateral triangle.

Now, we analyze the second family of central configurations obtained from the system of the equations (4.5),

$$\text{Family (II):}\begin{cases} y_1(\delta) &= k - \delta + O(2), \\ y_2(\delta) &= k - \delta + O(2), \\ y_3(\delta) &= k + \delta, \\ y_4(\delta) &= -4k + \left(\dfrac{3c_{11} + c_{12}}{f}\right)\delta^2 + O(3), \\ \epsilon(\delta) &= d\delta^3 + O(4), \end{cases}$$

where $d = -2d_1 + d_2 + d_3 - 2d_4 + 2d_5 - 2d_6$. In this case, the value of $\epsilon$ is uniquely determined by the value of $\delta$, and we have a single branch of solutions shooting off the centered tetrahedron.

Returning to the variables $x_i$, we get

$$\begin{cases} x_1(\delta) &= x_2(\delta) = k - \delta + O(2), \\ x_3(\delta) &= k + \delta, \\ x_4(\delta) &= -4k + \left(\dfrac{3c_{11} + c_{12}}{f}\right)\delta^2 + O(3), \\ x_5(\delta) &= k + \delta - \left(\dfrac{3c_{11} + c_{12}}{f}\right)\delta^2 + O(3). \end{cases}$$

Finally, we write the Taylor series of $s_{ij}$ around $\delta = 0$ up to third order as

$$s_{ij}(\delta) = s_{ij}^0 + v_{ij}\delta + \alpha_{ij}\delta^2 + w_{ij}\delta^3 + O(\delta^4),$$

where $v_{ij} = \left.\frac{ds_{ij}}{d\delta}\right|_{\delta=0}$, $\alpha_{ij} = \frac{1}{2}\left.\frac{d^2 s_{ij}}{d\delta^2}\right|_{\delta=0}$ and $w_{ij} = \frac{1}{3!}\left.\frac{d^3 s_{ij}}{d\delta^3}\right|_{\delta=0}$. The linear terms are

$$\begin{aligned}
v_{12} &= \frac{2\nu^{1-a}}{a}k < 0,\\
v_{13} &= v_{23} = v_{15} = v_{25} = 0,\\
v_{14} &= v_{24} = -\frac{4k}{am^*} > 0,\\
v_{34} &= v_{45} = \frac{4k}{am^*} < 0,\\
v_{35} &= -\frac{2}{a}\nu^{1-a}k > 0, \quad \forall\ a < -1.
\end{aligned}$$

The quadratic terms are

$$\begin{aligned}
\alpha_{12} &= \left(\frac{2\nu^{1-a}}{a^2}\right)\left[2k^2(1-a) - a\right],\\
\alpha_{13} &= \alpha_{23} = \frac{2}{a}\nu^{1-a},\\
\alpha_{15} &= \alpha_{25} = \left(\frac{2\nu^{1-a}}{a}\right)\left[1 + \frac{k(3c_{11} - c_{12})}{f}\right],\\
\alpha_{14} &= \alpha_{24} = \alpha_{34} = \left(\frac{2k}{a^2 m^{*2} f}\right)\left[8kf(1-a) - (3c_{11} - c_{12})am^*\right],\\
\alpha_{45} &= \left(\frac{2}{a^2 m^{*2} f}\right)\left[8f(1-a) - 5(3c_{11} - c_{12})am^*\right].
\end{aligned}$$

Finally,

$$\begin{aligned}
w_{12} =& \left(\frac{8k^3\nu^{1-3a}}{a^3}\right)(1 - 3a + 2a^2) + \left(\frac{12k\nu^{1-2a}}{a^2}\right)(1-a)\\
&- \left(\frac{6\nu^{1-a}}{a}\right)\left[\frac{3(m^* + 4\nu^a)}{(4+m^*)^2} - 2k^2\right]d,\\
w_{13} =& w_{23} = -6\left(\frac{\nu^{1-a}}{a}\right)\left(\frac{3(m^* + 4\nu^a)}{(4+m^*)^2} - 2k^2\right)d,\\
w_{15} =& w_{25} = -6\left(\frac{\nu^{1-a}}{a}\right)\left\{\left[\frac{3(m^* + 4\nu^a)}{(4+m^*)^2} - k^2\right]d + \frac{(3c_{11} - c_{12})}{f}\right\},\\
w_{14} =& w_{24} = -\left(\frac{64k^3}{a^3 m^{*3}}\right)(1 - 3a + 2a^2) + \left(\frac{24k^2}{a^2 m^{*2} f}\right)(3c_{11} - c_{12})(1-a)\\
&- \frac{6}{a}\left\{\left[\frac{3(m^* + 4\nu^a)}{(4+m^*)^2} + \frac{4k^2}{m^*}\right]d - \frac{3c_{11} - c_{12}}{fm^*}\right\},\\
w_{34} =& -\left(\frac{64k^3}{a^3 (m^*)^3}\right)(1 - 3a + 2a^2) - \left(\frac{24k^2}{a^2 (m^*)^2 f}\right)(3c_{11} - c_{12})(1-a)\\
&- \frac{6}{a}\left\{\left[\frac{3(m^* + 4\nu^a)}{(4+m^*)^2} + \frac{4k^2}{m^*}\right]d + \frac{3c_{11} - c_{12}}{fm^*}\right\},\\
w_{45} =& \left(\frac{64k^3}{a^3 m^{*3}}\right)(1 - 3a + 2a^2) - \left(\frac{120k^2}{a^2 m^{*2} f}\right)(3c_{11} - c_{12})(1-a)\\
&- \frac{6}{a}\left\{\left[3\frac{(m^* + 4\nu^a)}{(4+m^*)^2}\right]d + \frac{3c_{11} - c_{12}}{fm^*}\right\},
\end{aligned}$$

$$w_{35} = \left(\frac{8k^3\nu^{1-3a}}{a^3}\right)(-1+3a-2a^2) - \left(\frac{12k\nu^{1-2a}}{a^2}\right)\left[\frac{k(3c_{11}-c_{12})}{f} - 1\right](1-a)$$
$$- \left(\frac{6\nu^{1-a}}{a}\right)\left\{\left[\frac{3(m^*+4\nu^a)}{(4+m^*)^2} - k^2\right] d - \left(\frac{3c_{11}-c_{12}}{f}\right)\right\}.$$

Again comparing the solution above with the solution found in (SANTOS et al., 2017), we notice that they coincide up to terms of quadratic order and diverge for higher order terms.

The analysis for families (III) and (IV) follows the same path as family (II), noting that these solutions are basically solution (II) with the rearrangements of the mass indices.

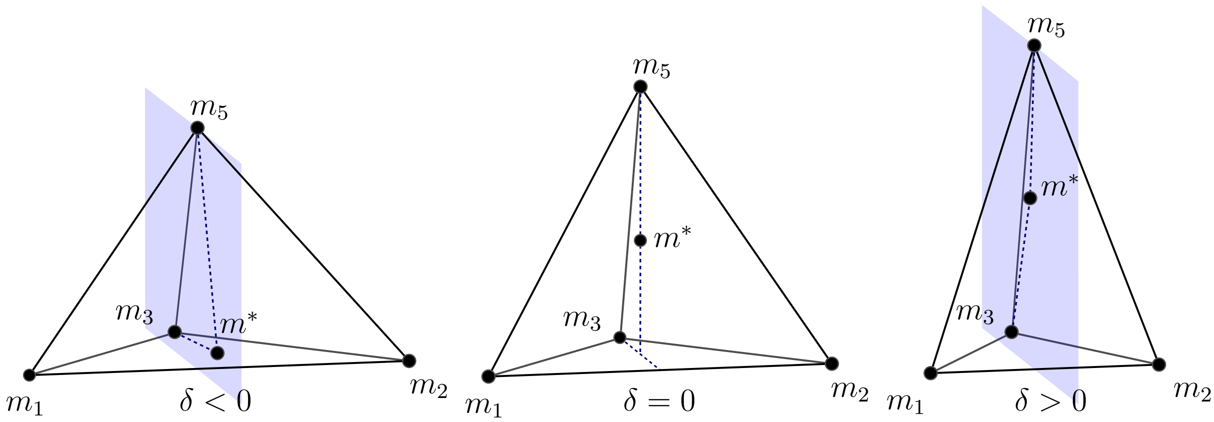


**Figure** 7: Bifurcation of the centered tetrahedron when $\epsilon > 0$ or $\epsilon < 0$ (corresponding to family (II)). The mass $m^*$ moves along a plane passing through the segment $q_3q_5$. When $\delta > 0$, $m^*$ is closer to the segment $q_3q_5$, and when $\delta < 0$, $m^*$ is closer to the segment $q_1q_2$.

To summarise the results obtained in this chapter, we state the following theorem.

**Theorem 4.10.** *Let $q(1,1,1,m,1)$ be a central configuration of five bodies in the space forming a centered regular tetrahedron. Denote by $m^* > 0$ the value of the central mass, which depends on the exponent $a < -1$, such that the configuration is degenerate. For each $\epsilon > 0$, if exactly three of the four vertices of the tetrahedron have equal masses in the interval $(1, 1+\epsilon)$, then there are two families of central configurations with axis-type symmetry that bifurcate from the degenerate configuration $q(1,1,1,m^*,1)$. Furthermore, three additional families of central configurations with plane-type symmetry bifurcate from the degenerate solution if the same three of the four vertices of the tetrahedron have equal masses in the interval $(1-\epsilon, 1+\epsilon)$. In all cases, there are no symmetrical central configurations bifurcating from $q(1,1,1,m^*,1)$.*

## 4.3 BIFURCATION PROBLEM WITH TWO EQUALLY VARYING MASSES

Next we are interested in studying the bifurcation problem for the centered tetrahedron with two vertex masses as bifurcation parameters. More precisely, we keep the masses $m_5 = m_3 = 1$ and $m_4 = m^*$ fixed and we approach the remaining two unit masses in the same way, $m_1 = m_2 = 1+\epsilon$. If we substitute these values in $(4.1)$ and consider $x_4 = -\left(\sum_{i=1}^{3} x_i + x_5\right)$, we get a system of four equations defined by $t_i - t_4 = 0$, four variables $X = (x_1, x_2, x_3, x_5)$ and the bifurcation parameter $\epsilon$. The pair $(X, \epsilon) = (X^0, 0) = (k, k, k, k, 0)$ is the degenerate configuration.

Define $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_5) : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4$ by

$$\Phi_1(X,\epsilon) = \sum_{\substack{i\neq 1\\ i\neq 4}}^{5} x_i \left[\frac{\lambda}{M} - \frac{x_1 x_i}{(1+\epsilon)m_i}\right]^{1/a} - \left(\sum_{i=1}^{3} x_i + x_5\right)\left[\frac{\lambda}{M} + \frac{x_1(\sum_{i=1}^{3} x_i + x_5)}{(1+\epsilon)m^*}\right]^{1/a} - t_4,$$

$$\Phi_2(X,\epsilon) = \sum_{\substack{i\neq 2\\ i\neq 4}}^{5} x_i \left[\frac{\lambda}{M} - \frac{x_2 x_i}{(1+\epsilon)m_i}\right]^{1/a} - \left(\sum_{i=1}^{3} x_i + x_5\right)\left[\frac{\lambda}{M} + \frac{x_2(\sum_{i=1}^{3} x_i + x_5)}{(1+\epsilon)m^*}\right]^{1/a} - t_4,$$

$$\Phi_3(X,\epsilon) = \sum_{\substack{i\neq 3\\ i\neq 4}}^{5} x_i \left(\frac{\lambda}{M} - \frac{x_3 x_i}{m_i}\right)^{1/a} - \left(\sum_{i=1}^{3} x_i + x_5\right)\left[\frac{\lambda}{M} + \frac{x_3(\sum_{i=1}^{3} x_i + x_5)}{m^*}\right]^{1/a} - t_4,$$

$$\Phi_5(X,\epsilon) = \sum_{i=1}^{3} x_i \left(\frac{\lambda}{M} - \frac{x_i x_5}{m_i}\right)^{1/a} - \left(\sum_{i=1}^{3} x_i + x_5\right)\left[\frac{\lambda}{M} + \frac{x_5(\sum_{i=1}^{3} x_i + x_5)}{m^*}\right]^{1/a} - t_4,$$

and

$$t_4 = \sum_{\substack{i=1\\ i\neq 4}}^{5} x_i \left[\frac{\lambda}{M} - \frac{x_i\left(\sum_{j=1}^{3} x_j + x_5\right)}{m_i m^*}\right]^{1/a},$$

where $M = 4 + m^* + 2\epsilon$, $\lambda = m^* + 4\nu^a$, $k = \sqrt{\dfrac{m^*}{4+m^*}(1-\nu^a)}$ and $a < -1$.

The problem is described by equation

$$\Phi(X,\epsilon) = 0, \tag{4.21}$$

with symmetry group $\Gamma = \{\text{id}, (12), (35), (12)(35)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The action of the group $\Gamma$ on $\mathbb{R}^4$, similarly to the previous problem, is defined by the representation given by $\rho_\gamma(x_1, x_2, x_3, x_5) = (x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)}, x_{\gamma(5)}),\ \forall\, \gamma \in \Gamma$, such that,

$$\Phi(\rho_\gamma \cdot X, \epsilon) = \rho_\gamma \cdot \Phi(X,\epsilon), \quad \forall\, \gamma \in \Gamma. \tag{4.22}$$

**Remark 4.11.** The matrices of the representation $\rho$ with respect to the canonical basis are

$$\rho_{(id)}:\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \rho_{(12)}:\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \rho_{(35)}:\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\ \rho_{(12)(35)}:\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}.$$

Therefore, the derivative of $\Phi$ at $X^0, \epsilon$ is a matrix of the form

$$D_X\Phi(X^0,\epsilon)=\begin{bmatrix}b&c&d&d\\c&b&d&d\\e&e&f&g\\e&e&g&f\end{bmatrix}.$$

Calculating the derivatives we get

$$\begin{aligned}b=f=&-2-\frac{k^2}{a}\left(3\nu^{1-a}+\frac{28}{m^*}\right),\\ c=d=g=&(\rho-2)-\frac{k^2}{a}\left(\nu^{1-a}+\frac{12}{m^*}\right).\end{aligned}$$

When $m_1=m_2=m_3=m_5=1$ and $m_4=m^*$, i.e., $\epsilon=0$, the derivative of $\Phi$ coincides with (4.3). So, at $(X^0,0)$ we have

$$D_X\Phi(X^0,0)=\begin{bmatrix}b&b&b&b\\b&b&b&b\\b&b&b&b\\b&b&b&b\end{bmatrix}.$$

Let us consider $\widetilde{L}=D_X\Phi(X^0,0)$ and from its structure, it is immediate to obtain its kernel and image

$$\ker(\widetilde{L})=\left\{(v_1,v_2,v_3,v_4)\in\mathbb{R}^4\,\middle|\,\sum_{i=1}^{4}v_i=0\right\}\text{ and }\mathrm{Im}(\widetilde{L})=\{(\mu,\mu,\mu,\mu)\mid\mu\in\mathbb{R}\},$$

and to check that both are $\rho$-invariant subspaces. Applying the Liapunov-Schmidt reduction process, we make the decomposition $\mathbb{R}^4=\ker(\widetilde{L})\oplus\mathrm{Im}(\widetilde{L})$ and choose convenient bases for the kernel and the image, respectively

$$\begin{aligned}&\beta_1=\{u_1=(-1,1,-1,1),u_2=(-1,-1,1,1),u_3=(1,-1,-1,1)\},\\&\beta_2=\{u_4=(1,1,1,1)\}.\end{aligned}$$

Because of the splitting of $\mathbb{R}^4$, we make the change of variables

$$(x_1, x_2, x_3, x_5) = \sum_{i=1}^{4} y_i u_i.$$

Explicitly,

$$\begin{aligned} x_1 &= -y_1 - y_2 + y_3 + y_4, \\ x_2 &= y_1 - y_2 - y_3 + y_4, \\ x_3 &= -y_1 + y_2 - y_3 + y_4, \\ x_5 &= y_1 + y_2 + y_3 + y_4. \end{aligned} \tag{4.23}$$

Substituting (4.23) in (4.21) we have a new equation

$$G(Y, \epsilon) = 0. \tag{4.24}$$

We know that $G(Y^0, 0) = 0$, where the point $Y^0 = (0, 0, 0, k(m^*))$ represents the degenerate configuration and was obtained by multiplying the inverse of the change of basis matrix by $X^0$. The functions $G_i$ are

$$\begin{aligned} G_1(Y,\epsilon) = & (y_1 - y_2 - y_3 + y_4)\left[\frac{\lambda}{M} - \frac{(-y_1 - y_2 + y_3 + y_4)(y_1 - y_2 - y_3 + y_4)}{(1+\epsilon)^2}\right]^{1/a} \\ & - (y_1 - y_2 + y_3 - y_4)\left[\frac{\lambda}{M} - \frac{(-y_1 - y_2 + y_3 + y_4)(-y_1 + y_2 - y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & + (y_1 + y_2 + y_3 + y_4)\left[\frac{\lambda}{M} - \frac{(-y_1 - y_2 + y_3 + y_4)(y_1 + y_2 + y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & - (4y_4)\left[\frac{\lambda}{M} + \frac{(4y_4)(-y_1 - y_2 + y_3 + y_4)}{(1+\epsilon)m^*}\right]^{1/a} - t_4, \\ G_2(Y,\epsilon) = & -(y_1 + y_2 - y_3 - y_4)\left[\frac{\lambda}{M} - \frac{(-y_1 - y_2 + y_3 + y_4)(y_1 - y_2 - y_3 + y_4)}{(1+\epsilon)^2}\right]^{1/a} \\ & - (y_1 - y_2 + y_3 - y_4)\left[\frac{\lambda}{M} - \frac{(y_1 - y_2 - y_3 + y_4)(-y_1 + y_2 - y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & + (y_1 + y_2 + y_3 + y_4)\left[\frac{\lambda}{M} - \frac{(y_1 - y_2 - y_3 + y_4)(y_1 + y_2 + y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & - (4y_4)\left[\frac{\lambda}{M} + \frac{(4y_4)(y_1 - y_2 - y_3 + y_4)}{(1+\epsilon)m^*}\right]^{1/a} - t_4, \\ G_3(Y,\epsilon) = & -(y_1 + y_2 - y_3 - y_4)\left[\frac{\lambda}{M} - \frac{(-y_1 - y_2 + y_3 + y_4)(-y_1 + y_2 - y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & + (y_1 - y_2 - y_3 + y_4)\left[\frac{\lambda}{M} - \frac{(y_1 - y_2 - y_3 + y_4)(-y_1 + y_2 - y_3 + y_4)}{(1+\epsilon)}\right]^{1/a} \\ & + (y_1 + y_2 + y_3 + y_4)\left[\frac{\lambda}{M} - (-y_1 + y_2 - y_3 + y_4)(y_1 + y_2 + y_3 + y_4)\right]^{1/a} \\ & - (4y_4)\left[\frac{\lambda}{M} + \frac{(4y_4)(-y_1 + y_2 - y_3 + y_4)}{m^*}\right]^{1/a} - t_4, \end{aligned}$$

$$\begin{aligned}G_4(Y,\epsilon) =& -(y_1+y_2-y_3-y_4)\left[\frac{\lambda}{M}-\frac{(-y_1-y_2+y_3+y_4)(y_1+y_2+y_3+y_4)}{(1+\epsilon)}\right]^{1/a}\\&+(y_1-y_2-y_3+y_4)\left[\frac{\lambda}{M}-\frac{(y_1-y_2-y_3+y_4)(y_1+y_2+y_3+y_4)}{(1+\epsilon)}\right]^{1/a}\\&-(y_1-y_2+y_3-y_4)\left[\frac{\lambda}{M}-(-y_1+y_2-y_3+y_4)(y_1+y_2+y_3+y_4)\right]^{1/a}\\&-(4y_4)\left[\frac{\lambda}{M}+\frac{(4y_4)(y_1+y_2+y_3+y_4)}{m^*}\right]^{1/a}-t_4,\end{aligned}$$

where

$$\begin{aligned}t_4 =& -(y_1+y_2-y_3-y_4)\left(\frac{\lambda}{M}+\frac{(4y_4)(-y_1-y_2+y_3+y_4)}{(1+\epsilon)m^*}\right)^{1/a}\\&+(y_1-y_2-y_3+y_4)\left(\frac{\lambda}{M}+\frac{(4y_4)(y_1-y_2-y_3+y_4)}{(1+\epsilon)m^*}\right)^{1/a}\\&-(y_1-y_2+y_3-y_4)\left(\frac{\lambda}{M}+\frac{(4y_4)(-y_1+y_2-y_3+y_4)}{m^*}\right)^{1/a}\\&+(y_1+y_2+y_3+y_4)\left(\frac{\lambda}{M}+\frac{(4y_4)(y_1+y_2+y_3+y_4)}{m^*}\right)^{1/a},\end{aligned}$$

$M=4+m^*+2\epsilon$, $\lambda=m^*+4\nu^a$, $k=\sqrt{\frac{m^*}{4+m^*}(1-\nu^a)}$ and $a<-1$.

**Remark 4.12.** The matrices of the representation of $\rho$ with respect to the basis $\beta=\beta_1\cup\beta_2$ are

$$\rho_{id}:\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \rho_{(12)}:\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\quad \rho_{(35)}:\begin{bmatrix}0&0&-1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{bmatrix},$$

$$\rho_{(12)(35)}:\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}.$$

The derivative of $G(Y,\epsilon)$ at the point $(Y^0,0)$ is

$$D_Y G(Y^0,0)=\begin{bmatrix} b & b & b & b\\ b & b & b & b\\ b & b & b & b\\ b & b & b & b\end{bmatrix}\begin{bmatrix} -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1\\ -1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 4b\\ 0 & 0 & 0 & 4b\\ 0 & 0 & 0 & 4b\\ 0 & 0 & 0 & 4b\end{bmatrix},$$

where $b=-\dfrac{4(1-2a)(1-\rho^a)}{a(4-3\nu^a)}$. Equation $(4.24)$ is equivalent to the pair of equations

$$(P\circ G)(Y,\epsilon)=\langle u_4,G(Y,\epsilon)\rangle u_4=(0,0,0,0), \tag{4.25}$$

$$((I-P)\circ G)(Y,\epsilon)=\sum_{i=1}^{3}\langle u_i,G(Y,\epsilon)\rangle u_i=(0,0,0,0), \tag{4.26}$$

where $P:\mathbb{R}^4\longrightarrow \text{Im}(\widetilde{L})$ is the canonical projection on $\text{Im}(\widetilde{L})$ with $\ker(P)=\ker(\widetilde{L})$ and $I-P$ is the complementary projection. Following the reduction of Liapunov-Schmidt, we want to show that the implicit function theorem is applicable to the function $(P\circ G)(Y,\epsilon)$ in order to solve equation $(4.25)$ for $y_4$ in terms of $(y_1,y_2,y_3,\epsilon)$. More precisely, we define the function $\Psi:\mathbb{R}^4\times\mathbb{R}\longrightarrow\mathbb{R}$ given by $\Psi(Y,\epsilon)=\sum_{i=1}^4 G_i(Y,\epsilon)$, and we verify that $\Psi(Y^0,0)=0$ and $\dfrac{\partial\Psi}{\partial y_4}(Y^0,0)=16b<0$, for all $a<-1$. Hence, by the implicit function theorem, there exists a unique analytic function $y_4=\widetilde{W}(y_1,y_2,y_3,\epsilon)$ such that $\Psi(y_1,y_2,y_3,\widetilde{W}(y_1,y_2,y_3,\epsilon),\epsilon)=0$, for all $(y_1,y_2,y_3,\epsilon)$ near of the origin. Since $\Psi$ is $\rho$-invariant, that is

$$\Psi(\rho_\gamma(Y),\epsilon)=\sum_{i=1}^{4}G_i(\rho_\gamma(Y),\epsilon)=\sum_{i=1}^{4}G_i(Y,\epsilon)=\Psi(Y,\epsilon),\ \forall\,\gamma\in\Gamma,$$

where the second equality follows from the equivariance of $G$, we conclude that $\widetilde{W}$ is $\rho$-invariant (See lemma 2.24 of chapter 2).

Substituting $\widetilde{W}$ in (4.26), we define the equation

$$g\left(\widetilde{Y},\epsilon\right)=0, \tag{4.27}$$

where $\widetilde{Y}=(y_1,y_2,y_3)$ and $g:\mathbb{R}^3\times\mathbb{R}\longrightarrow\mathbb{R}^3$ is an analytic function such that $g_i=\langle u_i,G(\widetilde{Y},\epsilon)\rangle$, for $u_i\in\ker(\widetilde{L})$, $i=1,2,3$.

Using an argument similar to the one used in lemma 4.6, we show that equation (4.27) does not admit a differentiable solution of the form $\widetilde{Y}(\epsilon)$. So, the parameter $\epsilon$ will be considered again as an additional variable of the problem.

Due to the symmetry of the reduced equation $(4.27)$, one way to find solutions is to restrict the function $g$ to the subspace of fixed points as discussed in chapter $2$. As we saw, this technique is interesting because it allows us to reduce the number of equations to be solved to dim(Fix($\Sigma$))-equations, where $\Sigma$ is a subgroup of $\Gamma$. We will briefly study how to apply this technique in our context, i.e., start by looking for solutions with symmetry. Consider

$$\Sigma_1 = \{\text{id}, (12)\},\ \ \Sigma_2 = \{\text{id}, (35)\} \text{ and } \Sigma_3 = \{\text{id}, (12)(35)\}$$

subgroups of $\Gamma$. If there exists a solution $\widetilde{Y}$ of (4.27), for some small $\epsilon$, which has symmetry $\Sigma_i$, then it must be in the Fix($\Sigma_i$), $i = 1, 2, 3$. Let us analyze each of the cases separately.

If $\widetilde{Y}$ is a solution with symmetry $\Sigma_3$, such that dim(Fix($\Sigma_3$)) $= 1$, then

$$\rho_{(12)(35)}(\widetilde{Y}) = \widetilde{Y} \ \Leftrightarrow\ y_1 = y_3 = 0.$$

this solution should be of the form $\widetilde{Y} = (0, y_2, 0)$. Hence, restricting the function $g$ to this subspace, we have the equation

$$g\Big|_{\text{Fix}(\Sigma_3)\times\mathbb{R}} = 0.$$

More precisely, we define $\tilde{g}_2(y_2, \epsilon) = g_2(0, y_2, 0, \epsilon)$ and the only equation to be solved is

$$\tilde{g}_2(y_2, \epsilon) = 0. \tag{4.28}$$

We can make use of the implicit function theorem in the function $\tilde{g}_2$ to solve equation (4.28) for $\epsilon = \epsilon(y_2)$. As a matter of fact, the function $\tilde{g}_2$ is smooth, $\tilde{g}_2(0) = 0$ and

$$\frac{\partial \tilde{g}_2}{\partial \epsilon}(0,0) = -\left[\frac{4(1-\nu^a)(m^*\nu^{1-a}+8)}{(4+m^*)a}\right]\sqrt{\frac{m^*}{4+m^*}(1-\nu^a)} > 0, \tag{4.29}$$

for all $a < -1$ (recall that $m^* > 0$ and $\nu = \frac{8}{3}$). It is worth noting that we can verify directly that $g_1(0, y_2, 0, \epsilon) = 0$ and $g_3(0, y_2, 0, \epsilon) = 0$.

For the solution $\widetilde{Y}$ whose symmetry is $\Sigma_1$ we have $y_1 = y_3$. So, restricting the function $g$ to Fix($\Sigma_1$) (whose dimension of Fix($\Sigma_1$) is equal to two) and setting $\tilde{g}_1(y_1, \epsilon) = g_1(y_1, y_2, y_1, \epsilon)$ and $\tilde{g}_2(y_1, \epsilon) = g_2(y_1, y_2, y_1, \epsilon)$, it is sufficient to solve the system of equations

$$\tilde{g}_1(y_1, y_2, \epsilon) = 0, \tag{4.30}$$

$$\tilde{g}_2(y_1, y_2, \epsilon) = 0. \tag{4.31}$$

Again, using that the derivative of $\tilde{g}_2$ with respect $\epsilon$ is different from zero, see (4.29), we can solve (4.31) for $\epsilon = \epsilon(y_1, y_2)$ for $(y_1, y_2)$ near of the origin $(0, 0)$. Substituting $\epsilon(y_1, y_2)$ in

(4.30) and noting that the solution found in (4.28) also admits symmetry of $\Sigma_1$, it follows that $(0, y_2, \epsilon(y_2))$ is a solution to (4.30) for $y_2$ near zero. Then, the function $\tilde{g}_1$ can be factored

$$\tilde{g}_1(y_1, y_2, \epsilon(y_1, y_2)) = y_1 f(y_1, y_2),$$

by theorem 4.7, where $f$ is an analytical function. The idea for exploring more solutions to equation (4.30) is by studying the zeros of the function $f$.

Finally, the solution whose symmetry is $\Sigma_2$ should be of the form $\widetilde{Y} = (y_1, y_2, -y_1)$. This case is similar to the previous one and follows the same analysis.

Our intention is not to study these particular cases. We will examine equation (4.27) without imposing any restrictions, and these cases will appear naturally, due of the symmetry of the equation.

The system of equations (4.27) is

$$g_1(y_1, y_2, y_3, \epsilon) = (G_2 - G_1 - G_3 + G_4)(y_1, y_2, y_3, \epsilon) = 0,$$
$$g_2(y_1, y_2, y_3, \epsilon) = (G3 + G4 - G_1 - G_2)(y_1, y_2, y_3, \epsilon) = 0,$$
$$g_3(y_1, y_2, y_3, \epsilon) = (G_1 - G_2 - G_3 + G_4)(y_1, y_2, y_3, \epsilon) = 0.$$

Since $\dfrac{\partial g_2}{\partial \epsilon}(0, 0) \neq 0$ by (4.29), we solve $g_2 = 0$ for $\epsilon = \epsilon(y_1, y_2, y_3)$ using the implicit function theorem. Substituting in the remaining equations, we have

$$\tilde{g}_1(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = 0,$$
$$\tilde{g}_3(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = 0,$$

which is equivalent to the system of equations

$$H_1(y_1, y_2, y_3) = (\tilde{g}_1 + \tilde{g}_3)(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = 0, \tag{4.32}$$
$$H_2(y_1, y_2, y_3) = (\tilde{g}_1 - \tilde{g}_3)(y_1, y_2, y_3, \epsilon(y_1, y_2, y_3)) = 0. \tag{4.33}$$

We check that if $y_1 = -y_3$, then $(-y_3, y_2, y_3)$ is a solution of (4.32). Moreover, if $y_1 = y_3$, then $(y_3, y_2, y_3)$ is a solution of (4.33). This follows directly from the equivariance of $g$, i.e.,

$$g_3(y_1, y_2, y_3) = g_1(y_3, y_2, y_1), \quad \text{(by element } (12) \text{ of group } \Gamma),$$
$$g_3(y_1, y_2, y_3) = -g_1(-y_3, y_2, -y_1), \quad \text{(by element } (35) \text{ of group } \Gamma).$$

Hence, we can write (4.32) and (4.33) in a factored form

$$H_1(y_1, y_2, y_3) = (y_1 + y_3) h_1(y_1, y_2, y_3) = 0, \tag{4.34}$$
$$H_2(y_1, y_2, y_3) = (y_1 - y_3) h_2(y_1, y_2, y_3) = 0, \tag{4.35}$$

where $h_1$ and $h_2$ are the analytic functions provided by theorem 4.7. Next, we will determine the analytical expressions of these functions $h_i$ using the analyticity of the function $G$.

### 4.3.1 Analytic Expressions

In this section, we use the analyticity of function $G$ to obtain analytic expressions for function $g$ and the solutions obtained implicitly, $\widetilde{W}(y_1, y_2, y_3, \epsilon)$ and $\epsilon(y_1, y_2, y_3)$.

Firstly, we make a translation $y_4 = y_4 + k$ in the expression of $G_i$, $i = 1, 2, 3, 4$, and we note that $G_1$ is invariant by $\rho_{(35)}$. So, its analytical expression is given by

$$\begin{aligned} G_1(Y, \epsilon) =& b_4 y_4 + b_5\epsilon + b_{11}(y_1^2 + y_2^2 + y_3^2) + b_{44}y_4^2 + b_{55}\epsilon^2 + b_{12}(y_1y_2 - y_1y_3 - y_2y_3) \\ &+ b_{11}(y_3 - y_2 - y_1)y_4 + b_{15}(y_1 - y_3)\epsilon + b_{25}y_2\epsilon + b_{45}y_4\epsilon + O(3). \end{aligned}$$

On the other hand, $G_3$ is invariant by $\rho_{(12)}$. Thus

$$\begin{aligned} G_3(Y, \epsilon) =& b_4 y_4 + c_5\epsilon + b_{11}(y_1^2 + y_2^2 + y_3^2) + b_{44}y_4^2 + c_{55}\epsilon^2 + b_{12}(y_1y_3 - y_1y_2 - y_2y_3) \\ &- b_{11}(y_1 - y_2 + y_3)y_4 + c_{15}(y_1 + y_3)\epsilon + c_{25}y_2\epsilon + c_{45}y_4\epsilon + O(3). \end{aligned}$$

Due to the $\Gamma$-equivariance of $G$, we must have

$$\begin{aligned} G_2(Y, \epsilon) =& b_4 y_4 + b_5\epsilon + b_{11}(y_1^2 + y_2^2 + y_3^2) + b_{44}y_4^2 + b_{55}\epsilon^2 + b_{12}(y_3y_2 - y_1y_3 - y_1y_2) \\ &+ b_{11}(y_1 - y_2 - y_3)y_4 + b_{15}(y_3 - y_1)\epsilon + b_{25}y_2\epsilon + b_{45}y_4\epsilon + O(3), \\ G_4(Y, \epsilon) =& b_4 y_4 + c_5\epsilon + b_{11}(y_1^2 + y_2^2 + y_3^2) + b_{44}y_4^2 + c_{55}\epsilon^2 + b_{12}(y_1y_3 + y_1y_2 + y_2y_3) \\ &+ b_{11}(y_1 + y_2 + y_3)y_4 - c_{15}(y_1 + y_3)\epsilon + c_{25}y_2\epsilon + c_{45}y_4\epsilon + O(3). \end{aligned}$$

The analytic expression of the function $\Psi$ is given by

$$\begin{aligned} \Psi(Y, \epsilon) =& 4b_4y_4 + 2(b_5 + c_5)\epsilon + 2b_{11}(y_1^2 + y_2^2 + y_3^2) + 2b_{44}y_4^2 + 2(b_{55} + c_{55})\epsilon^2 \\ &+ 2(b_{25} + c_{25})y_2\epsilon + 2(b_{45} + c_{45})y_4\epsilon + O(3). \end{aligned}$$

Since $\widetilde{W}$ is a solution to $\Psi = 0$ with $b_4 = -\dfrac{8k^2}{a}\left(3\nu^{1-a} + \dfrac{32}{m^*}\right)$, we substitute a generic Taylor expansion of $\widetilde{W}$ into the Taylor expression of $\Psi$, group the similar term and compare each term with the null series to obtain

$$\widetilde{W}(y_1, y_2, y_3, \epsilon) = -\left[w_5\epsilon + w_{11}(y_1^2 + y_2^2 + y_3^2) + w_{55}\epsilon^2 + w_{25}y_2\epsilon + O(3)\right],$$

where

$$\begin{aligned}
w_5 &= \frac{1}{2b_4}(b_5 + c_5),\\
w_{11} &= \frac{b_{11}}{b_4},\\
w_{55} &= \frac{1}{4b_4}\left[2(b_{55} + c_{55}) + \frac{b_{44}}{b_4}w_5 + 2b_4 w_5(b_{45} + c_{45})\right],\\
w_{25} &= \frac{1}{2b_4}(b_{25} + c_{25}).
\end{aligned}$$

Now, the expressions of the functions $g_i$ up to the third order are

$$\begin{aligned}
g_1(y_1, y_2, y_3, \epsilon) = {} & b_{12}y_2y_3 + (-b_{11}w_5 - b_{15} - c_{15})y_1\epsilon - w_{11}(b_{11} + b_{111})y_1^3\\
& - w_{11}(b_{11} + b_{112})y_1y_2^2 - (b_{11}w_{55} + 2b_{551} + 2c_{551} - b_{145}w_5 - c_{145}w_5)y_1\epsilon^2\\
& - (b_{11}w_{25} + b_{125} + c_{125})y_1y_2\epsilon + (b_{15} - c_{15})y_3\epsilon + (b_{112} - b_{11}w_{11})y_1y_3^2\\
& + (2b_{551} - 2c_{551} - b_{145}w_5 + c_{145})y_3\epsilon^2\\
& + (b_{125} - b_{123}w_5 - c_{125})y_2y_3\epsilon + O(4),
\end{aligned}$$

$$\begin{aligned}
g_2(y_1, y_2, y_3, \epsilon) = {} & (c_5 - b_5)\epsilon + \left[(c_{55} - b_{55}) - 2(c_{45} - b_{45})w_5\right]\epsilon^2 + b_{12}y_1y_3\\
& - (b_{11}w_5 + 2(b_{25} + c_{25}))y_2\epsilon - (b_{11}w_{11} + b_{112})y_2y_1^2 - b_{11}w_{11}y_2^3\\
& - (b_{11}w_{11} + b_{112})y_2y_3^2\\
& - \left[b_{11}w_{55} - b_{114}w_5^2 - (b_{45} - c_{45})w_{25} + 2(b_{552} + c_{552}) + 2(b_{245} + c_{245})\right] y_2\epsilon^2\\
& - \left[b_{11}w_{25} + 2(c_{45} - b_{45})w_{11} + 2(b_{225} - c_{225})\right] y_2^2\epsilon\\
& - 2\left[(c_{45} - b_{45})w_{11} + (b_{115} - c_{115})\right] y_1^2\epsilon + \left[(b_{45} - c_{45})w_{11} + c_{115} - b_{115}\right] y_3^2\epsilon\\
& + \left[(b_{45} - c_{45})w_{55} + (c_{555} - b_{555}) - (b_{554} - c_{554}) + (c_{445} - b_{445})\right]\epsilon^3\\
& + (b_{123}w_5 + c_{135} - b_{135})\, y_1y_3\epsilon + O(4),
\end{aligned}$$

$$\begin{aligned}
g_3(y_1, y_2, y_3, \epsilon) = {} & b_{12}y_1y_2 - (b_{11}w_5 + b_{15} + c_{15})y_3\epsilon - w_{11}(b_{11} + b_{111})y_3^3\\
& - w_{11}(b_{11} + b_{112})y_3y_2^2\\
& + (-b_{11}w_{55} - 2b_{551} - 2c_{551} + b_{145}w_5 + c_{145}w_5)y_3\epsilon^2\\
& - (b_{11}w_{25} + b_{125} - c_{125})y_2y_3\epsilon + (b_{15} - c_{15})y_1\epsilon + (b_{112} - b_{11}w_{11})y_3y_1^2\\
& + (2b_{551} - 2c_{551} - b_{145}w_5 + c_{145})y_1\epsilon^2 + (b_{125} - b_{123}w_5 - c_{125})y_1y_2\epsilon + O(4).
\end{aligned}$$

Recall $c_5 - b_5 = -\left[\frac{4(1-\nu^a)(m^*\nu^{1-a} + 8)}{(4 + m^*)a}\right]\sqrt{\frac{m^*}{4 + m^*}(1 - \nu^a)}$ is the derivative of $g_2$ with respect to $\epsilon$. Moreover, from the discussion in the last section, $\epsilon$ is an implicit solution of

$g_2 = 0$. Hence, we obtain the expression of $\epsilon$ in the same way as for $\widetilde{W}$.

$$\epsilon(y_1, y_2, y_3) = \left(\frac{1}{c_5 - b_5}\right)\left[-b_{12}y_1y_3 + b_{11}w_{11}y_2^3 + (b_{11}w_{11} + b_{112})(y_1^2 + y_3^2)\, y_2\right.$$
$$\left. - b_{12}\,(b_{11}w_5 + 2(b_{25} + c_{25}))y_1y_2y_3 + O(4)\right],$$

and replacing the above expression of $\epsilon$ in $g_1$ and $g_3$, it follows that

$$\begin{aligned}
H_1(y_1, y_2, y_3) = & \, b_{12}(y_1 + y_3)y_2 + \left(\frac{b_{12}}{c_5 - b_5}\right)(b_{11}w_5 + b_{15} + c_{15})(y_1 + y_3)y_1y_3 \\
& - w_{11}(b_{11} + b_{111})(y_1 + y_3)(y_1^2 - y_1y_3 + y_3^2) - w_{11}(b_{11} + b_{112})(y_1 + y_3)y_2^2 \\
& + \left[2\left(\frac{c_{15} - b_{15}}{c_5 - b_5}\right)b_{12} - b_{11}w_{11} + b_{112}\right](y_1 + y_3)y_1y_3 + O(4), \\
H_2(y_1, y_2, y_3) = & \, b_{12}(y_1 - y_3)y_2 + \left(\frac{b_{12}}{c_5 - b_5}\right)(b_{11}w_5 + b_{15} + c_{15})(y_1 - y_3)y_1y_3 \\
& - w_{11}(b_{11} + b_{111})(y_1 - y_3)(y_1^2 + y_1y_3 + y_3^2) - w_{11}(b_{11} + b_{112})(y_1 - y_3)y_2^2 \\
& - \left[2\left(\frac{c_{15} - b_{15}}{c_5 - b_5}\right)b_{12} - b_{11}w_{11} + b_{112}\right](y_1 - y_3)y_1y_3 + O(4).
\end{aligned}$$

We know that $y_1 = -y_3$ is a solution of $H_1 = 0$ and $y_1 = y_3$ is a solution of $H_2 = 0$, as explained in the previous section. Thus, we factor the term $(y_1 + y_3)$ from expansion of $H_1$ and $(y_1 - y_3)$ from $H_2$ and then, comparing with equations (4.34) and (4.35), we get

$$\begin{aligned}
h_1(y_1, y_2, y_3) = & \, b_{12}y_2 + \left[\left(\frac{b_{12}}{c_5 - b_5}\right)(b_{11}w_5 + 3c_{15} - b_{15}) + w_{11}b_{111} + b_{112}\right]y_1y_3 \qquad (4.36) \\
& - w_{11}\left[(b_{11} + b_{111})(y_1^2 + y_3^2) + (b_{11} + b_{112})y_2^2\right] + O(3), \\
h_2(y_1, y_2, y_3) = & \, b_{12}y_2 + \left[\left(\frac{b_{12}}{c_5 - b_5}\right)(b_{11}w_5 + 3b_{15} - c_{15}) - w_{11}(2b_{11} + b_{111}) + b_{112}\right]y_1y_3 \\
& - w_{11}\left[(b_{11} + b_{111})(y_1^2 + y_3^2) + (b_{11} + b_{112})y_2^2\right] + O(3). \qquad (4.37)
\end{aligned}$$

The derivative of $h_i$, $i = 1, 2$, with respect to $y_2$ is $b_{12} = \frac{6k}{a}\nu^{1-a} + \frac{64k^3}{a^2m^{*2}}(a-1)$, which is negative for all $a < -1$. Thus, it follows from the implicit function theorem that $h_i = 0, i = 1, 2$, can be solved for $y_2$ in terms of $(y_1, y_3)$ near the origin $(0, 0)$. Therefore, the expression of the implicit solution of $h_1 = 0$ is

$$y_2^1(y_1, y_3) = -\frac{1}{b_{12}}\left[\left(\frac{b_{12}}{c_5 - b_5}\Big(b_{11}w_5 + 3c_{15} - b_{15}\Big) + b_{112} + w_{11}b_{111}\right)y_1y_3\right.$$
$$\left. - w_{11}\,(b_{11} + b_{111})(y_1^2 + y_3^2) + O(3)\right].$$

On the other hand, the expression of the implicit solution of $h_2 = 0$ is

$$y_2^2(y_1, y_3) = -\frac{1}{b_{12}}\left[\left(\frac{b_{12}}{c_5 - b_5}\Big(b_{11}w_5 + 3b_{15} - c_{15}\Big) + b_{112} - w_{11}(b_{111} + 2b_{11})\right)y_1y_3\right.$$
$$\left. - w_{11}\,(b_{11} + b_{111})(y_1^2 + y_3^2) + O(3)\right].$$

**Remark 4.13.** We calculated all the coefficients $b$'s and $c$'s using the computer algebra system Maple.

### 4.3.2 Bifurcation Branches

Bifurcation branches arising from the centered tetrahedron are obtained by combining the solutions of equations (4.34) and (4.35). There are four possible cases.

Firstly, $y_1 = y_3 = 0$ is a solution of equations (4.34) and (4.35) for any $y_2$ near the origin.

The second solution is obtained substituting the solution $(-y_3, y_2, y_3)$ from equation (4.34) into (4.35). We have

$$(-2y_3)h_2(y_1, y_2) = 0, \tag{4.38}$$

whose solutions are $y_3 = 0$ and the zeros of the function $h_2$, namely $y_2^2(y_3)$.

Similarly, the third solution is obtained by substituting the solution $(y_3, y_2, y_3)$ from (4.35) into (4.34). We get

$$(2y_3)h_1(y_1, y_2) = 0, \tag{4.39}$$

whose solutions are $y_3 = 0$ and the zeros of $h_1$, namely $y_2^1(y_3)$.

Finally, we consider $y_1 \neq -y_3$ and substitute $y_2^1(y_1, y_3)$ in (4.34), assuming that $y_1 \neq y_3$, we obtain

$$\tilde{h}_2(y_1, y_2^1, y_3) = \left[\left(\frac{4b_{12}}{c_5 - b_5}\right)(b_{15} - c_{15}) - 2w_{11}(b_{111} + b_{11})\right] y_1 y_3 + O(3).$$

If we analyze the first term of $\tilde{h}_2$ numerically, we see that in the neighborhood of $a = -3/2$ this term is nonzero. Therefore, the solution of $\tilde{h}_2 = 0$ is $y_1 = y_3 = 0$, which results in $y_2 = 0$, i.e., the trivial solution.

To summarize the above discussion, we have obtained three families of solutions. In the family (I), the free variable is $y_2$, while in the families (II) and (III), the free variable is $y_3$. This means that we need to use different bifurcation parameters to represent distinct families, for different free variables lead to different relationships with the parameter $\epsilon$. Hence,

$$\text{Family (I):} \begin{cases} y_1 &= 0, \\ y_2 &=: \delta, \\ y_3 &= 0, \\ y_4 &= k - \dfrac{b_{11}}{b_4}\delta^2 + O(3), \\ \epsilon &= -\left[\dfrac{b_{11}^2}{b_4(c_5 - b_5)}\right]\delta^3 + O(4). \end{cases}$$

The term of third order in the above expansion for $\epsilon$ is negative, since $b_4 > 0$ and $c_5 - b_5 > 0$, for $a < -1$. The other two families one

$$\text{Family (II):} \begin{cases} y_1 &= -\mu, \\ y_2 &= \left\{\left[\dfrac{b_{112} - (3b_{111} + 4b_{11})w_{11}}{b_{12}}\right] - \left[\dfrac{b_{11}w_5 + 3b_{15} - c_{15}}{c_5 - b_5}\right]\right\}\mu^2 + O(3), \\ y_3 &=: \mu, \\ y_4 &= k - \left[\left(\dfrac{w_5 b_{12}}{c_5 - b_5}\right) + 2w_{11}\right]\mu^2 + O(3), \\ \epsilon &= \left(\dfrac{b_{12}}{c_5 - b_5}\right)\mu^2 + O(3). \end{cases}$$

$$\text{Family (III):} \begin{cases} y_1 &= \mu, \\ y_2 &= \left[\left(\dfrac{b_{112} + w_{11}b_{11}}{b_{12}}\right) - \left(\dfrac{b_{11}w_5 + 3c_{15} - b_{15}}{c_5 - b_5}\right)\right]\mu^2 + O(3), \\ y_3 &=: \mu, \\ y_4 &= k - \left[2w_{11} - \left(\dfrac{b_{12}w_5}{c_5 - b_5}\right)\right]\mu^2 + O(3), \\ \epsilon &= -\left(\dfrac{b_{12}}{c_5 - b_5}\right)\mu^2 + O(3). \end{cases}$$

In family (II) we have $\epsilon > 0$, since $b_{12} > 0$ and $c_5 - b_5 > 0$. Furthermore, for each value of $\epsilon$, we have $\epsilon(\mu) = \epsilon(-\mu)$, but $Y(\mu) \neq Y(-\mu)$. In other words, for each value of $\epsilon$, we have two central configurations shooting off the degenerate centered tetrahedron, when $\mu \to 0$, as for family (III), but in this case $\epsilon < 0$.

Returning to the variables $x_i$, we have

$$
\text{Family (I)}: \begin{cases} x_1 &= k - \delta - \dfrac{b_{11}}{b_4}\delta^2 + O(3), \\ x_2 &= k - \delta - \dfrac{b_{11}}{b_4}\delta^2 + O(3), \\ x_3 &= k + \delta - \dfrac{b_{11}}{b_4}\delta^2 + O(3), \\ x_4 &= -4k + \dfrac{4b_{11}}{b_4}\delta^2 + O(3), \\ x_5 &= k + \delta - \dfrac{b_{11}}{b_4}\delta^2 + O(3). \end{cases}
$$

$$
\text{Family (II)}: \begin{cases} x_1 &= k + 2\mu - (\alpha_2 + \beta_2)\mu^2 + O(3), \\ x_2 &= k - 2\mu - (\alpha_2 + \beta_2)\mu^2 + O(3), \\ x_3 &= k + (\alpha_2 - \beta_2)\mu^2 + O(3), \\ x_4 &= -4k + 4\beta_2\mu^2 + O(3), \\ x_5 &= k + (\alpha_2 - \beta_2)\mu^2 + O(3). \end{cases}
$$

$$
\text{Family (III)}: \begin{cases} x_1 &= k - (\alpha_3 + \beta_3)\mu^2 + O(3), \\ x_2 &= k - (\alpha_3 + \beta_3)\mu^2 + O(3), \\ x_3 &= k - 2\mu + (\alpha_3 - \beta_3)\mu^2 + O(3), \\ x_4 &= -4k + 4\beta_3\mu^2 + O(3), \\ x_5 &= k + 2\mu + (\alpha_3 - \beta_3)\mu^2 + O(3), \end{cases}
$$

where

$$
\alpha_2 = \frac{b_{112} - (3b_{111} + 4b_{11})w_{11}}{b_{12}} - \frac{b_{11}w_5 + 3b_{15} - c_{15}}{c_5 - b_5},\ \beta_2 = \frac{w_5 b_{12}}{c_5 - b_5} + 2w_{11},
$$
$$
\alpha_3 = \frac{b_{112} + w_{11}b_{11}}{b_{12}} - \frac{b_{11}w_5 + 3c_{15} - b_{15}}{c_5 - b_5},\ \beta_3 = 2w_{11} - \frac{b_{12}w_5}{c_5 - b_5}.
$$

Finally, the Taylor series of $s_{ij}$ of the first family around $\delta = 0$ up to second order

$$
s^I_{ij}(\delta) = s^0_{ij} + v^I_{ij}\delta + \alpha^I_{ij}\delta^2 + O(\delta^3), \quad 1 \le i, j \le 5, \tag{4.40}
$$

where $v^I_{ij} = \left.\dfrac{ds_{ij}}{d\delta}\right|_{\delta=0}$, and $\alpha^I_{ij} = \dfrac{1}{2}\left.\dfrac{d^2 s_{ij}}{d\delta^2}\right|_{\delta=0}$.

To understand how families (I), (II), and (III) behave, we will analyze the signs of linear terms of (4.40) to study the growth of $s_{ij}$.

The linear terms of the family (I) are

$$
\begin{aligned}
v_{12}^I &= \frac{2k}{a}\nu^{1-a} < 0,\\
v_{13}^I &= v_{23}^I = v_{15}^I = v_{25}^I = 0,\\
v_{35}^I &= -\frac{2k}{a}\nu^{1-a} > 0,\\
v_{14}^I &= v_{24}^I = -\frac{4k}{am^*} > 0,\\
v_{34}^I &= v_{45}^I = \frac{4k}{am^*} < 0, \quad \forall\, a \in (-\infty, -1).
\end{aligned}
$$

The behavior of the family (I) is shown in figure $8$. The symmetry is given by the intersection of two planes. One of the plane contains the midpoint of the segment $q_3q_5$ and the segment $q_1q_2$, and the other contains the midpoint of the segment $q_1q_2$ and the segment $q_3q_5$. The mass $m^*$ must be at the intersection of these two planes. If $\delta > 0$, then the mass $m^*$ is closer to the segment $q_3q_5$. Conversely, the mass $m^*$ is closer to the segment $q_1q_2$.

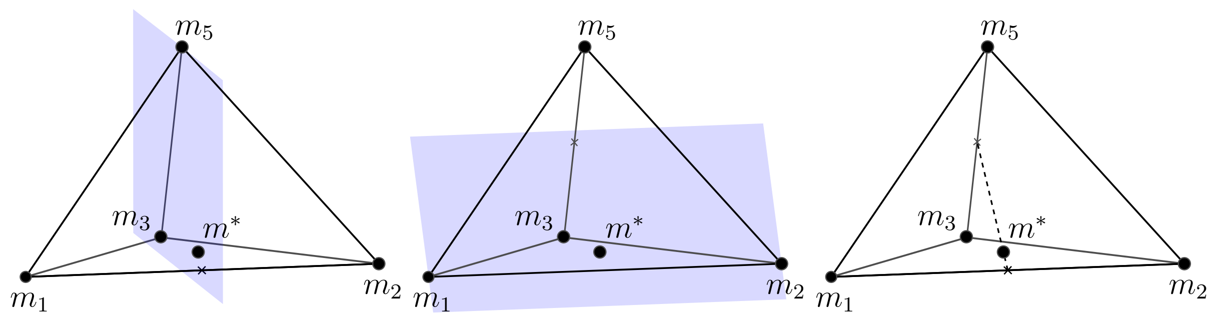


**Figure 8.** Bifurcation emerging from centered regular tetrahedron, which occurs when $\epsilon > 0$ ($\delta < 0$). The intersection of the two planes gives us the symmetry.

The Taylor series of $s_{ij}$ of the second and third families around $\mu = 0$ up to second order is

$$
s_{ij}^{II}(\mu) = s_{ij}^0 + v_{ij}^{II}\mu + \alpha_{ij}^{II}\mu^2 + O(\mu^3), \tag{4.41}
$$

where $v_{ij}^{II} = \left.\dfrac{ds_{ij}^{II}}{d\mu}\right|_{\mu=0}$, and $\alpha_{ij}^{II} = \left.\dfrac{1}{2}\dfrac{d^2 s_{ij}^{II}}{d\mu^2}\right|_{\mu=0}$.

The linear terms of the family (II) are

$$\begin{aligned}
v_{12}^{II} &= v_{34}^{II} = v_{45}^{II} = v_{35}^{II} = 0,\\
v_{13}^{II} &= v_{15}^{II} = -\frac{2k}{a}\nu^{1-a},\\
v_{14}^{II} &= \frac{8k}{am^*} < 0,\\
v_{23}^{II} &= v_{25}^{II} = \frac{2k}{a}\nu^{1-a} < 0,\\
v_{24}^{II} &= -\frac{8k}{am^*} > 0, \ \ \forall\, a \in (-\infty, -1).
\end{aligned}$$

In this case, the plane of symmetry intersects the segment $q_3q_5$ orthogonally and contains the segment $q_1q_2$. If $\mu > 0$, then the mass $m^*$ lies on this plane, but closer to the segment $q_1q_2$. Furthermore, $m^*$ is closer to $m_1$ than to $m_2$, meaning that it lies outside the plane of symmetry that intersects $q_1q_2$ at the midpoint. If $\mu < 0$, the opposite happens with the mass $m^*$.

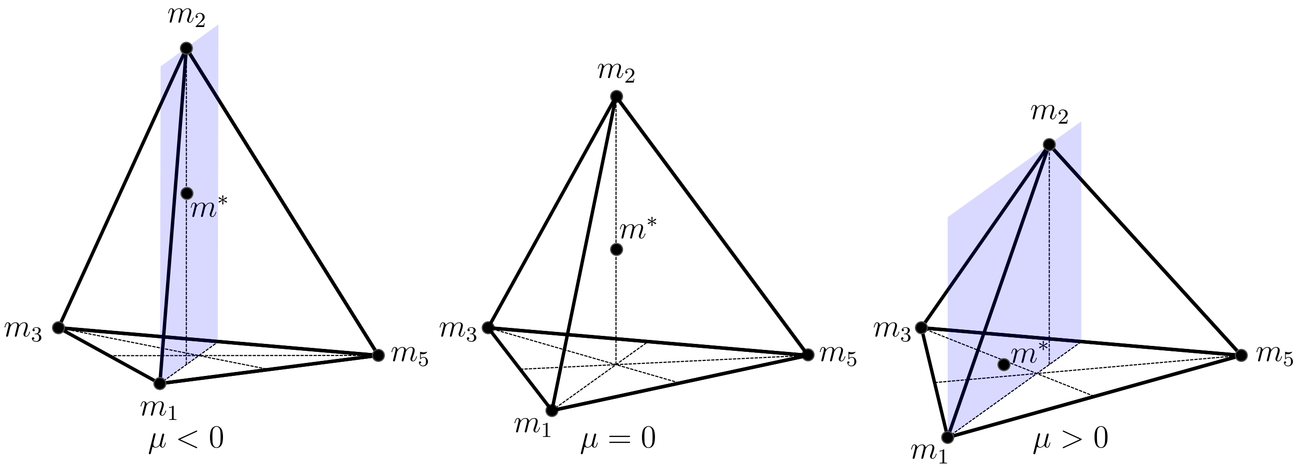


**Figure** 9: Bifurcation of family (II), which occurs when $\epsilon > 0$. The plane of symmetry intersects the midpoint of the segment $q_3q_5$.

Finally, the linear terms of the family (III) are

$$\begin{aligned}
v_{12}^{III} &= v_{14}^{III} = v_{24}^{III} = v_{35}^{III} = 0,\\
v_{13}^{III} &= v_{23}^{III} = \frac{2k}{a}\nu^{1-a} < 0,\\
v_{15}^{III} &= v_{25}^{III} = -\frac{2k}{a}\nu^{1-a} > 0,\\
v_{34}^{III} &= -\frac{8k}{am^*} > 0,\\
v_{45}^{III} &= \frac{8k}{am^*} < 0, \ \ \forall\, a \in (-\infty, -1).
\end{aligned}$$

The plane of symmetry intersects the segment $q_1q_2$ orthogonally and contains the segment $q_3q_5$. If $\mu > 0$, then the mass $m^*$ lies on this plane, but it is closer to $m_5$ than to $m_3$, meaning that it lies outside the plane of symmetry that intersects $q_3q_5$ at the midpoint. If $\mu < 0$, the opposite happens with the mass $m^*$.

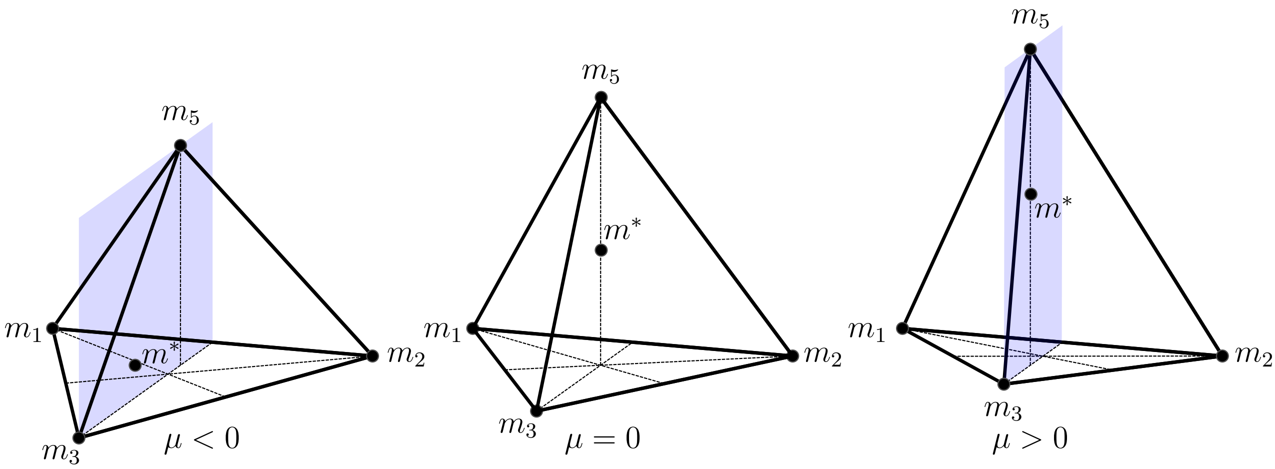


**Figure** 10: Bifurcation of family (III), which occurs when $\epsilon < 0$. The plane of symmetry intersects the midpoint of the segment $q_1q_2$.

**Theorem 4.14.** *Let $q(1, 1, 1, m, 1)$ be a central configuration of five bodies in the space forming a centered regular tetrahedron. Denote by $m^* > 0$ the value of the central mass, which depends on the exponent a<-1, such that the configuration is degenerate. If $\epsilon > 0$, for exactly two equal masses at the vertices of the tetrahedron either in the interval $(1, 1 + \epsilon)$ or $(1 - \epsilon, 1)$, then in both cases there exist two families of central configurations with plane-type symmetry which bifurcate from the degenerate solution $q(1, 1, 1, m^*, 1)$. Furthermore, there is only one family of central configurations with plane-type symmetry that bifurcates from the degenerate solution if exactly two equal masses are in the interval $(1 - \epsilon, 1 + \epsilon)$. In all cases, there are no symmetrical central configurations bifurcating from $q(1, 1, 1, m^*, 1)$.*

**Remark 4.15.** According to (GOLUBITSKY; SCHAEFFER, 1988), when a solution to a bifurcation problem has less symmetry than the equations that describe such problem, it is said that there has been a symmetry breaking. In both problems studied in this chapter, the symmetry-breaking phenomenon has been observed.

# 5 CONCLUSION AND PERSPECTIVES

In this thesis, we set out to find new central configurations from two degenerate central configurations of the four-body and five-body problems. We started by analyzing the bifurcations of a centered triangular configuration. We obtained three families of central configurations that are valid for any value of the exponent of the potential function in the interval $(-\infty, -1)$. However, we were only able to fully analyze the Newtonian case. We will continue this analysis in future projects in order to consider other particular cases.

For the five-body problem, we studied the bifurcation of a degenerate centered regular tetrahedron. We approached this degenerate central configuration following two different paths. Considering three equal bifurcation parameters, we obtained four families of central configurations. For two equal bifurcation parameters, there are three families of central configurations. We understand that for these cases the analysis was carried out completely and the solutions found are valid for any potential function with exponent $a < -1$.

The bifurcation analysis we used was heavily based on the symmetry of the problems. This indicates that we need to increasingly understand the role of symmetry in solving bifurcation problems.

Our expectation for future work is to study the bifurcations of the centered regular tetrahedron considering approaching the degenerate configuration through different paths than those already considered. At first, considering two different parameters and thus reducing the symmetry group of the problem seems like a natural way forward.

Another problem to be studied later are the bifurcations of a configuration that is not Dziobek. More precisely, a configuration of five bodies in the plane in the form of a square with equal masses at the vertices and an arbitrary mass in the center.

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# APPENDIX A – COMPARISON OF THE TERMS OF THE EXPANSION OF THE SQUARE OF THE DISTANCE

To facilitate a comparison the terms of the $s_{ij}$ presented in section 4.2.2 with the ones studied in (SANTOS et al., 2017), we display in this appendix the second-order terms of the family (I) and the second and third-order terms of family (II) found in (SANTOS et al., 2017).

The second order terms of family (I) are

$$\begin{aligned}
\alpha_{12} &= \left(\frac{4}{a^2}\nu^{1-2a}k^2\right)(1-a) + \frac{\nu^{1-a}}{a}\left\{-2+16\left[\frac{m^*+4\nu^a}{(4+m^*)^2}\right]\frac{b_{22}}{b_5}\right\},\\
\alpha_{15} &= \left(\frac{4}{a^2}\nu^{1-2a}k^2\right)(1-a)\\
&\quad + \frac{\nu^{1-a}}{a}\left\{6+16\left[\frac{m^*+4\rho^a}{(4+m^*)^2}-k^2\right]\frac{b_{22}}{b_5}+2k\left[\frac{-8b_{22}c_5+3b_5(c_{11}+c_{12})}{fb_5}\right]\right\},\\
\alpha_{14} &= \frac{16}{a^2(m^*)^2}k^2(1-a)\\
&\quad + \frac{1}{a}\left\{16\left[\frac{m^*+4\nu^a}{(4+m^*)^2}\right]\frac{b_{22}}{b_5}-2k\left[\frac{-8b_{22}c_5+3b_5(c_{11}+c_{12})}{fb_5m^*}\right]\right\},\\
\alpha_{45} &= \left(\frac{144}{a^2(m^*)^2}\right)k^2(1-a)\\
&\quad + \frac{1}{a}\left\{16\left[\frac{m^*+4\nu^a}{(4+m^*)^2}+\frac{4k^2}{m^*}\right]\frac{b_{22}}{b_5}-10k\left[\frac{8b_{22}c_5+3b_5(c_{11}+c_{12})}{fb_5m^*}\right]\right\}.
\end{aligned}$$

Since the second order terms of family (II) agree with whose was showed in section 4.2.2, we will show the third order terms

$$\begin{aligned}
w_{12} &= \left(\frac{8k^3\nu^{(1-3a)}}{a^3}\right)(1-3a+2a^2) - \left(\frac{12k\nu^{(1-2a)}}{a^2}\right)(1-a)\\
&\quad - \left(\frac{6\nu^{(1-a)}}{a}\right)\left(\frac{m^*+4\nu^a}{(4+m^*)^2}\right)d,\\
w_{13} = w_{23} &= -6\left(\frac{\nu^{(1-a)}}{a}\right)\left[\frac{(m^*+4\nu^a)}{(4+m^*)^2}\right]d,\\
w_{15} = w_{25} &= -6\left(\frac{\nu^{(1-a)}}{a}\right)\left\{\left[\frac{(m^*+4\nu^a)}{(4+m^*)^2}-k^2\right]d+\frac{(3c_{11}-c_{12})}{f}\right\}\\
w_{14} = w_{24} &= -\left(\frac{64k^3}{a^3m^{*3}}\right)(1-3a+2a^2) + \left(\frac{24k^2}{a^2m^{*2}f}\right)(3c_{11}-c_{12})(1-a)\\
&\quad - \frac{6}{a}\left\{\left[\frac{(m^*+4\nu^a)}{(4+m^*)^2}\right]d - \frac{3c_{11}-c_{12}}{fm^*}\right\},\\
w_{34} &= -\left(\frac{64k^3}{a^3m^{*3}}\right)(1-3a+2a^2) - \left(\frac{24k^2}{a^2m^{*2}f}\right)(3c_{11}-c_{12})(1-a)\\
&\quad - \frac{6}{a}\left\{\left[\frac{(m^*+4\nu^a)}{(4+m^{*2}}\right]d + \frac{3c_{11}-c_{12}}{fm^*}\right\},
\end{aligned}$$

$$
\begin{aligned}
w_{45} = &\left(\frac{64k^3}{a^3 m^{*3}}\right)(1 - 3a + 2a^2) - \left(\frac{120k^2}{a^2 m^{*2} f}\right)(3c_{11} - c_{12})(1 - a) \\
&- \frac{6}{a}\left\{\left[\frac{(m^* + 4\nu^a)}{(4 + m^*)^2}\right] d + \frac{3c_{11} - c_{12}}{f m^*}\right\}, \\
w_{35} = &\left(\frac{8k^3 \nu^{1-3a}}{a^3}\right)(-1 + 3a - 2a^2) - \left(\frac{12k\nu^{1-2a}}{a^2}\right)\left[\frac{k(3c_{11} - c_{12})}{f} - 1\right](1 - a) \\
&- \left(\frac{6\nu^{(1-a)}}{a}\right)\left\{\left[\frac{(m^* + 4\nu^a)}{(4 + m^*)^2} - k^2\right] d - \left(\frac{3c_{11} - c_{12}}{f}\right)\right\},
\end{aligned}
$$

where $d$ has an expression such as the one at the bottom of the page 58.