UNIVERSIDADE FEDERAL DE PERNAMBUCO
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA DE PRODUÇÃO

GROWTH MODELS INCORPORATING TECHNOLOGY AND A NEW POPULATION DYNAMICS EQUATION

DISSERTAÇÃO SUBMETIDA À UFPE

PARA OBTENÇÃO DO GRAU DE MESTRE

POR

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DE DEFESA DE DISSERTAÇÃO DE MESTRADO ACADÊMICO DE

LUÍS HENRIQUE DE SANTANA

"GROWTH MODELS INCORPORATING TECHNOLOGY AND A NEW POPULATION DYNAMICS EQUATION"

ÁREA DE CONCENTRAÇÃO: PESQUISA OPERACIONAL

A comissão examinadora, composta pelos professores abaixo, sob a presidência do primeiro, considera o candidato LUÍS HENRIQUE DE SANTANA **APROVADO.**

Recife, 28 de dezembro de 2006.

Campello

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Prof. ALEXANDRE STAMFORD DA SILVA, Doutor (UFPE)

"All theory depends on assumptions which are not quite true. That is what makes it theory."

Robert M. Solow (Solow, 1956)

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L. H. de Santana

Abstract

It is presented a new model of population growth and, in particular, the classical Sollow's economic growth model is analyzed when employed this new model for population growth. Furthermore, it is introduced the technology in Campello's macroeconomic model of optimal economic growth focusing on the energy sector. For this tasks, dynamic system tools – in special, the maximum principle – are employed. The results of the growth models with those modifications in the population dynamics maintain the classical results of the Sollow's model in the sense that they assert the existence of equilibrium points. The models of economic growth focusing of energy resources yield new results concerning price of energy sources, the dynamic of the shadow prices of technology and population, etc.

Keywords: Economic Growth, Population Growth, Technology, Energy Sources.

Resumo

Apresenta-se um novo modelo de crescimento populacional e, em particular, o modelo clássico de crescimento econômico de Sollow é analisado quando empregado este novo modelo. Além disto, introduz-se a tecnologia no modelo macroeconômico de crescimento econômico de Campello focando no setor energético. Para tanto, ferramentas de sistemas dinâmicos – em especial, o princípio do máximo – são aplicadas. Os resultados dos modelos de crescimento com a modificação na dinâmica populacional mantém os resultados clássicos do modelo de Sollow no sentido em que afirmam a existência de pontos de equilíbrio. Os modelos de crescimento focando no setor energético produziram novos resultados com respeito ao preço das fontes energéticas, a dinâmica do preço sombra da tecnologia e da população, etc.

Palavras-chave: Crescimento Econômico, Crescimento Populacional, Tecnologia, Fontes Energéticas.

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0.1 Objectives

0.1.1 General Objectives

1. To present a new model of population growth;
2. To introduce the technology in Campello's macroeconomic model[1] of optimal economic growth focusing on the energy sector.

0.1.2 Specific Objectives

1. to analyze the classical Sollow's economic growth model (Solow, 1956) when employed this new model for population growth;
2. To obtain results that give ideas for policies for the entire energy market.

0.2 Technology, economics, politics

> *"...eu pensava que os problemas tecnológicos pudessem ser resolvidos com engenharia apenas."*[2]

F. M. Campello de Souza.

In fact, it was a wrong idea from the young Campello around 1966. Indeed, technology, economics and politics are undoubtedly intertwined entities.

Technology projects arise as answer for provided needs of a society. However, why one chooses this or that technological approach? It is definitely not a simple questions and many aspects – such as, for instance, cultural features – may influence the decision making. Although the author realize the importance of this discussion, this work is not

[1]This model is presented in "Introdução do Aquecimento Solar na Matriz Energética", Recife, 1997, CNPq Resource Project, not published.

[2]"...I thought that the technological problems could be solved by using just engineering."

intended to go deeper in that subject. As discussed in (Strandberg, 2002), technology is, at first, a political decision – which is not made necessarily by politicians *per se* – among all available proposed projects. In contrast, one can think that this decision should be made by using decision theory!

Nevertheless, after the choice, it is certain that one must pay for the selected project, because

> *"Ideas can emerge from an individual, but capital is needed to bring the idea to fruition and production."*
>
> **Dr. Gene Strandberg** (Strandberg, 2002)

It is also important to point out that some researches do not result in an immediate application what characterizes technological investment as cost-intensive. According Strandberg (Strandberg, 2002), venture capital is essential to finance technology research and development what is not the Brazilian case. Roughly speaking, there is no venture capital in Brazilian economy. Furthermore, tax abatements, roads, and other incentives by local and state authorities play a fundamental role for the technological sector development.

Today, universities, large companies and governmental facilities are the entities that produce much of the new technology in the United States of America. Governmental and private company funds usually finance university research on account of the universities' highly competent human capital and available laboratories. Emerging or established technology companies also have used the human capital of American universities with the advantages of no costs in insurance and vacation time, equipping laboratories, etc. On other hand, the universities also gain both from indirect funds and prestige. In fact, all parts involved seem to win by reason of this cooperation.

It is obvious, since the humans are the "atoms" of any society, that most important capital investment is the human capital one.

> *"Part of a nation's wealth is in dollars, but more of its wealth is in human knowledge and application."*
>
> **Dr. Gene Strandberg** (Strandberg, 2002)

Brazil spend a lot of money with the universities, but nevertheless it is common that some of the great Brazilian's minds leave the country and never return. Many people have been worried with our natural resources explored by other nations over the history, while an important Brazilian's loss have been in human capital over the recent years.

Human capital investment means both education in general and specifical knowledge – such as engineering, chemistry, physic, philosophy, mathematics, economics, etc. The principle is simple: to produce a collective improvement, one must enhance the individuals and then these better individuals together will compound a much better society. As said by Aristotle

> *"the whole is more than the sum of its parts"*

Aristotle

In particular, human capital evolution have provided clear effects in the technological sector in both developed countries and the developing countries such as China, Korea, Singapore, etc.

Brazil must as soon as possible improve his politics concerning the research and development. One can perceive that the most advanced countries are expanding politically, economically, technologically and hence leaving the developing countries more dependent upon their technologies. As result, Brazil remains without global effectiveness and changed from Portugal's colony to be colony of this technologically advanced countries.

In the scientific view, the inclusion of technology into some macroeconomic models can explain why the capital *per* worker, or yield *per* worker can rise over time. Therefore, although there is no consensus concerning the way to insert technology in economic models, it is fundamental either to apply or suggest some reasonable manner to model the technological role.

0.3 Energy and economic growth

According Stamford da Silva (Stamford da Silva, 1999), the industrial revolution brought new habits concerning the consumption behavior of modern societies, in particular, the energy consumption behavior. Today, any economy requires energy for several ends such as to manufacture goods, provide transportation, run electronic devices and others. Since the means of production changed from animal power to steam power and after to the internal combustion engines and electricity, i.e., the manual work gave place to the machine, one cannot think about economic growth without take account the energy capacities because as well as a man needs food, the machine needs energy. Indeed, the way in which a nation manage its energy sector – in other words, the way how a nation feeds its machine – is a important feature of its economy.

Stamford da Silva also asserts that it is common to associate economic growth with increase of energy consumption. It is important to note that an increase on the energy consumption do not necessarily implies on development, in special for the developed countries. Some aspects must be considered to understand the interaction between economic growth and energy consumption (Fideles da Silva, 1997), namely:

- novel technological arrangements can allow more efficiency in the energy system, in special in the final consumption.
- politics for conservation of energy, in special, the electric energy provide reduction in consumption and attenuate the need of future amplification of energy production.
- Furthermore, the developed countries can be benefited from transporting its energy-intensive industrial park to sub-developed countries, etc.

According Edmonds and Reilly (Edmonds & Reilly, 1985), the main influences in the use of energy are:

- demography;
- productivity of work;
- yield;
- productivity of work;
- energy productivity;
- uncertainty.

"A incerteza é a marca indelével do universo"[3]

F. M. Campello de Souza.

For some phenomena, the uncertainty can be neglected without significant damage, however it is not the case for the use of energy. The demography influences the energy demand as well by the number of individuals that consomme energy as by the individuals that use energy producing goods and for its transport. In such way, the residential and commercial sector, the transport and industrial sector are influenced by the population growth. The work productivity is defined as

$$WP = \frac{GDP}{L}$$

where GDP is the gross domestic product and L is the labor force, i.e, the gross domestic product *per* worker. It is an index of yield level. Therefore, high WP rate are often associated to high growth rate of energy use. Empiric studies have shown that the yield influences in a non-proportional way due the heterogeneity of the economics. The yield-energy elasticity is employed to measure the change *per* cent of the of the used percentage of yield growth. The energy price influences the level and composition, and this influence is measured by using the elasticity of the energy price. Since an energy price augment lead to reduction in energy use, this elasticity should be negative. The term energy productivity refers to the production level obtained *per* energy employed. Technological and managemental changes may provide variations in this measure.

The rate energy consumption by gross domestic product which is called energy intensity (EI),

$$EI = \frac{GDP}{E}$$

where E denotes energy consumption, is the yardstick for the interaction between economic growth and energy consumption. The graphic 1 shows the behavior of the energy intensities over time. Notice that each energy intensity grows until reach a maximum point and then decline. The energy intensities of the developed countries are already decreasing probably due the politics regarding energy conservation and the technological

[3] The uncertainty is the indestructible feature of the universe

advances, while the developing countries' IE's are still growing even. However, the technological transfers among the globalized world might reduce the way for the maximum point for the developing countries.

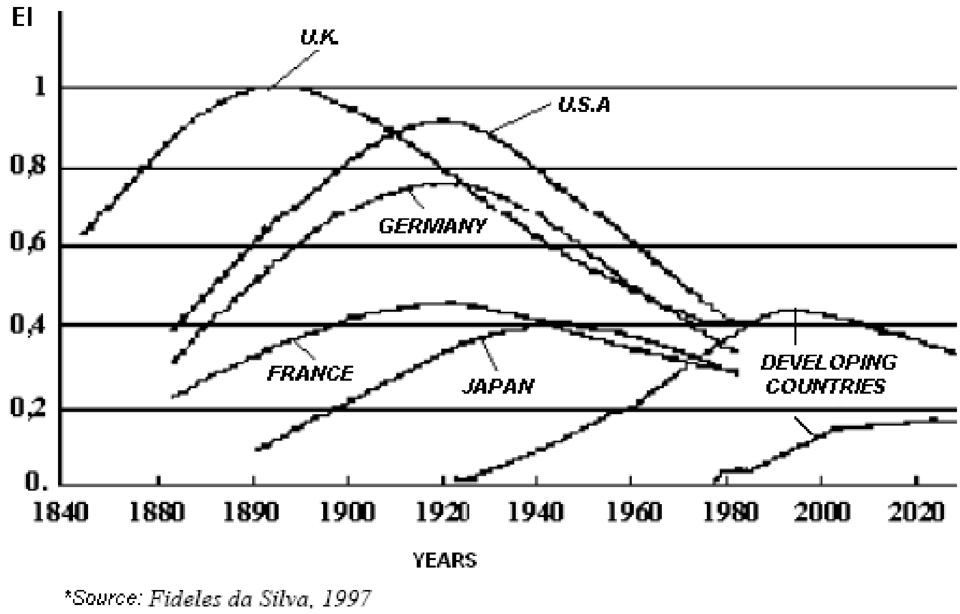

Figure 1: Energy Intensity

The primary sources and the way how they are employed constitutes compound an energy system. The choice of an energy system, as any technological one, is naturally complex. In Brazil, after the oil crisis, the hydroelectric generation was intensified to become the main source of energy due to the adequate nature conditions.

The energetic balance is made by the Balanço Energético Nacional (BEN) from the Ministério das Minas e Energia. In Brazil, over the years, the hydroelectric power remains as the most important primary energy source in both production and consumption while the oil remain as the source among the non-renewable ones. In 2006, Brazil still produces and consumes more electricity than oil, and the non-conventional sources have contributed few in the production and consumption of energy yet[4].

One must be worried because the hydroelectric energy is bounded since hydro-electric plants cannot either be construct in anywhere and produce any level of power. Therefore, as asserted by Endress and Roumasset (Endress & Roumasset, 1994), when the limit be reached, a substitute should be employed to keep the growth.

[4]http://ben.epe.gov.br/BEN2006/

0.4 Paradigm

"All models are wrong, but some are useful."

George Box

According Campello de Souza (Campello de Souza, 2002), in general, a system is a connection of united objects through either an interaction or interdependence. For a system engineer, a system is a device, a process or a scheme, that can behave in a predictable manner. The physic hypothesis and analogies which represent the "real world", whatever it means, must allow an immediate view or lead suitable structures and schemes for investigations.

That structures and schemes which either facilitate the discussions and logic constructions or allow that experiment are made to determine more accurately the nature of phenomenon are said to be models. Therefore, a model is an construct that corresponds to an investigated object or some feature of it; it represents essential characteristics of a process or system and can provide information about the system in an useful way. The models discussed in the this research will follow the Pierce's scientific thought presented in (Pierce, 2000).

0.5 Organization

The next chapter shall treat some aspects concerning optimal economic growth. The chapter 1 contains a brief review about the classical Sollow's model and also presents a modification in that model regarding the population dynamics. The chapter 2 presents economic growth models focusing on energy resources. The models are variations of the work Stamford da Silva (Stamford da Silva, 1999) which is, by its side, based on Campello's model. At last, the chapter three contains the conclusions and suggestions.

1 Optimal Economic Growth

1.1 Introduction

Economic growth, the rate at which national income is growing, is the most fundamental indicator of an economy's health. It is measured, usually, by the annual percentage rate change in a nation's gross domestic product (GDP), which is simply the economy's total income accruing from output; the market value of all goods and services produced within an economic area over a given period of time. Other measures of economic growth include gross national product (GNP), which measures the total output of a country's citizens regardless of where they are living and working. In order to give a more easily comparable picture of a countries' economic health, one uses the per capita GDP, usually measured in dollars (the final sales of goods and services in a country per person, adjusted for inflation). However GDP is the preferred measure for growth, as it indicates the amount of economic activity within a nation's borders.

Economic growth is to be understood as a sustained rise in a nation's production of goods and services. It results from investments in human and physical capital, research and development, technological change, and improved institutional arrangements and incentives. When individuals, regions, and nations specialize in what they can produce at the lowest cost and then trade with others, both production and consumption increase.

Among the signs of economic growth, which largely affects the material well-being of a country, one can mention, for instance, eager buyers crowding checkout lanes, cranes erecting buildings or help-wanted signs filling store windows. When the economy expands, jobs are created and goods and services to meet people's needs increase. It is important, so, to analyze and understand the causes of growth and what countries can do to maintain or enhance it.

Some inherent traits are responsible for some differences in economic growth. It is well known that throughout history, some economies have expanded faster than others. Amongst such inherent factors are climate and geography. People living near navigation routes or in temperate climates, at times, have fared better than people living far away from coastlines or in frigid climates. One can think also that culture plays a role in growth.

Notwithstanding these inherent factors, government and central bank policies also play a role. Policies affecting access to technology, sound money and banking practices, and prudent taxing and spending can improve or stifle economic growth.

In general terms, the expression "economic development" is thought of as an overall improvement in the quality of life in a given country. This includes, typically, a better health care, a cleaner environment and more freedom in terms of choosing work and leisure activities. In a period of economic growth, the overall wealth of a country increases, as do the variety and abundance of goods and services.

The phenomenon is complex, but some factors that influence economic growth have been identified. These include government, international trade and finance, technology and investment, political, social and geographical conditions, and money and banking.

The terms "sustainable" and "sustainability" appeared in the 1980s and made people increasingly aware of the growing global problems of overpopulation, drought, famine, and environmental degradation that had been the subject of Limits to Growth in the early 1970s, (Meadows *et al.*, 1972). The enormous problems and suffering that are being experienced with growing intensity every day throughout the underdeveloped world became more evident. A new era of economic growth started — growth that is forceful and at the same time socially and environmentally sustainable. Forceful, here, is in the sense of rapid, and there appears to be a conflict between forceful and sustainable. That is not the case. Sustainable development can be pursued if population size and growth are in harmony with the changing productive potential of the ecosystem.

Economies grow because there are more people, more machines, or more natural resources. They also grow because they find better ways to put things together, i.e., technology. The technological level, in general, advances over time. The man improves upon or replace the known technologies by using research and development activities and for this aim he employs all kinds of knowledge, namely, physics, chemistry, engineering, mathematical, etc.

One may mention lasers, holography, virtual reality, genomics, telecommunications, telematics, optics, photonics, computational biology, integration technology, biotechnology, nanotechnology, wireless, materials science, global positioning systems, robotics, cognitive science, etc.

In a free market democratic regime, three conditions are important in order for an economy to be able to grow:

1. Establishing and protecting individual property rights;

2. Entrepeneurialism, markets, and public policies that offer economic incentives;

3. Demographics.

On the other hand, economic development means economic growth accompanied by some other factors that — ensure a sustainable growth and — enhance level of overall economic welfare resulting from the growth. Some of such factors that should accompany growth include appropriate changes in output distribution (in favour of the poorer segment of the population) and economic structure (e.g., away from primary production).

However, it is not intended here to discuss the inadequacy of level of per capita income as an economic development indicator. The main essence of economic development is economic welfare. But, for a number of reasons, the level of per capita income is not a perfect measure of the level of development, just as its growth too is not a perfect indicator of rate of economic development — thus, they are both imperfect yardsticks for comparing (both the level and growth of) development (and, hence, economic welfare) over time and across countries.

The sources of development are:

- Natural resources; some countries benefit immensely, and other countries are stagnant despite plentiful resources. Others have none, yet enjoy high income. It can be said, thus, that natural resources are neither necessary nor sufficient for economic development;

- Population; population growth increases GDP, but may decrease GDP per capita because of required investments in human and physical capital. What happens is that as development occurs, people choose to have smaller families.

- Investment;
- Technological Innovation;
- Economic Policy (monetary policy, fiscal policy, regulation, government ownership, international trade and finance.

Indeed, there exists an intricate interplay between factors like labor force, technology, institutional arrangements, and capital that makes economic models often a great challenge.

The Reverend Thomas Malthus, on his *"An Essay on the Principle of Population"* modeled population growth as an exponential growth model. In classical mathematical models of economic growth, it is usually assumed that the labor force, L has an independent growth equation as employed by Malthus:

$$\frac{dL}{dt} = \beta L.$$

Malthus's population model predicts population growth without bound although it is obvious that the human population cannot grow at a constant rate indefinitely. What is often observed instead is that as the population grows, some members interfere with each other in competition for some critical resource. That competition diminishes the growth rate, until the population ceases to grow. It seems reasonable that a good population model must therefore reproduce this behavior. The logistic growth model, that was proposed by Pierre Francois Verhulst in 1838, is just such a model.

Letting L represent population size and t represent time, the logistic growth model is given by:

$$\frac{dL}{dt} = \beta L \left[1 - \frac{L}{B} \right], \tag{1.1.1}$$

where the parameter β defines the growth rate and B is the carrying capacity.

A modified labor population dynamics which introduces a natural dependence on the yield of the economy is proposed here. On account of this new approach, a larger level of golden rule capital per worker will be obtained .

1.2 The classical economic growth model

The simple one sector model, as presented, for instance, in (Intriligator, 1971), will be discussed in this section. In that model, the economy produces a single homogeneous good which represents Gross Domestic Product (GDP). The variables are the following:

- K is the capital;
- L is the labor force;
- Y is the economy output (yield, income);
- C is the consumption;
- I is the investment;
- F is the production function.

The basic hypothesis are:

The income identity:

$$Y = I + C, \tag{1.2.1}$$

which states that Gross National Product (GNP) can be either consumed or invested.

The output (GNP), is represented by an aggregated production function which depends on capital and labor:

$$Y = F(K, L) \tag{1.2.2}$$

That production function is assumed invariant over time and twice differentiable, the Inada conditions, (Intriligator, 1971), where for all positive factor inputs:

$$\begin{gathered} F(K,L) > 0; \quad \frac{\partial F}{\partial K}(K,L) > 0; \quad \frac{\partial F}{\partial L}(K,L) > 0; \\ \frac{\partial^2 F}{\partial K^2}(K,L) < 0; \quad \frac{\partial^2 F}{\partial L^2}(K,L) < 0; \quad \text{for all} \quad K, L > 0 \end{gathered} \tag{1.2.3}$$

and, taking limits:

$$\lim_{K\to 0}\frac{\partial F}{\partial K}(K,L)=\infty;\quad \lim_{L\to 0}\frac{\partial F}{\partial L}(K,L)=\infty;\quad \lim_{K\to\infty}\frac{\partial F}{\partial K}(K,L)=0;\quad \lim_{L\to\infty}\frac{\partial F}{\partial L}(K,L)=0; \tag{1.2.4}$$

The production function, as an assumption, exhibits constant returns of scale:

$$F(\alpha K,\alpha L)=\alpha F(K,L), \tag{1.2.5}$$

where α is a positive real number[1].

The investment, I, is used both to accumulate capital and to recover the depreciation of capital:

$$\frac{dK}{dt}+\mu K=I, \tag{1.2.6}$$

where μ is the capital depreciation rate. Another way to representing the dynamic of capital is by

$$\frac{dK}{dt}+\mu K=sF(K,L) \tag{1.2.7}$$

where $sF(K,L)=I$, i.e., the investment represents a fraction of the whole yield.

The equation for the labor force growth is (exponential growth):

$$\frac{dL}{dt}=\beta L. \tag{1.2.8}$$

Considering the variables per capita and from the assumption of the constant returns of scale of the production function, one arrives at the following equation called the fundamental differential equation of neoclassical economic growth:

$$\frac{dk}{dt}=-(\mu+\beta)k+f(k)-c, \tag{1.2.9}$$

[1] In other words, it is not assumed that “the whole is more than the sum of its parts”

or

$$\frac{dk}{dt} = -(\mu + \beta)k + sf(k), \tag{1.2.10}$$

where the small cap letters stand for their capital letters divided by the labor force L.

Equation 1.2.9 has, for no consumption per capita ($c = 0$), two equilibrium points, namely, $k = 0$ and $k = \tilde{k}$.

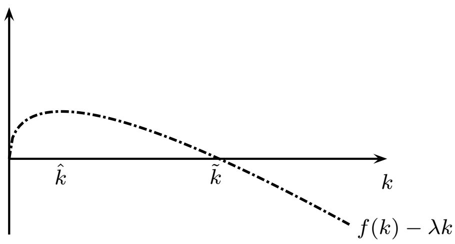

Figure 1.1: $f(k) - \lambda k$

Augmenting the level of consumption per capita, one can reach a level which is known as the golden rule level of capital per worker. It is the greatest level of capital per worker for which there is still an equilibrium point, $\dot{k} = 0$. In the sequel, $\dot{x}$ stands for dx/dt for every x.

The problem is then to choose a piecewise continuous trajectory c satisfying the Equation 1.2.9, the income identity and a boundary condition that maximizes a welfare function

$$J = \int_0^\infty e^{-\delta t} u(c)dt, \tag{1.2.11}$$

where δ is the discount rate, and u is a utility function. The utility function is assumed twice differentiable, and where, for all positive values of c:

$$\frac{\partial u}{\partial c}(c) > 0; \quad \frac{\partial^2 u}{\partial c^2}(c) < 0, \quad \text{for all} \quad c > 0. \tag{1.2.12}$$

It is also assumed that:

$$\lim_{c \to 0} \frac{\partial u}{\partial c}(c) = \infty; \quad \lim_{c \to \infty} \frac{\partial u}{\partial c}(c) = 0 \tag{1.2.13}$$

The problem is solved using the Pontryagin maximum principle, which leads to the following system of differential equations:

$$\begin{cases} \dot{c} = \dfrac{1}{\sigma(c)} \left[-(\lambda + \delta) + \dfrac{df}{dk} \right] c \\ \dot{k} = f(k) - \lambda k - c \end{cases} \tag{1.2.14}$$

where $\lambda = \mu + \beta$ and

$$\sigma(c) = -c \cdot \frac{\partial^2 u / \partial c^2}{\partial u / \partial c} \tag{1.2.15}$$

is defined as the elasticity of marginal utility.

By linearizing the system at the equilibrium point

$$\frac{dk}{dt} = \frac{dc}{dt} = 0 \tag{1.2.16}$$

corresponding to the higher value of the capital per capita, K, one gets:

$$\begin{cases} \dfrac{dk}{dt} = -(c - c^*) + \delta(k - k^*) \\ \dfrac{dc}{dt} = \dfrac{c^* \dfrac{d^2 f}{dk^2}}{\sigma(c^*)} (k - k^*) \end{cases} \tag{1.2.17}$$

The eigenvalues of this linear system are:

$$\frac{1}{2} \left(\delta \pm \sqrt{\delta^2 - \frac{4c^* \dfrac{d^2 f}{dk^2}}{\sigma(c^*)}} \right) \tag{1.2.18}$$

The equilibrium point is then a saddle point, whose stable branch consists of all points that eventually reach a balanced growth equilibrium.

1.3 A proposed model

Instead of considering an exponential growth, the labor force is supposed to follow a logistic equation:

$$\frac{dL}{dt} = \beta L \left[1 - \frac{\gamma L}{F(K,L)}\right]. \tag{1.3.1}$$

The rational for this model is as follows. For small values of L, the growth rate is small. The maximum value of dL/dt is attained when

$$\frac{d}{dL}\left(L\left[1 - \frac{\gamma L}{F(K,L)}\right]\right) = 0$$

As L increases, the term inside brackets will be decreasing, up to a point where it vanishes. If

$$\left[1 - \frac{\gamma L}{F(K,L)}\right] < 0,$$

the growth rate of the population will be negative in accordance with Malthusian thinking. The worker population then stabilizes, at a value that depends upon the income, given by $F(K,L)$. The larger the value of the income, the larger the value of the stabilized L.

The time rate of capital per worker will be given now by (details is appendix D):

$$\frac{dk}{dt} = -(\mu + \beta)k + f(k) + \frac{\beta\gamma k}{f(k)} - c \tag{1.3.2}$$

From the properties of the production function $f(k)$, one can prove some interesting properties for the function $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, given by

$$g(k) = \begin{cases} f(k) + \dfrac{\beta\gamma k}{f(k)}, & \text{if } k > 0 \\ 0, & \text{if } k = 0. \end{cases}$$

Since g is continuous at 0,

$$\lim_{k\to 0} f(k) + \frac{\beta\gamma k}{f(k)} = g(0) = 0,$$

it is clear that g is continuous.

Proposition 1.3.1 *If f is a nonnegative concave function defined on $\mathbb{R}^+ \cup \{0\}$ and f(0)=0, then g is monotonic-increasing.*

Proof: It is sufficient to prove that $h : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, given by $h(k) = k/f(k)$ and $h(0) = 0$ is monotonic-increasing. Let k_1 and k_2 be two positive real numbers such that $k_1 < k_2$. Therefore $k_1 = \lambda k_2$, where $0 < \lambda < 1$. Since f is concave,

$$f(0(1-\lambda) + k_2\lambda) > (1-\lambda)f(0) + \lambda f(k_2)$$
$$f(k_1) > \lambda f(k_2)$$
$$f(k_1) > \frac{k_1}{k_2} f(k_2)$$

Thus, since $f(k) > 0$ for all $k > 0$,

$$\frac{k_1}{f(k_1)} < \frac{k_2}{f(k_2)}$$

Q.E.D.

Proposition 1.3.2 *If $\lim_{k\to 0} f'(k) = \infty$, f is a nonnegative concave function defined on $\mathbb{R}^+ \cup \{0\}$ and f(0)=0, then $\lim_{k\to 0} g'(k) = \infty$.*

Proof: From the previous proposition,

$$\frac{d}{dk}\left(\frac{k}{f(k)}\right) > 0, \quad \forall k > 0$$

Therefore, since $\lim_{k\to 0} f'(k) = \infty$, it is clear that $\lim_{k\to 0} g'(k) = \infty$.

Q.E.D.

Further, it is assumed that $\lim_{k\to\infty} f(k) = \infty$.

Proposition 1.3.3 *If f is a nonnegative concave function defined on $\mathbb{R}^+ \cup \{0\}$, f(0)=0, $\lim_{k\to\infty} f(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$, then $\lim_{k\to\infty} g'(k) = 0$.*

Proof: It has been proved that

$$\frac{d}{dk}\left(\frac{k}{f(k)}\right) = \frac{f(k) - kf'(k)}{f^2(k)} > 0.$$

It is also clear that

$$\frac{1}{f(k)} = \frac{f(k)}{f^2(k)} > \frac{f(k) - kf'(k)}{f^2(k)} > 0.$$

Since $\lim_{k\to\infty} f(k) = \infty$ and by employing the sandwich theorem,

$$\lim_{k\to\infty} \frac{d}{dk}\left(\frac{k}{f(k)}\right) = 0.$$

Thus,

$$\lim_{k\to\infty} \frac{d}{dk}\left(f(k) + \frac{\beta k}{f(k)}\right) = \lim_{k\to\infty} \frac{df(k)}{dk} + \beta \lim_{k\to\infty} \frac{d}{dk}\left(\frac{k}{f(k)}\right) = 0$$

Q.E.D.

Moreover, if f is a concave Cobb-Douglas function (i.e., with decreasing return of scale), then g is also concave.

One can show that the income identity, the gross investment identity, the production function and the Expression1.3.1 together implies a new fundamental differential equation of economic growth:

$$\frac{dk}{dt} = -(\mu + \beta)k + g(k) - c. \tag{1.3.3}$$

The Figure 1.2 shows the *per* worker production function $f(k)$, $g(k)$ and λk

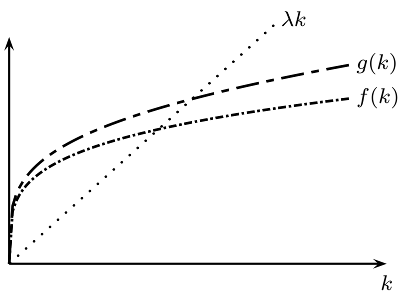

Figure 1.2: Production function f, g and λk.

Figure 1.3 shows the function $h(k) = g(k) - \lambda k$ which has two important points, namely, a unique maximum point $\hat{k}$ and a root $\tilde{k}$, i.e.,

$$h(\hat{k}) \geq h(k), \quad \forall k > 0$$

and

$$h(\tilde{k}) = 0.$$

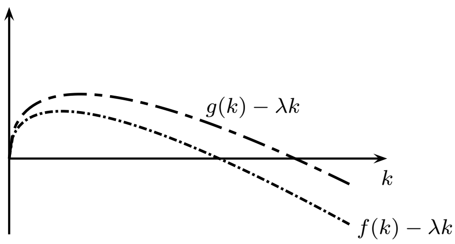

Figure 1.3: $f(k) - \lambda k$ and $g(k) - \lambda k$.

Observe that $\dot{k} = h(k) - c$, therefore the stability properties of this differential equation depend upon the level of consumption per worker. Remember that the capital per worker is a nonnegative value, that is, $c \geq 0$. If there is no consumption then $\dot{k} = h(k)$ and hence there exists two equilibrium points, $k = 0$ (which is locally unstable) and $k = \tilde{k}$ (which is stable, as shown in Figure 1.4).

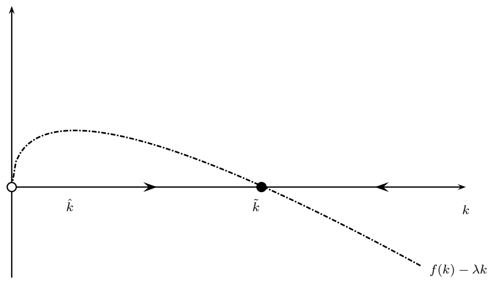

Figure 1.4: Equilibrium points

Augmenting the level of consumption per capita, one can again reach a so-called golden rule level of capital per worker. Notice that this new golden rule level is larger than the golden rule level of section 1.2.

Now, the optimal control model is

$$\operatorname*{Max}_{c} J = \int_0^{\infty} e^{-\delta t} u(c) dt \tag{1.3.4}$$

$$\dot{k} = -(\mu + \beta)k + g(k) - c \tag{1.3.5}$$

$$0 \le c \le f(k) \tag{1.3.6}$$

The maximum principle leads to the following system of differential equations (details in Appendix D):

$$\begin{cases} \dot{c} = \dfrac{1}{\sigma(c)}(-(\lambda + \delta) + g'(k))c \\ \dot{k} = -\lambda k + g(k) - c \end{cases} \tag{1.3.7}$$

By linearizing the system at the equilibrium point

$$\frac{dk}{dt} = \frac{dc}{dt} = 0,$$

corresponding to the higher value of capital per capita, k, one gets:

$$\begin{cases} \dfrac{dk}{dt} = -(c - c^*) + \delta(k - k^*) \\ \dfrac{dc}{dt} = \dfrac{c^* \dfrac{d^2 g}{dk^2}}{\sigma(c^*)}(k - k^*) \end{cases} \tag{1.3.8}$$

The eigenvalues of this linear system are:

$$\frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - \frac{4c^* \dfrac{d^2 g}{dk^2}}{\sigma(c^*)}}\right) \tag{1.3.9}$$

Therefore, since f is an Cobb-Douglas function, the equilibrium point is a saddle point.

In many countries, one observes that the capital per worker is growing over time and it is not explained by these models presented before. A common way to explain the capital per worker growth in macroeconomic growth models has been made by using technological progress.

The technological progress will be modeled by a continuous and smooth function of t as described in the following differential equation (Romer, 1994):

$$\dot{A} = \varepsilon A,$$

where ε is a positive constant.

A possible manner to introduce the technical progress into the production function is to add an "augmenting" factor to labor, analytically:

$$Y = F(K, AL).$$

This approach is known as Harrod-neutral or labor-augmenting technical progress.

Therefore, the model is now given by

$$\frac{dL}{dt} = \beta L \left[1 - \frac{\gamma AL}{F(K, AL)}\right].$$

In the appendix E there is the computations of that model.

1.4 Conclusions

The results of the growth models with those modifications in the population dynamics maintain the classical results of the Sollow's model in the sense that they assert the existence of equilibrium points, K/L in the first formulation and K/AL in the second one. Therefore, one should note that it is not necessary to be worried concerning the validation of the model. Validations of the Sollow's model also confirm those model's presented here.

2 Economic growth focusing on energy sources

According to Campello de Souza (Campello de Souza, 2006), the choice of an energy system (i.e., primary energy resources and their related technologies) has been a complex matter involving social, environmental and political issues. It is not clear how to argue pro or against this or that energy system option. The task of analyzing the future offer and demand of energy, and trying to harmonize them, as well as the establishing of prices for the various energy alternatives, is made more complex due to the effects of the technological progress, the availability of resources, the adopted regulation schemes (norms, rates, etc), the need of improving human capital and the particular national economic policy as a whole.

Energy resources would be allocated, under perfect competition, in an optimal way, that is, maximizing producer's profits and consumer's utilities. In some economies, however, many sectors, amongst them the energy sector, are perceived to exhibit many features which contradict then perfect competition condition, namely: oligopolist market structures, pollution, natural technical monopolies, non-renewability of some energy resources, etc. Since there is not a perfect competition environment, it is therefore interesting some kind of central planner which guides the behavior of the economic agents of the energy sector. For this task, this central planner should use a scientific approach.

This chapter investigates the economic growth problem treated in (Stamford da Silva, 1999) incorporating some changes in its models. Three models are analyzed and they contain, as in Stamford da Silva's work, factors that are often separately handled, namely: energy sources; capital and labor; and water applied to non-energetic ends.

Those macroeconomic models differ from the Stamford da Silva's work (Stamford da Silva, 1999) in interpreting the research and development ($R\&D$) effect and in modeling its evolution over time, as presented in (Romer, 2001). They are set in continuous time and deterministic. Moreover, they do not analyze monetary matters, and consumption distribution among the labor force is not modeled; instead an average individual is considered. It is also assumed that all individual generations have the same preferences. It is not considered energy source exports. Nonconventional energy technologies – such as solar, aeolian, nuclear, etc – are not presumed as backstop technologies. It is assumed that those technologies are ready for use and can replace conventional technologies. Among all assumptions, it seems the strongest and unusual one, but, indeed, it may occur soon.

An optimal economic growth model that incorporates several aspects of the energy resources, allowing the establishment of planning policies, is presented. The process of dynamic optimization can be understood as a centralizing planning where an authority maximizes an objective functional (the Welfare).

2.1 Notation

- J – welfare functional;
- δ – discount rate;
- L – labor force;
- u – utility *per* worker;
- c – consumption *per* worker of non-energy goods.
- t – time;
- α – consumption *per* worker of energy goods;
- β – population growth rate;
- F – production function of non-energy goods;
- E – consumption rate of aggregate energy resources for non-energy uses;
- A – technological level;
- W – consumption rate of water for non-energy ends;
- s – expenses concerning imported energy resources;

- D – water stock;

The subscripts in remaining variables have the following meanings:

- 0 – indicates non-energy goods, excluding the water for non-energy use;
- W – indicates non-energy water, i.e., water that is employed to produce non-energy goods;
- H – indicates hydroelectric resources;
- NR – indicates non-renewable energy resources;
- R – indicates renewable energy, namely, solar energy, eolian, nuclear, etc;
- A – indicates technology;
- E – indicates energy exports.

Thus,

- K_0 – capital employed to produce non-energy goods, excluding capital regarding non-energy water;
- L_0 – labor force employed to produce non-energy goods, excluding the labor force regarding non-energy water;
- I_0 – investment in the non-energy goods sector;
- μ_0 – depreciation rate of capital K_0;
- W – annual consumption rate of non-energy water;
- F_W – production function of non-energy water;
- K_W – capital employed to provide non-energy water;
- L_W – labor force employed to provide non-energy water;
- G_W – weighing function of non-energy water;
- I_W – investment in the non-energy water sector;

- μ_W – depreciation rate of capital K_W;
- E_H – annual consumption rate of hydroelectric resources;
- F_H – production function of hydroelectric resources;
- K_H – capital employed to produce hydroelectric energy;
- L_H – labor force employed to produce hydroelectric energy;
- G_H – weighing function of hydroelectric resources;
- I_H – investment in the hydroelectric sector;
- μ_H – depreciation rate of capital K_H;
- E_{NR} – annual consumption rate of non-renewable resources;
- F_{NR} – production function of non-renewable resources;
- K_{NR} – capital employed to produce non-renewable energy;
- L_{NR} – labor force employed to produce non-renewable energy;
- G_{NR} – weighing function of non-renewable resources;
- I_{NR} – investment in the non-renewable sector;
- μ_{NR} – depreciation rate of capital K_{NR};
- E_R – annual consumption rate of renewable resources;
- F_R – production function of renewable resources;
- K_R – capital employed to produce renewable energy;
- L_R – labor force employed to produce renewable energy;
- G_R – weighing function of renewable resources;
- I_R – investment in the renewable energy sector;
- μ_R – depreciation rate of capital K_R;

- F_A – production function of technology;
- K_A – capital employed to produce technology;
- L_R – labor force employed to produce technology;
- I_A – investment in the technological sector;
- μ_A – depreciation rate of capital K_A;

2.2 Optimal control Models

A plausible reason to explain how more output can be produced today from a given amount of labor and capital than could be produced in the past, seems to be, indeed, the technological evolution. Therefore, a research and development sector is introduced in the Stamford da Silva's work (Stamford da Silva, 1999) representing the production of new technologies.

The economy is partitioned into six sectors: a non-energetic goods-producing sector where non-energy output is produced; an *R&D* sector where additions to the stock of knowledge are made; a water sector, where water for non-energetic ends is provided; and three more sectors which compose the whole energy sector. The energy sector is decomposed into an hydroelectric sector, a renewable energy sector (excluding the hydroelectric one which has an specific sector), and a non-renewable energy sector.

Furthermore, another model is proposed, where there is not an *R&D* sector and technological progress is assumed to be exogenous.

2.2.1 Technological dynamics

It is assumed in one model that technology progress is exogenous; representing, for instance, an economy in which all its technology is brought from abroad. In that formulation, the dynamics of the technology is modeled by

$$\dot{A} = \epsilon A, \tag{2.2.1}$$

where ϵ is an exogenous parameter (in the technological sense, not the managerial one).

On the other hand, other models presented in this chapter consider technology growth as being endogenous. It is modeled by a conventional approach in which technology labor (L_A), technology capital (K_A), and technology itself (A) are combined to provide technology growth as

$$\dot{A} = F_A(K_A, L_A)A, \tag{2.2.2}$$

where $F_A : \{\mathbb{R}^+ \cup \{0\}\} \times \{\mathbb{R}^+ \cup \{0\}\} \to \mathbb{R}$.

2.2.2 Resource consumption dynamics

The consumption rate of the energy resource reserve (D_i) is given as in (Hotelling, 1931) by:

$$\dot{D}_i = -E_i \quad ; i = R, NR \tag{2.2.3}$$

where E_i is defined, as in section 2.1, as annual consumption rate of energy source i.

The water stock is diminished by both, the use as a productive input and the use in hydroelectric generation. The dynamic of the water stock (D) is represented by

$$\dot{D} = f(t) - (E_H + W) \tag{2.2.4}$$

where E_H is defined as the annual consumption rate of hydroelectricity, W is defined as the annual consumption rate of water for non-energy ends, $f(t)$ is assumed to be a continuous and differentiable function over t (maybe periodic) that models the water cycle.

2.2.3 Production of water and energy

It is supposed that the energy and water markets are in equilibrium. Furthermore, it is assumed that all energy and produced water are consumed on production of non-energy goods.

Each sector — hydroelectric, renewable, non-renewable, and water for energy production (hydro-electric energy) — has an annual consumption rate determined by using four factors of production, namely, its capital, its labor force, its technological level. These functions of annual consumption rate are assumed invariant over time.

Hydroelectric consumption rate is expressed by

$$E_H = F_H(K_H, AL_H)G_H(D_H) \tag{2.2.5}$$

where K_H is the capital employed in the hydroelectric sector, L_H is the labor employed in the hydroelectric sector, D is the water reserve, and A is the technological level.

Notice that the expression 2.2.5 is a product of two functions:

$$\begin{aligned} F_H : \{\mathbb{R}^+ \cup \{0\}\} \times \{\mathbb{R}^+ \cup \{0\}\} &\to \mathbb{R} \\ (K_H, AL_H) &\mapsto F_H(K_H, AL_H) \end{aligned} \tag{2.2.6}$$

which is defined as the production function for hydroelectric resources, and

$$\begin{aligned} G_H : \{\mathbb{R}^+ \cup \{0\}\} &\to \mathbb{R} \\ D &\mapsto G_H(D) \end{aligned} \tag{2.2.7}$$

which is called the weighing function of hydroelectric resources.

The production function of hydroelectric resources is assumed twice differentiable, where, for all positive factor inputs:

$$\frac{\partial F_H}{\partial K_H}(K_H, A.L_H) > 0; \quad \frac{\partial^2 F_H}{\partial K_H^2}(K_H, A.L_H) < 0 \tag{2.2.8}$$

$$\frac{\partial F_H}{\partial A.L_H}(K_H, A.L_H) > 0; \quad \frac{\partial^2 F_H}{\partial (A.L_H)^2}(K_H, A.L_H) < 0 \tag{2.2.9}$$

for all K_H, $A.L_H > 0$. Observe also that A and L_H enter into that function as a product. The factor AL_H is often referred as effective labor.

The weighing function of hydroelectric resources is assumed monotonically increasing, i.e.,

$$\frac{\partial G_H}{\partial D}(D) > 0, \tag{2.2.10}$$

differentiable, and it is assumed also that:

$$\lim_{D\to 0} G_H(D) = 0; \quad \lim_{D\to\infty} G_H(D) = 1; \tag{2.2.11}$$

$$\lim_{D\to 0} \frac{\partial G_H}{\partial D}(D) = \infty; \quad \lim_{D\to\infty} \frac{\partial G_H}{\partial D}(D) = 0; \tag{2.2.12}$$

The function G_H being monotonically increasing means diminished returns when the reserve is consumed.

Similarly, the annual consumption rate of water for non-energy ends is determined by

$$W = F_W(K_W, AL_W)G_W(D) \tag{2.2.13}$$

where K_W is the capital employed in the water sector, L_W is the labor employed in water sector, D is the water reserve, and A is the technological level. The functions $F_W(K_W, AL_W)$, $F_{NR}(K_{NR}, L_{NR})$ and $F_R(K_R, AL_R)$ are defined as production functions for non-energy water, non-renewable resources and renewable resources respectively, while $G_W(D)$, $G_{NR}(D_{NR})$, $G_R(D_R)$ are the weighing functions of non-energy water, non-renewable resources and renewable resources, respectively.

The functions $F_W(K_W, AL_W)$, $F_{NR}(K_{NR}, L_{NR})$ and $F_R(K_R, AL_R)$ are also assumed twice differentiable and for all positive inputs:

$$\frac{\partial F_W}{\partial K_W}(K_W, A.L_W) > 0; \quad \frac{\partial^2 F_W}{\partial K_W^2}(K_W, A.L_W) < 0 \tag{2.2.14}$$
$$\frac{\partial F_W}{\partial A.L_W}(K_W, A.L_W) > 0; \quad \frac{\partial^2 F_W}{\partial (A.L_W)^2}(K_W, A.L_W) < 0 \tag{2.2.15}$$
$$\frac{\partial F_{NR}}{\partial K_{NR}}(K_{NR}, A.L_{NR}) > 0; \quad \frac{\partial^2 F_{NR}}{\partial K_{NR}^2}(K_{NR}, A.L_{NR}) < 0 \tag{2.2.16}$$
$$\frac{\partial F_{NR}}{\partial A.L_{NR}}(K_W, A.L_W) > 0; \quad \frac{\partial^2 F_{NR}}{\partial (A.L_{NR})^2} < 0 \tag{2.2.17}$$
$$\frac{\partial F_R}{\partial K_R}(K_R, A.L_R) > 0; \quad \frac{\partial^2 F_R}{\partial K_R^2}(K_R, A.L_R) < 0 \tag{2.2.18}$$
$$\frac{\partial F_R}{\partial A.L_R}(K_R, A.L_R) > 0; \quad \frac{\partial^2 F_R}{\partial (A.L_R)^2}(K_R, A.L_R) < 0 \tag{2.2.19}$$

The weighing function of non-energy water and non-renewable resources are also assumed monotonically increasing, that is,

$$\frac{\partial G_W}{\partial D}(D) > 0, \quad \frac{\partial G_{NR}}{\partial D_{NR}}(D_{NR}) > 0, \tag{2.2.20}$$

differentiable and the following conditions are supposed to hold:

$$\lim_{D \to 0} G_W = 0; \quad \lim_{D \to \infty} G_W = 1; \tag{2.2.21}$$
$$\lim_{D_{NR} \to 0} \frac{\partial G_{NR}}{\partial D_{NR}} = \infty; \quad \lim_{D_{NR} \to \infty} \frac{\partial G_{NR}}{\partial D_{NR}} = 0. \tag{2.2.22}$$

For the renewable sector, R, one may consider

$$G_R(D_R) = 1 \tag{2.2.23}$$

2.2.4 The investment identity

The investment identities given by

$$\dot{K}_i = -\mu_i K_i + I_i \quad ; i = 0, R, NR, H, W, A \tag{2.2.24}$$

represent the fact that investments are employed both to augment the stock of capital and to replace depreciated capital.

2.2.5 The labor force

Among several possible ways of modeling the labor force growth, two of them are presented here.

The labor force growth can be described, as usual, as

$$\dot{L} = \beta L. \tag{2.2.25}$$

Robert Malthus in his *"Essay on the principle of population"*, in 1798, was the first economic thinker which states that limited resources implies a limited population growth. However, he did not forecast the technological development due mainly to the industrial revolution during the XVIII century. Indeed, technological progress has promoted an amazing augment of productivity. On account of the Malthus's idea, a different manner of representing population growth is proposed. The labor force growth is described as

$$\dot{L} = \beta L\Big(1 - \gamma F(K_0, A.L_0, E, W)^{-1} AL\Big), \tag{2.2.26}$$

where

$$\begin{aligned} &F : \{\mathbb{R}^+ \cup \{0\}\}^4 \to \mathbb{R} \\ &(K_0, AL_0, E, W) \mapsto F(K_0, A.L_0, E, W). \end{aligned} \tag{2.2.27}$$

It is a logistic function where the carrying capacity is F/γ (and γ is a constant). This model for the labor force growth means the more an economy produces, the more the labor force can grow.

Moreover, the labor force is allocated amongst the economic sectors, then defines the following identity for the models with an $R\&D$ sector:

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{2.2.28}$$

where

- the non-energy labor force is defined as L_0;
- hydroelectric labor force as L_H;
- renewable energy labor force as L_R;
- non-renewable energy labor force as L_{NR};
- technology labor force as L_A.

If there is not an $R\&D$ sector, the identity is

$$L = L_0 + L_R + L_{NR} + L_H + L_W \tag{2.2.29}$$

2.2.6 The income identity

This neoclassical growth model characterizes economics in an aggregative way. A single non-energetic good is produced, the output of which at time t is $Y(t)$, using four factor inputs, namely, non-energetic capital $K_0(t)$, the product of technology and non-energetic labor force $A(t)L(t)$, the annual consumption rate of aggregate energy resources $E(t)$ and the annual non-energy water consumption rate $W(t)$, where t is assumed to be continuous. In the case where the technological progress is considered as exogenous, the income identity is modeled by

$$Y = F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + s(E_E) + Lc. \tag{2.2.30}$$

On the other hand, when technological progress is assumed endogenous, the yield identity is expressed as

$$Y = F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc. \tag{2.2.31}$$

Notice that in the endogenous case there exists an $R\&D$ sector. The yield identities, as just defined, represent a basic identity of economic growth models. However, it is important to observe that the identity considers the expenses for the acquisition of imported energy resources. Therefore, it classifies that model as an open one concerning energetic resources, what is, in a certain way, a classical use.

It is supposed that the market is in equilibrium and that all output is either consumed or invested.

This approach provided a better comprehension concerning the trade-off between water for non-energy use and hydroelectric energy ends (Stamford da Silva, 1999). Furthermore, as shall be seen later in the energetic balance expression, the variable E depends on the variable W. It implies, after the chain rule, results that alters the usual analysis of marginal productivity of water.

The production function is assumed invariant over time and twice differentiable, the Inada conditions, (Intriligator, 1971), where for all positive factor inputs:

$$\frac{\partial F}{\partial K_0}(K_0, AL_0, E, W) > 0; \quad \frac{\partial F}{\partial AL_0}(K_0, AL_0, E, W) > 0; \tag{2.2.32}$$

$$\frac{\partial^2 F}{\partial K_0^2}(K_0, AL_0, E, W) < 0; \quad \frac{\partial^2 F}{\partial AL_0^2}(K_0, AL_0, E, W) < 0 \tag{2.2.33}$$

2.2.7 The energy balance

As in Stamford's model, the entire consumption rate of energy sources for non-energy ends are expressed by:

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{2.2.34}$$

where α denotes energetic resources consumption *per capita*, $-W$ represents the amount of hydroelectricity that would be reduced in the entire energy if the water was used for other ends. Notice that in an economy where the energy system does not have a great amount of hydroelectricity, this formulation is not essential. It is not the Brazilian case, where the hydroelectric power represents a great part of the total energy[1].

[1]http://ben.epe.gov.br/BEN2006/

2.2.8 Objective functional

The objective functional follows an utilitarian scheme which have been adopted by many researchers(Intriligator, 1971). It is assumed that the central planner, whatever it means, has a utility function that gives utility at any instant of time, denoted by u, as a function of consumption per worker, c, consumption per worker of energetic goods, α, and population growth rate, β. It means that all generations of individuals have the same preferences.

It is also assumed that utility at any instant of time is not directly dependent on c, α, β or utility at any other instant of time.

It is further assumed that utilities along the time can be added (integrated) but nevertheless these utilities are adequately discounted to represent the impatience of the central planner. In this research, it is considered an infinite time horizon, so, a Welfare functional, J, is defined as follows:

$$J = \int_0^\infty e^{-\delta t} L u(c, \alpha, \beta) dt, \tag{2.2.35}$$

where δ is a constant. The utility function is assumed monotonic-increasing regarding both c and α, that is,

$$\frac{\partial u}{\partial c}(c, \alpha, \beta) > 0, \quad \frac{\partial u}{\partial \alpha}(c, \alpha, \beta) > 0, \tag{2.2.36}$$

for every c, $\alpha > 0$. It is also assumed that the utility is quite flat for a wide range of values of β but it rises when β converges to zero or when β becomes too large.

2.3 The zero Model

The basic model to be analyzed is the following:

$$\operatorname{Max} W = \int_0^\infty e^{-\delta t} L u(c, \alpha, \beta) dt, \tag{2.3.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + s(E_E) + Lc \tag{2.3.2}$$
$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{2.3.3}$$
$$L = L_0 + L_R + L_{NR} + L_H + L_W \tag{2.3.4}$$
$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{2.3.5}$$
$$E_H = F_H(K_H, AL_H)G_H(D) \tag{2.3.6}$$
$$W = F_W(K_W, AL_W)G_W(D) \tag{2.3.7}$$
$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{2.3.8}$$
$$\dot{D}_i = -E_i; \quad i = R, NR \tag{2.3.9}$$
$$\dot{D} = f(t) - (E_H + W) \tag{2.3.10}$$
$$\dot{L} = \beta L \tag{2.3.11}$$
$$\dot{A} = \epsilon A \tag{2.3.12}$$

It is essentially the Stamford da Silva's work, but with an extra differential equation that represents the evolution of technology.

Notice that the state variables are:

- K_i, $i = 0, R, NR, H, W$;
- D_i, $i = R, NR$;
- D, L and A.

The control forces are:

- I_i, $i = 0, R, NR, H, W$;

- E_i, $i = R, NR, H$;
- W, E_E, α and β.

The Hamiltonian of this basic formulation is

$$\begin{aligned} H =& e^{-\delta t}\big\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A \epsilon A + q_L \beta L\big\} \end{aligned} \tag{2.3.13}$$

By applying the maximum principle one can obtain some interesting results. A detailed mathematical computation about the zero problem can be found in Appendix A.

2.3.1 The zero model's results

First, shadow prices q_i, where $i = 0, R, NR, H, W$, are all equal to $\partial u/\partial c$:

$$q_0 = q_W = q_H = q_R = q_{NR} = \frac{\partial u}{\partial c} \tag{2.3.14}$$

that is, in the optimal path, the marginal value of capital K_i must be equal to the marginal utility regarding the non-energy goods consumption per-worker.

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E} \tag{2.3.15}$$

Expression 2.3.15 means that the price of each domestic energy resource must be equal to the price of the imported energy resource. It means that there is a unique price for all primary energy resources because all energy resource contributes equally to the yield.

Another result is provided by the relation:

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c} = \frac{p}{q} \tag{2.3.16}$$

Expression 2.3.16 implies that the price of energy resources must be equal to the substitution rate between non-energy goods and energy goods. Moreover, it also contents that the substitution rate between non-energy goods and energy goods must be equal to the rate between the marginal value of energy resource in the reserve and the marginal value of capital.

$$q_L = -\frac{\partial u}{\partial \beta} \tag{2.3.17}$$

Equation 2.3.17 implies that the marginal value of labor force (the shadow price of labor force) in the economy must be equal to the negative marginal utility regarding labor force growth rate.

$$\frac{\dot{p}_R}{p_R} = \frac{\dot{p}_{NR}}{p_{NR}} = \frac{\dot{p}_D}{p_D} = \delta \tag{2.3.18}$$

This result known as the Hotelling rule which is an expected result when extraction rates are employed as control forces (notice that it is assumed that the extraction rates are equal to consumption rates). It means that the marginal value of energy resource in the reserve must rise according the interest rate, i.e., energy resources must be treated as any capital good.

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_0}{q_0} = \delta + \mu \tag{2.3.19}$$

Expression 2.3.19 implies, similarly as the previous result, that the marginal value of the capital K_i must rise according the interest rate δ plus depreciation rate μ_i, where $i = R, NR, H, W, 0$.

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \tag{2.3.20}$$

The identity 2.3.20 states that capital goods for every energy sectors and water sector must be homogenous, i.e., their depreciation rates must be the same.

$$\frac{\partial F}{\partial K_0} = \mu_0 - \mu \tag{2.3.21}$$

Since the depreciation rates μ_0 and μ are assumed constant, equation 2.3.21 asserts that the contribution of non-energy capital in producing non-energy goods is constant. In other words, the price of the non-energy capital in producing non-energy goods must be constant.

$$\mu_0 > \mu \tag{2.3.22}$$

The inequality 2.3.22 implies that depreciation of non-energy capital must be greater than depreciation rates μ_R, μ_{NR}, μ_H and μ_W.

$$p_i = \frac{\partial u}{\partial \alpha}; \quad i = D, R, NR. \tag{2.3.23}$$

Expression 2.3.23 contents that the marginal value of the energy resource i (shadow price for energy resource i) in the reserve must be equal to the marginal utility regarding energy consumption per worker.

$$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E} \tag{2.3.24}$$

The most important result of the work of Stamford da Silva isshown above in the expression 2.3.24. This result asserts that water contributes by a hundred percent more in producing non-energy goods than energy resources do. In other words, the price for water must be twice the price for energy resources.

The inclusion of the dynamics of technological change lead to the following new results:

$$\dot{p}_A =(\delta - \epsilon))p_A - q_0\frac{\partial F}{\partial AL_0}.L_0 \tag{2.3.25}$$

The expression 2.3.25 establish the dynamic of the marginal value of the technological level. When the other variables are maintained constant, *ceteris paribus*, the larger the discount rate value, the lower the technological growth rate, the lower the marginal value of the non-energy capital, the lower the labor force of the non-energy goods sector, the lower the marginal production regarding the effective labor, the larger shadow price of the technological level derivative.

$$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{2.3.26}$$

The expression 2.3.26 establish growth of the marginal value of the labor force. When the other variables are maintained constant, *ceteris paribus*, the larger the discount rate value, the lower the population growth rate, the larger the marginal value of the non-energy capital, the larger the consumption per worker of non-energy goods, the larger the consumption per worker of energy goods, the larger the price for energy, the lower the utility per worker, the larger the shadow price of the marginal value of the labor force derivative.

2.4 The first Model

In this model the technological change is supposed to be endogenous. The country (economy) develops its own technology.

$$\operatorname{Max} W = \int_0^\infty e^{-\delta t} Lu(c, \alpha, \beta) dt, \tag{2.4.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc \tag{2.4.2}$$

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{2.4.3}$$

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{2.4.4}$$

$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{2.4.5}$$

$$E_H = F_H(K_H, AL_H)G_H(D) \tag{2.4.6}$$

$$W = F_W(K_W, AL_W)G_W(D) \tag{2.4.7}$$

$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{2.4.8}$$

$$\dot{D}_i = -E_i; \quad i = R, NR \tag{2.4.9}$$

$$\dot{D} = f(t) - (E_H + W) \tag{2.4.10}$$

$$\dot{L} = \beta L \tag{2.4.11}$$

$$\dot{A} = F_A(K_A, L_A)A \tag{2.4.12}$$

Notice that state variables are:

- K_i, $i = 0, R, NR, H, W, A$;
- D_i, $i = R, NR$;
- D, L and A.

The control forces are:

- I_i, $i = 0, R, NR, H, W, A$;
- E_i, $i = R, NR, H$;

- W, E_E, α and β.

The Hamiltonian of this formulation is

$$\begin{aligned} H =& e^{-\delta t}\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L \beta L\} \end{aligned} \tag{2.4.13}$$

A detailed mathematical computation about the first model can be found in Appendix B.

2.4.1 The first model's results

The shadow prices q_i, where $i = 0, R, NR, H, W, A$, are all equal to $\partial u/\partial c$,

$$q_0 = q_W = q_H = q_R = q_{NR} = q_A = \frac{\partial u}{\partial c}, \tag{2.4.14}$$

that is, it implies, as said before in section 2.3.1, that the marginal value of the capital must be equal to the marginal utility of the non-energetic consumption (notice that in the first model there is another costate variable, namely, q_A regarding the technology sector).

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E} \tag{2.4.15}$$

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c} = \frac{p}{q} \tag{2.4.16}$$

$$q_L = -\frac{\partial u}{\partial \beta} \tag{2.4.17}$$

$$\frac{\dot{p}_R}{p_R} = \frac{\dot{p}_{NR}}{p_{NR}} = \frac{\dot{p}_D}{p_D} = \delta \tag{2.4.18}$$

Expressions 2.4.15, 2.4.16, 2.4.17, 2.4.18 are also results of the zero model, that remain here.

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_D}{q_D} = \frac{\dot{q}_A}{q_A} = \frac{\dot{q}_0}{q_0} = \delta + \mu \tag{2.4.19}$$

As in expression 2.3.19, the identity 2.4.19 implies that the marginal value of the capital K_i must rise according the interest rate δ plus depreciation rate μ_i, where $i = R, NR, H, W, 0, A$.

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \tag{2.4.20}$$

$$\frac{\partial F}{\partial K_0} = \mu_0 - \mu \tag{2.4.21}$$

$$\mu_0 > \mu \tag{2.4.22}$$

$$p_i = \frac{\partial u}{\partial \alpha}; \quad i = D, R, NR. \tag{2.4.23}$$

$$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E} \tag{2.4.24}$$

$$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{2.4.25}$$

Equations 2.4.20, 2.4.21, 2.4.23, 2.4.24, 2.4.25 and the inequality 2.4.22 are also repeated results of the zero model.

Now, the new result is:

$$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)} e^{(\delta+\mu)t} \tag{2.4.26}$$

This expression, 2.4.26, establishes the technology shadow price behavior along the time.

2.5 The second model

In this model, a substantial modification of the labor force growth equation is introduced.

$$\mathrm{Max}\, W = \int_0^\infty e^{-\delta t} Lu(c, \alpha, \beta) dt, \tag{2.5.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc \tag{2.5.2}$$

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{2.5.3}$$

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{2.5.4}$$

$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{2.5.5}$$

$$E_H = F_H(K_H, AL_H)G_H(D) \tag{2.5.6}$$

$$W = F_W(K_W, AL_W)G_W(D) \tag{2.5.7}$$

$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{2.5.8}$$

$$\dot{D}_i = -E_i; \quad i = R, NR \tag{2.5.9}$$

$$\dot{D} = f(t) - (E_H + W) \tag{2.5.10}$$

$$\dot{L} = \beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1} L) \tag{2.5.11}$$

$$\dot{A} = F_A(K_A, L_A)A \tag{2.5.12}$$

Recall the rational for Equation 2.5.11 presented in Section 2.2.5.

Notice that the state variables are:

- K_i, $i = 0, R, NR, H, W, A$;
- D_i, $i = R, NR$;
- D, L and A.

The control forces are:

- I_i, $i = 0, R, NR, H, W, A$;

- E_i, $i = R, NR, H$;
- W, E_E, α and β.

The Hamiltonian of this formulation is

$$\begin{aligned} H =& e^{-\delta t}\big\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L(\beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1}L))\big\} \end{aligned} \tag{2.5.13}$$

A detailed mathematical computation about the second model can be found in Appendix C.

2.5.1 Second model's results

Although many results of the previous models remain in the second model, namely,

$$q_0 = q_W = q_H = q_R = q_{NR} = q_A = \frac{\partial u}{\partial c}; \tag{2.5.14}$$

$$p_i = \frac{\partial u}{\partial \alpha}; \quad i = D, R, NR.; \tag{2.5.15}$$

$$\frac{\dot{p}_R}{p_R} = \frac{\dot{p}_{NR}}{p_{NR}} = \frac{\dot{p}_D}{p_D} = \delta; \tag{2.5.16}$$

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_D}{q_D} = \frac{\dot{q}_A}{q_A} = \frac{\dot{q}_0}{q_0} = \delta + \mu; \tag{2.5.17}$$

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu; \tag{2.5.18}$$

$$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E}; \tag{2.5.19}$$

there is also new results.

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial u/\partial c}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} \cdot \frac{\partial s}{\partial E_E} \tag{2.5.20}$$

Expression 2.5.20, on account of the logistic labor force growth equation, asserts that the price for domestic energy resources is no more equals to the price for imported energy resource as obtained in the zero and first models. Indeed, the price for the domestic energy resource is lower than the price for imported energy resource, according to the factor

$$\frac{\partial u/\partial c}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} \; .$$

The larger the value of $q_L \beta\gamma.(L/F)^2$, the larger will be the departure from the previous models.

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} = \frac{p}{q + q_L \beta\gamma.(L/F)^2} \tag{2.5.21}$$

Expression 2.5.21 implies that the price of energy resources is no more equals to the substitution rate between non-energy goods and energy goods.

$$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)} e^{(\delta+\mu)t} \tag{2.5.22}$$

$$q_L = \frac{C.\left(\mu_0 - \mu - \dfrac{\partial F}{\partial K_0}\right)}{\beta\gamma.\left(\dfrac{L}{F}\right)^2 \dfrac{\partial F}{\partial K_0}} \cdot e^{(\delta+\mu)t} \tag{2.5.23}$$

2.6 Conclusions

At first, it is important to observe that some classical results were obtained in those models, namely:

-
$$q_0 = q_W = q_H = q_R = q_{NR} = \frac{\partial u}{\partial c} \quad ; \tag{2.6.1}$$

- the Hotelling rule
$$\frac{\dot{p}_R}{p_R} = \frac{\dot{p}_{NR}}{p_{NR}} = \frac{\dot{p}_D}{p_D} = \delta \quad ; \tag{2.6.2}$$

Moreover, many results of the work of Stamford da Silva remains in the model zero and one , for instance:

-
$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E} \quad ; \tag{2.6.3}$$

-
$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c} = \frac{p}{q} \quad ; \tag{2.6.4}$$

-
$$q_L = -\frac{\partial u}{\partial \beta} \quad ; \tag{2.6.5}$$

-
$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_0}{q_0} = \delta + \mu \quad ; \tag{2.6.6}$$

- $$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \quad ; \tag{2.6.7}$$

- $$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E} \tag{2.6.8}$$

The fact that those results are preserved act as argument of validity of those models even.

For the zero model, the new result were:

- $$\dot{p}_A = (\delta - \epsilon))p_A - q_0\frac{\partial F}{\partial AL_0}.L_0 \quad ; \tag{2.6.9}$$

- $$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{2.6.10}$$

These results are, in special, difficult to interpret, but assuming an specific production function or solving (or simulating)those partial differential equations one may obtain a better comprehension about the phenomena.

For the one model, the new results were:

- $$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \quad ; \tag{2.6.11}$$

- $$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)}e^{(\delta+\mu)t}. \tag{2.6.12}$$

The first result remains from the zero model, and the second establish the dynamics of the technology shadow price. Since the shadow price must increase, or at leat not decrease, the depreciation of the technological capital must be larger than the depreciation of the other sector. Otherwise, if the shadow price decrease no one will want to develop technology.

For the two model, the new results were:

-
$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial u/\partial c}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} \cdot \frac{\partial s}{\partial E_E} \tag{2.6.13}$$

-
$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} = \frac{p}{q + q_L \beta\gamma.(L/F)^2} \tag{2.6.14}$$

whose are only adjustments of similar results of zero and one model;

-
$$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)} e^{(\delta+\mu)t} \quad ; \tag{2.6.15}$$

-
$$q_L = \frac{C.\left(\mu_0 - \mu - \dfrac{\partial F}{\partial K_0}\right)}{\beta\gamma.\left(\dfrac{L}{F}\right)^2 \dfrac{\partial F}{\partial K_0}} \cdot e^{(\delta+\mu)t} \quad . \tag{2.6.16}$$

.

The first remains form the one model and the second establishes the dynamic of the shadow price of the labor force. Therefore, since shadow price must increase over the time, implies that

$$\frac{\partial F}{\partial K_0} < \mu_0 - \mu \quad .$$

3 Concluding Remarks and Suggestions

3.1 Concluding remarks

The main conclusions are:

- The results of the chapter one's growth models with those modifications in the population dynamics maintain the classical results of the Sollow's model(Solow, 1956) in the sense that they assert the existence of equilibrium points, K/L in the first formulation and K/AL in the second one. Therefore, one should note that it is not necessary to be worried concerning the validation of the model. Validations of the Sollow's model also confirm those model's presented here.

- Classical results were obtained in those models of chapter two, namely:

  1.

$$q_0 = q_W = q_H = q_R = q_{NR} = \frac{\partial u}{\partial c} \quad ; \tag{3.1.1}$$

  2. the Hotelling rule

$$\frac{\dot{p}_R}{p_R} = \frac{\dot{p}_{NR}}{p_{NR}} = \frac{\dot{p}_D}{p_D} = \delta \quad ; \tag{3.1.2}$$

- many results of the work of Stamford da Silva(Stamford da Silva, 1999) remains in the model zero, one and two of chapter two. In sauch way, the fact that those results are preserved act as argument of validity of those models even.

- For the zero model,

  1.

$$\dot{p}_A = (\delta - \epsilon))p_A - q_0 \frac{\partial F}{\partial AL_0}.L_0 \quad ; \tag{3.1.3}$$

2.

$$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{3.1.4}$$

are, in special, difficult to interpret, but assuming an specific production function or solving (or simulating)those partial differential equations one may obtain a better comprehension about the phenomena.

- For the one model

$$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)} e^{(\delta+\mu)t}. \tag{3.1.5}$$

. Since the shadow price must increase, or at leat not decrease, the depreciation of the technological capital must be larger than the depreciation of the other sector. Otherwise, if the shadow price decrease no one will want to develop technology.

- For the two model, the new results were:

1.

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial u/\partial c}{\partial u/\partial c + q_L \beta\gamma.(L/F)^2} \cdot \frac{\partial s}{\partial E_E} \tag{3.1.6}$$

2.

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c + q_L\beta\gamma.(L/F)^2} = \frac{p}{q + q_L\beta\gamma.(L/F)^2} \tag{3.1.7}$$

whose are only adjustments of similar results of zero and one model, and

1.

$$q_L = \frac{C.\left(\mu_0 - \mu - \dfrac{\partial F}{\partial K_0}\right)}{\beta\gamma.\left(\dfrac{L}{F}\right)^2 \dfrac{\partial F}{\partial K_0}} \cdot e^{(\delta+\mu)t} \quad . \tag{3.1.8}$$

.

Since shadow price must increase over the time, implies that

$$\frac{\partial F}{\partial K_0} < \mu_0 - \mu \quad .$$

3.2 Suggestions

As said in chapter 0, section 0.3:

> *"... as well as a man needs food, the machine needs energy."*

This simple idea can inspire different macroeconomic models since the energy would be the "consumption" of the machine.

For the model presented in chapter 1 section 1.3, one can assume a particular production function and then it may yield new results. Further, one can try to solve or to simulate the partial differential equations that appear due the maximum principle in the chapter 2.

In this dissertation, it was only analyzed the macroeconomic aspects. Therefore, it is natural to suggest a microeconomic study concerning the energy sector. In particular, one should study why the solar energy is not used in Brazil. One should, for instance, try to attain the following objectives:

1. To study the tradeoff between prices for solar water heating system (SWHS) device and its lifetime cycle;

2. To study the consumer's decision problem: to invest or not to invest in a Solar Water Heating System, SWHS, device?

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A Appendix

A.1 Optimal control problem 0

$$\mathrm{Max}\, W = \int_0^\infty e^{-\delta t} Lu(c, \alpha, \beta) dt, \tag{A.1.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc \tag{A.1.2}$$

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{A.1.3}$$

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{A.1.4}$$

$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{A.1.5}$$

$$E_H = F_H(K_H, AL_H)G_H(D) \tag{A.1.6}$$

$$W = F_W(K_W, AL_W)G_W(D) \tag{A.1.7}$$

$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{A.1.8}$$

$$\dot{D}_i = -E_i; \quad i = R, NR \tag{A.1.9}$$

$$\dot{D} = f(t) - (E_H + W) \tag{A.1.10}$$

$$\dot{L} = \beta L \tag{A.1.11}$$

$$\dot{A} = \epsilon A \tag{A.1.12}$$

A.2 Hamiltonian

$$\begin{aligned} H =& e^{-\delta t}\big\{Lu(c, \alpha, \beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A \epsilon A + q_L \beta L\big\} \end{aligned} \tag{A.2.1}$$

From expression A.1.2 the variable c can be represented as

$$c = \frac{1}{L}\big[F(K_0, AL_0, E, W) - I_0 - I_W - I_H - I_R - I_{NR} - s(E_E)\big]. \tag{A.2.2}$$

Therefore, one obtains:

$$\frac{\partial c}{\partial F} = \frac{1}{L} \quad ; \frac{\partial c}{\partial I_i} = -\frac{1}{L} \quad (i = 0, R, NR, H, W) \quad ; \frac{\partial c}{\partial s} = -\frac{1}{L} \quad . \tag{A.2.3}$$

From $E = E_R + E_{NR} + E_H + E_E - W - L\alpha$, one gets then

$$\frac{\partial E}{\partial E_i} = 1 \quad (i = R, NR, H, E) \quad ; \frac{\partial E}{\partial W} = -1 \quad ; \frac{\partial E}{\partial \alpha} = -L \quad . \tag{A.2.4}$$

A.3 Control forces

- I_i, $i = 0, R, NR, H, W$;
- E_i, $i = R, NR, H$;
- W, E_E, α and β.

To maximize the Hamiltonian:

$$\frac{\partial H}{\partial I_i} = 0 \quad ; i = 0, W, R, NR, H. \tag{A.3.1}$$

$$\begin{aligned}\frac{\partial H}{\partial I_i} =& e^{-\delta t}\frac{\partial}{\partial I_i}\big\{Lu(c(I_i)) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial I_i} + q_i\Big\} = 0\end{aligned}$$

Thus,

$$q_i = \frac{\partial u}{\partial c} \quad ; i = 0, W, R, NR, H. \tag{A.3.2}$$

$$\frac{\partial H}{\partial E_H} = 0 \tag{A.3.3}$$

$$\begin{aligned}
\frac{\partial H}{\partial E_H} =& e^{-\delta t}\frac{\partial}{\partial E_H}\big\{Lu(c(F(E(E_H)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\
=& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H} - p_D\Big\} = 0
\end{aligned}$$

Thus,

$$p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} = 0 \tag{A.3.4}$$

From A.3.2, and assuming $\partial u/\partial c > 0$:

$$\frac{p_D}{q} = \frac{\partial F}{\partial E} \tag{A.3.5}$$

$$\frac{\partial H}{\partial E_i} = 0 \quad ; i = R, NR \tag{A.3.6}$$

$$\begin{aligned}
\frac{\partial H}{\partial E_i} =& e^{-\delta t}\frac{\partial}{\partial E_i}\big\{Lu(c(F(E(E_i)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\
=& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} - p_i\Big\} = 0
\end{aligned}$$

Thus,

$$p_i - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} = 0; \quad i = R, NR \tag{A.3.7}$$

From A.3.2:

$$\frac{p_i}{q} = \frac{\partial F}{\partial E}; \quad i = R, NR. \tag{A.3.8}$$

$$\frac{\partial H}{\partial E_E} = 0 \tag{A.3.9}$$

$$\begin{aligned}
\frac{\partial H}{\partial E_E} =& e^{-\delta t}\frac{\partial}{\partial E_E}\big\{Lu(c(F(E(E_E)), s(E_E))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\
=& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_E} + \frac{\partial c}{\partial s}.\frac{\partial s}{\partial E_E}\Big)\Big\} \\
=& e^{-\delta t}\Big\{\frac{\partial u}{\partial c}\Big(\frac{\partial F}{\partial E} - \frac{\partial s}{\partial E_E}\Big)\Big\} = 0
\end{aligned}$$

Thus, assuming $\partial u/\partial c > 0$:

$$\frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E} \tag{A.3.10}$$

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E}; \quad i = R, NR, H \tag{A.3.11}$$

$$\frac{\partial H}{\partial W} = 0 \tag{A.3.12}$$

$$\begin{aligned}
\frac{\partial H}{\partial W} =& e^{-\delta t}\frac{\partial}{\partial W}\big\{Lu(c(F(E(W), W))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\
=& e^{-\delta t}\Big\{L.\Big[\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W} + \frac{\partial F}{\partial W}\Big)\Big] - p_D\Big\} = 0
\end{aligned}$$

$$p_D - \frac{\partial u}{\partial c}.\Big(\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big) = 0 \tag{A.3.13}$$

From A.3.4, A.3.13 and assuming $\partial u/\partial c > 0$:

$$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E} \tag{A.3.14}$$

$$\frac{\partial H}{\partial \alpha} = 0 \tag{A.3.15}$$

$$\begin{aligned}\frac{\partial H}{\partial \alpha} =& e^{-\delta t}\frac{\partial}{\partial \alpha}\big\{Lu(c(F(E(\alpha))),\alpha) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A\epsilon A + q_L\beta L\big\} \\ =& e^{-\delta t}\Big\{L.\Big[\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial \alpha} + \frac{\partial u}{\partial \alpha}\Big]\Big\} = 0\end{aligned}$$

Thus,

$$\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} = \frac{\partial u}{\partial \alpha}$$

From A.3.7, A.3.4:

$$p_i = \frac{\partial u}{\partial \alpha}; \quad i = D, R, NR \tag{A.3.16}$$

And assuming $\partial u/\partial c > 0$:

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial c} \tag{A.3.17}$$

$$\frac{\partial H}{\partial \beta} = 0 \tag{A.3.18}$$

$$\begin{aligned}\frac{\partial H}{\partial \beta} =& e^{-\delta t}\frac{\partial}{\partial \beta}\{Lu(\beta)+\sum q_i(-\mu_i K_i+I_i)+\sum p_i(-E_i)+\\ & p_D(f(t)-(E_H+W))+p_A\epsilon A+q_L\beta L\}\\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial \beta}+L.q_L\Big\}=0\end{aligned}$$

Thus,

$$q_L=-\frac{\partial u}{\partial \beta} \tag{A.3.19}$$

For the co-state variables:

$$\frac{de^{-\delta t}.p_i}{dt}=-\frac{\partial H}{\partial D_i}\quad ; i=R,NR \tag{A.3.20}$$

$$\begin{aligned}e^{-\delta t}(\dot{p}_i-\delta p_i)=&-e^{-\delta t}\frac{\partial}{\partial D_i}\{Lu(c(F(E(E_i(D_i)))))+\sum q_i(-\mu_i K_i+I_i)+\\ &\sum p_i(-E_i(D_i))+p_D(f(t)-(E_H+W))+p_A\epsilon A+\\ &q_L\beta L\}\\ =&-e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial D_i}-p_i.\frac{\partial E_i}{\partial D_i}\Big\}\end{aligned}$$

Therefore,

$$\dot{p}_i=\delta p_i+\Big(p_i-\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial D_i}\quad ; i=R,NR.$$

From expression A.3.7:

$$\frac{\dot{p}_i}{p_i}=\delta;\quad i=R,NR. \tag{A.3.21}$$

$$\frac{de^{-\delta t}.p_D}{dt}=-\frac{\partial H}{\partial D} \tag{A.3.22}$$

$$
\begin{aligned}
e^{-\delta t}(\dot{p}_D - \delta p_D) = &- e^{-\delta t}\frac{\partial}{\partial D}\Big\{Lu(c(F(E(E_H(D), W(D)), W(D)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H(D) + W(D))) + p_A \epsilon A + \\
&q_L \beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}\Big[\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial D} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial D}\Big] + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial D}\Big) \\
&- p_D\Big[\frac{\partial E_H}{\partial D} + \frac{\partial W}{\partial D}\Big]\Big\}
\end{aligned}
$$

$$
\dot{p}_D = \delta p_D + \Big(p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial D} + \Big(p_D + \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial W}\Big).\frac{\partial W}{\partial D}
$$

From expressions A.3.4 and A.3.13:

$$
\frac{\dot{p}_D}{p_D} = \delta \tag{A.3.23}
$$

$$
\frac{de^{-\delta t}.q_i}{dt} = -\frac{\partial H}{\partial K_i} \quad ; i = R, NR. \tag{A.3.24}
$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_i - \delta q_i) = &- e^{-\delta t}\frac{\partial}{\partial K_i}\Big\{Lu(c(F(E(E_i(K_i))))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i(K_i)) + p_D(f(t) - (E_H + W)) + p_A \epsilon A + \\
&q_L \beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial K_i} + q_i(-\mu_i) + p_i\Big(- \frac{\partial E_i}{\partial K_i}\Big)\Big\}
\end{aligned}
$$

$$
\dot{q}_i = (\delta + \mu_i)q_i + \Big(p_i - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial K_i} \quad ; i = R, NR.
$$

From expression A.3.7:

$$
\frac{\dot{q}_i}{q_i} = \delta + \mu_i; \quad i = R, NR. \tag{A.3.25}
$$

$$\frac{de^{-\delta t}.q_H}{dt} = -\frac{\partial H}{\partial K_H} \tag{A.3.26}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_H - \delta q_H) =& - e^{-\delta t}\frac{\partial}{\partial K_H}\Big\{Lu(c(F(E(E_H(K_H))))) + \sum q_i(-\mu_i K_i + I_i) + \\
& \sum p_i(-E_i) + p_D(f(t) - (E_H(K_H) + W)) + p_A \epsilon A + \\
& q_L \beta L\Big\} \\
=& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial K_H} + q_H(-\mu_H) + p_D\Big(-\frac{\partial E_H}{\partial K_H}\Big)\Big\}
\end{aligned}$$

$$\dot{q}_H = (\delta + \mu_H)q_H + \Big(p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial K_H}$$

From expression A.3.4:

$$\frac{\dot{q}_H}{q_H} = \delta + \mu_H \tag{A.3.27}$$

$$\frac{de^{-\delta t}.q_W}{dt} = -\frac{\partial H}{\partial K_W} \tag{A.3.28}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_W - \delta q_W) =& - e^{-\delta t}\frac{\partial}{\partial K_W}\Big\{Lu(c(F(E(W(K_W)), W(K_W)))) + \sum q_i(-\mu_i K_i + I_i) + \\
& \sum p_i(-E_i) + p_D(f(t) - (E_H + W(K_W))) + p_A \epsilon A + \\
& q_L \beta L\Big\} \\
=& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W}.\frac{\partial W}{\partial K_W} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial K_W}\Big) + q_W(-\mu_W) \\
& + p_D\Big(-\frac{\partial W}{\partial K_W}\Big)\Big\}
\end{aligned}$$

$$\dot{q}_W = (\delta + \mu_W)q_W + \Big(p_D - \frac{\partial u}{\partial c}\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big]\Big).\frac{\partial W}{\partial K_W}$$

From expression A.3.13:

$$\frac{\dot{q}_W}{q_W} = \delta + \mu_W \tag{A.3.29}$$

From expressions A.3.2, A.3.25, A.3.27 and A.3.29:

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \tag{A.3.30}$$

and

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_D}{q_D} = \frac{\dot{q}_0}{q_0} = \delta + \mu. \tag{A.3.31}$$

$$\frac{de^{-\delta t}.q_0}{dt} = -\frac{\partial H}{\partial K_0} \tag{A.3.32}$$

$$\begin{aligned} e^{-\delta t}(\dot{q}_0 - \delta q_0) =& - e^{-\delta t}\frac{\partial}{\partial K_0}\Big\{Lu(c(F(K_0))) + \sum q_i(-\mu_i K_i + I_i) + \\ & \sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A \epsilon A + \\ & q_L \beta L\Big\} \\ =& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial K_0} + q_0(-\mu_0)\Big\} \end{aligned}$$

$$\dot{q}_0 = (\delta + \mu_0)q_0 - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial K_0}$$

From expression A.3.2:

$$\frac{\dot{q}_0}{q_0} = \delta + \mu_0 - \frac{\partial F}{\partial K_0} \tag{A.3.33}$$

From expression A.3.31:

$$\frac{\partial F}{\partial K_0} = \mu_0 - \mu \tag{A.3.34}$$

Assuming $\partial F/\partial K_0 > 0$:

$$\mu_0 > \mu \tag{A.3.35}$$

$$\frac{de^{-\delta t}.p_A}{dt} = -\frac{\partial H}{\partial A} \tag{A.3.36}$$

$$\begin{aligned}
e^{-\delta t}(\dot{p}_A - \delta p_A) =& -e^{-\delta t}\frac{\partial}{\partial A}\Big\{Lu(c(F(AL_0, E(E_H(F_H(AL_H)), E_R(F_R(AL_R)) \\
&, E_{NR}(F_{NR}(AL_{NR})), W(F_W(AL_W))), W(F_W(AL_W))))) \\
&+ \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i(F_i(AL_i))) \\
&+ p_D(f(t) - (E_H(F_H(AL_H)) + W(F_W(AL_W))))+ \\
&p_A.\epsilon A + q_L \beta L\Big\} \\
=& -e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}\Big[\frac{\partial F}{\partial AL_0}.\frac{\partial AL_0}{\partial A} + \frac{\partial F}{\partial E}\Big\{\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}. \\
&\frac{\partial AL_H}{\partial A} + \frac{\partial E}{\partial E_R}.\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A} + \frac{\partial E}{\partial E_{NR}}.\frac{\partial E_{NR}}{\partial F_{NR}}.\frac{\partial F_{NR}}{\partial AL_{NR}}. \\
&\frac{\partial AL_{NR}}{\partial A} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big\} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}. \\
&\frac{\partial AL_W}{\partial A}\Big]\Big) + p_R\Big(-\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A}\Big) + p_{NR}\Big(-\frac{\partial E_{NR}}{\partial F_{NR}}. \\
&\frac{\partial F_{NR}}{\partial AL_{NR}}.\frac{\partial AL_{NR}}{\partial A}\Big) + p_D\Big(-\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\frac{\partial AL_H}{\partial A} - \frac{\partial W}{\partial F_W}. \\
&\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big) + p_A\epsilon
\end{aligned}$$

$$\dot{p}_A = (\delta - \epsilon)p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0 - G_H(D).\frac{\partial F_H}{\partial AL_H}.L_H\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_D\Big) - G_R(D_R).\frac{\partial F_R}{\partial AL_R}.L_R\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_R\Big) - G_{NR}(D_{NR}).\frac{\partial F_{NR}}{\partial AL_{NR}}.L_{NR}\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_{NR}\Big) - G_W(D).\frac{\partial F_W}{\partial AL_W}.L_W\Big(- \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_D + \frac{\partial u}{\partial c}.\frac{\partial F}{\partial W}\Big)$$

$$\dot{p}_A = (\delta - \epsilon)p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0$$

$$\dot{p}_A = (\delta - \epsilon))p_A - q_0\frac{\partial F}{\partial AL_0}.L_0 \tag{A.3.37}$$

$$\frac{de^{-\delta t}.q_L}{dt} = -\frac{\partial H}{\partial L} \tag{A.3.38}$$

$$\begin{aligned} e^{-\delta t}(\dot{q}_L - \delta q_L) = & - e^{-\delta t}\frac{\partial}{\partial L}\Big\{Lu(c(L, F(E(L)))) + \sum q_i(-\mu_i K_i + I_i) + \\ & \sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A.\epsilon A + \\ & q_L \beta L\Big\} \\ = & - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial L} + \frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial L}\Big) + u + q_L\beta\Big\} \\ = & - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(- \frac{c}{L} - \alpha.\frac{\partial F}{\partial E}\Big) + u + q_L\beta\Big\} \end{aligned}$$

$$\dot{q}_L = (\delta - \beta)q_L + \frac{\partial u}{\partial c}\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u$$

$$\dot{q}_L = (\delta - \beta)q_L + q_0\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{A.3.39}$$

B Appendix

B.1 Optimal control problem 1

$$\text{Max}\, W = \int_0^\infty e^{-\delta t} Lu(c, \alpha, \beta) dt, \tag{B.1.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc \tag{B.1.2}$$

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{B.1.3}$$

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{B.1.4}$$

$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{B.1.5}$$

$$E_H = F_H(K_H, AL_H)G_H(D) \tag{B.1.6}$$

$$W = F_W(K_W, AL_W)G_W(D) \tag{B.1.7}$$

$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{B.1.8}$$

$$\dot{D}_i = -E_i; \quad i = R, NR \tag{B.1.9}$$

$$\dot{D} = f(t) - (E_H + W) \tag{B.1.10}$$

$$\dot{L} = \beta L \tag{B.1.11}$$

$$\dot{A} = F_A(K_A, L_A)A \tag{B.1.12}$$

B.2 Hamiltonian

$$\begin{aligned} H =& e^{-\delta t}\big\{Lu(c, \alpha, \beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L \beta L\big\} \end{aligned} \tag{B.2.1}$$

From expression B.1.2 the variable c can be represented as

$$c = \frac{1}{L}\big[F(K_0, AL_0, E, W) - I_0 - I_W - I_H - I_R - I_{NR} - I_A - s(E_E)\big]. \tag{B.2.2}$$

Therefore, one obtains:

$$\frac{\partial c}{\partial F} = \frac{1}{L} \quad ; \frac{\partial c}{\partial I_i} = -\frac{1}{L} \quad (i = 0, R, NR, H, W, A) \quad ; \frac{\partial c}{\partial s} = -\frac{1}{L} \quad . \tag{B.2.3}$$

From $E = E_R + E_{NR} + E_H + E_E - W - L\alpha$, one gets then

$$\frac{\partial E}{\partial E_i} = 1 \quad (i = R, NR, H, E) \quad ; \frac{\partial E}{\partial W} = -1; \quad \frac{\partial E}{\partial \alpha} = -L \quad . \tag{B.2.4}$$

B.3 Control forces

- I_i, $i = 0, R, NR, H, W, A$;
- E_i, $i = R, NR, H$;
- W, E_E, α and β.

To maximize the Hamiltonian:

$$\frac{\partial H}{\partial I_i} = 0; \quad i = 0, W, R, NR, H, A. \tag{B.3.1}$$

$$\begin{aligned}\frac{\partial H}{\partial I_i} =& e^{-\delta t}\frac{\partial}{\partial I_i}\big\{Lu(c(I_i)) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L \beta L\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial I_i} + q_i\Big\} = 0\end{aligned}$$

Thus,

$$q_i = \frac{\partial u}{\partial c}; \quad i = 0, W, R, NR, H, A. \tag{B.3.2}$$

$$\frac{\partial H}{\partial E_H} = 0 \tag{B.3.3}$$

$$\begin{aligned}\frac{\partial H}{\partial E_H} =& e^{-\delta t}\frac{\partial}{\partial E_H}\{Lu(c(F(E(E_H)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H} - p_D\Big\} = 0\end{aligned}$$

Thus,

$$p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} = 0 \tag{B.3.4}$$

From B.3.2, and assuming $\partial u/\partial c > 0$:

$$\frac{p_D}{q} = \frac{\partial F}{\partial E} \tag{B.3.5}$$

$$\frac{\partial H}{\partial E_i} = 0; \quad i = R, NR \tag{B.3.6}$$

$$\begin{aligned}\frac{\partial H}{\partial E_i} =& e^{-\delta t}\frac{\partial}{\partial E_i}\{Lu(c(F(E(E_i)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} - p_i\Big\} = 0\end{aligned}$$

Thus,

$$p_i - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} = 0; \quad i = R, NR \tag{B.3.7}$$

From B.3.2 and assuming $\partial u/\partial c > 0$:

$$\frac{p_i}{q} = \frac{\partial F}{\partial E}; \quad i = R, NR. \tag{B.3.8}$$

$$\frac{\partial H}{\partial E_E} = 0 \tag{B.3.9}$$

$$\begin{aligned}
\frac{\partial H}{\partial E_E} =& e^{-\delta t}\frac{\partial}{\partial E_E}\{Lu(c(F(E(E_E)), s(E_E))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L\} \\
=& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_E} + \frac{\partial c}{\partial s}.\frac{\partial s}{\partial E_E}\Big)\Big\} \\
=& e^{-\delta t}\Big\{\frac{\partial u}{\partial c}\Big(\frac{\partial F}{\partial E} - \frac{\partial s}{\partial E_E}\Big)\Big\} = 0
\end{aligned}$$

Thus, assuming $\partial u/\partial c > 0$:

$$\frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E} \tag{B.3.10}$$

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{\partial s}{\partial E_E}; \quad i = R, NR, H \tag{B.3.11}$$

$$\frac{\partial H}{\partial W} = 0 \tag{B.3.12}$$

$$\begin{aligned}
\frac{\partial H}{\partial W} =& e^{-\delta t}\frac{\partial}{\partial W}\{Lu(c(F(E(W), W))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L\} \\
=& e^{-\delta t}\Big\{L.\Big[\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W} + \frac{\partial F}{\partial W}\Big)\Big] - p_D\Big\} = 0
\end{aligned}$$

$$p_D - \frac{\partial u}{\partial c} . \Big(\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E} \Big) = 0 \tag{B.3.13}$$

From B.3.4, B.3.13 and assuming $\partial u / \partial c > 0$:

$$\frac{\partial F}{\partial W} = 2 \frac{\partial F}{\partial E} \tag{B.3.14}$$

$$\frac{\partial H}{\partial \alpha} = 0 \tag{B.3.15}$$

$$\begin{aligned} \frac{\partial H}{\partial \alpha} =& e^{-\delta t} \frac{\partial}{\partial \alpha} \big\{ Lu(c(F(E(\alpha))), \alpha) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L \beta L \big\} \\ =& e^{-\delta t} \Big\{ L . \Big[\frac{\partial u}{\partial c} . \frac{\partial c}{\partial F} . \frac{\partial F}{\partial E} . \frac{\partial E}{\partial \alpha} + \frac{\partial u}{\partial \alpha} \Big] \Big\} = 0 \end{aligned}$$

Thus,

$$\frac{\partial u}{\partial c} . \frac{\partial F}{\partial E} = \frac{\partial u}{\partial \alpha}$$

From B.3.7, B.3.4:

$$p_i = \frac{\partial u}{\partial \alpha}; \quad i = D, R, NR. \tag{B.3.16}$$

And, assuming $\partial u / \partial c > 0$:

$$\frac{\partial F}{\partial E} = \frac{\partial u / \partial \alpha}{\partial u / \partial c} \tag{B.3.17}$$

$$\frac{\partial H}{\partial \beta} = 0 \tag{B.3.18}$$

$$\begin{aligned}\frac{\partial H}{\partial \beta} =& e^{-\delta t}\frac{\partial}{\partial \beta}\big\{Lu(\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial \beta} + L.q_L\Big\} = 0\end{aligned}$$

Thus,

$$q_L = -\frac{\partial u}{\partial \beta} \tag{B.3.19}$$

For the co-state variables:

$$\frac{de^{-\delta t}.p_i}{dt} = -\frac{\partial H}{\partial D_i} \quad ; i = R, NR \tag{B.3.20}$$

$$\begin{aligned}e^{-\delta t}(\dot{p}_i - \delta p_i) =& -e^{-\delta t}\frac{\partial}{\partial D_i}\big\{Lu(c(F(E(E_i(D_i))))) + \sum q_i(-\mu_i K_i + I_i) + \\ & \sum p_i(-E_i(D_i)) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L\beta L\big\} \\ =& -e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial D_i} - p_i.\frac{\partial E_i}{\partial D_i}\Big\}\end{aligned}$$

Therefore,

$$\dot{p}_i = \delta p_i + \Big(p_i - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial D_i} \quad ; i = R, NR.$$

From expression B.3.7:

$$\frac{\dot{p}_i}{p_i} = \delta; \quad i = R, NR. \tag{B.3.21}$$

$$\frac{de^{-\delta t}.p_D}{dt} = -\frac{\partial H}{\partial D} \tag{B.3.22}$$

$$
\begin{aligned}
e^{-\delta t}(\dot{p}_D - \delta p_D) = &- e^{-\delta t}\frac{\partial}{\partial D}\Big\{Lu(c(F(E(E_H(D),W(D)),W(D)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H(D) + W(D))) + p_A(F_A(K_A,L_A)A) + \\
&q_L\beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}\Big[\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial D} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial D}\Big] + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial D}\Big) \\
&- p_D\Big[\frac{\partial E_H}{\partial D} + \frac{\partial W}{\partial D}\Big]\Big\}
\end{aligned}
$$

$$
\dot{p}_D = \delta p_D + \Big(p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial D} + \Big(p_D + \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial W}\Big).\frac{\partial W}{\partial D}
$$

From expressions B.3.4 and B.3.13:

$$
\frac{\dot{p}_D}{p_D} = \delta \tag{B.3.23}
$$

$$
\frac{de^{-\delta t}.q_i}{dt} = -\frac{\partial H}{\partial K_i}; \quad i = R, NR. \tag{B.3.24}
$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_i - \delta q_i) = &- e^{-\delta t}\frac{\partial}{\partial K_i}\Big\{Lu(c(F(E(E_i(K_i))))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i(K_i)) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A,L_A)A) + \\
&q_L\beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial K_i} + q_i(-\mu_i) + p_i\Big(-\frac{\partial E_i}{\partial K_i}\Big)\Big\}
\end{aligned}
$$

$$
\dot{q}_i = (\delta + \mu_i)q_i + \Big(p_i - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial K_i} \quad ; i = R, NR.
$$

From expression B.3.7:

$$
\frac{\dot{q}_i}{q_i} = \delta + \mu_i; \quad i = R, NR. \tag{B.3.25}
$$

$$\frac{de^{-\delta t}.q_H}{dt} = -\frac{\partial H}{\partial K_H} \tag{B.3.26}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_H - \delta q_H) = &- e^{-\delta t}\frac{\partial}{\partial K_H}\Big\{Lu(c(F(E(E_H(K_H))))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H(K_H) + W)) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial K_H} + q_H(-\mu_H) + p_D\Big(-\frac{\partial E_H}{\partial K_H}\Big)\Big\}
\end{aligned}$$

$$\dot{q}_H = (\delta + \mu_H)q_H + \Big(p_D - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial K_H}$$

From expression B.3.4:

$$\frac{\dot{q}_H}{q_H} = \delta + \mu_H \tag{B.3.27}$$

$$\frac{de^{-\delta t}.q_W}{dt} = -\frac{\partial H}{\partial K_W} \tag{B.3.28}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_W - \delta q_W) = &- e^{-\delta t}\frac{\partial}{\partial K_W}\Big\{Lu(c(F(E(W(K_W)), W(K_W)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H + W(K_W))) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W}.\frac{\partial W}{\partial K_W} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial K_W}\Big) + q_W(-\mu_W) + p_D\Big(-\frac{\partial W}{\partial K_W}\Big)\Big\}
\end{aligned}$$

$$\dot{q}_W = (\delta + \mu_W)q_W + \Big(p_D - \frac{\partial u}{\partial c}\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big]\Big).\frac{\partial W}{\partial K_W}$$

From expression B.3.13:

$$\frac{\dot{q}_W}{q_W} = \delta + \mu_W \tag{B.3.29}$$

From expressions B.3.2, B.3.25, B.3.27 and B.3.29:

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \tag{B.3.30}$$

and

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_D}{q_D} = \frac{\dot{q}_A}{q_A} = \frac{\dot{q}_0}{q_0} = \delta + \mu. \tag{B.3.31}$$

$$\frac{de^{-\delta t}.q_0}{dt} = -\frac{\partial H}{\partial K_0} \tag{B.3.32}$$

$$\begin{aligned} e^{-\delta t}(\dot{q}_0 - \delta q_0) = & - e^{-\delta t}\frac{\partial}{\partial K_0}\Big\{Lu(c(F(K_0))) + \sum q_i(-\mu_i K_i + I_i) + \\ & \sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L \beta L\Big\} \\ = & - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial K_0} + q_0(-\mu_0)\Big\} \end{aligned}$$

$$\dot{q}_0 = (\delta + \mu_0)q_0 - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial K_0}$$

From expression B.3.2:

$$\frac{\dot{q}_0}{q_0} = \delta + \mu_0 - \frac{\partial F}{\partial K_0} \tag{B.3.33}$$

From expression B.3.31 and B.3.33:

$$\frac{\partial F}{\partial K_0} = \mu_0 - \mu \tag{B.3.34}$$

Assuming $\partial F/\partial K_0 > 0$:

$$\mu_0 > \mu \tag{B.3.35}$$

$$\frac{de^{-\delta t}.q_A}{dt} = -\frac{\partial H}{\partial K_A} \tag{B.3.36}$$

$$\begin{aligned} e^{-\delta t}(\dot{q}_A - \delta q_A) =& - e^{-\delta t}\frac{\partial}{\partial K_A}\Big\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \\ & \sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L \beta L\Big\} \\ =& - e^{-\delta t}\Big\{q_A(-\mu_A) + p_A A.\frac{\partial F_A}{\partial K_A}\Big\} \end{aligned}$$

$$\dot{q}_A = (\delta + \mu_A)q_A - p_A A.\frac{\partial F_A}{\partial K_A} \tag{B.3.37}$$

From B.3.31:

$$q_A = C.e^{(\delta+\mu)t} \tag{B.3.38}$$

Thus, form B.3.37 and assuming $A, \partial F_A/\partial K_A > 0$:

$$p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)}e^{(\delta+\mu)t} \tag{B.3.39}$$

$$\frac{de^{-\delta t}.p_A}{dt} = -\frac{\partial H}{\partial A} \tag{B.3.40}$$

$$\begin{aligned}
e^{-\delta t}(\dot{p}_A - \delta p_A) =& - e^{-\delta t}\frac{\partial}{\partial A}\Big\{Lu(c(F(AL_0, E(E_H(F_H(AL_H)), E_R(F_R(AL_R))\\
&, E_{NR}(F_{NR}(AL_{NR})), W(F_W(AL_W))), W(F_W(AL_W)))))\\
&+ \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i(F_i(AL_i)))\\
&+ p_D(f(t) - (E_H(F_H(AL_H)) + W(F_W(AL_W))))+\\
&p_A(F_A(K_A, L_A)A) + q_L\beta L\Big\}\\
=& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}\Big[\frac{\partial F}{\partial AL_0}.\frac{\partial AL_0}{\partial A} + \frac{\partial F}{\partial E}\Big\{\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\\
&\frac{\partial AL_H}{\partial A} + \frac{\partial E}{\partial E_R}.\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A} + \frac{\partial E}{\partial E_{NR}}.\frac{\partial E_{NR}}{\partial F_{NR}}.\frac{\partial F_{NR}}{\partial AL_{NR}}.\\
&\frac{\partial AL_{NR}}{\partial A} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big\} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\\
&\frac{\partial AL_W}{\partial A}\Big]\Big) + p_R\Big(-\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A}\Big) + p_{NR}\Big(-\frac{\partial E_{NR}}{\partial F_{NR}}.\\
&\frac{\partial F_{NR}}{\partial AL_{NR}}.\frac{\partial AL_{NR}}{\partial A}\Big) + p_D\Big(-\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\frac{\partial AL_H}{\partial A} - \frac{\partial W}{\partial F_W}.\\
&\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big) + p_A F_A(K_A, L_A)
\end{aligned}$$

$$\begin{aligned}
\dot{p}_A =&(\delta - F_A(K_A, L_A))p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0 - G_H(D).\frac{\partial F_H}{\partial AL_H}.L_H\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_D\Big)\\
&- G_R(D_R).\frac{\partial F_R}{\partial AL_R}.L_R\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_R\Big) - G_{NR}(D_{NR}).\frac{\partial F_{NR}}{\partial AL_{NR}}.L_{NR}\Big(\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E}\\
&- p_{NR}\Big) - G_W(D).\frac{\partial F_W}{\partial AL_W}.L_W\Big(-\frac{\partial u}{\partial c}.\frac{\partial F}{\partial E} - p_D + \frac{\partial u}{\partial c}.\frac{\partial F}{\partial W}\Big)
\end{aligned}$$

$$\dot{p}_A =(\delta - F_A(K_A, L_A))p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0$$

$$\dot{p}_A =(\delta - F_A(K_A, L_A))p_A - q_A\frac{\partial F}{\partial AL_0}.L_0 \tag{B.3.41}$$

$$\frac{de^{-\delta t}.q_L}{dt} = -\frac{\partial H}{\partial L} \tag{B.3.42}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_L - \delta q_L) = &- e^{-\delta t}\frac{\partial}{\partial L}\Big\{Lu(c(L, F(E(L)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial L} + \frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial L}\Big) + u + q_L\beta\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(-\frac{c}{L} - \alpha.\frac{\partial F}{\partial E}\Big) + u + q_L\beta\Big\}
\end{aligned}$$

$$\dot{q}_L = (\delta - \beta)q_L + \frac{\partial u}{\partial c}\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u$$

$$\dot{q}_L = (\delta - \beta)q_L + q_A\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u \tag{B.3.43}$$

C Appendix

C.1 Optimal control problem 2

$$\mathrm{Max}\, W = \int_0^\infty e^{-\delta t} L u(c, \alpha, \beta) dt, \tag{C.1.1}$$

subject to

$$F(K_0, AL_0, E, W) = I_0 + I_W + I_H + I_R + I_{NR} + I_A + s(E_E) + Lc \tag{C.1.2}$$

$$E = E_R + E_{NR} + E_H + E_E - W - L\alpha \tag{C.1.3}$$

$$L = L_0 + L_R + L_{NR} + L_H + L_W + L_A \tag{C.1.4}$$

$$E_i = F_i(K_i, AL_i)G_i(D_i); \quad i = R, NR \tag{C.1.5}$$

$$E_H = F_H(K_H, AL_H)G_H(D) \tag{C.1.6}$$

$$W = F_W(K_W, AL_W)G_W(D) \tag{C.1.7}$$

$$\dot{K}_i = -\mu_i K_i + I_i; \quad i = 0, R, NR, H, W, A \tag{C.1.8}$$

$$\dot{D}_i = -E_i; \quad i = R, NR \tag{C.1.9}$$

$$\dot{D} = f(t) - (E_H + W) \tag{C.1.10}$$

$$\dot{L} = \beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1} L) \tag{C.1.11}$$

$$\dot{A} = F_A(K_A, L_A)A \tag{C.1.12}$$

C.2 Hamiltonian

$$
\begin{aligned}
H = & e^{-\delta t}\big\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
& q_L(\beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1}L))\big\}
\end{aligned} \tag{C.2.1}
$$

From expression C.1.2 the variable c can be represented as

$$c = \frac{1}{L}\big[F(K_0, AL_0, E, W) - I_0 - I_W - I_H - I_R - I_{NR} - I_A - s(E_E)\big]. \tag{C.2.2}$$

Therefore, one obtains:

$$\frac{\partial c}{\partial F} = \frac{1}{L} \quad ; \frac{\partial c}{\partial I_i} = -\frac{1}{L} \quad (i = 0, R, NR, H, W, A) \quad ; \frac{\partial c}{\partial s} = -\frac{1}{L} \quad . \tag{C.2.3}$$

From $E = E_R + E_{NR} + E_H + E_E - W - L\alpha$, one gets then

$$\frac{\partial E}{\partial E_i} = 1 \quad (i = R, NR, H, E) \quad ; \frac{\partial E}{\partial W} = -1 \quad ; \frac{\partial E}{\partial \alpha} = -L \quad . \tag{C.2.4}$$

C.3 Control forces

- I_i, $i = 0, R, NR, H, W, A$;
- E_i, $i = R, NR, H$;
- W, E_E, α and β.

To maximize the Hamiltonian:

$$\frac{\partial H}{\partial I_i} = 0; \quad i = 0, W, R, NR, H, A. \tag{C.3.1}$$

$$\begin{aligned}\frac{\partial H}{\partial I_i} =& e^{-\delta t}\frac{\partial}{\partial I_i}\big\{Lu(c(I_i)) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L(\beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1}L))\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial I_i} + q_i\Big\} = 0\end{aligned}$$

Thus,

$$q_i = \frac{\partial u}{\partial c}; \quad i = 0, W, R, NR, H, A. \tag{C.3.2}$$

$$\frac{\partial H}{\partial E_H} = 0 \tag{C.3.3}$$

$$\begin{aligned}\frac{\partial H}{\partial E_H} =& e^{-\delta t}\frac{\partial}{\partial E_H}\big\{Lu(c(F(E(E_H)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ & q_L(\beta L(1 - \gamma F(E(E_H))^{-1}L))\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H} - p_D + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H}\Big\} = 0\end{aligned}$$

Thus,

$$p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E} = 0 \tag{C.3.4}$$

From C.3.2:

$$\frac{p_D}{q + q_L\beta\gamma.(L/F)^2} = \frac{\partial F}{\partial E} \tag{C.3.5}$$

$$\frac{\partial H}{\partial E_i} = 0; \quad i = R, NR. \tag{C.3.6}$$

$$\begin{aligned}\frac{\partial H}{\partial E_i} =& e^{-\delta t}\frac{\partial}{\partial E_i}\big\{Lu(c(F(E(E_i)))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L(\beta L(1 - \gamma F(E(E_i))^{-1}L))\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} - p_i + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}\Big\} = 0\end{aligned}$$

Thus,

$$p_i - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E} = 0; \quad i = R, NR. \tag{C.3.7}$$

From C.3.2:

$$\frac{p_i}{q + q_L\beta\gamma.(L/F)^2} = \frac{\partial F}{\partial E} \tag{C.3.8}$$

$$\frac{\partial H}{\partial E_E} = 0 \tag{C.3.9}$$

$$\begin{aligned}\frac{\partial H}{\partial E_E} =& e^{-\delta t}\frac{\partial}{\partial E_E}\big\{Lu(c(F(E(E_E)), s(E_E))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\ & p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L(\beta L(1 - \gamma F(E(E_E))^{-1}L))\big\} \\ =& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_E} + \frac{\partial c}{\partial s}.\frac{\partial s}{\partial E_E}\Big) + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_E}\Big\} = 0\end{aligned}$$

Thus,

$$\Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E} - \frac{\partial u}{\partial c}.\frac{\partial s}{\partial E_E} = 0 \tag{C.3.10}$$

Assuming $\partial u/\partial c + q_L\beta\gamma(L/F)^2 > 0$:

$$\frac{\partial F}{\partial E} = \frac{q}{q + q_L\beta\gamma(L/F)^2}.\frac{\partial s}{\partial E_E} \tag{C.3.11}$$

$$\frac{\partial F}{\partial E_i} = \frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i} = \frac{\partial F}{\partial E} = \frac{q}{q + q_L\beta\gamma(L/F)^2}.\frac{\partial s}{\partial E_E} \tag{C.3.12}$$

$$\frac{\partial H}{\partial W} = 0 \tag{C.3.13}$$

$$\begin{aligned}
\frac{\partial H}{\partial W} =& e^{-\delta t}\frac{\partial}{\partial \alpha}\big\{Lu(c(F(E(W),W))) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L(1 - \gamma F(E(W),W)^{-1}L)\big\} \\
=& e^{-\delta t}\Big\{L.\Big(\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big[\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W} + \frac{\partial F}{\partial W}\Big]\Big) - p_D + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\Big[\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W} + \frac{\partial F}{\partial W}\Big]\Big\} = 0
\end{aligned}$$

Thus,

$$p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big] = 0 \tag{C.3.14}$$

From C.3.4, C.3.14:

$$\frac{\partial F}{\partial W} = 2\frac{\partial F}{\partial E} \tag{C.3.15}$$

$$\frac{\partial H}{\partial \alpha} = 0 \tag{C.3.16}$$

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} =& e^{-\delta t}\frac{\partial}{\partial \alpha}\big\{Lu(c(F(E(\alpha)),\alpha)) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L(\beta L(1 - \gamma F(E(\alpha))^{-1}L))\big\} \\
=& e^{-\delta t}\Big\{L.\Big(\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial \alpha} + \frac{\partial u}{\partial \alpha}\Big) + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial \alpha}\Big\} \\
=& L.\frac{\partial u}{\partial \alpha} - L.\frac{\partial F}{\partial E}\Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big) = 0
\end{aligned}$$

Thus,

$$\frac{\partial F}{\partial E} = \frac{\partial u/\partial \alpha}{\partial u/\partial \alpha + q_L\beta\gamma.(L/F)^2} \tag{C.3.17}$$

$$\frac{\partial H}{\partial \beta} = 0 \tag{C.3.18}$$

$$\begin{aligned}
\frac{\partial H}{\partial \beta} =& e^{-\delta t}\frac{\partial}{\partial \beta}\{Lu(\beta) + \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i) + \\
& p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + q_L\beta L(1 - \gamma F(K_0, AL_0, E, W)^{-1}L)\} \\
=& e^{-\delta t}\Big\{L.\frac{\partial u}{\partial \beta} + L.q_L(1 - \gamma F(K_0, AL_0, E, W)^{-1}L)\Big\} = 0
\end{aligned}$$

Thus,

$$\frac{\partial u}{\partial \beta} + q_L\Big(1 - \gamma\frac{L}{F}\Big) = 0 \tag{C.3.19}$$

For the co-state variables:

$$\frac{de^{-\delta t}.p_i}{dt} = -\frac{\partial H}{\partial D_i} \quad ; i = R, NR \tag{C.3.20}$$

$$\begin{aligned}
e^{-\delta t}(\dot{p}_i - \delta p_i) =& -e^{-\delta t}\frac{\partial}{\partial D_i}\{Lu(c(F(E(E_i(D_i))))) + \sum q_i(-\mu_i K_i + I_i) + \\
& \sum p_i(-E_i(D_i)) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
& q_L\beta L(1 - \gamma F(E(E_i(D_i))))^{-1}L)\} \\
=& -e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial D_i} - p_i.\frac{\partial E_i}{\partial D_i} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial D_i}\Big\}
\end{aligned}$$

Therefore,

$$\dot{p}_i = \delta p_i + \Big(p_i - \Big[\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big]\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial D_i} \quad ; i = R, NR.$$

From expression C.3.7:

$$\dot{p}_i = \delta p_i; \quad i = R, NR. \tag{C.3.21}$$

$$\frac{de^{-\delta t}.p_D}{dt} = -\frac{\partial H}{\partial D} \tag{C.3.22}$$

$$\begin{aligned}
e^{-\delta t}(\dot{p}_D - \delta p_D) = & - e^{-\delta t}\frac{\partial}{\partial D}\Big\{Lu(c(F(E(E_H(D), W(D)), W(D)))) + \sum q_i(-\mu_i K_i + I_i) + \\
& \sum p_i(-E_i) + p_D(f(t) - (E_H(D) + W(D))) + p_A(F_A(K_A, L_A)A) + \\
& q_L \beta L(1 - \gamma F(E(E_H(D), W(D))^{-1}L)\Big\} \\
= & - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}\Big[\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial D} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial D}\Big] + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial D}\Big) \\
& - p_D\Big[\frac{\partial E_H}{\partial D} + \frac{\partial W}{\partial D}\Big] + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2.\Big(\frac{\partial F}{\partial E}\Big[\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial D} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial D}\Big] + \\
& \frac{\partial F}{\partial W}.\frac{\partial W}{\partial D}\Big)\Big\}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_D = & \delta p_D + \Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial D} + \\
& \Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big]\Big).\frac{\partial W}{\partial D}
\end{aligned}$$

From expressions C.3.4 and C.3.14:

$$\frac{\dot{p}_D}{p_D} = \delta \tag{C.3.23}$$

$$\frac{de^{-\delta t}.q_i}{dt} = -\frac{\partial H}{\partial K_i} \quad ; i = R, NR. \tag{C.3.24}$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_i - \delta q_i) = &- e^{-\delta t}\frac{\partial}{\partial K_i}\Big\{Lu(c(F(E(E_i(K_i))))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i(K_i)) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L(1 - \gamma F(E(E_i(K_i)))^{-1} L)\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial K_i} + q_i(-\mu_i) + p_i\Big(-\frac{\partial E_i}{\partial K_i}\Big) + \\
&+ q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_i}.\frac{\partial E_i}{\partial K_i}\Big\}
\end{aligned}
$$

$$
\dot{q}_i = (\delta + \mu_i)q_i + \Big(p_i - \Big[\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big]\frac{\partial F}{\partial E}\Big).\frac{\partial E_i}{\partial K_i} \quad ; i = R, NR.
$$

From expression C.3.7:

$$
\frac{\dot{q}_i}{q_i} = \delta + \mu_i; \quad i = R, NR. \tag{C.3.25}
$$

$$
\frac{de^{-\delta t}.q_H}{dt} = -\frac{\partial H}{\partial K_H} \tag{C.3.26}
$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_H - \delta q_H) = &- e^{-\delta t}\frac{\partial}{\partial K_H}\Big\{Lu(c(F(E(E_H(K_H))))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H(K_H) + W)) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L(1 - \gamma F(E(E_H(K_H)))^{-1} L)\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial K_H} + q_H(-\mu_H) + p_D\Big(-\frac{\partial E_H}{\partial K_H}\Big) + \\
&q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial E}.\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial K_H}\Big\}
\end{aligned}
$$

$$
\dot{q}_H = (\delta + \mu_H)q_H + \Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E}\Big).\frac{\partial E_H}{\partial K_H}
$$

From expression C.3.4:

$$\frac{\dot{q}_H}{q_H} = \delta + \mu_H \tag{C.3.27}$$

$$\frac{de^{-\delta t}.q_W}{dt} = -\frac{\partial H}{\partial K_W} \tag{C.3.28}$$

$$\begin{aligned}
e^{-\delta t}(\dot{q}_W - \delta q_W) = &- e^{-\delta t}\frac{\partial}{\partial K_W}\Big\{Lu(c(F(E(W(K_W)), W(K_W)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H + W(K_W))) + p_A(F_A(K_A, L_A)A) + \\
&q_L \beta L(1 - \gamma F(E(W(K_W)), W(K_W))^{-1} L)\Big\} \\
= &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W}.\frac{\partial W}{\partial K_W} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial K_W}\Big) + q_W(-\mu_W) + \\
&p_D\Big(-\frac{\partial W}{\partial K_W}\Big) + q_L \beta \gamma.\Big(\frac{L}{F}\Big)^2\Big(\frac{\partial F}{\partial E}.\frac{\partial E}{\partial W}.\frac{\partial W}{\partial K_W} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial K_W}\Big)\Big\}
\end{aligned}$$

$$\dot{q}_W = (\delta + \mu_W)q_W + \Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L \beta \gamma.\Big(\frac{L}{F}\Big)^2\Big)\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big]\Big).\frac{\partial W}{\partial K_W}$$

From expression C.3.14:

$$\frac{\dot{q}_W}{q_W} = \delta + \mu_W \tag{C.3.29}$$

From expressions C.3.2, C.3.25, C.3.27, C.3.29:

$$\mu_R = \mu_{NR} = \mu_H = \mu_W = \mu \tag{C.3.30}$$

and

$$\frac{\dot{q}_R}{q_R} = \frac{\dot{q}_{NR}}{q_{NR}} = \frac{\dot{q}_H}{q_H} = \frac{\dot{q}_W}{q_W} = \frac{\dot{q}_D}{q_D} = \frac{\dot{q}_A}{q_A} = \frac{\dot{q}_0}{q_0} = \delta + \mu. \tag{C.3.31}$$

$$\frac{de^{-\delta t}.q_0}{dt} = -\frac{\partial H}{\partial K_0} \tag{C.3.32}$$

$$\begin{aligned} e^{-\delta t}(\dot{q}_0 - \delta q_0) = &- e^{-\delta t}\frac{\partial}{\partial K_0}\Big\{Lu(c(F(K_0))) + \sum q_i(-\mu_i K_i + I_i) + \\ &\sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\ &q_L \beta L(1 - \gamma F(K_0)^{-1} L)\Big\} \\ = &- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}.\frac{\partial c}{\partial F}.\frac{\partial F}{\partial K_0} + q_0(-\mu_0) + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial K_0}\Big\} \end{aligned}$$

$$\dot{q}_0 = (\delta + \mu_0)q_0 - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial K_0} - q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial K_0}$$

By expression C.3.2:

$$\dot{q}_0 = \Big(\delta + \mu_0 - \frac{\partial F}{\partial K_0}\Big)q_0 - q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial K_0} \tag{C.3.33}$$

From expression C.3.31:

$$q_0 = C.e^{(\delta+\mu)t} \tag{C.3.34}$$

Thus, from C.3.33:

$$q_L = \frac{C.\left(\mu_0 - \mu - \dfrac{\partial F}{\partial K_0}\right)}{\beta\gamma.\Big(\dfrac{L}{F}\Big)^2\dfrac{\partial F}{\partial K_0}} \cdot e^{(\delta+\mu)t} \tag{C.3.35}$$

$$\frac{de^{-\delta t}.q_A}{dt} = -\frac{\partial H}{\partial K_A} \tag{C.3.36}$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_A - \delta q_A) = & - e^{-\delta t}\frac{\partial}{\partial K_A}\Big\{Lu(c,\alpha,\beta) + \sum q_i(-\mu_i K_i + I_i) + \\
& \sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
& q_L \beta L(1 - F(K_0, AL_0, E, W)^{-1}L)\Big\} \\
= & - e^{-\delta t}\Big\{q_A(-\mu_A) + p_A A.\frac{\partial F_A}{\partial K_A}\Big\}
\end{aligned}
$$

$$
\dot{q}_A = (\delta + \mu_A)q_A - p_A A.\frac{\partial F_A}{\partial K_A} \tag{C.3.37}
$$

From C.3.31:

$$
q_A = C.e^{(\delta+\mu)t} \tag{C.3.38}
$$

Thus, form C.3.37:

$$
p_A = \frac{C.(\mu_A - \mu)}{A.(\partial F_A/\partial K_A)}e^{(\delta+\mu)t} \tag{C.3.39}
$$

$$\frac{de^{-\delta t}.p_A}{dt} = -\frac{\partial H}{\partial A} \tag{C.3.40}$$

$$\begin{aligned}
e^{-\delta t}(\dot{p}_A - \delta p_A) = &- e^{-\delta t}\frac{\partial}{\partial A}\Big\{Lu(c(F(AL_0, E(E_H(F_H(AL_H)), E_R(F_R(AL_R))\\
&, E_{NR}(F_{NR}(AL_{NR})), W(F_W(AL_W))), W(AL_W))))\\
&+ \sum q_i(-\mu_i K_i + I_i) + \sum p_i(-E_i(F_i(AL_i)))\\
&+ p_D(f(t) - (E_H(F_H(AL_H)) + W(F_W(AL_W))))+\\
&p_A(F_A(K_A, L_A)A) + q_L\beta L(1 - \gamma F(AL_0, E(E_H(F_H(AL_H)),\\
&E_R(F_R(AL_R)), E_{NR}(F_{NR}(AL_{NR})), W(F_W(AL_W))), W(AL_W))^{-1}L)\Big\}\\
=&- e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial F}\Big[\frac{\partial F}{\partial AL_0}.\frac{\partial AL_0}{\partial A} + \frac{\partial F}{\partial E}\Big\{\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\\
&\frac{\partial AL_H}{\partial A} + \frac{\partial E}{\partial E_R}.\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A} + \frac{\partial E}{\partial E_{NR}}.\frac{\partial E_{NR}}{\partial F_{NR}}.\frac{\partial F_{NR}}{\partial AL_{NR}}.\\
&\frac{\partial AL_{NR}}{\partial A} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big\} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\\
&\frac{\partial AL_W}{\partial A}\Big]\Big) + p_R\Big(-\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A}\Big) + p_{NR}\Big(-\frac{\partial E_{NR}}{\partial F_{NR}}.\\
&\frac{\partial F_{NR}}{\partial AL_{NR}}.\frac{\partial AL_{NR}}{\partial A}\Big) + p_D\Big(-\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\frac{\partial AL_H}{\partial A} - \frac{\partial W}{\partial F_W}.\\
&\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big) + p_A F_A(K_A, L_A) + q_L\beta\gamma\Big(\frac{L}{F}\Big)^2\Big[\frac{\partial F}{\partial AL_0}.\frac{\partial AL_0}{\partial A}+\\
&\frac{\partial F}{\partial E}\Big\{\frac{\partial E}{\partial E_H}.\frac{\partial E_H}{\partial F_H}.\frac{\partial F_H}{\partial AL_H}.\frac{\partial AL_H}{\partial A}+\\
&\frac{\partial E}{\partial E_R}.\frac{\partial E_R}{\partial F_R}.\frac{\partial F_R}{\partial AL_R}.\frac{\partial AL_R}{\partial A} + \frac{\partial E}{\partial E_{NR}}.\frac{\partial E_{NR}}{\partial F_{NR}}.\frac{\partial F_{NR}}{\partial AL_{NR}}.\\
&\frac{\partial AL_{NR}}{\partial A} + \frac{\partial E}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\frac{\partial AL_W}{\partial A}\Big\} + \frac{\partial F}{\partial W}.\frac{\partial W}{\partial F_W}.\frac{\partial F_W}{\partial AL_W}.\\
&\frac{\partial AL_W}{\partial A}\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
\dot{p}_A =&(\delta - F_A(K_A, L_A))p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0 + \\
&G_H(D).\frac{\partial F_H}{\partial AL_H}.L_H\Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E}\Big) + \\
&G_R(D_R).\frac{\partial F_R}{\partial AL_R}.L_R\Big(p_R - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E}\Big) + \\
&G_{NR}(D_{NR}).\frac{\partial F_{NR}}{\partial AL_{NR}}.L_{NR}\Big(p_{NR} - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\frac{\partial F}{\partial E}\Big) + \\
&G_W(D).\frac{\partial F_W}{\partial AL_W}.L_W\Big(p_D - \Big(\frac{\partial u}{\partial c} + q_L\beta\gamma.\Big(\frac{L}{F}\Big)^2\Big)\Big[\frac{\partial F}{\partial W} - \frac{\partial F}{\partial E}\Big]\Big) - \\
&q_L\beta\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial AL_0}L_0
\end{aligned}
$$

$$
\dot{p}_A =(\delta - F_A(K_A, L_A))p_A - \frac{\partial u}{\partial c}.\frac{\partial F}{\partial AL_0}.L_0 - q_L\beta\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial AL_0}L_0
$$

$$
\dot{p}_A =(\delta - F_A(K_A, L_A))p_A - q_A\frac{\partial F}{\partial AL_0}.L_0 - q_L\beta\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial AL_0}L_0 \tag{C.3.41}
$$

$$
\frac{de^{-\delta t}.q_L}{dt} = -\frac{\partial H}{\partial L} \tag{C.3.42}
$$

$$
\begin{aligned}
e^{-\delta t}(\dot{q}_L - \delta q_L) =& - e^{-\delta t}\frac{\partial}{\partial L}\Big\{Lu(c(L, F(E(L)))) + \sum q_i(-\mu_i K_i + I_i) + \\
&\sum p_i(-E_i) + p_D(f(t) - (E_H + W)) + p_A(F_A(K_A, L_A)A) + \\
&q_L\beta L(1 - F(E(L))^{-1}L)\Big\} \\
=& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(\frac{\partial c}{\partial L} + \frac{\partial c}{\partial F}.\frac{\partial F}{\partial E}.\frac{\partial E}{\partial L}\Big) + u + \\
&q_L\beta\Big(1 - 2\gamma\Big(\frac{L}{F}\Big) - \alpha\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial E}\Big)\Big\} \\
=& - e^{-\delta t}\Big\{L.\frac{\partial u}{\partial c}\Big(- \frac{c}{L} - \alpha.\frac{\partial F}{\partial E}\Big) + u + \\
&q_L\beta\Big(1 - 2\gamma\Big(\frac{L}{F}\Big) - \alpha\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial E}\Big)\Big\}
\end{aligned}
$$

$$
\dot{q}_L = \Big(\delta - \beta\Big[1 - 2\gamma\Big(\frac{L}{F}\Big) - \alpha\gamma\Big(\frac{L}{F}\Big)^2\frac{\partial F}{\partial E}\Big]\Big)q_L + \frac{\partial u}{\partial c}\Big(c + \alpha.\frac{\partial F}{\partial E}\Big) - u
$$

$$\dot{q}_L = \left(\delta - \beta\left[1 - 2\gamma\left(\frac{L}{F}\right) - \alpha\gamma\left(\frac{L}{F}\right)^2 \frac{\partial F}{\partial E}\right]\right) q_L + q_A\left(c + \alpha.\frac{\partial F}{\partial E}\right) - u \tag{C.3.43}$$

D Appendix

D.1 Model 1

The income identity,

$$Y = C + I = F(K, L). \tag{D.1.1}$$

The first modified labor force growth model,

$$\dot{L} = \beta L \left(1 - \frac{L}{F(K, L)}\right). \tag{D.1.2}$$

The gross investment identity,

$$\dot{K} = -\mu K + I. = -\mu K + F(K, L) - C \tag{D.1.3}$$

From D.1.1 and D.1.3:

$$\dot{K} = -\mu K + F(K, L) - C \tag{D.1.4}$$

Remember that the production function exhibits constant returns of scale, that is,

$$F(\alpha K, \alpha L) = \alpha F(K, L), \tag{D.1.5}$$

where α is a positive real number.

Thus,

$$\left(\frac{\dot{K}}{L}\right) = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}.\beta\left(1 - \frac{L}{F(K,L)}\right) \tag{D.1.6}$$

$$= -\mu k + f(k) - c - k.\beta\left(1 - \frac{1}{f(k)}\right) \tag{D.1.7}$$

$$= -(\mu + \beta)k + f(k) + \frac{\beta.k}{f(k)} - c \tag{D.1.8}$$

$$= -\lambda k + f(k) + \frac{\beta.k}{f(k)} - c \tag{D.1.9}$$

$$= -\lambda k + g(k) - c \tag{D.1.10}$$

where $g(k) = f(k) + \dfrac{\beta.k}{f(k)}$ and $\lambda = \mu + \beta$.

$$J = \int_c^\infty e^{-\delta t} u(c)dt \tag{D.1.11}$$

$$H = e^{-\delta t}\left\{u(c) + y\left(-\lambda k + g(k) - c\right)\right\} \tag{D.1.12}$$

$$\frac{\partial H}{\partial c} = \frac{\partial u}{\partial c} - y = 0 \tag{D.1.13}$$

$$y = \frac{\partial u}{\partial c} \tag{D.1.14}$$

$$\frac{de^{-\delta t}y}{dt} = -\frac{\partial H}{\partial k} = e^{-\delta t}y\left(\lambda - g'(k)\right) \tag{D.1.15}$$

$$\frac{\frac{dy}{dt}}{y} = \lambda + \delta - g'(k) = -\sigma(c).\frac{\dot{c}}{c} \tag{D.1.16}$$

$$\dot{c} = \frac{1}{\sigma(c)} \left(-(\lambda + \delta) + g'(k) \right) c \tag{D.1.17}$$

Thus,

$$\begin{cases} \dot{c} = \dfrac{1}{\sigma(c)}(-(\lambda + \delta) + g'(k))c \\ \dot{k} = -\lambda k + g(k) - c \end{cases} \tag{D.1.18}$$

E Appendix 5

E.1 Model 1

The income identity,

$$Y = C + I = F(K, AL). \tag{E.1.1}$$

The first modified labor force growth model,

$$\dot{L} = \beta L \left(1 - \frac{\gamma AL}{F(K, AL)}\right). \tag{E.1.2}$$

The gross investment identity,

$$\dot{K} = -\mu K + I. = -\mu K + F(K, AL) - C \tag{E.1.3}$$

From E.1.1 and E.1.3:

$$\dot{K} = -\mu K + F(K, AL) - C \tag{E.1.4}$$

Remember that the production function exhibits constant returns of scale, that is,

$$F(\alpha K, \alpha AL) = \alpha F(K, AL), \tag{E.1.5}$$

where α is a positive real number.

Thus,

$$\left(\frac{\dot{K}}{AL}\right) = \frac{AL\dot{K} - K\dot{AL}}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL} \cdot \frac{A}{A} \cdot \frac{\dot{L}}{L} - \frac{K}{AL} \cdot \frac{L}{L} \cdot \frac{\dot{A}}{A} \tag{E.1.6}$$

$$= -\frac{\mu K}{AL} + \frac{F(K, AL)}{AL} - \frac{C}{AL} - \frac{K}{AL} \cdot \frac{\dot{L}}{L} - \frac{K}{AL} \cdot \frac{\dot{A}}{A} \tag{E.1.7}$$

$$= -\mu k + f(k) - c - k.\beta\left(1 - \frac{1}{f(k)}\right) - k\gamma \tag{E.1.8}$$

$$= -(\mu + \beta + \gamma)k + f(k) + \frac{\beta.k}{f(k)} - c \tag{E.1.9}$$

$$= -\lambda k + f(k) + \frac{\beta.k}{f(k)} - c \tag{E.1.10}$$

$$= -\lambda k + g(k) - c \tag{E.1.11}$$

where $g(k) = f(k) + \dfrac{\beta.k}{f(k)}$ and $\lambda = \mu + \beta\gamma$.

$$J = \int_c^{\infty} e^{-\delta t} u(c) dt \tag{E.1.12}$$

$$H = e^{-\delta t} \left\{u(c) + y\left(-\lambda k + g(k) - c\right)\right\} \tag{E.1.13}$$

$$\frac{\partial H}{\partial c} = \frac{\partial u}{\partial c} - y = 0 \tag{E.1.14}$$

$$y = \frac{\partial u}{\partial c} \tag{E.1.15}$$

$$\frac{de^{-\delta t} y}{dt} = -\frac{\partial H}{\partial k} = e^{-\delta t} y\left(\lambda - g'(k)\right) \tag{E.1.16}$$

$$\frac{\frac{dy}{dt}}{y} = \lambda + \delta - g'(k) = -\sigma(c).\frac{\dot{c}}{c} \tag{E.1.17}$$

$$\dot{c} = \frac{1}{\sigma(c)} \left(-(\lambda + \delta) + g'(k) \right) c \tag{E.1.18}$$

Thus,

$$\begin{cases} \dot{c} = \dfrac{1}{\sigma(c)}(-(\lambda + \delta) + g'(k))c \\ \dot{k} = -\lambda k + g(k) - c \end{cases} \tag{E.1.19}$$