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LÁZARO RANGEL SILVA DE ASSIS

ON THE GENERALIZED FRACTIONAL SOBOLEV SPACES AND APPLICATIONS

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LÁZARO RANGEL SILVA DE ASSIS

ON THE GENERALIZED FRACTIONAL SOBOLEV SPACES AND APPLICATIONS

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LÁZARO RANGEL SILVA DE ASSIS

On the generalized fractional Sobolev spaces and applications

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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Não se perturbar com a grandeza das dores e problemas é um teste da nossa resiliência interior. A adversidade é uma parte inerente da vida, e nossa atitude perante ela define nosso caráter. Ao encarar as dificuldades com serenidade, fortalecemos nossa alma e nos tornamos mais capazes de enfrentar os desafios. Lembre-se de que a magnitude das dores é uma oportunidade para o crescimento e o autodomínio, permitindo-nos transcender as dificuldades e emergir mais fortes (AURÉLIO et al., 2023, p. 29).

RESUMO

Nesta tese, estudamos algumas generalizações dos espaços de Sobolev de ordem fracionária e aplicações. Especificamente, no caso dos espaços de Orlicz-Sobolev fracionários, apresentamos uma visão geral dos desenvolvimentos recentes na teoria, com foco em propriedades qualitativas e resultados de imersão. Em seguida, aplicamos esses resultados, juntamente com o método do quociente de Rayleigh não linear e o método de minimização na variedade de Nehari, para investigar condições que garantem a existência de soluções não triviais para uma classe de problemas do tipo Φ-Laplaciano fracionário superlinear com dois parâmetros. No contexto dos espaços de Musielak-Sobolev fracionários, estendemos e complementamos os resultados teóricos existentes. Mais precisamente, estabelecemos alguns resultados abstratos, como convexidade uniforme, a propriedade Radon-Riesz com relação à função modular, a propriedade (S_+), um lema do tipo Brezis-Lieb para a função modular e resultados de monotonicidade. Além disso, aplicamos a teoria desenvolvida para estudar a existência de soluções para uma classe de problemas envolvendo um operador não local e não linear geral do tipo Φ-Laplaciano fracionário. Por fim, estudamos o comportamento assintótico de funções modulares e seminormas associadas a espaços fracionários de Musielak-Sobolev quando o parâmetro fracionário se aproxima de 1, sem exigir a condição Δ_2 na função de Musielak ou em sua função complementar. Esta investigação culmina em uma fórmula do tipo Bourgain-Brezis-Mironescu para uma família muito geral de funcionais. É importante enfatizar que a obtenção desses resultados exigiu a introdução de hipóteses específicas sobre as funções de Musielak envolvidas.

Palavras-chaves: Espaços de Orlicz-Sobolev fracionários. Espaços de Musielak-Sobolev fracionários. Problemas não locais. Método do quociente do Rayleigh não linear. Fórmula do tipo Bourgain-Brezis-Mironescu.

ABSTRACT

In this thesis, we study some generalizations of the fractional order Sobolev spaces and applications. Specifically, in the case of the fractional Orlicz-Sobolev spaces, we present an overview of recent developments in the theory, focusing on qualitative properties and embedding results. We then apply these results, along with the nonlinear Rayleigh quotient method and the minimization method on the Nehari manifold, to investigate conditions that ensure the existence of nontrivial solutions to a class of superlinear fractional Φ-Laplacian type problems with two parameters. In the context of fractional Musielak-Sobolev spaces, we extend and complement the existing theoretical results. More precisely, we establish some abstract results, such as uniform convexity, the Radon-Riesz property with respect to the modular function, the (S_+)-property, a Brezis-Lieb type lemma for the modular function, and monotonicity results. Moreover, we apply the developed theory to study the existence of solutions to a class of problems involving a general nonlocal nonlinear operator of the fractional Φ-Laplacian type. Finally, we study the asymptotic behavior of modular functions and seminorms associated with fractional Musielak-Sobolev spaces as the fractional parameter approaches 1, without requiring the Δ_2-condition on the Musielak function or its complementary function. This investigation culminates in a Bourgain-Brezis-Mironescu type formula for a very general family of functionals. It is important to emphasize that the achieving these results required the introduction of specific assumptions regarding the Musielak functions involved.

Keywords: Fractional Orlicz-Sobolev spaces. Fractional Musielak-Sobolev spaces. Nonlocal problems. Nonlinear Rayleigh quotient method. Bourgain-Brezis-Mironescu type formula.

LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^N$	N-dimensional Euclidean space
Ω	Open subset of $\mathbb{R}^N$
$\partial\Omega$	Boundary of Ω
x	$(x_1, \dots, x_n)$ point in $\mathbb{R}^N$
$\|x\|$	Euclidean norm of a point x in $\mathbb{R}^N$
$B_r(x)$	Open ball in $\mathbb{R}^N$ with center x and radius $r > 0$
$\mathbb{S}^{n-1}$	$(n-1)$-dimensional unit sphere in $\mathbb{R}^N$
w_n	N-th coordinate of a point in $\mathbb{S}^{N-1}$
ω_N	Volume of the unit ball in $\mathbb{R}^N$
$\|A\|$	Lebesgue measure of the subset A of $\mathbb{R}^N$
dx	$dx_1, \dots, dx_N$ Lebesgue measure in $\mathbb{R}^N$
a.e.	Almost everywhere
χ_A	Characteristic function of a subset A
$\nearrow$	Limit of an increasing sequence
$\to$	Strong convergence
$\rightharpoonup$	Weak convergence
$\hookrightarrow$	Continuous embedding
$\to a^-$	Tends to a from the left
$\to a^+$	Tends to a from the right
C, C_i	Positive constants (possibly different)

∇u	Gradient of a function u
X	Arbitrary vector space
X^*	Dual space of the vector space X
$C^\infty(\Omega)$	Space of infinitely differentiable functions on Ω
$C_0^\infty(\Omega)$	Infinitely differentiable functions with compact support on Ω
(A, Σ, μ)	Measure space
L^0	Set of measurable functions
L^r	Classical Lebesgue space
$\|\cdot\|_r$	Usual norm in the space L^r, $r \in [1, \infty]$
r'	Dual exponent, $\frac{1}{r} + \frac{1}{r'} = 1$
S_r	Denotes the best constant for the continuous embedding $X \hookrightarrow L^r$
$W^{1,p}$	Classical Sobolev space
$W^{1,1}_{loc}$	Space of locally integrable functions whose first-order weak derivatives are also locally integrable
$W^{s,p}$	Fractional Sobolev space
L^Φ	Orlicz space
$W^{1,\Phi}$	Orlicz Sobolev space
$W^{s,\Phi}$	Fractional Orlicz-Sobolev space
L^{Φ_x}	Musielak space
W^{1,Φ_x}	Musielak-Sobolev space
$W^{s,\Phi_{x,y}}$	Fractional Musielak-Sobolev space

CONTENTS

1 INTRODUCTION

The evolution of Banach function spaces theory over the last century has seen a fascinating broadening of spaces beyond the classical Lebesgue spaces L^p. This journey began in the 1930s with the pioneering work of Birnbaum and Orlicz (1931), who laid down the initial framework for generalizing Lebesgue spaces. The foundation for these advancements was already being set by Young (1911), whose work on convex functions provided essential tools for the development of these new spaces.

In the second half of 1930, the concept of variable exponent Lebesgue spaces was introduced by Orlicz (1931), marking an important step in expanding the theory. Shortly thereafter, Orlicz (1932) shifted focus from variable exponent spaces to develop the theory of what are now known as Orlicz spaces, with an additional condition called Δ_2-condition. A few years later, Orlicz (1936) presents your spaces in full generality. This theory was further developed and systematized in the influential book by Krasnosel'skii and Rutickii (1961), which treated Orlicz spaces in connection with N-functions and Lebesgue measurable subsets in $\mathbb{R}^N$. The general framework for Young functions and arbitrary non-atomic measures was later formalized in the book by Rao and Ren (1985), providing a comprehensive understanding of these spaces and their applications.

The Orlicz spaces and Lebesgue variable exponent spaces have different nature, and neither of them is a subset of the other. In general terms, the former are obtained by replacing the function t^p by a function $t^{p(x)}$, where the exponent $p(\cdot)$ is allowed to depend on the spatial variable x. The latter arise when the role of the power t^p is played by a more general Young function $\Phi(\cdot)$, that is, a non-negative convex function that vanishes at 0. However, Nakano (1950) succeeds in encompassing these two spaces by abstracting certain central properties of the Young function, one is led to a more general class of so-called modular spaces.

Following the work of Nakano, a generalization of modular spaces were formalized by Musielak and Orlicz (1959). In this paper, the authors also presented some example of modular spaces generated by special functions, what would later be called Musielak-Orlicz spaces (also known as generalized Orlicz spaces). Somewhat later, the theory of these spaces were systematically presented in the comprehensive monograph by Musielak (1983), playing a key role in the functional analysis of modular spaces.

These notable spaces are built upon measure space (Ω, Σ, μ) and generalized Young functions $\Phi : \Omega \times [0, \infty) \to [0, \infty)$, also known as Musielak function, namely, are measurable functions in the variable $x \in \Omega$ for each $t \in [0, \infty)$ fixed, and Young functions in the variable t for μ-a.e. $x \in \Omega$ fixed. In precisely terms, the Musielak-Orlicz space L^{Φ_x} is defined as

$$L^{\Phi_x}(\Omega, \mu) = \{u : \Omega \to \mathbb{R}\ \mu\text{-measurable}\colon J_\Phi(\lambda u) < \infty \ \text{ for some } \lambda > 0\}$$

and equipped with the Luxemburg norm

$$\|u\|_{\Phi_x} = \inf \left\{ \lambda > 0\colon J_\Phi\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

associated to modular function J_Φ given by

$$J_\Phi(u) = \int_\Omega \Phi(x, |u(x)|)\, dx.$$

At this point, two possible extensions of the classical Sobolev spaces could be considered. The first one is the Orlicz-Sobolev spaces. A systematic study of these spaces in connection with the analysis of nonlinear partial differential equations without a polynomial growth was initiated by important works by Donaldson (1971), Donaldson and Trudinger (1971) and Adams (1977). A second possible extension is Sobolev spaces with variable exponent, which began only in the nineties with the work of Kováčik and Rákosník (1991), where some basic properties are proved. On the other hand, the Musielak–Orlicz setting unifies both spaces, but inherit technical difficulties resulting from general growth and inhomogeneity.

For convenience of the reader, when Ω is an open set in $\mathbb{R}^N$, the Musielak-Orlicz-Sobolev spaces $W^{1,\Phi_x}(\Omega)$ generalize the standard Sobolev spaces and consist of those functions whose weak derivatives belong to $L^{\Phi_x}(\Omega)$. More precisely, they are defined as

$$W^{1,\Phi_x}(\Omega) = \left\{ u \in W^{1,1}_{loc}(\Omega)\colon u, |\nabla u| \in L^{\Phi_x}(\Omega) \right\}$$

and endowed with the norm

$$\|u\|_{1,\Phi_x} = \|u\|_{\Phi_x} + \|\nabla u\|_{\Phi_x}.$$

The first systematic approach on general Musielak-Orlicz-Sobolev spaces appeared in the seventies with a series of the papers by H. Hudzik (HUDZIK, 1976a; HUDZIK, 1976b; HUDZIK, 1976c).

In parallel with the research on Muselak-Orlicz Sobolev spaces, those with variable exponents have received significant interest. Over the past 30 years, these spaces have been studied in more than a thousand papers. One of the reasons for this comes from the fact that they have a direct application to the study of differential equations associated with fluid mechanics models, image restoration processing and elasticity theory. For a good theoretical basis on this topic, we cite only the monographs of Rădulescu and Repovš (2015) and Diening et al. (2017).

Now we turn our attention to the nonlocal problems, which received great attention in the last decades, a fundamental tool to treat these type of problems is the so-called fractional order Sobolev spaces. The interest in these spaces is especially motivated by applications to the study of nonlocal problems driven by the fractional operators that arises naturally in different context, such as thin obstacle problem, flame propagation, anomalous diffusion, chemical reactions of liquids, population and fluid dynamics, water waves, crystal dislocations, nonlocal phase transitions, nonlocal minimal surfaces and many others. For more information and comprehensive applications, we refer the reader to works (LASKIN, 2000; LASKIN, 2002; DI NEZZA; PALATUCCI; VALDINOCI, 2012; BISCI; RĂDULESCU; SERVADEI, 2016) and references therein.

Several definitions of fractional Sobolev have been proposed in the literature. Recently, a natural fractional version of Orlicz-Sobolev spaces were introduced by Fernández Bonder and Salort (2019), where some basic properties of this space are analyzed under the Δ_2-condition on Φ and its conjugate. The first results referring to fractional version of Sobolev spaces with variable exponents were obtained by Kaufmann, Rossi and Vidal (2017).

Using the same spirit of local Sobolev spaces, a natural question can be posed. Can we encompass all the previous spaces in one definition? As far as we know, the first answers about the previous question are obtained in Azroul et al. (2020), by introducing the fractional version of the Musielak-Sobolev spaces. To be more precise, for a given Musielak function $\Phi : \Omega \times \Omega \times [0, \infty) \to [0, \infty)$, the fractional Musielak-Sobolev space of order $s \in (0, 1)$ is defined as

$$W^{s,\Phi_{x,y}}(\Omega) = \left\{ u \in L^{\widehat{\Phi}_x}(\Omega) \colon J_{s,\Phi}(\lambda u) < \infty \text{ for some } \lambda > 0 \right\}, \tag{1.1}$$

where $L^{\widehat{\Phi}_x}$ is the usual Musielak-Orlicz space associated with the function $\widehat{\Phi}(x,t) := \Phi(x,x,t)$, $(x,t) \in \Omega \times [0,\infty)$, and the modular $J_{s,\Phi}$ is determined in the following form

$$J_{s,\Phi}(u) = \int_\Omega \int_\Omega \Phi \left(x, y, \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N}.$$

These spaces are endowed with the norm

$$\|u\|_{s,\Phi_{x,y}} = \|u\|_{\widehat{\Phi}_x} + [u]_{s,\Phi_{x,y}},$$

where $[\cdot]_{s,\Phi}$ is the Luxemburg type seminorm given by

$$[u]_{s,\Phi_{x,y}} = \inf\left\{\lambda > 0\colon J_{\Phi_{x,y}}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Due to the above definitions, when Φ is independent of spatial variable (x, y), the corresponding space is the fractional Orlicz-Sobolev space. In particular, when $\Phi(t) = t^p$, for some $p \in [1, \infty)$, these definitions recover the classical fractional Sobolev spaces

$$W^{s,p}(\Omega) = \left\{u \in L^p(\Omega)\colon \frac{u(x) - u(y)}{|x - y|^{s+\frac{N}{p}}} \in L^p(\Omega \times \Omega)\right\}$$

equipped with norm $\|u\|_{s,p} = \|u\|_p + [u]_{s,p}$, where

$$[u]_{s,p} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{sp+N}}\, dxdy\right)^{\frac{1}{p}}.$$

is the (s, p)-Gagliardo seminorm. For more details and applications on fractional Orlicz-Soboleve spaces, we refer the readers to papers (BAHROUNI; OUNAIES, 2020; BAHROUNI; OUNAIES; TAVARES, 2020; ALBERICO et al., 2021a; SILVA et al., 2021; DE NÁPOLI; FERNÁNDEZ BONDER; SALORT, 2021) and references therein. The common goal in those researches is to extend and complement the theory of the fractional Sobolev spaces. More specifically, topological and qualitative properties were proved, such as completeness, reflexivity, density and embedding theorems.

In the case that Φ depends on (x, y), the theory considers also fractional Sobolev spaces with variable exponents $W^{s,p(\cdot,\cdot)}(\Omega)$, which is defined by function $\Phi(x, y, t) = t^{p(x,y)}$ where $p\colon \Omega \times \Omega \to \mathbb{R}$ is a measurable function lower and upper bounded by constants such that $1 \leq p^- \leq p(x, y) \leq p^+ < \infty$. Other results that complement this theory were obtained in (DEL PEZZO; ROSSI, 2017; BAHROUNI; RĂDULESCU, 2018; HO; KIM, 2020) and (BAHROUNI; OUNAIES, 2021), where is proved some further qualitative properties and continuous and compact embedding theorems.

On the other hand, a natural direction to relaxes power-type growth is considering functions that has double-phase growth, namely, $\Phi(x, y, t) = t^q + a(x, y)t^p$, with $1 \leq q < p < \infty$ and $a \in L^\infty(\Omega \times \Omega)$ a non-negative function. The main feature of the associated modular function $J_{s,\Phi}$ is the change of ellipticity and growth properties on the set where the weight function $a(\cdot, \cdot)$ vanishes. Another particular scenario where the space does not coincide with any of the aforementioned cases can be taken into account when $\Phi(x, y, t) = t^{p(x,y)} \log(1 + t)$, where $p(\cdot, \cdot)$ is as in the variable exponent case. These examples are special cases of functions with non-standard growth conditions, according to the terminology introduced by Marcellini (1989).

These examples suggest that the fractional Musielak-Sobolev space extends and unifies the standard fractional Sobolev space, the fractional Sobolev space with variable exponents and the fractional Orlicz-Sobolev space. In this sense, it is natural to study if the known results in the classical cases can be extended to this new setting. As far as we know, the only results on these spaces were obtained by Azroul et al. (2020) and Azroul et al. (2021). In summary, the authors proved some qualitative properties and embedding results for the fractional Musielak-Sobolev space in the case bounded domain.

Motivated by the above reasons and by a very recent trend in the fractional framework, the main goal in this thesis is extend and complement the previous results on the perspective of the generalized fractional Sobolev spaces. For this purpose, we perform a survey of some recent results and advances in the theory concerning these spaces.

The organization of the thesis is as follows. The Chapter 2 is dedicated to a survey of some recent results on the theory of Orlicz–Sobolev fractional spaces. They concern qualitative properties and Sobolev type embeddings for these spaces with an optimal Orlicz target and criteria for compact embeddings on the weighted fractional Orlicz-Sobolev. These results are based on recent papers by Fernández Bonder and Salort (2019), Alberico et al. (2021a), Silva et al. (2021) and references therein, where additional properties and proofs can be found.

In the Chapter 3, we apply the results from Chapter 2 together with a general method to analyze a wide class of nonlocal elliptic problems driven by the fractional Φ-Laplacian defined in the whole space, where the nonlinearity has growth superlinear at infinity and at the origin. More specifically, we study the following nonlinear fractional elliptic problem

$$\begin{cases}(-\Delta_\Phi)^s u + V(x)\varphi(|u|)u = \nu a(x)|u|^{q-2}u - \lambda|u|^{p-2}u \ \text{ in } \mathbb{R}^N, \\ u \in W^{s,\Phi}(\mathbb{R}^N),\end{cases} \tag{1.2}$$

where $s \in (0,1)$, $1 < \ell \leq m < q < p < \ell_s^* = N\ell/(N - \ell s)$, $N \geq 2$ and $\lambda, \nu > 0$. Furthermore, we assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ fulfills the Bartsch-Wang conditions and $a : \mathbb{R}^N \to \mathbb{R}$ is non-negative measurable function satisfying some suitable hypotheses. Here, $(-\Delta_\Phi)^s$ denoted the nonlinear fractional Φ-Laplacian operator introduced by Fernández Bonder and Salort (2019) and defined as follows

$$(-\Delta_\Phi)^s u(x) := \text{ p.v.} \int_{\mathbb{R}^N} \varphi(|D_s u(x,y)|) D_s u(x,y) \frac{dy}{|x-y|^{N+s}},$$

where $\varphi(|t|)t = \Phi'(t)$, $t \in \mathbb{R}$, and Φ is an N-function that satisfies some suitable assumptions. This operator can be identified as the Fréchet derivative of modulars functions defined in the appropriated fractional Orlicz-Sobolev type space $W^{s,\Phi}(\mathbb{R}^N)$.

It is worth emphasizing that many special types of nonlinear elliptic problems with general nonlinearities have also studied by several authors. In this line of research, the seminal work of Ambrosetti and Rabinowitz (1973) was the trigger for the development of an extensive literature related to existence, nonexistence and multiplicity results about it is class of problems.

In their celebrated work, Rabinowitz (1986) studied the equation (1.2) in bounded domains $\Omega \subset \mathbb{R}^N$ for the Laplacian operator, that is, for the local case $s = 1$ and $\varphi \equiv 1$. To be more precise, the author used the Mountain Pass Theorem combined with minimization arguments and some truncation techniques in nonlinearities, proving the existence of at least two weak solutions whenever the parameters $\nu = \lambda$ and $\lambda > 0$ are large enough. In the important works (BARTSCH; WANG, 1995; BERESTYCKI; LIONS, 1983a; BERESTYCKI; LIONS, 1983b), the authors studied the previous equation with general nonlinearities, taking into account the superlinear case $\lambda = 0$ and $\nu > 0$. For more results on this subject, we refer the interested reader to papers by Silva (2020) and Faraci and Silva (2021).

We also point out that several results regarding the solvability of local elliptic problems involving non-standard growth operators and a more general class of nonlinear terms has made great progress in the last years. For instance, Carvalho, Silva and Goulart (2017), Silva et al. (2019) and Silva et al. (2024b) considered quasilinear problems driven by the Φ-Laplacian operator with concave-convex nonlinearities. In the papers of Carvalho, Silva and Goulart (2021), Silva, Rocha and Silva (2024), Carvalho et al. (2024), semilinear and superlinear problems defined in the whole space $\mathbb{R}^N$ were studied, considering subcritical and nonlocal nonlinearities with some parameters. We also refer to the works of Il'yasov (2005), Silva and Macedo (2018) and Mishra, Silva and Tripathi (2023)

Similar results for nonlocal elliptic problems defined in bounded domains and in the whole space $\mathbb{R}^N$ have also been widely studied by several researchers in recent years. In the works (FELMER; QUAAS; TAN, 2012; CHANG; WANG, 2013; DIPIERRO; PALATUCCI; VALDINOCI, 2013; SECCHI, 2013; SECCHI, 2016), the authors employed the Mountain Pass Theorem, the Nehari manifold, and other appropriate minimization techniques to establish various results concerning the existence, nonexistence, multiplicity, and asymptotic behavior of solutions to fractional Laplacian type problems, under some appropriate conditions on the powers p, q, and the parameters λ and ν.

Regarding the study of problem (1.2) involving the classical fractional operator ($\varphi \equiv 1$), we would like to present the very recent work of Silva et al. (2024a). More precisely, the authors considered the following class of problem

$$\begin{cases}(-\Delta)^s u + V(x)u = \nu a(x)|u|^{q-2}u - \lambda |u|^{p-2}u \text{ in } \mathbb{R}^N,\\ u \in H^1(\mathbb{R}^N),\end{cases} \tag{1.3}$$

where $s \in (0,1), s < N/2, N \geq 2$ and $\nu, \lambda > 0$. By considering suitable assumptions on the potentials $V, a : \mathbb{R}^N \to \mathbb{R}$ and following the approaches of Nehari method and nonlinear Rayleigh quotient employed by Il'yasov (2017), the authors found sharp conditions on the parameters λ and ν such that the problem (1.3) admits at least two nontrivial solutions. Furthermore, they proved a nonexistence result under some appropriate conditions on $\lambda > 0$ and $\nu > 0$.

Additionally, a great attention has been devoted to the study of solutions for equations driven by fractional (p, q)-Laplacian type operator. For example, the existence and multiplicity of nontrivial solutions, ground state and nodal solutions, and among other qualitative properties of solutions were investigated by Alves, Ambrosio and Isernia (2019), Ambrosio and Rădulescu (2020), Zhang, Tang and Rădulescu (2021) and Silva, Oliveira and Goulart (2023) by using some topological and variational arguments. In the context of the fractional Φ-Laplacian type problems, we mention the interesting works (AZROUL; BENKIRANE; SHIMI, 2020; BAHROUNI; BAHROUNI; XIANG, 2020a; SALORT, 2020; FERNÁNDEZ BONDER; PÉREZ-LLANOS; SALORT, 2022; SALORT; VIVAS, 2022; MISSAOUI; OUNAIES, 2023) and (OCHOA; SILVA; MARZIANI, 2024).

Given this research scenario, we are directed to the following natural question: How the appearance of fractional Φ-Laplacian operator will affect the existence, multiplicity, and asymptotic behavior of solutions for problem (1.2)?

Motivated by the above question, the main goal in the Chpater 3 is to investigate the existence and multiplicity of solutions for problem (1.2). To this end, we use the Nehari method together with the Nonlinear Rayleigh quotient employed by recent works of Silva et al. (2024a) to find sharp conditions on the parameters λ and ν in order to guarantee the existence of weak solutions in the $\mathcal{N}^-_{\lambda,\nu}$ and $\mathcal{N}^+_{\lambda,\nu}$. Since we are taking into account the existence and multiplicity of solutions to superlinear elliptic problems with two parameters involving the fractional Φ-Laplacian operator that, to our knowledge, it is was not considered in the literature before, our work extends and complements the aforementioned results.

The Chapter 4 is dedicated to review of some recent results and advances on the general Musielak-Orlicz and fractional Musielak-Sobolev spaces. We also filled some gaps in the theory of fractional Musielak–Sobolev spaces. The gaps were mainly related to some details in respect to proof of basic results that seem to have missed in the previous works. Moreover, we establish some abstract results on the perspective of the fractional Musielak-Sobolev spaces, such as: uniform convexity, Radon-Riesz property with respect to the modular function, (S_+)-property, Brezis-Lieb type Lemma to the modular function and monotonicity results. Finally, we apply the theory developed to study the existence of solutions to a wide class of nonlocal problems.

In Chapter 5 and last, we study the asymptotic behavior for anisotropic nonlocal non-standard growth seminorms and modulars as the fractional parameter goes to 1 without requiring the Δ_2-condition on the Musielak function or its complementary function. This kind of result provides a so-called Bourgain-Brezis-Mironescu type formula for a very general family of functionals. In the classical fractional Sobolev space, this result allows recovering classical L^p norms for the gradientes from limits of nonlocal energie functionals.

The analysis of the limit of the fractional parameter in fractional-order Sobolev type spaces has received some attention in the last years. The seminal work of Bourgain, Brezis and Mironescu (2001) paved the way to the development of an extensive literature related with the limit study of fractional parameters in several functional spaces. In this work, the authors consider the classical fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$, $s \in (0,1)$, $p \in [1,\infty)$ and study the behavior of the corresponding Gagliardo-Slobodeckij seminorm as s approaches 1. More precisely, they prove that, if $u \in W^{1,p}(\mathbb{R}^N)$, $p \in [1,\infty)$, then

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy = K(n,p)\int_{\mathbb{R}^N}|\nabla u|^p\,dx,$$

where $K(N,p)=\frac{1}{p}\int_{\mathbb{S}^{N-1}}|w_N|^p\,d\mathcal{H}^{N-1}(w)$, being $\mathbb{S}^{N-1}$ the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$, $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure and w_N is the N-th coordinate of $w \in \mathbb{S}^{N-1}$.

The case in which $\mathbb{R}^N$ is replaced by a bounded regular domain was considered by Bourgain, Brezis and Mironescu (2001) and Dávila (2002). The case of bounded extension domains was treated by Bal, Mohanta and Roy (2020), while Drelichman and Durán (2022) deals with arbitrary bounded domains. Similar results were proved to hold in more general fractional Sobolev spaces. The extension to the so-called magnetic fractional Sobolev spaces was dealt by Squassina and Volzone (2016) and Pinamonti, Squassina and Vecchi (2019), and the case of spaces with anisotropic structure was studied by Fernández Bonder and Salort (2022) and Fernández Bonder and Dussel (2023).

Recently, these types of results have been extended to a broader class of functionals allowing a behavior more general behavior than a power associated with fractional Orlicz-Sobolev space

$W^{s,G}(\mathbb{R}^N)$ defined in terms of a Young function G, namely, a convex function from $[0,\infty)$ into $[0,\infty]$ vanishing at 0. When both Young function G and its complementary function satisfy an appropriated growth behavior known as the Δ_2-condition and $u \in W^{1,G}(\mathbb{R}^N)$, in Fernández Bonder and Salort (2019) the following limit behavior of the modulars was proved:

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N} = \int_{\mathbb{R}^N} G_0(|\nabla u|)\,dx,$$

where, for $t \geq 0$, $G_0(t) = \int_0^1 \int_{\mathbb{S}^{N-1}} G(t|w_N|r)\,d\mathcal{H}^{N-1}(w)\frac{dr}{r}$ is a Young function equivalent to G. The case of G with a general growth behavior was covered by Alberico et al. (2020) and Alberico et al. (2021a), where the same result was obtained without assuming the Δ_2- condition on G. A further extension to Carnot groups can be found in the work of Capolli et al. (2021). The case of the magnetic fractional Orlicz-Sobolev spaces was studied by Fernández Bonder and Salort (2021).

It is important to stress that Young functions include as typical examples power functions, i.e. $G(t) = t^p$, $p \geq 1$, and logarithmic perturbation of powers such as $G(t) = t^p \log(1+t)$, $p \geq 1$. Nevertheless, anisotropic and double-phase behaviors are not contemplated in this class. Likewise, functions of the type $t^{p(x,y)}$ with a suitable function $p(\cdot,\cdot)$, which are related with the fractional Sobolev spaces with variable exponent, are not covered by the previous results. In this line of research, in the work of Kim (2023) is answered whether a Bourgain-Brezis-Mironescu (hereinafter BBM) type result is true in the fractional Sobolev spaces with variable exponent $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$, when $p\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is such that $p(x,\cdot)$ is $\log$-Hölder continuous for any fixed $x \in \mathbb{R}^N$ and there are constants $p^{\pm}$ such that $1 < p^- \leq p(x,y) \leq p^+ < \infty$. The main result in Kim (2023) establishes that for sufficiently smooth functions, let us say $u \in C_0^2(\mathbb{R}^N)$, it holds that

$$\lim_{s\to 1^-} s(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy = \int_{\mathbb{R}^n} K_{N,\overline{p}(x)}|\nabla u(x)|^{\overline{p}(x)}\,dx,$$

where $\overline{p}(x) := p(x,x)$ and $K_{N,\overline{p}(x)} = \frac{1}{\overline{p}(x)}\int_{\mathbb{S}^{N-1}} |w_N|^{\overline{p}(x)}\,d\mathcal{H}^{N-1}(w)$.

Although the previous result holds for smooth functions, Kim (2023) proves that it does not hold for all functions in $W^{1,\overline{p}(\cdot)}(\mathbb{R}^N)$, even when the variable exponent p is smooth. This is in sharp contrast to the case when p is constant. The reason for this is that the target space $W^{1,\overline{p}(\cdot)}$ is too large for the previous BBM type expression to be true in general.

Motivated by the above discussion, the aim of the Chapter 5 is to study the asymptotic behavior, as $s \to 1^-$, of modular functions and seminorms related to general fractional Musielak-Sobolev space $W^{s,\Phi_{x,y}}(\mathbb{R}^N)$, where $s \in (0,1)$ and $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is a

Musielak function fulfilling some structural hypotheses. These spaces include fractional Orlicz-Sobolev spaces and fractional Sobolev spaces with variable exponents, which are examples of the aforementioned case.

The dependence of the energy functional $J_{s,\Phi}$ on both x and y adds an extra level of difficulty when dealing with the limit behavior on the fractional parameter s. Given that our results include the case of fractional Sobolev spaces with variable exponents, we cannot expect to obtain a BBM type formula beyond C_0^2 functions. Keeping these considerations in mind, our main result states that for any $u \in C_0^2(\mathbb{R}^N)$, there exists $\lambda_0 > 0$ such that

$$\lim_{s\to 1^-}(1-s)J_{s,\Phi}\left(\frac{u}{\lambda}\right) = \int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u(x)|}{\lambda}\right)dx, \quad \text{for all } \lambda \geq \lambda_0,$$

where the function H_0 is given by

$$H(x,t) = \int_0^1 \int_{\mathbb{S}^{N-1}} \Phi(x,x,t|w_N|r)\, d\mathcal{H}^{N-1}(w)\frac{dr}{r}$$

and w_N is the N-th coordinate of any point in $\mathbb{S}^{N-1}$. Furthermore, it is proved that the limit function $H(x,t)$ is in fact a Musielak function equivalent to $\widehat{\Phi}(x,t) := \Phi(x,x,t)$. As a consequence, we obtain a BBM type inequality for seminorms. It is worthwhile to mention that we do not require that neither Φ nor its conjugate function fulfill the Δ_2-condition.

More recently, there has been consideration of fractional anisotropic spaces where the functions have different fractional regularity and integrability in each coordinate direction, as seen by Chaker, Kim and Weidner (2023) and Fernández Bonder and Dussel (2023). The techniques used in our main result enable us to study energy functionals where the s-Hölder quotient depends solely on a direction, that is,

$$D_s^k u(x,h) := \frac{u(x-he_k)-u(x)}{|h|^s}, \quad \text{with } k \in \{1,\dots,N\},$$

being e_k the k-th canonical vector in $\mathbb{R}^N$. More precisely, we prove that, for any $u \in C_0^2(\mathbb{R}^N)$, there exists $\lambda_0 > 0$ such that

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}} \Phi\left(x, x-he_k, \frac{|D_s^k u(x,y)|}{\lambda}\right)\frac{dhdx}{|h|} = \int_{\mathbb{R}^N} H_0\left(x, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_k}\right|\right)dx,$$

for all $\lambda \geq \lambda_0$, where in this case the limit function H_0 is given by

$$H_0(x,t) = 2\int_0^1 \Phi(x,x,tr)\frac{dr}{r}.$$

2 A SURVEY ON FRACTIONAL ORLICZ-SOBOLEV SPACES

In this chapter, we collect preliminary concepts of the theory of Orlicz spaces and fractional Orlicz-Sobolev spaces, which will be used throughout the Chapter 3. For a more complete discussion on this subject, we refer the readers to (KRASNOSEL'SKII; RUTICKII, 1961; RAO; REN, 1985; ADAMS; FOURNIER, 2003; PICK et al., 2012) and (FUKAGAI; ITO; NARUKAWA, 2006).

The starting point of the theory of these spaces is the notions of a N*-function*.

2.1 N-FUNCTIONS

Definition 2.1.1. *A function $\Phi : \mathbb{R} \to \mathbb{R}$ is said to be an N-function (or Orlicz function) if satisfies the following conditions:*

(i) Φ *is even and convex.*

(iv) $\Phi(t) = 0$ *if and only if* $t = 0$.

(iii) $\displaystyle\lim_{t\to 0} \frac{\Phi(t)}{t} = 0.$

(iv) $\displaystyle\lim_{t\to \infty} \frac{\Phi(t)}{t} = \infty.$

Equivalently, an N-function can be represented as follows

$$\Phi(t) = \int_0^{|t|} \phi(\tau)\, d\tau,$$

where $\phi \colon [0, \infty) \to [0, \infty)$ is the right-hand derivative of Φ satisfying

(i) $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$.

(ii) ϕ is non-decreasing.

(iii) ϕ is right-continuous and $\lim_{t\to\infty} \phi(t) = \infty$.

See for instance Lemma 4.2.2 in Pick et al. (2012).

Definition 2.1.2. *Let Φ be an N-function. The function $\widetilde{\Phi} : \mathbb{R} \to [0, \infty)$ defined by the following Legendre's transformation*

$$\widetilde{\Phi}(t) := \sup_{s\geq 0}\{ts - \Phi(s)\}, \quad \textit{for } t \in \mathbb{R}, \tag{2.1}$$

is called the conjugate function of Φ in the sense of Young.

It is not hard to see that $\widetilde{\Phi}$ is well-defined, it is also an N-function and can be represented as follows:

$$\widetilde{\Phi}(t) = \int_0^t \widetilde{\phi}(\tau)\,\mathrm{d}\tau,$$

where

$$\widetilde{\phi}(t) = \sup\{s : \phi(s) \le t\}, \quad \text{for } t \in \mathbb{R}$$

If ϕ is continuous and strictly increasing in $[0,\infty)$, then $\widetilde{\phi}$ is the inverse of ϕ. Furthermore, in view of (2.1) one may deduce that $\widetilde{\widetilde{\Phi}} = \Phi$ a and the Young's type inequality

$$st \le \Phi(s) + \widetilde{\Phi}(t), \quad \text{for all } s,t \ge 0. \tag{2.2}$$

where equality holds (Young's equality) if, and only if, $s = \widetilde{\phi}(t)$ or $t = \widetilde{\phi}(s)$.

Example 2.1.3. *The following are examples of N-functions and it is conjugate functions:*

(i) Let $\Phi(t) = \frac{|t|^p}{p}$, $p \in (1,\infty)$. Then, $\widetilde{\Phi}(t) = \frac{|t|^q}{q}$, with $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let $\Psi(t) = e^{|t|} - |t| - 1$. Then, $\widetilde{\Psi}(t) = (1+|t|)\ln(1+|t|) - |t|$.

In the sequel, we recall some growth conditions related to N-function.

Definition 2.1.4. *We say that an N-function Φ satisfies the Δ_2-condition if there exists a constant $K > 0$ such that*

$$\Phi(2t) \le K\Phi(t), \quad \textit{for all } t \ge 0.$$

When this inequality holds only for $t \ge t_0$ and some $t_0 > 0$, Φ is said to satisfy the $\Delta_2(\infty)$-condition near infinity.

According to Theorem 4.4.4 present in Pick et al. (2012), an N-function Φ satisfies the Δ_2-condition if, and only if,

$$\sup_{t \ne 0} \frac{\phi(t)t}{\Phi(t)} < \infty.$$

Proceeding similarly as in Lemma 2.5 obtained by Fukagai, Ito and Narukawa (2006), if there exist constants $m \ge \ell > 1$ such that

$$\ell \le \frac{\phi(t)t}{\Phi(t)} \le m \quad \text{for all } t \ne 0,$$

we deduce that

$$\frac{m}{m-1} \le \frac{\widetilde{\phi}(t)t}{\widetilde{\Phi}(t)} \le \frac{\ell}{\ell-1}, \quad \text{for all } t \ne 0.$$

Thus, $\widetilde{\Phi}$ also satisfies the Δ_2-condition if, and only if, $\ell > 1$.

Arguing as in the proof of Lemma 2.1 in Azroul et al. (2020), one can prove that Φ and $\widetilde{\Phi}$ satisfy the following inequality

$$\widetilde{\Phi}(\phi(t)) \leq \Phi(2t), \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Next, we present some inequalities involving an N-function and it is conjugate.

Lemma 2.1.5. *(FUKAGAI; ITO; NARUKAWA, 2006, Lemma 2.1 and Lemma 2.5) Assume that*

$$1 < \ell := \inf_{t \neq 0} \frac{\phi(t)t}{\Phi(t)} \leq \sup_{t \neq 0} \frac{\phi(t)t}{\Phi(t)} =: m < \infty. \tag{2.4}$$

Henceforth, we use the following notation:

$$\xi_0^-(t) = \min\{t^\ell, t^m\}, \quad \xi_0^+(t) = \max\{t^\ell, t^m\},$$
$$\xi_1^-(t) = \min\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\}, \quad \xi_1^+(t) = \max\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\}, \quad t \geq 0,$$

where $\widetilde{\ell} = \frac{\ell}{\ell-1}$ and $\widetilde{m} = \frac{m}{m-1}$. Then, Φ and $\widetilde{\Phi}$ satisfy the following estimates:

(i) $\xi_0^-(\sigma)\Phi(t) \leq \Phi(\sigma t) \leq \xi_0^+(\sigma)\Phi(t)$, for all $\sigma, t \in \mathbb{R}$.

(ii) $\xi_1^-(\sigma)\widetilde{\Phi}(t) \leq \widetilde{\Phi}(\sigma t) \leq \xi_1^+(\sigma)\widetilde{\Phi}(t)$, for all $\sigma, t \in \mathbb{R}$.

2.2 ORLICZ SPACES

In this section, we introduce the basic concepts and some results of Orlicz spaces on arbitrary measure spaces. Such spaces are more general than the classical Lebesgue spaces, the topological vector space properties depended both on the N-function and the measure space.

We start by recalling some concepts on measure theory. Hereafter, (Ω, Σ, μ) usually denotes an abstract complete measure space. Namely, Σ is a σ-algebra of subsets on a nonempty set Ω and $\mu : \Sigma \to [0, \infty]$ is a measure, that is, a σ-additive set function satisfying $\mu(\emptyset) = 0$. We denote by $L^0(\Omega, \mu)$ the set of μ-measurable functions on Ω. We also assume that (Ω, Σ, μ) satisfy the natural assumption that our measure μ is not identically zero or infinity.

Definition 2.2.1. *Let (Ω, Σ, μ) be a complete measure space. The measure μ is called atomless or diffuse if, for any measurable set $A \in \Sigma$ with $\mu(A) > 0$, there exists $B \in \Sigma$ such that $B \subset A$ and $\mu(A) > \mu(B) > 0$.*

A measure space (Ω, Σ, μ) is said to be *(i)* *σ-finite if there exists an increasing sequence $(\Omega_k)_{k\in\mathbb{N}} \in \Sigma$ such that $\mu(\Omega_k) < \infty$ for all $k \in \mathbb{N}$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$.*

(ii) *separable if there exists a sequence $(A_k)_{k\in\mathbb{N}} \subset \Sigma$, with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$, such that for every $A \in \Sigma$ with $\mu(A) < \infty$ and $\varepsilon > 0$, there exists k_0 satisfying $\mu(A\Delta A_{k_0}) < \varepsilon$, where Δ denotes the symmetric difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$.*

Given an N-function Φ, the *Orlicz* class associated to Φ and (Ω, Σ, μ) is defined by

$$K_\Phi(\Omega,\mu) = \left\{u \in L^0(\Omega,\mu) : \int_\Omega \Phi(|u(x)|)\, d\mu < \infty\right\},$$

We point out that if Φ satisfies the $\Delta_2(\infty)$-condition (Δ_2-condition), then $K_\Phi(\Omega,\mu)$ is a vector space when $\mu(\Omega) < \infty$ ($\mu(\Omega) = \infty$).The converse holds if μ is diffuse on a set of positive measure

The *Orlicz space* is defined as follows:

$$L^\Phi(\Omega,\mu) = \left\{u \in L^0(\Omega,\mu) : \int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu < \infty \text{ for some } \lambda > 0\right\}.$$

Now, we introduce the called *modular function* $J_\Phi : L^0(\Omega,\mu) \to \mathbb{R}$ defined by

$$J_\Phi(u) = \int_\Omega \Phi(|u(x)|)\, d\mu.$$

The space $L^\Phi(\Omega,\mu)$ is a Banach space when endowed with the called *Luxemburg norm*

$$\|u\|_\Phi := \inf\left\{\lambda > 0 : J_\Phi\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

or the equivalent norm (*the Orlicz norm*) given by

$$\|u\|_{(\Phi)} = \sup\left\{\left|\int_\Omega u(x)v(x)\, d\mu\right| : v \in L^{\widetilde{\Phi}}(\Omega,\mu) \text{ and } J_{\widetilde{\Phi}}(v) \leq 1\right\}.$$

Using the Young's type inequality (2.2) for Φ and $\widetilde{\Phi}$, one may deduce the following Hölder's type inequality

$$\left|\int_\Omega u(x)v(x)\, d\mu\right| \leq 2\|u\|_\Phi \|v\|_{\widetilde{\Phi}},$$

for all $u \in L^\Phi(\Omega,\mu)$ and $v \in L^{\widetilde{\Phi}}(\Omega,\mu)$.

Definition 2.2.2. *Given an Orlicz space $L_\Phi(\Omega,\mu)$, the N-function Φ is determined Δ_2-regular if either:*

(i) *Φ satisfies $\Delta_2(\infty)$-condition when $\mu(\Omega) < \infty$, or*

(ii) Φ *satisfies* Δ_2*-condition when* $\mu(\Omega) = \infty$.

Remark 2.2.3. *The Fatou's lemma give us*

$$J_\Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right) \leq 1,$$

for all $0 \neq u \in L^\Phi(\Omega, \mu)$*. The equality holds if* Φ *is* Δ_2*-regular. In general, we have that* $J_\Phi(u) \leq 1$ *is equivalent to* $\|u\|_\Phi \leq 1$*. Therefore,*

$$\|u\|_\Phi \leq J_\Phi(u) + 1.$$

for all $u \in L^\Phi(\Omega, \mu)$*.*

In order to analyze the linear structure of $L^\Phi(\Omega, \mu)$, we introduce the *Morse-Transue space* defined as follows:

$$E^\Phi(\Omega, \mu) = \left\{u \in L^0(\Omega, \mu) \colon \lambda u \in K^\Phi(\Omega, \mu) \text{ for all } \lambda > 0\right\}.$$

We would like to mention that $E^\Phi(\Omega, \mu)$ is a maximal closed subspace of $L^\Phi(\Omega, \mu)$ which is contained in $K^\Phi(\Omega, \mu)$. Precisely, the interrelations between the spaces above are given by

$$E^\Phi(\Omega, \mu) \subseteq K^\Phi(\Omega, \mu) \subseteq L^\Phi(\Omega, \mu).$$

The equality holds between the first two if Φ is Δ_2-regular, and the converse holding when μ is diffuse on a set of positive measure. Moreover, there exists the equality between the last two if, and only if, Φ is Δ_2-regular.

The following properties of this space are also required.

Proposition 2.2.4. *Let* (Ω, Σ, μ) *be a measure space and* Φ *an* N*-function. Then,*

(i) $E^\Phi(\Omega, \mu)$ *is separable if and only if* (Ω, Σ, μ) *is separable.*

(ii) $L^\Phi(\Omega, \mu)$ *is separable if and only if* $L^\Phi(\Omega, \mu) = E^\Phi(\Omega, \mu)$*. In particular, if* Φ *is* Δ_2*-regular, then* $L^\Phi(\Omega, \mu)$ *is separable if and only if* (Ω, Σ, μ) *is separable.*

Remark 2.2.5. *When* Ω *is an open set of* $\mathbb{R}^N$ *and* μ *is the Lebesgue measure on the Borel* σ*-algebra of subsets of* Ω*, we omit* μ *in the above definitions. Namely, we use the notations* $E^\Phi(\Omega)$*,* $K^\Phi(\Omega)$ *and* $L^\Phi(\Omega)$*. In this case,* $L^\Phi(\Omega)$ *is separable if and only if* Φ *is* Δ_2*-regular.*

The next result is a generalization of the well known Riesz's representation theorem of the classical Lebesgue spaces. Its proof for the particular case in which Ω is an open set of $\mathbb{R}^N$ and μ is the Lebesgue measure can be found in (ADAMS; FOURNIER, 2003) and (PICK et al., 2012). We emphasize that in the general case can found in (RAO; REN, 1985, Theorem 7, section 4.1).

Proposition 2.2.6. *(Riesz's representation theorem) Let (Ω, Σ, μ) be a measure space and Φ a N-function. If $v \in L^{\widetilde{\Phi}}(\Omega, \mu)$, then the linear functional $T_v : E^{\Phi}(\Omega, \mu) \to \mathbb{R}$ defined by*

$$T_v(u) = \int_\Omega uv \, d\mu \tag{2.5}$$

is bounded and

$$\|v\|_{\widetilde{\Phi}} \leq \|T_v\|_{(L_\Phi(\Omega))^*} \leq 2\|v\|_{\widetilde{\Phi}}.$$

Conversely, every bounded linear functional in $(E^{\Phi}(\Omega, \mu))^*$ is of the form (2.5)*. In other words, the map $v \mapsto T_v$ defines an isomorphism of $L^{\widetilde{\Phi}}(\Omega, \mu)$ onto $(E^{\Phi}(\Omega, \mu))^*$.*

In particular, the following assertions hold:

(i) $L^{\widetilde{\Phi}}(\Omega, \mu) \cong (L^{\Phi}(\Omega, \mu))^*$ if and only if Φ is Δ_2-regular.

(ii) $L^{\Phi}(\Omega, \mu) \cong (L^{\widetilde{\Phi}}(\Omega, \mu))^*$ if and only if $\widetilde{\Phi}$ is Δ_2-regular.

Consequently, $L^{\Phi}(\Omega, \mu)$ is a reflexive space if and only if Φ and $\widetilde{\Phi}$ are Δ_2-regular.

We end this section by presenting an important result that relates the norm and modular function.

Lemma 2.2.7. *Let (Ω, Σ, μ) be a measure space and Φ a N-function. Assume that* (2.4) *holds. Then, the following estimates hold:*

(i) $\xi_0^-(\|u\|_\Phi) \leq J_\Phi(u) \leq \xi_0^+(\|u\|_\Phi)$, for all $u \in L_\Phi(\Omega, \mu)$.

(ii) $\xi_1^-(\|u\|_{\widetilde{\Phi}}) \leq J_{\widetilde{\Phi}}(u) \leq \xi_1^+(\|u\|_{\widetilde{\Phi}})$, for all $u \in L_{\widetilde{\Phi}}(\Omega, \mu)$.

2.3 FRACTIONAL ORLICZ-SOBOLEV SPACES

We consider an open set $\Omega \subset \mathbb{R}^N$, an N-function Φ and a parameter $s \in (0,1)$. The *fractional Orlicz-Sobolev space* is defined as follows:

$$W^{s,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega) : J_{s,\Phi}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\},$$

where the semimodular function $J_{s,\Phi}$ is defined by

$$J_{s,\Phi}(u) := \int_\Omega \int_\Omega \Phi\left(|D_s u(x,y)|\right) d\mu, \quad \text{for } s \in (0,1),$$

and the s-Hölder quotient $D_s u$ and the measure μ are defined as

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^s} \quad \text{and} \quad d\mu := \frac{dxdy}{|x-y|^N}.$$

The space $W^{s,\Phi}(\Omega)$ is endowed with the norm

$$\|u\|_{s,\Phi} := \|u\|_\Phi + [u]_{s,\Phi},$$

where the term $[\cdot]_{s,\Phi_{x,y}}$ is the so called (s,Φ)-*Gagliardo seminorm* defined by

$$[u]_{s,\Phi} := \inf\left\{\lambda > 0 : J_{s,\Phi}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Remark 2.3.1. *Let $u \in L^\Phi(\Omega)$. Note that $u \in W^{s,\Phi}(\Omega)$ if and only if $D_s u \in L^\Phi(d\mu) := L^\Phi(\Omega \times \Omega, d\mu)$ and $[u]_{s,\Phi} = \|D_s u\|_{L^\Phi(d\mu)}$. Namely, we have that*

$$W^{s,\Phi}(\Omega) = \left\{u \in L^\Phi(\Omega) : D_s u \in L^\Phi(\Omega \times \Omega, d\mu)\right\}.$$

Next, we list some remarks about the measure μ.

Remark 2.3.2. *It is important to emphasize that μ is not a regular Borel measure on the set $\mathbb{R}^N \times \mathbb{R}^N$. In fact, let (x_0, x_0) be a fixed point on the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$. For each $y \in B_R(x_0)$ with $R > 0$, consider $\varepsilon > 0$ small enough such that $B_\varepsilon(y) \subset B_R(x_0)$. Thus,*

$$\int_{\overline{B}_R(x_0)} \frac{dx}{|x-y|^N} \geq \int_{B_\varepsilon(y)} \frac{dx}{|x-y|^N} = N\omega_N \int_0^\varepsilon \frac{dr}{r} = \infty.$$

This implies that for every closed ball $\overline{B}_R(x_0) \subset \mathbb{R}^N$ with $R > 0$, we have

$$\mu(\overline{B}_R(x_0) \times \overline{B}_R(x_0)) = \int_{\overline{B}_R(x_0)} \int_{\overline{B}_R(x_0)} \frac{dxdy}{|x-y|^N} = \infty.$$

Therefore, even within a compact set containing (x_0, x_0) as an interior point, the measure μ is infinite. This shows that μ is not Borel regular.

On the other hand, if $K \subset (\mathbb{R}^N \times \mathbb{R}^N) \setminus D$ is a compact set, where $D \subset \mathbb{R}^N \times \mathbb{R}^N$ is a diagonal $D = \{(x,x) : x \in \mathbb{R}^N\}$, then $\mu(K) < \infty$.

Moreover, we do not know whether the measure μ is σ-finite. This phenomenon presents certain challenges regarding the application of standard arguments and results from real analysis. Fortunately, any subset of a set of measure zero will also be measurable and have measure zero, which guarantees that the measure is complete. For this reason, we can apply the results for Orlicz spaces given in Section 2.2.

It is also worthwhile to recall that if Φ and $\widetilde{\Phi}$ satisfy the Δ_2-condition, then $L^{\Phi}(\Omega)$ and $L^{\Phi}(\Omega \times \Omega, d\mu)$ are reflexive Banach spaces. For this reason, $W^{s,\Phi}(\Omega)$ is also a reflexive Banach space. Moreover, the space $C_0^{\infty}(\mathbb{R}^N)$ is dense in $W^{s,\Phi}(\mathbb{R}^N)$ if Φ satisfies the Δ_2-condition. For further details on this property, we refer the readers to (DE NÁPOLI; FERNÁNDEZ BONDER; SALORT, 2021, Proposition 2.9).

We also emphasize that the fractional Orlicz-Sobolev space is the appropriated one for studying nonlocal elliptic problems driven by *fractional Φ-Laplacian operator,*

$$(-\Delta_{\Phi})^s u(x) = 2\,\text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(D_s u(x,y)) \frac{dy}{|x-y|^{N+s}},$$

where p.v. is a general abbreviation used in the principle value sense. Indeed, according to Fernández Bonder and Salort (2019), if Φ satisfies Δ_2-condition, this operator is well-defined between $W^{s,\Phi}(\mathbb{R}^N)$ and it is topological dual space $W^{-s,\widetilde{\Phi}}(\mathbb{R}^N)$ and the following representation formula holds

$$\langle (-\Delta_{\Phi})^s u, v \rangle = J'_{s,\Phi}(u)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi\left(D_s u\right) D_s v \, d\mu, \ \text{ for all } \ u, v \in W^{s,\Phi}(\mathbb{R}^N),$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $W^{s,\Phi}(\mathbb{R}^N)$ and $W^{-s,\widetilde{\Phi}}(\mathbb{R}^N)$.

In the sequel, we present some definition and results that play a key role when dealing with embedding results of fractional Orlicz-Sobolev spaces.

Definition 2.3.3. *Let Φ an N-function satisfying*

$$\int_0^1 \left(\frac{t}{\Phi(t)} \right)^{\frac{s}{N-s}} dt < \infty \qquad \textit{and} \qquad \int_1^{\infty} \left(\frac{t}{\Phi(t)} \right)^{\frac{s}{N-s}} dt = \infty. \tag{2.6}$$

The optimal N-function (or Sobolev's critical function of Φ) is defined as follows:

$$\Phi_*(t) = (\Phi \circ H^{-1})(t), \ \ \textit{for all } \ t \geq 0,$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{\Phi(\tau)} \right)^{\frac{s}{N-s}} d\tau \right)^{\frac{N-s}{N}}, \ \ \textit{for all } \ t \geq 0.$$

We consider Φ_* extended to $\mathbb{R}$ by $\Phi_*(t) = \Phi_*(-t)$ for $t < 0$.

Remark 2.3.4. *The condition* (2.6) *amounts to requiring that Φ has subcritical growth with respect to the fractional parameter s. For instance, if $\Phi(t) = |t|^p, 1 < p < \infty$, then* (2.6) *holds if $p < N/s$. Hence, it generalizes the condition required for classical fractional Sobolev embedding.*

Definition 2.3.5. *Let Ψ, Φ be two N-functions. We say that Ψ is stronger than Φ near infinity, and we write $\Phi < \Psi$, if*

$$\Phi(t) \leq \Psi(kt), \quad \textit{for all } t \geq t_0,$$

for some $k > 0$ and $t_0 > 0$ fixed.

If for each $k > 0$, there exists $t_k > 0$ depending on k such that

$$\Phi(t) \leq \Psi(kt), \quad \textit{for all } t \geq t_k,$$

we say that Ψ is essentially stronger than Φ or equivalently that Φ grows essentially more slowly near infinity than Ψ, and we write $\Phi \ll \Psi$.

Remark 2.3.6. *It is important to mention that $\Phi \ll \Psi$ is equivalent to the following condition:*

$$\lim_{t\to\infty} \frac{\Phi(kt)}{\Psi(t)} = 0, \quad \textit{for all } k > 0.$$

See section 8.5 in Adams and Fournier (2003). Moreover, if $\Phi \ll \Psi$ and $\Omega \subset \mathbb{R}^N$ has finite measure, then $L^\Psi(\Omega) \hookrightarrow L^\Phi(\Omega)$.

Under the condition (2.6), we have the following optimal fractional Orlicz-Sobolev embedding proved by Alberico et al. (2021a) (Theorem 6.1).

Proposition 2.3.7. *Assume that Φ is an N-function satisfying the condition* (2.6)*. Then,*

$$W^{s,\Phi}(\mathbb{R}^N) \hookrightarrow L^{\Phi_*}(\mathbb{R}^N), \tag{2.7}$$

and $L^{\Phi_*}(\mathbb{R}^N)$ is the optimal Orlicz target space such that (2.7) *holds. Moreover, there exists a constant $C := C(n, s) > 0$ such that*

$$\|u\|_{\Phi_*} \leq C[u]_{s,\Phi},$$

for all $u \in W^{s,\Phi}(\mathbb{R}^N)$.

Remark 2.3.8. *The above embedding is optimal in the sense that if the embedding holds for an N-function Ψ, then the space $L^{\Phi_*}(\mathbb{R}^N)$ is continuously embedded into $L^\Psi(\mathbb{R}^N)$.*

The next result is a criterion for the compactness of a fractional Orlicz–Sobolev embedding into an Orlicz space, as established by Alberico et al. (2021b) (Theorem 3.5).

Proposition 2.3.9. *Let Φ be an N-function satisfying the condition* (2.6)*. Assume that Ψ is an N-function. The following properties are equivalent:*

(i) $\Psi \ll \Phi_*$.

(ii) *The embedding* $W^{s,\Phi}(\mathbb{R}^N) \hookrightarrow L^\Psi_{loc}(\mathbb{R}^N)$ *is compact.*

(iii) *The embedding* $W^{s,\Phi}(\Omega) \hookrightarrow L^\Psi(\Omega)$ *is compact for each bounded Lipschitz domain* Ω *in* $\mathbb{R}^N$.

Remark 2.3.10. *The claim that the embedding* $W^{s,\Phi}(\mathbb{R}^N) \hookrightarrow L^\Psi_{loc}(\mathbb{R}^N)$ *is compact means that every bounded sequence in* $W^{s,\Phi}(\mathbb{R}^N)$ *has a subsequence whose restriction to any bounded measurable set* K *converges in* $L^\Phi(K)$.

According to Missaoui and Ounaies (2023) (Lemmas 4.3 and 4.5), we obtain the following growth behavior:

Lemma 2.3.11. *Assume that the conditions* (2.4) *and* (2.6) *hold and let*

$$\xi_*^-(t) = \min\{t^{\ell_s^*}, t^{m_s^*}\} \quad \text{and} \quad \xi_*^+(t) = \max\{t^{\ell_s^*}, t^{m_s^*}\}, \quad \text{for } t \geq 0,$$

where $\ell, m \in (1, N/s)$, $\ell_s^* = \frac{N\ell}{N-s\ell}$ *and* $m_s^* = \frac{Nm}{N-sm}$. *Then,* Φ_* *satisfies the following estimates:*

(i) $\ell_s^* \leq \dfrac{\phi_*'(t)t}{\Phi_*(t)} \leq m_s^*$, *for all* $t > 0$, *where* $\Phi_*(t) = \displaystyle\int_0^t \phi_*(\tau)\,d\tau$. *Namely,* Φ_* *satisfies* Δ_2*-condition.*

(ii) $\xi_*^-(t)\Phi_*(\rho) \leq \Phi_*(\rho t) \leq \xi_*^+(t)\Phi(\rho)$, *for all* $\rho, t > 0$.

(iii) $\xi_*^-(\|u\|_{\Phi_*}) \leq J_{\Phi_*}(u) \leq \xi_*^+(\|u\|_{\Phi_*})$, *for all* $u \in L^{\Phi_*}(\mathbb{R}^N)$.

In the paper by Bahrouni and Ounaies (2020), the continuous embedding in Orlicz spaces defined on $\mathbb{R}^N$ for functions with subcritical growth is discussed. However, it is not clear that the embedding $L^{\Phi_*}(\mathbb{R}^N) \hookrightarrow L^\Psi(\mathbb{R}^N)$ holds for any $\Psi \ll \Phi_*$. In order to state a continuous embedding between these Orlicz spaces an additional hypothesis is needed besides the condition of growth at infinity.

Proposition 2.3.12. *Let* Φ *be an* N*-function satisfying the condition* (2.6). *Assume that* Ψ *is an* N*-function satisfying* $\Psi \ll \Phi_*$ *and*

$$\limsup_{t\to 0} \frac{\Psi(t)}{\Phi(t)} < \infty. \tag{2.8}$$

Then, the embedding $W^{s,\Phi}(\mathbb{R}^N) \hookrightarrow L^\Psi(\mathbb{R}^N)$ *is continuous.*

Proof. Let $u \in W^{s,\Phi}(\mathbb{R}^N)$ with $\|u\|_{s,\Phi} = 1$. In particular, $u \in L^{\Phi}(\mathbb{R}^N)$ and $\|u\|_{\Phi} \leq 1$. Moreover, by Proposition 2.3.7, $u \in L^{\Phi_*}(\mathbb{R}^N)$ and $\|u\|_{\Phi_*} \leq C_0\|u\|_{s,\Phi} \leq C_0$. Without loss of generality, assume that $C_0 > 1$. These assertions and Remark 2.2.3, we have that

$$J_{\Phi}(u) \leq 1 \quad \text{and} \quad J_{\Phi_*}\left(\frac{u}{C_0}\right) \leq 1. \tag{2.9}$$

On the other hand, by using $\Psi \ll \Phi_*$ and (2.8), there exist $C, T > 0$ and $\delta > 0$ such that

$$\Psi(t) \leq \Phi_*(t), \quad \text{for all } t \geq T$$

and

$$\Psi(t) \leq C\Phi(t), \quad \text{for all } 0 \leq t \leq \delta.$$

Hence, by using convexity of Φ and (2.9), we infer that

$$\begin{aligned}
\int_{\mathbb{R}^n} \Psi\left(\frac{|u(x)|}{C_0}\right) dx &\leq C \int_{\left\{\frac{|u(x)|}{C_0} \leq \delta\right\}} \Phi\left(\frac{|u(x)|}{C_0}\right) dx + \int_{\left\{\delta < \frac{|u(x)|}{C_0} < T\right\}} \Psi\left(\frac{|u(x)|}{C_0}\right) dx \\
&\quad + \int_{\left\{\frac{|u(x)|}{C_0} \geq T\right\}} \Phi_*\left(\frac{|u(x)|}{C_0}\right) dx \\
&\leq \frac{C}{C_0} \int_{\mathbb{R}^N} \Phi(|u(x)|)\, dx + \int_{\mathbb{R}^N} \Phi_*\left(\frac{|u(x)|}{C_0}\right) dx \\
&\quad + \Psi(T) \left|\left\{\delta < \frac{|u(x)|}{C_0} < T\right\}\right|. \\
&\leq \frac{C}{C_0} + 1 + \Psi(T) \left|\left\{\delta < \frac{|u(x)|}{C_0} < T\right\}\right|.
\end{aligned}$$

Now, using the monotonicity and convexity of Φ and (2.9), we have that

$$\left|\left\{\delta < \frac{|u(x)|}{C_0} < T\right\}\right| \leq \frac{1}{\Phi(\delta)} \int_{\mathbb{R}^n} \Phi\left(\frac{|u(x)|}{C_0}\right) dx \leq \frac{1}{\Phi(\delta)C_0} \int_{\mathbb{R}^n} \Phi(|u(x)|)\, dx \leq \frac{1}{\Phi(\delta)C_0}.$$

Thereby,

$$\int_{\mathbb{R}^n} \Psi\left(\frac{|u(x)|}{C_0}\right) dx \leq \frac{C}{C_0} + \frac{\Psi(T)}{\Phi(\delta)C_0} + 1 < \infty.$$

This yields that $u \in L^{\Psi}(\mathbb{R}^N)$.

In order to complete the proof, we consider $u \in W^{s,\Phi}(\mathbb{R}^N)$ arbitrary and $v = \frac{u}{\|u\|_{s,\Phi}}$. Therefore, by Remark 2.2.3, we obtain that

$$\|v\|_{\Psi} \leq \int_{\mathbb{R}^N} \Psi(|v(x)|)\, dx + 1 \leq C + 1,$$

which implies that $\|u\|_{\Psi} \leq (C+1)\|u\|_{s,\Phi}$. This ends the proof. □

Another functional space to study variationally problems involving the fractional Φ-Laplacian operator is the weighted fractional Orlicz-Sobolev space defined as follows

$$X := \left\{ u \in W^{s,\Phi}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)\Phi(|u|)\, dx < \infty \right\},$$

equipped with the following norm

$$\|u\|_{s,\Phi,V} = \|u\|_{\Phi,V} + [u]_{s,\Phi},$$

where

$$\|u\|_{\Phi,V} = \inf\left\{ \lambda > 0 \colon \int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\},$$

and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous function and satisfies the following Bartsch and Wang (1995) type assumptions:

(V_0) There exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$.

(V_1) For each $M > 0$, it holds that the set $\{x \in \mathbb{R}^N : V(x) \leq M\}$ has finite Lebesgue measure.

It is important to mention that if Φ and $\widetilde{\Phi}$ satisfy the Δ_2-condition, then X is a reflexive Banach space, and it is a closed subset of $W^{s,\Phi}(\mathbb{R}^N)$, see (BAHROUNI; OUNAIES, 2020). Furthermore, $\|\cdot\|_{s,\Phi,V}$ is equivalent to Luxemburg's norm given by

$$\|u\| := \inf\left\{ \lambda > 0 : \mathcal{J}_{s,\Phi,V}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

where $\mathcal{J}_{s,\Phi,V} : X \to \mathbb{R}$ is defined by

$$\mathcal{J}_{s,\Phi,V}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi(|D_s u|)\, d\mu + \int_{\mathbb{R}^N} V(x)\Phi(|u|)\, dx.$$

In light of Lemma 13 obtained by Bahrouni and Ounaies (2020), a corresponding version of Lemma 2.2.7 for $\mathcal{J}_{s,\Phi,V}$ can be stated as follows:

Lemma 2.3.13. *Assume that Φ satisfies Δ_2-condition and (V_0)-(V_1) hold. Then,*

$$\xi_0^-(\|u\|) \leq \mathcal{J}_{s,\Phi,V}(u) \leq \xi_0^+(\|u\|),\ u \in X.$$

Recently, Silva et al. (2021) established some continuous and compact embedding results for the space X. These results are summarized in the following propositions:

Proposition 2.3.14. *Assume that Φ satisfies Δ_2-condition and (V_0)-(V_1) hold. Then, the embedding $X \hookrightarrow L^{\Phi}(\mathbb{R}^N)$ is compact.*

Proposition 2.3.15. *Assume that Φ satisfies Δ_2-condition and (V_0)-(V_1) hold. Suppose also that $\Phi < \Psi \ll \Phi_*$ and at least one of the following conditions are satisfied:*

(i) The following limit holds

$$\limsup_{|t|\to 0} \frac{\Psi(|t|)}{\Phi(|t|)} < \infty.$$

(ii) The function Ψ satisfies Δ_2-condition and there exist an N-function R and $b \in (0,1)$ such that $\Psi \circ R < \Phi_*$ and

$$\Psi(\widetilde{R}(|t|^{1-b})) \leq C\Phi(|t|) \text{ for } |t| \leq 1,$$

where $\widetilde{R}$ is the conjugate function of R.

Then, the space X is compactly embedded into $L^{\Psi}(\mathbb{R}^N)$.

As a consequence of Propositions 2.3.12 and 2.3.15, we can prove the following result:

Lemma 2.3.16. *Assume that Φ satisfies the condition* (2.4) *and (V_0)-(V_1) hold. Then, the embedding $X \hookrightarrow L^r(\mathbb{R}^n)$ is continuous for all $r \in [m, \ell_s^*)$ and compact for all $r \in (m, \ell_s^*)$.*

Proof. Firstly, using the assumption (V_0), we have that $J_\Phi(u) \leq V_0^{-1} J_{\Phi,V}(u)$ for all $u \in X$. Without loss of generality, we can assume that $V_0 < 1$. Then, by convexity of Φ and definition of the Luxemburg's norm, we obtain that

$$J_\Phi\left(\frac{V_0 u}{\|u\|_{\Phi,V}}\right) \leq V_0^{-1} J_{\Phi,V}\left(\frac{V_0 u}{\|u\|_{\Phi,V}}\right) \leq V_0^{-1} V_0 J_{\Phi,V}\left(\frac{u}{\|u\|_{\Phi,V}}\right) \leq 1,$$

for all $x \in X \setminus \{0\}$. This yields that $\|u\|_\Phi \leq \frac{1}{V_0}\|u\|_{\Phi,V}$. Thereby, $X \hookrightarrow W^{s,\Phi}(\mathbb{R}^N)$.

We will now verify that the hypotheses of the Proposition 2.3.12 and Theorem 2.3.15 are satisfied for the N-function $\Psi(t) = |t|^r$ for all $r \in [m, \ell_*^*)$. In fact, It follows from Lemma 2.1.5 that

$$\Phi(t) \leq \Phi(1)|t|^m \leq \Phi(1)|t|^r = \Phi(1)\Psi(t), \quad \text{for all } |t| \geq 1,$$

which implies that $\Phi < \Psi$. Moreover, by using Lemma 2.1.5 and Lemma 2.3.11, we obtain that

$$\lim_{t\to\infty} \frac{\Psi(kt)}{\Phi_*(t)} \leq \frac{k^r}{\Phi_*(1)} \lim_{t\to\infty} \frac{t^r}{t^{\ell_s^*}} = 0, \quad \text{for all } k > 0,$$

and

$$\lim_{t\to 0} \frac{\Psi(t)}{\Phi(t)} \leq \frac{1}{\Phi(1)} \lim_{t\to 0} \frac{t^r}{t^m} < \infty.$$

This concludes the proof. □

Finally, proceeding as in Theorem 3.14 of Albuquerque et al. (2023), we have that $\mathcal{J}'_{s,\Phi,V}$ is of type (S_+). The proof of this result, including a more general operator, will be presented in Section 4.5.

Proposition 2.3.17. *Assume that Φ satisfies Δ_2-condition and (V_0)-(V_1) hold. Then, $\mathcal{J}'_{s,\Phi,V} : X \to X'$ satisfies the (S_+)-property, that is, if for a given $(u_k)_{n\in\mathbb{N}} \subset X$ satisfying $u_k \rightharpoonup u$ weakly in X and*

$$\limsup_{k\to\infty} \langle \mathcal{J}'_{s,\Phi,V}(u_k), u_k - u \rangle \leq 0,$$

then $u_k \to u$ strongly in X.

3 ON A SUPERLINEAR FRACTIONAL Φ-LAPLACIAN TYPE PROBLEM

In the present chapter, we study the following nonlinear fractional elliptic problem

$$\begin{cases} (-\Delta_\Phi)^s u + V(x)\varphi(|u|)u = \nu a(x)|u|^{q-2}u - \lambda|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ \qquad u \in W^{s,\Phi}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_{\lambda,\nu}$}$$

where $s \in (0,1)$, $1 < \ell \leq m < q < p < \ell_s^* = N\ell/(N-\ell s)$, $N \geq 2$ and $\lambda, \nu > 0$. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $a : \mathbb{R}^N \to \mathbb{R}$ is a non-negative measurable function satisfying some additional hypotheses. Furthermore, $\Phi : \mathbb{R} \to \mathbb{R}$ is an N-function defined by

$$\Phi(t) = \int_0^{|t|} \varphi(\tau)\tau \, d\tau,$$

where $\varphi : (0,\infty) \to (0,\infty)$ fulfills some structural assumptions.

The main goal in the present chapter is to investigate the existence and multiplicity of solutions for problem ($\mathcal{P}_{\lambda,\nu}$). Specifically, we find sharp conditions on the parameters λ and ν in order to guarantee the existence of weak solutions in the Nehari sets $\mathcal{N}_{\lambda,\nu}^-$ and $\mathcal{N}_{\lambda,\nu}^+$.

It is important to emphasize that when we deal to these types of problems via variational methods, some difficulties arise. The first one come from loss of homogeneity on the left side of the equation ($\mathcal{P}_{\lambda,\nu}$) inherited from the fractional Φ-Laplacian operator, which implies that we cannot explicitly establish the critical point for fibering map of the nonlinear Rayleigh quotient. Furthermore, an extra level of difficulty arises when dealing with the coercivity of the energy functional associated to the problem restricted to the Nehari sets $\mathcal{N}_{\lambda,\nu}^\pm$. These kinds of difficulties are overcome by using some precisely properties and some extra assumptions on Φ, the powers ℓ, m, q, p, and the weight function involved.

The second difficulty arises from the fact that the nonlinearity

$$f_{\lambda,\nu}(x,t) = \nu a(x)|t|^{q-2}t - \lambda|t|^{p-2}t, x \in \mathbb{R}^N, t \in \mathbb{R},$$

is a sign-changing function which does not satisfy the well-known Ambrosetti-Rabinowitz condition. This fact does permit us to conclude in general that any Palais-Smale sequence is bounded. However, using the Nehari method, we are able to prove some fine estimates showing that a suitable minimization problem has a solution. Here, the main difficulty is to prove that there exists a real number $\nu_n(\lambda) > 0$ such that the Nehari manifold $\mathcal{N}_{\lambda,\nu}$ is empty for each $\nu < \nu_n(\lambda)$ and $\lambda > 0$. Another factor inherent to our work environment is that the Nehari manifold can splited as $\mathcal{N}_{\lambda,\nu} = \mathcal{N}_{\lambda,\nu}^+ \cup \mathcal{N}_{\lambda,\nu}^0 \cup \mathcal{N}_{\lambda,\nu}^-$ with $\mathcal{N}_{\lambda,\nu}^0$ nonempty for each $\nu \geq \nu_n(\lambda)$ and $\lambda > 0$. In addition, we have that $\overline{\mathcal{N}^{\pm}_{\lambda,\nu}} \subseteq \mathcal{N}^{\pm}_{\lambda,\nu} \cup \mathcal{N}^{0}_{\lambda,\nu}$ (see Lemma 3.2.24), that is, the Nehari sets $\mathcal{N}^{-}_{\lambda,\nu}$ and $\mathcal{N}^{+}_{\lambda,\nu}$ are not necessarily closed. Consequently, the minimizing sequences for the energy functional associated to the problem $(\mathcal{P}_{\lambda,\nu})$ restricted to $\mathcal{N}^{\pm}_{\lambda,\nu}$ can strongly converge to a function which belongs to $\mathcal{N}^{0}_{\lambda,\nu}$ where the Lagrange Multipliers Theorem does not apply anymore. The strategy to overcome such difficulties it is based on arguments employed by Silva et al. (2024a). More precisely, we use the Nehari method (NEHARI, 1960; NEHARI, 1961) combined with the Pohozhaev fibering method (POHOZAEV, 1990) and nonlinear Rayleigh Quotient method (IL'YASOV, 2017; IL'YASOV, 2005) to prove the existence of the parameters $\lambda_*, \lambda^* > 0$ in such a way that, for each $\nu > \nu_n(\lambda)$ and $\lambda \in (0, \lambda_*)$ or $\lambda \in (0, \lambda^*)$, the minimizing functions for the energy functional restricted to $\mathcal{N}^{\pm}_{\lambda,\nu}$ does not belong to $\mathcal{N}^{0}_{\lambda,\nu}$. Therefore, this statement provides us two weak solutions, since the Nehari sets $\mathcal{N}^{\pm}_{\lambda,\nu}$ are natural constraints for the Problem $(\mathcal{P}_{\lambda,\nu})$.

The remainder of this chapter is organized as follows: In the forthcoming section, we present our assumptions and the main results of this chapter. In Section 3.2, we establish some results concerning on the Nehari method and nonlinear Rayleigh Quotient method for our main problem. In Section 3.3 is proved our main results by analyzing the energy levels for each minimizer in the Nehari manifolds $\mathcal{N}^{\pm}_{\lambda,\nu}$. The Section 3.4 is devoted to the asymptotic behavior of solutions obtained in the Theorems 3.1.1 and 3.1.2. Finally, in Section 3.5 is studied the cases in which the parameters λ and ν are equal to λ_*, λ^* and ν_n, respectively.

3.1 ASSUMPTIONS AND STATEMENT OF THE MAIN THEOREMS

As mentioned in the introduction, following Silva et al. (2024a), we consider existence and multiplicity of nontrivial weak solutions for the Problem $(\mathcal{P}_{\lambda,\nu})$ for suitable parameters $\lambda > 0$ and $\nu > 0$. The main idea is to ensure sharp conditions on the parameters λ and ν such that the Nehari method can be applied. For this purpose, throughout this chapter, we assume we assume the following hypotheses:

(φ_1) $\varphi : (0, +\infty) \to (0, +\infty)$ is a C^2-function, $\lim_{t\to 0^+} t\varphi(t) = 0$ and $\lim_{t\to\infty} t\varphi(t) = \infty$.

(φ_2) $t \mapsto t\varphi(t)$ is strictly increasing in $(0, \infty)$.

(φ_3) There exist $\ell, m \in (1, N/s)$ such that

$$\ell - 2 := \inf_{t>0} \frac{(\varphi(t)t)''t}{(\varphi(t)t)'} \leq \sup_{t>0} \frac{(\varphi(t)t)''t}{(\varphi(t)t)'} =: m - 2.$$

(φ_4) The following embedding conditions hold

$$\int_0^1 \left(\frac{t}{\Phi(t)}\right)^{\frac{s}{N-s}} dt < \infty \quad \text{and} \quad \int_1^\infty \left(\frac{t}{\Phi(t)}\right)^{\frac{s}{N-s}} dt = \infty.$$

In order to apply the nonlinear Rayleigh quotient method in our framework, we also suppose the following hypotheses:

(H_1) It holds that $\nu, \lambda > 0$, $\ell \leq m < q < p < \ell_s^* = N\ell/(N - \ell s)$ and $m(q-\ell) < p(q-m)$.

(H_2) It holds that $a \in L^r(\mathbb{R}^N)$ with $r = (p/q)' = p/(p-q)$ and $a(x) > 0$ a.e. in $x \in \mathbb{R}^N$.

Furthermore, since we study $(\mathcal{P}_{\lambda,\nu})$ by using variational methods in the spirit of Bartsch and Wang (1995), we assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(V_0) There exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$.

(V_1) For each $M > 0$, it holds that the set $\{x \in \mathbb{R}^N : V(x) \leq M\}$ has finite Lebesgue measure.

Due to the presence of the potential V, our working space is give by

$$X := \left\{ u \in W^{s,\Phi}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)\Phi(|u|)\, dx < \infty \right\},$$

endowed with the Luxemburg's norm as follows:

$$\|u\| := \inf \left\{ \lambda > 0 : \mathcal{J}_{s,\Phi,V}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

where $W^{s,\Phi}(\mathbb{R}^N)$ is the fractional Orlicz-Sobolev type space and the modular function $\mathcal{J}_{s,\Phi,V} : X \to \mathbb{R}$ is determined in the following form:

$$\mathcal{J}_{s,\Phi,V}(u) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \Phi(|D_s u|)\, d\mu + \int_{\mathbb{R}^N} V(x)\Phi(|u|)\, dx,$$

being the measure μ defined by $d\mu = \frac{dxdy}{|x-y|^N}$.

It is important to mention that X is a reflexive Banach space, see (BAHROUNI; OUNAIES, 2020). Moreover, the energy functional $\mathcal{I}_{\lambda,\nu} : X \to \mathbb{R}$ associated to Problem $(\mathcal{P}_{\lambda,\nu})$ is given by

$$\mathcal{I}_{\lambda,\nu}(u) = \mathcal{J}_{s,\Phi,V}(u) - \frac{\nu}{q}\|u\|_{q,a}^q + \frac{\lambda}{p}\|u\|_p^p, \ u \in X, \tag{3.1}$$

where

$$\|u\|_{q,a}^q = \int_{\mathbb{R}^N} a(x)|u|^q dx \text{ and } \|u\|_p^p = \int_{\mathbb{R}^N} |u|^p dx,\ u \in X.$$

Under our hypotheses, usual computations and the Sobolev embedding show that $\mathcal{I}_{\lambda,\nu}$ belongs to $C^2(X, \mathbb{R})$ for all $\lambda > 0$ and $\nu > 0$, and their Fréchet derivative $\mathcal{I}'_{\lambda,\nu} \colon X \to X^*$ is given by

$$\mathcal{I}'_{\lambda,\nu}(u)v = \mathcal{J}'_{s,\Phi,V}(u)v - \nu \int_{\mathbb{R}^N} a(x)|u|^{q-2}uv\, dx + \lambda \int_{\mathbb{R}^N} |u|^{p-2}uv\, dx, \text{ for all } u, v \in X,$$

where

$$\mathcal{J}'_{s,\Phi,V}(u)v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(|D_s u|) D_s u D_s v\, d\mu + \int_{\mathbb{R}^N} \varphi(|u|)uv\, dx, \text{ for all } u, v \in X.$$

Furthermore, a function $u \in X$ is a critical point for the functional $\mathcal{I}_{\lambda,\nu}$ if and only if u is a weak solution to the elliptic Problem $(\mathcal{P}_{\lambda,\nu})$. Precisely, a function $u \in X$ is said to be a weak solution for Problem $(\mathcal{P}_{\lambda,\nu})$ if and only if

$$\mathcal{J}'_{s,\Phi,V}(u)v - \nu \int_{\mathbb{R}^N} a(x)|u|^{q-2}uv\, dx + \lambda \int_{\mathbb{R}^N} |u|^{p-2}uv\, dx = 0 \text{ for all } v \in X.$$

Now, by using the same ideas introduced by Nehari (NEHARI, 1960; NEHARI, 1961), we consider the Nehari set associated to our main Problem $(\mathcal{P}_{\lambda,\nu})$ as follows

$$\mathcal{N}_{\lambda,\nu} := \{u \in X \setminus \{0\} : \mathcal{I}'_{\lambda,\nu}(u)u = 0\} = \left\{u \in X \setminus \{0\} : \mathcal{J}'_{s,\Phi,V}(u)u + \lambda\|u\|_p^p = \nu\|u\|_{q,a}^q\right\}. \tag{3.2}$$

Under these conditions, by using the same ideas employed by Tarantello (1992), we can split the Nehari set $\mathcal{N}_{\lambda,\nu}$ into three disjoint subsets in the following way:

$$\begin{aligned}
\mathcal{N}_{\lambda,\nu}^+ &= \{u \in \mathcal{N}_{\lambda,\nu} : \mathcal{I}''_{\lambda,\nu}(u)(u,u) > 0\},\\
\mathcal{N}_{\lambda,\nu}^- &= \{u \in \mathcal{N}_{\lambda,\nu} : \mathcal{I}''_{\lambda,\nu}(u)(u,u) < 0\},\\
\mathcal{N}_{\lambda,\nu}^0 &= \{u \in \mathcal{N}_{\lambda,\nu} : \mathcal{I}''_{\lambda,\nu}(u)(u,u) = 0\}.
\end{aligned}$$

The main feature in the present chapter is to find weak solutions for our main problem using for the following minimization problems

$$\mathcal{E}_{\lambda,\nu}^- := \inf\{\mathcal{I}_{\lambda,\nu}(u) : u \in \mathcal{N}_{\lambda,\nu}^-\} \tag{3.3}$$

and

$$\mathcal{E}_{\lambda,\nu}^+ := \inf\{\mathcal{I}_{\lambda,\nu}(u) : u \in \mathcal{N}_{\lambda,\nu}^+\}. \tag{3.4}$$

Namely, we prove that $\mathcal{E}_{\lambda,\nu}^-$ and $\mathcal{E}_{\lambda,\nu}^+$ are attained by some specific functions.

In order to apply the nonlinear Rayleigh quotient, we also need to consider others definitions. Firstly, we define the following set

$$\mathcal{E}_{\lambda,\nu} = \{u \in X \setminus \{0\} : \mathcal{I}_{\lambda,\nu}(u) = 0\}. \tag{3.5}$$

In the sequel, we introduce the nonlinear generalized Rayleigh quotients which have been extensively explored in the last years, see (IL'YASOV, 2005; IL'YASOV, 2017; IL'YASOV; SILVA, 2018; CARVALHO; SILVA; GOULART, 2021; SILVA; OLIVEIRA; GOULART, 2023; CARVALHO et al., 2024; SILVA; ROCHA; SILVA, 2024). More specifically, we define the functionals $R_n, R_e : X \setminus \{0\} \to \mathbb{R}$ associated with the parameter $\nu > 0$ in the following form:

$$R_n(u) := R_{n,\lambda}(u) = \frac{\mathcal{J}'_{s,\Phi,V}(u)u + \lambda||u||_p^p}{||u||_{q,a}^q}, \quad \text{for } u \in X \setminus \{0\},\ \lambda > 0 \tag{3.6}$$

and

$$R_e(u) := R_{e,\lambda}(u) = \frac{\mathcal{J}_{s,\Phi,V}(u) + \frac{\lambda}{p}||u||_p^p}{\frac{1}{q}||u||_{q,a}^q}, \quad \text{for } u \in X \setminus \{0\},\ \lambda > 0. \tag{3.7}$$

The sets given in (3.2) and (3.5) is linked to the nonlinear generalized Rayleigh quotients. Precisely, given $u \in X \setminus \{0\}$, we have the following assertions:

$$u \in \mathcal{N}_{\lambda,\nu} \text{ if and only if } \nu = R_n(u) \tag{3.8}$$

and

$$u \in \mathcal{E}_{\lambda,\nu} \text{ if and only if } \nu = R_e(u). \tag{3.9}$$

We also define the extremal values:

$$\nu_n(\lambda) := \inf_{u \in X \setminus \{0\}} \inf_{t>0} R_n(tu) \qquad \text{and} \qquad \nu_e(\lambda) := \inf_{u \in X \setminus \{0\}} \inf_{t>0} R_e(tu). \tag{3.10}$$

It is worthwhile to mention that under our assumptions, the functionals R_n and R_e belong to $C^2(X \setminus \{0\}, \mathbb{R})$ for each $\lambda > 0$. This can be verified using standard arguments and the Sobolev embeddings, together with the fact that $m < q < p < \ell_s^*$.

Now we are in position to present our main results. Firstly, taking into account that $\mathcal{I}_{\lambda,\nu}$ is bounded from below in $\mathcal{N}^-_{\lambda,\nu}$, we can consider the minimization problem given in (3.3). Thus, our first main theorem can be stated as follows:

Theorem 3.1.1. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, for each* $\lambda > 0$ *we have that* $0 < \nu_n(\lambda) < \nu_e(\lambda) < \infty$ *and there exists* $\lambda_* > 0$ *such that the Problem* $(\mathcal{P}_{\lambda,\nu})$ *admits at least one weak solution* $u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu}$ *for each* $\lambda \in (0, \lambda_*)$ *and* $\nu > \nu_n(\lambda)$. *Moreover, the following statements are satisfied:*

(i) $\mathcal{E}^{-}_{\lambda,\nu} = \mathcal{I}_{\lambda}(u_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^{-}_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w)$.

(ii) *There exists* $D_{\nu} > 0$ *such that* $\mathcal{E}^{-}_{\lambda,\nu} \geq D_{\nu}$.

Next, we consider the minimization problem given in (3.4). It is worthwhile to mention that a ground state solution is a nontrivial solution which has the minimal energy level among any other nontrivial solutions. Hence, we stated our next main result in the following form:

Theorem 3.1.2. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, there exists* $\lambda^* > 0$ *such that the Problem* $(\mathcal{P}_{\lambda,\nu})$ *admits at least one weak solution* $v_{\lambda,\nu} \in \mathcal{N}^{+}_{\lambda,\nu}$ *if one of the following conditions is satisfied:*

(i) $\lambda \in (0, \lambda^*)$ *and* $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$.

(ii) $\lambda > 0$ *and* $\nu \in [\nu_e(\lambda), \infty)$.

(iii) $\lambda > 0$ *and* $\nu \in (\nu_e(\lambda) - \varepsilon, \nu_e(\lambda))$, *where* $\varepsilon > 0$ *is small enough.*

Furthermore, the weak solution $v_{\lambda,\nu}$ *is a ground state solution with the following properties:*

(i) *For each* $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$ *we obtain that* $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) > 0$.

(ii) *For* $\nu = \nu_e(\lambda)$ *we have that* $\mathcal{I}_{\lambda}(v_{\lambda,\nu}) = 0$.

(iii) *For each* $\nu > \nu_e(\lambda)$ *we have also that* $\mathcal{I}_{\lambda}(v_{\lambda,\nu}) < 0$.

As a consequence, by using Theorems 3.1.1 and 3.1.2, we obtain the following result:

Corollary 3.1.3. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\lambda \in (0, \min(\lambda_*, \lambda^*))$ *and* $\nu > \nu_n(\lambda)$. *Then, the Problem* $(\mathcal{P}_{\lambda,\nu})$ *admits at least two weak solutions.*

Finally, we prove a nonexistence result for Problem $(\mathcal{P}_{\lambda,\nu})$.

Theorem 3.1.4. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\lambda > 0$ *and* $\nu \in (-\infty, \nu_n(\lambda))$. *Then, the Problem* $(\mathcal{P}_{\lambda,\nu})$ *does not admit any nontrivial solution.*

Next, we present some examples of functions Φ for which the previous results may be applied.

Example 3.1.5. *The N-function $\Phi(t) = \frac{|t|^r}{r}$, which produces the well-known fractional r-Laplacian operator $(-\Delta_r)^s$ with $1 < r < N/s$, is a relevant example of how our assumptions are satisfied. In this case, we have $\varphi(t) = t^{r-2}$, and it satisfies the assumptions (φ_1)-(φ_4) with $\ell = m = r$. Moreover, the hypothesis (H_1) is trivially verified for any $1 < r < q < p < r_s^*$.*

As a second example, we can consider the function $\varphi(t) = t^{r_1-2} + t^{r_2-2}$, with $1 < r_2 < r_2 < N/s$. For this function, the problem $(\mathcal{P}_{\lambda,\nu})$ read as

$$\begin{cases} (-\Delta_{r_1})^s u + (-\Delta_{r_2})^s u + V(x)(|u|^{r_1-2}u + |u|^{r_2-2}u) = \nu a(x)|u|^{q-2}u - \lambda|u|^{p-2}u \ \text{ in } \mathbb{R}^N, \\ \qquad\qquad u \in W^{s,r_1}(\mathbb{R}^N) \cap W^{s,r_2}(\mathbb{R}^N). \end{cases}$$

This problem is named in the literature as fractional (r_1, r_2)-Laplacian type problem. We observe that φ satisfies the hypotheses (φ_1)-(φ_4) with $\ell = r_1$ e $m = r_2$. Furthermore, (H_1) is verified whenever $\frac{r_2(q-r_1)}{q-r_2} < p < \frac{Nr_1}{N-sr_1} = (r_1)_s^*$.

Example 3.1.6. *Another special case that can be considered is the function $\varphi(t) = \log(1+t)$, $t \geq 0$. In this case, by direct computations, we obtain the following N-function:*

$$\Phi(t) = \int_0^t \varphi(\tau)\tau \, d\tau = \frac{t^2}{2}\log(1+t) - \frac{t^2}{4} + \frac{t}{2} - \frac{1}{2}\log(1+t), \ t \geq 0.$$

It is clear that φ satisfies the assumptions (φ_1) and (φ_2). In addition, using the elementary inequality $\frac{t}{1+t} \leq \log(1+t), \ t \geq 0$ we obtain that

$$(\varphi(t)t)' = \log(1+t) + \frac{t}{1+t} \geq \frac{2t}{1+t},$$

and

$$(\varphi(t)t)''t = \frac{2t}{1+t} - \frac{t^2}{(1+t)^2} = \left(2 - \frac{t}{1+t}\right)\frac{t}{1+t} > 0.$$

As a result, we deduce that

$$0 < \frac{(\varphi(t)t)''t}{(\varphi(t)t)'} \leq \frac{1}{2}\left(2 - \frac{t}{1+t}\right) < 1, \ t > 0.$$

Hence, the assumption (φ_3) is satisfied for $\ell = 2$ and $m = 3$. Moreover, (H_1) holds whenever $\frac{3(q-2)}{q-3} < p < \frac{2N}{N-2s} = 2_s^*$.

Remark 3.1.7. *Under the assumptions (φ_1)-(φ_3), we prove that the function*

$$t \mapsto \frac{(2-q)\varphi(t) + \varphi'(t)t}{t^{p-2}}$$

is strictly increasing for all $t > 0$. This fact implies that we can apply the nonlinear Raleigh quotient due to the fact that the map $t \mapsto R_n(tu), u \neq 0$ has a unique critical point, see Proposition 3.2.3 ahead. Under these conditions, we are able to prove that any function in a cone set has projection in the Nehari set.

Remark 3.1.8. *The hypotheses (H_1) and (H_2) play a crucial role in the development of the main results. They ensure the necessary conditions for applying variational methods and tools, as well as compactness results.*

The condition $m(q-\ell) < p(q-m)$ in (H_1) is used for the first time to establish the positivity of the energy level $\mathcal{E}^-_{\lambda,\nu}$ for each $\lambda > 0$ and $\nu > \nu_n(\lambda)$. This condition also guarantees the existence of parameters $\lambda_*, \lambda^* > 0$ in Theorem 3.1.1 and 3.1.2, respectively. Furthermore, it is utilized to obtain qualitative properties of the solution for problem $(\mathcal{P}_{\lambda,\nu})$, such as continuity and asymptotic behavior with respect to parameters λ and ν. As for hypothesis (H_2), it is used to derive the lower bound for the function Λ_n and to establish the coercivity of the functional $\mathcal{I}_{\lambda,\nu}$ restricted to Nehari manifold.

3.2 THE NEHARI AND NONLINEAR RAYLEIGH QUOTIENT METHODS

In this section, we follow some ideas discusse by Silva et al. (2024a). The main goal here is to ensure existence of weak solutions for our main problem using the Nehari method together with the nonlinear Rayleigh quotient. In order to do that, we also consider the fibration method described below.

The Nehari manifold has an intrinsic connection with the behavior of the so-called fibering map $\gamma_u : [0,\infty) \to \mathbb{R}$ defined by

$$\gamma_u(t) = \mathcal{I}_{\lambda,\nu}(tu),$$

for each $u \in X \setminus \{0\}$ fixed. Under our assumptions, $\gamma_u \in C^2(0,\infty)$ and $u \in \mathcal{N}_{\lambda,\nu}$ if and only if $\gamma_u'(1) = 0$. More generally, we have that $tu \in \mathcal{N}_{\lambda,\nu}$ if and only if $\gamma_u'(t) = 0$ where $t > 0$. Therefore, the geometric analysis of the fibering maps plays a key role in our arguments. For further details on this subject, we refer the reader to the important works (POHOZAEV, 1990; DRÁBEK; POHOZAEV, 1997; BROWN; WU, 2007; BROWN; WU, 2009).

Next, we present some useful results related to the energy functional $\mathcal{I}_{\lambda,\nu}$ and the nonlinear Rayleigh quotient. Firstly, we mention that the functional $\mathcal{I}_{\lambda,\nu}$ is in C^2 class due to the fact that $m < q < p < \ell_s^*$. Furthermore, we obtain that

$$\mathcal{I}''_{\lambda,\nu}(u)(u,u) = \mathcal{J}''_{s,\Phi,V}(u)(u,u) - \nu(q-1)\|u\|^q_{a,q} + \lambda(p-1)\|u\|^p_p, \;\; u \in X, \tag{3.11}$$

where

$$\begin{aligned}\mathcal{J}''_{s,\Phi,V}(u)(u,u) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[\phi'(|D_s u|)|D_s u|^3 + \phi(|D_s u|)|D_s u|^2\right] d\mu \\ &+ \int_{\mathbb{R}^N} V(x)\left[\phi'(|u|)|u|^3 + \phi(|u|)|u|^2\right] dx.\end{aligned}$$

Observe that when $u \in \mathcal{N}_{\lambda,\nu}$, using 3.2, we have

$$\begin{aligned}\mathcal{I}''_{\lambda,\nu}(u)(u,u) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\varphi'(|D_s u|)|D_s u|^3 + (2-q)\varphi(|D_s u|)|D_s u|^2\right] d\mu \\ &+ \int_{\mathbb{R}^N} V(x)\left[\varphi'(|u|)|u|^3 + (2-q)\varphi(|u|)|u|^2\right] dx + \lambda(p-q)\|u\|_p^p\end{aligned} \tag{3.12}$$

or equivalently

$$\begin{aligned}\mathcal{I}''_{\lambda,\nu}(u)(u,u) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\varphi'(|D_s u|)|D_s u|^3 + (2-p)\varphi(|D_s u|)|D_s u|^2\right] d\mu \\ &+ \int_{\mathbb{R}^N} V(x)\left[\varphi'(|u|)|u|^3 + (2-p)\varphi(|u|)|u|^2\right] dx + (p-q)\nu\|u\|_{a,q}^q.\end{aligned} \tag{3.13}$$

From now on, we study the behavior of the fibering maps associated to the functionals R_n and R_e. It is important to emphasize that, in the spirit of the works (CARVALHO; SILVA; GOULART, 2021; SILVA et al., 2024a), a major challenge in the present chapter is to consider implicitly the critical values of the fibering maps due to the non-homogeneity of the functionals R_n and R_e. In order to overcome this obstacle, we will employ the assumption (φ_3) to prove that the functions $t \mapsto R_n(tu)$ and $t \mapsto R_e(tu)$ admits exactly one critical point for each $u \in X \setminus \{0\}$ fixed. This property is described by the following results:

Lemma 3.2.1. *Assume that (φ_1)-(φ_3) hold. Then, the following assertions hold:*

(i) It holds that

$$-1 < \ell - 2 := \inf_{t>0} \frac{\varphi'(t)t}{\varphi(t)} \leq \sup_{t>0} \frac{\varphi'(t)t}{\varphi(t)} =: m-2.$$

Consequently, we have that

$$1 < \ell \leq \frac{\varphi(t)t^2}{\Phi(t)} \leq m < \infty, \quad t > 0.$$

(ii) The function

$$t \mapsto \frac{(2-q)\varphi(t) + \varphi'(t)t}{t^{p-2}}$$

is strictly increasing for all $t > 0$.

Proof. Firstly, by using (φ_3), we have that $(\ell-2)(\varphi(t)t)' \leq (\varphi(t)t)''t \leq (m-2)(\varphi(t)t)'$. Then, by using integration by parts, we obtain that

$$(\ell-2)\varphi(t)t \leq t(\varphi(t)t)' - \int_0^t (\varphi(s)s)' ds \leq (m-2)\varphi(t)t.$$

This implies that

$$(\ell-1)\varphi(t)t \leq t(\varphi(t)t)' \leq (m-1)\varphi(t)t \tag{3.14}$$

Since $t(\varphi(t)t)' = \varphi'(t)t^2 + \varphi(t)t$, it follows from (3.14) that

$$(\ell-2)\varphi(t)t \leq \varphi'(t)t^2 \leq (m-2)\varphi'(t)t.$$

This shows that (i) holds.

Now, we will prove that (ii) holds. Indeed, we define the auxiliary function

$$\Theta(t) = \frac{(2-q)\varphi(t) + \varphi'(t)t}{t^{p-2}}.$$

Note that $\varphi'(t)t = (\varphi(t)t)' - \varphi(t)$. Then, we can rewrite $\Theta(t)$ as

$$\Theta(t) = \frac{(1-q)\varphi(t)t + (\varphi(t)t)'t}{t^{p-1}}, \quad t > 0.$$

By differentiating Θ and using (φ_3), we deduce that

$$\begin{aligned}
\Theta'(t) &= \frac{t^{p-1}\left[(1-q)(\varphi(t)t)' + (\varphi(t)t)''t + (\varphi(t)t)'\right] - (p-1)t^{p-2}\left[(1-q)\varphi(t)t + (\varphi(t)t)'t\right]}{t^{2(p-1)}} \\
&= \frac{t^{p-1}}{t^{2(p-1)}}\left[(2-q)(\varphi(t)t)' + (\varphi(t)t)''t + (p-1)(q-1)\varphi(t) - (p-1)(\varphi(t)t)'\right] \\
&\geq \frac{t^{p-1}}{t^{2(p-1)}}\left[(\ell-q)(\varphi(t)t)' + (p-1)(q-1)\varphi(t) - (p-1)(\varphi(t)t)'\right].
\end{aligned}$$

Since $1 < \ell \leq m < q < p$, we conclude from inequality (3.14) that

$$\begin{aligned}
\Theta'(t) &\geq \frac{t^{p-1}}{t^{2(p-1)}}\left[(\ell-q)(m-1)\varphi(t) + (p-1)(q-1)\varphi(t) - (p-1)(m-1)\varphi(t)\right] \\
&\geq \frac{t^{p-1}}{t^{2(p-1)}}\left[(\ell-q)(p-1)\varphi(t) + (p-1)(q-1)\varphi(t) - (p-1)(m-1)\varphi(t)\right] \\
&= \frac{t^{p-1}}{t^{2(p-1)}}[(q-m)(p-m)]\varphi(t) > 0.
\end{aligned}$$

Here, we used that $(\ell-q)(p-1) + (p-1)(q-1) - (p-1)(m-1) = (q-m)(p-m) > 0$. Therefore, Θ is strictly increasing, which proves (ii). □

Proposition 3.2.2. *Assume that* (φ_1)-(φ_3), (H_1)-(H_2) *and* (V_0) *hold. Let* $u \in X \setminus \{0\}$ *be fixed. Then, the function* $t \mapsto R_n(tu)$ *satisfies the following properties:*

(i) It holds that

$$\lim_{t\to 0^+} \frac{R_n(tu)}{t^{m-q}} > 0 \quad \textit{and} \quad \lim_{t\to 0^+} \frac{\frac{d}{dt}R_n(tu)}{t^{m-q-1}} < 0.$$

(ii) It holds that

$$\lim_{t\to \infty} \frac{R_n(tu)}{t^{p-q}} > 0 \quad \textit{and} \quad \lim_{t\to \infty} \frac{\frac{d}{dt}R_n(tu)}{t^{p-q-1}} > 0.$$

Proof. (i) First, since

$$R_n(tu) = \frac{t^{2-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right) + \lambda t^{p-q}\|u\|_p^p}{\|u\|_{q,a}^q},$$

we infer that

$$\frac{R_n(tu)}{t^{m-q}} = \frac{t^{2-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}\varphi(t|u|)|u|^2\,dx\right) + \lambda t^{p-m}\|u\|_p^p}{\|u\|_{q,a}^q}.$$

This implies that

$$\lim_{t\to 0^+}\frac{R_n(tu)}{t^{m-q}} \geq \lim_{t\to 0^+}\frac{t^{2-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\|u\|_{q,a}^q}.$$

The last inequality together with Lemma 3.2.1 and Lemma 2.1.5 imply that

$$\begin{aligned}\lim_{t\to 0^+}\frac{R_n(tu)}{t^{m-q}} &\geq \lim_{t\to 0^+}\frac{t^{-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Phi(t|D_su|)\,d\mu + \int_{\mathbb{R}^N}V(x)\Phi(t|u|)\,dx\right)}{\|u\|_{q,a}^q}\\ &\geq \lim_{t\to 0^+}\frac{\mathcal{J}_{s,\Phi,V}(u)}{\|u\|_{q,a}^q} > 0.\end{aligned}$$

Now, we observe that

$$\begin{aligned}\frac{d}{dt}R_n(tu) = {} & \frac{(2-q)t^{1-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\|u\|_{q,a}^q}\\ &+\frac{t^{2-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi'(t|D_su|)|D_su|^3d\mu + \int_{\mathbb{R}^N}V(x)\varphi'(t|u|)|u|^3\,dx\right)}{\|u\|_{q,a}^q}\\ &+\frac{\lambda(p-q)t^{p-q-1}\|u\|_p^p}{\|u\|_{q,a}^q}.\end{aligned} \tag{3.15}$$

Then,

$$\begin{aligned}\frac{\frac{d}{dt}R_n(tu)}{t^{m-q-1}} = {} & \frac{(2-q)t^{2-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\|u\|_{q,a}^q}\\ &+\frac{t^{3-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi'(t|D_su|)|D_su|^3d\mu + \int_{\mathbb{R}^N}V(x)\varphi'(t|u|)|u|^3\,dx\right)}{\|u\|_{q,a}^q}\\ &\frac{+\lambda(p-q)t^{p-m}\|u\|_p^p}{\|u\|_{q,a}^q}.\end{aligned}$$

By using Lemma 3.2.1, we deduce that

$$\begin{aligned}\frac{\frac{d}{dt}R_n(tu)}{t^{m-q-1}} \leq {} & \frac{(2-q)t^{2-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\|u\|_{q,a}^q}\\ &+\frac{t^{2-m}(m-2)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu + \int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\|u\|_{q,a}^q}\\ &+\frac{\lambda(p-q)t^{p-m}\|u\|_p^p}{\|u\|_{q,a}^q}\\ \leq {} & \frac{\ell(m-q)t^{-m}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Phi(t|D_su|)\,d\mu + \int_{\mathbb{R}^N}V(x)\Phi(t|u|)\,dx\right)}{\|u\|_{q,a}^q}\\ &\frac{+\lambda(p-q)t^{p-m}\|u\|_p^p}{\|u\|_{q,a}^q}.\end{aligned}$$

Therefore, the last inequality jointly with Lemma 2.1.5 and (H_1) give us

$$\lim_{t\to 0^+}\frac{\frac{d}{dt}R_n(tu)}{t^{m-q-1}} \leq \ell(m-q)\frac{\mathcal{J}_{s,\Phi,V}(u)}{\|u\|_{q,a}^q} + \lim_{t\to 0^+}\frac{\lambda(p-q)t^{p-m}\|u\|_p^p}{\|u\|_{q,a}^q} = \ell(m-q)\frac{\mathcal{J}_{s,\Phi,V}(u)}{\|u\|_{q,a}^q} < 0.$$

This ends the proof of item (i).

(ii) Similarly, taking into account Lemma 3.2.1, we obtain that

$$\lim_{t\to\infty}\frac{R_n(tu)}{t^{p-q}} \geq \lim_{t\to\infty}\frac{t^{-p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Phi(t|D_s u|)\,d\mu + \int_{\mathbb{R}^N}V(x)\Phi(t|u|)\,dx\right)}{\|u\|_{q,a}^q} + \frac{\lambda\|u\|_p^p}{\|u\|_{q,a}^q}.$$

Then, by Lemma 2.1.5 and (H_1), we conclude that

$$\lim_{t\to\infty}\frac{R_n(tu)}{t^{p-q}} \geq \frac{\lambda\|u\|_p^p}{\|u\|_{q,a}^q} + \lim_{t\to\infty}\frac{t^{\ell-p}\mathcal{J}_{s,\Phi,V}(u)}{\|u\|_{q,a}^q} = \frac{\lambda\|u\|_p^p}{\|u\|_{q,a}^q} > 0.$$

Finally, using the expression (3.15), (φ_3) and proceeding as above, we infer that

$$\lim_{t\to\infty}\frac{\frac{d}{dt}R_n(tu)}{t^{p-q-1}} \geq \frac{\lambda(p-q)\|u\|_p^p}{\|u\|_{q,a}^q} + \lim_{t\to\infty}\frac{(\ell-q)t^{\ell-q}\mathcal{J}_{s,\Phi,V}(u)}{\|u\|_{q,a}^q} = \frac{\lambda(p-q)\|u\|_p^p}{\|u\|_{q,a}^q} > 0.$$

This finishes the proof. □

Now, by using the previous proposition, we prove the following result:

Proposition 3.2.3. *Assume that (φ_1)-(φ_3), (H_1)-(H_2) and (V_0)-(V_1) hold. Then, for each $u \in X\setminus\{0\}$ and $\lambda > 0$, there exists and unique $\mathsf{t}(u) := \mathsf{t}_\lambda(u) > 0$ satisfying*

$$\frac{d}{dt}R_n(tu) = 0 \quad \textit{for} \quad t = \mathsf{t}(u). \tag{3.16}$$

Proof. First, according to Proposition 3.2.2 and Bolzano's theorem, there exists at least one real value $\mathsf{t}(u) > 0$ such that the equation (3.16) is verified. Furthermore, by expression (3.15), we have that

$$\frac{d}{dt}R_n(tu) = 0 \quad \text{for} \quad t > 0$$

is equivalent to following identity

$$\begin{aligned}-\lambda(p-q)\|u\|_p^p = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[(2-q)\varphi(t|D_s u|) + \varphi'(t|D_s u|)|tD_s u|]\,|D_s u|^2}{t^{p-2}}d\mu\\ &+ \int_{\mathbb{R}^N}V(x)\frac{[(2-q)\varphi(t|u|) + \varphi'(t|u|)|tu|]\,|u|^2}{t^{p-2}}dx,\end{aligned}$$

which is also equivalent to

$$\begin{aligned}-\lambda(p-q)\|u\|_p^p = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(2-q)\varphi(t|D_s u|) + \varphi'(t|D_s u|)|tD_s u|}{|tD_s u|^{p-2}}|D_s u|^p d\mu\\ &+ \int_{\mathbb{R}^N}V(x)\frac{(2-q)\varphi(t|u|) + \varphi'(t|u|)|tu|}{|tu|^{p-2}}|u|^p dx.\end{aligned} \tag{3.17}$$

On the other side, the Lemma 3.2.1 and (V_0) guarantee us that the function $\mathcal{K}_u\colon (0,\infty)\to\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{K}_u(t) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(2-q)\varphi(|tD_s u|)+\varphi'(|tD_s u|)|tD_s u|}{|tD_s u|^{p-2}}|D_s u|^p d\mu \\ &+\int_{\mathbb{R}^N} V(x)\frac{(2-q)\varphi(|tu|)+\varphi'(|tu|)|tu|}{|tu|^{p-2}}|u|^p dx\end{aligned} \tag{3.18}$$

is strictly increasing for each $u\in X\setminus\{0\}$ fixed. Moreover, it is not difficult to verify that the hypotheses (φ_1)-(φ_3) and (H_1) imply that

$$\lim_{t\to 0^+}\mathcal{K}_u(t) = -\infty \quad \text{and} \quad \lim_{t\to\infty}\mathcal{K}_u(t)=0.$$

Therefore, the equation 3.17 admits only one root $\mathsf{t}(u)>0$ for each $u\in X\setminus\{0\}$. □

Remark 3.2.4. *In the paper (SILVA et al., 2024a), the function $u\mapsto \mathsf{t}(u)$ is obtained explicitly, which allows to prove the continuity directly. However, in the present work, this function is obtained only implicitly, which requires a more delicate approach. In this case, we shall prove the continuity by taking into account the Implicit Function Theorem.*

Proposition 3.2.5. *Assume that (φ_1)-(φ_3), (H_1)-(H_2) and (V_0)-(V_1) hold. Let $u\in X\setminus\{0\}$ be fixed. Then, the following properties are verified:*

(i) There exists a constant $c:=c(\ell,m,p,q,N,s,V_0,\lambda)>0$ such that $\|\mathsf{t}(u)u\|>c$ for all $u\in X\setminus\{0\}$.

(ii) The functional $\mathsf{t}: X\setminus\{0\}\to(0,\infty)$ is of class C^1.

Proof. (i) Firstly, we will prove that $\|\mathsf{t}(u)u\|\geq c$ for some positive constant c. By using Proposition 3.2.3, we have that

$$\begin{aligned}0 &= \|\mathsf{t}(u)u\|_{q,a}^q \mathsf{t}(u)R_n'(\mathsf{t}(u)u)u \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[(2-q)\varphi(|\mathsf{t}(u)D_s u|)+\varphi'(|\mathsf{t}(u)D_s u|)|\mathsf{t}(u)D_s u|\right]|\mathsf{t}(u)D_s u|^2\,d\mu \\ &\quad+\int_{\mathbb{R}^N}V(x)\left[(2-q)\varphi(|\mathsf{t}(u)u|)+\varphi'(|\mathsf{t}(u)u|)|\mathsf{t}(u)u|\right]|\mathsf{t}(u)u|^2\,dx \\ &\quad+\lambda(p-q)\|\mathsf{t}(u)u\|_p^p.\end{aligned} \tag{3.19}$$

The identity (3.19) combined with (φ_3) yield that

$$\begin{aligned}0 &\leq (m-q)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(|\mathsf{t}(u)D_s u|)|\mathsf{t}(u)D_s u|^2\,d\mu+\int_{\mathbb{R}^N}V(x)\varphi(|\mathsf{t}(u)u|)|\mathsf{t}(u)u|^2\,dx\right) \\ &\quad+\lambda(p-q)\|\mathsf{t}(u)u\|_p^p \\ &= (m-q)\mathcal{J}_{s,\Phi,V}'(\mathsf{t}(u)u)(\mathsf{t}(u)u)+\lambda(p-q)\|\mathsf{t}(u)u\|_p^p.\end{aligned}$$

Combining this inequality with the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$, (φ_3) and Lemma 2.3.13, we obtain that

$$\|\mathsf{t}(u)u\|^p \geq \frac{\ell(q-m)}{\lambda(p-q)S_p^p} \min\{\|\mathsf{t}(u)u\|^\ell, \|\mathsf{t}(u)u\|^m\}. \tag{3.20}$$

Since $\lambda > 0$, the assumption (H_1) and estimates (3.20) gives us the desired.

(iii) Now let us prove that $\mathsf{t} : X\backslash\{0\} \to (0,\infty)$ is of class C^1. For this end, let $u_0 \in X\backslash\{0\}$ be fixed. Then, $\mathsf{t}(u_0) > 0$ is well-defined and $R_n'(\mathsf{t}(u_0)u_0)u_0 = 0$. Considering the function $\mathcal{F}\colon X \setminus \{0\} \times (0,\infty) \to \mathbb{R}$ defined by $\mathcal{F}(v,t) = R_n'(tv)(tv)$, it follows that

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{F}(v,t)\Big|_{(v,t)=(u_0,\mathsf{t}(u_0))} &= R_n''(\mathsf{t}(u_0)u_0)(\mathsf{t}(u_0)u_0,u_0) + R_n'(\mathsf{t}(u_0)u_0)u_0 \\ &= \frac{R_n''(\mathsf{t}(u_0)u_0)(\mathsf{t}(u_0)u_0,\mathsf{t}_0(u_0)u_0)}{\mathsf{t}(u_0)} > 0. \end{aligned}$$

In the last inequality was used that $\mathsf{t}(u_0)$ is a global minimum point of $R_n(tu_0)$. Hence, by Implicit Function Theorem (DRÁBEK; MILOTA, 2013), there exists a neighborhood $\mathcal{U}$ of u_0 and a C^1-functional $\eta\colon \mathcal{U} \to (0,\infty)$ such that

$$\mathcal{F}(v,\eta(v)) = R_n'(\eta(v)v)(\eta(v)v) = 0, \quad \text{for all} \quad v \in \mathcal{U}.$$

Since $\mathsf{t}(v)$ is the unique real value that satisfies $R_n'(\mathsf{t}(v)v)(\mathsf{t}(v)v) = 0$, we deduce that $\eta(v) = \mathsf{t}(v)$ for all $v \in \mathcal{U}$. Finally, we conclude from arbitrariness of u_0 that $\mathsf{t} : X \setminus \{0\} \to (0,\infty)$ is a functional of class C^1. This ends the proof. □

Remark 3.2.6. *Let $t > 0$ and $u \in X\backslash\{0\}$ be fixed. Then, using (3.6), the following assertions hold:*

(i) $R_n(tu) = \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}'(tu)tu = 0.$

(ii) $R_n(tu) > \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}'(tu)tu > 0.$

(iii) $R_n(tu) < \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}'(tu)tu < 0.$

As a consequence of Propositions 3.2.2 and 3.2.3, we deduce that the function $Q_n\colon (0,\infty) \to \mathbb{R}$ defined by $Q_n(t) = R_n(tu)$ satisfies $Q_n'(t) = 0$ if and only if $t = \mathsf{t}(u)$. Furthermore, $Q_n'(t) < 0$ if and only if $t \in (0,\mathsf{t}(u))$ and $Q_n'(t) > 0$ if and only if $t > \mathsf{t}(u)$. In the other words, $\mathsf{t}(u)$ is a global minimum point for Q_n. Hence, we can consider the auxiliary functional $\Lambda_n : X \setminus \{0\} \to \mathbb{R}$ defined by

$$\Lambda_n(u) = \min_{t>0} Q_n(tu) = R_n(\mathsf{t}(u)u). \tag{3.21}$$

Moreover, the following extreme parameter is well-defined

$$\nu_n(\lambda) = \inf_{u \in X \setminus \{0\}} \Lambda_n(u).$$

Under these conditions, we have the following result:

Proposition 3.2.7. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, the functional* Λ_n *satisfies the following properties:*

(i) Λ_n *is* 0*-homogeneous, that is,* $\Lambda_n(tu) = \Lambda_n(u)$ *for all* $t > 0$, $u \in X \setminus \{0\}$.

(ii) Λ_n *is continuous and weakly lower semicontinuous.*

(iii) *There exists* $C := C(\ell, m, p, q, N, s, V_0, a, \lambda) > 0$ *such that* $\Lambda_n(u) \geq C > 0$ *for all* $u \in X \setminus \{0\}$. *Furthermore, the function* Λ_n *is unbounded from above.*

(iv) *There exists* $u^* \in X \setminus \{0\}$ *such that* $\nu_n(\lambda) = \Lambda_n(u^*) > 0$. *In particular,* u^* *is a critical point for the functional* Λ_n.

(v) *The function* $v^* := \mathsf{t}(u^*)u^*$ *is a weakly solution of the following nonlocal elliptic problem:*

$$\begin{cases} \mathcal{L}^s_\Phi u + V(x)\left[2\varphi(|u|) + \varphi'(|u|)|u|\right]u = q\nu_n(\lambda)a(x)|u|^{q-2}u - p\lambda|u|^{p-2}u \\ u \in W^{s,\Phi}(\mathbb{R}^N), \end{cases} \qquad (\mathcal{P}_{\nu_n(\lambda)})$$

where the operator $\mathcal{L}^s_\Phi$ *is given by*

$$\mathcal{L}^s_\Phi u = 2(-\Delta_\Phi)^s u + (-\Delta_\varphi)^s u,$$

being

$$(-\Delta_\varphi)^s u := \text{p.v.} \int_{\mathbb{R}^N} \varphi'(|D_s u|)|D_s u| D_s u \frac{dy}{|x-y|^{n+s}}.$$

Proof. (i) Let $s > 0$ and $u \in X \setminus \{0\}$ be fixed. We know that

$$R_n'\left(\frac{\mathsf{t}(u)}{s}(su)\right)\left(\frac{\mathsf{t}(u)}{s}(su)\right) = 0.$$

Then, it follows from Proposition 3.2.3 that $\mathsf{t}(su) = \dfrac{\mathsf{t}(u)}{s}$. Therefore,

$$\Lambda_n(su) = R_n(\mathsf{t}(su)su) = R_n(\mathsf{t}(u)u) = \Lambda_n(u).$$

This proves (i).

(ii) First, the continuity of Λ_n it follows from Proposition 3.2.5 and the continuous embedding $X \hookrightarrow L^q(\mathbb{R}^N)$. Now, we consider a sequence $(u_k)_{k\in\mathbb{N}} \subset X$ such that $u_k \rightharpoonup u \neq 0$.

Since the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact, $u_k \to u$ a.e. in $\mathbb{R}^n$. Defining the function $H(t) = (2-q)\varphi(|t|)|t|^2 + \varphi'(|t|)|t|^3$, the last assertion implies that

$$H(|D_s u_k|)|x-y|^{-N} \to H(|D_s u|)|x-y|^{-N} \quad \text{a.e. in} \quad \mathbb{R}^n \times \mathbb{R}^N.$$

We also have that $H(t) \leq \ell(m-q)\Phi(|t|) \leq 0$ by assumptions (φ_3) and (H_1). Thus, by Fatou's Lemma, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(t|D_s u|)\,d\mu &\geq -\liminf_{k\to\infty}\Big(-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(t|D_s u_k|)\,d\mu\Big)\\ &= \limsup_{k\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(t|D_s u_k|)\,d\mu,\end{aligned}$$

and similarly

$$\int_{\mathbb{R}^N} H(t|u|)\,dx \geq \limsup_{k\to\infty}\int_{\mathbb{R}^N} H(t|u_k|)\,dx,$$

for each $t > 0$ fixed. This and the compact embedding $X \hookrightarrow L^r(\mathbb{R})$, for each $r \in (m, \ell_s^*)$, shows that

$$v \mapsto R_n'(tv)(tv) = \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(t|D_s v|)\,d\mu + \int_{\mathbb{R}^N} V(x)H(t|v|)\,dx + \lambda(p-q)\|tv\|_p^p}{\|tv\|_{q,a}^q}$$

is weakly upper semicontinuous for each $t > 0$ fixed. Hence, for k enough larger, it holds

$$0 = R_n'(\mathsf{t}(u)u)(\mathsf{t}(u)u) \geq R_n'(\mathsf{t}(u)u_k)(\mathsf{t}(u)u_k),$$

which implies that $\mathsf{t}(u) \leq \mathsf{t}(u_k)$ for $k \gg 1$, that is, $u \mapsto \mathsf{t}(u)$ is weakly lower semicontinuous. Using this fact and that $v \mapsto R_n(tv)$ is also weakly lower semicontinuous, we conclude that

$$\Lambda_n(u) = R_n(\mathsf{t}(u)u) \leq \liminf_{k\to\infty} R_n(\mathsf{t}(u)u_k) \leq \liminf_{k\to\infty} R_n(\mathsf{t}(u_k)u_k) = \liminf_{k\to\infty}\Lambda_n(u_k),$$

proving that Λ_n is weakly lower semicontinuous.

(iii) Assume first that $\|\mathsf{t}(u)u\| \leq 1$. Hence, by (3.6), (φ_3) and Lemma 2.3.13, we obtain that

$$\begin{aligned}\Lambda_n(u) = R_n(\mathsf{t}(u)u) &\geq \frac{\mathcal{J}_{s,\Phi,V}'(\mathsf{t}(u)u)(\mathsf{t}(u)u)}{\|\mathsf{t}(u)u\|_{q,a}^q}\\ &\geq \ell\frac{\min\{\|\mathsf{t}(u)u\|^\ell, \|\mathsf{t}(u)u\|^m\}}{\|\mathsf{t}(u)u\|_{q,a}^q}\\ &= \ell\frac{\|\mathsf{t}(u)u\|^m}{\|\mathsf{t}(u)u\|_{q,a}^q}.\end{aligned} \tag{3.22}$$

By using assumption (H_2) and Hölder inequality, we have that

$$\|v\|_{q,a}^q \leq \|a\|_r\|v\|_p^q, \quad \text{for all } v \in L^p(\mathbb{R}^N), \tag{3.23}$$

where $r = (p/q)'$. Putting together (3.22), (3.23) and the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$, we deduce that

$$\Lambda_n(u) = R_n(\mathsf{t}(u)u) \geq \frac{\ell}{S_p^q \|a\|_r} \|\mathsf{t}(u)u\|^{m-q} \geq \frac{\ell}{S_p^q \|a\|_r} =: C > 0.$$

Suppose now that $\|\mathsf{t}(u)u\| > 1$. Then, arguing as in (3.20), we can see that

$$\|\mathsf{t}(u)u\|_p^p \geq \frac{\ell(q-m)}{\lambda(p-q)} \|\mathsf{t}(u)u\|^\ell,$$

which implies that

$$\mathsf{t}(u) \geq \left[\frac{(q-m)}{\lambda(p-q)} \frac{\|u\|^\ell}{\|u\|_p^p} \right]^{\frac{1}{p-\ell}}. \tag{3.24}$$

Moreover, by using (3.19) and (φ_3), we have that

$$0 \geq (\ell - q)\mathcal{J}'_{s,\Phi,V}(\mathsf{t}(u)u)(\mathsf{t}(u)u) + \lambda(p-q)\|\mathsf{t}(u)u\|_p^p. \tag{3.25}$$

Thence, by using (3.24) and (3.25), we infer that

$$\begin{aligned}
\Lambda_n(u) = R_n(\mathsf{t}(u)u) &= \frac{\mathcal{J}'_{s,\Phi,V}(\mathsf{t}(u)u)(\mathsf{t}(u)u) + \lambda\|\mathsf{t}(u)u\|_p^p}{\|\mathsf{t}(u)u\|_{q,a}^q} \geq \frac{\lambda\frac{(p-q)}{(q-\ell)}\|\mathsf{t}(u)u\|_p^p + \lambda\|\mathsf{t}(u)u\|_p^p}{\|\mathsf{t}(u)u\|_{q,a}^q} \\
&\geq \frac{\lambda \frac{p-\ell}{q-\ell} \left[\frac{(q-m)}{\lambda(p-q)} \frac{\|u\|^\ell}{\|u\|_p^p} \right]^{\frac{p-q}{p-\ell}} \|u\|_p^p}{\|u\|_{q,a}^q} \\
&= \frac{\frac{p-\ell}{q-\ell} \left[\frac{(q-m)}{(p-q)} \right]^{\frac{p-q}{p-\ell}} \lambda^{\frac{q-\ell}{p-\ell}} \|u\|^{\ell\frac{p-q}{p-\ell}} \|u\|_p^{p\frac{q-\ell}{p-\ell}}}{\|u\|_{q,a}^q} \\
&= C_{\ell,m,p,q} \lambda^{\frac{q-\ell}{p-\ell}} \frac{\|u\|^{\ell\frac{p-q}{p-\ell}} \|u\|_p^{p\frac{q-\ell}{p-\ell}}}{\|u\|_{q,a}^q}.
\end{aligned}$$

Since the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, we deduce from inequality (3.23) that

$$\Lambda_n(u) \geq \frac{C_{\ell,m,p,q}\lambda^{\frac{q-\ell}{p-\ell}}}{\|a\|_r} \|u\|_p^{\left[\ell\frac{p-q}{p-\ell} + p\frac{q-\ell}{p-\ell} - q\right]} = \frac{C_{\ell,m,p,q}\lambda^{\frac{q-\ell}{p-\ell}}}{\|a\|_r} =: C > 0,$$

where have used that $\ell\frac{p-q}{p-\ell} + p\frac{q-\ell}{p-\ell} - q = 0$. It remains to prove that the function Λ_n is unbounded from above. Since X is a reflexive space, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in X such that $\|w_k\| = 1$ for all $k \in \mathbb{N}$ and $w_k \rightharpoonup 0$. By item (i), we can assume without loss of generality that $\mathsf{t}(w_k) = 1$. Then, proceeding as in (3.22) and using the compact embedding $X \hookrightarrow L^q(\mathbb{R}^n)$, we conclude that

$$\Lambda_n(w_k) \geq \ell \frac{\|\mathsf{t}(w_k)w_k\|^m}{\|\mathsf{t}(w_k)w_k\|_{q,a}^q} \geq \frac{\ell}{\|a\|_\infty} \frac{1}{\|w_k\|_q^q} \to \infty, \quad \text{as } k \to \infty.$$

This proves (iii).

(iv) Since Λ_n is 0-homogeneous, we can take a sequence $(u_k)_{k\in\mathbb{N}}$ in X such that

$$\Lambda_n(u_k) \to \nu_n(\lambda) \quad \text{as} \quad k \to \infty, \quad \|u_k\| = 1 \quad \text{and} \quad \mathsf{t}(u_k) = 1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Due to reflexivity of X, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X. Then, by embedding compact $X \hookrightarrow L^q(\mathbb{R}^N)$ and the fact that $\|\cdot\|$ is weakly lower semicontinuous, we have that

$$\|u\| \leq \liminf_{k\to\infty} \|u_k\| \quad \text{and} \quad \lim_{k\to\infty} \|u_k\|_q^q = \|u\|_q^q.$$

We claim that $u \neq 0$. Otherwise, by using the last assertion and (3.22), we infer that

$$\Lambda_n(u_k) \geq \frac{\ell}{\|a\|_\infty} \frac{1}{\|u_k\|_q^q} \to \infty.$$

This contradiction implies that $u \neq 0$, which shows the claim above. Hence, since Λ_n is weakly lower semicontinuous, we conclude that

$$\nu_n(\lambda) = \lim_{k\to\infty} \Lambda_n(u_k) \geq \Lambda_n(u) \geq \nu_n(\lambda),$$

which proves (iv).

(v) First, by using the fact that u^* is a critical point of Λ_n, we have that

$$\begin{aligned} 0 = (\Lambda_n)'(u^*)w &= (R_n(\mathsf{t}(u^*)u^*))'w \\ &= R_n'(\mathsf{t}(u^*)u^*)\,[\mathsf{t}'(u^*)w]\,u^* + R_n'(\mathsf{t}(u^*)u^*)\mathsf{t}(u^*)w, \end{aligned} \tag{3.26}$$

for all $w \in X$. Since $R_n'(\mathsf{t}(u^*)u^*)u^* = 0$ by Proposition 3.2.3, it follows from (3.26) that $R_n'(v^*)w = R_n'(\mathsf{t}(u^*)u^*)w = 0$ for all $w \in X$. Now, we consider functions $\mathcal{F}, \mathcal{G} : X \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(|D_s u|)|D_s u|^2\,d\mu + \int_{\mathbb{R}^N} V(x)\varphi(|u|)|u|^2\,dx + \lambda\|u\|_p^p$$

and

$$\mathcal{G}(u) = \|u\|_{q,a}^q.$$

It is easy to check that $R_n(u) = \frac{\mathcal{F}(u)}{\mathcal{G}(u)}$. Hence, using that $\nu_n(\lambda) = \frac{\mathcal{F}(v^*)}{\mathcal{G}(v^*)}$, we deduce that

$$\begin{aligned} 0 = R_n'(v^*)w = \left[\frac{\mathcal{F}(v^*)}{\mathcal{G}(v^*)}\right]' w &= \frac{\mathcal{G}(v^*)\mathcal{F}'(v^*)w - \mathcal{F}(v^*)\mathcal{G}'(v^*)w}{\mathcal{G}(v^*)^2} \\ &= \frac{1}{\mathcal{G}(v^*)}\left[\mathcal{F}'(v^*)w - \nu_n(\lambda)\mathcal{G}'(v^*)w\right], \end{aligned}$$

holds for all $w \in X$. On the other hand, by standard calculation, we obtain that

$$\begin{aligned}\mathcal{F}'(v^*)w &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[2\varphi(|D_s v^*|) + \varphi'(|D_s v^*|)|D_s v^*|\right] D_s v^* D_s w \, d\mu \\ &\quad + \int_{\mathbb{R}^N} V(x) \left[2\varphi(|v^*|) + \varphi'(|v^*|)|v^*|\right] v^* w \, dx + \lambda p \int_{\mathbb{R}^N} |v^*|^{p-2} v^* w \, dx \\ &= 2\langle (-\Delta_\Phi)^s v^*, w\rangle + \langle (-\Delta_{\varphi'})^s v^*, w\rangle + \int_{\mathbb{R}^N} V(x) \left[2\varphi(|v^*|) + \varphi'(|v^*|)|v^*|\right] v^* w \, dx \\ &\quad + \lambda p \int_{\mathbb{R}^N} |v^*|^{p-2} v^* w \, dx.\end{aligned}$$

and

$$\mathcal{G}'(v^*)w = q \int_{\mathbb{R}^N} a(x)|v^*|^{q-2} v^* w \, dx.$$

Therefore, combining the tree last identity, we conclude the proof. □

In the sequel, we present similar results involving the functional R_e.

Proposition 3.2.8. *Assume that* (φ_1)-(φ_3)*,* (H_1) *and* (V_0)-(V_1) *hold. Then, the fibering map* $t \mapsto R_e(tu)$ *satisfies the following properties:*

(i) It holds that

$$\lim_{t\to 0^+} \frac{R_e(tu)}{t^{m-q}} > 0 \quad \textit{and} \quad \lim_{t\to 0^+} \frac{\frac{d}{dt}R_e(tu)}{t^{m-q-1}} < 0.$$

(ii) It holds that

$$\lim_{t\to \infty} \frac{R_e(tu)}{t^{p-q}} > 0 \quad \textit{and} \quad \lim_{t\to \infty} \frac{\frac{d}{dt}R_e(tu)}{t^{p-q-1}} > 0.$$

Proof. The proof is similar to the Proposition 3.2.2, for this reason we omit the details. □

In order to prove the uniqueness of the critical point of $t \mapsto R_e(tu)$ we need the following auxiliary result:

Lemma 3.2.9. *Assume that* (φ_1)-(φ_3) *holds. Then, the function*

$$t \mapsto \frac{\varphi(t)t^2 - q\Phi(t)}{t^p}, \quad t > 0,$$

is strictly increasing.

Proof. Let $0 < s < t < \infty$ be fixed. For each $\zeta_1, \zeta_2 \in (0,t)$ such that $\zeta_1 < \zeta_2$, we deduce from Lemma 3.2.1 that

$$\left[(2-q)\varphi(\zeta_1)\zeta_1 + \varphi'(\zeta_1)\zeta_1^2\right]\zeta_2^{p-1} < \left[(2-q)\varphi(\zeta_2)\zeta_2 + \varphi'(\zeta_2)\zeta_2^2\right]\zeta_1^{p-1}.$$

On the one hand, integrating the last inequality with respect to variable ζ_2 over interval $[0,t]$, we obtain that

$$\left[(2-q)\varphi(\zeta_1)\zeta_1+\varphi'(\zeta_1)\zeta_1^2\right]\frac{t^p}{p}<\left[-q\Phi(t)+\varphi(t)t^2\right]\zeta_1^{p-1}.$$

On the other hand, integrating with respect to variable ζ_1 over interval $[0,s]$, we have that

$$\left[-q\Phi(s)+\varphi(s)s^2\right]\frac{t^p}{p}<\left[-q\Phi(t)+\varphi(t)t^2\right]\frac{s^p}{p}.$$

This proves the desired result. □

Proposition 3.2.10. *Assume that* (φ_1)-(φ_3), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, for each* $u\in X\setminus\{0\}$, *there exists and unique* $\mathsf{s}(u)>0$ *satisfying*

$$\frac{d}{dt}R_e(tu)=0 \quad \textit{for} \quad t=\mathsf{s}(u). \tag{3.27}$$

In addition, the functional $\mathsf{s}:X\setminus\{0\}\to(0,\infty)$ *is of class* C^1 *and there exists* $c>0$ *such that* $\|\mathsf{s}(u)u\|>c$ *for all* $u\in X\setminus\{0\}$.

Proof. Firstly, thanks to the Proposition 3.2.8 and Bolzano's theorem, the equation (3.27) admits at least one root $\mathsf{s}(u)>0$. On the other hand, we recall that

$$R_e(tu)=\frac{t^{-q}\mathcal{J}_{s,\Phi,V}(tu)+\lambda\left(\frac{t^{p-q}}{p}\right)\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q}$$

Then, your derivative is given by

$$\begin{aligned}\frac{d}{dt}R_e(tu)&=\frac{-qt^{-q-1}\mathcal{J}_{s,\Phi,V}(tu)+t^{-q}\mathcal{J}'_{s,\Phi,V}(tu)u+\lambda(p-q)\left(\frac{t^{p-q-1}}{p}\right)\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q}\\&=\frac{-qt^{-q-1}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Phi(t|D_su|)\,d\mu+\int_{\mathbb{R}^N}V(x)\Phi(t|u|)\,dx\right)}{\frac{1}{q}\|u\|_{q,a}^q}\\&\quad+\frac{t^{-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(t|D_su|)|D_su|^2\,d\mu+\int_{\mathbb{R}^N}V(x)\varphi(t|u|)|u|^2\,dx\right)}{\frac{1}{q}\|u\|_{q,a}^q}\\&\quad+\lambda(p-q)\frac{t^{p-q-1}}{p}\frac{\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q}.\end{aligned}$$

Hence, the equation (3.27) is equivalent to

$$\begin{aligned}-\lambda\frac{(p-q)}{p}\|u\|_p^p&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi(|tD_su|)|tD_su|^2-q\Phi(|tD_su|)}{|tD_su|^p}|D_su|^p d\mu\\&\quad+\int_{\mathbb{R}^N}V(x)\frac{\varphi(|tu|)|tu|^2-q\Phi(|tu|)}{|tu|^p}|u|^p dx.\end{aligned}$$

But according to Lemma 3.2.9 and (V_0), the function $\mathcal{L}_u : (0,\infty) \to \mathbb{R}$ defined by

$$\mathcal{L}_u(t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(|tD_su|)|tD_su|^2 - q\Phi(|tD_su|)}{|tD_su|^p}|D_su|^p d\mu + \int_{\mathbb{R}^N} V(x)\frac{\varphi(|tu|)|tu|^2 - q\Phi(|tu|)}{|tu|^p}|u|^p dx$$

is strictly increasing. Therefore, $\mathsf{s}(u) > 0$ is unique for each $u \in X \setminus \{0\}$. The last assertion about $\mathsf{s}(u)$ follows using the same ideas discussed in the proof Proposition 3.2.5. $\square$

Remark 3.2.11. *Let $u \in X \setminus \{0\}$ and $t > 0$ be fixed. Then, taking into account* (3.7)*, we have the following assertions:*

(i) $R_e(tu) = \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}(tu) = 0$.

(ii) $R_e(tu) > \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}(tu) > 0$.

(iii) $R_e(tu) < \nu$ *if and only if* $\mathcal{I}_{\lambda,\nu}(tu) < 0$.

According to Proposition 3.2.8 and Proposition 3.2.10, we obtain that the function $Q_e \colon (0,\infty) \to \mathbb{R}$ defined by $Q_e(t) = R_e(tu)$ satisfies $Q_e'(t) = 0$ if and only if $t = \mathsf{s}(u)$. Moreover, $Q_e'(t) < 0$ if and only if $t \in (0, \mathsf{s}(u))$ and $Q_e'(t) > 0$ if and only if $t > \mathsf{s}(u)$, that is, $\mathsf{s}(u)$ is a global minimum point for Q_e. Therefore, we can also consider the auxiliary functional $\Lambda_e : X \setminus \{0\} \to \mathbb{R}$ given by

$$\Lambda_e(u) = \min_{t>0} Q_e(t) = R_e(\mathsf{s}(u)u). \tag{3.28}$$

As consequence, the following extreme parameter is well-defined

$$\nu_e(\lambda) = \inf_{u \in X\setminus\{0\}} \Lambda_e(u).$$

Under these notations, a version of Proposition 3.2.7 for Λ_e can be stated as follows:

Proposition 3.2.12. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Then, the functional Λ_e satisfies the following properties:*

(i) Λ_e *is* 0*-homogeneous, that is,* $\Lambda_e(tu) = \Lambda_e(u)$ *for all* $t > 0$, $u \in X \setminus \{0\}$.

(ii) Λ_e *is differentiable and weakly lower semicontinuous.*

(iii) *There exists* $C \colon = C(\ell, m, p, q, s, V_0, a, \lambda) > 0$ *such that* $\Lambda_e(u) \geq C > 0$ *for all* $u \in X \setminus \{0\}$.

(iv) *There exists* $u_* \in X \setminus \{0\}$ *such that* $\nu_n(\lambda) = \Lambda_e(u_*) > 0$. *In particular,* u_* *is a critical point for the functional* Λ_e.

(v) *The function* $v_* := \mathsf{s}(u_*)u_*$ *is a weakly solution of the nonlocal elliptic problem* $(\mathcal{P}_{\lambda,\nu})$ *for* $\nu = \Lambda_e(u_*)$.

Proof. The proof of items $(i) - (iv)$ follows the same argument discussed in the proof of Proposition 3.2.7.

Now, we will prove the item (v). Since $\mathsf{s}(u) > 0$ is a critical point of $Q_e(t) := R_e(tu)$, we have that $0 = R_e'(\mathsf{s}(u)u)u$. On the other side, using that u_* is a critical point for the functional Λ_e, we obtain that

$$0 = \Lambda_e'(u_*)w = [R_e(\mathsf{s}(u_*)u_*)]'w = \Lambda_e'(\mathsf{s}'(u_*)u_*)[\mathsf{s}(u_*)w]u_* + R_e'(\mathsf{s}(u_*)u_*)\mathsf{s}(u_*)w, \quad w \in X. \tag{3.29}$$

It follows from (3.29) that $R_e'(\mathsf{s}(u_*)u_*)w = 0$ holds for all $w \in X$. Hence, by using the function $v_* = \mathsf{s}(u_*)u_*$ and the fact that $\Lambda_e(\mathsf{s}(u_*)u_*) = \Lambda_e(u_*)$, we obtain the following identities

$$0 = R_e'(v_*)w = \frac{1}{\frac{1}{q}\|v_*\|_{q,a}^q}\mathcal{I}'_{\lambda,\Lambda_e(u_*)}(v_*)w, \quad \text{for all} \quad w \in X.$$

Here, also used an argument similar to the proof of the Proposition 3.2.7(v). The last assertion says that $\mathcal{I}'_{\lambda,\Lambda_e(u_*)}(v_*)w = 0$ for all $w \in X$. This ends the proof. □

Next, we point out others fundamental properties regarding the nonlinear Rayleigh quotients previously defined.

Lemma 3.2.13. *Assume that* (φ_1)-(φ_4), (H_1), (H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $R_n(tu) = \nu$ *for some* $t > 0$ *and* $u \in X \setminus \{0\}$. *Then,*

$$R_n'(tu)u = \frac{d}{dt}R_n(tu) = \frac{1}{t}\frac{\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu)}{||tu||_{q,a}^q}. \tag{3.30}$$

Consequently, the following assertions are verified:

(i) $R_n'(tu)u > 0$ *if and only if* $\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu) > 0$.

(ii) $R_n'(tu)u < 0$ *if and only if* $\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu) < 0$.

(iii) $R_n'(tu)u = 0$ *if and only if* $\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu) = 0$.

Proof. We consider the auxiliary functional $\mathcal{H} : X \to \mathbb{R}$ defined by $\mathcal{H}(u) = ||u||_{q,a}^q$. By definition (3.6), we have

$$\mathcal{H}(tu)R_n(tu) = \mathcal{J}'_{s,\Phi,V}(tu)(tu) + \lambda\|tu\|_p^p.$$

By differentiating the equation above with respect to t and multiplying the result by $t > 0$, we obtain that

$$tR_n(tu)\frac{d}{dt}\mathcal{H}(tu) + t\mathcal{H}(tu)\frac{d}{dt}R_n(tu) = \mathcal{J}''_{s,\Phi,V}(tu)(tu,tu) + \mathcal{J}'_{s,\Phi,V}(tu)(tu) + \lambda p\|tu\|_p^p.$$

It is not hard to see that $t\frac{d}{dt}\mathcal{H}(tu) = q\|tu\|_{q,a}^q$ for all $t > 0$. This condition, along with $R_n(tu) = \nu$, ensures that

$$\begin{aligned} t\mathcal{H}(tu)\frac{d}{dt}R_n(tu) &= \mathcal{J}''_{s,\Phi,V}(tu)(tu,tu) + \mathcal{J}'_{s,\Phi,V}(tu)(tu) + \lambda p\|tu\|_p^p - \nu q\|tu\|_{q,a}^q \\ &= \mathcal{J}''_{s,\Phi,V}(tu)(tu,tu) - \nu(q-1)\|tu\|_{q,a}^q + \lambda(p-1)\|tu\|_p^p. \end{aligned}$$

As a consequence, we have that

$$\begin{aligned} \frac{d}{dt}R_n(tu) &= \frac{\mathcal{J}''_{s,\Phi,V}(tu)(tu,tu) - \nu(q-1)\|tu\|_{q,a}^q + \lambda(p-1)\|tu\|_p^p}{t\mathcal{H}(tu)} \\ &= \frac{1}{t}\frac{\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu)}{\mathcal{H}(tu)}, \end{aligned}$$

where we have used the identity (3.11). This finishes the proof. □

Using the same ideas discussed in the previous Lemma, we deduce a similar result for R_e.

Lemma 3.2.14. *Assume that (φ_1)-(φ_4), (H_1), (H_2) and (V_0)-(V_1) hold. Suppose also that $R_e(tu) = \nu$ for some $t > 0$ and $u \in X \setminus \{0\}$. Then,*

$$R'_e(tu)u = \frac{d}{dt}R_e(tu) = \frac{1}{t}\frac{\mathcal{I}'_{\lambda,\nu}(tu)tu}{\frac{1}{q}||tu||_{q,a}^q}. \tag{3.31}$$

Consequently, the following assertions are verified:

(i) $R'_e(tu)u > 0$ *if and only if* $\mathcal{I}'_{\lambda,\nu}(tu)tu > 0$.

(ii) $R'_e(tu)u < 0$ *if and only if* $\mathcal{I}'_{\lambda,\nu}(tu)tu < 0$.

(iii) $R'_e(tu)u = 0$ *if and only if* $\mathcal{I}'_{\lambda,\nu}(tu)tu = 0$.

The next result establishes a relationship between the extremal values ν_n and ν_e.

Proposition 3.2.15. *Assume that (φ_1)-(φ_4), (H_1), (H_2) and (V_0)-(V_1) hold. Then, the following assertions are verified:*

(i) It holds that $\mathsf{t}(u) < \mathsf{s}(u)$ *for all* $u \in X \setminus \{0\}$.

(ii) It holds that $\Lambda_n(u) < \Lambda_e(u)$ *for all* $u \in X \setminus \{0\}$.

(iii) It holds that $0 < \nu_n(\lambda) < \nu_e(\lambda)$ *for all* $\lambda > 0$.

Proof. By (3.6), (3.7), and straightforward calculation, we deduce that

$$\begin{aligned} R_n(tu) - R_e(tu) &= \frac{t^{-q}\mathcal{J}'_{s,\Phi,V}(tu)(tu) + \lambda t^{p-q}\|u\|_p^p}{\|u\|_{q,a}^q} - \frac{t^{-q}\mathcal{J}_{s,\Phi,V}(tu) + \frac{\lambda}{p}t^{p-q}\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q} \\ &= \frac{1}{q}\frac{t^{-q}\mathcal{J}'_{s,\Phi,V}(tu)(tu) + \lambda t^{p-q}\|u\|_p^p - qt^{-q}\mathcal{J}_{s,\Phi,V}(tu) - \lambda\frac{q}{p}t^{p-q}\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q} \\ &= \frac{t}{q}\ \frac{t^{-q}\mathcal{J}'_{s,\Phi,V}(tu)u - qt^{-q-1}\mathcal{J}_{s,\Phi,V}(tu) + \lambda\frac{p-q}{p}t^{p-q-1}\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q}. \end{aligned}$$

From where it follows that

$$R_n(tu) - R_e(tu) = \frac{t}{q}\frac{d}{dt}R_e(tu).$$

for all $t > 0$ and $u \in X \setminus \{0\}$. This identity implies that $R_n(tu) < R_e(tu)$ for all $t \in (0, \mathsf{s}(u))$ and $R_n(tu) > R_e(tu)$ for all $t \in (\mathsf{s}(u), \infty)$. Moreover, $R_n(tu) = R_e(tu)$ if and only if $t = \mathsf{s}(u)$. As a consequence, we obtain that $\mathsf{t}(u) < \mathsf{s}(u)$ for all $u \in X \setminus \{0\}$. Indeed, if $\mathsf{t}(u) > \mathsf{s}(u)$ for some $u \in X \setminus \{0\}$, then

$$R_n(\mathsf{s}(u)u) \geq R_n(\mathsf{t}(u)u) > R_e(\mathsf{t}(u)u) \geq R_e(\mathsf{s}(u)u).$$

In the case which $\mathsf{t}(u) = \mathsf{s}(u)$, we obtain that $R_n(\mathsf{t}(u)u) = R_e(\mathsf{t}(u)u) > R_n(tu)$ for all $t < \mathsf{t}(u)$. In both cases we have a contradiction, which proves the item (i). Now, using the item (i), we obtain that

$$\Lambda_n(u) = R_n(\mathsf{t}(u)u) < R_n(\mathsf{s}(u)u) = R_e(\mathsf{s}(u)u) = \Lambda_e(u), \quad \text{for all} \quad u \in X \setminus \{0\}.$$

This proves the item (ii). Finally, in view to Proposition 3.2.12, there exists $u_* \in X \setminus \{0\}$ such that $\Lambda_e(u_*) = \nu_e(\lambda)$. Therefore, it follows from (ii) that

$$\nu_n(\lambda) \leq \Lambda_n(u_*) < \Lambda_e(u_*) = \nu_e(\lambda).$$

This finishes the proof. □

Our objective now is to describe the behavior of the fibering map $\gamma_u(t) = \mathcal{I}_{\lambda,\nu}(tu)$ according to the ν parameter. For this purpose, we consider the following open set:

$$\mathcal{U}_{\lambda,\nu} = \{u \in X \setminus \{0\} : \nu > \Lambda_n(u)\}.$$

This set plays an important role in the application of Nehari method. Precisely, we prove that for any function $u \in \mathcal{U}_{\lambda,\nu}$ it is projections onto the Nehari manifolds $\mathcal{N}^{\pm}_{\lambda,\nu}$ are unique, as stated in the following proposition.

Proposition 3.2.16. *Assume that* (φ_1)-(φ_4)*,* (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\nu > \nu_n(\lambda)$ *. Then, for each* $u \in \mathcal{U}_{\lambda,\nu}$*, the fibering map* $\gamma_u(t) = \mathcal{I}_{\lambda,\nu}(tu)$ *has exactly two critical points* $\mathsf{t}^-_\nu(u), \mathsf{t}^+_\nu(u) > 0$ *such that* $\mathsf{t}^-_\nu(u) < \mathsf{t}(u) < \mathsf{t}^+_\nu(u)$*. Furthermore, we have the following properties:*

(i) It is hold that $t^-_\nu(u)$ *is a local maximum point for the fibering map* γ_u *and* $\mathsf{t}^-_\nu(u)u \in \mathcal{N}^-_{\lambda,\nu}$*.*

(ii) It is hold that $\mathsf{t}^+_\nu(u)$ *is a local minimum point for the fibering map* γ_u *and* $\mathsf{t}^+_\nu(u)u \in \mathcal{N}^+_{\lambda,\nu}$*. Furthermore, if* $\nu > \Lambda_e(u)$*, then* $\mathsf{t}^+_\nu(u)$ *is a global minimum point for* γ_u*.*

(iii) The functionals $u \mapsto \mathsf{t}^+_\nu(u)$ *and* $u \mapsto \mathsf{t}^-_\nu(u)$ *belong to* $C^1(\mathcal{U}_{\lambda,\nu}, \mathbb{R})$*.*

Proof. Let $u \in X \setminus \{0\}$ be a fixed. Since $\nu > \Lambda_n(u)$, we have that

$$\nu > \Lambda_n(u) = \min_{t>0} Q_n(t) = R_n(\mathsf{t}(u)u). \tag{3.32}$$

By Proposition 3.2.2, the following limits

$$\lim_{t\to 0^+} Q_n(t) = \lim_{t\to\infty} Q_n(t) = \infty. \tag{3.33}$$

hold true. Taking into account (3.32) and (3.33), we deduce that the identity $Q_n(t) = R_n(tu) = \nu$ admits exactly two roots in the following form $0 < \mathsf{t}^-_\nu(u) < \mathsf{t}(u) < \mathsf{t}^+_\nu(u)$. By using Remark 3.2.6, the roots $\mathsf{t}^-_\nu(u)$ and $\mathsf{t}^+_\nu(u)$ are critical points for the fibering map γ_u. Moreover,

$$Q'_n(\mathsf{t}^-_\nu) < 0 \quad \text{and} \quad Q'_n(\mathsf{t}^+_\nu) > 0. \tag{3.34}$$

Hence, by using(3.34) and Lemma 3.2.13, we also have that

$$\mathcal{I}''(\mathsf{t}^-(u)u)(\mathsf{t}^-(u)u, \mathsf{t}^-(u)u) < 0 \quad \text{and} \quad \mathcal{I}''(\mathsf{t}^+(u)u)(\mathsf{t}^+(u)u, \mathsf{t}^+(u)u) > 0. \tag{3.35}$$

which proves that $\mathsf{t}^+_\nu(u)u \in \mathcal{N}^+_{\lambda,\nu}$ and $\mathsf{t}^-_\nu(u)u \in \mathcal{N}^-_{\lambda,\nu}$. The first inequality in (3.35) implies that $\gamma''_u(\mathsf{t}^-_\nu(u)) < 0$. As a consequence, $\mathsf{t}^-(u)$ is a local maximum point for γ_u. Similarly, the second inequality in (3.35) proves that $\mathsf{t}^+(u)$ is a local minimum point for γ_u. Now, assume that $\nu > \Lambda_e(u)$. Then,

$$Q_n(\mathsf{t}^+_\nu(u)) = \nu > \Lambda_e(u) = R_e(\mathsf{s}(u)u) = R_n(\mathsf{s}(u)u) = Q_n(\mathsf{s}(u)).$$

This fact and Proposition 3.2.15 imply that $\mathsf{s}(u) < \mathsf{t}_\nu^+(u)$ for all $u \in X \setminus \{0\}$ since Q_n it is strictly increasing in $(\mathsf{t}(u), \infty)$. As consequence, $R_e(\mathsf{t}_\nu^+(u)u) < R_n(\mathsf{t}_\nu^+(u)u) = \nu$. Moreover, it is not hard to see that $\nu \mapsto \mathcal{I}_{\lambda,\nu}(u)$ is a strictly decreasing function for each $\lambda > 0$ and $u \in X \setminus \{0\}$ fixed. Hence, these statements and Remark 3.2.11 give us

$$\mathcal{I}_{\lambda,\nu}(\mathsf{t}_\nu^+(u)u) < \mathcal{I}_{\lambda,R_e(\mathsf{t}_\nu^+(u)u)}(\mathsf{t}_\nu^+(u)u) = 0,$$

proving that $\mathsf{t}_\nu^+(u)$ is a global minimum point for γ_u for each $\nu > \Lambda_e(u)$. This ends the proof of items (i) and (ii).

Now, let us prove the (iii). We consider the function $\mathcal{F}^\pm : (0,\infty)\times(X\setminus\{0\}) \to \mathbb{R}$ defined by $\mathcal{F}^\pm(t,u) = \mathcal{I}'_{\lambda,\nu}(tu)tu$. Note that $\mathcal{F}^\pm(t,u) = 0$ if and only if $tu \in \mathcal{N}_{\lambda,\nu}$. Furthermore, we have that

$$\frac{\partial}{\partial t}\mathcal{F}^\pm(t,u) = \frac{1}{t}(\mathcal{I}''_{\lambda,\nu}(tu)(tu,tu) + \mathcal{I}'_{\lambda,\nu}(tu)tu) \neq 0,$$

for all $(t,u) \in (0,\infty)\times(X\setminus\{0\})$ such that $tu \in \mathcal{N}_{\lambda,\nu}^\pm$. Therefore, it follows from Implicit Function Theorem that functions $u \mapsto \mathsf{t}_\nu^+(u)$ and $u \mapsto \mathsf{t}_\nu^-(u)$ belong to $C^1(\mathcal{U}_{\lambda,\nu},\mathbb{R})$. This finishes the proof. □

Remark 3.2.17. *It is important to mention that Proposition 3.2.16 provides us that $\mathcal{N}_{\lambda,\nu} = \mathcal{N}_{\lambda,\nu}^+ \cup \mathcal{N}_{\lambda,\nu}^- \cup \mathcal{N}_{\lambda,\nu}^0$ where $\mathcal{N}_{\lambda,\nu}^+$ and $\mathcal{N}_{\lambda,\nu}^-$ are nonempty sets whenever $\nu > \nu_n(\lambda)$ and $\lambda > 0$. In the other words, the fibering map $t \mapsto \gamma_u(t) = \mathcal{I}_{\lambda,\nu}(tu)$ always intersects the Nehari set in two distinct points.*

In the next result it is established that the fibering map γ_u does not intercept the Nehari set in the case $\nu < \Lambda_n(u)$ and intercept it only in a point of $\mathcal{N}_{\nu,\lambda}^0$ taking into account the extremal case $\nu = \Lambda_n(u)$.

Proposition 3.2.18. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $u \in X \setminus \{0\}$. Then, the following assertions are verified:*

(i) Assume that $\nu < \Lambda_n(u)$. Then, the fibering map γ_u does not admit any critical point, that is, $tu \notin \mathcal{N}_{\lambda,\nu}$ for all $t > 0$.

(ii) Assume that $\nu = \Lambda_n(u)$. Then, the fibering map $\gamma_u(t) = \mathcal{I}_{\lambda,\nu}(tu)$ has a unique critical point $\mathsf{t}(u) > 0$ such that $\mathsf{t}(u)u \in \mathcal{N}_{\lambda,\nu}^0$.

Proof. (i) Since $\nu < \Lambda_n(u) = Q(\mathsf{t}(u)) = R_n(\mathsf{t}(u)u)$ and $\mathsf{t}(u)$ is a global minimum point for Q_n, it follows that $Q_n(t) = \nu$ does not admit any root, which is equivalent to say that

$\gamma_u'(t) = \mathcal{I}_{\lambda,\nu}'(tu)u \neq 0$ for all $t > 0$ and for each $u \in X \setminus \{0\}$ by Remark 3.2.6. Consequently, $tu \notin \mathcal{N}_{\lambda,\nu}$ for all $t > 0$, proving the item (i).

(ii) Assume that $\nu = \Lambda_n(u) = Q_n(\mathsf{t}(u))$. Then, $\gamma_u'(\mathsf{t}(u)) = \mathcal{I}_{\lambda,\nu}'(\mathsf{t}(u)u)u = 0$ by Remark 3.2.6. Moreover, for each $u \in X \setminus \{0\}$, we also have that $Q_n'(\mathsf{t}(u)) = 0$ since $\mathsf{t}(u) > 0$ is the unique critical point of Q_n. Hence, due to Lemma 3.2.13, we conclude that $\mathcal{I}_{\lambda,\nu}''(\mathsf{t}(u)u)(\mathsf{t}(u)u, \mathsf{t}(u)u) = 0$, that is, $\mathsf{t}(u)u \in \mathcal{N}_{\lambda,\nu}^0$. This finishes the proof. □

Lemma 3.2.19. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $\nu > \nu_n(\lambda)$ and $\lambda > 0$. Then, the sets $\mathcal{N}_{\lambda,\nu}^+$ and $\mathcal{N}_{\lambda,\nu}^-$ are C^1-submanifolds in X.*

Proof. Let $u \in \mathcal{U}_{\lambda,\nu}$ be a fixed function. Then, by Proposition 3.2.16, there exist $\mathsf{t}_\nu^-(u) < \mathsf{t}_\nu^+(u)$ such that $\mathsf{t}_\nu^-(u)u \in \mathcal{N}_{\lambda,\nu}^-$ and $\mathsf{t}_\nu^+(u)u \in \mathcal{N}_{\lambda,\nu}^+$. Consequently, the sets $\mathcal{N}_{\lambda,\nu}^-$ and $\mathcal{N}_{\lambda,\nu}^+$ are nonempty. We then define the C^1-functional $F_{\lambda,\nu} : X \to \mathbb{R}$ defined by $F_{\lambda,\nu}(u) = \mathcal{I}_{\lambda,\nu}'(u)u$. By assumptions (φ_1)-(φ_4), we have that $F_{\lambda,\nu}$ is of class C^1 and it is derivative is given by

$$F_{\lambda,\nu}'(u)v = \mathcal{I}_{\lambda,\nu}''(u)(u,v) + \mathcal{I}_{\lambda,\nu}'(u)v, \quad u, v \in X.$$

In particular, $F_{\lambda,\nu}'(u)u = \mathcal{I}_{\lambda,\nu}''(u)(u,u) \neq 0$ for all $u \in \mathcal{N}_{\lambda,\nu}^- \cup \mathcal{N}_{\lambda,\nu}^+$. Namely, zero is a regular value to $F_{\lambda,\nu}$ restrict to sets $\mathcal{N}_{\lambda,\nu}^\pm$. Therefore, the desired results are obtained from the Implicit Function Theorem (DRÁBEK; MILOTA, 2013). □

Using the uniqueness of projection on the Nehari set, we can prove that $\mathcal{N}_{\lambda,\nu}^0$ is nonempty for all $\nu \geq \nu_n(\lambda)$. Moreover, Nehari set $\mathcal{N}_{\lambda,\nu}$ is empty for all $\nu \in (-\infty, \nu_n(\lambda))$. Summarizing, we have the following result:

Lemma 3.2.20. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Then, the following assertions hold:*

(i) The Nehari set $\mathcal{N}_{\lambda,\nu}^0$ is nonempty for all $\nu \in [\nu_n(\lambda), \infty)$ and $\lambda > 0$.

(ii) Furthermore, $\mathcal{N}_{\lambda,\nu}$ is empty for all $\nu \in (-\infty, \nu_n(\lambda))$ and $\lambda > 0$.

Proof. (i) Initially, we consider $\nu = \nu_n(\lambda)$. In this case, by Proposition 3.2.7, there exists $u \in X \setminus \{0\}$ such that $\nu = \nu_n = \Lambda_n(u)$. Then, the Proposition 3.2.18 guarantee that there exists a unique $\mathsf{t}(u) > 0$ in such way that $\mathsf{t}(u)u \in \mathcal{N}_{\lambda,\nu}^0$. Now, we assume that $\nu > \nu_n(\lambda)$. Using once more the Proposition 3.2.7, we obtain that $\Lambda_n(u) = \nu_n(\lambda) < \nu$ for some $u \in X \setminus \{0\}$. On the other hand, since by Proposition 3.2.13 the function Λ_n is unbounded from above, there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset X \setminus \{0\}$ such that $\Lambda_n(v_k) \to \infty$ as $k \to \infty$. We can assume without any loss of generality that $\Lambda_n(v_k) > \nu$ for all $k \in \mathbb{N}$. In this case, we have that $u \neq \alpha v_k$ for all $\alpha > 0$ and $k \in \mathbb{N}$. In fact, we assume that $u \neq \alpha v_k$ for some α. Then, using that Λ_n is 0-homogeneous, we obtain that $\Lambda_n(u) = \Lambda_n(\alpha v_k) = \Lambda_n(v_k) > \nu$, which is a contradiction. Thereby, we obtain that $tu + (1-t)v_k \neq 0$ for each $t \in [0,1]$. Now, we consider the auxiliary function $f : [0,1] \to \mathbb{R}$ defined by $f(t) = \Lambda_n(tu + (1-t)v_k)$. Since Λ_n is a continuous function, it follows that f is also continuous. Moreover, observe that $f(0) = \Lambda_n(u) < \nu$ and $f(1) = \Lambda_n(v_k) > \nu$. Applying the Intermediate Value Theorem, we obtain $t_0 \in (0,1)$ such that $\Lambda_n(t_0 u + (1-t_0)v_k) = \nu$. We then consider the function $w_k = t_0 u + (1-t_0)v_k$. Therefore, by Proposition 3.2.18, there exists a unique $\mathsf{t}(w_k) > 0$ in such way $\mathsf{t}(w_k)w_k \in \mathcal{N}^0_{\lambda,\nu}$. This proves that $\mathcal{N}^0_{\lambda,\nu}$ is nonempty for all $\nu \in [\nu_n(\lambda), \infty)$.

(ii) Assume by contradiction that there exists $u \in \mathcal{N}_{\lambda,\nu}$ for some $\lambda > 0$ and $\nu \in (-\infty, \nu_n(\lambda))$. Thus, by Remark 3.2.6, we have that $R_n(u) = \nu$. On the other hand, we deduce from (3.10) that

$$\nu < \nu_n(\lambda) = \inf_{w \in X\setminus\{0\}} \inf_{t>0} R_n(tw) \leq \inf_{t>0} R_n(tu) \leq R_n(u),$$

which is a contradiction. Therefore, $\mathcal{N}_{\lambda,\nu} = \emptyset$ for each $\nu \in (-\infty, \nu_n(\lambda))$ and $\lambda > 0$. This finishes the proof. □

Proposition 3.2.21. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\nu > \nu_n(\lambda)$ *and* $\lambda > 0$. *Then, the following assertions are satisfied:*

(i) There exist $c_\nu > 0$ *such that* $\|u\| \geq c_\nu$ *for all* $u \in \mathcal{N}_{\lambda,\nu}$.

(ii) The Nehari manifold $\mathcal{N}_{\lambda,\nu}$ *and* $\mathcal{N}^0_{\lambda,\nu}$ *are closed sets.*

Proof. (i) Let $u \in \mathcal{N}_{\lambda,\nu}$ be fixed. By definition of Nehari manifold (3.2),

$$\nu||u||^q_{q,a} = \mathcal{J}'_{s,\Phi,V}(u)u + \lambda||u||^p_p.$$

Now, using the Lemma 2.3.13, continuous embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ for all $r \in [m, \ell_s^*)$ and the last identity, we have

$$\ell \min\{\|u\|^\ell, \|u\|^m\} \leq \ell \mathcal{J}_{s,\Phi,V}(u) \leq \nu||u||^q_{q,a} \leq \nu S_q^q \|a\|_\infty \|u\|^q.$$

As a consequence, we deduce that

$$\|u\| \geq c_\nu := \min\left\{ \left(\frac{\ell}{\nu S_q^q \|a\|_\infty}\right)^{\frac{1}{q-\ell}}, \left(\frac{\ell}{\nu S_q^q \|a\|_\infty}\right)^{\frac{1}{q-m}} \right\}. \tag{3.36}$$

This proves item (i).

(ii) By item (i), the Nehari manifold $\mathcal{N}_{\lambda,\nu}$ is away from zero. Hence, by using a standard argument, $\mathcal{N}_{\lambda,\nu}$ is a closed set. Now, let us prove that $\mathcal{N}^0_{\lambda,\nu}$ is also closed. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{N}^0_{\lambda,\nu}$ such that $u_k \to u$ in X for some $u \in X$. Since $\mathcal{N}_{\lambda,\nu}$ is closed, we have that $u \neq 0$ and $u \in \mathcal{N}_{\lambda,\nu}$. Finally, using the strong converge and the fact that $\mathcal{I}_{\lambda,\nu} \in C^2(X,\mathbb{R})$, we conclude that

$$\mathcal{I}''_{\lambda,\nu}(u)(u,u) = \mathcal{J}''_{s,\Phi,V}(u)(u,u) - \nu(q-1)\|u\||^q_{q,a} + \lambda(p-1)\|u\|^p_p = \lim_{k\to\infty} \mathcal{I}''_{\lambda,\nu}(u_k)(u_k,u_k) = 0.$$

Therefore, $u \in \mathcal{N}^0_{\lambda,\nu}$, which proves that $\mathcal{N}^0_{\lambda,\nu}$ is closed. This ends the proof. □

Remark 3.2.22. *There holds that any sequence $(u_k)_{k\in\mathbb{N}}$ in the Nehari set $\mathcal{N}_{\lambda,\nu}$ such that $u_k \rightharpoonup u$ for some $u \in X$ satisfies $u \neq 0$. Indeed, arguing by contradiction and assuming that $u_k \rightharpoonup 0$, it follows from the compact embeddings $X \hookrightarrow L^r(\mathbb{R}^N), r \in [2, 2^*_s)$, that $\|u_k\|^q_{q,a}, \|u_k\|^p_p \to 0$ as $k \to \infty$. In view of Lemma 2.3.13 and Proposition 3.2.21 (i), we have that*

$$\ell \min\{c^\ell_\nu, c^m_\nu\} \leq \ell \min\{\|u\|^\ell, \|u\|^m\}\ell\mathcal{J}_{s,\Phi,V}(u) \leq \mathcal{J}'_{s,\Phi,V}(u)u = \nu\|u_k\|^q_{q,a} - \lambda\|u_k\|^p_p \to 0.$$

This is a contradiction, proving that $u \neq 0$ as was mentioned before.

Lemma 3.2.23. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $\nu > \nu_n(\lambda)$ and $\lambda > 0$. Then, the following assertions are satisfied:*

(i) It holds that $\overline{\mathcal{N}^-_{\lambda,\nu}} \subseteq \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$.

(ii) It holds that $\overline{\mathcal{N}^+_{\lambda,\nu}} \subseteq \mathcal{N}^+_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$.

Proof. Firstly, we shall consider the Nehari set $\mathcal{N}^-_{\lambda,\nu}$. We consider a sequence $(u_k)_k \subset \mathcal{N}^-_{\lambda,\nu}$ such that $u_k \to u$ in X holds for some $u \in X$. Using the fact that $\mathcal{N}_{\lambda,\nu}$ is closed, we obtain that u is in $\mathcal{N}_{\lambda,\nu}$. Furthermore, using that $u_k \in \mathcal{N}^-_{\lambda,\nu}$, the following identity holds

$$\mathcal{I}''_{\lambda,\nu}(u_k)(u_k,u_k) = \mathcal{J}''_{s,\Phi,V}(u_k)(u_k,u_k) - \nu(q-1)\|u_k\||^q_{q,a} + \lambda(p-1)\|u_k\|^p_p < 0.$$

Since $\mathcal{I}_{\lambda,\nu}$ is of class C^2, taking the limit as $k \to \infty$ in the last inequality and taking, we conclude that $\mathcal{I}''_{\lambda,\nu}(u)(u,u) \leq 0$. Namely, $u \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$. The proof of item (ii) follows the same ideas. □

It is important to emphasize that the inclusion $\overline{\mathcal{N}^\pm_{\lambda,\mu}} \subseteq \mathcal{N}^\pm_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$ must not be strict in order to achieve our proposed objectives. For the reason, the hypothesis (φ_3) is necessary.

Proposition 3.2.24. *Assume that* (φ_1)-(φ_4)*,* (H_1)*,* (H_2) *and* (V_0)-(V_1) *hold. Suppose also* $\nu > \nu_n(\lambda)$ *and* $\lambda > 0$*. Then,* $\overline{\mathcal{N}^{\pm}_{\lambda,\nu}} = \mathcal{N}^{\pm}_{\lambda,\nu} \cup \mathcal{N}^{0}_{\lambda,\nu}$.

Proof. By Lemma 3.2.23, it remains to prove that $\mathcal{N}^{\pm}_{\lambda,\nu} \cup \mathcal{N}^{0}_{\lambda,\nu} \subseteq \overline{\mathcal{N}^{\pm}_{\lambda,\nu}}$. Since $\mathcal{N}^{\pm}_{\lambda,\nu} \subset \overline{\mathcal{N}^{\pm}_{\lambda,\nu}}$, it is sufficient to show that $\mathcal{N}^{0}_{\lambda,\nu} \subset \overline{\mathcal{N}^{\pm}_{\lambda,\nu}}$. Let u be any fixed function in $\mathcal{N}^{0}_{\lambda,\nu}$. Then, by Lemma 3.2.13, we have that $\mathrm{t}(u) = 1$. In this case, we obtain

$$R_n(u) = \nu \quad \text{and} \quad \frac{d}{dt} R_n(tu)\Big|_{t=1} = 0. \tag{3.37}$$

We recall that

$$\begin{aligned}\frac{d}{dt} R_n(tu) = {} & \frac{(2-q)t^{1-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(t|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(t|u|)|u|^2\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{t^{2-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(t|D_s u|)|D_s u|^3 d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(t|u|)|u|^3\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{\lambda(p-q)t^{p-q-1}\|u\|^p_p}{\|u\|^q_{q,a}}.\end{aligned}$$

Then, $\frac{d}{dt} R_n(tu)\Big|_{t=1} = 0$ if, and only if,

$$\begin{aligned}\lambda(p-q)\|u\|^p_p = {} & (q-2)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(|u|)|u|^2\, dx\right) \\ & - \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(|D_s u|)|D_s u|^3 d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(|u|)|u|^3\, dx\right).\end{aligned} \tag{3.38}$$

Moreover, by Proposition 3.2.3 we know that the map $u \mapsto R_n(tu)$ has a unique critical point which corresponds to a global minimum point. In particular, since φ is a C^2-function, we have $\frac{d^2}{dt^2} R_n(tu)\Big|_{t=1} > 0$ for each $u \in \mathcal{N}^{0}_{\lambda,\nu}$. We mention that

$$\begin{aligned}\frac{d^2}{dt^2} R_n(tu) = {} & \frac{(2-q)(1-q)t^{-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(t|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(t|u|)|u|^2\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{(2-q)t^{1-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(t|D_s u|)|D_s u|^3\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(t|u|)|u|^3\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{(2-q)t^{1-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(t|D_s u|)|D_s u|^3 d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(t|u|)|u|^3\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{t^{2-q}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi''(t|D_s u|)|D_s u|^4 d\mu + \int_{\mathbb{R}^N} V(x)\varphi''(t|u|)|u|^4\, dx\right)}{\|u\|^q_{q,a}} \\ & + \frac{\lambda(p-q)(p-q-1)t^{p-q-2}\|u\|^p_p}{\|u\|^q_{q,a}},\end{aligned}$$

Hence, we obtain that

$$\begin{aligned} 0 < {} & (q-2)(q-1)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(|u|)|u|^2\, dx\right) \\ & + 2(2-q)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(|D_s u|)|D_s u|^3\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(|u|)|u|^3\, dx\right) \\ & + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi''(|D_s u|)|D_s u|^4 d\mu + \int_{\mathbb{R}^N} V(x)\varphi''(|u|)|u|^4\, dx\right) \\ & + \lambda(p-q)(p-q-1)\|u\|^p_p.\end{aligned} \tag{3.39}$$

At this stage, we will prove that there exists $v \in (B_R(u) \cap \mathcal{N}_{\lambda,\nu}) \setminus \mathcal{N}^0_{\lambda,\nu}$, where $B_R(u)$ denotes the open ball centered at u with radius $R > 0$. In fact, we assume by contradiction that $B_R(u) \cap \mathcal{N}_{\lambda,\nu} \subset \mathcal{N}^0_{\lambda,\nu}$. We also consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\varepsilon_k \to 0$ as $k \to \infty$. Up to a subsequence, we will consider the following cases:

Case 1**:** Assume that $\Lambda_n(u + \varepsilon_k w) < \nu$ holds for any $w \in X \setminus \{0\}$ fixed. In this case, by Proposition 3.2.16, there exist $0 < \mathsf{t}^-_\nu(u + \varepsilon_k w) < \mathsf{t}(u + \varepsilon_k w) < \mathsf{t}^+_\nu(u + \varepsilon_k w) < \infty$ such that $\mathsf{t}^\pm_\nu(u + \varepsilon_k w)(u + \varepsilon_k w) \in \mathcal{N}^\pm_{\lambda,\nu}$. Furthermore, $v \mapsto \mathsf{t}^\pm_\nu(v)$ are continuous functions. Thus, $\mathsf{t}^\pm_\nu(u + \varepsilon_k w)(u + \varepsilon_k w) \in B_R(u) \cap \mathcal{N}_{\lambda,\nu} \subset \mathcal{N}^0_{\lambda,\nu}$ for k large enough. However, the Lemma 3.2.13 assures us that $\mathsf{t}^\pm_\nu(u + \varepsilon_k w)$ are critical points for fibering map $t \mapsto R_n(t(u + \varepsilon_k w))$ for k large enough, contradicting the uniqueness of $\mathsf{t}(u + \varepsilon_k w)$. This shows that the Case 1 is impossible.

Case 2**:** Assume $\Lambda_n(u+\varepsilon_k w) = \nu$ holds for any $w \in X\setminus\{0\}$ fixed. Then, $u+\varepsilon_k w \in \mathcal{N}^0_{\lambda,\nu}$, that is, $\mathsf{t}(u + \varepsilon_k w) = 1$. This fact gives us the following identity:

$$\begin{aligned}\lambda(p-q)\|u+\varepsilon_k w\|_p^p = {}& (q-2)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(|D_s(u+\varepsilon_k w)|)|D_s(u+\varepsilon_k w)|^2\, d\mu \\ & + (q-2)\int_{\mathbb{R}^N} V(x)\varphi(|u+\varepsilon_k w|)|u+\varepsilon_k w|^2\, dx \\ & - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi'(|D_s(u-\varepsilon_k w)|)|D_s(u+\varepsilon_k w)|^3 d\mu \\ & - \int_{\mathbb{R}^N} V(x)\varphi'(|u+\varepsilon_k w|)|u+\varepsilon_k w|^3\, dx.\end{aligned} \tag{3.40}$$

By subtracting the identity (3.38) from (3.40) and dividing by ε_k, we obtain that

$$\begin{aligned}&\lambda(p-q)\frac{\|u+\varepsilon_k w\|_p^p - \|u\|_p^p}{\varepsilon_k} \\ &\quad = (q-2)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(|D_s(u+\varepsilon_k w)|)|D_s(u+\varepsilon_k w)|^2 - \varphi(|D_s u|)|D_s u|^2}{\varepsilon_k}\, d\mu \\ &\quad + (q-2)\int_{\mathbb{R}^N} V(x)\frac{\varphi(|u+\varepsilon_k w|)|u+\varepsilon_k w|^2 - \varphi(|u|)|u|^2}{\varepsilon_k}\, dx \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi'(|D_s(u-\varepsilon_k w)|)|D_s(u+\varepsilon_k w)|^3 - \varphi'(|D_s u|)|D_s u|^3}{\varepsilon_k}\, d\mu \\ &\quad - \int_{\mathbb{R}^N} V(x)\frac{\varphi'(|u+\varepsilon_k w|)|u+\varepsilon_k w|^3 - \varphi'(|u|)|u|^3}{\varepsilon_k}\, dx.\end{aligned}$$

Since φ is a C^2-function, taking the limit as $k \to \infty$ in the above identity, we deduce that

$$\begin{aligned}
&\lambda(p-q)p \int_{\mathbb{R}^N} |u|^{p-2} u\, w\, dx \\
&\qquad = (q-2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[2\varphi(|D_s u|) D_s u D_s w + \varphi'(|D_s u|)|D_s u| D_s u D_s w\right] d\mu \\
&\qquad\quad + (q-2) \int_{\mathbb{R}^N} V(x) \left[2\varphi(|u|) u w + \varphi'(|u|)|u| u w\right] dx \\
&\qquad\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big[3\varphi'(|D_s u|)|D_s u| D_s u D_s w + \varphi''(|D_s u|)|D_s u|^2 D_s u D_s w\Big] d\mu \\
&\qquad\quad - \int_{\mathbb{R}^N} V(x) \Big[3\varphi'(|u|)|u| u w + \varphi''(|u|)|u|^2 u w\Big] dx.
\end{aligned} \tag{3.41}$$

In particular, taking $w = u$ in (3.41), we have that

$$\begin{aligned}
\lambda(p-q)p\|u\|_p^p &= (q-2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big[2\varphi(|D_s u|)|D_s u|^2 + \varphi'(|D_s u|)|D_s u|^3\Big] d\mu \\
&\quad + (q-2) \int_{\mathbb{R}^N} V(x) \Big[2\varphi(|u|)|u|^2 + \varphi'(|u|)|u|^3\Big] dx \\
&\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big[3\varphi'(|D_s u|)|D_s u|^3 + \varphi''(|D_s u|)|D_s u|^4\Big] d\mu \\
&\quad - \int_{\mathbb{R}^N} V(x) \Big[3\varphi'(|u|)|u|^3 + \varphi''(|u|)|u|^4\Big] dx.
\end{aligned}$$

This identity is equivalent to

$$\begin{aligned}
\lambda(p-q)p\|u\|_p^p &= 2(q-2) \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(|u|)|u|^2\, dx \right) \\
&\quad + (q-5) \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi'(|D_s u|)|D_s u|^3 d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(|u|)|u|^3\, dx \right) \\
&\quad - \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi''(|D_s u|)|D_s u|^4\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi''(|u|)|u|^4\, dx. \right).
\end{aligned} \tag{3.42}$$

In light of (3.38),(3.39), and (3.42), we arrive at the following linear system:

$$\begin{cases}
x - y - w = 0 \\
2x + (q-5)y - z - pw = 0 \\
(q-1)x + 2(2-q)y + z + (p-q-1)w > 0,
\end{cases} \tag{3.43}$$

where we denote

$$\begin{aligned}
x &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(|D_s u|)|D_s u|^2\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi(|u|)|u|^2\, dx, \\
y &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi'(|D_s u|)|D_s u|^3 d\mu + \int_{\mathbb{R}^N} V(x)\varphi'(|u|)|u|^3\, dx, \\
z &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi''(|D_s u|)|D_s u|^4\, d\mu + \int_{\mathbb{R}^N} V(x)\varphi''(|u|)|u|^4\, dx, \\
w &= \lambda(p-q)\|u\|_p^p.
\end{aligned}$$

Thanks to system (3.43), we define the number $\beta = (q-1)x+2(2-q)y+z+(p-q-1)w > 0$. Hence, we have the following equivalent system:

$$\begin{cases} x - y - w = 0 \\ 2x + (q-5)y - z - pw = 0 \\ (q-1)x + 2(2-q)y + z + (p-q-1)w = \beta. \end{cases} \tag{3.44}$$

However, by applying the Gauss method to system (3.44), we obtain that $\beta = 0$. This contradiction shows that Case 2 is impossible.

Caso 3**:** Suppose that $\Lambda_n(u+\varepsilon_k w) > \nu$ holds for any $w \in X\setminus\{0\}$ fixed. Define a sequence $\nu_k = \Lambda_n(u+\varepsilon_k w)$ for all $k \in \mathbb{N}$. It follows from Proposition 3.2.7 (i) that $\nu_k \to \nu$ as $k \to \infty$. Furthermore, $\nu_k = \Lambda_n(u+\varepsilon_k w)$ implies that $u + \varepsilon_k w \in \mathcal{N}^0_{\lambda,\nu_k}$, that is, $\mathsf{t}(u+\varepsilon_k w) = 1$. This fact implies that the function $u+\varepsilon_k w$ also satisfies the identity (3.40). Thereby, by proceeding as in the Case 2, we obtain the same system established in (3.43). This also leads to a contradiction.

Therefore, it follows from three cases that there exists $v \in B_R(u) \cap (\mathcal{N}_{\lambda,\nu} \setminus \mathcal{N}^0_{\lambda,\nu})$ for each $R > 0$. In this case, $v \in \mathcal{N}^+_{\lambda,\nu}$ or $v \in \mathcal{N}^-_{\lambda,\nu}$. This implies that $v \in \mathcal{U}_{\lambda,\nu} = \{v \in X \setminus \{0\} : \Lambda_n(v) < \nu\}$. Thence, the Proposition 3.2.16 guarantees that there exist $0 < \mathsf{t}^-_\nu(v) < \mathsf{t}(v) < \mathsf{t}^+_\nu(v) < \infty$ such that $\mathsf{t}^-_\nu(v)(v) \in \mathcal{N}^-_{\lambda,\nu}$ and $\mathsf{t}^+_\nu(v)(v) \in \mathcal{N}^+_{\lambda,\nu}$. Thus, considering the sequence $R_k = \frac{1}{k}$, we obtain a sequence $(v_k)_{k\in\mathbb{N}} \subset B_R(u)\cap(\mathcal{N}_{\lambda,\nu}\setminus\mathcal{N}^0_{\lambda,\nu})$ with $\mathsf{t}^-_\nu(v_k)(v_k) \in \mathcal{N}^-_{\lambda,\nu}$ and $\mathsf{t}^+_\nu(v_k)(v_k) \in \mathcal{N}^+_{\lambda,\nu}$. Since $R_k \to 0$ as $k \to \infty$, we infer that $v_k \to u$ in X. In addition, because of the continuity of $u \mapsto \mathsf{t}^\pm_\nu(u)$ and $u \mapsto \mathsf{t}(u)$, we conclude that

$$\mathsf{t}^\pm_\nu(v_k) \to \mathsf{t}^\pm_\nu(u) = \mathsf{t}(u) = 1, \quad \text{as} \quad k \to \infty.$$

This condition enables us to define the sequences $(f_k)_{k\in\mathbb{N}} \subset \mathcal{N}^-_{\lambda,\nu}$ and $(g_k)_{k\in\mathbb{N}} \subset \mathcal{N}^+_{\lambda,\nu}$ given by $f_k := \mathsf{t}^-_\nu(v_k)v_k$ and $g_k := \mathsf{t}^+_\nu(v_k)v_k$ such that $f_k \to u$ and $g_k \to u$ in X for each $u \in \mathcal{N}^0_{\lambda,\nu}$. Therefore, $\mathcal{N}^0_{\lambda,\nu} \subset \overline{\mathcal{N}^\pm_{\lambda,\nu}}$, which proves that $\overline{\mathcal{N}^\pm_{\lambda,\nu}} = \mathcal{N}^\pm_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$. □

Remark 3.2.25. *By using Proposition 3.2.24, we deduce that*

$$\mathcal{E}^-_{\lambda,\nu} = \inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) = \inf_{u\in\overline{\mathcal{N}^-_{\lambda,\nu}}} \mathcal{I}_{\lambda,\nu}(u) \le \inf_{u\in\mathcal{N}^-_{\lambda,\nu}\cup\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) \le \inf_{u\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) =: \mathcal{E}^0_{\lambda,\nu}$$

and

$$\mathcal{E}^+_{\lambda,\nu} = \inf_{u\in\mathcal{N}^+_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) = \inf_{u\in\overline{\mathcal{N}^+_{\lambda,\nu}}} \mathcal{I}_{\lambda,\nu}(u) \le \inf_{u\in\mathcal{N}^+_{\lambda,\nu}\cup\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) \le \inf_{u\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) =: \mathcal{E}^0_{\lambda,\nu}.$$

Therefore, the energy levels $\mathcal{E}^-_{\lambda,\nu}, \mathcal{E}^+_{\lambda,\nu}$ and $\mathcal{E}^0_{\lambda,\nu}$ can not be distinct with $\lambda > 0$ and $\nu \geq \nu_n(\lambda)$. However, in the sequel we prove that $\mathcal{E}^-_{\lambda,\nu}, \mathcal{E}^+_{\lambda,\nu}$ are attained in the sets $\mathcal{N}^-_{\lambda,\nu}, \mathcal{N}^+_{\lambda,\nu}$, respectively.

In the following, we prove that $\mathcal{I}_{\lambda,\nu}$ is coercive and bound below on the Nehari manifolds $\mathcal{N}^-_{\lambda,\nu}$ and $\mathcal{N}^+_{\lambda,\nu}$, which allow us to find finding minimizers for the problem given by (3.3) and (3.4), respectively. More precisely, we obtain the following result:

Proposition 3.2.26. *Assume that (φ_1)-(φ_4), (H_1), (H_2) and (V_0)-(V_1) hold. Then, $\mathcal{I}_{\lambda,\nu}$ is coercive in the Nehari manifold $\mathcal{N}_{\lambda,\nu}$.*

Proof. Let $u \in \mathcal{N}_{\lambda,\nu}$ be fixed. It follows from continuous embedding $X \hookrightarrow L^r(\Omega)$ for each $r \in (m, \ell_s^*)$, assumption (H_2) and Hölder inequality that $||u||^q_{q,a} \leq \|a\|_r \|u\|^q_p, \;\; u \in X,$ holds true where $r = (p/q)'$. As a consequence, we obtain by Lemma 2.3.13 that

$$\begin{aligned}\mathcal{I}_{\lambda,\nu}(u) &\geq \mathcal{J}_{s,\Phi,V}(u) - \frac{\nu}{q}\|a\|_r\|u\|^q_p + \frac{\lambda}{p}||u||^p_p \\ &\geq \min\{\|u\|^\ell, \|u\|^m\} - \left[\frac{\nu}{q}\|a\|_r\|u\|^{q-p}_p - \frac{\lambda}{p}\right]||u||^p_p.\end{aligned} \tag{3.45}$$

Now, assume that $\|u\| \to \infty$. On the one hand, if $||u||_p \to \infty$, then since $m < q < p < \ell^*_s$ by assumption (φ_1), the inequality (3.45) implies that $\mathcal{I}_{\lambda,\nu}$ is coercive in the Nehari manifold $\mathcal{N}_{\lambda,\nu}$. On the other hand, if $||u||_p \leq C$ holds true for some constant $C > 0$, then using (3.45) once more, we deduce that

$$\mathcal{I}_{\lambda,\nu} \geq \min\{\|u\|^\ell, \|u\|^m\} - \left[\frac{\nu}{q}\|a\|_r C - \frac{\lambda}{p}\right] C \to \infty \;\text{ as }\; \|u\| \to \infty.$$

Therefore, $\mathcal{I}_{\lambda,\nu}$ is coercive in the Nehari manifold $\mathcal{N}_{\lambda,\nu}$. □

The next result assures us that $\mathcal{N}^-_{\lambda,\nu}$ and $\mathcal{N}^+_{\lambda,\nu}$ are natural constraints for our main problem.

Lemma 3.2.27. *Assume that (φ_1)-(φ_4), (H_1), (H_2) and (V_0)-(V_1) hold. Let $u \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^+_{\lambda,\nu}$ be a local minimum for $\mathcal{I}_{\lambda,\nu}$ in $\mathcal{N}_{\lambda,\nu}$. Then, u is a critical point of $\mathcal{I}_{\lambda,\nu}$ in X, that is, $\mathcal{I}'_{\lambda,\nu}(u)v = 0$ for all $v \in X$.*

Proof. We consider the functional $\mathcal{F} : X \setminus \{0\} \to \mathbb{R}$ defined by $\mathcal{F}(u) = \mathcal{I}'_{\lambda,\nu}(v)v$. Thence, $\mathcal{N}_{\lambda,\nu} = \mathcal{F}^{-1}(0)$ and

$$\mathcal{F}'(v)w = \mathcal{I}''_{\lambda,\nu}(v)(v,w) + \mathcal{I}'_{\lambda,\nu}(v)w, \quad \text{for all } v \in X \setminus \{0\} \text{ and } w \in X.$$

Since $v \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^+_{\lambda,\nu}$ be a local minimum for $\mathcal{I}_{\lambda,\nu}$ in $\mathcal{N}_{\lambda,\nu}$, we can apply the Lagrange Multiplier Theorem (DRÁBEK; MILOTA, 2013) to obtain $\nu \in \mathbb{R}$ such that

$$\mathcal{I}'_{\lambda,\nu}(u)w = \nu\mathcal{F}'(u)w, \quad w \in X.$$

In particular, we have that $\mathcal{F}'(u)u = \mathcal{I}''_{\lambda,\nu}(u)(u,u) < 0$ or $\mathcal{F}'(u)u = \mathcal{I}''_{\lambda,\nu}(u)(u,u) > 0$. Therefore, these assertions together with the last identity yield that $\nu = 0$, which ends the proof. $\square$

At this stage, we apply standard arguments used in the Nehari method.

Proposition 3.2.28. *Assume that* (φ_1)*-*(φ_4)*,* (H_1)*-*(H_2) *and* (V_0)*-*(V_1) *hold. Let* $\lambda > 0$ *be fixed and* $\nu > \nu_n(\lambda)$*. Then, there exists a constant* $D_\nu > 0$ *such that* $\mathcal{I}_{\lambda,\nu}(u) \geq D_\nu$ *holds for all* $u \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$*. In particular,* $\mathcal{E}^-_{\lambda,\nu} \geq D_\nu$*, where* $\mathcal{E}^-_{\lambda,\nu}$ *was defined in* (3.3)*.*

Proof. Let $u \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ be a fixed function. As a consequence, by using (3.2), it is easy to verify that

$$\begin{aligned}\mathcal{I}_{\lambda,\nu}(u) = \mathcal{I}_{\lambda,\nu}(u) - \frac{1}{q}\mathcal{I}'_{\lambda,\nu}(u)u &\geq \left(1 - \frac{m}{q}\right)\mathcal{J}_{s,\Phi,V}(u) + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_p^p \\ &= \frac{q-m}{q}\mathcal{J}_{s,\Phi,V}(u) + \lambda\frac{q-p}{pq}\|u\|_p^p.\end{aligned}$$

Taking into account (3.12) and (φ_3), we have that

$$\lambda(q-p)\|u\|_p^p \geq (\ell - q)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(|D_s u|)|D_s u|^2\, d\mu \geq m(\ell - q)\mathcal{J}_{s,\Phi,V}(u).$$

Then, combining the last two inequality and Lemma 2.3.13, we deduce that

$$\begin{aligned}\mathcal{I}_{\lambda,\nu}(u) &\geq \left(\frac{q-m}{q}\right)\mathcal{J}_{s,\Phi,V}(u) + m\left(\frac{\ell - q}{pq}\right)\mathcal{J}_{s,\Phi,V}(u) \\ &\geq \frac{p(q-m) - m(q-\ell)}{pq}\min\{\|u\|^\ell, \|u\|^m\}.\end{aligned} \tag{3.46}$$

In view of Proposition 3.2.21 and (3.46), we infer that

$$\mathcal{I}_{\lambda,\nu}(u) \geq \frac{p(q-m) - m(q-\ell)}{pq}\min\{c_\nu^\ell, c_\nu^m\} =: D_\nu > 0.$$

In the last inequality we have used that $m(q-\ell) < p(q-m)$. Therefore, by using (3.3), we deduce that $\mathcal{E}^-_{\lambda,\nu} \geq D_\nu > 0$. This finishes the proof. $\square$

Proposition 3.2.29. *Assume that* (φ_1)*-*(φ_4)*,* (H_1)*-*(H_2) *and* (V_0)*-*(V_1) *hold. Suppose also that* $\lambda > 0$ *and* $\nu > \nu_n(\lambda)$*. Let* $(u_k)_{k\in\mathbb{N}} \subset \mathcal{N}^-_{\lambda,\nu}$ *be a minimizer sequence for* $\mathcal{I}_{\lambda,\nu}$ *in* $\mathcal{N}^-_{\lambda,\nu}$*. Then, there exists* $u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ *such that, up to a subsequence,* $u_k \to u_{\lambda,\nu}$ *in* X*. Consequently, there exists a constant* $D_\nu > 0$ *such that* $\mathcal{E}^-_{\lambda,\nu} = \mathcal{I}_\lambda(u_{\lambda,\nu}) \geq D_\nu$*.*

Proof. Firstly, by Proposition 3.2.26 (i) the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded. Then, up to a subsequence, we can assume that $u_k \rightharpoonup u_{\lambda,\nu}$ in X as $k \to \infty$. In virtue of Remark 3.2.22 we have also that $u_{\lambda,\nu} \neq 0$. Since $\overline{\mathcal{N}^-_{\lambda,\nu}} \subset \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ by Proposition 3.2.24, it is sufficient to prove the strong convergence $u_k \to u_{\lambda,\nu}$ in X as $k \to \infty$. To this end, suppose by contradiction that (u_k) does not converges to $u_{\lambda,\nu}$ in X. Hence, since Λ_n is weakly lower semicontinuous (by Proposition 3.2.7) and $u_k \in \mathcal{N}^-_{\lambda,\nu}$, we obtain that

$$\Lambda_n(u_{\lambda,\nu}) \leq \liminf_{k\to\infty} \Lambda_n(u_k) < \limsup_{k\to\infty} \Lambda_n(u_k) = \limsup_{k\to\infty} R_n(\mathsf{t}(u_k)u_k) \leq \liminf_{k\to\infty} R_n(u_k) = \nu.$$

Then, by Proposition 3.2.16, there exist unique $\mathsf{t}^-_\nu(u_{\lambda,\nu}) < \mathsf{t}(u_{\lambda,\nu}) < \mathsf{t}^+_\nu(u_{\lambda,\nu})$ such that $\mathsf{t}^-_\nu(u_{\lambda,\nu})u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu}$ and $\mathsf{t}^+_\nu(u_{\lambda,\nu})u_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu}$. Now, using that $u \mapsto R_n(tu)$ is lower semicontinuous for each $t > 0$, we infer that

$$R_n(tu_{\lambda,\nu}) \leq \liminf_{k\to\infty} R_n(tu_k) < \limsup_{k\to\infty} R_n(tu_k), \tag{3.47}$$

which implies that $R_n(tu_{\lambda,\nu}) < R_n(tu_k)$ for each k large enough. This assertion shows that $\mathsf{t}^-_\nu(u_{\lambda,\nu}) < \mathsf{t}^-_\nu(u_k) = 1$ for each $k \gg 1$. Indeed, arguing by contradiction, assume that $\mathsf{t}^-_\nu(u_{\lambda,\nu}) \geq 1$ for some $k \gg 1$. Then, since $t \mapsto R_n(tu_{\lambda,\nu})$ is strictly decreasing in $(0, \mathsf{t}(v_{\lambda,\nu}))$, we have by above assertion that $R_n(\mathsf{t}^-_\nu(u_{\lambda,\nu})u_{\lambda,\nu}) \leq R_n(u_{\lambda,\nu}) < R_n(u_k) = \nu$, which is a contradiction.

On the other hand, using that the functional $u \mapsto \mathcal{I}_{\lambda,\nu}(tu)$ is also weakly lower semicontinuous for each $t > 0$, we obtain the following estimate

$$\mathcal{I}_{\lambda,\nu}(tu_{\lambda,\nu}) \leq \liminf_{k\to\infty} \mathcal{I}_{\lambda,\nu}(tu_k), \quad \text{for all} \quad t > 0 \tag{3.48}$$

Since $(u_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{N}^-_{\lambda,\nu}$, we have that $t \mapsto \mathcal{I}_{\lambda,\nu}(tu_k)$ is a strictly increasing function in $(0,1)$. Thence, by using (3.48), we conclude that

$$\mathcal{I}_{\lambda,\nu}(\mathsf{t}^-_\nu(u_{\lambda,\nu})u_{\lambda,\nu}) < \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) \leq \liminf_{k\to\infty} \mathcal{I}_{\lambda,\nu}(u_k) = \mathcal{E}_{\lambda,\nu}.$$

This is a contradiction, which proves $u_k \to u_{\lambda,\nu}$ in X as $k \to \infty$. Hence, using the strong convergence, we conclude that $\mathcal{E}^-_{\lambda,\nu} = \mathcal{I}_\lambda(u_{\lambda,\nu})$. Moreover, it follows from Proposition 3.2.28 that $\mathcal{E}^-_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) \geq D_\nu > 0$. Finally, by Proposition 3.2.24, we obtain also that $u_\lambda \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$. □

Proposition 3.2.30. *Assume that* (φ_1)-(φ_4)*,* (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\lambda > 0$ *and* $\nu > \nu_n(\lambda)$*. Let* $(v_k) \subset \mathcal{N}^+_{\lambda,\nu}$ *be a minimizer sequence for* $\mathcal{I}_{\lambda,\nu}$ *in* $\mathcal{N}^+_{\lambda,\nu}$*. Then, there exists* $v_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ *such that, up to a subsequence,* $v_k \to v_{\lambda,\nu}$ *in* X*. Consequently,* $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_\lambda(v_{\lambda,\nu})$ *where* $\mathcal{E}^+_{\lambda,\nu}$ *was given in* (3.4).

Proof. In virtue of Proposition 3.2.26 (ii) the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded. Hence, up to a subsequence, it follows that $v_k \rightharpoonup v_{\lambda,\nu} \neq 0$ in X as $k \to \infty$. Now let us prove the strong convergence $v_k \to v_{\lambda,\nu}$ in X. Assume by contradiction that $(v_k)_{k\in\mathbb{N}}$ does not converge to $v_{\lambda,\nu}$ in X. On the one hand, by using Proposition 3.2.7 and that $u_k \in \mathcal{N}^+_{\lambda,\nu}$, we obtain that

$$\Lambda_n(v_{\lambda,\nu}) \leq \liminf_{k\to\infty} \Lambda_n(v_k) < \limsup_{k\to\infty} \Lambda_n(v_k) = \limsup_{k\to\infty} R_n(\mathsf{t}(v_k)v_k) \leq \nu.$$

Then, by Proposition 3.2.16, there exist unique $\mathsf{t}^-_\nu(v_{\lambda,\nu}) < \mathsf{t}(v_{\lambda,\nu}) < \mathsf{t}^+_\nu(v_{\lambda,\nu})$ such that $\mathsf{t}^-_\nu(u_{\lambda,\nu})u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu}$ and $\mathsf{t}^+_\nu(u_{\lambda,\nu})u_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu}$.

On the other hand, using that the functionals $u \mapsto R_n'(u)u$ is weakly upper semicontinuous, we deduce from Lemma 3.2.13 that

$$\frac{d}{dt}R_n(tv_{\lambda,\nu})\Big|_{t=1} = R_n'(v_{\lambda,\nu})v_{\lambda,\nu} \geq \limsup_{k\to\infty} R_n'(v_k)v_k > \liminf_{k\to\infty} R_n'(v_k)v_k \geq 0, \tag{3.49}$$

Consequently, by using (3.49), we also deduce that $\mathsf{t}(v_{\lambda,\nu}) < 1$. Now, using the fact that R_n is weakly lower semicontinuous and $v_k \in \mathcal{N}^+_{\lambda,\nu}$, we infer that

$$R_n(v_{\lambda,\nu}) \leq \liminf_{k\to\infty} R_n(v_k) < \limsup_{k\to\infty} R_n(v_k) = \nu. \tag{3.50}$$

Since $t \mapsto R_n(tv_{\lambda,\nu})$ is strictly increasing in $(\mathsf{t}(v_{\lambda,\nu}), \infty)$ and $R_n(\mathsf{t}^+_\nu(v_{\lambda,\nu})v_{\lambda,\nu}) = \nu$, then the inequality (3.50) implies $\mathsf{t}^-_\nu(v_{\lambda,\nu}) < 1 < \mathsf{t}^+_\nu(u_{\lambda,\nu})$. Hence, using that fibering map $t \mapsto \mathcal{I}_{\lambda,\nu}(tv_{\lambda,\nu})$ is a strictly decreasing function in $(\mathsf{t}^-_\nu(v_{\lambda,\nu}), \mathsf{t}^+_\nu(v_{\lambda,\nu}))$, we conclude that

$$\mathcal{I}_{\lambda,\nu}(\mathsf{t}^+_\nu(v_{\lambda,\nu})v_{\lambda,\nu}) < \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) \leq \liminf_{k\to\infty} \mathcal{I}_{\lambda,\nu}(v_k) = \mathcal{E}^+_{\lambda,\nu}.$$

This is a contradiction, which proves $v_k \to v_{\lambda,\nu}$ in X as $k \to \infty$. Therefore, using the strong convergence, we conclude that $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_\lambda(v_{\lambda,\nu})$. It follows from Proposition 3.2.28 that $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) \geq D_\nu > 0$. Finally, according to Proposition 3.2.24 we obtain also that $v_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$. This finishes the proof. □

From this point onward, our objective is to ensure that the functional $\mathcal{I}_{\lambda,\nu}$ has at least two critical points for each $\lambda > 0$ and $\nu \in (\nu_n(\lambda), \infty)$. However, it is known that $\mathcal{N}^0_{\lambda,\nu}$ is nonempty for each $\lambda > 0$ and $\nu \in (\nu_n(\lambda), +\infty)$. As a result, the minimizer sequences on the Nehari manifolds $\mathcal{N}^-_{\lambda,\nu}$ and $\mathcal{N}^+_{\lambda,\nu}$ may strongly converge to a function in $\mathcal{N}^0_{\lambda,\nu}$, where the Lagrange Multipliers Theorem does not apply. In order to overcome this phenomenon, we explore some fine properties which are crucial in proving that any minimizers for the functional $\mathcal{I}_{\lambda,\nu}$ restricted to the Nehari manifold $\mathcal{N}^-_{\lambda,\nu}$ or $\mathcal{N}^+_{\lambda,\nu}$ does not belong to $\mathcal{N}^0_{\lambda,\nu}$. As a first auxiliary result, we point out the following statement:

Lemma 3.2.31. *Assume that* (φ_1)-(φ_4)*,* (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Let* $\lambda > 0$ *and* $\nu \in (\nu_n(\lambda), \infty)$ *be fixed. Then, the following assertions hold:*

(i) There holds that

$$\mathcal{E}^-_{\lambda,\nu} := \inf_{w\in\mathcal{U}_{\lambda,\nu}} \left[\sup_{t\in[0,\mathsf{t}(w)]} \mathcal{I}_{\lambda,\nu}(tw)\right].$$

(ii) There holds that

$$\mathcal{E}^+_{\lambda,\nu} := \inf_{w\in\mathcal{U}_{\lambda,\nu}} \left[\inf_{t\in[\mathsf{t}(w),\infty)} \mathcal{I}_{\lambda,\nu}(tw)\right].$$

Proof. Let $w \in \mathcal{U}_{\lambda,\nu}$ be a fixed function. According to Proposition 3.2.16 there exist unique $0 < \mathsf{t}^-_\nu(w) < \mathsf{t}(w) < \mathsf{t}^+_\nu(w) < \infty$ such that $\mathsf{t}^-_\nu(w)w \in \mathcal{N}^-_{\lambda,\nu}$ and $\mathsf{t}^+_\nu(w)w \in \mathcal{N}^+_{\lambda,\nu}$. In particular,

$$\sup_{t\in[0,\mathsf{t}(w)]} \mathcal{I}_{\lambda,\nu}(tw) = \mathcal{I}_{\lambda,\nu}(t^{n,-}_\nu(w)w) \geq \inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u), w \in \mathcal{U}_{\lambda,\nu}.$$

Consequently, we infer that

$$\inf_{w\in\mathcal{U}_{\lambda,\nu}} \left[\sup_{t\in[0,\mathsf{t}(w)]} \mathcal{I}_{\lambda,\nu}(tw)\right] \geq \inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u).$$

On the other side, for all $\varepsilon > 0$, there exists $w_\epsilon \in \mathcal{N}^-_{\lambda,\nu}$ such that

$$\inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) \leq \mathcal{I}_{\lambda,\nu}(w_\epsilon) \leq \inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) + \epsilon.$$

Moreover, we recall that $\mathcal{I}_{\lambda,\nu}(w_\epsilon) = \sup_{t\in[0,\mathsf{t}(w_\epsilon)]} \mathcal{I}_{\lambda,\nu}(tw_\epsilon)$. Therefore,

$$\inf_{w\in\mathcal{U}_{\lambda,\nu}} \left[\sup_{t\in[0,\mathsf{t}(w)]} \mathcal{I}_{\lambda,\nu}(tw)\right] \leq \sup_{t\in[0,\mathsf{t}(w_\epsilon)]} \mathcal{I}_{\lambda,\nu}(tw_\epsilon) \leq \inf_{u\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude the proof for the item (i). The proof of item (ii) follows the same ideas discussed above by using $\mathsf{t}^+_\nu(w)$ instead of $\mathsf{t}^-_\nu(w)$. □

It is important to emphasize that the function Λ_n defined in (3.21) depend on parameter λ. In this case, we can consider the function $\lambda \mapsto \Lambda_{n,\lambda}(u)$ given by

$$\Lambda_{n,\lambda}(u) = R_{n,\lambda}(\mathsf{t}_\lambda(u)u)$$

for each $u \in X \setminus \{0\}$ fixed, where $\mathsf{t}_\lambda(u)$ is obtained in Proposition 3.2.3.

Remark 3.2.32. *A fundamental property for obtaining the results is the continuity and monotonicity of the function* $\lambda \mapsto \mathsf{t}_\lambda(u)$*. In the paper by Silva et al. (2024a), this property is easily observed due to the availability of an explicit expression for this function. However, in the present work, we cannot derive such an expression explicitly. Therefore, a more careful analysis is required. For this reason, we introduce the following proposition:*

Proposition 3.2.33. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Let $u \in X \setminus \{0\}$ be fixed. Then, the following assertions hold:*

(i) $\lambda \mapsto \Lambda_{n,\lambda}(u)$ *is an increasing function.*

(ii) $\lambda \mapsto \mathsf{t}_\lambda(u)$ *is a decreasing continuous function.*

(iii) *For each $\nu \in (\Lambda_{n,\lambda}(u), \infty)$ fixed, the functions $\lambda \mapsto \mathsf{t}^{\pm}_{\lambda,\nu}(u)$ are of class C^1. Furthermore, $\lambda \mapsto \mathsf{t}^{-}_{\lambda,\nu}(u)$ is increasing while $\lambda \mapsto \mathsf{t}^{+}_{\lambda,\nu}(u)$ is decreasing.*

(iv) *For each $\lambda > 0$ and $u \in \mathcal{U}_{\lambda,\nu}$ fixed, the functions $\nu \mapsto \mathsf{t}^{\pm}_{\lambda,\nu}(u)$ are class C^1. Moreover, $\nu \mapsto \mathsf{t}^{-}_{\lambda,\nu}(u)$ is decreasing while $\nu \mapsto \mathsf{t}^{+}_{\lambda,\nu}(u)$ is increasing.*

Proof. (i) Let $0 < \lambda < \lambda_0$ be fixed. By direct calculation, we obtain that

$$R_{n,\lambda}(tu) = R_{n,\lambda_0}(tu) + (\lambda - \lambda_0)\frac{\|tu\|_p^p}{\|tu\|_{q,a}^q} < R_{n,\lambda_0}(tu).$$

for all $t > 0$. In particular, $\Lambda_{n,\lambda}(u) < \Lambda_{n,\lambda_0}(u)$, which proves that $\lambda \mapsto \Lambda_{n,\lambda}(u)$ is an increasing function.

(ii) Firstly, we will prove the continuity of $\lambda \mapsto t_\lambda(u)$. Let $\lambda_0 > 0$ and $(\lambda_k)_{k\in\mathbb{N}} \subset (0, \infty)$ be a sequence such that $\lambda_k \to \lambda_0$. We show that $(\mathsf{t}_{\lambda_k}(u))_{k\in\mathbb{N}}$ is bounded for each $u \in X \setminus \{0\}$ fixed. Indeed, assume by contradiction that, up to subsequence, $\mathsf{t}_{\lambda_k}(u) \to \infty$. Since

$$\lim_{t\to\infty} \left(\mathcal{K}_u(t) + \lambda(p-q)\|u\|_p^p\right) = \lambda(p-q)\|u\|_p^p > 0,$$

there exists $t' > 0$ such that $\mathcal{K}_u(t') + \lambda(p-q)\|u\|_p^p > 0$. Moreover, $\mathsf{t}_{\lambda_k}(u) > t'$ for k enough larger. On the other hand, by Lemma 3.2.1, we know that $t \mapsto \mathcal{K}_u(t) + \lambda(p-q)\|u\|_p^p$ is a strictly increasing continuous function. Thus, for k enough larger, it holds that

$$0 < \mathcal{K}_u(t') + \lambda(p-q)\|u\|_p^p \le \mathcal{K}_u(t_{\lambda_k}(u)) + \lambda(p-q)\|u\|_p^p = 0,$$

which is a contradiction. Hence, $(\mathsf{t}_\lambda(u))_{k\in\mathbb{N}}$ is bounded. By this assertion, there exists $t_0 \ge 0$ such that, up to a subsequence, $\mathsf{t}_{\lambda_k}(u) \to t_0$. But by proceeding as in (3.20), we obtain that

$$\|\mathsf{t}_{\lambda_k}(u)u\|^p \ge \frac{\ell(q-m)}{\lambda_k(p-q)S_p^p}\min\{\|\mathsf{t}_{\lambda_k}(u)u\|^\ell, \|\mathsf{t}_{\lambda_k}(u)u\|^m\}.$$

This implies that $t_0 > 0$. Then, $\mathsf{t}_{\lambda_k}(u)u \to t_0 u$ in X. Since R_n is of class C^1, we deduce from Proposition 3.2.3 that

$$\frac{d}{dt}R_n(tu)\Big|_{t=t_0} = R_n'(t_0u)u = \lim_{k\to\infty} R_n'(\mathsf{t}_{\lambda_k}(u)u)u = 0,$$

which implies that $t_0 = \mathsf{t}(u)$. This proves that $\lambda \mapsto \mathsf{t}_\lambda(u)$ is continuous.

The monotonicity it follows directly of identity (3.17). Indeed, we known that

$$-\lambda(p-q)\|u\|_p^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(2-q)\varphi(\mathsf{t}_\lambda(u)|D_s u|) + \varphi'(\mathsf{t}_\lambda(u)|D_s u|)|\mathsf{t}_\lambda(u) D_s u|}{|\mathsf{t}_\lambda(u) D_s u|^{p-2}} |D_s u|^p d\mu + \int_{\mathbb{R}^N} V(x) \frac{(2-q)\varphi(\mathsf{t}_\lambda(u)|u|) + \varphi'(\mathsf{t}_\lambda(u)|u|)|\mathsf{t}_\lambda(u) u|}{|\mathsf{t}_\lambda(u) u|^{p-2}} |u|^p dx.$$

Hence, by Lemma 3.2.1, if $\lambda < \lambda_0$, then $\mathsf{t}_{\lambda_0}(u) < \mathsf{t}_\lambda(u)$.

(iii) We consider the function $\mathcal{F}_u^\pm \colon (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by $\mathcal{F}_u^\pm(\lambda, t) = \mathcal{I}'_{\lambda,\nu}(tu)(tu)$. Then, since $\mathsf{t}_{\lambda,\nu}(u)u \in \mathcal{N}_{\lambda,\nu}$ for each $\lambda > 0$ and $\nu > \Lambda_{n,\lambda}(u)$ fixed, we have that $\mathcal{F}_u^\pm(\lambda, \mathsf{t}^\pm_{\lambda,\nu}(u)) = 0$. Moreover, by Lemma 3.2.13, we obtain that

$$\frac{\partial \mathcal{F}_u^\pm}{\partial t}(\lambda, \mathsf{t}^\pm_{\lambda,\nu}(u)) = \mathcal{I}''_{\lambda,\nu}(\mathsf{t}^\pm_{\lambda,\nu}(u)u)(\mathsf{t}^\pm_{\lambda,\nu}(u)u, u) \neq 0.$$

Hence, it follows from Implicit Function Theorem that $(0,\infty) \ni \lambda \mapsto \mathsf{t}^\pm_{\lambda,\nu}(u)$ is of class C^1 and

$$\frac{\partial}{\partial \lambda}\mathsf{t}^\pm_{\lambda,\nu}(u) = -\frac{\frac{\partial \mathcal{F}_u^\pm}{\partial \lambda}(\lambda, \mathsf{t}^\pm_{\lambda,\nu}(u))}{\frac{\partial \mathcal{F}_u^\pm}{\partial t}(\lambda, \mathsf{t}^\pm_{\lambda,\nu}(u))} = -\frac{\mathsf{t}^\pm_{\lambda,\nu}(u)\|\mathsf{t}^\pm_{\lambda,\nu}(u)u\|_p^p}{\mathcal{I}''_{\lambda,\nu}(\mathsf{t}^\pm_{\lambda,\nu}(u)u)(\mathsf{t}^\pm_{\lambda,\nu}(u)u, \mathsf{t}^\pm_{\lambda,\nu}(u)u)}, \quad \text{for all} \quad \lambda > 0.$$

Therefore, $\frac{\partial}{\partial \lambda}\mathsf{t}^-_{\lambda,\nu}(u) > 0$ and $\frac{\partial}{\partial \lambda}\mathsf{t}^+_{\lambda,\nu}(u) < 0$ for all $\lambda > 0$. This proves the item (iii).

(iv) In the same way we can prove the functions $\nu \mapsto \mathsf{t}^\pm_{\lambda,\nu}(u)$ are class C^1 and

$$\frac{\partial}{\partial \nu}\mathsf{t}^\pm_{\lambda,\nu}(u) = \frac{\mathsf{t}^\pm_{\lambda,\nu}(u)\|\mathsf{t}^\pm_{\lambda,\nu}(u)u\|_{q,a}^q}{\mathcal{I}''_{\lambda,\nu}(\mathsf{t}^\pm_{\lambda,\nu}(u)u)(\mathsf{t}^\pm_{\lambda,\nu}(u)u, \mathsf{t}^\pm_{\lambda,\nu}(u)u)}, \quad \text{for all} \quad \nu > \Lambda_{n,\lambda}(u),$$

which implies that $\frac{\partial}{\partial \nu}\mathsf{t}^-_{\lambda,\nu}(u) < 0$ and $\frac{\partial}{\partial \nu}\mathsf{t}^+_{\lambda,\nu}(u) > 0$ for all $\nu > \Lambda_{n,\lambda}(u)$. This finishes the proof. □

From the Proposition 3.2.33, we also obtain the following result:

Proposition 3.2.34. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, the functions* $\mathcal{E}^\pm_{\lambda,\nu} : [0,\infty) \times (\nu_n(\lambda), \infty) \to \mathbb{R}$ *defined in* (3.3) *satisfies the following properties:*

(ii) The functions $\lambda \mapsto \mathcal{E}^\pm_{\lambda,\nu}$ *are non-decreasing, that is, it holds that* $\mathcal{E}^\pm_{\lambda_1,\nu} \le \mathcal{E}^\pm_{\lambda_2,\nu}$ *for each* $\lambda_1 \in (0, \lambda_2)$ *and* $\nu > \nu_n(\lambda_2)$ *fixed.*

(iii) For each $\lambda > 0$ *fixed, the function* $\nu \mapsto \mathcal{E}^\pm_{\lambda,\nu}$ *are non-increasing, that is, it holds that* $\mathcal{E}^\pm_{\lambda,\nu_2} \le \mathcal{E}^\pm_{\lambda,\nu_1}$ *for each* $\nu_1 < \nu_2$.

Proof. (i) Firstly, let us prove that $\lambda \mapsto \mathcal{E}^-_{\lambda,\nu}$ is increasing. By Proposition 3.2.33, we obtain that $\nu > \nu_n(\lambda_2) \geq \nu_n(\lambda_1)$ for each $0 < \lambda_1 < \lambda_2$, which implies $\mathcal{U}_{\lambda_2,\nu} \subset \mathcal{U}_{\lambda_1,\nu}$. Moreover, it is not hard to see that the function $\lambda \mapsto \mathcal{I}_{\lambda,\nu}(u)$ is increasing for each $\nu > 0$ and $u \in X \setminus \{0\}$ fixed. Then, according to Lemma 3.2.31, we have that

$$\mathcal{E}^-_{\lambda_1,\nu} \leq \sup_{t\in[0,\mathsf{t}_{\lambda_1}(w)]} \mathcal{I}_{\lambda_1,\nu}(tw) = \mathcal{I}_{\lambda_1,\nu}(\mathsf{t}^-_{\lambda_1,\nu}(w)w) \leq \mathcal{I}_{\lambda_2,\nu}(\mathsf{t}^-_{\lambda_1,\nu}(w)w), \tag{3.51}$$

for all $w \in \mathcal{U}_{\lambda_2,\nu}$. On the other hand, by using (3.51), Proposition 3.2.16 and that $\mathsf{t}^-_{\lambda_1,\nu}(w) < \mathsf{t}^-_{\lambda_2,\nu}(w) < \mathsf{t}_{\lambda_2}(w)$, we deduce that

$$\mathcal{E}^-_{\lambda_1,\nu} \leq \mathcal{I}_{\lambda_2,\nu}(\mathsf{t}^-_{\lambda_2,\nu}(w)w) = \sup_{t\in[0,\mathsf{t}_{\lambda_2}(w)]} \mathcal{I}_{\lambda_2,\nu}(tw), \quad \text{for all } w \in \mathcal{U}_{\lambda_2,\nu}.$$

Therefore,

$$\mathcal{E}^-_{\lambda_1,\nu} \leq \inf_{w\in\mathcal{U}_{\lambda_2,\nu}} \left[\sup_{t\in[0,\mathsf{t}_{\lambda_2}(w)]} \mathcal{I}_{\lambda_1,\nu}(tw) \right] = \mathcal{E}^-_{\lambda_2,\nu}.$$

Now, let us prove that $\lambda \mapsto \mathcal{E}^+_{\lambda,\nu}$ is increasing. By Lemma 3.2.31,

$$\mathcal{E}^+_{\lambda_1,\nu} \leq \inf_{t\in[\mathsf{t}_{\lambda_1}(w),\infty)} \mathcal{I}_{\lambda_1,\nu}(tw) = \mathcal{I}_{\lambda_1,\nu}(\mathsf{t}^+_{\lambda_1,\nu}(w)w), \quad \text{for all } w \in \mathcal{U}_{\lambda_2,\nu}. \tag{3.52}$$

On the other side, we have that $\mathsf{t}^-_{\lambda_1,\nu}(w) < \mathsf{t}^-_{\lambda_2,\nu}(w) < \mathsf{t}_{\lambda_2}(w) < \mathsf{t}^+_{\lambda_2,\nu}(w) < \mathsf{t}^+_{\lambda_1,\nu}(w)$ by Proposition 3.2.33. This fact combined with the Proposition 3.2.16 imply that

$$\mathcal{I}_{\lambda_1,\nu}(\mathsf{t}^+_{\lambda_1,\nu}(w)w) \leq \mathcal{I}_{\lambda_1,\nu}(\mathsf{t}^+_{\lambda_2,\nu}(w)w) \leq \mathcal{I}_{\lambda_2,\nu}(\mathsf{t}^+_{\lambda_2,\nu}(w)w) = \inf_{t\in[\mathsf{t}_{\lambda_2}(w),\infty)} \mathcal{I}_{\lambda_2,\nu}(tw), \tag{3.53}$$

for all $w \in \mathcal{U}_{\lambda_2,\nu}$. From inequalities (3.52) and (3.53), we conclude that

$$\mathcal{E}^+_{\lambda_1,\nu} \leq \inf_{w\in\mathcal{U}_{\lambda_2,\nu}} \left[\inf_{t\in[\mathsf{t}_{\lambda_2}(w),\infty)} \mathcal{I}_{\lambda_2,\nu}(tw) \right] = \mathcal{E}_{\lambda_2,\nu}.$$

This finishes the proof of item (i). The proof of item (ii) follows the same argument as that of item (i). □

Finally, the next propositions ensure that any minimizers for the functional $\mathcal{I}_{\lambda,\nu}$ in the Nehari manifold $\mathcal{N}^-_{\lambda,\nu}$ or $\mathcal{N}^+_{\lambda,\nu}$ not belong to $\mathcal{N}^0_{\lambda,\nu}$.

Proposition 3.2.35. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, there exists* $\lambda_* > 0$ *such that the minimizer* $u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ *obtained in Proposition 3.2.29 belongs to* $\mathcal{N}^-_{\lambda,\nu}$ *for each* $\lambda \in (0, \lambda_*)$ *and* $\nu \in (\nu_n(\lambda), \infty)$ *be fixed. Furthermore,* $u_{\lambda,\nu}$ *is a critical point for the functional* $\mathcal{I}_{\lambda,\nu}$.

Proof. By Proposition 3.2.29, there exists $u_{\lambda,\nu} \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$ such that

$$\mathcal{E}^-_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w).$$

Arguing by contradiction, we assume that $u_{\lambda,\nu} \in \mathcal{N}^0_{\lambda,\nu}$ for $\lambda > 0$ and $\nu > \nu_n(\lambda)$. It follows from Remark 3.2.25 that

$$\mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w) \leq \inf_{w\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w) \leq \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}).$$

As a consequence, we obtain that

$$\mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^-_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w) = \inf_{w\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w).$$

On the other hand, by using the fact that $u_{\lambda,\nu} \in \mathcal{N}^0_{\lambda,\nu}$ and arguing as in (3.46), we also have that

$$\mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) \geq \frac{p(q-m) - m(q-\ell)}{pq} \min\{\|u_{\lambda,\nu}\|^\ell, \|u_{\lambda,\nu}\|^m\}. \tag{3.54}$$

Let $\lambda_0 > 0$ and $\nu_0 \in (\nu_n(\lambda_0), \nu)$ be fixed. Since $\mathcal{E}^-_{\lambda_0,\nu_0} \geq \mathcal{E}^-_{\lambda,\nu}$ for all $\nu \geq \nu_0$ and $\lambda \in (0, \lambda_0)$ by Proposition 3.2.34, we deduce that

$$\min\{\|u_{\lambda,\nu}\|^\ell, \|u_{\lambda,\nu}\|^m\} \leq \frac{pq}{p(q-m) - m(q-\ell)} \mathcal{E}^-_{\lambda_0,\nu_0}. \tag{3.55}$$

Moreover, by using (3.12) and (φ_3) together with the Lemma 2.3.13, we obtain that

$$\begin{aligned}\lambda(p-q)\|w\|_p^p &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (q-m)\phi(|D_s w|)|D_s w|^2 d\mu + \int_{\mathbb{R}^N} V(x)(q-m)\phi(|w|)|w|^2 dx \\ &\geq \ell(q-m)\mathcal{J}_{s,\Phi,V}(w) \\ &\geq (q-m)\min\{\|w\|^\ell, \|w\|^m\},\end{aligned}$$

for all $w \in \mathcal{N}^0_{\lambda,\nu}$. Now, using the Sobolev embedding $X \hookrightarrow L^p(\mathbb{R}^N)$, we can see that $\lambda(p-q)||w||_p^p \leq \lambda(p-q)S_p^p||w||^p$. The two last inequalities give us

$$||w||^p \geq \frac{q-m}{\lambda(p-q)S_p^p} \min\{\|w\|^\ell, \|w\|^m\}, \quad \text{for all } w \in \mathcal{N}^0_{\lambda,\nu}. \tag{3.56}$$

Hence, by using (3.55) and (3.56), we deduce that

$$\left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{1/(p-\ell)} \leq \left[\frac{pq}{p(q-m) - m(q-\ell)}\mathcal{E}^-_{\lambda_0,\nu_0}\right]^{\frac{1}{\ell}} \tag{3.57}$$

or

$$\left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{1/(p-m)} \leq \left[\frac{pq}{p(q-m) - m(q-\ell)}\mathcal{E}^-_{\lambda_0,\nu_0}\right]^{\frac{1}{m}}. \tag{3.58}$$

hold true for $\lambda \in (0, \lambda_0)$. In particular, the inequalities (3.57) and (3.58) are satisfied for all $\lambda \in (0, \lambda_*)$ with $\lambda_* > 0$ given by

$$\lambda_* = \min\left\{\left[\frac{p(q-m)-m(q-\ell)}{pq\mathcal{E}^-_{\lambda_0,\nu_0}}\right]^{\frac{p-\ell}{\ell}}\frac{(q-m)}{(p-q)S_p^p}, \left[\frac{p(q-m)-m(q-\ell)}{pq\mathcal{E}^-_{\lambda_0,\nu_0}}\right]^{\frac{p-m}{m}}\frac{(q-m)}{(p-q)S_p^p}, \lambda_0\right\},$$

and we have a contradiction establishing that $u_{\lambda,\nu}$ belongs to $\mathcal{N}^-_{\lambda,\nu}$ for each $\lambda \in (0, \lambda_*)$ and $\nu > \nu_n(\lambda)$. Therefore, by using Lemma 3.2.27, we conclude that $u_{\lambda,\nu}$ is a critical point for $\mathcal{I}_{\lambda,\nu}$ whenever $\lambda \in (0, \lambda_*)$ and $\nu \in (\nu_n(\lambda), \infty)$. □

The next results state that, depending on the values of the parameters λ and ν, the energy level $\mathcal{E}^+_{\lambda,\nu}$ can be negative, zero or positive.

Proposition 3.2.36. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\lambda > 0$ *and* $\nu > \nu_n(\lambda)$. *Let* $v_{\lambda,\nu} \in X$ *be solution obtained in the Proposition 3.2.30. Then, the following assertions are satisfied:*

(i) Assume that $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$. *Then,* $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) > 0$.

(ii) Assume that $\nu = \nu_e(\lambda)$. *Then,* $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = 0$.

(iii) Assume that $\nu \in (\nu_e(\lambda), \infty)$. *Then,* $\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) < 0$.

Proof. (i) Assume that $\nu_n(\lambda) < \nu < \nu_e(\lambda)$ and let $u \in \mathcal{N}_{\lambda,\nu}$ be fixed. By definition of ν_e , we can see that $\nu < \nu_e(\lambda) \leq R_e(u)$. Moreover, for each $u \in \mathcal{N}_{\lambda,\nu}$, we have by assumption (φ_3) that

$$\nu\|u\|^q_{q,a} \geq \ell\mathcal{J}_{s,\Phi,V}(u) + \lambda\|u\|^p_p > \min\{\|u\|^\ell, \|u\|^m\}.$$

On the other hand, since $\mathcal{I}_{\lambda,R_e(u)}(u) = 0$, for each $u \in \mathcal{N}_{\lambda,\nu}$, it follows from Proposition 3.2.21 that

$$\mathcal{I}_{\lambda,\nu}(u) = \frac{R_e(u)-\nu}{q}||u||^q_{q,a} \geq \frac{\nu_e(\lambda)-\nu}{q}||u||^q_{q,a} \geq \frac{\nu_e(\lambda)-\nu}{\nu q}\min\{\|u\|^\ell, \|u\|^m\} \geq C_\nu,$$

where $C_\nu > 0$. Therefore,

$$\mathcal{E}^+_{\lambda,\nu} = \inf_{u\in\mathcal{N}^+_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(u) > 0.$$

This ends the proof for the item (i).

(ii) Assume that $\nu = \nu_e(\lambda)$. Using the Proposition 3.2.12 we can consider $u_e \in X$ such that $\Lambda_e(u_e) = \nu_e(\lambda)$. Since $\Lambda_e(tu_e) = \Lambda_e(u_e)$ for all $t > 0$, we can suppose without any loss of generality that s$(u_e) = 1$, that is,

$$\Lambda_e(u_e) = R_e(u_e) = \inf_{w\in X\setminus\{0\}} \Lambda_e(w).$$

Then, since Λ_e is differentiable, we have that $R_e'(u_e)v = 0$ for all $v \in X$. By using (3.9) and Lemma 3.2.14, we obtain that u_e is a critical point for $\mathcal{I}_{\lambda,\nu}$ with zero energy. Furthermore, the Proposition 3.2.15 give us that

$$\Lambda_n(u_e) < \Lambda_e(u_e) = R_e(u_e) = \nu_e(\lambda) = \nu,$$

which implies that $u_e \in \mathcal{U}_{\lambda,\nu}$. Thus, by Proposition 3.2.16, there exist $0 < \mathsf{t}_\nu^-(u_e) < \mathsf{t}(u_e) < \mathsf{t}_\nu^+(u_e)$ such that $\mathsf{t}_\nu^-(u_e)u_e \in \mathcal{N}_{\lambda,\nu}^-$ and $\mathsf{t}_\nu^+(u_e)u_e \in \mathcal{N}_{\lambda,\nu}^+$. Since $\mathsf{t}_\nu^-(u_e)$ and $\mathsf{t}_\nu^+(u_e)$ are the only roots of the equation $R_n(tu) = \nu$ and $R_n(u_e) = R_e(u_e) = \nu_e(\lambda) = \nu$, it follows from Proposition 3.2.15 that $\mathsf{t}_{\nu_e}^-(u_e) < \mathsf{t}_{\nu_e}^+(u_e) = \mathsf{s}(u_e) = 1$. As a consequence, we obtain that $u_e \in \mathcal{N}_{\lambda,\nu}^+$. Thence, using this fact we deduce that $\mathcal{E}_{\lambda,\nu}^+ = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) \leq \mathcal{I}_{\lambda,\nu}(u_e) = 0$. On the other side, thanks to Remark 3.2.11 and the fact that $\nu = \nu_e(\lambda) \leq R_e(v)$, we also have that $\mathcal{I}_{\lambda,\nu}(v) \geq 0$, for all $v \in X \setminus \{0\}$. This finishes the proof of item (ii).

(iii) Assume that $\nu > \nu_e = R_e(u_e)$. Since $\nu \mapsto \mathcal{I}_{\lambda,\nu}(u)$ is a strictly decreasing function for each $\lambda > 0$ and $u \in X \setminus \{0\}$ fixed, then $\mathcal{I}_{\lambda,\nu}(u_e) < \mathcal{I}_{\lambda,R_e(u_e)}(u_e) = 0$. On the other hand, since $\Lambda_n(u_e) = R_n(u_e) = R_e(u_e) = \nu_e(\lambda) < \nu$, we infer that $u_e \in \mathcal{U}_{\lambda,\nu}$. Hence, by Proposition 3.2.16 there exists $\mathsf{t}_\nu^+(u_e) \in (0,\infty)$ such that $\mathsf{t}_\nu^+(u_e)u_e \in \mathcal{N}_{\lambda,\nu}^+$ and

$$\mathcal{E}_{\lambda,\nu}^+ \leq \mathcal{I}_{\lambda,\nu}(\mathsf{t}_\nu^+(u_e)u_e) = \inf_{t>0} \mathcal{I}_{\lambda,\nu}(tu_e) \leq \mathcal{I}_{\lambda,\nu}(u_e) < 0.$$

This finishes the proof. □

Proposition 3.2.37. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold.Suppose also that* $\lambda > 0$ *and* $\nu > \nu_n(\lambda)$. *Then, there exists* $\lambda^* > 0$ *such that the minimizer* $v_{\lambda,\nu} \in \mathcal{N}_{\lambda,\nu}^+ \cup \mathcal{N}_{\lambda,\nu}^0$ *obtained in Proposition 3.2.30 belongs to* $\mathcal{N}_{\lambda,\nu}^+$. *Furthermore,* $v_{\lambda,\nu}$ *is a critical point for the functional* $\mathcal{I}_{\lambda,\nu}$ *if one of the following conditions is satisfied:*

(i) $\nu \in [\nu_e(\lambda), \infty)$ *and* $\lambda > 0$.

(ii) $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$ *and* $\lambda \in (0, \lambda^*)$.

(iii) $\lambda > 0$ *and* $\nu \in (\nu_e(\lambda) - \varepsilon, \nu_e(\lambda))$, *where* $\varepsilon > 0$ *is small enough.*

Proof. By Proposition 3.2.29, there exists $v_{\lambda,\nu} \in \mathcal{N}_{\lambda,\nu}^+ \cup \mathcal{N}_{\lambda,\nu}^0$ such that

$$\mathcal{E}_{\lambda,\nu}^+ = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = \inf_{w \in \mathcal{N}_{\lambda,\nu}^+} \mathcal{I}_{\lambda,\nu}(w).$$

Firstly, let us assume that (i) holds, that is, $\nu \in [\nu_e(\lambda), \infty)$ and $\lambda > 0$. Then, by Proposition 3.2.36, we have that $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) \leq 0$. On the other hand, by using the same ideas discussed in the proof of Proposition 3.2.28, we deduce that the inequality

$$\inf_{w\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w) > 0 \geq \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^+_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w)$$

holds true for all $\lambda > 0$ and $\nu \geq \nu_e(\lambda)$. This implies that $v_{\lambda,\nu}$ is in $\mathcal{N}^+_{\lambda,\nu}$. Therefore, by using Lemma 3.2.27, we conclude that $v_{\lambda,\nu}$ is a critical point for $\mathcal{I}_{\lambda,\nu}$ for all $\nu \in [\nu_e(\lambda), \infty)$ and $\lambda > 0$.

For the item (ii), we proceed by contradiction. We assume that $v_{\lambda,\nu} \in \mathcal{N}^0_{\lambda,\nu}$ for $\lambda > 0$ and $\nu \in (\nu_n(\lambda), \nu_n(\lambda))$. As a consequence of Remark 3.2.25, we obtain that

$$\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = \inf_{w\in\mathcal{N}^+_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w) = \inf_{w\in\mathcal{N}^0_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w).$$

Let $\lambda_0 > 0$ and $\nu_0 \in (\nu_n(\lambda), \nu)$ be fixed. By Proposition 3.2.34 and Proposition 3.2.36, we have that $\mathcal{E}^+_{\lambda_0,\nu_0} \geq \mathcal{E}^+_{\lambda,\nu} > 0$ for all $\nu \geq \nu_0$ and $\lambda \in (0, \lambda_0)$. Thus, using the same ideas discussed in the proof of Proposition 3.2.35, we deduce that

$$\left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{1/(p-\ell)} \leq \left[\frac{pq}{p(q-m)-m(q-\ell)}\mathcal{E}^+_{\lambda_0,\nu_0}\right]^{\frac{1}{\ell}} \tag{3.59}$$

or

$$\left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{1/(p-m)} \leq \left[\frac{pq}{p(q-m)-m(q-\ell)}\mathcal{E}^+_{\lambda_0,\nu_0}\right]^{\frac{1}{m}}. \tag{3.60}$$

holds true for $\lambda \in (0, \lambda_0)$. In particular, the inequalities (3.59) and (3.60) are satisfied for all $\lambda \in (0, \lambda^*)$ with $\lambda^* > 0$ given by

$$\lambda^* = \min\left\{\left[\frac{p(q-m)-m(q-\ell)}{pq\mathcal{E}^+_{\lambda_0,\nu_0}}\right]^{\frac{p-\ell}{\ell}} \frac{(q-m)}{(p-q)S_p^p}, \left[\frac{p(q-m)-m(q-\ell)}{pq\mathcal{E}^+_{\lambda_0,\nu_0}}\right]^{\frac{p-m}{m}} \frac{(q-m)}{(p-q)S_p^p}, \lambda_0\right\},$$

which is a contradiction. Therefore, $v_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu}$ which implies that $v_{\lambda,\nu}$ is a critical point for the functional $\mathcal{I}_{\lambda,\nu}$ by Lemma 3.2.27.

It remains to consider the item (iii). In this case, it is sufficient to prove that $\mathcal{E}^+_{\lambda,\nu} < \mathcal{E}^0_{\lambda,\nu}$. Since $\mathcal{N}^0_{\lambda,\nu}$ is closed and $\mathcal{I}_{\lambda,\nu}|_{\mathcal{N}^0_{\lambda,\nu}}$ is coercive, there exists $w_{\lambda,\nu} \in \mathcal{N}^0_{\lambda,\nu}$ such that $\mathcal{I}_{\lambda,\nu}(w_{\lambda,\nu}) = \mathcal{E}^0_{\lambda,\nu}$. By using the estimates (3.36) and (3.46), we deduce the following inequality

$$\mathcal{E}^0_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(w_{\lambda,\nu}) \geq \frac{p(q-m)-m(q-\ell)}{pq}\min\left\{\left(\frac{\ell}{\nu S_q^q\|a\|_\infty}\right)^{\frac{\ell}{q-\ell}}, \left(\frac{\ell}{\nu S_q^q\|a\|_\infty}\right)^{\frac{m}{q-m}}\right\}.$$

Assuming that $\nu < \nu_e(\lambda)$, we obtain that

$$\mathcal{E}^0_{\lambda,\nu} > \frac{p(q-m)-m(q-\ell)}{pq}\min\left\{\left(\frac{\ell}{\nu_e(\lambda) S_q^q\|a\|_\infty}\right)^{\frac{\ell}{q-\ell}}, \left(\frac{\ell}{\nu_e(\lambda) S_q^q\|a\|_\infty}\right)^{\frac{m}{q-m}}\right\} =: C_{\nu_e(\lambda)}.$$

Now, using the Proposition 3.2.36, we have that $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = \mathcal{E}^{+}_{\lambda,\nu} = 0$ when $\nu = \nu_e$. Moreover, $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = \mathcal{E}^{+}_{\lambda,\nu} > 0$ for each $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$. More precisely, we employ the same estimates used in the proof of Proposition 3.2.36 (i), we obtain that

$$\mathcal{E}^{+}_{\lambda,\nu} \geq \frac{\nu_e(\lambda) - \nu}{\nu q} \min\left\{ \left(\frac{\ell}{\nu_e(\lambda) S_q^q \|a\|_\infty} \right)^{\frac{\ell}{q-\ell}}, \left(\frac{\ell}{\nu_e(\lambda) S_q^q \|a\|_\infty} \right)^{\frac{m}{q-m}} \right\}.$$

Hence, by two last inequality, there exists $\varepsilon > 0$ small enough such that $\mathcal{E}^{+}_{\lambda,\nu} < C_{\nu_e(\lambda)} < \mathcal{E}^{0}_{\lambda,\nu}$ for each $\nu \in (\nu_e(\lambda) - \varepsilon, \nu_e(\lambda))$. This ends the proof. □

3.3 THE PROOF OF MAIN RESULTS

The purpose of this section is to present the proof of our main results.

The proof of Theorem 3.1.1. According to Proposition 3.2.15 we have that $0 < \nu_n(\lambda) < \nu_e(\lambda)$ for all $\lambda > 0$. Now, by using Proposition 3.2.29 we find a function $u_{\lambda,\nu} \in \mathcal{N}^{-}_{\lambda,\nu} \cup \mathcal{N}^{0}_{\lambda,\nu}$ that solve the minimization problem given by (3.3), that is,

$$\mathcal{E}^{-}_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) = \inf_{w \in \mathcal{N}^{-}_{\lambda,\nu}} \mathcal{I}_{\lambda,\nu}(w).$$

But thanks to Proposition 3.2.35, there exists $\lambda_* > 0$ such that the function $u_{\lambda,\nu} \in \mathcal{N}^{+}_{\lambda,\nu}$ for each $\lambda \in (0, \lambda_*)$ and $\nu \in (\nu_n(\lambda), \infty)$. Therefore, it follows from Lemma 3.2.27 that $u_{\lambda,\nu}$ is a weak solution for problem $(\mathcal{P}_{\lambda,\nu})$ Moreover, according to Proposition 3.2.28 we also obtain that $\mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) \geq D_\nu$ holds true for some $D_\nu > 0$. This finishes the proof. □

The proof of Theorem 3.1.2. Firstly, using the Proposition 3.2.30, we find a minimizer $v_{\lambda,\nu} \in \mathcal{N}^{+}_{\lambda,\nu} \cup \mathcal{N}^{0}_{\lambda,\nu}$ for the minimization problem given by (3.4). However, by Proposition 3.2.37, there exists $\lambda^* > 0$ in such way that $v_{\lambda,\nu} \notin \mathcal{N}^{0}_{\lambda,\nu}$ for each $\lambda \in (0, \lambda^*)$ and $\nu > \nu_n(\lambda)$. More preciselly, we obtain that $v_{\lambda,\nu}$ is a weak solution of problem $(\mathcal{P}_{\lambda,\nu})$ if one of the following conditions is satisfied:

(i) $\nu \in [\nu_e(\lambda), \infty)$ and $\lambda > 0$.

(ii) $\nu \in (\nu_n(\lambda), \nu_e(\lambda))$ and $\lambda \in (0, \lambda^*)$.

(iii) $\lambda > 0$ and $\nu \in (\nu_e(\lambda) - \varepsilon, \nu_e(\lambda))$, where $\varepsilon > 0$ is small enough.

In order to prove that $v_{\lambda,\nu}$ is a ground state solution, it is sufficient to verify that $\mathcal{E}^{+}_{\lambda,\nu} < \mathcal{E}^{-}_{\lambda,\nu}$. Let $u_{\lambda,\nu}$ be the weak solution obtained in Theorem 3.1.1. We know that $1 = \mathsf{t}^{-}_{\nu}(u_{\lambda,\nu}) < \mathsf{t}^{+}_{\nu}(u_{\lambda,\nu})$

with $\mathsf{t}_\nu^+(u_{\lambda,\nu})u_{\lambda,\nu} \in \mathcal{N}_{\lambda,\nu}^+$. Hence, since $t \mapsto \mathcal{I}_{\lambda,\nu}(tu_{\lambda,\nu})$ is a decreasing function on $[1, \mathsf{t}_\nu^+(u_{\lambda,\nu})]$, we conclude that

$$\mathcal{E}_{\lambda,\nu}^+ = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) \leq \mathcal{I}_{\lambda,\nu}(\mathsf{t}_\nu^+(u_{\lambda,\nu})u_{\lambda,\nu}) < \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) = \mathcal{E}_{\lambda,\nu}^-.$$

Therefore, $v_{\lambda,\nu}$ is a ground state solution. Finally, according to Proposition 3.2.36 we also obtain that $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) > 0$ whenever $\nu \in (\nu_e, \infty)$. Furthermore, by using Proposition 3.2.36, we observe that $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) = 0$ for $\nu = \nu_e$. In the same way, assuming that $\nu \in (\nu_n, \nu_e)$, the solution $v_{\lambda,\nu}$ satisfies $\mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) < 0$, see Proposition 3.2.36. This ends the proof. □

The proof of Corollary 3.1.3. The proof is an immediate consequence of the Theorems 3.1.1 and 3.1.2. □

The proof of Theorem 3.1.4. The proof it follows directly from Lemma 3.2.20, which states that the Nehari set $\mathcal{N}_{\lambda,\nu}$ is empty for each $\nu \in (-\infty, \nu_n(\lambda))$ and $\lambda > 0$. □

3.4 THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS

In the section, we study the asymptotic behavior of solutions $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ obtained from Propositions 3.2.29 and 3.2.30 as $\lambda \to 0$ or $\nu \to \infty$.

We start proving some properties on the energy $\mathcal{E}_{\lambda,\nu}^-$ as well as the solution $u_{\lambda,\nu}$ when the parameter ν is fixed.

Proposition 3.4.1. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, the function* $\mathcal{E}_{\lambda,\nu}^- : [0, \lambda_*) \times (\nu_n(\lambda), \infty) \to \mathbb{R}$ *given by* (3.3) *possesses the following properties:*

(i) It holds that $\lambda \mapsto \mathcal{E}_{\lambda,\nu}^-$ *is a continuous function for each* $\nu > \nu_n(\lambda)$ *fixed.*

(ii) For each λ_0 *and* $\nu > \nu_n(\lambda_0)$ *fixed, there exists* $C > 0$ *independent on* λ *such that* $0 < \mathcal{E}_{\lambda,\nu}^- \leq C$ *for all* $\lambda \in [0, \lambda_0)$.

(iii) For each $\lambda_0 > 0$ *and* $\nu > \nu_n(\lambda_0)$ *fixed, the sequence* $(u_{\lambda,\nu})_{\lambda<\lambda_0}$ *obtained as minimizer in* $\mathcal{N}_{\lambda,\nu}^-$ *is bounded in* X.

(iv) In addition, $u_{\lambda,\nu} \to u_\nu$ *in* X *for some* $u_\nu \in X \setminus \{0\}$ *as* $\lambda \to 0$. *Consequently, we obtain that* $\mathcal{E}_{\lambda,\nu}^- \to \mathcal{E}_{0,\nu}^-$ *as* $\lambda \to 0$ *and* $\mathcal{E}_{0,\nu}^- = \mathcal{I}_{0,\nu}(u_\nu)$, *where* $\mathcal{E}_{0,\nu}^- := \inf_{u \in \mathcal{N}_{0,\nu}} \mathcal{I}_{0,\nu}(u)$.

Proof. (i) It is sufficient to prove that the function $\lambda \mapsto \mathcal{E}^-_{\lambda,\nu}$ is sequentially continuous. To this end, let $\lambda \in [0, \lambda_*)$ and $\nu > \nu_n(\lambda)$ be fixed and consider a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lambda_k \to \lambda$. First, we know from Proposition 3.2.7 that, for each $\lambda > 0$, it holds $\nu_n(\lambda) = \inf_{u\in X\setminus\{0\}} \Lambda_{n,\lambda}(u) = R_{n,\lambda}(\mathsf{t}_\lambda(u_\lambda)u_\lambda)$ for some $u_\lambda \in X \setminus \{0\}$. Then, it follows from Proposition 3.2.33 that

$$\limsup_{k\to\infty} \nu_n(\lambda_k) \leq \limsup_{k\to\infty} R_{n,\lambda_k}(\mathsf{t}_{\lambda_k}(u_\lambda)u_\lambda) = R_{n,\lambda}(\mathsf{t}_\lambda(u_\lambda)u_\lambda) = \nu_n(\lambda) < \nu,$$

which implies that $\nu_n(\lambda_k) < \nu$ for all $k \in \mathbb{N}$ enough large. Thus, according to Proposition 3.2.29 and Proposition 3.2.35, we have that $\mathcal{E}^-_{\lambda_k,\nu}$ is attained by a function $u_k \in \mathcal{N}^-_{\lambda_k,\nu} \subset X$ which is a critical point for the functional $\mathcal{I}_{\lambda_k,\nu}$ for all $k \in \mathbb{N}$ enough large. Consequently, we obtain that

$$\mathcal{E}^-_{\lambda_k,\nu} = \mathcal{I}_{\lambda_k,\nu}(u_k) \leq \mathcal{I}_{\lambda_k,\nu}(w), \quad \text{for all } w \in \mathcal{N}^-_{\lambda_k,\nu}, \tag{3.61}$$

$$\mathcal{I}'_{\lambda_k,\nu}(u_k)w = 0, \quad \text{for all } w \in X \quad \text{and} \quad \mathcal{I}''_{\lambda_k,\nu}(u_k)(u_k, u_k) < 0, \tag{3.62}$$

for all $k \in \mathbb{N}$ enough large. Now, we will show that $(u_k)_{k\in\mathbb{N}}$ is bounded. Indeed, by Proposition 3.2.33,

$$\limsup_{k\to\infty} \mathcal{E}^-_{\lambda_k,\nu} \leq \limsup_{k\to\infty} \mathcal{I}_{\lambda_k,\nu}(\mathsf{t}^-_{\lambda_k,\nu}(u_{\lambda,\nu})u_{\lambda,\nu}) = \mathcal{I}_{\lambda,\nu}(\mathsf{t}^-_{\lambda,\nu}(u_{\lambda,\nu})u_{\lambda,\nu}) = \mathcal{E}^-_{\lambda,\nu},$$

where $u_{\lambda,\nu}$ is obtained in Proposition 3.2.29. Hence, by using the same ideas discussed in the proof of inequality (3.46), we deduce that

$$\mathcal{E}^-_{\lambda,\nu} \geq \limsup_{k\to\infty} \mathcal{E}^-_{\lambda_k,\nu} \geq \limsup_{k\to\infty} \left[\frac{p(q-m) - m(q-\ell)}{pq} \min\{\|u_k\|^\ell, \|u_k\|^m\} \right],$$

that is, $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in X. Thence, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X. It is not hard to verify that $u \neq 0$, see Remark 3.2.22. Moreover, by using (3.62), Hölder inequality and compact embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ for each $r \in (m, \ell_s^*)$, we have that

$$\mathcal{J}_{s,\Phi,V}(u_k)(u_k - u) = \nu_k \int_{\mathbb{R}^N} a(x)|u_k|^{q-2}u_k(u_k - u)dx - \lambda_k \int_{\mathbb{R}^N} |u_k|^{p-2}u_k(u_k - u)dx = o_k(1).$$

Then, by (S_+)-condition (see Proposition 2.3.17), we infer that $u_k \to u$ in X. This fact combined with (3.62) imply that $\mathcal{I}'_{\lambda,\nu}(u)w = 0$ for all $w \in X$ and $\mathcal{I}''_{\lambda,\nu}(u)(u, u) \leq 0$, that is, $u \in \mathcal{N}^-_{\lambda,\nu} \cup \mathcal{N}^0_{\lambda,\nu}$. Now, proceeding as in proof of Proposition 3.2.35, we obtain that $u \notin \mathcal{N}^0_{\lambda,\nu}$. Finally, by using (3.61) and strong converge, we have also that $\mathcal{I}_{\lambda,\nu}(u) \leq \mathcal{I}_{\lambda,\nu}(w)$ for all $w \in \mathcal{N}^-_{\lambda,\nu}$, which implies that

$$\lim_{k\to\infty} \mathcal{E}^-_{\lambda_k,\nu} = \lim_{k\to\infty} \mathcal{I}_{\lambda_k,\nu}(u_k) = \mathcal{I}_{\lambda,\nu}(u) = \mathcal{E}^-_{\lambda,\nu}.$$

This finishes the proof of item (i).

(ii) Let λ_0 and $\nu > \nu_n(\lambda_0)$ be fixed. It follows from Proposition 3.2.34 that

$$\mathcal{E}^-_{\lambda,\nu} \leq \mathcal{E}^-_{\lambda_0,\nu} = \inf_{w\in\mathcal{N}^-_{\lambda_0,\nu}} \mathcal{I}_{\lambda_0,\nu}(w) = C < \infty, \quad \text{for all } \lambda \in (0,\lambda_0),$$

where $C := C(q,p,\lambda_0,\nu,N)$ is independent on λ.

(iii) Let $(u_{\lambda,\nu})_{\lambda<\lambda_0}$ be the sequence obtained as minimizers in $\mathcal{N}^-_{\lambda,\nu}$ for the functional $\mathcal{I}_{\lambda,\nu}$, see Proposition 3.2.29 and Proposition 3.2.35. Arguing as in (3.46), we deduce also that

$$C \geq \mathcal{E}^-_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(u_{\lambda,\nu}) \geq \frac{p(q-m)-m(q-\ell)}{pq}\min\{\|u_{\lambda,\nu}\|^\ell, \|u_{\lambda,\nu}\|^m\},$$

for all $\lambda \in [0,\lambda_0)$, where $C > 0$ is independent on λ. The last assertion implies that

$$\|u_{\lambda,\nu}\| \leq \min\left\{\left[\frac{pqC}{p(q-m)-m(q-\ell)}\right]^{\frac{1}{\ell}}, \left[\frac{pqC}{p(q-m)-m(q-\ell)}\right]^{\frac{1}{m}}\right\}.$$

Consequently, the sequence $(u_{\lambda,\nu})_{\lambda<\lambda_0}$ is bounded in X with respect to $\lambda > 0$. Therefore, there exists $u_\nu \in X$ such that $u_{\lambda,\nu} \rightharpoonup u_\nu$ in X as $\lambda \to 0$. Arguing as in the proof of Proposition 3.4.1 (i), we obtain also that $u_{\lambda,\nu} \to u_\nu$ in X as $\lambda \to 0$. By using Proposition 3.2.28, we have that $\|u_\nu\| \geq D_\nu > 0$, that is, $u_\nu \neq 0$. Finally, by using item (i), we conclude that $0 < \mathcal{E}^-_{\lambda,\nu} \to \mathcal{E}^-_{0,\nu}$ as $\lambda \to 0$, where $\mathcal{E}^-_{0,\nu} = \mathcal{I}_{0,\nu}(u_\nu)$. This finishes the proof. □

The next result is useful in order to study the asymptotic behavior of solutions obtained from Theorems 3.1.1 and 3.1.2 as $\lambda \to 0$.

Proposition 3.4.2. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then,* $\nu_e(\lambda) \to 0$ *and* $\nu_n(\lambda) \to 0$ *as* $\lambda \to 0$.

Proof. Let $u \in X \setminus \{0\}$ be fixed. First, by using Proposition 3.2.10 and assumption (φ_3), we obtain that

$$\begin{aligned} 0 = \frac{1}{q}\|\mathsf{s}(u)u\|^q_{q,a}\mathsf{s}(u)R_e(\mathsf{s}(u)u)u &= -q\mathcal{J}_{s,\Phi,V}(\mathsf{s}(u)u) + \mathcal{J}'_{s,\Phi,V}(\mathsf{s}(u)u)(\mathsf{s}(u)u) \\ &\quad + \lambda\frac{p-q}{p}\|\mathsf{s}(u)u\|^p_p \\ &\geq (\ell-q)\mathcal{J}_{s,\Phi,V}(\mathsf{s}(u)u) + \lambda\frac{p-q}{p}\|\mathsf{s}(u)u\|^p_p. \end{aligned} \tag{3.63}$$

The last inequality combined with Lemma 2.3.13 imply that

$$\|\mathsf{s}(u)u\|^p_p \leq \frac{p(q-\ell)}{\lambda(p-q)}\max\{\|\mathsf{s}(u)u\|^\ell, \|\mathsf{s}(u)u\|^m\}. \tag{3.64}$$

On the other hand, combining (3.63) and (φ_3) once more, we deduce that

$$0 \leq (m-q)\mathcal{J}_{s,\Phi,V}(\mathsf{s}(u)u) + \lambda\frac{p-q}{p}\|\mathsf{s}(u)u\|_p^p. \tag{3.65}$$

Assume that $\|\mathsf{s}(u)u\| \leq 1$. Hence, combining (3.64) and (3.65), we infer that

$$\begin{aligned}
0 < \nu_e(\lambda) \leq \Lambda_e(u) &= \frac{\mathcal{J}_{s,\Phi,V}(\mathsf{s}(u)u) + \frac{\lambda}{p}\|\mathsf{s}(u)u\|_p^p}{\frac{1}{q}\|\mathsf{s}(u)u\|_{q,a}^q} \leq \frac{\lambda\frac{p-q}{p(q-m)}\|\mathsf{s}(u)u\|_p^p + \frac{\lambda}{p}\|\mathsf{s}(u)u\|_p^p}{\frac{1}{q}\|\mathsf{s}(u)u\|_{q,a}^q} \\
&\leq \frac{\lambda\frac{(p-m)}{p(q-m)}\left[\frac{p(q-\ell)}{\lambda(p-q)}\frac{\|u\|^\ell}{\|u\|_p^p}\right]^{\frac{p-q}{p-\ell}}\|u\|_p^p}{\frac{1}{q}\|u\|_{q,a}^q} \\
&= \frac{\frac{q(p-m)}{p(q-m)}\left[\frac{p(q-\ell)}{(p-q)}\right]^{\frac{p-q}{p-\ell}}\lambda^{\frac{q-\ell}{p-\ell}}\|u\|^{\ell\frac{p-q}{p-\ell}}\|u\|_p^{p\frac{q-\ell}{p-\ell}}}{\|u\|_{q,a}^q} \\
&= C_{\ell,m,q,p}\lambda^{\frac{q-\ell}{p-\ell}}\frac{\|u\|^{\ell\frac{p-q}{p-\ell}}\|u\|_p^{p\frac{q-\ell}{p-\ell}}}{\|u\|_{q,a}^q}.
\end{aligned}$$

In the case of $\|\mathsf{s}(u)u\| > 1$, we obtain that

$$0 < \nu_e(\lambda) \leq C_{\ell,m,q,p}\lambda^{\frac{q-m}{p-q}}\frac{\|u\|^{m\frac{p-q}{p-m}}\|u\|_p^{p\frac{q-m}{p-m}}}{\|u\|_{q,a}^q}.$$

Summarizing, we have that

$$0 < \nu_e(\lambda) \leq \Lambda_e(u) \leq C_{\ell,m,q,p}\max\left\{\lambda^{\frac{q-\ell}{p-q}}\frac{\|u\|^{\ell\frac{p-q}{p-\ell}}\|u\|_p^{p\frac{q-\ell}{p-\ell}}}{\|u\|_{q,a}^q}, \lambda^{\frac{q-m}{p-q}}\frac{\|u\|^{m\frac{p-q}{p-m}}\|u\|_p^{p\frac{q-m}{p-m}}}{\|u\|_{q,a}^q}\right\}$$

The last inequality implies that $\nu_e(\lambda) \to 0$ as $\lambda \to 0$. Finally, by using Proposition 3.2.15, we also obtain that $\nu_n(\lambda) \to 0$ as $\lambda \to 0$. □

Remark 3.4.3. *As a consequence of the previous result, for each fixed $\nu > 0$, there exists a sufficiently small $\lambda_0 > 0$ such that $\nu > \nu_n(\lambda)$ and $\nu > \nu_e(\lambda)$ hold for each $\lambda \in (0, \lambda_0)$. For this reason, we can analyze the asymptotic behavior of the solutions $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ as $\lambda \to 0$.*

Now we are ready to prove the following result.

Proposition 3.4.4. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Then, the weak solutions $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ obtained respectively in Propositions 3.2.29 and 3.2.30 has the following asymptotic behavior:*

(i) It holds that $u_{\lambda,\nu} \to u_\nu$ in X as $\lambda \to 0$, where u_ν is ground state solution to the following nonlocal elliptic problem

$$(-\Delta_\Phi)^s w + V(x)\varphi(|w|)w = \nu a(x)|w|^{q-2}w \textit{ in } \mathbb{R}^N, w \in W^{s,\Phi}(\mathbb{R}^N). \tag{3.66}$$

(ii) It holds that $\|v_{\lambda,\nu}\| \to \infty$ *as* $\lambda \to 0$.

Proof. (i) In view from Proposition 3.4.1, there exists $u_\nu \in X \setminus \{0\}$ such that $u_{\lambda,\nu} \to u_\nu$ in X as $\lambda \to 0$ and $\mathcal{E}^-_{0,\nu} = \mathcal{I}_{0,\nu}(u_\nu)$. Since $\mathcal{I}'_{\lambda,\nu}(u_{\lambda,\nu})w = 0$ for all $w \in X$, using the compact embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ for each $r \in (m, \ell^*_s)$, we conclude that $\mathcal{I}'_{0,\nu}(u_\nu)w = 0$ for all $w \in X$. Therefore, the function u_ν is a weak nontrivial solution with minimal energy level for the nonlocal elliptic problem (3.66). This proves the item (i).

(ii) Firstly, it follows from (3.12), (φ_3) and Lemma 2.3.13 that

$$\lambda(p-q)\|v_{\lambda,\nu}\|_p^p \geq \ell(q-m)\mathcal{J}_{s,\Phi,V}(v_{\lambda,\nu}) \geq (q-m)\min\{\|v_{\lambda,\nu}\|^\ell, \|v_{\lambda,\nu}\|^m\},$$

Now, combining the last assertion with Sobolev embedding $X \hookrightarrow L^p(\mathbb{R}^N)$, we infer that

$$\|v_{\lambda,\nu}\| \geq \min\left\{\left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{\frac{1}{p-\ell}}, \left[\frac{q-m}{\lambda(p-q)S_p^p}\right]^{\frac{1}{p-m}}\right\} \to \infty \text{ as } \lambda \to 0.$$

This ends the proof. □

In the sequel, we study the behavior of solutions $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ as $\nu \to \infty$. The idea is to guarantee that the sequences $(u_{\lambda,\nu})_\nu$ and $(v_{\lambda,\nu})_\nu$ remain bounded in X for each $\nu > 0$ large enough. As a starting point, we consider the following two result:

Lemma 3.4.5. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, the function* $\mathcal{E}^-_{\lambda,\nu} : [0, \lambda_*) \times (\nu_n(\lambda), \infty) \to \mathbb{R}$ *given by* (3.3) *possesses the following properties:*

(i) For each $\lambda > 0$ *and* $\nu_0 > \nu_n(\lambda)$ *fixed, there exists* $C > 0$ *independent on* ν *such that* $0 < \mathcal{E}^-_{\lambda,\nu} \leq C$ *for all* $\nu > \nu_0$.

(ii) For each $\lambda > 0$ *and* $\nu_0 > \nu_n(\lambda)$ *fixed, the sequence* $(u_{\lambda,\nu})_{\nu>\nu_0}$ *obtained as minimizer in* $\mathcal{N}^-_{\lambda,\nu}$ *is bounded in* X.

Proof. (i) Let $\lambda > 0$ and $\nu_0 > \nu_n(\lambda)$ be fixed. By Proposition 3.2.34, it follows that

$$\mathcal{E}^-_{\lambda,\nu} \leq \mathcal{E}^-_{\lambda,\nu_0} = \inf_{w \in \mathcal{N}^-_{\lambda,\nu_0}} \mathcal{I}_{\lambda,\nu}(w) = C < \infty, \quad \text{for all } \nu > \nu_0,$$

where $C := C(p, q, \lambda, \nu_0, N) > 0$ is independent on ν. This proves the item (i).

The proof of item (ii) follows the same ideas as Proposition 3.4.1 (iii). □

Lemma 3.4.6. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Then, the function* $\mathcal{E}^+_{\lambda,\nu} : [0, \lambda^*) \times (\nu_n(\lambda), \infty) \to \mathbb{R}$ *given by* (3.3) *possesses the following properties:*

(i) *It holds that* $\nu \mapsto \mathcal{E}^{+}_{\lambda,\nu}$ *is a continuous function for* $\lambda > 0$ *fixed.*

(ii) *For each* $\lambda > 0$ *and* $\nu_0 > \nu_n(\lambda)$*, there exists* $C > 0$ *independent on* ν *such that* $\mathcal{E}^{+}_{\lambda,\nu} \le C$ *for all* $\nu > \nu_0$*.*

(iii) *It holds that* $\mathcal{E}^{+}_{\lambda,\nu} \to -\infty$ *as* $\nu \to \infty$ *for each* $\lambda > 0$ *fixed.*

Proof. The proof of item (i) and (ii) follows using the same ideas discussed in the proof of Proposition 3.4.1.

(iii) Let $w \in X\backslash\{0\}$ be fixed. Taking $\nu > 0$ large enough, we can assume that $\nu > \Lambda_n(w)$. Then, by Proposition 3.2.16, there exists $\mathsf{t}_{\lambda,\nu} := \mathsf{t}^{+}_{\lambda,\nu}(w) > 0$ such that $\mathsf{t}_{\lambda,\nu}w \in \mathcal{N}^{+}_{\lambda,\nu}$. Consequently, it follows from assumption (φ_3) and Lemma 2.3.13 that

$$\begin{aligned} \nu t^{q}_{\lambda,\nu}\|w\|^{q}_{q,a} &= \mathcal{J}'_{s,\Phi,V}(\mathsf{t}_{\lambda,\nu}w)(\mathsf{t}_{\lambda,\nu}w) + \lambda t^{p}_{\lambda,\nu}\|w\|^{p}_{p} \\ &\le m\max\{\|\mathsf{t}_{\lambda,\nu}w\|^{\ell}, \|\mathsf{t}_{\lambda,\nu}w\|^{m}\} + \lambda t^{p}_{\lambda,\nu}\|w\|^{p}_{p}. \end{aligned} \tag{3.67}$$

On the other hand, by using (3.12) combined with assumption (φ_3) and Lemma 2.3.13, we obtain that

$$\lambda(p-q)\|\mathsf{t}_{\lambda,\nu}w\|^{p}_{p} > (q-m)\min\{\|\mathsf{t}_{\lambda,\nu}w\|^{\ell}, \|\mathsf{t}_{\lambda,\nu}w\|^{m}\}.$$

We consider the proof for the case $\|\mathsf{t}_{\lambda,\nu}w\| > 1$. The proof for the case $\|\mathsf{t}_{\lambda,\nu}w\| \le 1$ is analogous. In view of last estimate, we have that $\lambda(p-q)\mathsf{t}^{p}_{\lambda,\nu}\|w\|^{p}_{p} > (q-m)\mathsf{t}^{m}_{\lambda,\nu}\|w\|^{m}$. This inequality implies that $t_{\lambda,\nu} \ge \delta$ for some $\delta := \delta(m,q,p,\lambda,w) > 0$ independent on $\nu > 0$. Thus, by inequality (3.67) and the continuous embedding $X \hookrightarrow L^{p}(\mathbb{R}^N)$, we deduce that

$$\nu t^{q-m}_{\lambda,\nu}\|w\|^{q}_{q,a} \le m\|w\|^{m} + \lambda t^{p-2}_{\lambda,\nu}\|w\|^{p} \le t^{p-m}_{\lambda,\nu}\left(m\frac{\|w\|^{m}}{\delta^{p-m}} + \lambda S^{p}_{p}\|w\|^{p}\right).$$

As a consequence, we infer that $\nu \le t^{p-q}_{\lambda,\nu}C$ for some $C := C(\lambda,p,q,w) > 0$ which is independent on $\nu > 0$. Since $q < p$, we conclude that $t_{\lambda,\nu} \to \infty$ as $\nu \to \infty$. Finally, using once more (φ_3), Lemma 2.3.13 and the fact that $\mathsf{t}_{\lambda,\nu}w \in \mathcal{N}^{+}_{\lambda,\nu}$, we obtain that

$$\begin{aligned} \mathcal{E}^{+}_{\lambda,\nu} \le \mathcal{I}_{\lambda,\nu}(\mathsf{t}_{\lambda,\nu}w) &= \mathcal{I}_{\lambda,\nu}(\mathsf{t}_{\lambda,\nu}w) - \frac{1}{p}\mathcal{I}'_{\lambda,\nu}(\mathsf{t}_{\lambda,\nu}w)(\mathsf{t}_{\lambda,\nu}w) \\ &\le \left(1-\frac{\ell}{p}\right)\mathsf{t}^{m}_{\lambda,\nu}\|w\|^{m} + -\nu\mathsf{t}^{q}_{\lambda,\nu}\left(\frac{1}{q}-\frac{1}{p}\right)\|w\|^{q}_{q,a}, \end{aligned} \tag{3.68}$$

for all $\nu > \nu_n(\lambda)$ with $\lambda > 0$ fixed. Therefore, since $\ell \le m < q < p < \ell^{*}_{s}$ and $t_{\lambda,\nu} \to \infty$ as $\nu \to \infty$, we deduce from estimate (3.68) that $\mathcal{E}^{+}_{\lambda,\nu} \to -\infty$ as $\nu \to \infty$ for each $\lambda > 0$ fixed, which ends the proof of item (iii). $\square$

Now we are in position to prove the asymptotic behavior for solution $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ as $\nu \to \infty$.

Proposition 3.4.7. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Then, the weak solutions $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ obtained respectively in Propositions 3.2.29 and 3.2.30 has the following asymptotic behavior:*

(i) For each $\lambda > 0$ fixed, it holds that $u_{\lambda,\nu} \to 0$ in X as $\nu \to \infty$. In particular, $\mathcal{E}^-_{\lambda,\nu} \to 0$ as $\nu \to \infty$.

(ii) For each $\lambda > 0$ fixed, it holds that $\|v_{\lambda,\nu}\| \to \infty$ as $\nu \to \infty$.

Proof. (i) By Proposition 3.4.5 (ii) there exists $u_\lambda \in X$ in such that $u_{\lambda,\nu} \rightharpoonup u_\lambda$ in X. Proceeding as in the proof of Proposition 3.4.1 (i), we also obtain that $u_{\lambda,\nu} \to u_\lambda$ in X as $\nu \to \infty$. But in view of (φ_3), Lemma 2.3.13, the continuous embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ and Lemma 3.4.5 (ii), we have that

$$\nu\|u_{\lambda,\nu}\|^q_{q,a} = \mathcal{J}'_{s,\Phi,V}(u_{\lambda,\nu})u_{\lambda,\nu} + \lambda\|u_{\lambda,\nu}\|^p_p \leq m\max\{\|u_{\lambda,\nu}\|^\ell, \|u_{\lambda,\nu}\|^m\} + \lambda S^p_p\|u_{\lambda,\nu}\|^p \leq C_1,$$

where $C_1 > 0$ is independent on $\nu > \nu_n(\lambda)$. Then, the last inequality gives us that $\|u_{\lambda,\nu}\|_{q,a} \to 0$ as $\nu \to \infty$, proving that $u_\lambda = 0$. Therefore, $u_{\lambda,\nu} \to 0$ in X as $\nu \to \infty$, which ends the proof of item (i).

(ii) By using (φ_3), Lemma 2.3.13 and the fact that $v_{\lambda,\nu} \in \mathcal{N}^+_{\lambda,\nu}$, we infer that

$$\mathcal{E}^+_{\lambda,\nu} = \mathcal{I}_{\lambda,\nu}(v_{\lambda,\nu}) - \frac{1}{q}\mathcal{I}'_{\lambda,\nu}(v_{\lambda,\nu})(v_{\lambda,\nu}) \geq \left(1 - \frac{m}{p}\right)\mathcal{J}_{s,\Phi,V}(v_{\lambda,\nu}) + \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\|v_{\lambda,\nu}\|^p_p, \tag{3.69}$$

for all $\nu > \nu_n(\lambda)$ and $\lambda > 0$ fixed. Then, since $m < q < p < \ell^*_s$, it follows from continuous embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ and estimate (3.69) that

$$\mathcal{E}^+_{\lambda,\nu} \geq \lambda S^p_p\left(\frac{1}{p} - \frac{1}{q}\right)\|v_{\lambda,\nu}\|^p.$$

Therefore, combining the last estimate with the Lemma 3.4.6 (iii) we conclude that $\|v_{\lambda,\nu}\| \to \infty$ as $\nu \to \infty$ for each $\lambda > 0$. This finishes the proof. □

3.5 THE EXTREMAL CASES

In this section, our main goal is study the existence of solution for the problem $(\mathcal{P}_{\lambda,\nu})$ taking into account the extremal cases $\nu = \nu_n$ and $\lambda = \lambda_*(= \lambda^*)$. We start recalling that $\mathcal{N}^0_{\lambda,\nu_n}$ is not empty for all $\lambda > 0$.

The case $\nu = \nu_n$

The first existence result when $\nu = \nu_n$ is state as follows:

Proposition 3.5.1. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $\lambda \in (0, \lambda_*)$ and $\nu = \nu_n$. Then, the problem $(\mathcal{P}_{\lambda,\nu})$ has at least one nontrivial solution $u \in \mathcal{N}^0_{\lambda,\nu_n}$.*

Proof. We consider a sequence $(\nu_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\nu_k > \nu_n$ for each $k \in \mathbb{N}$ and $\nu_k \to \nu_n$ as $k \to \infty$. Applying the Theorem 3.1.1, we obtain a sequence $(u_k)_{k\in\mathbb{N}} \in \mathcal{N}^-_{\lambda,\nu_k}$ which is a critical point for the functional $\mathcal{I}_{\lambda,\nu_k}$ and $\mathcal{E}^-_{\lambda,\nu_k} = \mathcal{I}_{\lambda,\nu_k}(u_k)$ for all $k \in \mathbb{N}$. Precisely,

$$\mathcal{I}'_{\lambda,\nu_k}(u_k)w = 0, \quad \text{for all } w \in X \quad \text{and} \quad \mathcal{I}''_{\lambda,\nu_k}(u_k)(u_k, u_k) < 0, \tag{3.70}$$

for all $k \in \mathbb{N}$. Now, by using Proposition 3.2.34 and inequality (3.46), we obtain

$$\mathcal{E}^-_{\lambda,\nu_1} \geq \mathcal{E}^-_{\lambda,\nu_k} = \mathcal{I}_{\lambda,\nu_k}(u_k) \geq \frac{p(q-m) - m(q-\ell)}{pq} \min\{\|u_k\|^\ell, \|u_k\|^m\},$$

Consequently, $(u_k)_{k\in\mathbb{N}}$ is bounded in X. Hence, there exists $u \in X$ such that, up to a subsequence, $u_k \rightharpoonup u$ in X. Using the same ideas employed in the Remark 3.2.22, we deduce that $u \neq 0$. Moreover, by using (3.70), Hölder inequality and compact embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ for each $r \in (m, \ell^*_s)$, we have that

$$\mathcal{J}_{s,\Phi,V}(u_k)(u_k - u) = \nu_k \int_{\mathbb{R}^N} a(x)|u_k|^{q-2}u_k(u_k - u)dx - \int_{\mathbb{R}^N} |u_k|^{p-2}u_k(u_k - u)dx = o_k(1).$$

Thence, by (S_+)-condition (see Proposition 2.3.17), we infer that $u_k \to u$ in X. Therefore, since $\mathcal{I}_{\lambda,\nu}$ is of class C^2, the strong convergence combined with (3.70) imply that $\mathcal{I}''_{\lambda,\nu_n}(u)(u,u) \leq 0$ and $\mathcal{I}'_{\lambda,\nu_n}(u)w = 0$ for all $w \in X$, that is, $u \in \mathcal{N}^-_{\lambda,\nu_n} \cup \mathcal{N}^0_{\lambda,\nu_n}$ and is a critical point for the functional $\mathcal{I}_{\lambda,\nu_n}$. But according to Proposition 3.2.18, we know that $\mathcal{N}^-_{\lambda,\nu_n}$ is empty. This finishes the proof. □

Analogously, we also obtain the following result:

Proposition 3.5.2. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $\lambda \in (0, \lambda^*)$ and $\nu = \nu_n$. Then, the problem $(\mathcal{P}_{\lambda,\nu})$ has at least one nontrivial solution $v \in \mathcal{N}^0_{\lambda,\nu_n}$.*

Proof. Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence such that $\nu_k > \nu_n$ for each $k \in \mathbb{N}$ and $\nu_k \to \nu_n$ as $k \to \infty$. According to Theorem 3.1.2, there exists a sequence $(v_k)_{k\in\mathbb{N}} \in \mathcal{N}^-_{\lambda,\nu_k}$ which is a critical point for the functional $\mathcal{I}_{\lambda,\nu_k}$ and $\mathcal{E}^+_{\lambda,\nu_k} = \mathcal{I}_{\lambda,\nu_k}(v_k)$ for all $k \in \mathbb{N}$. Using the Proposition 3.2.34 and proceeding as in (3.45), we obtain that

$$\mathcal{E}^+_{\lambda,\nu_1} \geq \mathcal{E}^+_{\lambda,\nu_k} = \mathcal{I}_{\lambda,\nu_k}(v_k) \geq \min\{\|u_k\|^\ell, \|u_k\|^m\} - \left[\frac{\nu_k}{q}\|a\|_r\|v_k\|_p^{q-p} - \frac{\lambda}{p}\right]\|v_k\|_p^p.$$

which proves that $(v_k)_{k\in\mathbb{N}}$ is bounded in X. The rest of the proof follows the same ideas discussed in the proof of Proposition 4.2. For this reason, we omit the details. □

Cases $\lambda = \lambda_*$ and $\lambda = \lambda^*$

In this section, we investigate the existence of solution for the problem $(\mathcal{P}_{\lambda,\nu})$ taking into account the cases $\lambda = \lambda_*$ and $\lambda = \lambda^*$. The main strategy is to consider a sequence $(\nu_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\nu_k > \nu_n(\lambda_*)$ for each $k \in \mathbb{N}$ and $\nu_k \to \nu_n$ as $k \to \infty$, in order to apply theorem 3.1.1, and finally control the behavior of $\mathcal{I}_{\lambda,\nu_k}$ when ν_k approaches ν_n.

Proposition 3.5.3. *Assume that (φ_1)-(φ_4), (H_1)-(H_2) and (V_0)-(V_1) hold. Suppose also that $\lambda = \lambda_*$ and $\nu > \nu_n(\lambda_*)$. Then, the problem $(\mathcal{P}_{\lambda,\nu})$ has at least one nontrivial solution $u \in \mathcal{N}^-_{\lambda_*,\nu} \cup \mathcal{N}^0_{\lambda_*,\nu}$.*

Proof. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence such that $\lambda_k < \lambda_*$ and $\lambda_k \to \lambda_*$ as $k \to \infty$. Firstly, arguing as in the proof of Proposition 3.4.1, we deduce that

$$\limsup_{k\to\infty} \nu_n(\lambda_k) \leq \nu_n(\lambda_*) < \nu,$$

which implies that $\nu_n(\lambda_k) < \nu$ for all $k \in \mathbb{N}$ enough large. Then, we can apply the Proposition 3.2.35 to obtain a sequence $(u_k)_{k\in\mathbb{N}} \in \mathcal{N}^-_{\lambda_k,\nu}$ which is a critical point for the functional $\mathcal{I}_{\lambda_k,\nu}$ and $\mathcal{E}^-_{\lambda_k,\nu} = \mathcal{I}_{\lambda_k,\nu}(u_k)$ for all $k \in \mathbb{N}$ enough large. Now, using once more the Proposition 3.2.33, we have that

$$\limsup_{k\to\infty} \mathcal{E}^-_{\lambda_k,\nu} \leq \limsup_{k\to\infty} \mathcal{I}_{\lambda_k,\nu}(\mathsf{t}^-_{\lambda_k,\nu}(u_{\lambda_*,\nu})u_{\lambda_*,\nu}) = \mathcal{I}_{\lambda_*,\nu}(\mathsf{t}^-_{\lambda_*,\nu}(u_{\lambda_*,\nu})u_{\lambda_*,\nu}) = \mathcal{E}^-_{\lambda_*,\nu},$$

where $u_{\lambda_*,\nu} \in \mathcal{N}^-_{\lambda_k,\nu} \cup \mathcal{N}^0_{\lambda_k,\nu}$ is obtained in Proposition 3.2.29. Therefore, by using the same ideas employed in the proof of Proposition 3.5.1, we deduce that there exists $u \in X \setminus \{0\}$ such that $u_k \to u$ in X as $k \to \infty$. Furthermore, taking into account that $\mathcal{I}_{\lambda,\nu}$ is of class C^2, we conclude that $u \in \mathcal{N}^-_{\lambda_*,\nu} \cup \mathcal{N}^0_{\lambda_*,\nu_k}$ and is a critical point for the functional $\mathcal{I}_{\lambda_*,\nu}$. This finishes the proof. □

A similar result is obtained considering the case $\lambda = \lambda^*$.

Proposition 3.5.4. *Assume that* (φ_1)-(φ_4), (H_1)-(H_2) *and* (V_0)-(V_1) *hold. Suppose also that* $\lambda = \lambda^*$ *and* $\nu > \nu_n(\lambda^*)$*. Then, the problem* $(\mathcal{P}_{\lambda,\nu})$ *has at least one nontrivial solution* $v \in \mathcal{N}^+_{\lambda^*,\nu} \cup \mathcal{N}^0_{\lambda^*,\nu}$.

4 A SURVEY ON FRACTIONAL MUSIELAK-SOBOLEV SPACES

In this chapter, we first review some recent results on generalized Φ-functions. We then focus on the relevant functional space structure provided by Musielak-Orlicz spaces. Furthermore, we describe the basic tools and properties used to deal with the general fractional Musielak-Sobolev spaces $W^{s,\Phi_{x,y}}(\Omega)$ associated with Musielak functions

$$\Phi : \Omega \times \Omega \times [0, \infty) \to [0, \infty)$$

that satisfy some suitable assumptions, which will be mentioned later. Finally, we establish some abstract results on these spaces, and then we apply the developed theory to study the existence of solutions to a very general class of nonlocal problem.

In the work of Azroul et al. (2020) it was proved that $W^{s,\Phi_{x,y}}(\Omega)$ is a reflexive Banach space. However, it seems that some important details were forgotten. Moreover, if Φ satisfies the Δ_2-condition and

$$t \mapsto \Phi(x, y, \sqrt{t}) \text{ is convex for all } (x, y) \in \Omega \times \Omega, \tag{4.1}$$

then $W^{s,\Phi_{x,y}}(\Omega)$ is a uniformly convex space. The proofs are inspired by results obtained by Mihăilescu and Rădulescu (2008) for generalized Orlicz-Sobolev spaces.

In the works by Bahrouni, Ounaies and Tavares (2020) and Bahrouni, Bahrouni and Xiang (2020a), the convexity assumption (4.1) has been used by to obtain the (S_+)-property for a suitable functional (see Definition 4.5.7). The (S_+)- property plays an important role in the study of solutions for differential equations in Orlicz-Sobolev and Musielak-Sobolev spaces, as evidenced in the works by Mihăilescu and Rădulescu (2008), Fan (2012), Liu and Zhao (2015) and Liu and Dai (2018). Recently, the (S_+)-property was obtained by Bahrouni, Ounaies and Tavares (2020) and Bahrouni, Bahrouni and Xiang (2020a) for a very wide class of operators associated to the fractional Orlicz-Sobolev spaces. In these works, in order to prove that the space is uniformly convex and the (S_+)-property, the authors assumed Δ_2-condition and the convexity assumption (4.1).

It is important to mention that hypothesis (4.1) is restrictive, since it excludes classical examples of functions with balanced growth, for instance, $\Phi(t) = t^p$ with $1 < p < 2$, which satisfies Δ_2-condition, but does not satisfy (4.1). In this chapter, we will extend these results assuming only that Φ and its conjugate function satisfy the Δ_2-condition.

The aim of this chapter is to extend and complement the previous results on the perspective of the class of fractional Musielak-Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$. In the abstract point of view, our main contributions are the following:

(i) We prove that $W^{s,\Phi_{x,y}}(\Omega)$ is uniformly convex and the Radon-Riesz property with respect to the modular function.

(ii) We prove the (S_+)-property.

(iii) We prove a Brezis-Lieb type Lemma to the modular function and other convergence results.

(iv) We establish several monotonicity properties.

(v) We obtain an existence and uniqueness result for a very general class of nonlocal problems.

It is important to emphasize that items (i)-(iii) are obtained without assuming the convexity assumption (4.1).

It is worth noting that these spaces enable the presence of many particular functions. For instance, we can consider the following special cases:

- *Double-phase growth*: $\Phi_{x,y}(t) = t^p + a(x,y)t^q$ with $1 \leq p < q < \infty$ and $a \in L^\infty(\Omega \times \Omega)$ a non-negative function.
- *Variable exponent*: $\Phi_{x,y}(t) = t^{p(x,y)}$, where $p : \Omega \times \Omega \to (0, \infty)$ is a measurable function lower and upper bounded by constants $1 \leq p^- \leq p^+ < \infty$.
- *Logarithmic perturbation of power*: $\Phi_{x,y}(t) = t^{p(x,y)} \log(1+t)$, where $p \in L^\infty(\Omega \times \Omega)$ is as in the previous example.
- *Anisotropic case*: $\Phi_{x,y}(t) = a(x,y)t^p$, with $1 \leq p < \infty$ and $a \in L^\infty(\Omega \times \Omega)$ satisfying $0 < a^- \leq a(x,y) \leq a^+ < \infty$.

The chapter is organized as follows. In the forthcoming section, we collect some preliminary results about Φ-function and we establish some notation that will be used throughout this chapter. The Section 4.2 is devoted to a review of Musielak-Orlicz spaces and their main properties. In Section 4.3, we introduce Musielak-Sobolev fractional spaces and show some standard results that require additional assumptions about the function Φ. The Section 4.4 is dedicated to Uniform convexity e Radon-Riesz property with respect to the modular function. In the Section 4.5, we prove Brezis-Lieb type Lemma to the modular function, monotonicity results and (S_+)-property. In the Section 4.6, we apply a monotonicity result to establish existence and uniqueness of solutions for a very general class of nonlocal problems. Finally, in Section 4.7, we present some examples of functions Φ for which the existence result may be applied.

4.1 MUSIELAK FUNCTIONS

Next, we present a generalization of the concept of N-functions in such way they can depend on the spatial variable. Let (Ω, Σ, μ) be a complete measure space satisfying the natural assumption that our measure μ is not identically zero or infinity.

We start by recalling some definitions and properties employed by Diening et al. (2017), Harjulehto and Hästö (2019a) and Musielak (1983), which are needed to prove the uniform convexity of the fractional Musielak-Sobolev spaces.

Definition 4.1.1. *A function $g : (0, \infty) \to \mathbb{R}$ is said to be almost increasing if there exists a constant $a \geq 1$ such that $g(s) \leq ag(t)$, for all $0 < s < t$. Similarly, we define almost decreasing.*

Let $p, q > 0$. The function g is said to satisfy the condition:

(aInc)$_p$ if $t \mapsto \frac{g(t)}{t^p}$ is almost increasing.

(aDec)$_q$ if $t \mapsto \frac{g(t)}{t^q}$ is almost decreasing.

When $a = 1$, we use the notation and (Inc)$_p$ and (Dec)$_q$.

Definition 4.1.2. *We say that a function $\Phi : \Omega \times [0, \infty) \to [0, \infty]$ is a generalized Φ-prefunction if $x \mapsto \Phi(x, |u(x)|)$ is μ-measurable for all measurable function $u : \Omega \to \mathbb{R}$,*

$$\lim_{t \to 0^+} \Phi(x, t) = \Phi(x, 0) = 0 \quad \textit{and} \quad \lim_{t \to \infty} \Phi(x, t) = \infty, \quad \mu\text{-a.e. } x \in \Omega.$$

A generalized Φ-prefunction Φ is said to be a

(i) generalized weak Φ-function if $t \mapsto \Phi(x, t)$ satisfies (aInc)$_1$ for μ-a.e. $x \in \Omega$.

(ii) generalized Φ-function if $t \mapsto \Phi(x, t)$ is left-continuous and convex for μ-a.e. $x \in \Omega$.

(iii) Musielak function if $t \mapsto \Phi(x,t)$ it is finite, convex and positive for $t > 0$ and μ-a.e. $x \in \Omega$.

When Φ is independent of the spatial variable x, that is, $\Phi(x,t) = \Phi(t)$, Φ is called a weak Φ-function, Φ-function or Orlicz function, respectively. The sets of generalized weak Φ-function, Φ-function and Musielak function are denoted by $\Phi_w(\Omega,\mu)$ and $\Phi(\Omega,\mu)$ and $\Phi_M(\Omega,\mu)$, respectively.

Remark 4.1.3. *Next, we present some remarks regarding these classes of functions.*

(i) If $\Phi(x,\cdot)$ is convex and $\Phi(x,0) = 0$, then it is not hard to see that $\Phi(x,\cdot)$ is increasing for μ-a.e $x \in \Omega$ and satisfies (Inc)$_1$. Therefore, we have that

$$\Phi_M(\Omega,\mu) \subset \Phi(\Omega,\mu) \subset \Phi_w(\Omega,\mu).$$

(ii) Equivalently, a Musielak function can be represented for μ-almost every point as follows:

$$\Phi(x,t) = \int_0^t \phi(x,\tau)\, d\tau,$$

where $\phi(x,\cdot)$ is the right-hand derivative of $\Phi(x,\cdot)$. Moreover, $\phi(x,\cdot)$ is positive, right-continuous and increasing in $(0,\infty)$.

(iii) In some references, the generalized Φ-function is also called of the generalized Young function. Note that it can be zero for $t > 0$ and μ-a.e. $x \in \Omega$.

Example 4.1.4. *Here are some examples of functions.*

(i) Let $\Phi_p, \Phi_\infty : [0,\infty) \to [0,\infty]$ be functions given by

$$\Phi_p(t) = \begin{cases} \dfrac{t^p}{p}, & \text{if } p \in [1,\infty), \\ \infty \cdot \chi_{(1,\infty)}(t), & \text{if } p = \infty. \end{cases}$$

We have that Φ_p is a Φ-function for all $p \in [1,\infty]$. However, for each $p \in [1,\infty)$, Φ_p is continuous and positive, while Φ_∞ is only left-continuous but not positive. Namely, Φ_p with $p \in [0,\infty)$ is a Orlicz function, while Φ_∞ is only a Φ-function.

(ii) Let $p : \Omega \to [0,\infty]$ and $a : \Omega \to [0,\infty)$ be measurable functions and let $1 \le r < q < \infty$. We denote $p_\infty := \limsup_{|x|\to\infty} p(x)$ and $t^\infty := \infty \cdot \chi_{(1,\infty)}(t)$. For $t \geq 1$, we define the following functions:

(a) $\Phi_1(x,t) = a(x) t^{p(x)}$.

(b) $\Phi_2(x,t) = t^{p(x)} \log(1+t)$.

(c) $\Psi_1(x,t) = \min\{t^{p(x)}, t^{p_\infty}\}$.

(d) $\Psi_2(x,t) = t^r + a(x)t^q$.

We point out that $\Psi_1 \in \Phi_w(\Omega,\mu)\backslash\Phi(\Omega,\mu)$ *if* p *is non-constant. Moreover,* $\Phi_2, \Psi_2 \in \Phi_M(\Omega,\mu)$ *if* p *is finite* μ*-a.e., while* $\Phi_1, \Phi_2 \in \Phi(\Omega,\mu) \setminus \Phi_M(\Omega,\mu)$ *when* a *positive and* $p = \infty$ μ*-a.e..*

Definition 4.1.5. *Two weak* Φ*-functions* $\Phi, \Psi : \Omega \times [0,\infty) \to [0,\infty]$ *are said to be equivalent in the sense of Young, denoted by* $\Phi \simeq \Psi$*, if there exists* $L \geq 1$ *such that*

$$\Phi(x, t/L) \leq \Psi(x,t) \leq \Phi(x, Lt), \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

Definition 4.1.6. *We say that* $\Phi \in \Phi_w(\Omega,\mu)$ *satisfies the* Δ_2*-condition if there exists a constant* $K > 0$ *such that*

$$\Phi(x, 2t) \leq K\Phi(x,t), \quad \text{for } \mu\text{-a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.$$

We emphasize that $(\text{aDec})_q$ is a qualitative version of Δ_2-condition.

Proposition 4.1.7. *If* $\Phi \in \Phi_w(\Omega,\mu)$*, then* Δ_2 *is equivalent to* $(\text{aDec})_q$ *for some* $q \geq 1$*. In particular, if* $\Phi \in \Phi(\Omega,\mu)$*, then* Δ_2 *is equivalent to* $(\text{Dec})_q$ *for some* $q \geq 1$*.*

Definition 4.1.8. *For* $\Phi \in \Phi_w(\Omega,\mu)$*, the function* $\widetilde{\Phi} : \Omega \times [0,\infty) \to [0,\infty]$ *defined by*

$$\widetilde{\Phi}(x,t) := \sup_{s \geq 0}(ts - \Phi(x,s)), \quad \text{for } \mu\text{-a.e. } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{4.2}$$

is called the complementary or conjugate function of Φ *in the sense of Young.*

Remark 4.1.9. *Here are some observations regarding the conjugate function.*

(i) We point out that although the function $\widetilde{\Phi}(x,\cdot)$ *is convex, it does not necessarily have finite values and can vanish at* $t > 0$*. For example, if* $\Phi(x,t) = t$*, then* $\widetilde{\Phi}(x,t) = \infty \cdot \chi_{(1,\infty)}(t)$*.*

(ii) If $\widetilde{\Phi}(x,\cdot)$ *is finite for* μ*-a.e.* $x \in \Omega$*, it is also a Musielak function and can be represented as follows:*

$$\widetilde{\Phi}(x,t) = \int_0^t \widetilde{\phi}(x,\tau)\, d\tau,$$

where

$$\widetilde{\phi}(x,t) = \sup\{s : \phi(x,s) \leq t\}, \quad \text{for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.$$

If $\phi(x,\cdot)$ *it is strictly increasing and continuous, then* $\widetilde{\phi}(x,\cdot)$ *is the inverse of* $\phi(x,\cdot)$ *for* μ*-a.e.* $x \in \Omega$*. Furthermore, in view of definition* (4.2) *one may deduce the Young's type inequality*

$$st \leq \Phi(x,s) + \widetilde{\Phi}(x,t), \quad \text{for } \mu\text{-a.e. } x \in \Omega \text{ and } s,t \geq 0. \tag{4.3}$$

In general, we have the following results that can be found in (HARJULEHTO; HÄSTÖ, 2019b).

Proposition 4.1.10. *Let $\Phi \in \mathit{\Phi}_w(\Omega, \mu)$. Then,*

(i) $\widetilde{\Phi} \in \mathit{\Phi}(\Omega, \mu)$ *and* $\widetilde{\widetilde{\Phi}} \simeq \Phi$. *In particular, if* $\Phi \in \mathit{\Phi}(\Omega, \mu)$*, then* $\widetilde{\widetilde{\Phi}} = \Phi$ *and*

$$\Phi(x,t) := \sup_{s \geq 0}(ts - \widetilde{\Phi}(x,s)), \quad \textit{for } \mu\textit{-a.e. } x \in \Omega \textit{ and } t \geq 0.$$

(ii) if $t \mapsto \Phi(x,t)$ *satisfies (aInc)*$_p$ *with* $p \geq 1$*, then there exists* $\Psi \in \mathit{\Phi}(\Omega, \mu)$ *such that* $\Phi \simeq \Psi$ *and* $\Psi^{\frac{1}{p}}$ *is convex.*

Another especially useful function, called generalized N-function (N *stands for nice*), is defined below.

Definition 4.1.11. *Let Ω be an open set in $\mathbb{R}^N$. A function $\Phi : \Omega \times \mathbb{R} \to [0, \infty)$ is said to be a generalized N-function if satisfies the following conditions:*

(i) $\Phi(\cdot, t)$ *is measurable for all* $t \geq 0$.

(ii) $\Phi(x, \cdot)$ *is* N*-function for* μ*-a.e.* $x \in \Omega$.

We denote by $\mathcal{N}(\Omega, \mu)$ the set of all generalized N-functions defined on Ω. If Ω is an open subset of $\mathbb{R}^N$, Σ is the Borel σ-algebra and μ is the Lebesgue measure, we abbreviate this notation as $\mathcal{N}(\Omega)$ or we simply say that it is a generalized N-function on Ω.

Remark 4.1.12. *It is important to stress that, if we consider the even extension to $\mathbb{R}$ of a function $\Phi \in \mathit{\Phi}_M(\Omega, \mu)$, that is, $\Phi(x,t) = \Phi(x,-t)$ for all $t < 0$, then any generalized N-function is also a Musielak function. Moreover, $\Phi \in \mathcal{N}(\Omega, \mu)$ implies $\widetilde{\Phi} \in \mathcal{N}(\Omega, \mu)$.*

Although the presence of spatial variable adds several technical difficulties that will be mentioned throughout this chapter, some definitions and properties that were presented in Chapter 2 are extended point-wise uniformly to the generalized N-function case.

The following result establishes conditions equivalent to the Δ_2-condition.

Lemma 4.1.13. *Let $\Phi, \widetilde{\Phi} \in \mathit{\Phi}_M(\Omega, \mu)$ be generalized conjugate N-functions. Then, the following statements are equivalents:*

(i) Φ *satisfies* Δ_2*-condition.*

(ii) There exists $1 < m < \infty$ *such that*

$$\operatorname*{ess\,sup}_{x\in\Omega} \frac{\phi(x,t)t}{\Phi(x,t)} \leq m, \quad \textit{for all } t > 0.$$

(iii) There exists $1 < \widetilde{\ell} < \infty$ *such that*

$$\widetilde{\ell} \leq \operatorname*{ess\,inf}_{x\in\Omega} \frac{\widetilde{\phi}(x,t)t}{\widetilde{\Phi}(x,t)}, \quad \textit{for all } t > 0.$$

Summarizing, Φ and $\widetilde{\Phi}$ also satisfy the Δ_2-condition if, and only if, there exist $\ell, m \in (1,\infty)$ such that

$$\ell \leq \frac{\phi(x,t)t}{\Phi(x,t)} \leq m \quad \text{for a.e. } x \in \Omega \text{ and } t \neq 0. \tag{4.4}$$

4.2 MUSIELAK-ORLICZ SPACES

In this section, we present the Musielak-Orlicz spaces. Such spaces represent, at the same time, a generalization of Orlicz spaces presented in the chapter 2 and Lebesgue spaces with variable exponent employed in the monograph of Diening et al. (2017).

For the sake of completeness, we first present general semimodular spaces investigated by Nakano (1950), as well as a fundamental concept in the theory of Banach function spaces. After this, we will consider spaces where modular is generated by the appropriate $\varPhi$-function, the called Musielak-Orlicz space.

Definition 4.2.1. *Let X be a vector space defined over $\mathbb{R}$.*

(i) A function $J : X \to [0,\infty]$ is said to be a semimodular on X if the following properties hold:

(a) $\mathcal{J}$ is even, convex and left-continuous, that is, $\lim\limits_{\lambda\to 1^-} J(\lambda x) = J(x)$ for all $x \in X$.

(b) $J(0) = 0$ and if $J(\lambda x) = 0$ for all $\lambda > 0$, then $x = 0$.

(ii) A semimodular J is said to be continuous if $\lim\limits_{\lambda\to 1} J(\lambda x) = J(x)$ for all $x \in X$.

(iii) A semimodular J is said to be a modular if $J(x) = 0$ implies $x = 0$.

(iv) A semimodular J satisfies the Δ_2-condition if there exists a constant $K > 0$ such that $J(2x) \leq KJ(x)$ for all $x \in X$.

(v) If J is a semimodular or modular on X, then the set

$$X_J := \{x \in X : \mathcal{J}(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

is called a semimodular space or modular space, respectively.

Proposition 4.2.2. *Let J be a semimodular on X. Then, X_J is a normed space when endowed with the so-called Luxemburg norm defined by*

$$\|x\|_J := \inf\left\{\lambda > 0 : J\left(\frac{x}{\lambda}\right) \leq 1\right\},$$

where the infimum of the empty set is by definition infinity.

Definition 4.2.3. *Let $(X, \|\cdot\|_X)$ be a normed space with $X \subset L^0(\Omega, \mu)$. The set*

$$X' := \left\{v \in L^0(\Omega, \mu) : \sup_{u \in X, \|u\|_X \leq 1} \int_\Omega |uv|\, d\mu < \infty\right\},$$

with the norm

$$\|v\|_{X'} := \sup_{u \in X, \|u\|_X \leq 1} \int_\Omega |uv|\, d\mu,$$

is called the associate space of X.

The space X is said to be a Banach function space, if the following conditions hold:

(i) $(X, \|\cdot\|_X)$ is circular: $\|u\|_X = \||u|\|_X$ for all u.

(ii) $(X, \|\cdot\|_X)$ is Solid: If $u \in X, v \in L^0(\Omega, \mu)$ and $0 \leq |v| \leq |u|$, then $v \in X$ and $\|v\|_X \leq \|u\|_X$.

(iii) $(X, \|\cdot\|_X)$ satisfies Fatou property: If $|u_k| \nearrow |u|$ μ-a.e. with $(u_k)_{k\in\mathbb{N}} \subset X$ and $\sup_{k\in\mathbb{N}} \|u_k\|_X < \infty$, then $u \in X$ and $\|u_k\|_X \nearrow \|u\|_X$.

(iv) If $A \in \Sigma$ and $\mu(A) < \infty$, then $\chi_A \in X$

(v) If $A \in \Sigma$ and $\mu(A) < \infty$, then $\chi_A \in X'$.

Given a function $\Phi \in \mathit{\Phi}(\Omega, \mu)$, the *Musielak-Orlicz space* is defined as the modular space

$$L^{\Phi_x}(\Omega, \mu) := (L^0(\Omega, \mu))_{J_\Phi} = \left\{u \in L^0(\Omega, \mu) : J_\Phi(\lambda u) < \infty \text{ for some } \lambda > 0\right\},$$

where the modular function $J_\Phi : L^0(\Omega, \mu) \to [0, \infty]$ is defined by

$$J_\Phi(u) = \int_\Omega \Phi(x, |u(x)|)\, d\mu.$$

The Musielak-Orlicz spaces are also called *generalized Orlicz spaces* in the literature. They provide a good framework when endowed with the Luxemburg norm

$$\|u\|_{\Phi_x} := \inf\left\{\lambda > 0 : J_\Phi\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Using the Young's type inequality (4.3) for Φ and $\widetilde{\Phi}$, one may deduce the following Hölder's type inequality

$$\left|\int_\Omega uv\, dx\right| \leq 2\|u\|_{\Phi_x}\|v\|_{\widetilde{\Phi}_x},$$

for all $u \in L^{\Phi_x}(\Omega)$ and $v \in L^{\widetilde{\Phi}_x}(\Omega)$, see (DIENING et al., 2017, Lemma 2.6.5).

On the other hand, additional hypotheses are needed on $\mathit{\Phi}$-functions for several classical Real Analysis results to be valid in the associated Musielak-Orlicz spaces. In fact, in several references, we can find that $L^{\Phi_x}(\Omega, \mu)$ is a Banach space (complete space) if μ is σ-finite. See, for instance, (MUSIELAK, 1983) and (DIENING et al., 2017). However, if this assumption is relaxed, it is needed to require the *proper* condition on the generalized $\mathit{\Phi}$-function defined below. In this case, $L^{\Phi_x}(\Omega, \mu)$ will be a Banach function space, see (MÉNDEZ; LANG, 2019).

Definition 4.2.4. *A function $\Phi \in \mathit{\Phi}(\Omega, \mu)$ (or the space $L^{\Phi_x}(\Omega, \mu)$) is said to be proper if the following conditions hold:*

(i) If $A \in \Sigma$ and $\mu(A) < \infty$, then $\chi_A \in L^{\Phi_x}(\Omega, \mu)$.

(ii) If $A \in \Sigma$ and $\mu(A) < \infty$, then there exists a constant $C(A) > 0$ such that

$$\int_\Omega \chi_A(x)u(x)\, d\mu \leq C(A)\|u\|_{\Phi_x},$$

for all $u \in L^{\Phi_x}(\Omega, \mu)$.

Under the above condition, we can declare the following result:

Proposition 4.2.5. *Let (Ω, Σ, μ) be a measure space and $\Phi \in \mathit{\Phi}(\Omega, \mu)$. Then, the following assertions hold:*

(i) If (Ω, Σ, μ) is σ-finite, then $L^{\Phi_x}(\Omega, \mu)$ is a Banach space.

(ii) If $\Phi \in \mathit{\Phi}(\Omega, \mu)$ is proper, then $L^{\Phi_x}(\Omega, \mu)$ is a Banach function space. Consequently, it is a Banach space.

In general, we have that the properties $(i), (ii), (iii)$ in Definition 4.2.3 always hold. Actually, as pointed out by Harjulehto and Hästö (2019b), a sufficient condition for Φ to be proper, that is, the other two hold, is the *weight condition* (A_0) defined as follows.

Definition 4.2.6. *A function $\Phi \in \mathit{\Phi}(\Omega,\mu)$ is said to satisfy the condition* (A$_0$) *if there exists $\beta \in (0,1]$ such that*

$$\beta \le \Phi^{-1}(x,1) \le \frac{1}{\beta} \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{A$_0$}$$

where $\Phi^{-1}(x,\cdot)$ is the left-continuous inverse of $\Phi(x,\cdot)$ defined by

$$\Phi^{-1}(x,t) = \inf\{s \ge 0 : \Phi(x,s) \ge t\} \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

One has that

$$\Phi(x,\Phi^{-1}(x,t)) = t \quad \text{and} \quad \Phi^{-1}(x,\Phi(x,t)) \le t \quad \text{for } t \ge 0 \text{ and } \mu\text{-a.e. } x \in \Omega.$$

Next, we present an equivalent formulation of the condition (A$_0$) and some consequences.

Proposition 4.2.7. *Let $\Phi \in \mathit{\Phi}(\Omega,\mu)$. Then, the following assertions hold:*

(i) Φ satisfies (A$_0$) *if and only if there exists $\alpha \in (0,1]$ such $\Phi(x,\alpha) \le 1 \le \Phi(x,1/\alpha)$ for μ-a.e. $x \in \Omega$.*

(ii) If there exists $\lambda > 0$ such that $0 < C_1 \le \Phi(x,\lambda) \le C_2 < \infty$, then Φ satisfies (A$_0$).

(iii) If Φ satisfies (A$_0$)*, then $\widetilde{\Phi}$ satisfies* (A$_0$).

(iv) If Ω has measure finite and Φ satisfies (A$_0$) *and (aInc)$_p$ with $p \in [1,\infty)$, Then, $L^{\Phi_x}(\Omega,\mu) \hookrightarrow L^p(\Omega,\mu)$.*

The results in the following Proposition are well known and can be found in (HARJULEHTO; HÄSTÖ, 2019b).

Lemma 4.2.8. *Let $\Phi, \Psi \in \mathit{\Phi}(\Omega,\mu)$. Then, the following statements hold:*

(i) $J_\Phi(u) \le 1$ if and only if $\|u\|_{\Phi_x} \le 1$.

(ii) If $\Phi \simeq \Psi$, then $L^{\Phi_x}(\Omega,\mu) = L^{\Psi_x}(\Omega,\mu)$ and the norms $\|\cdot\|_{\Phi_x}$ and $\|\cdot\|_{\Psi_x}$ are equivalents.

From now on, we present some basic properties of Musielak-Orlicz spaces that require some additional structure. It is natural that some restrictions are necessary, since these properties do not even hold for the full of classical Lebesgue spaces.

We first need the following concept of uniform convexity for the $\mathit{\Phi}$-function and semimodular.

Definition 4.2.9. *A function $\Phi \in \mathit{\Phi}(\Omega,\mu)$ is said to be uniformly convex if, for given $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) \in (0,1)$ such that*

$$\Phi\left(x, \frac{s+t}{2}\right) \leq (1-\delta)\frac{\Phi(x,s)+\Phi(x,t)}{2} \quad \text{for } \mu\text{-a.e. } x \in \Omega,$$

and for all $s,t \in [0,\infty)$ with $|s-t| > \varepsilon \max\{|s|,|t|\}$.

This definition can be formulated for values in $\mathbb{R}$ as follows:

Lemma 4.2.10. *Let $\Phi \in \mathit{\Phi}(\Omega,\mu)$ be uniformly convex. Then, for all $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that*

$$\Phi\left(x, \left|\frac{s+t}{2}\right|\right) \leq (1-\delta)\frac{\Phi(x,|s|)+\Phi(x,|t|)}{2} \quad \text{for } \mu\text{-a.e. } x \in \Omega,$$

and for all $s,t \in \mathbb{R}$ with $|s-t| > \varepsilon \max\{|s|,|t|\}$.

Example 4.2.11. *It is well-known that if $\Phi(x,t) = t^p$ with $p \in (1,\infty)$, then Φ is uniformly convex, see (DIENING et al., 2017, Remark 2.4.6).*

Definition 4.2.12. *A semimodular function J on a space X is called uniformly convex if for all $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) \in (0,1)$ such that*

$$J\left(\frac{u-v}{2}\right) \leq \varepsilon\frac{J(u)+J(v)}{2} \quad \text{or} \quad J\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{J(u)+\mathcal{J}(v)}{2}, \tag{4.5}$$

for all $u,v \in X_J$.

The following results provide a relationship between the uniform convexity of the space and the semimodular function, and can be found in (DIENING et al., 2017).

Proposition 4.2.13. *Let J be a uniformly convex semimodular on X satisfying Δ_2-condition. Then, the following statements hold:*

(i) X_J is uniformly convex with respect to Luxemburg norm.

(ii) If $x_k \rightharpoonup x$ in X_J, $J(x_k) \to J(x)$ and $J(x) < \infty$, then $J(\lambda(x_k - x)) \to 0$ for all $\lambda > 0$.

Proposition 4.2.14. *Let $\Phi \in \mathit{\Phi}(\Omega,\mu)$ be uniformly convex. Then, J_Φ is uniformly convex. Consequently, $(L^{\Phi_x}(\Omega,\mu), \|\cdot\|_{\Phi_x})$ is uniformly convex if Φ satisfies Δ_2-condition.*

Another paramount result obtained by Harjulehto and Hästö (2019b) describes that, even though non-convex in general, every weak $\mathit{\Phi}$ function can be described in terms of an equivalent $\mathit{\Phi}$-function uniformly convex under some conditions.

Proposition 4.2.15. *Let $\Phi \in \mathit{\Phi}_w(\Omega, \mu)$. Then, the following conditions are equivalent:*

(i) Φ *is equivalent to a uniformly convex* $\Psi \in \mathit{\Phi}(\Omega, \mu)$.

(ii) $t \mapsto \Phi(x,t)$ *satisfies (aInc)$_p$ for some* $p > 1$.

(ii) $\widetilde{\Phi}$ *satisfies* Δ_2*-condition.*

As a consequence of Propositions 4.2.14, 4.2.15 and 4.2.5, we deduce that for a measurable space not necessarily σ-finite, every Musielak space is a Reflexive Banach space if Φ and it is conjugate function satisfy Δ_2-condition.

Proposition 4.2.16. *Let* $\Phi \in \mathit{\Phi}(\Omega, \mu)$ *be proper. If* Φ *and* $\widetilde{\Phi}$ *satisfy* Δ_2*-condition, then* $L^{\Phi_x}(\Omega, \mu)$ *is a Reflexive Banach space.*

We now present conditions that guarantee the separability of spaces. Such property is tied to both the structure of the measure and of the Musielak function.

Firstly, we point out that under Δ_2-condition, the separability of the measure implies separability of the space, but it is necessary an additional property under the generalized $\mathit{\Phi}$-function (MÉNDEZ; LANG, 2019, Theorem 2.3.2), called local integrability and is defined as follows.

Definition 4.2.17. *A function* $\Phi \in \mathit{\Phi}(\Omega, \mu)$ *is said to be locally integrable if, for every* $\lambda > 0$ *and every subset* $A \subset \Omega$ *with* $\mu(A) < \infty$*, one has*

$$\int_A \Phi(x, \lambda)\, d\mu < \infty.$$

Remark 4.2.18. *Below are some key observations about the Integrability local.*

(i) In general, a simple function does not necessarily belong to $L^{\Phi_x}(\Omega, \mu)$*. For example, if* $\Omega = (0,1)$ *and* $\Phi : (0,1) \times [0, \infty) \to [0, \infty)$ *is defined by* $\Phi(x,t) = \frac{t}{|x|}$*, then* Φ *is a* $\mathit{\Phi}$*-function, but* $u \equiv 1 \notin L^{\Phi_x}(0,1)$*. Indeed, we have*

$$\int_0^1 \Phi(x, \lambda|f|)\, dx = \int_0^1 \frac{\lambda}{x}\, dx = \infty,$$

for all $\lambda > 0$.

(ii) If $\Phi \in \mathit{\Phi}(\Omega, \mu)$ *satisfies* Δ_2*-condition and there exists* $\alpha \in (0,1]$ *such that* $\Phi(x, \alpha) \leq 1$ *for* μ*-a.e.* $x \in \Omega$*, then* Φ *is locally integrable.*

(iii) When $\Phi \in \mathit{\Phi}(\Omega, \mu)$ *is locally integrable, we have that the set of simple functions is contained in* $L^{\Phi_x}(\Omega, \mu)$ *if* Φ*-satisfies* Δ_2*-condition. Moreover, this property is used to approximate a function in* $L^{\Phi_x}(\Omega, \mu)$ *by simple functions.*

On the other hand, if we drop local integrability, we have to assume that the measure is σ-finite, see (HARJULEHTO; HÄSTÖ, 2019b). In summary, we can state the following result:

Proposition 4.2.19. *Let (Ω, Σ, μ) be a measure space separable. If $\Phi \in \mathit{\Phi}(\Omega, \mu)$ satisfies Δ_2-condition, then, $L^{\Phi}(\Omega, \mu)$ is separable if one of the following conditions holds:*

(i) Φ is locally integrable.

(ii) μ is σ-finite.

The next result is a generalization of the well known Riesz representation theorem to the classical Lebesgue spaces.

Proposition 4.2.20. *(MÉNDEZ; LANG, 2019, Theorem 2.4.4) Let (Ω, Σ, μ) be σ-finite and let $\Phi \in \mathit{\Phi}_M(\Omega, \mu)$ be proper and locally integrable. Then, the linear functional*

$$\begin{aligned} \mathcal{T} : L^{\widetilde{\Phi}_x}(\Omega, \mu) &\to \left(L^{\Phi_x}(\Omega, \mu)\right)^* \\ \langle \mathcal{T}(v), u \rangle &= \int_\Omega u(x)v(x)\, d\mu \end{aligned}$$

is an isomorphism if and only if Φ satisfies Δ_2-condition.

Remark 4.2.21. *It is worthwhile to mention that the Musielak-Orlicz space includes as examples the classical Lebesgue spaces $L^p(\Omega, \mu)$ with $1 \leq p \leq \infty$. However, most of the time we work with the class of Musielak or generalized N-functions that have better properties. On the other hand, the special cases $p = 1$ and $p = \infty$ are not covered, because, just like in the classical theory, it is also often treated differently.*

We finish this section with a version of Lemmas 2.1.5 and 2.2.7 for Musielak-Orlicz case that will be used ahead.

Lemma 4.2.22. *Assume that $\Phi \in \Phi_M(\Omega, \mu)$ satisfies (4.4). Then, the following estimates hold:*

(i) $\xi_0^-(\sigma)\Phi(x,t) \leq \Phi(x, \sigma t) \leq \xi_0^+(\sigma)\Phi(x,t)$, for a.e. $x \in \Omega$ and $\sigma, t \geq 0$.

(ii) $\xi_0^-(\|u\|_{\Phi_x}) \leq J_\Phi(u) \leq \xi_0^+(\|u\|_{\Phi_x})$, for all $u \in L^{\Phi_x}(\Omega)$.

(iii) $\xi_1^-(\sigma)\widetilde{\Phi}(x,t) \leq \widetilde{\Phi}(x, \sigma t) \leq \xi_1^+(\sigma)\widetilde{\Phi}(x,t)$, for a.e. $x \in \Omega$ and $\sigma, t \geq 0$.

(iv) $\xi_1^-(\|u\|_{\widetilde{\Phi}_x}) \leq J_{\widetilde{\Phi}}(u) \leq \xi_1^+(\|u\|_{\widetilde{\Phi}_x})$, for all $u \in L^{\widetilde{\Phi}_x}(\Omega)$.

Recall that we use the following notation:

$$\xi_0^-(t) = \min\{t^\ell, t^m\}, \quad \xi_0^+(t) = \max\{t^\ell, t^m\},$$
$$\xi_1^-(t) = \min\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\}, \quad \xi_1^+(t) = \max\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\}, \quad t \geq 0,$$

where $\widetilde{\ell} = \frac{\ell}{\ell-1}$ *and* $\widetilde{m} = \frac{m}{m-1}$.

4.3 FRACTIONAL MUSIELAK-SOBOLEV SPACES

In this section, we introduce some preliminary concepts about the fractional Musielak-Sobolev spaces. For a more complete discuss on this subject, we refer the readers to works (AZROUL et al., 2020; AZROUL et al., 2021).

Given a Musielak function $\Phi \in \mathit{\Phi}_M(\Omega\times\Omega)$, we consider the Musielak-function $\widehat{\Phi} : \Omega\times\mathbb{R} \to [0,\infty)$ given by

$$\widehat{\Phi}(x,t) := \Phi(x,x,t) = \int_0^t \widehat{\phi}(x,\tau)\, d\tau,$$

where $\widehat{\phi}(x,t) = \phi(x,x,t)$ for any $(x,t) \in \Omega \times \mathbb{R}$. Given a parameter $s \in (0,1)$, the *fractional Musielak-Sobolev* space is defined as follows

$$W^{s,\Phi_{x,y}}(\Omega) = \left\{ u \in L^{\widehat{\Phi}_x}(\Omega) : J_{s,\Phi}\left(\lambda u\right) < \infty \text{ for some } \lambda > 0 \right\},$$

where the semimodular function $J_{s,\Phi}$ is defined by

$$J_{s,\Phi}(u) := \int_\Omega \int_\Omega \Phi\left(x, y, |D_s u(x,y)|\right) d\mu, \quad \text{for } s \in (0,1),$$

and the s-Hölder quotient $D_s u$ and the measure μ are defined as

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^s} \quad \text{and} \quad d\mu := \frac{dxdy}{|x-y|^N}.$$

We have known that μ is not a regular Borel measure on the set $\Omega \times \Omega$.

The space $W^{s,\Phi_{x,y}}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{s,\Phi_{x,y}}(\Omega)} := \|u\|_{\widehat{\Phi}_x} + [u]_{s,\Phi_{x,y}},$$

where the term $[\cdot]_{s,\Phi_{x,y}}$ is the so called $(s,\Phi_{x,y})$-*Gagliardo seminorm* defined by

$$[u]_{s,\Phi_{x,y}} := \inf\left\{ \lambda > 0 : J_{s,\Phi}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

In light of Proposition 2.1 in Azroul et al. (2020), $\|\cdot\|_{W^{s,\Phi_{x,y}}(\Omega)}$ is equivalent to Luxemburg norm given by

$$\|u\|_{(\Omega)} := \inf\left\{ \lambda > 0 : \mathcal{J}_{s,\Phi}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

with the relation

$$\frac{1}{2}\|u\|_{W^{s,\Phi_{x,y}}(\Omega)} \leq \|u\|_{(\Omega)} \leq 2\|u\|_{W^{s,\Phi_{x,y}}(\Omega)}, \quad \text{for all } u \in W^{s,\Phi_{x,y}}(\Omega),$$

where the modular function $\mathcal{J}_{s,\Phi} : L^{\widehat{\Phi}_x}(\Omega) \to [0,\infty]$ is defined by

$$\mathcal{J}_{s,\Phi}(u) = J_{\widehat{\Phi}}(u) + J_{s,\Phi}(u).$$

Furthermore, It follows from definition that a function $u \in L_{\widehat{\Phi}_x}(\Omega)$ belongs to $W^{s,\Phi_{x,y}}(\Omega)$ if and only if $D_s u \in L^{\Phi_{x,y}}(d\mu) := L^{\Phi_{x,y}}(\Omega \times \Omega, d\mu)$ and $[u]_{s,\Phi_{x,y}} = \|D_s u\|_{L^{\Phi_{x,y}}(d\mu)}$.

Remark 4.3.1. *Note that for the case $\Phi_{x,y}(t) = \Phi(t)$, that is, when Φ is independent of the spatial variables x and y, we have that $L^{\Phi}(\Omega)$ and $W^{s,\Phi}(\Omega)$ are Orlicz spaces and fractional Orlicz-Sobolev spaces, respectively.*

It is worthwhile to mention that some details in respect to proof of basic results seems to be missed in the previous works on the subject in the literature. For instance, in the paper of Azroul et al. (2020) it is pointed out that it is not necessary to suppose that $\inf_{x,y\in\Omega} \Phi_{x,y}(t) > 0$ to prove that $W^{s,\Phi_{x,y}}(\Omega)$ is Banach space. However, it seems not clear that the measure μ has the σ-finite property. For this reason, we introduce an alternative proof by assuming that the Musielak-function satisfies the weighted condition (A_0).

Proposition 4.3.2. *Assume that $\Phi \in \mathit{\Phi}_M(\Omega \times \Omega)$ satisfies* (A_0)*. Then, $W^{s,\Phi_{x,y}}(\Omega)$ is a Banach space. Moreover, $W^{s,\Phi_{x,y}}(\Omega)$ is reflexive if Φ and $\widetilde{\Phi}$ satisfy the Δ_2-condition.*

Proof. In order to deduce these properties, let us consider the linear operator $T : W^{s,\Phi_{x,y}}(\Omega) \to L_{\widehat{\Phi}_x}(\Omega) \times L_{\Phi_{x,y}}(d\mu)$ defined by $T(u) = (u, D_s u)$. It is not hard to see that T is well-defined and it is an isometry. Since $L_{\widehat{\Phi}_x}(\Omega)$ and $L_{\Phi_{x,y}}(d\mu)$ are Banach spaces by Proposition 4.2.3, it is sufficiently to prove that $T\left(W^{s,\Phi_{x,y}}(\Omega)\right)$ is a closed in $L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu)$. For this end, assume that $(u_k, D_s u_k) \to (u, v)$ in $L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu)$. By Proposition 4.2.7, we know that $L^{\widehat{\Phi}_x}(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ and $L^{\Phi_{x,y}}(d\mu) \hookrightarrow L^1_{loc}(d\mu)$. Then, passing to a subsequence, we have that $u_k \to u$ and $D_s u_k \to v$ a.e. Thus, $D_s u_k \to D_s u$ a.e., which implies that $v = D_s u$ This proves that $T\left(W^{s,\Phi_{x,y}}(\Omega)\right)$ is closed. Therefore, $W^{s,\Phi_{x,y}}(\Omega)$ is a Banach space. The reflexivity follows from Proposition 4.2.16. □

The following definition was employed by Azroul et al. (2020).

Definition 4.3.3. *A function $\Phi \in \mathit{\Phi}_M(\Omega\times\Omega)$ is said to be satisfies the fractional boundedness condition if*

$$\Phi_{x,y}(1) \leq C, \quad \text{for a.e. } (x,y) \in \Omega \times \Omega, \tag{$\mathcal{B}_f$}$$

for some constant $C > 0$.

Remark 4.3.4. *Due to condition ($\mathcal{B}_f$), for any $s \in (0,1)$ and an open set Ω with finite measure, we have that $C_0^2(\Omega) \subset W^{s,\Phi_{x,y}}(\Omega)$ without to use the local integrability condition* *which is necessary in the local case, see (AZROUL et al., 2020, Theorem 2.2). Therefore, the boundedness condition is used to obtain a well definition of fractional Muselak-Sobolev spaces.*

In the following, we prove that $C_0^1(\Omega) \subset W^{s,\Phi_{x,y}}(\Omega)$ hold true for arbitrary domain.

Proposition 4.3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $\Phi \in \mathit{\Phi}_M(\Omega \times \Omega)$ satisfies $(\mathcal{B}_f)$, then $C_0^1(\Omega) \subset W^{s,\Phi_{x,y}}(\Omega)$.*

Proof. We can assume that $\operatorname{supp}(u) \subset B_R(0) \cap \Omega$ with $R > 1$. Let $\lambda = \frac{1}{2\|u\|_{C^1(\Omega)}}$. By monotonicity of $\widehat{\Phi}_x$ and $(\mathcal{B}_f)$, we have

$$\int_\Omega \widehat{\Phi}_x\left(\lambda|u(x)|\right)\, dx \leq \int_{B_R(0)} \widehat{\Phi}_x(1)\, dx \leq C|B_R(0)| = CN\omega_N R^N < \infty.$$

Now, we estimate $J_{s,\Phi}$. Firstly, observe that for each $x \in \Omega$, we can write

$$\begin{aligned}\int_\Omega \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} &= \left(\int_{\Omega\cap B_1(x)} + \int_{\Omega\setminus B_1(x)}\right) \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} \\ &=: J_1 + J_2.\end{aligned}$$

Since $u \in C_0^1(\mathbb{R}^N)$, we have that

$$|u(x) - u(y)| \leq \int_0^1 |\nabla u(y + t(x-y)) \cdot (x-y)| dt \leq \|\nabla u\|_\infty |x-y|.$$

This inequality together with the convexity and monotonicity of $\Phi_{x,y}(\cdot)$ and $(\mathcal{B}_f)$ gives us

$$\begin{aligned}J_1 &\leq \int_{\Omega\cap B_1(x)} \Phi_{x,y}\left(\lambda\|\nabla u\|_\infty |x-y|^{1-s}\right) \frac{dy}{|x-y|^N} \\ &\leq \int_{B_1(0)} \Phi_{x,y}(1) \frac{dh}{|h|^{N+s-1}} \\ &\leq C\frac{N\omega_N}{1-s}.\end{aligned} \tag{4.6}$$

Similarly, it follows from $(\mathcal{B}_f)$ that

$$J_2 \leq \int_{\Omega\setminus B_1(x)} \Phi_{x,y}\left(\lambda\frac{2\|u\|_\infty}{|x-y|^s}\right) \frac{dy}{|x-y|^N} \leq C\int_{\mathbb{R}^N\setminus B_1(0)} \frac{dh}{|h|^{N+s}} = C\frac{N\omega_N}{s}. \tag{4.7}$$

Combining (4.6) and (4.7), we deduce

$$\int_\Omega \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} = J_1 + J_2 \leq C(n,s), \quad \text{for all } x \in \Omega. \tag{4.8}$$

Observe that this ends the proof when Ω has finite measure.

In the sequence, we consider the case when Ω has infinite measure. For this end, assume that $x \in \Omega \setminus B_{2R}(0)$. Then, $u(x) = 0$ and we have that

$$\int_\Omega \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} = \int_{\Omega\cap B_R(0)} \Phi_{x,y}\left(\lambda\frac{|u(y)|}{|x-y|^s}\right) \frac{dy}{|x-y|^N}.$$

Since $|x-y| \geq |x|-|y| \geq |x|-R \geq \frac{1}{2}|x|$, from the monotonicity and convexity of $\Phi_{x,y}(\cdot)$, we infer that

$$\begin{aligned}
\int_{\Omega} \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} &\leq \frac{2^N}{|x|^N} \int_{\Omega \cap B_R(0)} \Phi_{x,y}\left(\lambda \frac{2^s|u(y)|}{|x|^s}\right) dy \\
&\leq \frac{2^N}{|x|^{N+s}} \int_{B_R(0)} \Phi_{x,y}(1)\, dy \qquad (4.9)\\
&\leq CN\omega_N R^N \frac{2^N}{|x|^{N+s}}.
\end{aligned}$$

Hence, it follows from (4.8) and (4.9) that

$$\int_{\mathbb{R}^n} \Phi_{x,y}\left(\lambda|D_s u(x,y)|\right) \frac{dy}{|x-y|^N} \leq C(N,s)\left(\chi_{B_{2R}(0)}(x) + \frac{1}{|x|^{N+s}}\chi_{B_{2R}(0)^c}(x)\right) \in L^1(\Omega).$$

Therefore, by Tonelli's Theorem, we conclude that $J_{s,\Phi}(\lambda u) < \infty$. □

In the remainder of this chapter, unless otherwise stated, we will consider an open set $\Omega \subset \mathbb{R}^N, N \geq 1$ and $\Phi : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function defined by

$$\Phi_{x,y}(t) := \Phi(x,y,t) = \int_0^{|t|} \phi(x,y,\tau)\, d\tau,$$

where $\phi : \Omega \times \Omega \times [0,\infty) \to [0,\infty)$ is given by

$$\phi(x,y,t) = \begin{cases} t\varphi(x,y,t), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0, \end{cases}$$

being $\varphi : \Omega \times \Omega \times (0,\infty) \to (0,\infty)$ a Carathéodory function satisfying the following assumptions:

(ϕ_1) $\lim_{t\to 0^+} t\varphi_{x,y}(t) = 0$ and $\lim_{t\to\infty} t\varphi_{x,y}(t) = \infty$ for a.e. $(x,y) \in \Omega \times \Omega$, where $\varphi_{x,y}(t) := \varphi(x,y,t)$.

(ϕ_2) for a.e. $(x,y) \in \Omega \times \Omega$, $t \mapsto t\varphi_{x,y}(t)$ is increasing on $(0,\infty)$.

(ϕ_3) there exist $1 < \ell \leq m < \infty$ such that

$$\ell \leq \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)} \leq m, \quad \text{for a.e. } (x,y) \in \Omega \times \Omega \text{ and } t > 0.$$

(ϕ_4) there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \Phi_{x,y}(1) \leq C_2 \quad \text{for a.e. } (x,y) \in \Omega \times \Omega.$$

We also consider the function $\widehat{\Phi} : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{\Phi}_x(t) := \widehat{\Phi}(x,t) = \int_0^{|t|} \widehat{\phi}(x,\tau)\, d\tau, \tag{4.10}$$

where $\widehat{\phi}_x(t) := \widehat{\phi}(x,t) = \phi(x,x,t)$ for all $(x,t) \in \Omega \times [0,\infty)$. The assumption (ϕ_3) implies that

$$\ell \leq \frac{t^2 \widehat{\varphi}_x(t)}{\widehat{\Phi}_x(t)} \leq m, \quad \text{for all } x \in \Omega \text{ and } t > 0,$$

with $\widehat{\varphi}_x(t) := \widehat{\varphi}(x,t) = \varphi(x,x,t)$ for all $(x,t) \in \Omega \times (0,\infty)$.

Next, we list some remarks on our assumptions.

Remark 4.3.6. *In contrast to the assumption (ϕ_3), if (4.1) holds, then*

$$\Phi_{x,y}(\sqrt{s}) \geq \Phi_{x,y}(\sqrt{t}) + \frac{d}{d\tau}\Big|_{\tau=t} \Phi_{x,y}(\sqrt{\tau})(s-t), \quad \textit{for a.e. } (x,y) \in \Omega \times \Omega \textit{ and } s,t > 0.$$

In particular,

$$\Phi_{x,y}(s) \geq \Phi_{x,y}(t) + \frac{\varphi_{x,y}(t)}{2}(s^2 - t^2), \quad \textit{for a.e. } (x,y) \in \Omega \times \Omega \textit{ and } s,t > 0.$$

By fixing $t > 0$ and making $s \to 0^+$, we get

$$2 \leq \frac{t^2 \varphi_{x,y}(t)}{\Phi_{x,y}(t)}, \quad \textit{for a.e. } (x,y) \in \Omega \times \Omega \textit{ and } t > 0.$$

Thus, if the condition (4.1) is required, then the case $1 < \ell \leq m < 2$ is not contemplated. Therefore, assumption (ϕ_3) is more general.

Remark 4.3.7. *In light of assumptions $(\phi_1) - (\phi_2)$, $\Phi_{x,y}$ and $\widehat{\Phi}_x$, as well as their conjugate functions, are generalized N-functions and, due to (ϕ_3), satisfy the Δ_2-condition. Using the convexity and differentiability of Φ, one can prove that Φ and $\widetilde{\Phi}$ satisfy the following inequality*

$$\widetilde{\Phi}_{x,y}(t\varphi_{x,y}(t)) \leq \Phi_{x,y}(2t), \quad \textit{for a.e. } (x,y) \in \Omega \times \Omega \textit{ and } t \geq 0. \tag{4.11}$$

We observe that (ϕ_4) is equivalent to weight condition (A_0) thanks to Δ_2-condition.

A fractional version of Lemma 4.2.22 can be stated as follows:

Lemma 4.3.8. *(AZROUL et al., 2020, Lemma 2.2 and Proposition 2.3) Let $\Phi \in \mathit{\Phi}_M(\Omega \times \Omega)$ be satisfying (ϕ_3) and let $s \in (0,1)$. Then, the following assertions hold:*

(i) $\xi_0^-(\sigma)\Phi_{x,y}(t) \leq \Phi_{x,y}(\sigma t) \leq \xi_0^+(\sigma)\Phi_{x,y}(t)$*, for all $(x,y) \in \Omega \times \Omega$ and $\sigma, t \geq 0$.*

(ii) $\xi_0^-([u]_{s,\Phi}) \leq J_{s,\Phi}(u) \leq \xi_0^+([u]_{s,\Phi})$*, for all $u \in W^{s,\Phi_{x,y}}(\Omega)$.*

(iii) $\xi_0^-(\|u\|_{(\Omega)}) \leq \mathcal{J}_{s,\Phi}(u) \leq \xi_0^+(\|u\|_{(\Omega)})$, *for all* $u \in W^{s,\Phi_{x,y}}(\Omega)$.

In particular, this lemma together with (ϕ_4) gives that

$$\begin{aligned} \Phi_{x,y}(t) &\leq \xi_0^+(t)\Phi_{x,y}(1) \leq \xi_0^+(t)\sup_{x,y}\Phi_{x,y}(1) < \infty, \\ \Phi_{x,y}(t) &\geq \xi_0^-(t)\Phi_{x,y}(1) \geq \xi_0^-(t)\inf_{x,y}\Phi_{x,y}(1) > 0,\ t > 0. \end{aligned} \tag{4.12}$$

In the article by Azroul et al. (2021), some embedding results and a version of the Poincaré inequality for fractional Musielak-Sobolev spaces are presented. However, we believe that certain steps in the proofs lack clarity and require additional justification. Specifically, the estimates employed in the proofs are not evidently uniform due to their dependence on spatial variables. Controlling these spatial variables represents one of the significant challenges in the study of fractional Musielak-Sobolev spaces. Unfortunately, we have not yet been able to produce a convincing proof.

On the other hand, some results obtained by these authors remain valid and are summarized as follows:

Proposition 4.3.9. *(AZROUL et al., 2021, Lemma 2.3) Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set, and let* $0 < s' < s < 1$. *Assume that* (ϕ_1)-(ϕ_4) *hold. Then, the space* $W^{s,\Phi_{x,y}}(\Omega)$ *is continuously embedded in* $W^{s',q}(\Omega)$ *for all* $q \in [1,\ell]$. *Consequently,* $W^{s,\Phi}(\mathbb{R}^N) \hookrightarrow W^{s',q}_{loc}(\mathbb{R}^N)$.

As a consequence of the classical theory of fractional Sobolev spaces, the following embedding result holds:

Corollary 4.3.10. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set with* $C^{0,1}$*-regularity, and let* $0 < s' < s < 1$. *We define*

$$\ell^*_{s'} = \begin{cases} \frac{N\ell}{N-s'\ell} & \text{if } N > s'\ell, \\ \infty & \text{if } N \leq s'\ell. \end{cases}$$

Assume that (ϕ_1)-(ϕ_4) *hold. Then, the following embeddings are valid:*

(i) If $s'\ell < N$, *then* $W^{s,\Phi}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $r \in [1,\ell^*_{s'}]$, *and the embedding of* $W^{s,\Phi}(\Omega)$ *into* $L^r(\Omega)$ *is compact for all* $r \in [1,\ell^*_{s'})$.

(ii) If $s'\ell = N$, *then* $W^{s,\Phi}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $r \in [1,\infty]$, *and the embedding of* $W^{s,\Phi}(\Omega)$ *into* $L^r(\Omega)$ *is compact for all* $r \in [1,\infty)$.

(iii) If $s'\ell > N$, *then embedding* $W^{s,\Phi}(\Omega) \hookrightarrow L^\infty(\Omega)$ *is compact.*

We also consider the following work space

$$W_0^{s,\Phi_{x,y}}(\Omega) := \{u \in W^{s,\Phi_{x,y}}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

where $\Omega \subset \mathbb{R}^N$ is a domain (bounded or not).

Proposition 4.3.11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that (ϕ_1)-(ϕ_4) hold. Then $W_0^{s,\Phi_{x,y}}(\Omega)$ is a closed subspace of $W^{s,\Phi_{x,y}}(\Omega)$.*

Proof. Firstly, let us prove that $W_0^{s,\Phi_{x,y}}(\Omega)$ is a subset of $W^{s,\Phi_{x,y}}(\Omega)$. Given $u \in W_0^{s,\Phi_{x,y}}(\Omega)$, since $u \in W^{s,\Phi_{x,y}}(\mathbb{R}^N)$ and $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we deduce

$$\int_\Omega \widehat{\Phi}_x(u(x))\,dx = \int_{\mathbb{R}^N} \widehat{\Phi}_x(u(x))\,dx < \infty$$

and

$$\int_\Omega \int_\Omega \Phi_{x,y}(D_s u(x,y))\,d\mu \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi_{x,y}(D_s u(x,y))\,d\mu < \infty.$$

Then, $u \in W^{s,\Phi_{x,y}}(\Omega)$, and the claim is proved. Clearly, $W_0^{s,\Phi_{x,y}}(\Omega)$ is a subspace of $W^{s,\Phi_{x,y}}(\Omega)$. Now, let $(u_n)_{n\in\mathbb{N}} \subset W_0^{s,\Phi_{x,y}}(\Omega)$ be a sequence such that $u_n \to u$ in $W^{s,\Phi_{x,y}}(\Omega)$. Then, by Proposition 4.3.9, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By using the property that $u_n = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ for all $n \in \mathbb{N}$, it is not hard to see that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Therefore, $u \in W_0^{s,\Phi_{x,y}}(\Omega)$, and this ends the proof. $\square$

4.4 UNIFORM CONVEXITY

In this section, we are interested in the uniform convexity of $\left(W^{s,\Phi_{x,y}}(\Omega), \|\cdot\|_{(\Omega)}\right)$. For this purpose, we assume that

$(\phi_3)'$ for a.e. $(x,y) \in \Omega \times \Omega$, $t \mapsto \varphi_{x,y}(t)$ is a differentiable function on $(0,\infty)$ and there exist $1 < \ell \leq m < \infty$ such that

$$\ell - 1 \leq \frac{\frac{d}{dt}\left(t\varphi_{x,y}(t)\right)}{\varphi_{x,y}(t)} \leq m - 1, \quad \text{for a.e. } (x,y) \in \Omega \times \Omega \text{ and } t > 0.$$

By straightforward computations, we can verify that $(\phi_3)'$ implies (ϕ_3).

Lemma 4.4.1. *If $(\phi_1), (\phi_2)$ and $(\phi_3)'$ hold, then Φ is uniformly convex function.*

Proof. First, let us prove that the function $t \mapsto \Phi_{x,y}(t^{\frac{1}{\ell}})$ is convex for $t > 0$ and a.e. $(x,y) \in \Omega \times \Omega$. Indeed, by direction calculation, we have that

$$\begin{aligned}
\frac{d^2}{dt^2}\left[\Phi_{x,y}\left(t^{\frac{1}{\ell}}\right)\right] &= \frac{d}{dt}\left[\frac{d}{dt}\Phi_{x,y}\left(t^{\frac{1}{\ell}}\right)\right] \\
&= \frac{d}{dt}\left[\frac{1}{\ell}t^{\frac{1}{\ell}}\varphi_{x,y}\left(t^{\frac{1}{\ell}}\right)t^{\frac{1}{\ell}-1}\right] \\
&= \frac{1}{\ell}\left[\frac{d}{ds}(s\varphi_{x,y}(s))\Big|_{s=t^{\frac{1}{\ell}}}\frac{1}{\ell}\left(t^{\frac{1}{\ell}-1}\right)\left(t^{\frac{1}{\ell}-1}\right) + t^{\frac{1}{\ell}}\varphi_{x,y}\left(t^{\frac{1}{\ell}}\right)\left(\frac{1}{\ell}-1\right)t^{\frac{1}{\ell}-2}\right] \\
&= \frac{1}{\ell}\left[\frac{1}{\ell}\frac{d}{ds}(s\varphi_{x,y}(s))\Big|_{s=t^{\frac{1}{\ell}}}\left(t^{\frac{1}{\ell}-1}\right)^2 + \varphi_{x,y}\left(t^{\frac{1}{\ell}}\right)\left(\frac{1}{\ell}-1\right)\left(t^{\frac{1}{\ell}-1}\right)^2\right] \\
&= \frac{1}{\ell}\left(t^{\frac{1}{\ell}-1}\right)^2\varphi_{x,y}(s)\Big|_{s=t^{\frac{1}{\ell}}}\left[\frac{1}{\ell}\frac{\frac{d}{ds}(s\varphi_{x,y}(s))}{\varphi_{x,y}(s)}\Big|_{s=t^{\frac{1}{\ell}}} + \frac{1}{\ell} - 1\right].
\end{aligned}$$

This and the hypothesis $(\phi_3)'$ imply

$$\frac{d^2}{dt^2}\left[\Phi_{x,y}\left(t^{\frac{1}{\ell}}\right)\right] \geq \frac{1}{\ell}\left(t^{\frac{1}{\ell}-1}\right)^2\varphi_{x,y}(s)\Big|_{s=t^{\frac{1}{\ell}}}\left[\frac{1}{\ell}(\ell-1) + \frac{1}{\ell} - 1\right] = 0,$$

which shows the statement.

It remains to show that the N-function Φ is uniformly convex. To this end, let $\varepsilon > 0$ and let $t, s \geq 0$ be such that $|t-s| > \varepsilon \max\{t,s\}$. We know that the function $t \mapsto t^\ell$ is uniformly convex, that is, there exists $\delta_\Phi := \delta_\Phi(\varepsilon, \ell) > 0$ such that

$$\left(\frac{t+s}{2}\right)^\ell \leq (1-\delta_\Phi)\frac{t^\ell + s^\ell}{2}$$

This inequality together with the convexity of $t \mapsto \Phi_{x,y}(t^{\frac{1}{\ell}})$ gives us

$$\Phi_{x,y}\left(\frac{t+s}{2}\right) \leq \Phi_{x,y}\left(\left((1-\delta_\Phi)\frac{t^\ell+s^\ell}{2}\right)^{\frac{1}{\ell}}\right) \leq (1-\delta_\Phi)\frac{\Phi_{x,y}(t) + \Phi_{x,y}(s)}{2}.$$

This completes the proof of the uniform convexity of Φ. □

Finally, we are able to prove our first main result, which can be stated as follows.

Theorem 4.4.2. *Let $s \in (0,1)$ and assume that (ϕ_1), (ϕ_2) and $(\phi_3)'$ hold. Then, the following assertions hold:*

(i) $W^{s,\Phi_{x,y}}(\Omega)$ is a uniformly convex space with respect to norm $\|\cdot\|_{(\Omega)}$;

(ii) If $u_n \rightharpoonup u$ in $W^{s,\Phi_{x,y}}(\Omega)$ and $\mathcal{J}_{s,\Phi}(u_n) \to \mathcal{J}_{s,\Phi}(u)$, then $u_n \to u$ in $W^{s,\Phi_{x,y}}(\Omega)$.

Proof. (i) To prove the uniform convexity of the space $W^{s,\Phi_{x,y}}(\Omega)$, let us consider the product space $L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu)$ with Luxemburg equivalent norm $\|\cdot\|_{\widehat{\Phi},\Phi}$ generated by the modular function

$$\mathcal{J}_{\widehat{\Phi},\Phi}(u,v) = \int_\Omega \widehat{\Phi}_x(|u(x)|)\,dx + \int_\Omega\int_\Omega \Phi_{x,y}(|v(x,y)|)\,d\mu = J_{\widehat{\Phi}}(u) + J_\Phi(v),$$

for all $u \in L^{\widehat{\Phi}_x}(\Omega)$ and $v \in L^{\Phi_{x,y}}(d\mu)$. Now, consider the linear operator

$$T : \left(W^{s,\Phi_{x,y}}(\Omega), \|\cdot\|_{(\Omega)}\right) \to \left(L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu), \|\cdot\|_{\widehat{\Phi},\Phi}\right)$$

defined by $T(u) = (u, D_s u)$. It is not hard to see that T is well-defined and it is an isometric embedding. Thus, $T\left(W^{s,\Phi_{x,y}}(\Omega)\right)$ is a closed subspace of $\left(L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu), \|\cdot\|_{\widehat{\Phi},\Phi}\right)$. Since $\left(L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu), \|\cdot\|_{\widehat{\Phi},\Phi}\right)$ is Banach space, in order to prove that $W^{s,\Phi_{x,y}}(\Omega)$ is uniformly convex space, it is sufficient to show that $\left(L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu), \|\cdot\|_{\widehat{\Phi},\Phi}\right)$ is uniformly convex space.

Firstly, by using Lemma 4.4.1 the N-funtions $\widehat{\Phi}$ and Φ are uniformly convex. Let us fix $\varepsilon > 0$. For $\varepsilon/2$, let $\delta_0 > 0$ as in Lemma 4.2.10 for $\widehat{\Phi}$ and Φ. Assume that $(u_1, v_1), (u_2, v_2) \in L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu)$ satisfying

$$\mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(u_1, v_1) - (u_2, v_2)}{2}\right) > \varepsilon \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2}. \tag{4.13}$$

We prove that the second inequality of Definition 4.2.12 holds for $\delta = \frac{\delta_0 \varepsilon}{2}$. For this, we define the sets

$$U = \left\{x \in \Omega : |u_1(x) - u_2(x)| > \frac{\varepsilon}{2} \max\{|u_1(x)|, |u_2(x)|\}\right\}$$

and

$$V = \left\{(y, z) \in \Omega \times \Omega : |v_1(y, z) - v_2(y, z)| > \frac{\varepsilon}{2} \max\{|v_1(y, z)|, |v_2(y, z)|\}\right\}.$$

For a.e. $x \in \Omega \setminus U$, it follows from convexity $\widehat{\Phi}$ that

$$\begin{aligned}\widehat{\Phi}_x\left(\frac{|u_1(x) - u_2(x)|}{2}\right) &\leq \widehat{\Phi}_x\left(\frac{\varepsilon}{2} \frac{|u_1(x)| + |u_2(x)|}{2}\right) \\ &\leq \frac{\varepsilon}{2} \frac{\widehat{\Phi}_x(|u_1(x)|) + \widehat{\Phi}_x(|u_2(x)|)}{2}.\end{aligned}$$

Then,

$$J_{\widehat{\Phi}}\left(\chi_{\Omega\setminus U} \frac{u_1 - u_2}{2}\right) \leq \frac{\varepsilon}{2} \frac{J_{\widehat{\Phi}}(\chi_{\Omega\setminus U} u_1) + J_{\widehat{\Phi}}(\chi_{\Omega\setminus U} u_2)}{2}. \tag{4.14}$$

Analogously, for a.e. $(y, z) \in (\Omega \times \Omega) \setminus V$, we have

$$J_\Phi\left(\chi_{(\Omega\times\Omega)\setminus V} \frac{v_1 - v_2}{2}\right) \leq \frac{\varepsilon}{2} \frac{J_\Phi(\chi_{(\Omega\times\Omega)\setminus V} v_1) + J_\Phi(\chi_{(\Omega\times\Omega)\setminus V} v_2)}{2}. \tag{4.15}$$

By the additivity of the integral, (4.14) and (4.15), we obtain

$$\begin{aligned}&\mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_{\Omega\setminus U} u_1, \chi_{(\Omega\times\Omega)\setminus V} v_1) - (\chi_{\Omega\setminus U} u_2, \chi_{(\Omega\times\Omega)\setminus V} v_2)}{2}\right) \\ &= \mathcal{J}_{\widehat{\Phi},\Phi}\left(\chi_{\Omega\setminus U} \frac{u_1 - u_2}{2}, \chi_{(\Omega\times\Omega)\setminus V} \frac{v_1 - v_2}{2})\right) \\ &\leq \frac{\varepsilon}{2} \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(\chi_{\Omega\setminus U} u_1, \chi_{(\Omega\times\Omega)\setminus V} v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(\chi_{\Omega\setminus U} u_2, \chi_{(\Omega\times\Omega)\setminus V} v_2)}{2} \\ &\leq \frac{\varepsilon}{2} \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2}.\end{aligned}$$

This inequality jointly with (4.13) and aditivity of the integral give us

$$\begin{aligned}
&\mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_U u_1, \chi_V v_1) - (\chi_U u_2, \chi_V v_2)}{2}\right)\\
&= \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(u_1, v_1) - (u_2, v_2}{2}\right) - \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_{\Omega\setminus U} u_1, \chi_{(\Omega\times\Omega)\setminus V} v_1) - (\chi_{\Omega\setminus U} u_2, \chi_{(\Omega\times\Omega)\setminus V} v_2)}{2}\right)\\
&> \varepsilon \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2} - \frac{\varepsilon}{2}\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2}\\
&= \frac{\varepsilon}{2}\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2}.
\end{aligned} \tag{4.16}$$

On the other side, it follows from uniform convexity of Φ and definition of U and V that

$$\begin{aligned}
&\mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_U u_1, \chi_V v_1) + (\chi_U u_2, \chi_V v_2)}{2}\right) = J_{\widehat{\Phi}}\left(\chi_U \frac{u_1 + u_2}{2}\right) + J_{\Phi}\left(\chi_V \frac{v_1 + v_2}{2}\right)\\
&\leq (1-\delta_0)\frac{J_{\widehat{\Phi}}(\chi_U u_1) + J_{\widehat{\Phi}}(\chi_U u_2)}{2} + (1-\delta_0)\frac{J_{\Phi}(\chi_V v_1) + J_{\Phi}(\chi_V v_2)}{2}\\
&= (1-\delta_0)\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_1, \chi_V v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_2, \chi_V v_2)}{2}.
\end{aligned} \tag{4.17}$$

Since the following inequality hold

$$\frac{1}{2}\left[\widehat{\Phi}_x(\chi_{\Omega\setminus U} u_1) + \widehat{\Phi}_x(\chi_{\Omega\setminus U} u_2)\right] - \widehat{\Phi}_x\left(\chi_{\Omega\setminus U}(x)\frac{u_1(x) + u_2(x)}{2}\right) \geq 0$$

and

$$\frac{1}{2}\left[\Phi_{x,y}(\chi_{(\Omega\times\Omega)\setminus V} v_1) + \Phi_{y,z}(\chi_{(\Omega\times\Omega)\setminus V} v_2)\right] - \Phi_{x,y}\left(\chi_{(\Omega\times\Omega)\setminus V}\frac{v_1 + v_2}{2}\right) \geq 0,$$

we deduce

$$\begin{aligned}
&\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2} - \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(u_1, v_1) + (u_2, v_2)}{2}\right)\\
&\geq \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_1, \chi_V v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_2, \chi_V v_2)}{2} - \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_U u_1, \chi_V v_1) + (\chi_U u_2, \chi_V v_2)}{2}\right).
\end{aligned}$$

This inequality, (4.16), (4.17) and convexity imply

$$\begin{aligned}
&\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2} - \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(u_1, v_1) + (u_2, v_2)}{2}\right)\\
&\geq \delta_0 \frac{\mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_1, \chi_V v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(\chi_U u_2, \chi_V v_2)}{2}\\
&\geq \delta_0 \mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(\chi_U u_1, \chi_V v_1) - (\chi_U u_2, \chi_V v_2)}{2}\right)\\
&\geq \delta_0 \frac{\varepsilon}{2}\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2}.
\end{aligned}$$

Hence,

$$\mathcal{J}_{\widehat{\Phi},\Phi}\left(\frac{(u_1, v_1) + (u_2, v_2)}{2}\right) \leq \left(1 - \frac{\delta_0 \varepsilon}{2}\right)\frac{\mathcal{J}_{\widehat{\Phi},\Phi}(u_1, v_1) + \mathcal{J}_{\widehat{\Phi},\Phi}(u_2, v_2)}{2},$$

which proves the uniform convexity of the modular $\mathcal{J}_{\widehat{\Phi},\Phi}$. Finally, by Proposition 4.2.13, we conclude that $\left(L^{\widehat{\Phi}_x}(\Omega) \times L^{\Phi_{x,y}}(d\mu), \|\cdot\|_{\widehat{\Phi},\Phi}\right)$ is uniformly convex.

(ii) By recalling that Φ satisfies Δ_2-condition, the required property also follows from Proposition 4.2.13. □

Remark 4.4.3. *In Theorem 4.4.2 the domain Ω could be unbounded or $\mathbb{R}^N$ itself.*

4.5 MONOTONICITY AND CONVERGENCE RESULTS

Inspired by Bahrouni, Bahrouni and Xiang (2020b), we introduce a version of the classical Lemma of Brézis and Lieb (1983), to modular functions.

Proposition 4.5.1 (Brezis-Lieb type Lemma). *Assume that $(\phi_1)-(\phi_3)$ hold. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{s,\Phi_{x,y}}(\Omega)$ such that $u_n(x) \to u(x)$ a.e. in Ω. Then, $u \in W^{s,\Phi_{x,y}}(\Omega)$ and*

$$\lim_{n\to\infty} \left(\mathcal{J}_{s,\Phi}(u_n) - \mathcal{J}_{s,\Phi}(u_n - u)\right) = \mathcal{J}_{s,\Phi}(u).$$

Proof. Firstly, by the boundedness of $(u_n)_{n\in\mathbb{N}}$, Fatou's Lemma and Lemmas 4.2.22 (ii) and 4.3.8 (ii), we have

$$\int_\Omega \widehat{\Phi}_x(|u|)\, dx \leq \liminf_{n\to\infty} \int_\Omega \widehat{\Phi}_x(|u_n|)\, dx < \infty,$$

that is, $u \in L^{\widehat{\Phi}_x}(\Omega)$, and

$$\int_\Omega \int_\Omega \Phi_{x,y}(|D_s u|)\, d\mu \leq \liminf_{n\to\infty} \int_\Omega \int_\Omega \Phi_{x,y}(|D_s u_n|)\, d\mu < \infty.$$

Hence, $u \in W^{s,\Phi_{x,y}}(\Omega)$.

In order to complete the proof, it only remains to show that

$$\lim_{n\to\infty} \left(J_{s,\Phi}(u_n) - J_{s,\Phi}(u_n - u)\right) = J_{s,\Phi}(u) \tag{4.18}$$

and

$$\lim_{n\to\infty} \left(J_{\widehat{\Phi}}(u_n) - J_{\widehat{\Phi}}(u_n - u)\right) = J_{\widehat{\Phi}}(u). \tag{4.19}$$

In view of the Mean Value Theorem, for each $(x,y) \in \Omega\times\Omega$, there exists $z_n := z_n(x,y)$ between $|D_s u_n(x,y) - D_s u(x,y)|$ and $|D_s u_n(x,y)|$ such that

$$|\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|)| = z_n \varphi_{x,y}(z_n) ||D_s u_n| - |D_s u_n - D_s u||,$$

where we have used that $\Phi'_{x,y}(t) = t\varphi_{x,y}(t)$, for all $t \geq 0$. Thus, by using (φ_2), we have

$$\begin{aligned}|\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|)| &\leq z_n \varphi_{x,y}(z_n)|D_s u| \\ &\leq (|D_s u_n| + |D_s u_n - D_s u|)\varphi_{x,y}(|D_s u_n| + |D_s u_n - D_s u|)|D_s u|.\end{aligned}$$

For any $\varepsilon \in (0,1)$, the Young's inequality (4.3) and (4.11) imply in

$$\begin{aligned}&(|D_s u| + |D_s u_n - D_s u|)\varphi_{x,y}(|D_s u| + |D_s u_n - D_s u|)|D_s u| \\ &\quad\leq \varepsilon\widetilde{\Phi}_{x,y}\Big((|D_s u| + |D_s u_n - D_s u|)\varphi_{x,y}(|D_s u| + |D_s u_n - D_s u|)\Big) + C_\varepsilon \Phi_{x,y}(|D_s u|) \\ &\quad\leq \varepsilon 2^m \Phi_{x,y}(|D_s u| + |D_s u_n - D_s u|) + C_\varepsilon \Phi_{x,y}(|D_s u|) \\ &\quad\leq \varepsilon C_m \Phi_{x,y}(|D_s u_n - D_s u|) + C_{\varepsilon,m}\Phi_{x,y}(|D_s u|),\end{aligned}$$

where $C_m := 2^{2m-1}$ and $C_{\varepsilon,m} := \varepsilon 2^{2m-1} + C_\varepsilon$. Therefore, we obtain

$$|\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|)| \leq \varepsilon C_m \Phi_{x,y}(|D_s u_n - D_s u|) + C_{\varepsilon,m}\Phi_{x,y}(|D_s u|). \quad (4.20)$$

Next, for $n \in \mathbb{N}$, we define

$$\mathcal{W}_{\varepsilon,n}(x,y) = \Big[\,|\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|) - \Phi_{x,y}(|D_s u|)| - \varepsilon C_m \Phi_{x,y}(|D_s u_n - D_s u|)\Big]^+,$$

where $a^+ := \max\{a, 0\}$, for all $a \in \mathbb{R}$. Note that $\mathcal{W}_{\varepsilon,n}(x,y) \to 0$, as $n \to \infty$, a.e. in $\Omega \times \Omega$. Moreover, it follows from (4.20) that

$$\begin{aligned}&|\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|) - \Phi_{x,y}(|D_s u|)| \\ &\quad\leq |\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|)| + |\Phi_{x,y}(|D_s u|)| \\ &\quad\leq \varepsilon C_m \Phi_{x,y}(|D_s u_n - D_s u|) + (C_{\varepsilon,m} + 1)\Phi_{x,y}(|D_s u|),\end{aligned}$$

which implies that

$$\mathcal{W}_{\varepsilon,n}(x,y)|x-y|^{-N} \leq (C_{\varepsilon,m} + 1)\Phi_{x,y}(|D_s u|)|x-y|^{-N} \in L^1(\Omega \times \Omega).$$

Hence, in light of Lebesgue's Dominated Convergence Theorem, there holds

$$\int_\Omega \int_\Omega \mathcal{W}_{\varepsilon,n}(x,y)\, d\mu \to 0, \quad \text{as } n \to \infty.$$

This fact and the following inequality

$$\begin{aligned}&|J_{s,\Phi}(u_n) - J_{s,\Phi}(u_n - u) - J_{s,\Phi}(u)| \\ &\quad\leq \int_\Omega \int_\Omega |\Phi_{x,y}(|D_s u_n|) - \Phi_{x,y}(|D_s u_n - D_s u|) - \Phi_{x,y}(|D_s u|)|\ d\mu \\ &\quad\leq \int_\Omega \int_\Omega (\mathcal{W}_{\varepsilon,n}(x,y) + \varepsilon C_m \Phi_{x,y}(|D_s u_n - D_s u|))\ d\mu \\ &\quad\leq \int_\Omega \int_\Omega \mathcal{W}_{\varepsilon,n}(x,y)\, d\mu + \varepsilon C_m J_{s,\Phi}(u_n - u),\end{aligned}$$

imply that

$$\lim_{n\to\infty} |J_{s,\Phi}(u_n) - J_{s,\Phi}(u_n - u) - J_{s,\Phi}(u)| \leq \varepsilon C_m K,$$

for some constant $K > 0$. Therefore, by making $\varepsilon \to 0$ we obtain the assertion (4.18). Similarly, we can show that (4.19) is valid. This prove the desired result. □

Due to Lemma 4.3.8 (ii) and Brezis-Lieb-type Lemma we obtain the following convergence result:

Corollary 4.5.2. *Assume that $(\phi_1) - (\phi_3)$ hold. Let $u, u_n \in W^{s,\Phi_{x,y}}(\Omega)$, $n \in \mathbb{N}$. Then, the following assertions are equivalent:*

(i) $\lim\limits_{n\to\infty} \|u_n - u\|_{(\Omega)} = 0.$

(ii) $\lim\limits_{n\to\infty} \mathcal{J}_{s,\Phi}(u_n - u) = 0.$

(iii) $u_n(x) \to u(x)$ *for a.e* $x \in \Omega$ *and* $\lim\limits_{n\to\infty} \mathcal{J}_{s,\Phi}(u_n) = \mathcal{J}_{s,\Phi}(u).$

Now, we recall some definitions of operators of monotone type that we will use throughout this section.

Definition 4.5.3. *Let X be a reflexive Banach space with norm $\|\cdot\|_X$ and let $A : X \to X^*$ be an operator. Then, A is said to be*

(i) monotone (strictly monotone) if $\langle Au - Av, u - v\rangle \geq 0$ (> 0), for all $u, v \in X$ with $u \neq v$;

(ii) uniformly monotone if $\langle Au - Av, u - v\rangle \geq \alpha(\|u - v\|)\|u - v\|$ for all $u, v \in X$, where $\alpha : [0, \infty) \to [0, \infty)$ is strictly increasing with $\alpha(0) = 0$ and $\alpha(t) \to \infty$, as $t \to \infty$;

(iii) pseudomonotone if $u_n \rightharpoonup u$ weakly in X and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$ imply

$$\langle Au, u - v\rangle \leq \liminf_{n\to\infty} \langle Au_n, u_n - v\rangle, \quad \text{for all } v \in X;$$

(iv) coercive if there exists a function $\beta : [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} \beta(t) = \infty$ and

$$\frac{\langle Au, u\rangle}{\|u\|_X} \geq \beta(\|u\|_X), \quad \text{for all } u \in X.$$

Under the assumptions $(\phi_1) - (\phi_3)$, the property listed in the following lemma is valid.

Lemma 4.5.4. *Let $s \in (0,1)$ and assume that $(\phi_1)-(\phi_4)$ hold. Then, $\mathcal{J}_{s,\Phi}$ belongs to $C^1\Big(W^{s,\Phi_{x,y}}(\Omega),\mathbb{R}\Big)$ and its Gâteaux derivative is given by*

$$\begin{aligned}\langle \mathcal{J}'_{s,\Phi}(u),v\rangle &= \int_\Omega \widehat{\varphi}_x(|u(x)|)u(x)v(x)\,dx + \int_\Omega\int_\Omega \varphi_{x,y}\left(|D_su(x,y)|\right)D_su(x,y)D_sv(x,y)\,d\mu \\ &= \langle J'_{\widehat{\Phi}}(u),v\rangle + \langle J'_{s,\Phi}(u),v\rangle,\end{aligned}$$

for all $u,v\in W^{s,\Phi_{x,y}}(\Omega)$.

Proof. The proof is similar to Lemma 3.1 of Azroul et al. (2020) and we omit here. □

Next, we shall prove some monotonicity properties of the operator $\mathcal{J}'_{s,\Phi}: X\to X^*$, where $X=W^{s,\Phi_{x,y}}(\Omega)$ or $X=W_0^{s,\Phi_{x,y}}(\Omega)$ if Ω is a bounded domain.

Proposition 4.5.5. *Assume that $(\phi_1)-(\phi_4)$ hold. The operator $\mathcal{J}'_{s,\Phi}: X\to X^*$ satisfies the following properties:*

(i) *$\mathcal{J}'_{s,\Phi}$ is bounded, coercive and monotone;*

(ii) *$\mathcal{J}'_{s,\Phi}$ is pseudomonotone.*

Proof. (i) Since $\Phi_{x,y}$ is convex, it follows that $\mathcal{J}_{s,\Phi}$ is convex. Then, $\mathcal{J}'_{s,\Phi}$ is a monotone operator. Next, we shall prove that $\mathcal{J}'_{s,\Phi}$ is bounded. For this, let $u,v\in W^{s,\Phi_{x,y}}(\Omega)\setminus\{0\}$. It follows from Young's inequality (4.3), (4.11), Lemma 4.3.8 (i) and the definition of the Luxemburg norm that

$$\begin{aligned}\left|\left\langle \mathcal{J}'_{s,\Phi}(u),\frac{v}{\|v\|_{(\Omega)}}\right\rangle\right| &\le \int_\Omega \widehat{\varphi}_x(|u|)|u|\left|\frac{v}{\|v\|_{(\Omega)}}\right|dx + \int_\Omega\int_\Omega \varphi_{x,y}(|D_su|)|D_su|\left|\frac{D_sv}{\|v\|_{(\Omega)}}\right|d\mu \\
&\le \int_\Omega\left[\widetilde{\widehat{\Phi}}_x\Big(\widehat{\varphi}_x(|u|)|u|\Big)+\widehat{\Phi}_x\left(\frac{|v|}{\|v\|_{(\Omega)}}\right)\right]dx \\
&\quad+\int_\Omega\int_\Omega\left[\widetilde{\Phi}_{x,y}\Big(\varphi_{x,y}(|D_su|)|D_su|\Big)+\Phi_{x,y}\left(\frac{|D_sv|}{\|v\|_{(\Omega)}}\right)\right]d\mu \\
&\le \int_\Omega\left[2^m\widehat{\Phi}_x\left(\|u\|_{(\Omega)}\frac{|u|}{\|u\|_{(\Omega)}}\right)+\widehat{\Phi}_x\left(\frac{|v|}{\|v\|_{(\Omega)}}\right)\right]\mathrm{d}x \\
&\quad+\int_\Omega\int_\Omega\left[2^m\Phi_{x,y}\left(\|u\|_{(\Omega)}\frac{|D_su|}{\|u\|_{(\Omega)}}\right)+\Phi_{x,y}\left(\frac{|D_sv|}{\|v\|_{(\Omega)}}\right)\right]d\mu \\
&\le 2^m\xi_0^+(\|u\|_{(\Omega)})\mathcal{J}'_{s,\Phi}\left(\frac{u}{\|u\|_{(\Omega)}}\right)+\mathcal{J}'_{s,\Phi}\left(\frac{v}{\|v\|_{(\Omega)}}\right) \\
&\le 2^m\xi_0^+(\|u\|_{(\Omega)})+1 \\
&\le 2^m\left(\xi_0^+(\|u\|_{(\Omega)})+1\right).\end{aligned}$$

Hence,

$$\|\mathcal{J}'_{s,\Phi}(u)\|_* = \sup_{v\, \in W^{s,\Phi_{x,y}}(\Omega)\setminus\{0\}} \frac{\left\langle \mathcal{J}'_{s,\Phi}(u), v\right\rangle}{\|v\|_{(\Omega)}} \leq 2^m \left(\xi_0^+(\|u\|_{(\Omega)}) + 1\right),$$

which implies that $\mathcal{J}'_{s,\Phi}$ is bounded. It remains to prove that $\mathcal{J}'_{s,\Phi}$ is coercive. For each $u \in W^{s,\Phi_{x,y}}(\Omega) \setminus \{0\}$, it follows from condition (ϕ_3) and Lemma 4.3.8 (ii) that

$$\begin{aligned}
\frac{\langle \mathcal{J}'_{s,\Phi}(u), u\rangle}{\|u\|} &= \frac{1}{\|u\|_{(\Omega)}}\left[\int_\Omega \widehat{\varphi}_x(|u|)(u)^2\,dx + \int_\Omega\int_\Omega \varphi_{x,y}(|D_s u|)(D_s u)^2\,d\mu\right] \\
&\geq \frac{\ell}{\|u\|_{(\Omega)}}\left[\int_\Omega \widehat{\Phi}_x(|u|)\,dx + \int_\Omega\int_\Omega \Phi_{x,y}\left(|D_s u|\right)d\mu\right] \\
&\geq \frac{\ell}{\|u\|_{(\Omega)}}\min\{\|u\|^\ell_{(\Omega)}, \|u\|^m_{(\Omega)}\} \\
&= \ell\min\{\|u\|^{\ell-1}_{(\Omega)}, \|u\|^{m-1}_{(\Omega)}\}.
\end{aligned}$$

Hence, since $m \geq \ell > 1$, we conclude that

$$\lim_{\|u\|_{(\Omega)}\to\infty} \frac{\langle \mathcal{J}'_{s,\Phi}(u), u\rangle}{\|u\|_{(\Omega)}} = \infty,$$

which proves that $\mathcal{J}'_{s,\Phi}(u)$ is coercive.

(ii) By Lemma 4.5.4, $\mathcal{J}'_{s,\Phi}$ is continuous. Thus, since $\mathcal{J}'_{s,\Phi}$ is monotone, it follows from Proposition 27.6 of Zeidler (2013) that $\mathcal{J}'_{s,\Phi}$ is pseudomonotone. □

Now, let us assume the following conditions for a.e. $(x,y) \in \Omega\times\Omega$:

(ϕ_5) $t \mapsto \varphi_{x,y}(t)$ is a C^1-function on $(0,\infty)$.

(ϕ_6) $t \mapsto \varphi_{x,y}(t)$ is increasing in $(0,\infty)$.

Under these conditions, we can state another monotonicity property of the operator $\mathcal{J}'_{s,\Phi}$, which is motivated by the work of Montenegro (1999).

Proposition 4.5.6. *Suppose that $(\phi_1), (\phi_3), (\phi_4), (\phi_5)$ and (ϕ_6) hold. Then, $\mathcal{J}'_{s,\Phi}$ is uniformly monotone.*

Proof. Let $a_{x,y}(t) = \varphi_{x,y}(|t|)t$. In view of (ϕ_5) and (ϕ_6), we obtain

$$a'_{x,y}(t) = \varphi'_{x,y}(|t|)\frac{t^2}{|t|} + \varphi_{x,y}(|t|) \geq \varphi_{x,y}(|t|), \quad \text{for all } t \neq 0. \tag{4.21}$$

For any $\xi, \eta \in \mathbb{R}$ and $0 < t \leq \frac{1}{4}$ there holds

$$\frac{1}{4}|\xi-\eta| \leq |t\xi + (1-t)\eta|.$$

This fact combined with $(\phi_3),(\phi_6)$, (4.21) and Lemma 4.3.8 (i), imply that

$$\begin{aligned}(\varphi_{x,y}(|\xi|)\xi-\varphi_{x,y}(|\eta|)\eta)\,(\xi-\eta) &= \int_0^1 \frac{d}{dt}\Big(a_{x,y}(t\xi+(1-t)\eta)\Big)(\xi-\eta)\;dt\\ &= \int_0^1 a'_{x,y}(t\xi+(1-t)\eta)(\xi-\eta)^2\;dt\\ &\geq \int_0^1 \varphi_{x,y}(|t\xi+(1-t)\eta|)(\xi-\eta)^2\;dt\\ &\geq \int_0^{\frac14} \varphi_{x,y}(|t\xi+(1-t)\eta|)(\xi-\eta)^2\;dt\\ &\geq \int_0^{\frac14} 16\varphi_{x,y}\left(\frac14|\xi-\eta|\right)\left(\frac14|\xi-\eta|\right)^2\;dt\\ &\geq 4\ell\Phi_{x,y}\left(\frac14|\xi-\eta|\right)\\ &\geq 4^{1-m}\ell\Phi_{x,y}\left(|\xi-\eta|\right).\end{aligned}$$

Thus, using the above inequality and Lemma 4.3.8 (ii), we have

$$\begin{aligned}\Big\langle \mathcal{J}'_{s,\Phi}(u)-\mathcal{J}'_{s,\Phi}(v),u-v\Big\rangle &= \int_\Omega \left(\widehat{\varphi}_x(|u|)u-\widehat{\varphi}_x(|v|)v\right)(u-v)\;\mathrm{d}x\\ &\quad+\int_\Omega\int_\Omega \left(\varphi_{x,y}(|D_su|)D_su-\varphi_{x,y}(|D_sv|)D_sv\right)(D_su-D_sv)\;d\mu\\ &\geq 4^{1-m}\ell\left[\int_\Omega \widehat{\Phi}_x\left(|u-v|\right)\,\mathrm{d}x+\int_\Omega\int_\Omega \Phi_{x,y}\left(|D_su-D_sv|\right)d\mu\right]\\ &\geq 4^{1-m}\ell\min\{\|u-v\|_{(\Omega)}^{\ell},\|u-v\|_{(\Omega)}^{m}\}\\ &= 4^{1-m}\ell\min\{\|u-v\|_{(\Omega)}^{\ell-1},\|u-v\|_{(\Omega)}^{m-1}\}\|u-v\|_{(\Omega)}.\end{aligned}$$

Therefore, considering the function $\alpha(t)=4^{1-m}\ell\min\{t^{\ell-1},t^{m-1}\}$ for $t\geq 0$, we conclude that $\mathcal{J}'_{s,\Phi}$ is uniformly monotone. □

Definition 4.5.7. *We say that $\mathcal{J}'_{s,\Phi}$ satisfies the (S_+)-property if for a given $(u_n)_{n\in\mathbb{N}}\subset W^{s,\Phi_{x,y}}(\Omega)$ satisfying $u_n\rightharpoonup u$ weakly in $W^{s,\Phi_{x,y}}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle \mathcal{J}'_{s,\Phi}(u_n),u_n-u\rangle\leq 0,$$

there holds $u_n\to u$ strongly in $W^{s,\Phi_{x,y}}(\Omega)$.

Theorem 4.5.8. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with $C^{0,1}$-regularity and assume that $(\phi_1)-(\phi_4)$ hold. Then, $\mathcal{J}'_{s,\Phi}$ satisfies the (S_+)-property.*

Proof. Suppose that $u_n\rightharpoonup u$ weakly in $W^{s,\Phi_{x,y}}(\Omega)$ and

$$\limsup_{n\to\infty}\langle \mathcal{J}'_{s,\Phi}(u_n),u_n-u\rangle\leq 0.$$

By Corollary 4.5.2, in order to prove that $u_n \to u$ strongly in $W^{s,\Phi_{x,y}}(\Omega)$, it is sufficient to show that

$$\lim_{n\to\infty} J_{s,\Phi}(u_n - u) = \lim_{n\to\infty} \int_\Omega \int_\Omega \Phi_{x,y}(|D_s u_n - D_s u|)\, d\mu = 0 \tag{4.22}$$

and

$$\lim_{n\to\infty} J_{\widehat{\Phi}}(u_n - u) = \lim_{n\to\infty} \int_\Omega \widehat{\Phi}_x(|u_n - u|)\, dx = 0. \tag{4.23}$$

Since the embedding $W^{s,\Phi_{x,y}}(\Omega) \hookrightarrow L^1(\Omega)$ is compact (see Corollary 4.3.10), we have that $u_n(x) \to u(x)$ a.e in Ω. Then, $D_s u_n(x,y) \to D_s u(x,y)$ a.e. in $\Omega \times \Omega$, which implies that

$$\lim_{n\to\infty} \Phi_{x,y}(|D_s u_n(x,y) - D_s u(x,y)|)|x-y|^{-N} = 0, \quad \text{a.e. in } \Omega \times \Omega. \tag{4.24}$$

In view from (4.24) and Vitali's Theorem (RAO, 2004), to prove (4.22), it is sufficient to prove that the sequence

$$h_n(x,y) := \Phi_{x,y}(|D_s u_n(x,y) - D_s u(x,y)|)|x-y|^{-N}, \quad n \in \mathbb{N}$$

is equi-integrable over $\Omega \times \Omega$, that is, the following conditions hold:

(i) for all $\varepsilon > 0$, there $\delta_\varepsilon > 0$ such that, if $A \times B \subset \Omega \times \Omega$ and $|A \times B| < \delta_\varepsilon$, then

$$\int_A \int_B h_n(x,y)\, dxdy < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

(ii) for all $\varepsilon > 0$, there exists a subset $A_\varepsilon \times B_\varepsilon \subset \Omega \times \Omega$ such that $|A_\varepsilon \times B_\varepsilon| < \infty$ and

$$\int_{\Omega\setminus A_\varepsilon} \int_{\Omega\setminus B_\varepsilon} h_n(x,y)\, dxdy < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

Firstly, note that Lemma 4.5.4 and the weak convergence $u_n \rightharpoonup u$ in $W_0^{s,\Phi_{x,y}}(\Omega)$ imply that

$$\lim_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle = 0.$$

Thus,

$$\limsup_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle \le 0.$$

Hence, the monotonicity of operator $\mathcal{J}'_{s,\Phi}$ jointly with the limit just above implies that

$$0 \le \liminf_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle \le \limsup_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle \le 0,$$

that is,

$$\lim_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle = 0. \tag{4.25}$$

For $n \in \mathbb{N}$, define

$$f_n(x,y) := \Big(\varphi_{x,y}(|D_s u_n|)D_s u_n - \varphi_{x,y}(|D_s u|)D_s u\Big)\left(D_s u_n - D_s u\right).$$

Since (ϕ_2) holds, a direct computation infers

$$(t\varphi_{x,y}(|t|) - s\varphi_{x,y}(|s|))(t-s) \geq 0, \quad \text{for all } (x,y) \in \Omega \times \Omega \text{ and } s,t \in \mathbb{R}, \tag{4.26}$$

see for instance Lemma 7.5 of Alves, Gonçalves and Santos (2014) or Proposition 2.5 obtained by Carvalho, Goncalves and Silva (2015). The inequality (4.26) combined with the limit (4.25) imply that the sequence $(f_n(x,y)|x-y|^{-N})_{n\in\mathbb{N}}$ converges to 0 in $L^1(\Omega\times\Omega)$. Thus, by converse Vitali's Theorem, $(f_n(x,y)|x-y|^{-N})_{n\in\mathbb{N}}$ is equi-integrable over $\Omega \times \Omega$.

Now, observe that

$$\begin{aligned} f_n(x,y) &= \varphi_{x,y}(|D_s u_n|)(D_s u_n)^2 + \varphi_{x,y}(|D_s u|)(D_s u)^2 \\ &\quad - \varphi_{x,y}(|D_s u_n|)D_s u_n D_s u - \varphi_{x,y}(|D_s u|)D_s u D_s u_n. \end{aligned} \tag{4.27}$$

For each $\varepsilon \in (0,1)$, using Young's inequality (4.3), (4.11), (4.27), Lemma 4.3.8 (i) and (φ_3), we obtain

$$\begin{aligned} \varphi_{x,y}(|D_s u_n|)(D_s u_n)^2 &= f_n(x,y) - \varphi_{x,y}(|D_s u|)(D_s u)^2 + \varphi_{x,y}(|D_s u_n|)D_s u_n D_s u \\ &\quad + \varphi_{x,y}(|D_s u|)D_s u D_s u_n \\ &\leq f_n(x,y) + \varepsilon\widetilde{\Phi}_{x,y}(\varphi_{x,y}(|D_s u_n|)|D_s u_n|) + C_\varepsilon \Phi_{x,y}(|D_s u|) \\ &\quad + \widetilde{C}_\varepsilon \widetilde{\Phi}_{x,y}(\varphi_{x,y}(|D_s u|)|D_s u|) + \varepsilon\Phi_{x,y}(|D_s u_n|) \\ &\leq f_n(x,y) + (C_\varepsilon + 2^m\widetilde{C}_\varepsilon)\Phi_{x,y}(|D_s u|) \\ &\quad + \varepsilon(1+2^m)\ell^{-1}\varphi_{x,y}(|D_s u_n|)(D_s u_n)^2. \end{aligned}$$

Thus, by choosing $0 < \varepsilon < \frac{\ell}{1+2^m}$ sufficiently small and using (φ_3), we obtain $C := C(\varepsilon, \ell, m) > 0$ such that

$$\begin{aligned} \Phi_{x,y}(|D_s u_n|) &\leq \ell^{-1}\varphi_{x,y}(|D_s u_n|)(D_s u_n)^2 \\ &\leq C\left(f_n(x,y) + \Phi_{x,y}(|D_s u|)\right). \end{aligned} \tag{4.28}$$

Hence, using that $\Phi_{x,y}$ is convex, Lemma 4.3.8 (i) and (4.28), we obtain

$$\begin{aligned} h_n(x,y) &= \Phi_{x,y}\Big(|D_s u_n - D_s u|\Big)|x-y|^{-N} \leq \Phi_{x,y}\left(\frac{2|D_s u_n| + 2|D_s u|}{2}\right)|x-y|^{-N} \\ &\leq \Big(2^{m-1}\Phi_{x,y}(|D_s u_n|) + 2^{m-1}\Phi_{x,y}(|D_s u|)\Big)|x-y|^{-N} \\ &\leq 2^{m-1}C\left(f_n(x,y) + \Phi_{x,y}(|D_s u|)\right)|x-y|^{-N} + 2^{m-1}\Phi_{x,y}(|D_s u|)|x-y|^{-N}, \end{aligned}$$

which implies that the sequence $(h_n)_{n\in\mathbb{N}}$ is also equi-integrable. Therefore, (4.22) holds by Vitali's Theorem.

Using arguments similar to those above, we can prove that (4.23) holds. □

Remark 4.5.9. *We point out that in Theorem 4.5.8 we can consider the condition* (ϕ_3) *with* $\ell \geq 1$ *and therefore the space* $W^{s,\Phi_{x,y}}(\Omega)$ *is non-reflexive. Also, it is not used that* $t \mapsto \Phi_{x,y}(\sqrt{t})$ *is convex. Thus, Theorem 4.5.8 can be seen as a generalization of the result obtained by Bahrouni, Ounaies and Tavares (2020) in Lemma 3.4.*

In the sequel, we will give an alternative proof for the (S_+)-property in domain arbitrary assuming a stronger hypothesis than (ϕ_2), namely:

$(\phi_2)'$ $t \mapsto t\varphi_{x,y}(t)$ is strictly increasing.

Theorem 4.5.10. *Assume that* $(\phi_1), (\phi_2)', (\phi_3)$ *and* (ϕ_4) *hold. Then,* $\mathcal{J}'_{s,\Phi}$ *satisfies* (S_+)-*property.*

Proof. Using the same techniques as in the proof of Theorem 4.5.8, we conclude that the sequence $(f_n(x,y))_{n\in\mathbb{N}}$ defined by

$$f_n(x,y) := (\varphi_{x,y}(|D_s u_n|)D_s u_n - \varphi_{x,y}(|D_s u|)D_s u)\,(D_s u_n - D_s u)$$

converges to 0 in $L^1(\Omega\times\Omega, d\mu)$. Then, by conversely of Lebesgue's Theorem, there exist $g \in L^1(\Omega\times\Omega, d\mu)$ and a subsequence, still denoted by $(f_n)_{n\in\mathbb{N}}$, such that $|f_n| \leq g$ and $f_n \to 0$ μ-a.e. in $\Omega\times\Omega$. Thus, by inequality (4.28), we obtain $C := C(\varepsilon, \ell, m) > 0$ such that

$$\begin{aligned}\Phi_{x,y}(|D_s u_n(x,y)|) &\leq \ell^{-1}\varphi_{x,y}(|D_s u_n(x,y)|)(D_s u_n)^2(x,y)\\ &\leq C\,(g(x,y) + \Phi_{x,y}(|D_s u(x,y)|)) \in L^1(\Omega\times\Omega, d\mu).\end{aligned} \tag{4.29}$$

On the other hand, since $(\phi_2)'$ holds, the inequality (4.26) becomes

$$(t\varphi_{x,y}(|t|) - s\varphi_{x,y}(|s|))(t-s) > 0, \quad \text{for a.e. } (x,y)\in\Omega\times\Omega \text{ and } t \neq s. \tag{4.30}$$

We proceed as in Lemma 6 present in Dal Maso and Murat (1998) to prove that $D_s u_n(x,y)$ converges to $D_s u(x,y)$ μ-a.e. in $\Omega\times\Omega$. Indeed, we assume by contradiction that there exist $\varepsilon > 0$ and a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that $|D_s u_n - D_s u| \geq \varepsilon$ for all n. Let $t_n = \varepsilon|D_s u_n - D_s u|^{-1}$ and we consider the sequence

$$v_n(x,y) = t_n D_s u_n(x,y) + (1-t_n)D_s u(x,y), \quad (x,y)\in\Omega\times\Omega.$$

Observe that $|v_n - D_s u| = t_k |D_s u_n - D_s u| = \varepsilon$. This means that $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Thus, a subsequence, still denoted by $(v_n)_{n\in\mathbb{N}}$, converges to some measurable function $v : \Omega \times \Omega \to \mathbb{R}$ where $|v - D_s u| = \varepsilon$. Since (4.30) holds and $0 < t_n \leq 1$, it follows that

$$\begin{aligned} f_n(x,y) &= \left(\varphi_{x,y}(|D_s u_n|) D_s u_n - \varphi_{x,y}(|v_n|) v_n\right)(D_s u_n - D_s u) \\ &\quad + \left(\varphi_{x,y}(|v_n|) v_n - \varphi_{x,y}(|D_s u|) D_s u\right)(D_s u_n - D_s u) \\ &\geq \left(\varphi_{x,y}(|v_n|) v_n - \varphi_{x,y}(|D_s u|) D_s u\right)(D_s u_n - D_s u) \\ &\geq \left(\varphi_{x,y}(|v_n|) v_n - \varphi_{x,y}(|D_s u|) D_s u\right)(v_n - D_s u) \geq 0. \end{aligned}$$

Thereby, this inequality and convergence $f_n \to 0$ μ-a.e. imply

$$\lim_{n\to\infty} \left(\varphi_{x,y}(|v_n|) v_n - \varphi_{x,y}(|D_s u|) D_s u\right)(v_n - D_s u) = 0.$$

On the other side, from continuity of $\varphi_{x,y}(\cdot)$ we have

$$\lim_{n\to\infty} \left(\varphi_{x,y}(|v_n|) v_n - \varphi_{x,y}(|D_s u|) D_s u\right)(v_n - D_s u) = \left(\varphi_{x,y}(|v|) v - \varphi_{x,y}(|D_s u|) D_s u\right)(v - D_s u),$$

which implies that $\left(\varphi_{x,y}(|v|) v - \varphi_{x,y}(|D_s u|) D_s u\right)(v - D_s u) = 0$ μ-a.e. Since $|v - D_s u| = \varepsilon$, this contradicts the strict monotonicity (4.30), proving the convergence.

Consequently, by Lebesgue's Dominated Convergence Theorem, $J_{s,\Phi}(u_n) \to J_{s,\Phi}(u)$. Analogously, we have $J_{\widehat{\Phi}}(u_n) \to J_{\widehat{\Phi}}(u)$. This shows that $\mathcal{J}_{s,\Phi}(u_n) \to \mathcal{J}_{s,\Phi}(u)$. Therefore, by Corollary 4.5.2, we obtain $\|u_n - u\|_{(\Omega)} \to 0$, i.e., $u_n \to u$ strongly in $W^{s,\Phi_{x,y}}(\Omega)$. □

The next result characterizes the strong convergence in the space $W^{s,\Phi_{x,y}}(\Omega)$ under the assumption $(\phi_2)'$.

Proposition 4.5.11. *Assume that $(\phi_1), (\phi_2)', (\phi_3)$ and (ϕ_4) hold. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{s,\Phi_{x,y}}(\Omega)$. Then, $u_n \to u$ in $W^{s,\Phi_{x,y}}(\Omega)$ if and only if*

$$\lim_{n\to\infty} \left\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u), u_n - u \right\rangle = 0. \tag{4.31}$$

Proof. If $u_n \to u$, then by Proposition 4.5.4 the limit (4.31) holds. Conversely, assuming (4.31) and arguing as in proof Theorem 4.5.10, we obtain the desired result. □

Next, we introduce another monotonicity result in the presence of hypothesis $(\phi_2)'$.

Proposition 4.5.12. *Assume that $(\phi_1), (\phi_2)', (\phi_3)$ and (ϕ_4) hold. Then, $\mathcal{J}'_{s,\Phi}$ is a homeomorphism strictly monotone.*

Proof. The strict monotonicity of $\mathcal{J}'_{s,\Phi}$ follows from (4.30). Thus, by Proposition 4.5.5 (i) and Minty-Browder Theorem (ZEIDLER, 2013, Theorem 26.A), $\mathcal{J}'_{s,\Phi}$ is invertible and $(\mathcal{J}'_{s,\Phi})^{-1}$ is strictly monotone and bounded. Therefore, in order to complete the proof of (ii) we only need to show that $(\mathcal{J}'_{s,\Phi})^{-1}$ is continuous. For this purpose, let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $g_n \to g$ strongly in $\left(W^{s,\Phi_{x,y}}(\Omega)\right)^*$. By taking $u_n = (\mathcal{J}'_{s,\Phi})^{-1}(g_n)$ and $u = (\mathcal{J}'_{s,\Phi})^{-1}(g)$, it follows from the strong convergence of $(g_n)_{n\in\mathbb{N}}$ and the boundedness of $(\mathcal{J}'_{s,\Phi})^{-1}$ that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{s,\Phi_{x,y}}(\Omega)$. Thus, up to subsequence $u_n \rightharpoonup u_0$ in $W^{s,\Phi_{x,y}}(\Omega)$. Consequently,

$$\begin{aligned}\lim_{n\to\infty}\langle \mathcal{J}'_{s,\Phi}(u_n) - \mathcal{J}'_{s,\Phi}(u_0), u_n - u_0\rangle &= \lim_{n\to\infty}\langle \mathcal{J}'_{s,\Phi}(u_n) - g, u_n - u_0\rangle \\ &\quad + \lim_{n\to\infty}\langle g - \mathcal{J}'_{s,\Phi}(u_0), u_n - u_0\rangle \\ &= 0.\end{aligned}$$

This fact jointly with Theorem 4.5.10 imply that $u_n \to u_0$. Hence, by the continuity of the operator $\mathcal{J}'_{s,\Phi}$ we obtain

$$\mathcal{J}'_{s,\Phi}(u_0) = \lim_{n\to\infty}\mathcal{J}'_{s,\Phi}(u_n) = \lim_{n\to\infty} g_n = g = \mathcal{J}'_{s,\Phi}(u),$$

i.e., $u = u_0$. Therefore, $(\mathcal{J}'_{s,\Phi})^{-1}$ is continuous. □

4.6 APPLICATION TO A NONLOCAL FRACTIONAL TYPE PROBLEM

In this section, we investigate the existence of nontrivial solution for the following class of fractional type problems

$$\begin{cases} \mathcal{L}^s_{\varphi_{x,y}} u + \widehat{\varphi}_x(|u|)u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.32}$$

where $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $f : \Omega \to \mathbb{R}$ is a measurable function that belongs to a suitable Musielak-Orlicz space.

Here, we consider a general nonlocal nonlinear operator of the Φ-Laplacian type $\mathcal{L}^s_{\Phi_{x,y}}$: $W_0^{s,\Phi_{x,y}}(\Omega) \to \left(W_0^{s,\Phi_{x,y}}(\Omega)\right)^*$ defined as follows:

$$\langle \mathcal{L}^s_{\Phi_{x,y}} u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi_{x,y}(|D_s u|) D_s u D_s v \, d\mu.$$

where $s \in (0,1)$, $\Phi_{x,y}(t) = \int_0^t \varphi_{x,y}(\tau)\tau \, d\tau$ is a Musielak function that satisfies the conditions $(\phi_1) - (\phi_4)$ and $\widehat{\varphi}_x(t) = \varphi_{x,x}(t)$. It is not hard to see that the operator $\mathcal{L}^s_{\Phi_{x,y}}$ is well-defined and coincide with Fréchet derivative of modular function $J_{s,\Phi} : W_0^{s,\Phi_{x,y}}(\Omega) \to \mathbb{R}$ defined in Section 4.3. See also Lemma 4.5.4.

Remark 4.6.1. *Due to the absence of the Poincaré inequality in the considered context, it is not possible to address the problem without including the nonlinear term* $\widehat{\varphi}_x(|t|)t$.

In order to define the notion of solution for problem (4.32), we need to require a symmetry assumption in x and y for the function $\varphi(x,y,t)$, precisely,

$$\varphi(x,y,t)=\varphi(y,x,t), \quad \text{for all } (x,y)\in\Omega\times\Omega \text{ and } t\geq 0.$$

Definition 4.6.2. *A function* $u\in W_0^{s,\Phi_{x,y}}(\Omega)$ *is said to be a weak solution for Problem* (4.32) *if satisfies*

$$\langle\mathcal{L}^s_{\Phi_{x,y}}u,v\rangle+\int_\Omega\widehat{\varphi}_x(|u(x)|)uv\,dx=\int_\Omega f(x)v\,dx,$$

for all $v\in W_0^{s,\Phi_{x,y}}(\Omega)$.

The following result is an immediate consequence of the Proposition 4.5.12.

Proposition 4.6.3. *Assume that* $(\phi_1),(\phi_2)',(\phi_3)$ *and* (ϕ_4) *hold. If* $f\in L^{\widetilde{\widehat{\Phi}}_x}(\Omega)$, *then Problem* (4.32) *has a unique weak solution.*

Proof. By Hölder's inequality it can be seen that

$$\langle f,v\rangle:=\int_\Omega f(x)v(x)\;dx,\quad v\in W_0^{s,\Phi_{x,y}}(\Omega),$$

defines a continuous linear functional on $W_0^{s,\Phi_{x,y}}(\Omega)$, i.e., $f\in\left(W_0^{s,\Phi_{x,y}}(\Omega)\right)^*$. It follows from Proposition 4.5.12 that $J'_{s,\Phi}$ is bijective. Therefore, Problem (4.32) has a unique weak solution. □

4.7 SOME CLASSES OF PROBLEMS

In this section, we present some examples of functions $\Phi_{x,y}$ for which the existence result Theorem 4.6.3 may be applied.

Double phase problem

For $1<p<q<\infty$ and $a\in L^\infty(\Omega\times\Omega)$ a non-negative symmetric function, we consider the Musielak function given by

$$\Phi_{x,y}(t)=\frac{|t|^p}{p}+a(x,y)\frac{|t|^q}{q}.$$

This gives the operator

$$\mathcal{L}^s_{\Phi_{x,y}}u = \mathcal{L}^s_p u + \mathcal{L}^s_{q,a}u,$$

where (up to multiplicative constant) $\mathcal{L}^s_p := (-\Delta)^s_p$ is the so called fractional p-Laplacian operator and $\mathcal{L}^s_{q,a} := (-\Delta)^s_{q,a}$ is the anisotropic fractional p-Laplacian defined as

$$\langle \mathcal{L}^s_{q,a}u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} a(x,y)\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sq}}\,dxdy. \quad (4.33)$$

In this case, the nonlocal operator present in (4.32) is associated with the energy functional

$$J_{s,\Phi}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi_{x,y}(|D_s u|)\,\mathrm{d}\mu = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{|D_s u|^p}{p} + a(x,y)\frac{|D_s u|^q}{q}\right)d\mu, \quad (4.34)$$

whose integrand $\Phi_{x,y}(t)$ shows an unbalanced growth, precisely

$$C_1|t|^p \le \Phi_{x,y}(t) \le C_2(|t|^p + |t|^q), \text{ for a.e. } (x,y)\in\Omega\times\Omega \text{ and for all } t\in\mathbb{R},$$

with $C_1, C_2 > 0$. It is not hard to see that $\Phi_{x,y}(1) \le C$ for a.e. $(x,y)\in\Omega\times\Omega$ and

$$\Phi_{x,y}(2t) = |2t|^p + a(x,y)|2t|^q \le 2^q\Phi_{x,y}(t), \quad (x,y)\in\Omega\times\Omega,\ t\in\mathbb{R},$$

that is, Φ satisfies the (Δ_2)-condition. We note that by direct computation,

$$t\varphi_{x,y}(t) = \Phi'_{x,y}(t) = |t|^{p-2}t + a(x,y)|t|^{q-2}t,\ t\neq 0,$$

and thus, the conditions $(\phi_1)-(\phi_4)$ are satisfied with $\ell = p$ and $m = q$.

The main feature of the functional integral (4.34) is the change of ellipticity and growth properties on the set where the weight function $a(\cdot,\cdot)$ vanishes. More precisely, the energy density of $J_{s,\Phi}$ is controlled by $D_s u$ at an order q in the set $\{(x,y)\in\Omega\times\Omega:\ a(x,y)\neq 0\}$ and at an order p in the set $\{(x,y)\in\Omega\times\Omega:\ a(x,y)=0\}$. For this reason, this operator is known as fractional double phase operator. Moreover, it is a special case of functional with non-standard growth conditions, according to Marcellini (1989) terminology. Due to the unbalanced growth of $\Phi_{x,y}(\cdot)$, the classical fractional Sobolev space is not suitable to analyze (4.32) and so we have to use the general abstract setting of the new fractional Musielak-Sobolev spaces.

Fractional double phase problems are motivated by numerous local and nonlocal models arising in many fields of mathematical physics, for instance, composite materials, fractional quantum mechanics in the study of particles on stochastic fields, fractional superdiffusion, fractional white-noise limit and several equations that appear in the electromagnetism, electrostatics, and electrodynamics as a model based on a modification of Maxwell's Lagrangian density. For more details, see the works (AMBROSIO; RĂDULESCU, 2020; ZHANG; TANG; RĂDULESCU, 2021) and the references therein.

We point out that, in the local case $s = 1$, Zhikov (1987) was the first to investigate the integral functional of the double phase type in the context of the theory of elasticity and calculus of variations to describe models of strongly anisotropic materials. For instance, in the elasticity theory, the weight function $a(\cdot,\cdot)$ is called modulating coefficient and dictates the geometry of composites made of two different materials with distinct power hardening p and q. This terminology, including the name double phase, was introduced by Colombo and Mingione (2015) when they first studied elliptic equations driven by double phase operator and their regularity properties.

Fractional $p(x,\cdot)$-Laplacian operator

Given the function $\Phi_{x,y}(t) = \frac{1}{p(x,y)}|t|^{p(x,y)}$ with $p \in L^0(\Omega \times \Omega)$ symmetric and satisfying the following condition

$$1 < p^- \leq p(x,y) \leq p^+ < \infty, \quad (x,y) \in \Omega \times \Omega,$$

as a second example, we have the well-known fractional $p(x,y)$-Laplacian operator

$$\langle \mathcal{L}^s_{p_{x,y}} u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\, dxdy.$$

It is not hard to see that $\Phi_{x,y}$ satisfy conditions $(\varphi_1)-(\varphi_3)$ with $\ell = p^-$ and $m = p^+$. This type of operator has applications in several fields of physics and mathematics, for example, filtration of fluids in porous media, restricted heating, elastoplasticity, image processing, optimal control and financial mathematics (ABERQI et al., 2022). For a more comprehensive study of nonlocal problems of this nature, we refer the reader to works (KAUFMANN; ROSSI; VIDAL, 2017; BAHROUNI; RĂDULESCU, 2018) and (AZROUL; BENKIRANE; SHIMI, 2019)

Logarithmic perturbation of the $p(x,\cdot)$-Laplacian operator

Given the function $\Phi_{x,y}(t) = |t|^{p(x,y)}\log(1+|t|)$ with $p \in L^0(\Omega \times \Omega)$ symmetric and satisfying the following condition

$$1 < p^- \leq p(x,y) \leq p^+ < \infty, \quad (x,y) \in \Omega \times \Omega,$$

as a third example, we can consider the operator

$$\mathcal{L}^s_{\Phi_{x,y}} u(x) := \int_{\mathbb{R}^N} \left(p(x,y)|D_s u|^{p(x,y)-2} \log(1+|D_s u|) + \frac{|D_s u|^{p(x,y)-1}}{1+|D_s u|} \right) D_s u \, \frac{dy}{|x-y|^{N+s}},$$

By direct computations, we have

$$t\varphi_{x,y}(t) = \Phi'_{x,y}(t) = p(x,y)t^{p(x,y)-1}\log(1+t) + \frac{t^{p(x,y)}}{1+t}, \quad (x,y) \in \Omega \times \Omega,\ t > 0, \tag{4.35}$$

When $p(x,y) \equiv p$, $(-\Delta)^s_{\Phi_{x,y}}$ is a fractional version of the logarithmic perturbation of the classical fractional p-Laplacian problem. It is clear that $\Phi_{x,y}$ satisfies the assumptions (φ_1) and (φ_2). It remains to show that (φ_3) holds. Indeed, note that

$$p^- \leq p(x,y) \leq p(x,y) + \frac{t}{(1+t)\log(1+t)} = \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)}, \quad (x,y) \in \Omega \times \Omega,\ t > 0.$$

In addition,

$$\lim_{t\to 0^+} \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)} = p(x,y) + 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)} = p(x,y), \quad (x,y) \in \Omega \times \Omega.$$

Thus, since $\frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)}$ is continuous on $\Omega \times \Omega \times (0,\infty)$, it follows that

$$p^- \leq \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)} \leq p^+ + 1, \quad (x,y) \in \Omega \times \Omega,\ t > 0.$$

This we conclude that the condition (φ_3) is satisfied for $\ell = p^-$ and $m = p^+ + 1$.

5 A BOURGAIN-BREZIS-MIRONESCU TYPE FORMULA

In this chapter, we study the asymptotic behavior of anisotropic nonlocal nonstandard growth seminorms and modulars related to general fractional Musielak-Sobolev spaces as the fractional parameter goes to 1 without assuming the Δ_2-condition. This provides a so-called Bourgain-Brezis-Mironescu type formula for a very general family of functionals. Precisely, given $s \in (0,1)$ and a Musielak function $\Phi(x,y,t) = \int_0^t \phi(x,y,\tau)\,d\tau$, we consider the energy functional

$$J_{s,\Phi}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi(x,y,|D_s u(x,y)|)\,d\mu,$$

where the s-Hölder quotient $D_s u$ and the measure μ are defined as

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^s}, \qquad d\mu := \frac{dxdy}{|x-y|^N}.$$

The functional $J_{s,\Phi}$ is well-defined when u belongs to the fractional Musielak-Sobolev space $W^{s,\Phi_{x,y}}(\mathbb{R}^N)$ presented in Chapter 4. We emphasize that the Δ_2-condition on Φ or on its complementary function is not required in our results.

In order to prove our results, we assume some structural hypotheses on the Musielak function Φ. First, we impose a boundedness condition on Φ with respect to $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$: there exist constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \le \inf_{x,y\in\mathbb{R}^N} \Phi(x,y,1) \le \Phi(x,y,1) \le \sup_{x,y\in\mathbb{R}^N} \Phi(x,y,1) \le C_2. \tag{H_1}$$

In order to analyze the behavior as $s \to 1^-$, we impose the following condition:

$$y \mapsto \Phi(x,y,t) \text{ is continuous.} \tag{H_2}$$

Our main result, stated in Theorem 5.1.3, establishes that for any $u \in C_0^2(\mathbb{R}^N)$, there exists $\lambda_0 > 0$ such that

$$\lim_{s\to 1^-}(1-s)J_{s,\Phi}\left(\frac{u}{\lambda}\right) = \int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u(x)|}{\lambda}\right)dx, \quad \text{for all } \lambda \ge \lambda_0,$$

where the function H is given by

$$H(x,t) = \int_0^1 \int_{\mathbb{S}^{N-1}} \Phi(x,x,t|w_N|r)\,d\mathcal{H}^{N-1}(w)\frac{dr}{r}$$

and w_N is the N-th coordinate of any point in $\mathbb{S}^{N-1}$. In Proposition 5.1.2 it is proved that the limit function $H(x,t)$ is in fact equivalent to the Musielak function $\widehat{\Phi}(x,t) := \Phi(x,x,t)$.

As a consequence, we obtain in Corollary 5.1.6 a BBM type inequality for seminorms. Namely, we prove that there exists $\lambda_0 \geq 1$ such that for any $u \in C_0^2(\mathbb{R}^N)$ and $\lambda \geq \lambda_0$ it holds that

$$\limsup_{s\to 1^-}[[u]]_{s,\Phi} \leq \lambda\|\nabla u\|_H.$$

Examples of functions fulfilling our hypotheses include:

(i) $\Phi(x,y,t) = A(t)$ for any Young function A, that is, a convex function $A\colon [0,\infty) \to [0,\infty)$ with $A(0) = 0$. In particular, for $A(t) = t^p$ and $A(t) = t^p \log(1+t)$ with $p \in [1,\infty)$, $A(t) = e^t - t - 1$ and $A(t) = e^{t^\alpha} - t$ with $\alpha \in (1,\infty)$.

(ii) $\Phi(x,y,t) = t^p + a(x,y)t^q$ where $t \geq 0$, $1 \leq p < q < \infty$ and $a(x,y)$ is a non-negative bounded and continuous function in the second variable.

(iii) $\Phi(x,y,t) = t^{p(x,y)}$ with $t \geq 0$, and $p(x,y)$ is a continuous function in the second variable such that $1 \leq p^- \leq p(x,y) \leq p^+ < \infty$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(iv) $\Phi(x,y,t) = a(x,y)(e^t - t - 1)$ where $a(x,y)$ is defined as in (ii).

See Section 5.2 for the precise statement of our results in each particular example.

These examples include equations defined in the fractional Orlicz-Sovolev spaces (ALBERICO et al., 2021a; ALBERICO et al., 2020; FERNÁNDEZ BONDER; SALORT, 2019; FERNÁNDEZ BONDER; SALORT, 2021), double phase problems (BARONI; COLOMBO; MINGIONE, 2015; BARONI; COLOMBO; MINGIONE, 2018) and Sobolev spaces of variable exponent (BAHROUNI; RĂDULESCU, 2018; KIM, 2023; KAUFMANN; ROSSI; VIDAL, 2017).

The limit formula that we obtain in Theorem 5.1.3 is not valid for the entire local Musielak-Sobolev space $W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$ for an arbitrary Musielak function Φ that satisfies (H_1) and (H_2), as demonstrated by some counterexamples in Kim (2023). However, in Section 5.2, we prove its validity for certain particular classes. In fact, for any Musielak function, the limit formula holds in the classical Sobolev space as shown in Corollary 5.1.5, which is determined by the powers ℓ, m when assuming the following growth behavior for $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty)$: there exist constants $1 \leq \ell \leq m < \infty$ such that

$$\ell \leq \frac{t\phi(x,y,t)}{\Phi(x,y,t)} \leq m. \tag{H_3}$$

We have known that $\Phi(\cdot,\cdot,t)$ and its complementary function satisfy the Δ_2-condition if and only if (H_3) holds with $\ell > 1$.

It is worthwhile to mention that we have extended and complemented the case of the fractional Sobolev space with a variable exponent (i.e., example (iii) above). Our result imposes a weaker assumption on $\Phi(x,\cdot,t)$ compared to those required by Kim (2023). Furthermore, we do not require that neither Φ nor its conjugate function fulfill the Δ_2-condition.

The techniques used in our results enable us to study energy functionals where the s-Hölder quotient depends only on a direction, that is,

$$D_s^k u(x,h) := \frac{u(x-he_k)-u(x)}{|h|^s}, \quad \text{with } k \in \{1,\dots,N\},$$

being e_k the k-th canonical vector in $\mathbb{R}^N$. More precisely, in Theorem 5.3.1 we prove that, for $u \in C_0^2(\mathbb{R}^N)$,

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}}\Phi\left(x, x-he_k, \frac{|D_s^k u(x,y)|}{\lambda}\right)\frac{dhdx}{|h|} = \int_{\mathbb{R}^N} H\left(x, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_k}\right|\right)dx,$$

for all $\lambda \geq \lambda_0$, where in this case the limit function H is defined as

$$H(x,t) = 2\int_0^1 \Phi(x,x,tr)\frac{dr}{r}.$$

The chapter is organized as follows. The forthcoming section contains the proof of our main results. In Section 5.2, we introduce some application examples considered in this theory. In Section 5.3, we address the anisotropic case and provide some extensions.

5.1 THE BBM TYPE FORMULA

We begin this section by proving a technical lemma. These result plays a very important role in the proof of the BBM formula.

Lemma 5.1.1. *Assume* (H_1) *and* (H_2)*. Then, for any* $x \in \mathbb{R}^N$ *and* $0 \leq t \leq 1$*, it holds that*

$$\lim_{s\to 1^-}(1-s)\int_0^1\int_{\mathbb{S}^{N-1}}\Phi(x, x-rw, t|w_N|r^{1-s})\, dS_w\frac{dr}{r} = H(x,t),$$

where

$$H(x,t) := \int_0^1\int_{\mathbb{S}^{N-1}}\Phi(x,x,t|w_N|r)\, dS_w\frac{dr}{r} \tag{5.1}$$

and w_N *is the* N*-th coordinate of any point in* $\mathbb{S}^{N-1}$*.*

Proof. By performing the change of variables $\rho = r^{1-s}$, we deduce that

$$\begin{aligned}\int_0^1\int_{\mathbb{S}^{N-1}}&\Phi(x, x-rw, t|w_N|r^{1-s})\, dS_w\frac{dr}{r}\\ &= \frac{1}{1-s}\int_0^1\int_{\mathbb{S}^{N-1}} G\left(x, x-\rho^{\frac{1}{1-s}}w, t|w_N|\rho\right) dS_w\frac{d\rho}{\rho}.\end{aligned}$$

Since $0 < \rho < 1$ and $\Phi(x, \cdot, t)$ is continuous at x by (H_2), it follows that

$$\lim_{s \to 1^-} \Phi\left(x, x - \rho^{\frac{1}{1-s}} w, t|w_N|\rho\right) = \Phi\left(x, x, t|w_N|\rho\right).$$

Moreover, using the convexity of $\Phi(x, y, \cdot)$ and (H_1), we have that

$$\Phi\left(x, x - \rho^{\frac{1}{1-s}} w, t|w_N|\rho\right) \rho^{-1} \leq \sup_{x,y \in \mathbb{R}^N} \Phi(x, y, 1)|w_N| \leq C_2.$$

Therefore, the result follows by Lebesgue's Dominated Convergence Theorem. □

The next proposition ensures that $H(x,t)$ is a Musielak function and that it is equivalent to $\widehat{\Phi}(x,t)$. Thereby, the Musielak-Orlicz spaces $L^{\widehat{\Phi}_x}(\mathbb{R}^N)$ and $L^{H_x}(\mathbb{R}^N)$ are the same.

Proposition 5.1.2. *The function H defined in* (5.1) *is a Musielak function. Furthermore, there exist positive constants c_1 and c_2 such that*

$$\widehat{\Phi}(x, c_1 t) \leq H(x,t) \leq c_2 \widehat{\Phi}(x,t), \tag{5.2}$$

for any $(x,t) \in \mathbb{R}^N \times [0, \infty)$.

Proof. We prove first that H is a Musielak function. Note that, making use of the change of variables $\rho = tr$, we can write

$$H(x,t) = \int_0^t \int_{\mathbb{S}^{N-1}} \Phi(x, x, |w_N|\rho)\, dS_w \frac{d\rho}{\rho} = \int_0^t h(x, \rho)\, d\rho,$$

where

$$h(x,\rho) = \int_{\mathbb{S}^{N-1}} \frac{\Phi(x, x, |w_N|\rho)}{\rho}\, dS_w.$$

It is not hard to see that $h(x, \cdot) > 0$ for any $x \in \mathbb{R}^N$. Finally, since $\Phi(x, x, \cdot)$ is continuous and the function $t^{-1}\Phi(x, x, |w_N|t)$ is increasing in $t \in (0, \infty)$ for any $x \in \mathbb{R}^N$, we conclude that $h(x, \cdot)$ is right-continuous and increasing.

It remains to prove the equivalence (5.2). Given $x \in \mathbb{R}^N$ and $t \geq 0$, it follows from the monotonicity of $\Phi(x, y, t)$ at t that

$$\begin{aligned} H(x,t) &\leq \int_0^t \int_{\mathbb{S}^{N-1}} \frac{\Phi(x, x, \rho)}{\rho}\, dS_w d\rho \\ &\leq N\omega_N \int_0^t \frac{\Phi(x, x, \rho)}{\rho}\, d\rho \\ &\leq N\omega_N \Phi(x, x, t), \end{aligned}$$

where in the last inequality we have used that $t \mapsto \frac{\Phi(x,x,t)}{t}$ is increasing. On the other hand, using again this monotonicity property, we have

$$\begin{aligned} H(x,t) &\geq \int_0^t \int_{\left\{w\in\mathbb{S}^{N-1}:|w_N|\geq\frac{1}{2}\right\}} \Phi\left(x,x,\rho/2\right) dS_w \frac{d\rho}{\rho} \\ &\geq \left(\int_{\left\{w\in\mathbb{S}^{N-1}:|w_N|\geq\frac{1}{2}\right\}} dS_w\right)\left(\int_0^t \frac{\Phi\left(x,x,\rho/2\right)}{\rho}\,d\rho\right) \\ &= C(N)\int_0^{\frac{t}{2}} \frac{\Phi\left(x,x,r\right)}{r}\,dr \\ &\geq C(N)\int_{\frac{t}{4}}^{\frac{t}{2}} \frac{\Phi\left(x,x,r\right)}{r}\,dr \geq C(N)\Phi\left(x,x,t/4\right) \geq \Phi\left(x,x,c_1 t\right), \end{aligned}$$

where the last inequality hold with $c_1 = \frac{1}{4}\min\{1, C(N)\}$. This ends the proof. $\square$

In the following we establish our main result.

Theorem 5.1.3 (BBM type formula)**.** *Let $u \in C_0^2(\mathbb{R}^N)$. Assume (H_1) and (H_2). Then, there exists $\lambda_0 > 0$ such that*

$$\lim_{s\to 1^-}(1-s)J_{s,\Phi}\left(\frac{u}{\lambda}\right) = \int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u(x)|}{\lambda}\right) dx \tag{5.3}$$

for all $\lambda \geq \lambda_0$, where H was defined in (5.1)*. In particular, if Φ satisfies Δ_2-condition, then equation* (5.3) *holds for every $\lambda > 0$.*

Proof. Let $\lambda_0 = 2\|u\|_{C_2(\mathbb{R}^N)}$. Then, using (H_1), we have

$$\int_{\mathbb{R}^N} \widehat{\Phi}\left(x, \frac{|u(x)|}{\lambda_0}\right) dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} \widehat{\Phi}\left(x, \frac{|\nabla u(x)|}{\lambda_0}\right) dx < \infty.$$

Thus, by Proposition 5.2, we also have $\int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u(x)|}{\lambda_0}\right) dx < \infty$. For a fixed $x \in \mathbb{R}^N$ and $\lambda \geq \lambda_0$, we split the integral

$$\begin{aligned} \int_{\mathbb{R}^N} \Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right)\frac{dh}{|h|^N} &= \int_{|h|<1} \Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right)\frac{dh}{|h|^N} \\ &\quad + \int_{|h|\geq 1} \Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right)\frac{dh}{|h|^N} \\ &=: I_1 + I_2. \end{aligned}$$

Let us deal with I_2. By using the monotonicity and convexity of $G(x,y,t)$ in t and (H_1) , we have that

$$\begin{aligned} I_2 &\leq \int_{|h|\geq 1} \Phi\left(x, x-h, \frac{2\|u\|_\infty}{\lambda|h|^s}\right)\frac{dh}{|h|^N} \\ &\leq \int_{|h|\geq 1} \Phi\left(x, x-h, \frac{2\|u\|_\infty}{\lambda}\right)\frac{dh}{|h|^{n+s}} \\ &\leq C_2\frac{N\omega_N}{s}, \end{aligned} \tag{5.4}$$

from where we obtain

$$\lim_{s\to 1^-}(1-s)I_2 = 0. \tag{5.5}$$

Let us now estimate I_1. Since u is a C^2 function and $\Phi(x,y,t)$ is locally Lipschitz continuous in t, we deduce

$$\left|\Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right) - \Phi\left(x, x-h, \frac{1}{\lambda}\left|\nabla u(x)\cdot\frac{h}{|h|^s}\right|\right)\right| \\ \leq \frac{L}{\lambda}\frac{|u(x)-u(x-h)-\nabla u(x)\cdot h|}{|h|^s} \leq C|h|^{2-s},$$

where L is the Lipschitz constant of Φ on the interval $[0, \|\nabla u\|_\infty]$ and C depends on the C^2-norm of u. Since

$$\int_{|h|<1}|h|^{2-s-N}\,dh = \frac{N\omega_N}{2-s},$$

it follows that

$$\lim_{s\to 1^-}(1-s)I_1 = \lim_{s\to 1^-}(1-s)\int_{|h|<1}\Phi\left(x, x-h, \frac{1}{\lambda}\left|\nabla u(x)\cdot\frac{h}{|h|^s}\right|\right)\frac{dh}{|h|^N}.$$

Observe that, by using spherical coordinates, we can write

$$\begin{aligned}\int_{|h|<1}\Phi\left(x, x-h, \frac{1}{\lambda}\left|\nabla u(x)\cdot\frac{h}{|h|^s}\right|\right)\frac{dh}{|h|^N} &= \int_0^1\int_{|h|=r}\Phi\left(x, x-h, \frac{1}{\lambda}\left|\nabla u(x)\cdot\frac{h}{|r|^s}\right|\right)dS_h\frac{dr}{r^N} \\ &= \int_0^1\int_{\mathbb{S}^{N-1}} r^{N-1}\Phi\left(x, x-rw, \frac{|\nabla u(x)\cdot w|}{\lambda}r^{1-s}\right)dS_w\frac{dr}{r^N} \\ &= \int_0^1\int_{\mathbb{S}^{N-1}}\Phi\left(x, x-rw, \frac{|\nabla u(x)|}{\lambda}|w_N|r^{1-s}\right)dS_w\frac{dr}{r},\end{aligned}$$

where in the last equality we have performed a rotation such that $\nabla u(x) = |\nabla u(x)|e_N$. Thus, in view of Lemma 5.1.1, we have that

$$\lim_{s\to 1^-}(1-s)I_1 = H\left(x, \frac{|\nabla u(x)|}{\lambda}\right), \tag{5.6}$$

for any $x\in\mathbb{R}^N$. Gathering (5.5) and (5.6), we conclude that, for any $x\in\mathbb{R}^N$,

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right)\frac{dh}{|h|^N} = H\left(x, \frac{|\nabla u(x)|}{\lambda}\right).$$

In order to complete the proof, it only remains to show the existence of an integrable majorant for $(1-s)F_s$, where

$$F_s(x) := \int_{\mathbb{R}^N}\Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right)\frac{dh}{|h|^N}.$$

Since $u \in C_0^2(\mathbb{R}^N)$, we can assume without loss of generality that $\operatorname{supp}(u) \subset B_R(0)$ with $R > 1$. First we analyze the behavior of $F_s(x)$ for small values of x. When $|x| < 2R$ we can write

$$|F_s(x)| = \left(\int_{|h|<1} + \int_{|h|\geq 1}\right) \Phi\left(x, x-h, \frac{|D_s u(x,x-h)|}{\lambda}\right) \frac{dh}{|h|^N} := I_1 + I_2.$$

By using the expression

$$|u(x) - u(x-h)| \leq \int_0^1 |\nabla u(x-h+th) \cdot h| \leq \|\nabla u\|_\infty |h|$$

together with the convexity and monotonicity of Φ, and assumption (H_1), we obtain

$$\begin{aligned}
I_1 &\leq \int_{|h|<1} \Phi\left(x, x-h, \frac{\|\nabla u\|_\infty |h|^{1-s}}{\lambda}\right) \frac{dh}{|h|^N} \\
&\leq \int_{|h|<1} |h|^{1-s-n} \Phi\left(x, x-h, \frac{\|\nabla u\|_\infty}{\lambda}\right) dh \\
&\leq \int_{|h|<1} |h|^{1-s-n} \Phi(x, x-h, 1)\, dh \\
&\leq C_2 \frac{N\omega_N}{1-s}.
\end{aligned} \tag{5.7}$$

Furthermore, it is follows from (5.4) that

$$I_2 \leq \frac{C_2 N \omega_N}{s}. \tag{5.8}$$

On the other hand, when $|x| \geq 2R$, u vanishes and we have that

$$F_s(x) = \int_{B_R(0)} \Phi\left(x, y, \frac{|u(y)|}{\lambda |x-y|^s}\right) \frac{dy}{|x-y|^N}.$$

Since $|x-y| \geq |x| - |y| \geq \frac{1}{2}|x|$, from the monotonicity and convexity of $\Phi(x,y,\cdot)$, we deduce that

$$\begin{aligned}
|F_s(x)| &\leq \frac{2^N}{|x|^N} \int_{B_R(0)} \Phi\left(x, y, \frac{2^s |u(y)|}{\lambda |x|^s}\right) dy \\
&\leq \frac{2^N}{|x|^{N+s}} \int_{B_R(0)} \Phi\left(x, y, 1\right) dy \leq \frac{2^N}{|x|^{N+\frac{1}{2}}} C_2 N \omega_N R^N,
\end{aligned} \tag{5.9}$$

for any $s \geq \frac{1}{2}$ and $2^s \|u\|_\infty \leq \lambda$. Finally, from (5.7), (5.8) and (5.9) we obtain that

$$(1-s)|F_s(x)| \leq C\left(\chi_{B_{2R}(0)} + |x|^{-N-\frac{1}{2}} \chi_{B_{2R}(0)^c}\right) \in L^1(\mathbb{R}^N),$$

where $C > 0$ is a constant depending on n and u, but independent of s. Therefore, the result follows from the Lebesgue's Dominated Convergence Theorem for any $u \in C_0^2(\mathbb{R}^N)$. □

It is important to emphasize that Theorem 5.1.3 holds for any smooth function, but in general could be false in the space $W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$, as shown in the following example introduced by Kim (2023).

Example 5.1.4. *Let $p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that*

$$1 < p^- \leq p(x,y) \leq p^+ \quad \text{and} \quad p(x,y) = p(|x-y|) \text{ for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{5.10}$$

We also consider $\Phi(x,y,t) = t^{p(x,y)}$. In this case, $\widehat{\Phi}(x,t) = t^{\overline{p}}$ where $\overline{p} := p(x,y) = p(0)$. For $q \in (1,N)$, we also assume that

$$1 < \overline{p} < \frac{N}{q} \quad \text{and} \quad p(r) \geq \frac{N}{q-1} \text{ for } r = |x-y| \geq 1. \tag{5.11}$$

For instance, we can consider the following function

$$p(x,y) = N\tanh(|x-y|^2) + p^-, \quad 1 < p^- < \frac{N}{q} < \frac{N}{q-1} \leq p(1).$$

In this case, $p^- = \overline{p}$ and $p^+ = N + p^-$.

We consider a smooth decreasing function $\eta : (0,\infty) \to [0,\infty)$ such that

$$\eta(t) = \begin{cases} |t|^{1-q}, & \text{if } t \in (0,1] \\ 0 \leq \eta(t) \leq 1 & \text{if } t \in [1,2] \\ 0 & \text{if } t \in [2,\infty). \end{cases}$$

We shall show that the function $u : \mathbb{R}^N \to \mathbb{R}$ defined by $u(x) = \eta(|x|)$ belongs to $W^{1,\overline{p}}(\mathbb{R}^N) \setminus W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ for all $s \in (0,1)$. Indeed, since u vanishes in $\mathbb{R}^N \setminus B_2(0)$ and $(q-1)\overline{p} < q\overline{p} < N$, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^{\overline{p}}\,dx &= \int_{B_1(0)} |u(x)|^{\overline{p}}\,dx + \int_{B_2(0)\setminus B_1(0)} |u(x)|^{\overline{p}}\,dx \\ &\leq \int_{B_1(0)} \frac{1}{|x|^{(q-1)\overline{p}}}\,dx + |B_2(0) \setminus B_1(0)| < \infty \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u(x)|^{\overline{p}}\,dx &= \int_{B_1(0)} |\nabla u(x)|^{\overline{p}}\,dx + \int_{B_2(0)\setminus B_1(0)} |\nabla u(x)|^{\overline{p}}\,dx \\ &\leq (q-1)^{\overline{p}} \int_{B_1(0)} \frac{1}{|x|^{q\overline{p}}}\,dx + C|B_2(0) \setminus B_1(0)| < \infty. \end{aligned}$$

Hence, $u \in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N) = W^{1,\overline{p}}(\mathbb{R}^N)$.

It remains to show that $u \notin W^{s,\Phi_{x,y}}(\mathbb{R}^N) = W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ *for all* $s \in (0,1)$*. Since* u *vanishes outside* $B_2(0)$*, we obtain that*

$$\begin{aligned} J_{s,\Phi}(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(|x-y|)}}{|x-y|^{N+sp(|x-y|)}}\,dxdy \\ &\geq \int_{\mathbb{R}^N\setminus B_2(0)}\int_{B_1(0)} \frac{|u(x)-u(y)|^{p(|x-y|)}}{|x-y|^{N+sp(|x-y|)}}\,dxdy \\ &= \int_{\mathbb{R}^N\setminus B_2(0)}\int_{B_1(0)} \frac{|x|^{(1-q)p(|x-y|)}}{|x-y|^{N+sp(|x-y|)}}\,dxdy. \end{aligned}$$

Note that, for any $y \in \mathbb{R}^N \setminus B_2(0)$ *and* $x \in B_1(0)$*, we have* $1 \leq |x-y| \leq 1+|y| \leq \frac{3}{2}|y|$*. Then, using the assumptions* (5.10) *and* (5.11)*, we deduce that*

$$\begin{aligned} J_{s,\Phi}(u) &\geq \int_{\mathbb{R}^N\setminus B_2(0)}\int_{B_1(0)} \frac{|x|^{-N}}{(\frac{3}{2}|y|)^{N+sp^+}}\,dxdy \\ &= \left(\int_{B_1(0)} \frac{1}{|x|^N}\,dx\right)\left(\int_{\mathbb{R}^N\setminus B_2(0)} \frac{1}{(\frac{3}{2}|y|)^{N+sp^+}}\,dy\right) = \infty. \end{aligned}$$

Therefore, $u \notin W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ *for all* $s \in (0,1)$*.*

Although that space is too large for (5.3) to be true, as a direct consequence of Lemma 4.2.22 and the Theorem 5.1.3 in the usual fractional Sobolev spaces, we obtain the following result.

Corollary 5.1.5. *Assume* (H_1)*,* (H_2) *and* (H_3)*. Then,* (5.3) *holds for any* $u \in W^{1,\ell}(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N)$*.*

Proof. For any $u \in W^{1,\ell}(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N)$, we take a sequence $(u_k)_{k\in\mathbb{N}} \subset C_0^2(\mathbb{R}^N)$ such that $u_k \to u$ in $W^{1,\ell}(\mathbb{R}^N)$ and $W^{1,m}(\mathbb{R}^N)$. Without loss of generality, we may assume that $u_k \to u$ a.e. in $\mathbb{R}^N$. Observe that

$$\begin{aligned} |(1-s)J_{s,\Phi}(u) - J_{1,H}(|\nabla u|)| &\leq (1-s)|J_{s,\Phi}(u) - J_{s,\Phi}(u_k)| \\ &\quad + |(1-s)J_{s,\Phi}(u_k) - J_{1,H}(|\nabla u_k|)| \\ &\quad + |J_{1,H}(|\nabla u_k|) - J_{1,H}(|\nabla u|)| \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where $J_{1,H}(u) := \int_{\mathbb{R}^N} H(x,|u(x)|)\,dx$. Using Lemma 4.2.22 (i), we can verify that the inclusion $W^{1,\ell}(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N) \subset W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$ is valid and $u_k \to u$ in $W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$. This fact together with the Proposition 5.1.2 imply that $I_3 \to 0$ as $k \to \infty$.

By (H_1), Lemma 4.2.22 and Theorem 1 present in Bourgain, Brezis and Mironescu (2001), we have that

$$\begin{aligned} J_{s,\Phi}(v) &\leq C_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (|D_s v(x,y)|^\ell + |D_s v(x,y)|^m)\, d\mu \\ &\leq \frac{C_2 n\omega_N}{\ell}\left[\frac{1}{1-s}\left(\|\nabla v\|_\ell^\ell + \|\nabla v\|_m^m\right) + \frac{2^m}{s}\left(\|v\|_\ell^\ell + \|v\|_m^m\right)\right], \end{aligned}$$

for any $v \in W^{1,\ell}(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N)$. From where we deduce that $J_{s,\Phi}(u_k - u) \to 0$ as $k \to \infty$. Then, by Proposition 4.5.1, $I_1 \to 0$ as $k \to \infty$. Thus, for any $\varepsilon > 0$, we can take k enough large such that

$$|(1-s)J_{s,\Phi}(u) - J_{1,H}(|\nabla u|)| \leq \varepsilon + I_2.$$

Therefore, taking the limit as $s \uparrow 1$ and invoking Theorem 5.1.3, the result follows. □

The asymptotic behavior of modulars stated in Theorem 5.1.3 gives indeed a BBM type inequality formula for norms. For this purpose, instead of the seminorm $[\cdot]_{s,\Phi}$ defined in Section 4.3 as

$$[u]_{s,\Phi} := \inf\left\{\lambda > 0\colon J_{s,\Phi}\left(\frac{u}{\lambda}\right)\, dx \leq 1\right\},$$

we consider the equivalent one defined as

$$[[u]]_{s,\Phi} := \inf\left\{\lambda > 0\colon (1-s)J_{s,\Phi}\left(\frac{u}{\lambda}\right)\, dx \leq 1\right\}.$$

In this case, the definition of the seminorm gives that

$$(1-s)J_{s,\Phi}\left(\frac{u}{[[u]]_{s,\Phi}}\right) \leq 1. \tag{5.12}$$

The following result establishes a BBM type inequality formula for norms.

Corollary 5.1.6. *Assume that* (H_1) *and* (H_2) *hold. Then, there exists* $\lambda_0 \geq 1$ *such that for any* $u \in C_0^2(\mathbb{R}^N)$ *and* $\lambda \geq \lambda_0$ *it holds that*

$$\limsup_{s\to 1^-} [[u]]_{s,\Phi} \leq \lambda \|\nabla u\|_H,$$

where H *was defined in* (5.1)*. In particular, if* Φ *satisfies the* Δ_2*-condition, then the inequality holds for any* $\lambda \geq 1$*.*

Proof. Let $u \in C_0^2(\mathbb{R}^N)$. We prove first that bounded modular of H implies bounded norm. Indeed, without loss of generality assume that if $\int_{\mathbb{R}^N} H(x, |\nabla u|)\, dx \leq C$ for some $C \geq 1$, then using the convexity of $H(x,\cdot)$ we get that

$$\int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u|}{C}\right) dx \leq \frac{1}{C}\int_{\mathbb{R}^N} H(x, |\nabla u(x)|)\, dx \leq 1,$$

which, in light of the definition of the Luxemburg norm provides $\|\nabla u\|_H \leq C$.

In view of Theorem 5.1.3, the convexity of Φ and the definition of the norm $\|\cdot\|_H$, there exists $\lambda_0 \geq 1$ such for $\lambda \geq \lambda_0$ it holds that

$$\begin{aligned}\lim_{s\to 1^-}(1-s)J_{s,\Phi}\left(\frac{u}{\lambda\|\nabla u\|_H}\right) &= \int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u|}{\lambda\|\nabla u\|_H}\right)dx \\ &\leq \frac{1}{\lambda}\int_{\mathbb{R}^N} H\left(x, \frac{|u|}{\|\nabla u\|_H}\right)dx \\ &=: L \leq 1.\end{aligned}$$

Thus, by definition of limit, there exists $\varepsilon_s > 0$ such that $\varepsilon_s \to 0$ as $s \to 1^+$ and

$$\left|(1-s)J_{s,\Phi}\left(\frac{u}{\lambda\|\nabla u\|_H}\right) - L\right| \leq \varepsilon_s.$$

In particular, this gives that

$$(1-s)J_{s,\Phi}\left(\frac{u}{\lambda\|\nabla u\|_H}\right)\frac{1}{1+\varepsilon_s} \leq 1.$$

Observe that $\frac{1}{1+\varepsilon_s} < 1$, so, by the convexity of G, we have

$$(1-s)J_{s,\Phi}\left(\frac{u}{\lambda(1+\varepsilon_s)\|\nabla u\|_H}\right) \leq (1-s)J_{s,\Phi}\left(\frac{u}{\lambda\|\nabla u\|_H}\right)\frac{1}{1+\varepsilon_s} \leq 1.$$

Then, by definition of the norm, we obtain that

$$[[u]]_{s,G} \leq \lambda(1+\varepsilon_s)\|\nabla u\|_H,$$

from where, taking the limit in s, we conclude that

$$\limsup_{s\to 1^-}[[u]]_{s,G} \leq \lambda\|\nabla u\|_H.$$

This conclude the proof. $\square$

5.2 SOME SPECIAL CASES

Even though for an arbitrary Musielak function Φ, the Theorem 5.1.3 may not be extended beyond $C_0^2(\mathbb{R}^N)$ functions, in this section we illustrate some examples where Theorem 5.1.3 holds in a suitable space.

5.2.1 Convex functions

The Theorem 5.1.3 holds in particular when the Young function does not depend on the spatial variables, that is, for any convex function $G\colon [0,\infty) \to [0,\infty)$ with $G(0) = 0$ by writing $\Phi(x,y,t) = G(t)$. In this case, it is recovered the results from Alberico et al. (2021a) and Fernández Bonder and Salort (2019). In particular, this includes the case of powers given in Bourgain, Brezis and Mironescu (2001).

5.2.2 Double phase functions

Let $1 \le p < q < \infty$ and consider a function $a\colon \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ continuous in the second variable such that for some constants $a_\pm$,

$$0 < a_- \le a(x,y) \le a_+ < \infty, \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Under these assumptions, we consider the Musielak function $\Phi(x,y,t) = t^p + a(x,y)t^q,\ t \ge 0$. In this case,

$$J_{s,\Phi}(u)\colon = J_{s,p,q}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \right) dxdy$$

and we have the following result.

Proposition 5.2.1. *Let $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Then,*

$$\lim_{s\to 1^-} (1-s) J_{s,p,q}(u) = \mathcal{K}_{N,p} \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx + \mathcal{K}_{N,q} \int_{\mathbb{R}^N} a(x,x)|\nabla u(x)|^q \, dx$$

where

$$\mathcal{K}_{N,\kappa} := \frac{1}{\kappa}\int_{\mathbb{S}^{N-1}} |w_N|^\kappa \, dS_w, \quad \kappa \ge 1.$$

Conversely, if $p > 1$ and $u \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ is such that

$$\liminf_{s\to 1^-} (1-s) J_{s,p,q}(u) < \infty,$$

then $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

Proof. From Theorem 5.1.3 the result holds for any $u \in C_0^2(\mathbb{R}^N)$. By the boundedness of a and Theorem 1 of Bourgain, Brezis and Mironescu (2001),

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} a(x,y)\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\, dy &\le a_+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\, dy \\ &\le \frac{a_+ N\omega_N}{q}\left(\frac{1}{1-s}\|\nabla u\|_q^q + \frac{2^q}{s}\|u\|_q^q \right).\end{aligned}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sq}}\,dy \leq \frac{N\omega_N}{q}\left(\frac{1}{1-s}\|\nabla u\|_p^p+\frac{2^p}{s}\|u\|_p^p\right).$$

Therefore, arguing as in Theorem 2 obtained by Bourgain, Brezis and Mironescu (2001), the result is extended to an arbitrary $u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$. Conversely, it holds that if $p>1$ and

$$\liminf_{s\to 1^-}(1-s)J_{s,p,q}(u)<\infty,$$

then $u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$. □

5.2.3 Logarithmic perturbations of powers

Let $a(\cdot,\cdot)$ be as in the previous subsection. We consider the Musielak function $\Phi(x,y,t)=a(x,y)t^p(\log^+(t)+1)$, $t\geq 0$, where $p\in[1,\infty)$ and $\log^+(t):=\max\{0,\log t\}$. In this case,

$$J_{s,\Phi}(u)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}a(x,y)\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\left(\log^+\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)+1\right)dxdy$$

and we have the following result:

Proposition 5.2.2. *Let $u\in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$. Then it holds that*

$$\lim_{s\to 1^-}(1-s)J_{s,\Phi}(u)=\int_{\mathbb{R}^N}H(x,|\nabla u(x)|)\,dx,$$

where

$$H(x,t)=\begin{cases}a(x,x)t^p\mathcal{K}_{N,p}, & \text{when } t|w_N|\leq 1,\\ a(x,x)t^p\left[\mathcal{K}_{N,p}\left(\dfrac{p-1}{p}+\log t\right)+\mathcal{K}_{\log,N,p}\right]+\dfrac{a(x,x)}{p^2}, & \text{when } t|w_N|>1,\end{cases}$$

being w_N the $N-$th coordinate of any point in $\mathbb{S}^{N-1}$,

$$\mathcal{K}_{N,p}=\frac{1}{p}\int_{\mathbb{S}^{N-1}}|w_N|^p\,dS_w \qquad \text{and} \qquad \mathcal{K}_{\log,N,p}=\frac{1}{p}\int_{\mathbb{S}^{N-1}}|w_N|^p\log|w_N|\,dS_w.$$

Conversely, if $p>1$ and $u\in L^{\widehat{\Phi}}(\mathbb{R}^N)$ is such that

$$\liminf_{s\to 1^-}(1-s)J_{s,\Phi}(u)<\infty,$$

then $u\in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$.

Proof. In this case, we can split the following integral as

$$\int_0^1\Phi(x,x,t|w_N|r)\frac{dr}{r}=a(x,x)t^p|w_N|^p\left(\int_0^1 r^{p-1}\,dr+\int_0^1 r^{p-1}\log^+(t|w_N|r)\,dr\right).$$

Define $t_* := \frac{1}{t|w_N|}$. If $t_* \geq 1$, then $r \leq t_*$ for all $r \in (0,1)$ and in this case $\log^+(t|w_N|r) = 0$, giving that

$$\int_0^1 \Phi(x,x,t|w_N|r)\frac{dr}{r} = a(x,x)\frac{t^p}{p}|w_N|^p.$$

When $t_* < 1$, we have

$$\int_0^1 r^{p-1}\log^+(t|w_N|r)\,dr = \int_{t_*}^1 r^{p-1}\log(t|w_N|r)\,dr = \frac{1}{p^2}\left(p\log(t|w_N|) + \frac{1}{(t|w_N|)^p} - 1\right),$$

which implies

$$\int_0^1 \Phi(x,x,t|w_N|r)\frac{dr}{r} = a(x,x)\frac{t^p}{p}|w_N|^p\left(\frac{p-1}{p} + \frac{1}{p(t|w_N|)^p} + \log(t|w_N|)\right),$$

and the expression of $H(x,t)$ follows just by integrating the variable w in $\mathbb{S}^{N-1}$.

Now, from Theorem 5.1.3 the result holds for any $u \in C_0^2(\mathbb{R}^N)$. On the other hand, by the boundedness of a, we have that

$$J_{s,\Phi}(u) \leq a_+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N},$$

where $G\colon [0,\infty) \to [0,\infty)$ is the Young function given by $G(t) = t^p(\log^+(t)+1)$ with $p \geq 1$. Then, arguing as in the proof of Theorem 4.1 in Fernández Bonder and Salort (2019), the result holds for any $u \in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$ (space which is equal to $W^{1,G}(\mathbb{R}^n)$, since G and $\widehat{\Phi}$ are equivalent Musielak functions). Conversely, proceeding as in the last part of the proof of Theorem 4.1 obtained by Fernández Bonder and Salort (2019), it holds that if $p > 1$ and

$$\liminf_{s\to 1^-}(1-s)J_{s,\Phi}(u) < \infty,$$

then $u \in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$. □

5.2.4 Spaces with variable exponent

Given a continuous function in the second variable $p\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that

$$1 \leq p^- \leq p(x,y) \leq p^+ < \infty \quad \text{for all } x,y \in \mathbb{R}^N,$$

and a function $a(\cdot,\cdot)$ as in the previous example, consider the Musielak function $\Phi(x,y,t) = a(x,y)t^{p(x,y)}$. In this case,

$$J_{s,\Phi}(u) := J_{s,p(\cdot,\cdot)}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} a(x,y)\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy.$$

Proposition 5.2.3. *Let $u \in W^{1,p^+}(\mathbb{R}^N) \cap W^{1,p^-}(\mathbb{R}^N)$. Then it holds that*

$$\lim_{s\to 1^-}(1-s)J_{s,p(\cdot,\cdot)}(u) = \int_{\mathbb{R}^N} H(x, |\nabla u(x)|)\,dx,$$

where

$$H(x,t) = K_{N,p}t^{p(x,x)}, \quad \text{with} \quad K_{N,p} = \frac{a(x,x)}{p(x,x)}\int_{\mathbb{S}^{N-1}} |w_N|^{p(x,x)}\,dS_w.$$

Proof. In this case, the expression of H is immediate. From Theorem 5.1.3 the limit holds for any $u \in C_0^2(\mathbb{R}^N)$. Due to the assumptions on p, one has that

$$a_- \min\{t^{p^+}, t^{p^-}\} \le \Phi(x,y,t) \le a_+ \max\{t^{p^+}, t^{p^-}\}$$

for any $t \ge 0$ and $x, y \in \mathbb{R}^N$. Then, proceeding as in the proof of Corollary 5.1.5, the limit holds for any $u \in W^{1,p^+}(\mathbb{R}^N) \cap W^{1,p^-}(\mathbb{R}^N)$. □

5.2.5 Exponential growth

Our result applies to functions with growing faster than powers. Let $a(\cdot,\cdot)$ be as in Example 5.2.2. Consider the Young function $\Phi(x,y,t) = a(x,y)(e^t - 1)$. Then, in this case

$$J_{s,\Phi}(u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} a(x,y)\left(e^{\frac{|u(x)-u(y)|}{|x-y|^s}} - 1\right)\frac{dxdy}{|x-y|^N}.$$

Proposition 5.2.4. *Let $u \in W^{1,\widehat{\Phi}_x}(\mathbb{R}^N)$. Then, there exists $\lambda_0 > 0$ such that*

$$\lim_{s\to 1^-}(1-s)J_{s,\Phi}\left(\frac{u}{\lambda}\right) = \int_{\mathbb{R}^N} H\left(x, \frac{|\nabla u(x)|}{\lambda}\right)dx,$$

for all $\lambda \ge \lambda_0$, where H is defined as

$$H(x,t) = a(x,x)\int_{\mathbb{S}^{N-1}} \left(\mathrm{Chi}(t|w_N|) + \mathrm{Shi}\,(t|w_N|) - \log(t|w_N|) - \gamma\right)dS_w$$

where $\mathrm{Chi}(t)$ denotes the hiperbolic cosine integral function, $\mathrm{Shi}(t)$ denotes the hiperbolic sine integral function and γ stands for the Euler-Mascheroni constant.

5.3 ANISOTROPIC s-HÖLDER QUOTIENTS

Fractional anisotropic spaces in which in each coordinate direction the functions have different fractional regularity and different integrability have been considered recently, see for example Chaker, Kim and Weidner (2023) and Fernández Bonder and Dussel (2023).

In this section, we consider a family of functionals in which the s-Hölder quotients depend on only one direction. Given a Musielak function $\Phi\colon \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$, $s \in (0,1)$ and $k \in \{1,\ldots,n\}$, we consider the energy functional

$$J^k_{s,\Phi}(u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}} \Phi(x, x-he_k, |D^k_s u(x,h)|)\,\frac{dhdx}{|h|},$$

where e_k is the k-th canonical vector in $\mathbb{R}^N$ and the s-Hölder quotient $D^k_s u$ in the direction e_k is defined as

$$D^k_s u(x,h) := \frac{u(x-he_k)-u(x)}{|h|^s}.$$

These functionals naturally define the fractional Musielak-Sobolev-like spaces $W^{s,\Phi_{x,y}}_k(\mathbb{R}^N)$ as

$$W^{s,\Phi_{x,y}}_k(\mathbb{R}^N) = \left\{u \in L^{\widehat{\Phi}_x}(\mathbb{R}^N)\colon J^k_{s,\Phi}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0\right\}.$$

We also consider the local Musielak-Sobolev space

$$W^{1,\widehat{\Phi}_x}_k(\mathbb{R}^N) := \left\{u \in L^{\widehat{\Phi}_x}(\mathbb{R}^N)\colon \frac{\partial u}{\partial x_k} \in L^{\widehat{\Phi}}(\mathbb{R}^N)\right\}.$$

With the same technique as in Theorem 5.1.3, we prove a BBM result for smooth functions.

Theorem 5.3.1. *Let $u \in C^2_0(\mathbb{R}^N)$ and $k \in \{1,\ldots,N\}$. Assume (H_1) and (H_2). Then, there exists $\lambda_0 > 0$ such that*

$$\lim_{s\to 1^-}(1-s)J^k_{s,\Phi}\left(\frac{u}{\lambda}\right) = \int_{\mathbb{R}^N} H\left(x, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_k}\right|\right)\,dx, \tag{5.13}$$

for all $\lambda \geq \lambda_0$, where

$$H(x,t) = 2\int_0^1 \Phi(x,x,tr)\frac{dr}{r}.$$

Proof. Let $u \in C^2_0(\mathbb{R}^N)$ and let x be fixed. Without loss of generality, we can assume $k = 1$. Proceeding similarly as in the proof of Theorem 5.1.3, we can obtain the following

$$\begin{aligned}
\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}}\Phi&\left(x, x-he_1, \frac{D^1_s u(x,h)|}{\lambda}\right)\,\frac{dh}{|h|}\\
&= \lim_{s\to 1^-}(1-s)\int_{|h|<1}\Phi\left(x, x-he_1, \frac{1}{\lambda}\left|\nabla u(x)\cdot e_1 h|h|^{-s}\right|\right)\,\frac{dh}{|h|}\\
&= \lim_{s\to 1^-}(1-s)\int_{|h|<1}\Phi\left(x, x-he_1, \frac{1}{\lambda}|\nabla u(x)\cdot e_1||h|^{1-s}\right)\,\frac{dh}{|h|}\\
&= \lim_{s\to 1^-}(1-s)\int_{|h|<1}\Phi\left(x, x-he_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right||h|^{1-s}\right)\,\frac{dh}{|h|}\\
&=: \lim_{s\to 1^-}(1-s)I_s(x).
\end{aligned}$$

Now, observe that, similarly as in the proof of Lemma 5.1.1, performing the change of variables $h^{1-s}=\rho$ we get that $(1-s)\frac{dh}{h}=\frac{d\rho}{\rho}$ which yields

$$
\begin{aligned}
I_s(x) &= \int_0^1 \Phi\left(x, x-he_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| h^{1-s}\right)\frac{dh}{h} - \int_{-1}^0 \Phi\left(x, x-he_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| (-h)^{1-s}\right)\frac{dh}{h} \\
&= \int_0^1 \left(\Phi\left(x, x-he_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| h^{1-s}\right) + \Phi\left(x, x+he_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| h^{1-s}\right)\right)\frac{dh}{h} \\
&= \frac{1}{1-s}\int_0^1 \left(\Phi\left(x, x-\rho^{\frac{1}{1-s}}e_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| \rho\right) + \Phi\left(x, x+\rho^{\frac{1}{1-s}}e_1, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| \rho\right)\right)\frac{d\rho}{\rho}.
\end{aligned}
$$

Then, since we assume Φ continuous in the second parameter and $\rho\in(0,1)$,

$$
\lim_{s\to 1^-}(1-s)I_s(x) = 2\int_0^1 \Phi\left(x, x, \frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_1}\right| \rho\right)\frac{d\rho}{\rho}.
$$

Thus, arguing as in the last part of the proof of Theorem 5.1.3, the limit (5.13) holds. Finally, as in Proposition 5.1.2, the function $H(x,t)=2\int_0^1 \Phi(x,x,tr)\frac{dr}{r}$, up to constant, is comparable with $\overline{\Phi}(x,t)$. □

Remark 5.3.2. *In the case in which there is no dependence on the spatial variables and the s-Hölder quotient has a power behavior, i.e.,*

$$
\Phi(x,y,t)=t^p, \quad p\geq 1,\ s\in(0,1),
$$

for each $k\in\{1,\dots,n\}$, the limit function H is easily computed as

$$
H(x,t)=2t^p\int_0^1 r^{p-1}\,dr=\frac{2}{p}t^p,
$$

which implies that

$$
\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}}\frac{|u(x-he_k)-u(x)|^p}{|h|^{1+sp}}\,dhdx=\frac{2}{p}\int_{\mathbb{R}^N}\left|\frac{\partial u(x)}{\partial x_k}\right|^p\,dx
$$

holds for any $u\in C_0^2(\mathbb{R}^N)$. Thus, arguing as in Proposition 5.2.3, it holds for any $u\in W_k^{s,p}(\mathbb{R}^N)$. This recovers the limit result of Fernández Bonder and Dussel (2023).

Remark 5.3.3. *When the dependence of the variables x and y is removed in the anisotropic energy, much more information can be obtained. Indeed, in the case in which*

$$
J_{s,\Phi}^k(u)=\int_{\mathbb{R}^N}\int_{\mathbb{R}}\Phi(|D_s^k u(x,h)|)\frac{dh}{|h|}\,dx
$$

following the proof of Theorem 1.1 present in Alberico et al. (2020), it is not too hard to see that, for any $u\in W_k^{1,\Phi}(\mathbb{R}^N)$ and $k\in\{1,\dots,N\}$, there exists $\lambda_0>0$ such that

$$
\lim_{s\to 1^-}(1-s)J_{s,\Phi}^k\left(\frac{u}{\lambda}\right)=2\int_{\mathbb{R}^N}\int_0^1\Phi\left(\frac{1}{\lambda}\left|\frac{\partial u(x)}{\partial x_k}\right| r\right)\frac{dr}{r}\,dx, \quad \textit{for all } \lambda\geq\lambda_0.
$$

Moreover, when Φ satisfies Δ_2-condition, it is possible in this case to go further and prove a BBM type result for sequences of functions. A close inspection of Theorem 5.2 employed by Fernández Bonder and Salort (2019) reveals that in this case, if $0 \le s_n \to 1^-$ and $(u_n)_{n\in\mathbb{N}} \subset L^\Phi(\mathbb{R}^N)$ is such that $u_n \to u$ in $L^\Phi_{loc}(\mathbb{R}^N)$ and

$$\sup_{n\in\mathbb{N}} \left((1 - s_j) J^k_{s,\Phi}(u_n) + \int_{\mathbb{R}^N} \Phi(u_n)\, dx \right) < \infty,$$

then, $u \in W^{1,\Phi}_k(\mathbb{R}^N)$ and

$$2 \int_{\mathbb{R}^N} \int_0^1 \Phi\left(\left| \frac{\partial u(x)}{\partial x_k} \right| r \right) \frac{dr}{r} dx \le \liminf_{n\to\infty} (1 - s_n) J^k_{s,\Phi}(u_n).$$

6 EXPLORING FUTURE PATHS: PERSPECTIVES AND CHALLENGES IN THE FIELD

In this chapter, we will explore various perspectives on fractional Musielak-Sobolev theory, highlighting the key challenges that need to be addressed and the results we hope to achieve in the coming years.

In this thesis, we study some generalizations of the fractional order Sobolev spaces and applications. In the context of fractional Orlicz-Sobolev spaces, we present an overview of the developments in the theory, such as some qualitative properties and results on embedding. We then apply these results and the nonlinear Rayleigh quotient method to study conditions that guarantee the existence of weak solutions for a class of superlinear problems with two parameters involving the fractional Φ-Laplacian operator. It is important to mention that, in order to establish a continuous embedding into Orlicz spaces in $\mathbb{R}^N$ for functions with subcritical growth $\Psi \ll \Phi_*$, was required the additional hypothesis:

$$\limsup_{t\to 0} \frac{\Psi(t)}{\Phi(t)} < \infty.$$

This condition is crucial because it ensures that the growth of Ψ relative to Φ does not lead to an unbounded behavior near zero, which could otherwise affect the embedding results.

We also explore spaces more general than fractional Sobolev-Orlicz spaces. Specifically, we establish some abstract results within the framework of fractional Musielak-Sobolev spaces, such as uniform convexity, the Radon-Riesz property with respect to the modular function, the (S_+)-property, a Brezis-Lieb type lemma for the modular function, and monotonicity results. Furthermore, we apply the developed theory to investigate the existence of solutions to a class of problems involving a general nonlocal nonstandard growth operator of the Φ-Laplacian type.

Finally, we obtain a Bourgain-Brezis-Mironescu type result for a very general family of modular functions, without requiring the Δ_2-condition on the Musielak function or its complementary function. These results increase our understanding of fractional spaces and provide new perspectives on the analysis of nonlocal problems.

Throughout this work, several research questions have arisen, pointing to promising directions for future investigations. These questions include exploring additional results on embeddings and Poincaré type inequalities, as well as gaining a deeper understanding of the properties of fractional Musielak-Sobolev spaces and their applications.

The main difficulties arise from both the measure μ and the dependence on the spatial variables x and y in the Musielak space $L^{\Phi_{x,y}}(\Omega \times \Omega, d\mu)$. The measure μ presents additional challenges in analyzing functional properties and obtaining precise estimates, as it is neither Borel regular nor do we know if it is σ-finite. This lack of regularity and the possible absence of the σ-finite property further complicate the analysis. Additionally, the explicit dependence on spatial variables in the functions $\Phi_{x,y}$ makes the use of classical techniques more difficult, requiring a more sophisticated approach to handle variations at different points in the domain. Consequently, the study of results in fractional Musielak-Sobolev spaces becomes more delicate.

In the paper by Azroul et al. (2021), embedding results for fractional Musielak-Sobolev spaces and the Poincaré inequality are introduced. However, we believe that certain steps in the proofs lack clarity and require further justification. Specifically, the estimates employed in the proofs are not evidently uniform due to their dependence on spatial variables. Controlling these spatial variables represents one of the significant challenges in the study of fractional Musielak-Sobolev spaces. Unfortunately, we have not yet succeeded in providing a convincing proof.

At the end of Chapter 3, we outline some perspectives and open questions related to the study of solutions to the superlinear fractional Φ-Laplacian type problem $(\mathcal{P}_{\lambda,\nu})$. Due to the non-homogeneity of the operator, we are unable to establish a nonexistence result for the case $\nu = \nu_n(\lambda)$ as stated in Theorem 1.4 by Silva et al. (2024a), although we suspect that this may hold under appropriate conditions on the function Φ. Additionally, it is natural to question whether the results presented in this work can be extended to a broader class of nonlinearities, allowing for more general behavior than power-type functions. Another intriguing yet highly complex question involves understanding the behavior of solutions as the fractional parameter s approaches 0, in line with the work by Fernández Bonder and Salort (2019).

In the Chapter 5, we leave some interesting open questions related to the asymptotic behavior of these energies. Due to the high dependence on spatial coordinates, studying whether a BBM type formula holds for sequences of functions (depending on s) is a challenging task. This question remains unanswered even in the case of the fractional Laplacian with variable exponent. Due to the lack of this result, we were not able to obtain a BBM type formula for seminorms (see Corollary 5.1.6), although we suspect this is valid. Another interesting yet highly nontrivial point is understanding the behavior of the energies as the fractional parameter s approaches 0, in the spirit of the seminal work of Maz'ya and Shaposhnikova (2002).

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