(12) INTERNATIONAL APPLICATION PUBLISHED UNDER THE PATENT COOPERATION TREATY (PCT)

(19) World Intellectual Property Organization
International Bureau

(43) International Publication Date
05 August 2021 (05.08.2021)

WIPO | PCT

(10) International Publication Number
WO 2021/154273 A1

(51) International Patent Classification:
*G06T 17/30* (2006.01) *B29C 64/393* (2017.01)
*G06T 7/11* (2017.01)

(21) International Application Number:
PCT/US2020/016045

(22) International Filing Date:
31 January 2020 (31.01.2020)

(25) Filing Language: English

(26) Publication Language: English

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(81) Designated States *(unless otherwise indicated, for every kind of national protection available)*: AE, AG, AL, AM, AO, AT, AU, AZ, BA, BB, BG, BH, BN, BR, BW, BY, BZ, CA, CH, CL, CN, CO, CR, CU, CZ, DE, DJ, DK, DM, DO, DZ, EC, EE, EG, ES, FI, GB, GD, GE, GH, GM, GT, HN, HR, HU, ID, IL, IN, IR, IS, JO, JP, KE, KG, KH, KN, KP, KR, KW, KZ, LA, LC, LK, LR, LS, LU, LY, MA, MD, ME, MG, MK, MN, MW, MX, MY, MZ, NA, NG, NI, NO, NZ, OM, PA, PE, PG, PH, PL, PT, QA, RO, RS, RU, RW, SA, SC, SD, SE, SG, SK, SL, ST, SV, SY, TH, TJ, TM, TN, TR, TT, TZ, UA, UG, US, UZ, VC, VN, WS, ZA, ZM, ZW.

(84) Designated States *(unless otherwise indicated, for every kind of regional protection available)*: ARIPO (BW, GH, GM, KE, LR, LS, MW, MZ, NA, RW, SD, SL, ST, SZ, TZ, UG, ZM, ZW), Eurasian (AM, AZ, BY, KG, KZ, RU, TJ, TM), European (AL, AT, BE, BG, CH, CY, CZ, DE, DK, EE, ES, FI, FR, GB, GR, HR, HU, IE, IS, IT, LT, LU, LV, MC, MK, MT, NL, NO, PL, PT, RO, RS, SE, SI, SK, SM, TR), OAPI (BF, BJ, CF, CG, CI, CM, GA, GN, GQ, GW, KM, ML, MR, NE, SN, TD, TG).

**Declarations under Rule 4.17:**
— *as to the identity of the inventor (Rule 4.17(i))*

**Published:**
— *with international search report (Art. 21(3))*

**(54) Title:** GENERATING 3D PRINTING POINTS USING AN APPROXIMATE MATHEMATICAL EXPRESSION OF A RQB CURVE

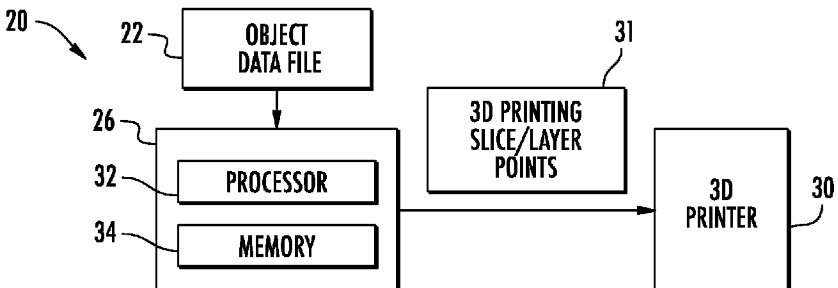


FIG. 1

**(57) Abstract:** A non-transitory computer-readable medium may contain instructions. The instructions may direct the processor to: obtain a mathematical expression of Steiner patch that is part of a tessellation approximation of a three-dimensional (3D) object to be printed by a 3D printer, together with a height value of a slicing plane, determine an approximate mathematical expression of a Rational Quartic Bézier (RQB) curve, the RQB curve being an intersection of the slicing plane and the Steiner patch and generate 3D printing points in Euclidean space for the object based upon the approximate mathematical expression of the RQB, the 3D printing points for 3D printing of the 3D object.

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# GENERATING 3D PRINTING POINTS USING AN APPROXIMATE MATHEMATICAL EXPRESSION OF A RQB CURVE

## BACKGROUND

**[0001]** Three-dimensional (3D) printing is sometimes used to print a three-dimensional object on a layer-by-layer basis. Each layer corresponds to a virtual slice of the three-dimensional object being printed. Each slice comprises printing points that are generated from a digital file of the three-dimensional object being printed. The generated printing points are used by a three-dimensional printer to print the three-dimensional object.

## BRIEF DESCRIPTION OF THE DRAWINGS

**[0002]** Figure 1 is a block diagram schematically altering portions of an example 3D printing system.

**[0003]** Figure 2 is a block diagram schematically illustrating an example non-transitory computer-readable medium containing instructions for the generation of 3D printing points.

**[0004]** Figure 3 is a flow diagram of a computer implemented 3D printing point generation method.

**[0005]** Figure 4 is a flow diagram of an example computer implemented 3D printing point generation method.

**[0006]** Figure 5A is an example of a sphere approximated by a mesh of Steiner patches.

**[0007]** Figure 5B of an individual Steiner patch of the example sphere of Figure 5A, together with its control points and edges.

[0008] Figure 6 is a diagram of an example Steiner patch intersected by slicing plane along a Rational Quartic Bezier curve in Euclidean space.

[0009] Figure 7 is a diagram of an example parametric space including a parametric curve corresponding to the RQB curve of Figure 6.

[00010] Figure 8 is a diagram of different examples of conic classifications.

[00011] Figure 9 is a diagram illustrating a rectification of the parametric curve of Figure 7 and the identification of conic intermediate points.

[00012] Figure 10 is a flow diagram of an example method for refining a set of control points of an approximate mathematical expression of an RQB curve.

[00013] Figure 11 is a flow diagram of an example method for refining a set of control points of an approximate mathematical expression of an RQB curve determined pursuant to the method of Figure 4.

[00014] Throughout the drawings, identical reference numbers designate similar, but not necessarily identical, elements. The figures are not necessarily to scale, and the size of some parts may be exaggerated to more clearly illustrate the example shown. Moreover, the drawings provide examples and/or implementations consistent with the description; however, the description is not limited to the examples and/or implementations provided in the drawings.

## DETAILED DESCRIPTION OF EXAMPLES

[00015] Generating and storing the printing points that form the slice of the three-dimensional object and that are used by the three-dimensional printer to print the three-dimensional object uses large amounts of memory, is often computationally burdensome and may be prone to inaccuracies.

Generating such printing points necessitates use of a computational device or computer as such computations may be mathematically intense and may involve thousands to millions of different printing points for each slice of the three-dimensional object being printed.

**[00016]** An object may be modeled in a computer-assisted design (CAD) application. The CAD application may create a non-uniform rational basis spline (NURBS) representation of the object. A NURBS representation format may offer flexibility and precision in modeling the object. The NURBS representation may capture the surface area of the object being represented. A mesh format may be used to represent the same object. The mesh format may comprise multiple polygons connected along their sides. For example, different sizes and shapes of triangles may be used to represent the surface area of the object. The NURBS representation may be able to better capture curves of the object, while the mesh may approximate curves by using small polygons. The mesh may be more easily manipulated for other purposes, such as determining an intersection with a plane.

**[00017]** 3D objects to be printed are sometimes defined by a tessellation or mesh of Steiner patches. The mesh of Steiner patches is sometimes referred to as a non-uniform rational quadratic splines (NURQS) model. A Steiner Patch, also known as a rational quadratic Bézier triangle, may be used to model the object to allow for more closely matching curves than planar triangles, with a still computationally simple intersection with a plane, while maintaining the advantages of a parametric representation just like in the original NURBS representation. A planar triangle may be represented by three points in Euclidean space, one per corner of the planar triangle. A point may include three numbers for coordinates, an x, y, and z. These coordinates may be represented by floating point numbers, or nine floating point numbers per planar triangle. A Steiner Patch may be represented by 21 floating point numbers, such as three corner points in Euclidean space, three intermediary control points in Euclidean space, and one weight per intermediary control point. As the Steiner Patch allows modeling of a 3D curved triangle, modeling a NURBS object using a mesh of Steiner Patches may use less storage as a whole than modeling a NURBS object using a mesh of planar triangles, while doing so with a higher accuracy. Other curved triangles may also be used, such as a rational cubic Bézier triangle, or other higher degrees of curved triangles. Although Steiner patches may offer enhanced accuracy (smoother edges) as compared to polygon meshes and may simplify the representation of the 3D object (simpler intersection calculation and less computational burden) as compared to NURBS, generation of 3D printing points from Steiner patches or NURQS models may not be adequate for repeated prototyping or object cloning due to excessive storage requirements. A more suitable approach is to store each slicing curve as a rational quartic Bézier Curve, since this involves just five control points and their weights as opposed to storing potentially thousands of sample points per curve. A rational quartic Bézier Curve is analytically the result of intersecting a plane with a Steiner Patch.

**[00018]** Disclosed are example 3D printing points determination instructions, computer implemented methods and 3D printing systems that determine an approximate mathematical expression of a rational quartic Bézier (RQB) curve that may be used to represent any point on the RQB curve, wherein the RQB curve is the intersection of a mathematical expression of a Steiner patch and a slicing plane. The points on the RQB curve correspond to 3D printing points in Euclidean space for a layer of a three-dimensional object being printed. For purposes of this disclosure, an "approximate mathematical expression of an RQB curve" refers to a mathematical expression that represents the approximate curve of the intersection of a particular Steiner patch of a 3D object and a particular slicing plane. This is in contrast to a generic or symbolic mathematical expression of an RQB curve which generally defines the form of an RQB curve with unknown coefficients, the coefficients varying from one another depending upon the particular characteristics of the particular Steiner patch and the height of the slicing plane that intersects Steiner patch, the approximate mathematical expression of an RQB curve includes such coefficients. As will be described hereafter, such coefficients may be in the form of control points and their associated weights.

**[00019]** In contrast to existing methods that attempt to identify such points directly from the intersection of the mathematical expression of the Steiner patch and the slicing plane, the disclosed example 3D printing points determination instructions, example computer implemented methods and example 3D printing systems may consume much less computational resources. In contrast to existing methods that attempt to sample individual points along the intersection of the Steiner patch and the slicing plane, the disclosed example 3D printing points determination instructions, example computer implemented methods and example 3D printing systems may store and provide a greater number of 3D printing points for a smoother curved surface of the object being printed while consuming less memory. The example computer implemented methods and example 3D printing systems may offer an efficient form of mathematically representing and storing the slicing curves for multiple usage.

**[00020]** In some implementations, the approximate mathematical expression of the RQB curve comprises control points and associated weights. The disclosed example 3D printing points determination instructions, example computer implemented methods and example 3D printing systems determine values for the control points and associated weights. The values for the control points and associated weights may be determined by identifying the endpoints of the RQB curve directly from the intersection of the mathematical expression of the Steiner patch and the slicing plane (the height of the slicing plane). The values for intermediate control points and their associated weights may be determined by determining a classification of a conic curve in parametric space that corresponds to the RQB curve, sampling points along the conic curve in determining a length estimate of the conic curve based upon the sample points and the classification. Three conic intermediate points along the conic curve may be determined based upon our using the estimated length of the conic curve. The three conic intermediate points may then be used to identify or determine three corresponding RQB curve intermediate points. The values for the three intermediate control points for the approximate mathematical expression of the RQB curve may be determined from the three RQB curve intermediate points and the endpoints with the differentials.

**[00021]** In one implementation, the values for the three intermediate control points for the approximate mathematical expression of the RQB curve are determined by interpolating the endpoints with the differentials and solving a system of linear equations (linear algebra). In one implementation, the linear equations comprise a mathematical expression for an RQB curve for each of the control points equated to the corresponding RQB curve intermediate point, where the values for the control points are unknown. The system of linear equations may further comprise the first order derivative of the RQB curve at the endpoints equated to the first order derivatives (differentials) of the surface of the first order derivatives (differentials) of a surface of the Steiner patch at the endpoints. This simply results in the assigning of values to some coefficients of the other equations. The values for the three intermediate control points are determined by solving the equations to identify the values for the unknown intermediate control points.

**[00022]** In some implementations, the values for the three intermediate control points for the approximate mathematical expression of the RQB curve are fine-tuned. During such fine-tuning, the values for the initially determined three intermediate control points are adjusted such that the 3D printing points determined from the approximate mathematical expression of the RQB curve with the set of control points more closely matches corresponding 3D printing points sampled directly from the intersection of the mathematical expression of the Steiner patch.

**[00023]** In one implementation, a plurality of candidate approximate mathematical expressions of the RQB curve with different sets of control point are determined. For each candidate approximate mathematical expression of the RQB curve, estimate printing points are determined based upon a set of parameter values and the candidate approximate mathematical expression of the RQB curve (including the values for the control points, which includes their associated weights). Corresponding actual printing points in Euclidean space are also determined for the set of parameter values. Such actual printing points may be determined by directly sampling particular points from the intersection of the mathematical expression of the Steiner patch and the slicing plane. The estimate printing points and the actual printing points may be compared for each candidate approximate mathematical expression. The candidate mathematical expression of the RQB curve having the least error, satisfying a predetermined relationship, or satisfying a predetermined threshold will be chosen for use in printing the 3D object.

**[00024]** In some implementations, the initial candidate approximate mathematical expression of the RQB curve may comprise values for intermediate control points derived from RQB intermediate curve points corresponding to three uniformly spaced conic intermediate points. Thereafter, the values (coordinates) for the three uniformly spaced conic intermediate points may be adjusted (may no longer be uniformly spaced) to identify three different RQB intermediate curve points which results in the identification of different values for the intermediate control points for the approximate mathematical expression of the RQB curve. This process of adjusting the values or coordinates for the three conic intermediate points to derive different values for the corresponding RQB intermediate curve points and to correspondingly identify different values for the intermediate control points may be iteratively carried out until the differences between the actual sampled printing points and the estimate printing points satisfy a predefined threshold or criteria. In some implementations, a predefined number of iterations may be budgeted for fine-tuning the values of the intermediate control points for the approximate mathematical expression of the RQB curve.

**[00025]** Figure 1 is a block diagram schematically illustrating portions of an example 3D printing system 20. 3D printing system 20 carries out printing of a three-dimensional structure or object based upon data contained in a received object data file 22. The data contained in the object data file 22 defines a model of the three-dimensional object to be printed. System 20 comprises 3D printing points 3D printing point generator 26 and 3D printer 30.

**[00026]** 3D printing point generator 26 generates the individual 3D printing slice/layer points 31 and transmits points 31 to 3D printer 30. 3D printer utilizes points 31 to form the layers of the 3D object defined by the object data file 22. In one implementation, 3D printer 30 selectively solidifies portions of 3D build material, in the form of powder or particulates, at each of the 3D printing points. In another implementation, 3D printer 30 selectively deposits or jets build material at each of the 3D printing points to form the layer of the object being printed. In some implementations, the 3D printing points may further serve to define where release materials or sacrificial materials are to be selectively deposited or formed by 3D printer 30. In one implementation, 3D printing points 3D printing point generator 26 is provided in a separate computing unit or device as 3D printer 30. In another implementation, 3D printing points 3D printing point generator 26 is incorporated into a single unit along with 3D printer 30.

**[00027]** 3D printing point generator 26 comprises processor 32 and memory 34. Processor 32 comprises electronics or circuitry that carries out instructions provided by memory 34. Memory 34 comprises a non-transitory computer-readable medium that contains such instructions. As will be described hereafter, such instructions may direct processor 32 to generate 3D printing slice points from Steiner patches or a NURQS model in a more efficient manner. Various implementations of the instructions contained in memory 34 may direct the processor 32 to reduce the computational burden while enhancing accuracy by determining an approximate mathematical expression, with coefficient values that represents an approximation of the intersection of a Steiner patch and a slicing plane (the intersection also referred as a RQB curve).

**[00028]** The approximate mathematical expression is a function of determined coefficient values and the variables which are the different parameter values of an RQB curve. Because all points of the RQB curve are represented by single approximate mathematical expression of the RQB curve, any point approximately along an RQB curve may be quickly determined by plugging in different parameter values into the approximate mathematical expression. Such a task may be more computationally efficient as compared to individually sampling points along the RQB curve by directly using a slicing plane and a mathematical expression of the Steiner patch itself.

**[00029]** Moreover, because all the approximate points of the RQB curve may be represented in a single approximate mathematical expression, all the values for the points may be stored in less space as compared to the storing of the individual coordinate values for the potentially thousands and millions of individual points individually sampled along the RQB curve using the slicing plane and the mathematical expression of the Steiner patch itself. The storage space for all the approximate points of the RQB curve is reduced to the storage space of a single approximate mathematical expression. As a result, given time, computer resource and storage constraints, 3D printing point generator 26 may economically and practically generate a greater number of approximate 3D printing points for the RQB curve and for the layer that is to form the surface of the 3D object being printed. This may result in an improved printed 3D object having smoother curved surfaces or higher resolution surfaces.

**[00030]** In one implementation, the “coefficient values” are in the form of control points of an approximate mathematical expression for the RQB curve. Each of the control point has an associated weight, which is also determined. In one implementation, the approximate mathematical expression of the RQB curve comprises a sybolic mathematical expression of the RQB curve (one whose coefficients are awaiting for valuing) supplemented with the determined coefficients, the determined control points and associated weights. In one implementation, the approximate mathematical expression of the RQB curve comprises five coefficients or control points: two endpoints and three intermediate control points. The two endpoints correspond to the endpoints of the intersection of the Steiner patch and the slicing plane (based upon the height value of the slicing plane). The three intermediate control points are determined solving a system of linear equations to identify the values of the intermediate control points.

**[00031]** Figure 2 is a block diagram schematically illustrating portions of the non-transitory computer-readable medium or persistent storage forming memory 34. In addition to storing the already determined approximate mathematical expression along with the associated control points for each layer of the 3D object, memory 34 may contain instructions for directing processor 32 to determine the control points for each layer of the 3D object. Such instructions comprise object data file instructions 38, mathematical expression value determination instructions 40 and 3D printing point instructions 42. The instructions contained in memory 34 cause processor 32 to carry out the example 3D printing point generation method 100 depicted in Figure 3.

**[00032]** Object data file instructions 38 direct processor 32 to carry out block 104 of method 100 (shown in Figure 3). Object data file instructions 38 direct processor 32 to obtain a Steiner patch that is part of a tessellation approximation of a 3D object to be printed by 3D printer 30. Object data file structure 38 further determine or obtain a height value for the slicing plane, the value that identifies what layer of the 3D object is to be printed from the 3D printing points generated by method 100.

**[00033]** Mathematical expression determination instructions 40 carry out block 108 of method 100 (shown in Figure 3). Instructions 40 direct processor 32 to determine values for an approximate mathematical expression of a RQB curve, the RQB curve being an intersection of the slicing plane and the Steiner patch. As described above, the values determined for the mathematical expression may be coefficients for various variables of the mathematical expression. With such determined coefficients, the mathematical expression may be used by processor 32 to economically and practically (given time, computing resource and storage constraints) generate any of the 3D printing points along the RQB curve and any number of 3D printing points for the 3D object to provide a chosen printing resolution for the 3D object given the printing resolution specifications of 3D printer 30. As described above, in some implementations, the approximate mathematical expression for an individual RQB curve may be the generic mathematical expression of an RQB curve provided with the determined values of the coefficients for the particular RQB curve (the particular layer of the 3D object to be printed given the mathematical expression of the particular Steiner patch and the height value of the slicing plane) in the form of two end control points and three intermediate control points. In other implementations, other approximate mathematical expressions with associated values or coefficients may be determined and used to mathematically represent all of the approximate 3D printing points of a layer of an object to be printed. In some implementations, multiple approximate mathematical expressions with associated values or coefficients may be used to represent the RQB curve, wherein different approximate mathematical expressions represent different segments or portions of the RQB curve. Different approximate mathematical expressions may also represent different forms that approximately reproduce independent properties of the RQB, such as curvature, smoothness and inflexion points.

**[00034]** 3D printing point instructions 42 direct processor 32 to carry out block 112 of method 100 (shown in Figure 3). Instructions 42 direct processor 32 to generate the 3D printing points 31 in Euclidean space for the object defined by the object data file 22. Such 3D printing points are generated based upon the determined values for the coefficients of the approximate mathematical expression of the RQB curve. In implementations where the approximate mathematical expression of the RQB curve comprises a combination of the generic mathematical representation of an RQB curve and the determined control points (including their associated weights) for the particular individual layer to be printed, instructions 42 may direct processor 32 to plug-in different parameter values (variable values) to generate different corresponding individual 3D printing points. The number and spacing of the generated 3D printing points along the RQB curve in Euclidean space may be chosen by a user given resolution targets and the resolution capabilities of the 3D printer 30 being used to print the 3D object.

**[00035]** Figure 4 is a flow diagram of an example computer implemented 3D printing point generation method 200. Method 200 may be carried out by processor 32 following instructions contained in memory 34. Blocks 104 and 112 correspond to blocks 104 and one earned 12 method 100 and are described above. Figures 5A, 5B and 6-9 illustrate one example application of method 200.

**[00036]** As indicated by block 104, 3D printing point generator 26 obtains an individual Steiner patch of a mesh of Steiner patches, the Steiner patch being part of a tessellation approximation of a three-dimensional object to be printed by 3D printer 30. Figure 5A illustrates an example of a mesh of Steiner patches, or a NURQS model, representing a three-dimensional object that may be printed. In the example illustrated in Figure 5A, the mesh of Steiner patches 128 approximates a sphere 130. As should be appreciated, any variety of three-dimensional objects may be represented by a greater or fewer number of such Steiner patches. Figure 5B illustrates an individual

Steiner patch 128 of the sphere 130 of Figure 5A and control points 134 which are the base ingredients of a mathematical expression to produce, along with parametric information, points of the Steiner patch 128. The NURQS model may be provided as part of the object data file 22 supplied to 3D printing point generator 26 or may itself be generated from a model in another format contained in object data file 22. For example, in one implementation, the object data file 22 may contain a NURBS model from a computer aided design device, wherein 3D printing point generator 26 generates NURQS model of Steiner patches from the NURBS model in accordance with any of various existing and future developed techniques.

**[00037]** End Points and Differentials Determinations (Blocks 214 and 216)

**[00038]** As indicated by block 214 in Figure 4, instructions 40 direct processor 32 to determine endpoints of a RQB curve. As described above, the RQB curve is the intersection of the particular slicing plane and the particular Steiner patch of the 3D object being printed. Figure 6 depicts the intersection of an example Steiner patch 310 and a particular intersection plane 312 intersecting Steiner patch 310 along an RQB curve 314 in Euclidean space. The Steiner patch obtained in block 104 is in the form of a mathematical expression of the Steiner patch. Given the mathematical expression of the Steiner patch and a height value of the slicing plane 312, the endpoints 316-1 and 316-2 (collectively referred to as endpoints 316) of the RQB curve 314 may be mathematically determined. Endpoints 316 serve as the corresponding end control points for the approximate mathematical expression of the RQB curve 314 being determined.

**[00039]** As indicated by block 216, when carrying out method 200, instructions 40 direct processor 32 to determine first order differentials of a surface of the Steiner patch 310 at endpoints 316.

**[00040]** Classification of the Conic Curve (Block 218)

**[00041]** As indicated by block 218, when carrying out method 200, instructions 40 direct processor 32 to determine a classification of a conic curve in parametric space that corresponds to the RQB curve 314. Figure 7 illustrates an example conic curve segment 320 in parametric space that corresponds to the example RQB curve 314 in Euclidean space. Figure 8 illustrates various non-degenerate classifications of a parametric curve. As shown by Figure 8, the parametric curve comprises a cross-section of a cone, comprising a conic. Example non-degenerate classifications for a parametric curve include a circle 450, an ellipse 452, a parabola 454 and a hyperbola 456. It should be noted that the illustrated example hyperbola appears also on the upper half of a double cone and is not seen in Figure 8. In some circumstances, the parametric curve may have a degenerate classification such as parallel lines, concurrent lines, a single-line, coincident lines or a single point. As will be described hereafter, the classification of the parametric curve determines how points are sampled along the conic curve and how the length of the conic curve is estimated.

**[00042]** In one implementation, processor 32 (1) receives as input the coefficients of a quadratic equation (obtained from the Steiner patch and the plane); and (2) returns as output the parametrization of the conic. The classification is internal to this method. The parametrization of the conic is a mathematical expression that does the following: when you input a value (ex: 0.25) it returns the corresponding (s,t) in the parametric space of the Steiner patch. This (s,t) is a point in the conic inside the triangle in the parametric space, which corresponds to one fourth of the length of the conic curve (the 0.25 value).

**[00043]** Below is a description of the mathematical foundations and computations that may be carried out by 3D printing point generator 26 in accordance with instructions contained in memory 34 when generating 3D printing points in accordance with the above described methods. It should be appreciated that the above described methods may be carried out using other mathematical computations or other transformations of Steiner patches provided in or derived from the data contained in the object data file 22. As noted above, a Steiner Patch in rational form is a function of parameters (s, t), given the control points $P_{20}, P_{02}, P_{10}, P_{01}, P_{11}$ and $P_{00}$, along with their intermediary weights $w_{10}, w_{01}$ and $w_{11}$, described as follows:

$$S(s,t) = \frac{A_{20}s^2 + A_{02}t^2 + A_{11}st + A_{10}s + A_{01}t + P_{00}}{D_{20}s^2 + D_{02}t^2 + D_{11}st + D_{10}s + D_{01}t + 1} \qquad (Eq.1)$$

where:

- $A_{20} = P_{20} + P_{00} - 2w_{10}P_{10}$ and $D_{20} = 2 - 2w_{10}$
- $A_{02} = P_{02} + P_{00} - 2w_{01}P_{01}$ and $D_{02} = 2 - 2w_{01}$
- $A_{10} = 2w_{10}P_{10} - 2P_{00}$ and $D_{10} = 2w_{10} - 2$
- $A_{01} = 2w_{01}P_{01} - 2P_{00}$ and $D_{01} = 2w_{01} - 2$
- $A_{11} = 2P_{00} + 2w_{11}P_{11} - 2w_{01}P_{01} - 2w_{10}P_{10}$ and $D_{11} = 2 + 2w_{11} - 2w_{10} - 2w_{01}$.

**[00044]** In order for the slicing process to occur, an intersection between such a patch and a plane parallel to the printing bay of the 3D printer is utilized. This process is described by the implicit equation: $z = k_p$, where $k_p$ designates a height of a certain layer, measured from the printing bay up. The intersection can be calculated by just taking the z-component of Eq. 1 and equating to $k_p$. If we let the z-component of $A_{ij}$ to be $z_{ij}$, then the intersection yields the following equation on s and t parameters:

$$(z_{20} - D_{20}k_p)s^2 + (z_{11} - D_{11}k_p)st + (z_{02} - D_{02}k_p)t^2 + (z_{10} - D_{10}k_p)s + (z_{01} - D_{01}k_p)t + (z_{00} - k_p) = 0$$

**[00045]** This equation can be further simplified by renaming the $s$ and $t$ terms' multipliers with $C_1, C_2, \ldots, C_5$ and naming the independent term $(z_{00} - k_p) = C_6$, thus making

$$C_1 s^2 + C_2 st + C_3 t^2 + C_4 s + C_5 t + C_6 = 0 \qquad (Eq.2)$$

**[00046]** Equation 2 describes a conic curve in parametric space, which is the pre-image of the intersection between the Steiner Patch and the plane (in the Euclidean space). To properly sample points in this curve, a determination is made as to which type of conic this equation refers to, and that means that a new system of coordinates (with new variables $s'$ and $t'$) in the parametric space should be established in such a way that the mixed term $s't'$ does not show up in the new equation.

**[00047]** In Equation 2 the $C_2st$ term being non zero is an indication that the new axes of the variables $s'$ and $t'$ are to be rotated of a certain angle $\alpha_s$ in relation to their counterparts $s$ and $t$ in order for the mixed term $s't'$ to become nullified in the yielded equation. The angle $\alpha_s$ is already known in Analytical Geometry as being such that $\tan(2\alpha_s) = \frac{C_2}{C_1 - C_3}$, in case $C_1 - C_3 \neq 0$, otherwise $\alpha_s = 45^0$. This is analogous to rotating the parametric space of the Steiner patch such that it aligns with the canonical axis as shown in Figure 3. The rotation of $\alpha_s$ produces the relations: $s = s' \, cos(\alpha_s) - t' \, sin(\alpha_s)$ and $t = s' \, sin(\alpha_s) + t' \, cos(\alpha_s)$ that can be plugged into Equation 2 to generate the following reduced equation of a conic in the variables $s'$ and $t'$:

$$K_1 \, s'^2 + K_2 t'^2 + K_3 s' + K_4 t' + C_6 = 0 \qquad (Eq. 3)$$

where

$$K_1 = C_1 \cos^2 \alpha_s + C_2 \cos \alpha_s \sin \alpha_s + C_3 \sin^2 \alpha_s$$

$$K_2 = C_1 \cos^2 \alpha_s - C_2 \cos \alpha_s \sin \alpha_s + C_3 \cos^2 \alpha_s$$

$$K_3 = C_4 \cos \alpha_s + C_5 \sin \alpha_s$$

$$K_4 = -C_4 \sin \alpha_s + C_5 \cos \alpha_s$$

**[00048]** With Equation 3 it is possible to classify the type of conic which is in the parametric space the pre-image of the slice curve that comes from the intersection between a Steiner patch and a plane. First, a check is made as to whether this is a degenerate conic (single point, intersecting lines, parallel lines and coincident lines). Consider the following symmetric matrix:

$$M = \frac{1}{2} \begin{pmatrix} 2C_1 & C_2 & C_4 \\ C_2 & 2C_3 & C_5 \\ C_4 & C_5 & 2C_6 \end{pmatrix}$$

**[00049]** If the determinant of M is zero, then the conic is degenerate. Otherwise, in the case of non-degeneracy, if on top of this $K_1$ and $K_2$ are both nonzero and present the same sign, then Equation 2 represents an Ellipse and If they present opposite signs, then Equation 2 represents a hyperbola. If either $K_1$ or $K_2$ is zero, then Equation 2 represents a parabola. Each type of conic uses its own form of parametrization, which has also to take into consideration the parametrization speed, in order to produce an equally spaced sampling, or at least as close as possible to this.

**[00050]** Unless explicitly defined otherwise, the calculation of the patch-plane intersection curve is a conic in parametric space which is further parameterized in terms of an angle $\theta_s$ covering the full length of the conic. Since, generally, a portion of the conic is contained within the parametric space of the patch, this angle $\theta_s$ will be bounded by the inferior and superior limits of the border points. The general algorithm for slice computation comprises finding the appropriate parameterization in terms of $\theta_s$, finding the points where the slice intersects with the patch's border, finding the correspondent parameters of $\theta_s$ for the border points and sampling the slice for the valid interval of $\theta_s$. The computed points are yielded in terms of the transformed parametric space $(s', t')$. A final transformation to $(s, t)$ is used and can be computed through

$$s = s' \, cos(\alpha_s) - t' \, sin(\alpha_s) \, , t = s' \, sin(\alpha_s) + t' \, cos(\alpha_s).$$

**[00051]** **General Notation**

**[00052]** The detailed process of computing the slices is broken down for each classification of the conic that is the pre-image of the intersection between a Steiner patch and a plane. These are the non-degenerate cases of the ellipse, parabola, hyperbola and the degenerate cases of intersecting lines, parallel lines, single line and point. There is also a special case – treated separately - where the conic arc segment is so far from the center that it is approximately straight for the portion that intersects with the patch.

**[00053]** To simplify the calculations for obtaining the slices, it is useful to define some previous notations.

Let $G_1 = \frac{1}{|K_1|}\left(\frac{1}{4}\frac{K_3^2}{K_1} + \frac{1}{4}\frac{K_4^2}{K_2} - K_5\right), G_2 = \frac{1}{|K_2|}\left(\frac{1}{4}\frac{{K_3}^2}{K_1} + \frac{1}{4}\frac{{K_4}^2}{K_2} - K_5\right), A_s = \sqrt{|G_1|}, A_t = \sqrt{|G_2|}, s_0' = \frac{-K_3}{2K_1}$ and $t_0' = \frac{-K_4}{2K_2}$.

**[00054]** **Ellipse Slicing**

**[00055]** As will be recurrent, the slicing method will first determine the parameter value $\theta_s^b$ for the border point and then iterate $\theta_s$ over the valid range. The parameter value $\theta_s^b$ can be found by inputting the values of $(s_b', t_b')$ (transformed from the point's corresponding $(s, t)$ parameters) for a border point $b$ in the following equations:

$$\theta_s^b = \begin{cases} arccos\left(\frac{s_b' - s_0'}{A_s}\right), \ if\ arcsin\left(\frac{t_b' - t_0'}{A_t}\right) < 0 \\ 2\pi - arccos\left(\frac{s_b' - s_0'}{A_s}\right), \text{ otherwise} \end{cases}.$$

**[00056]** Once the bounds for $\theta_s$ are defined, the actual slice can be computed by evaluating the parameter within the specified range such that

$$s' = s_0' + A_s\, cos(\theta_s),\ t' = t_0' + A_t\, sin(\theta_s).$$

**[00057]** **Hyperbola Slicing**

**[00058]** Hyperbola slicing differs from the ellipse slicing due to the caveat that there are potentially two segments which are accounted for. For this type of conic, it is useful to separate the calculations of the border point parameter $\theta_s^b$ by the quadrant in which the arc segment is located. The conditions for the value of $\theta_s^b$ are given in the table below and will depend on the values of $K_{cos}$ and $K_{tan}$ computed as:

$$K_{cos} = \begin{cases} \frac{A_s}{s_b' - s_0'}, \text{ if } G_2 < 0 \\ \frac{A_t}{t_b' - t_0'}, \text{ if } G_1 < 0 \end{cases}, K_{tan} = \begin{cases} \frac{t_b' - t_0'}{A_t}, \text{ if } G_2 < 0 \\ \frac{s_b' - s_0'}{A_s}, \text{ if } G_1 < 0 \end{cases}$$

| Quadrant | $arccos(K_{cos})$ | $arctan(K_{tan})$ | $\theta_s^b$ |
|---|---|---|---|
| First | $< \pi/2$ | $> 0$ | $arccos(K_{cos})$ |
| Second | $\geq \pi/2$ | $> 0$ | $arctan(K_{tan}) + \pi$ |
| Third | $\geq \pi/2$ | $\leq 0$ | $arccos(K_{cos})$ |
| Fourth | $< \pi/2$ | $\leq 0$ | $arctan(K_{tan}) + \pi$ |

[00059] Once the limits for $\theta_s$ have been established, the slicing can be evaluated from the $(s', t')$ parameters computed as:

$$s' = \begin{cases} s_0' + A_s\, cos^{-1}(\theta_s)\,, \text{ if } G_2 < 0 \\ s_0' + A_s\, tan(\theta_s)\,, \text{ otherwise} \end{cases}, \qquad t' = \begin{cases} t_0' + A_t\, tan(\theta_s)\,, \text{ if } G_2 < 0 \\ t_0' + A_t\, cos^{-1}(\theta_s)\,, \text{ otherwise} \end{cases}$$

[00060] **Parabola Slicing**

[00061] For the parabola-type slicing, a new definition for $s_0'$ and $t_0'$ is given. Besides, a new coefficient of $dv$ is also defined. Have $s_0' = \frac{1}{K_3}\left(K_5 - \frac{K_4^2}{4K_2}\right), t_0' = \frac{-K_4}{2K_2}$ $and\ dv = \frac{-K_3}{4K_2}$ if $K_1 = 0$ or $s_0' = \frac{-K_3}{4K_2}, t_0' = \frac{1}{K_4}\left(K_5 - \frac{K_3^2}{4K_1}\right)$ $and\ dv = \frac{-K_4}{2K_1}$ otherwise. Once again, the values for the parameters $\theta_s^b$ can be computed from the transformed $(s'_b, t'_b)$ parameters of the border points. Make

$$\theta_s^b = \begin{cases} \dfrac{t_b' - t_0'}{2dv}, \ if\ K_1 = 0 \\ \dfrac{s_b' - s_0'}{2dv}, \ otherwise \end{cases}.$$

[00062] Once $\theta_s^b$ has been defined, the values for $(s', t')$ of the slicing, points can be obtained by varying $\theta_s$ within the valid interval such that:

$$s' = \begin{cases} \theta_s^2 dv + s_0',\ if\ K_1 = 0 \\ 2\theta_s dv + s_0',\ otherwise \end{cases}, t' = \begin{cases} 2\theta_s dv + t_0',\ if\ K_1 = 0 \\ \theta_s^2 dv + t_0',\ otherwise \end{cases}.$$

[00063] **Degenerate Conic Curves**

[00064] Let

$$\zeta = det\begin{pmatrix} C_1 & \frac{1}{2}C_3 & \frac{1}{2}C_4 \\ \frac{1}{2}C_3 & C_2 & \frac{1}{2}C_5 \\ \frac{1}{2}C_4 & \frac{1}{2}C_5 & C_6 \end{pmatrix}, \quad \lambda = det\begin{pmatrix} C_1 & \frac{1}{2}C_3 \\ \frac{1}{2}C_3 & C_2 \end{pmatrix},$$

$$\Omega = det\begin{pmatrix} C_1 & \frac{1}{2}C_4 \\ \frac{1}{2}C_4 & C_6 \end{pmatrix}.$$

[00065] A conic that is the pre-image of the intersection between a Steiner Patch and a plane is considered degenerate if $\zeta = 0$. A further classification is given by the values of $\lambda$ and $\Omega$. If $\lambda > 0$ the conic is reduced to a point, and if $\lambda < 0$ it is reduced to a pair of intersecting lines. In the case of $\lambda = 0$, the conic will be classified as coincident lines if $\Omega = 0$, or parallel lines if $\Omega < 0$.

[00066] **Intersecting Lines Slicing**

[00067] The conic image will be classified as the degenerate type of intersecting lines if either $C_1$ or $C_2$ are zero. As this conic type is composed of two independent line segments which intersect at a point, it is natural to first identify the parameters for the limits of the valid region of the segments within the patch and then iterate over the segment. The value of the valid $(s, t)$ parameters for the points will depend on whether $C_1$ or $C_2$ is zero. Since this is not parameterized as a conic but rather as a line segment, the parameters for the limits of the conic are defined simply as the $(s, t)$ values of the border point. From the coefficients of the curve, it is possible to obtain $x_0$ and $y_0$.

$$y_0 = \frac{2C_1C_5 - C_3C_4}{4C_1C_2 - C_3^2}, \qquad x_0 = \begin{cases} \dfrac{C_4 - C_3y_0}{2C_1}, & \text{if } C_1 \neq 0 \\ \dfrac{C_5 - 2C_2y_0}{C_3}, & \text{otherwise} \end{cases}$$

[00068] The calculation for the $(s, t)$ parameter pair for the line segments will be achieved by expressing $s$ in terms of $t$ or vice-versa. There will be two equations, one for each segment. A straightforward approach is to

parameterize $s$ in terms of $t$ by the pairs of equations $s = \frac{(-C_3+\alpha_s)(t+y_0)}{2C_1} - x_0$ and $s = \frac{(-C_3-\alpha_s)(t+y_0)}{2C_1} - x_0$.

[00069] However, there is a possibility of $C_1 = 0$, breaking the equation. In that case, $y_0 = \frac{2C_1C_5-C_3C_4}{4C_1C_2-C_3^2}$ simplifies to $y_0 = \frac{C_4}{C_3}$ and $t$ is placed in terms of $s$ instead resulting in the following pair of equations: $t = \frac{-C_3(s+x_0)}{C_2-y_0}$ and $t = -y_0$ for all $s$ within the interval.

[00070] One should consider the possibility of $C_2 = 0$ along with $C_1$. In this case, one segment will be given in terms of $s = \frac{-C_5}{C_3}$ for all valid $t$ and the other given in terms of $t = \frac{-C_4}{C_3}$ for all valid $s$.

[00071] **Parallel Lines Slicing**

[00072] If the conic curve in the parametric space degenerates as a pair of parallel lines, then each segment is computed independently. Let $\delta = C_4^2 - 4C_1C_6$. If $C_1 \neq 0$ and $\delta \geq 0$, then the values of $s$ can be parameterized in terms of $t$ for both segments:

[00073] Let $s_1 = \frac{-C_3t_1-C_4+\sqrt{\delta}}{2C_1}$ for all valid $t_1$ on the first segment and let $s_2 = \frac{-C_3t_2-C_4-\sqrt{\delta}}{2C_1}$ for all valid $t_2$ on the second segment. If, however, $C_1 = 0$ but $C_2 \neq 0$, then instead make $\delta = C_5^2 - 4C_2C_6$ and compute $t_1 = \frac{-C_5+\delta}{2C_2}$ constant for all valid $s_1$ in the first segment and $t_2 = \frac{-C_5-\delta}{2C_2}$ also constant for every valid $s_2$ on the second segment. If none of the conditions are met the parameter pair will either be $\left(\frac{-C_6}{C_4}, t\right)$ for all valid $t$ if $C_4 > C_5$ or $\left(s, \frac{-C_6}{C_5}\right)$ otherwise.

[00074] **Single Line Slicing**

[00075] For the case where the conic curve has degenerated into a single line, the parameters for the slice will be given by $s = \frac{-C_5t-C_6}{C_4}$ for all valid

$t$ if $C_4 \neq 0$. If $C_4 = 0$ however, then the slicing will be put in terms of $t = \frac{-C_6}{C_5}$, with $s$ arbitrary within the interval.

[00076] **Coincident Lines Slicing**

[00077] In some instances, the conic curve in the parametric space will degenerate into a pair of coincident lines. Normally this case would be like the single line, however since these are limited line segments are treated specifically. Since there are several possible parameterizations for the values of $(s, t)$ depending on the values of $C_1, C_2, \ldots, C_6$, it is useful to display them as a table:

| | | | |
|---|---|---|---|
| $C_3 \neq 0$ | $C_6 = \frac{C_4 C_5}{2C_3}$ | $C_3 > 0$ | $\sqrt{C_1}s + \sqrt{C_2}t = -\sqrt{C_6}$ or $\sqrt{C_1}s + \sqrt{C_2}t = \sqrt{C_6}$ |
| | | $C_3 < 0$ | $\sqrt{C_1}s - \sqrt{C_2}t = \sqrt{C_6}$ or $\sqrt{C_1}s - \sqrt{C_2}t = -\sqrt{C_6}$ |
| $C_3 = 0$ | $C_1 = 0, \quad C_2 \neq 0$ | $C_6 = \frac{C_5^2}{4C_2}$ | $t = \frac{-C_5}{2C_2}$ for all $s$ |
| | $C_1 \neq 0, \quad C_2 = 0$ | $C_6 = \frac{C_4^2}{2C_1}$ | $s = \frac{-C_4}{2C_1}$ for all $t$ |

[00078] **Point Slicing**

[00079] Computing the slice for the conic which degenerated into a point is trivially accomplished by letting $s = \frac{C_3 C_4 - 2C_1 C_5}{4C_1 C_2 - C_3^2}$, $t = \begin{cases} \frac{C_3 y_0 - C_4}{2C_1}, & \text{if } C_1 \neq 0 \\ \frac{2C_2 y_0 - C_5}{C_3}, & \text{otherwise} \end{cases}$.

[00080] The above illustrate one example for classifying the particular conic curve 320. In other implementations, other techniques may be used for classifying the particular conic curve.

**[00081]** Length Estimation of Conic Curve (Block 220)

**[00082]** As indicated by block 220, based upon the classification of the conic curve 320, instructions 40 direct process 32 to sample points along the conic curve and to determine a length estimate for the conic curve using the sampled points. In one implementation, points are sampled along the conic curve in parametric space. The distance between each pair of contiguous points is measured. All of these distances are added together and the assumption is made that the total added distance is the total length of the conic curve in the parametric space.

**[00083]** Conic Intermediate Points Determinations (Block 222)

**[00084]** As indicated by block 222, instructions 40 direct processor 32 to determine three conic intermediate points based upon the length estimation for the conic curve. In the example illustrated, instructions 40 direct processor 32 to determine three equally spaced conic intermediate points. In the example illustrated, instructions 40 direct processor 32 to identify the parametric values for points along the conic curve 320 that are at the midpoint (0.5), at the quarter-point (0.25) and at the three-quarter point (0.75) of the total estimated length of the conic curve 320 between the conic curve end points 326-1 and 326-2 (which correspond to the RQB curve endpoints 316-1316-2, respectively). In one implementation, the length of the conic curve 320 is rectified or normalized into a straight line as shown in Figure 9, facilitating identification of the three intermediate conic points 328. Conic intermediate points 328-1, 328-2 and 328-3 have coordinates along the conic curve 320 that are spaced from the conic end point 326-1, along the conic curve 320, by distances 0.25X, 0.5X and 0.75 X, respectively, where X is the total estimated length of the conic curve 320.

**[00085]** RQB Curve Intermediate Points Determinations (Block 224)

**[00086]** As indicated by block 224, controller 40 directs processor 32 to determine the values or coordinates of three RQB curve intermediate points that correspond to three conic intermediate points determine in block 222. To do so, processor 32 plugs the three conic curve intermediate points into the mathematical expression of the Steiner patch to determine the three corresponding RQB curve intermediate points. The generic mathematical expression of a Steiner patch is shown below.

$$S(s,t) = \sum_{i+j\leq 2} \frac{\left(\frac{2!}{i!\,j!\,(2-i-j)!}\right) s^i t^j (1-s-t)^{2-i-j} w_{ij}}{\sum_{k+l\leq 2} \left(\frac{2!}{k!\,l!\,(2-k-l)!}\right) s^k t^l (1-s-t)^{2-k-l} w_{kl}} P_{ij}$$

where $P_{00}$, $P_{02}$ and $P_{20}$ are the corner control points, with their weights $w_{00}$, $w_{02}$ and $w_{20}$ set to 1; $P_{11}$, $P_{01}$ and $P_{10}$ are the intermediate control points, with arbitrary weights $w_{11}$, $w_{01}$ and $w_{10}$

(see Figure 5B).

**[00087]** Figure 6 illustrates three example RQB intermediate points 336-1, 336-2 and 336-3 which correspond to conic intermediate points 328-1, 328-2 and 328-3, respectively.

**[00088]** Control Point Determinations (Block 226)

**[00089]** As indicated by block 226 of Figure 4, instructions 40 (shown in Figure 2) direct processor 32 (shown in Figure 1) to determine values for the three intermediate control points 346 of the approximate mathematical expression 340 of the RQB curve 314. Figure 6 illustrates an example approximate control polygon 340 of the RQB curve 314 as represented by end control points 316-1, 316-2 and its intermediate control points 346-1, 346-2 and 346-3 (collectively referred to as intermediate control points 346).

**[00090]** The values for the intermediate control points 346 are determined by processor 32 interpolating the endpoints 316 with their differentials and solving a system of linear equations (linear algebra). In one implementation, the linear equations comprise a mathematical expression for an RQB curve equated to the RQB curve intermediate points corresponding to the three equally spaced parameter values, and having as unknowns the intermediate control points of the RQB.. The system of linear equations may further comprise the first order derivative of the RQB curve at the endpoints equated to the first order derivatives (differentials) of the surface of the first order derivatives (differentials) of a surface of the Steiner patch at the endpoints. This results in the assigning of values to some coefficients of the other equations. The values for the three intermediate control points are determined by solving the equations to identify the values for the unknown intermediate control points.

**[00091]** The generic expression or analytical form for an RQB curve is shown below.

$$\mathbf{b}_0^4(t) = \sum_{j=0}^{4} \frac{\binom{4}{j}(1-t)^{4-j}t^j w_j}{\sum_{k=0}^{4}\binom{4}{k}(1-t)^{4-k}\,t^k w_k}\mathbf{b}_j$$

**[00092]** with $w_i = 1$ for $i = 0$ and $i = 4$. Given that $S: \Delta \subset R^2 \rightarrow R^3$is the analytical form of the Steiner patch, and the determined endpoints $\mathbf{b}_0$ and $\mathbf{b}_4$ (endpoints 316) together with their tangent directions $\mathbf{t}_0$ and $\mathbf{t}_4$,, respectively, process 32 solves the following system of linear equations to determine the values for the intermediate control points $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, (control points 346-1, 346-2 and 346-3):

$$k_0\mathbf{t}_0 = \mathbf{b}_0\mathbf{b}_1$$
$$k_4\mathbf{t}_4 = \mathbf{b}_3\mathbf{b}_4$$
$$\mathbf{b}_0^4(1/4) = S(C(1/4))$$
$$\mathbf{b}_0^4(1/2) = S(C(1/2))$$
$$\mathbf{b}_0^4(3/4) = S(C(3/4))$$

where $C: [0,1] \rightarrow \Delta \subset R^2$ is the analytical form of the parametrization of the conic, with $\Delta$ being the triangle which is the pre-image of the Steiner patch, and $C(0)$ and $C(1)$ are the ends of the segment of the curve which is internal to the triangle $\Delta$; $C(¼)$, $C(½)$ and $C(¾)$ are the equally spaced points in the conic curve in parametric space.

[00093] Approximate Mathematical Expression Refinement

[00094] The values for the approximate mathematical expression determined in block 108 of method 100 by instructions 40 and processor 32 (shown in figures 1-3) are approximations of the actual RQB curve. Figure 10 is a flow diagram of an example computer implemented method 400 for refining the previously determined values such that the approximate mathematical expression more closely matches the actual RQB curve.

[00095] As indicated by block 404, instructions 40 may direct processor 32 to determine a plurality of candidate approximate mathematical expressions of the RQB curve, wherein each candidate approximate mathematical expression comprises a different set of determined values, such as a different set of coefficients. As indicated by block 408, for each of the different candidate approximate mathematical expressions of the RQB curve, instructions 40 may further direct processor 32 to (1) determine estimate printing points based upon a set of parameter values and the candidate approximate mathematical expression of the RQB curve (block 410); (2) determine actual printing points in the Euclidean space for the set of parameter values (block 412) and compare the estimate printing points to the corresponding actual printing points (block 414). The actual printing points may be determined by sampling points along the RQB curve using the same set of parameter values applied into the conic parametrization to produce (s,t) parameter values to be used in the mathematical expression of the Steiner patch. As indicated by block 418, instructions 40 may direct processor 32 to

select one of the candidate approximate mathematical expressions of the RQB curve (with associated values/coefficients) as the approximate mathematical expression of the RQB curve based upon the comparisons.

**[00096]** Figure 11 is a flow diagram of an example method 500 for refining the previously determined values for the control points determined pursuant to method 200 such that the approximate mathematical expression for the RQB curve more closely matches the actual RQB curve. Method 500 proceeds block 226. Method 500 illustrates how the determined control points may be iteratively adjusted to satisfy criteria such as a percent or degree of correspondence to the actual RQB curve or so as to reduce the errors or differences between the approximate 3D printing points and the actual points of the RQB curve to within predetermined acceptable differences.

**[00097]** As indicated by block 504, instructions 40 may direct processor 32 to sample a first set of point and Euclidean space using a current set of control points for the approximate mathematical expression of the RQB curve. The first set of points are approximate or estimate points of the RQB curve. Upon immediately following block 226, the current set of control points will be those control points determined in method 200 that were derived using RQB curve intermediate points corresponding to the three uniformly spaced conic intermediate points. During subsequent iterations, the “current set of control points” may be previously adjusted control points.

**[00098]** As indicated by block 508, instructions 40 may direct processor 32 to sample a second corresponding set of points in Euclidean space using the mathematical expression of the Steiner patch in the slicing plane. The second set of points are actual points along the RQB curve.

**[00099]** As indicated by block 512, the first set of points and the second set of points are compared to one another to determine a difference D. As indicated by block 514, instructions 40 direct processor 32 to compare the determined difference or differences D to a preestablished or predefined criterion. For example, the predetermined criterion may be a certain percent or degree of correspondence to the actual RQB curve or may be a certain predetermined acceptable amount or degree of difference between the actual and the estimate printing points. If the current set of control points for the approximate mathematical expression satisfy the criteria (any differences are small enough so as to be acceptable), instructions 40 may direct processor 32 to output the current set of control points for the generation of 3D printing points as indicated in block 418.

**[000100]** As indicated by block 516, in response to the differences D between the estimate printing points and the actual printing points being so great so if you do not satisfy the predefined criteria, instructions 40 may direct processor 32 to adjust the values of the conic intermediate points. For example, as shown in Figure 9, the values for conic intermediate 328-1, 328-2 and 328-3 may be adjusted, resulting in conic intermediate points 328-1', 328-2'and 328-3, respectively.

**[000101]** As indicated by block 518, instructions 40 may direct processor 32 to determine a new set of values for the three RQB curve intermediate points based upon the adjusted values of the conic intermediate points. Processor 32 may determine a new set of RQB intermediate points 336-1', 336-2'and 336-3' that correspond to the three new conic intermediate points 328-1', 328-2'and 328-3', respectively.

**[000102]** As indicated by block 520, instructions 40 may direct processor 32 to determine a new set of values for the three intermediate control points from the new set of three RQB curve intermediate points and the endpoints 316 along with their differentials. As described above with respect to block 226, the new set of values for the three intermediate control points may be determined by solving a series of linear equations. Once a new set of values for the three intermediate control points have been determined, resulting in a new approximate mathematical expression with the two end control points and the new set of three intermediate control points, block 504, 508, 512 and 514 are once again carried out with the new approximate mathematical expression to determine whether the new approximate expression more closely approximates the actual RQB curve or is within an acceptable range of the actual RQB curve.

**[000103]** Although the present disclosure has been described with reference to example implementations, workers skilled in the art will recognize that changes may be made in form and detail without departing from disclosure. For example, although different example implementations may have been described as including features providing various benefits, it is contemplated that the described features may be interchanged with one another or alternatively be combined with one another in the described example implementations or in other alternative implementations. Because the technology of the present disclosure is relatively complex, not all changes in the technology are foreseeable. The present disclosure described with reference to the example implementations and set forth in the following claims is manifestly intended to be as broad as possible. For example, unless specifically otherwise noted, the claims reciting a single particular element also encompass a plurality of such particular elements. The terms “first”, “second”, “third” and so on in the claims merely distinguish different elements and, unless otherwise stated, are not to be specifically associated with a particular order or particular numbering of elements in the disclosure.

WHAT IS CLAIMED IS:

1. A non-transitory computer-readable medium containing instructions to direct a processor to:

obtain a mathematical expression of Steiner patch that is part of a tessellation approximation of a three-dimensional (3D) object to be printed by a 3D printer, together with a height value of a slicing plane;

determine values for an approximate mathematical expression of a Rational Quartic Bézier (RQB) curve, the RQB curve being an intersection of the slicing plane and the Steiner patch; and

generate 3D printing points in Euclidean space for the object based upon the values for the approximate mathematical expression of the RQB curve, the 3D printing points for 3D printing of the 3D object.

2. The medium of claim 1, wherein the values for the approximate mathematical expression of the RQB curve comprises values for control points and associated weights.

3. The medium of claim 2, wherein the determining of values for the approximate mathematical expression of the RQB curve comprises:

determining end points of the RQB curve, the endpoints forming end control points;

determining differentials of a surface of the Steiner patch at the end points;

determining a classification of a conic curve in parametric space that corresponds to the RQB curve;

sample points along the conic curve;

determining a length estimate of the conic curve based on the sampled points and the classification;

determining three conic intermediate points along the conic curve based upon the length estimate;

determining three RQB curve intermediate points that correspond to the three conic intermediate points; and

determining values for three intermediate control points for the approximate mathematical expression of the RQB curve from the three RQB curve intermediate points and the endpoints with their differentials, wherein the approximate mathematical expression of the RQB curve comprises the values for the end control points and the three intermediate control points.

4. The medium of claim 3, wherein the three conic intermediate points are uniformly spaced between the endpoints.

5. The medium of claim 3, wherein each of the end control points and the three intermediate control points comprises an x-coordinate, a y-coordinate, a z-coordinate and a weight.

6. The medium of claim 2, wherein the determining of the values for the three intermediate control points for the approximate mathematical expression of the RQB curve comprises determining each intermediate control point and their weights from solving a system of linear equations that arises from (1) identifying the values for the three intermediate control points that results in an RQB mathematical expression with the three intermediate control points equaling the RQB mathematical expression with the three RQB curve intermediate sample points and (2) from interpolation of the endpoints with their differentials.

7. The medium of claim 1, wherein the determining of the values for the approximate mathematical expression of the RQB curve comprises:

determining a plurality of candidate approximate mathematical expressions of the RQB curve, each candidate approximate mathematical expression comprising a different set of values;

for each of the candidate approximate mathematical expressions of the RQB curve:

determining estimate printing points based upon a set of parameter values and the candidate approximate mathematical expression of the RQB curve;

determining actual printing points in Euclidean space for the set of parameter values; and

comparing the estimate printing points to the corresponding actual printing points; and

selecting one of the candidate approximate mathematical expressions of the RQB curve as the approximate mathematical expression of the RQB curve based upon the comparisons.

8. The medium of claim 7, wherein each of the candidate approximate mathematical expressions of the RQB curve comprises a different set of intermediate control points and a same set of end control points.

9. The medium of claim 7, wherein the candidate approximate mathematical expressions of the RQB curve comprise:

a first candidate approximate mathematical expression of the RQB curve comprising a first set of intermediate control point values based in part upon a first set of RQB intermediate points corresponding to a first set of uniformly spaced intermediate conic points; and

a second candidate approximate mathematical expression of the RQB curve comprising a second set of intermediate control point values based in part upon a first set of RQB intermediate points corresponding to a second set of intermediate conic points different than the first set of uniformly spaced intermediate conic points.

10. A computer implemented method for generating three-dimensional (3D) printing points for printing a 3D object, the method comprising:

obtaining a mathematical expression of Steiner patch that is part of a tessellation approximation of a three-dimensional (3D) object to be printed by a 3D printer, together with a height value of a slicing plane;

determining values for an approximate mathematical expression of a Rational Quartic Bézier (RQB) curve, the RQB curve being an intersection of the slicing plane and the Steiner patch; and

generating 3D printing points in Euclidian space for the object based upon the values for the approximate mathematical expression, the 3D printing points for using 3D printing of the 3D object.

11. The method of claim 10, wherein the determining the values for the approximate mathematical expression of the RQB curve comprises:

determining end points of the RQB curve, the endpoints forming end control points;

determining differentials of a surface of the Steiner patch at the end points;

determining a classification of a conic curve in parametric space that corresponds to the RQB curve;

sample points along the conic curve;

determining a length estimate of the conic curve based on the sampled points and the classification;

determining three conic intermediate points along the conic curve based upon the length estimate;

determining three RQB curve intermediate points that correspond to the three conic intermediate points; and

determining three intermediate control points for the approximate mathematical expression of the RQB curve from the three RQB curve intermediate points and the endpoints with their differentials, wherein the approximate mathematical expression of the RQB curve comprises the end control points and the three intermediate control points.

12. The method of claim 11, wherein the determining of the values for the three intermediate control points for the approximate mathematical expression of the RQB curve comprises determining each intermediate control point from a set of linear equations comprising the three RQB curve intermediate points and the endpoints with their differentials.

13. The method of claim 10, wherein the determining of the values for the approximate mathematical expression of the RQB curve comprises:

determining a plurality of candidate approximate mathematical expressions of the RQB curve, each candidate approximate mathematical expression comprising a different set of values;

for each of the candidate approximate mathematical expressions of the RQB curve:

determining estimate printing points based upon a set of parameter values and the candidate approximate mathematical expression of the RQB curve;

determining actual printing points in Euclidean space for the set of parameter values; and

comparing the estimate printing points to the corresponding actual printing points; and

selecting one of the candidate approximate mathematical expressions of the RQB curve as the approximate mathematical expression of the RQB curve based upon the comparisons.

14. The method of claim 11 further comprising printing the 3D object using the generated 3D printing points.

15. A three-dimensional (3D) printing system comprising:

a 3D printer;

a processor; and

a non-transitory computer-readable medium containing instructions for directing the processor, the instructions comprising:

object data file instructions to obtain a Steiner patch that is part of a tessellation approximation of a three-dimensional (3D) object to be printed by a 3D printer, together with a height value of a slicing plane;

mathematical expression determination instructions to determine values for an approximate mathematical expression of an RQB curve, the RQB curve being an intersection of the slicing plane and the Steiner patch; and

3D printing point instructions to generate 3D printing points in Euclidian space for the object based upon the determined values for the approximate mathematical expression, the 3D printing points for using 3D printing of the 3D object.

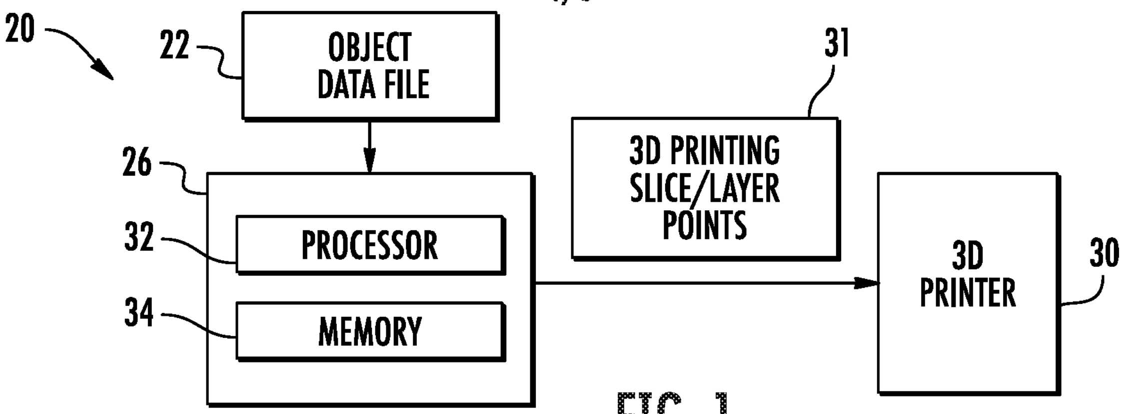


FIG. 1

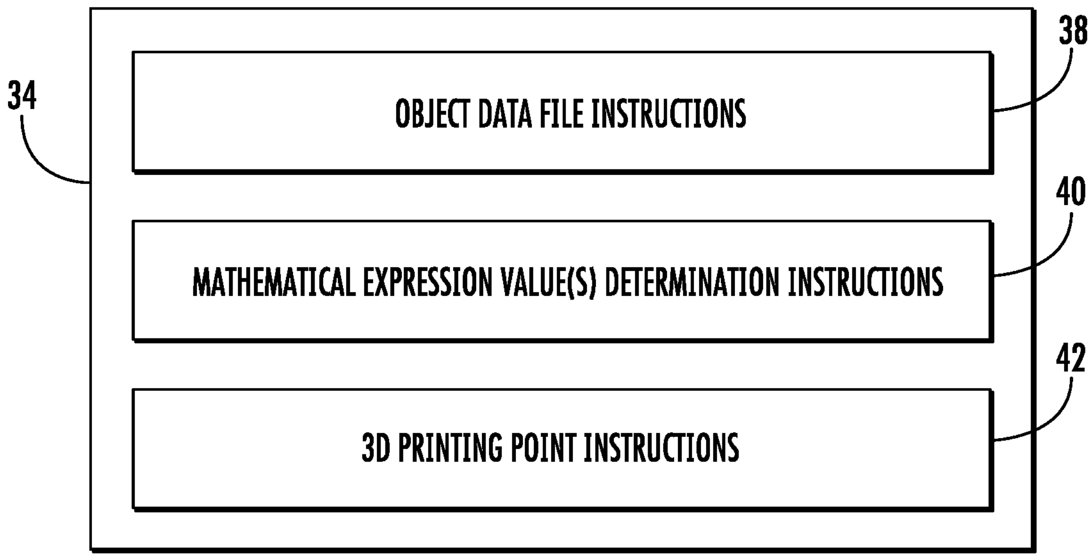


FIG. 2

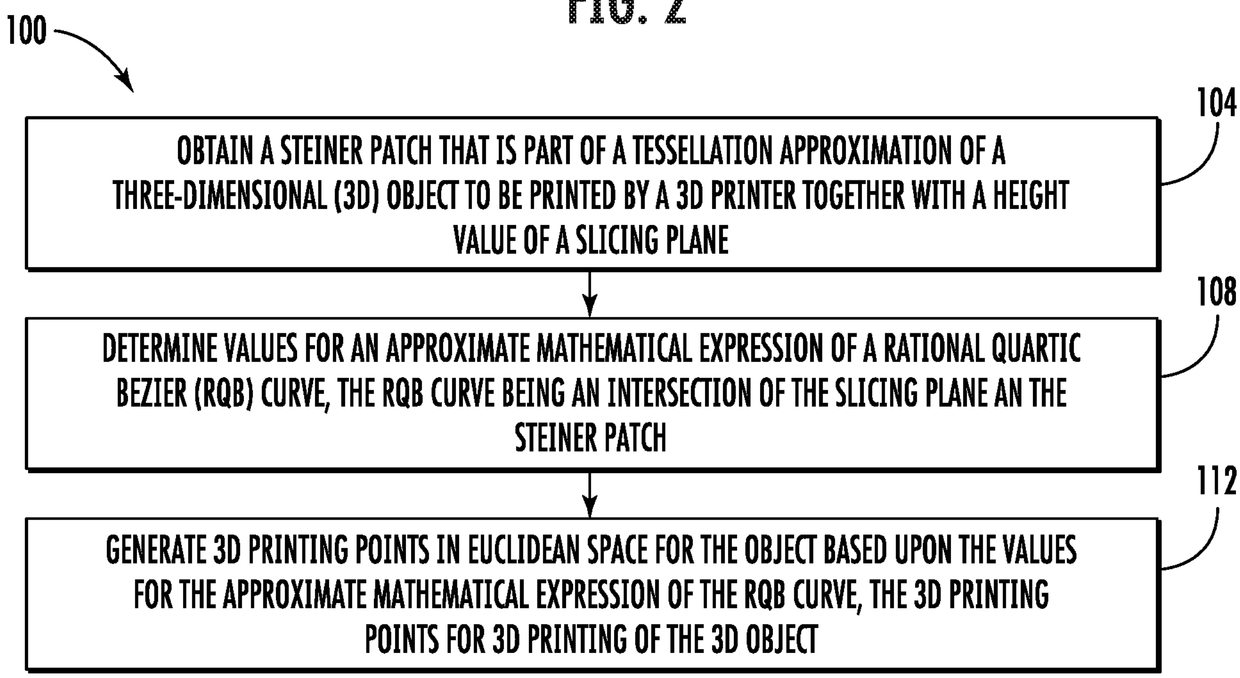


FIG. 3

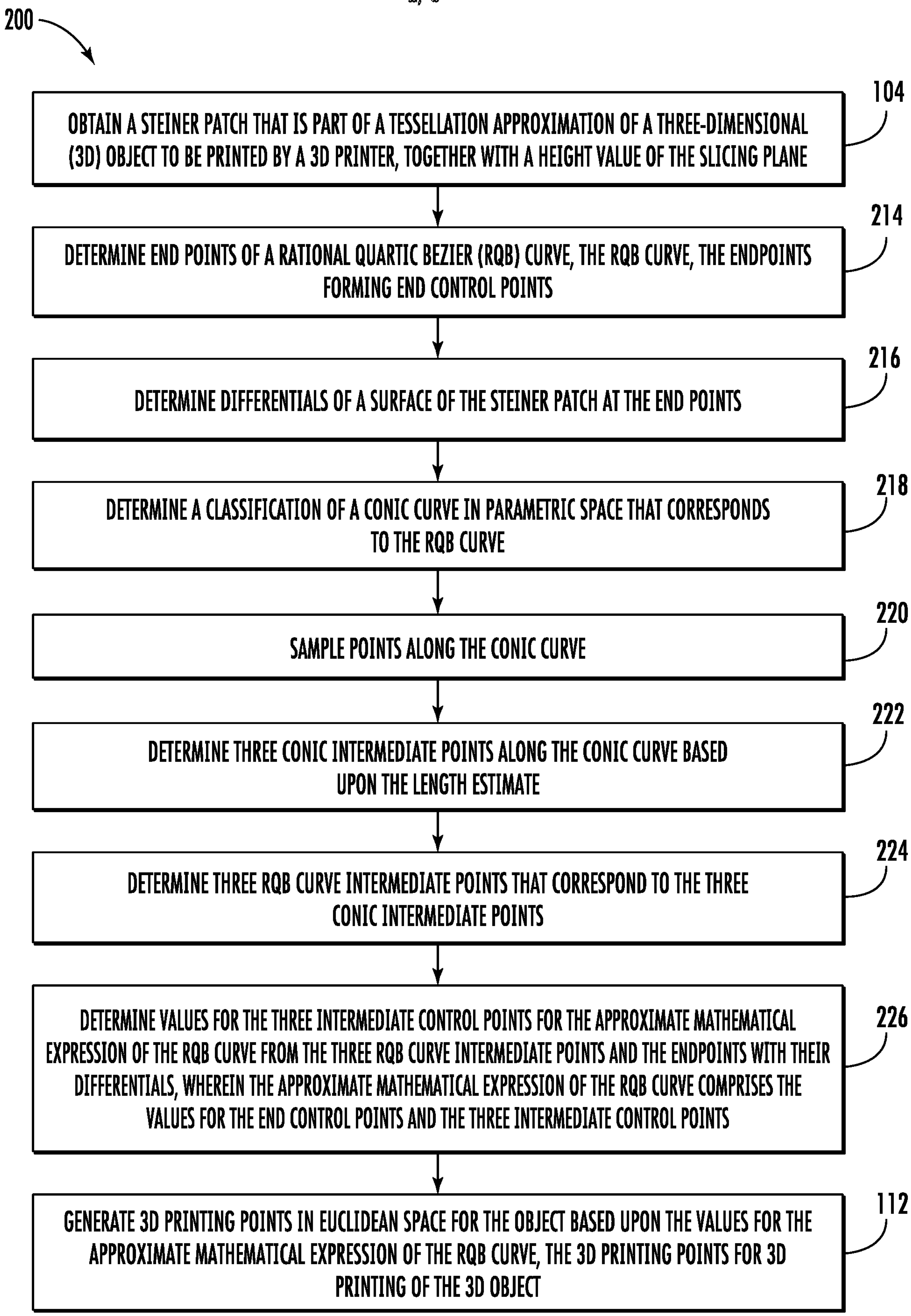
200
OBTAIN A STEINER PATCH THAT IS PART OF A TESSELLATION APPROXIMATION OF A THREE-DIMENSIONAL (3D) OBJECT TO BE PRINTED BY A 3D PRINTER, TOGETHER WITH A HEIGHT VALUE OF THE SLICING PLANE
104
DETERMINE END POINTS OF A RATIONAL QUARTIC BEZIER (RQB) CURVE, THE RQB CURVE, THE ENDPOINTS FORMING END CONTROL POINTS
214
DETERMINE DIFFERENTIALS OF A SURFACE OF THE STEINER PATCH AT THE END POINTS
216
DETERMINE A CLASSIFICATION OF A CONIC CURVE IN PARAMETRIC SPACE THAT CORRESPONDS TO THE RQB CURVE
218
SAMPLE POINTS ALONG THE CONIC CURVE
220
DETERMINE THREE CONIC INTERMEDIATE POINTS ALONG THE CONIC CURVE BASED UPON THE LENGTH ESTIMATE
222
DETERMINE THREE RQB CURVE INTERMEDIATE POINTS THAT CORRESPOND TO THE THREE CONIC INTERMEDIATE POINTS
224
DETERMINE VALUES FOR THE THREE INTERMEDIATE CONTROL POINTS FOR THE APPROXIMATE MATHEMATICAL EXPRESSION OF THE RQB CURVE FROM THE THREE RQB CURVE INTERMEDIATE POINTS AND THE ENDPOINTS WITH THEIR DIFFERENTIALS, WHEREIN THE APPROXIMATE MATHEMATICAL EXPRESSION OF THE RQB CURVE COMPRISES THE VALUES FOR THE END CONTROL POINTS AND THE THREE INTERMEDIATE CONTROL POINTS
226
GENERATE 3D PRINTING POINTS IN EUCLIDEAN SPACE FOR THE OBJECT BASED UPON THE VALUES FOR THE APPROXIMATE MATHEMATICAL EXPRESSION OF THE RQB CURVE, THE 3D PRINTING POINTS FOR 3D PRINTING OF THE 3D OBJECT
112

FIG. 4

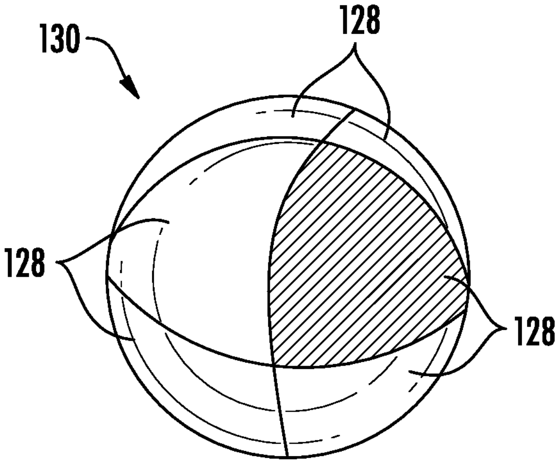


FIG. 5A

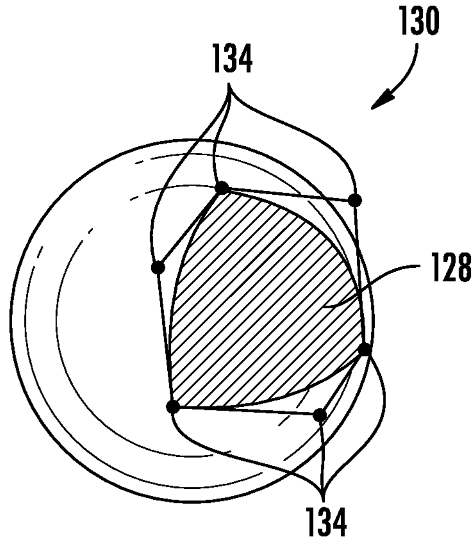


FIG. 5B

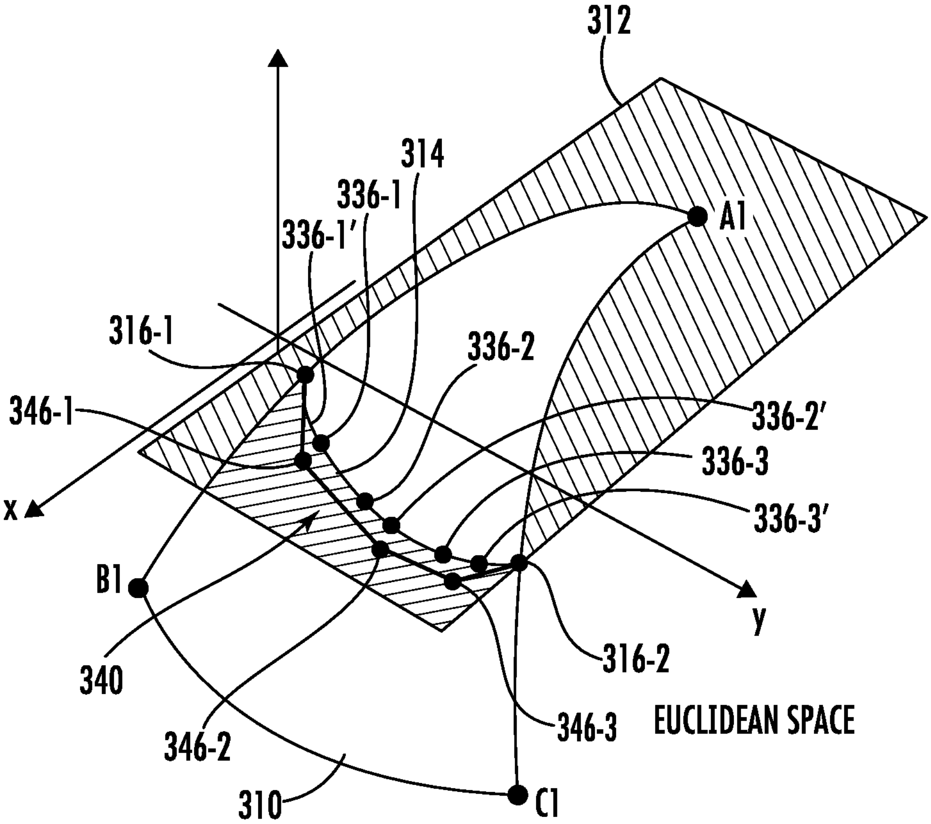


FIG. 6

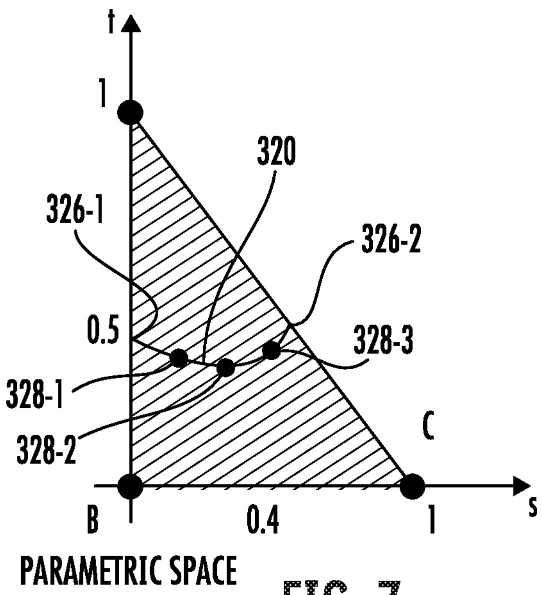


FIG. 7

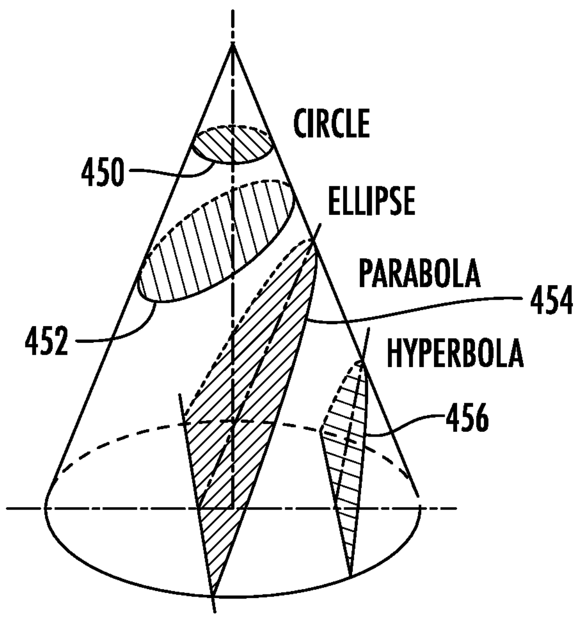


FIG. 8

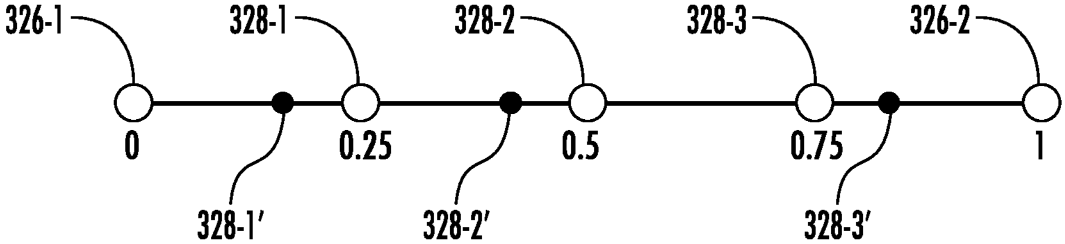


FIG. 9

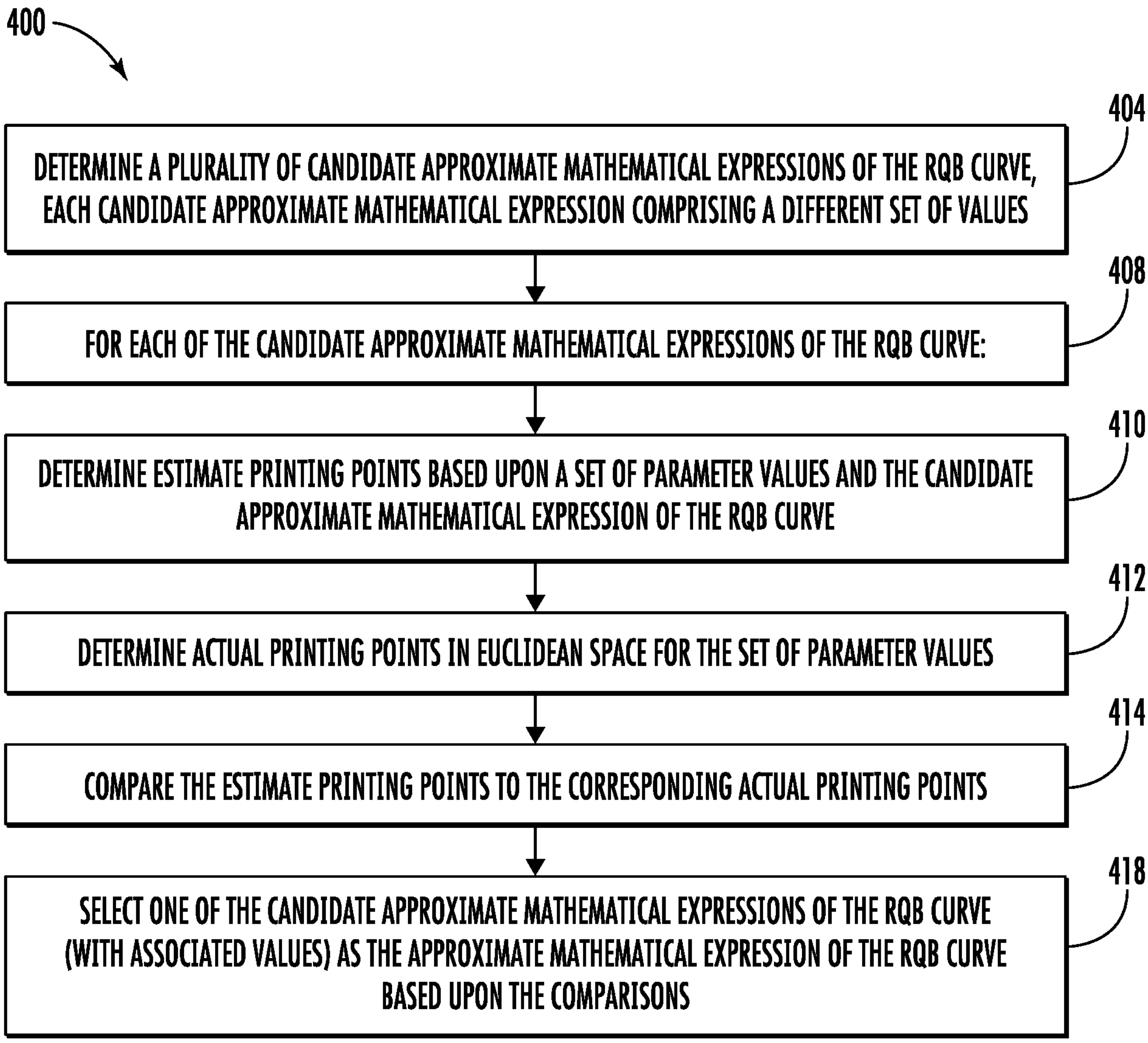
400
DETERMINE A PLURALITY OF CANDIDATE APPROXIMATE MATHEMATICAL EXPRESSIONS OF THE RQB CURVE, EACH CANDIDATE APPROXIMATE MATHEMATICAL EXPRESSION COMPRISING A DIFFERENT SET OF VALUES
404
FOR EACH OF THE CANDIDATE APPROXIMATE MATHEMATICAL EXPRESSIONS OF THE RQB CURVE:
408
DETERMINE ESTIMATE PRINTING POINTS BASED UPON A SET OF PARAMETER VALUES AND THE CANDIDATE APPROXIMATE MATHEMATICAL EXPRESSION OF THE RQB CURVE
410
DETERMINE ACTUAL PRINTING POINTS IN EUCLIDEAN SPACE FOR THE SET OF PARAMETER VALUES
412
COMPARE THE ESTIMATE PRINTING POINTS TO THE CORRESPONDING ACTUAL PRINTING POINTS
414
SELECT ONE OF THE CANDIDATE APPROXIMATE MATHEMATICAL EXPRESSIONS OF THE RQB CURVE (WITH ASSOCIATED VALUES) AS THE APPROXIMATE MATHEMATICAL EXPRESSION OF THE RQB CURVE BASED UPON THE COMPARISONS
418

FIG. 10

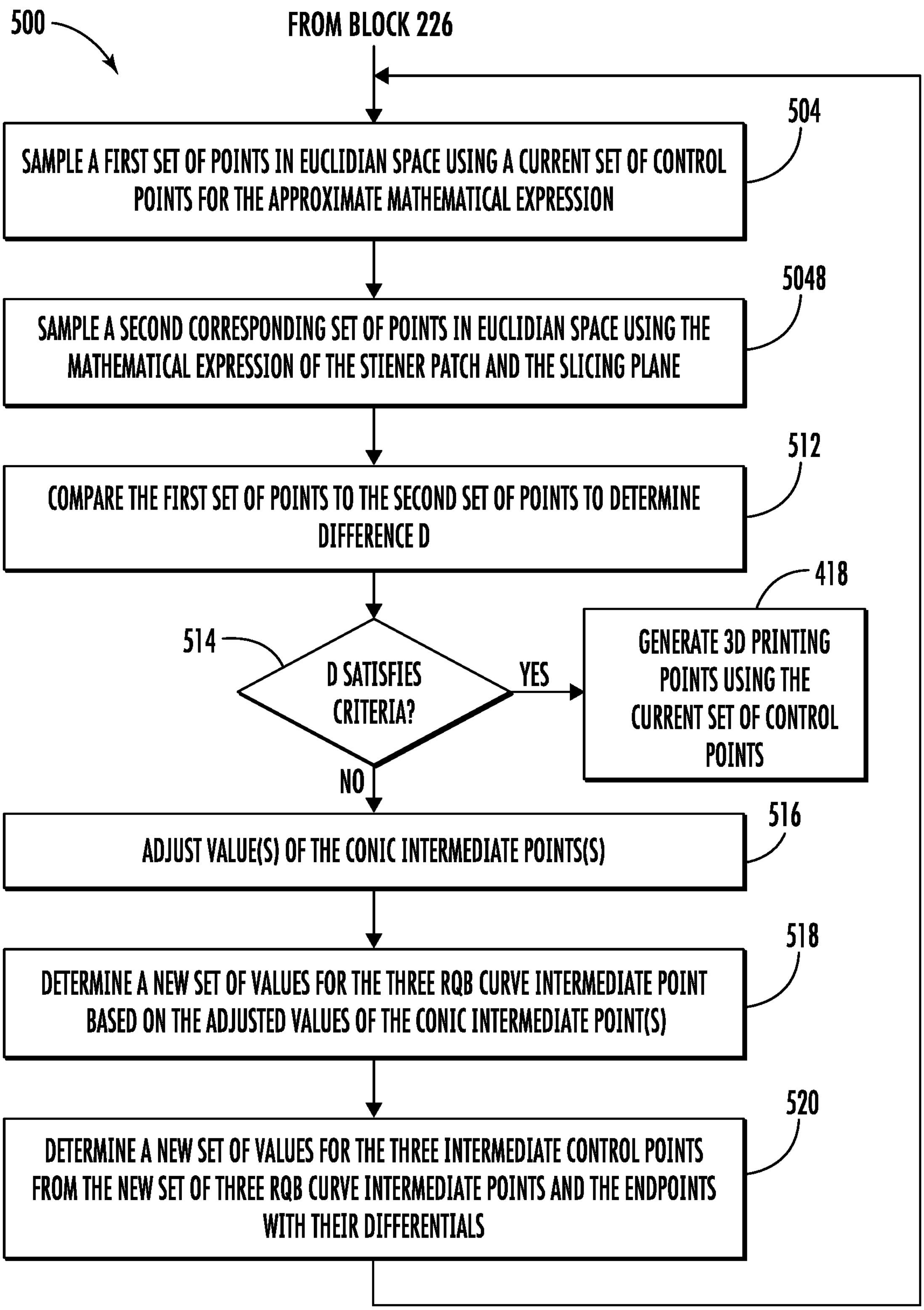
500
FROM BLOCK 226
504
SAMPLE A FIRST SET OF POINTS IN EUCLIDIAN SPACE USING A CURRENT SET OF CONTROL POINTS FOR THE APPROXIMATE MATHEMATICAL EXPRESSION
5048
SAMPLE A SECOND CORRESPONDING SET OF POINTS IN EUCLIDIAN SPACE USING THE MATHEMATICAL EXPRESSION OF THE STIENER PATCH AND THE SLICING PLANE
512
COMPARE THE FIRST SET OF POINTS TO THE SECOND SET OF POINTS TO DETERMINE DIFFERENCE D
514
D SATISFIES CRITERIA?
YES
418
GENERATE 3D PRINTING POINTS USING THE CURRENT SET OF CONTROL POINTS
NO
516
ADJUST VALUE(S) OF THE CONIC INTERMEDIATE POINTS(S)
518
DETERMINE A NEW SET OF VALUES FOR THE THREE RQB CURVE INTERMEDIATE POINT BASED ON THE ADJUSTED VALUES OF THE CONIC INTERMEDIATE POINT(S)
520
DETERMINE A NEW SET OF VALUES FOR THE THREE INTERMEDIATE CONTROL POINTS FROM THE NEW SET OF THREE RQB CURVE INTERMEDIATE POINTS AND THE ENDPOINTS WITH THEIR DIFFERENTIALS

FIG. 11

**INTERNATIONAL SEARCH REPORT**

International application No.

PCT/US 2020/016045

A. CLASSIFICATION OF SUBJECT MATTER

***G06T 17/30*** *(2006.01)*
***G06T 7/11*** *(2017/01)*
***B29C 64/393*** *(2017.01)*

According to International Patent Classification (IPC) or to both national classification and IPC

B. FIELDS SEARCHED

Minimum documentation searched (classification system followed by classification symbols)

G06T 7/00 - 19/20; B29C 64/00- 64/393

Documentation searched other than minimum documentation to the extent that such documents are included in the fields searched

Electronic data base consulted during the international search (name of data base and, where practicable, search terms used)

PatSearch (RUPTO internal), USPTO, PAJ, K-PION, Esp@cenet, Information Retrieval System of FIPS

C. DOCUMENTS CONSIDERED TO BE RELEVANT

| Category* | Citation of document, with indication, where appropriate, of the relevant passages | Relevant to claim No. |
|---|---|---|
| A | US 2017/0372480 A1 (UNIVERSITY OF CINCINNATI) 28.12.2017 | 1-15 |
| A | US 2010/0131251 A1 (FUJISU LIMITED) 27.05.2010 | 1-15 |
| A | US 6906718 B1 (MICROSOFT CORPORATION) 14.06.2005 | 1-15 |
| A | EP 0596667 B1 (CANON INC.) 03.06.1998 | 1-15 |

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"P" document published prior to the international filing date but later than the priority date claimed
"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention
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"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art
"&" document member of the same patent family

| Date of the actual completion of the international search | Date of mailing of the international search report |
|---|---|
| 29 September 2020 (29.09.2020) | 08 October 2020 (08.10.2020) |

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