Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
Programa de Pós-graduação em Matemática

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Control and Stabilization for the Benjamin–Bona–Mahony Equation on the One-Dimensional Torus

Recife

2025

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Master's thesis submitted to the Graduate Program in Mathematics at the Department of Mathematics of the Federal University of Pernambuco, as a partial requirement for the degree of Master in Mathematics.

Area of Concentration: Analysis

Advisor: Prof. Dr. Victor Hugo Gonzalez Martinez

Recife
2025

.Catalogação de Publicação na Fonte. UFPE - Biblioteca Central

Oliveira, Matheus Luiz da Silva.
Control and stabilization for the Benjamin-Bona-Mahony equation on the one-dimensional torus, 2025 / Matheus Luiz da Silva Oliveira. - Recife, 2025.
144f.: il.

Dissertação (Mestrado) - Universidade Federal de Pernambuco, Centro de Ciências Exatas e da Natureza, Programa de Pós-Graduação em Matemática, 2025.
Orientação: Victor Hugo Gonzalez Martinez.
Inclui referências.

1. Equação de Benjamin-Bona-Mahony; 2. Propriedade de continuação única; 3. Controlabilidade exata; 4. Estabilização; 5. Controle de ponto móvel; 6. Equação de Korteweg-de Vries. I. Gonzalez Martinez, Victor Hugo. II. Título.

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MATHEUS LUIZ DA SILVA OLIVEIRA

Control and Stabilization for the Benjamin–Bona–Mahony Equation on the One-Dimensional Torus

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestrado em Matemática.

Aprovado em: 19/02/2025

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ACKNOWLEDGMENTS

This section acknowledges the many unseen contributors to this work, each of whom has played a unique role in its completion. My first gratitude goes to my family, especially my parents, Jorge and Rosilene Oliveira, for all their support. I also offer my heartfelt apologies for the moments of absence; this dissertation is the result of those sacrifices. To Yasmim Brasil, my companion in love, thank you for your emotional support, constant encouragement, and for sharing the joy of this journey.

If your name appears on this list, it is because I am grateful to you in some way—whether for your support, encouragement, teachings, clarification of doubts, or sharing a template. The specific reason may vary, but each of you has contributed to this journey: Nelson Leal Júnior, Micael Estevão, Willikat Bezerra, Leonardo Tavares, Diego Teixeira, Jonatas Teodomiro, Antonio Marcos, Everlon Figueiroa, Daniel José, and Érick Caetano. I have learned from each of you.

Special thanks are extended to Rafael Cavalcanti, who supported my admission into the graduate mathematics program and has continued to support me since then; to Jandeilson Santos Silva, whose expertise, particularly in Distributions Theory, proved invaluable—perhaps the only one in the department to have studied it in a regular course; to Jefferson Bezerra for similar assistance; and to Matheus Nunes, whose encouragement and dedication have been a consistent source of inspiration.

As promised, special thanks are also owed to Matheus Nasi, who, instead of math-exchange, embodies the Mathew-Exchange.

I would also like to thank professors Ademir Fernando Pazoto and Miguel Dario Soto Vieira, who supported me in proving a Carleman estimate.

My deepest gratitude is reserved for Professor Victor Hugo Gonzalez Martinez, first for have accepted the role of advisor and subsequently exemplified patience, dedication, and an extensive sharing of mathematical knowledge. The passion for teaching was evident throughout. A close examination reveals that all the preliminary results of this work—the mathematical framework—were imparted by him. This includes functional analysis, distributions, and Sobolev spaces during regular coursework, as well as semigroup theory, control, and stabilization during the advisory period. It is safe to acknowledge that his guidance has shaped my path in mathematics, and for that, I am sincerely grateful.

I would also like to thank the members of my examination committee, Professor Márcio Cavalcante de Melo and Professor Roberto de Almeida Capistrano Filho, for their time and for accepting the invitation, as well as the entire Graduate Program in Mathematics at UFPE.

As Newton said, 'If I have seen further, it is by standing on the shoulders of giants.' To all of you, my giants, thank you for providing the foundation and support that have allowed me to take these steps forward in my journey.

Finally, I acknowledge CAPES for its financial support.

ABSTRACT

In the work *Unique continuation property and control for the Benjamin–Bona–Mahony equation on a periodic domain, Journal of Differential Equations*, (254), no. 1, 2013 by Lionel Rosier and Bing-Yu Zhang, the authors studied the Benjamin-Bona-Mahony (BBM) equation, a fundamental model for the propagation of long waves with small amplitude in nonlinear dispersive systems, on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. First, the authors showed that the initial-value problem associated with the BBM equation is globally well-posed in $H^s(\mathbb{T})$, for $s \geqslant 0$. Moreover, the mapping associating the solution to a given initial data is smooth and the solution is analytic in time. Subsequently, they establish a unique continuation property (UCP) for small data in $H^1(\mathbb{T})$ with nonnegative zero means. This result is further extended to certain BBM-like equations, including the equal width wave equation and the KdV-BBM equation, where, for the latter, some Carleman estimates are derived. Applications to stabilization are developed, showing that semiglobal exponential stabilization can be achieved in $H^s(\mathbb{T})$ for any $s \geqslant 1$ when an internal control acting on a moving interval is applied. Furthermore, they prove that the BBM equation with a moving control is locally exactly controllable in $H^s(\mathbb{T})$ for $s \geqslant 0$ and globally exactly controllable in $H^s(\mathbb{T})$ for $s \geqslant 1$ over sufficiently large times, depending on the H^s-norms of the initial and terminal states. The results of this article are explored in detail in this master's thesis.

Keywords: Benjamin–Bona–Mahony equation; unique continuation property; exact controllability; stabilization; moving point control; Korteweg–de Vries equation.

RESUMO

No trabalho *Unique continuation property and control for the Benjamin–Bona–Mahony equation on a periodic domain, Journal of Differential Equations*, (254), no. 1, 2013, de Lionel Rosier e Bing-Yu Zhang, os autores estudaram a equação de Benjamin–Bona–Mahony (BBM), um modelo fundamental para a propagação de ondas longas com pequena amplitude em sistemas dispersivos não lineares, no toro unidimensional $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Primeiramente, os autores demonstraram que o problema de valor inicial associado à equação BBM é globalmente bem-posto em $H^s(\mathbb{T})$, para $s \geqslant 0$. Além disso, mostra-se que a aplicação que associa a solução ao dado inicial é suave e que a solução é analítica no tempo. Subsequentemente, eles estabelecem a Propriedade de Continuação Única (PCU) para dados pequenos em $H^1(\mathbb{T})$ com média zero não negativa. Esse resultado é então estendido para certas equações do tipo BBM, incluindo a equação de ondas de largura igual e a equação KdV-BBM, para a qual algumas estimativas de Carleman são derivadas. Aplicações à estabilização também são desenvolvidas, mostrando que a estabilização exponencial semiglobal pode ser alcançada em $H^s(\mathbb{T})$ para qualquer $s \geqslant 1$, quando um controle interno atuando em um intervalo móvel é aplicado. Além disso, eles provam que a equação BBM com controle móvel é localmente exatamente controlável em $H^s(\mathbb{T})$ para $s \geqslant 0$ e globalmente exatamente controlável em $H^s(\mathbb{T})$ para $s \geqslant 1$, em tempos suficientemente grandes, dependendo das normas H^s dos estados iniciais e finais. Os resultados deste artigo são explorados e detalhados nesta dissertação de mestrado.

Palavras-Chave: Equação de Benjamin–Bona–Mahony; propriedade de continuação única; controlabilidade exata; estabilização; controle de ponto móvel; equação de Korteweg–de Vries.

CONTENTS

1 INTRODUCTION

This work builds on the foundational results of the paper *Unique continuation property and control for the Benjamin–Bona–Mahony equation on a periodic domain*, by Lionel Rosier and Bing-Yu Zhang, published in *Journal of Differential Equations* in 2013. Before embarking on the mathematical development, it is worth putting the Benjamim-Bona-Mahony (BBM) equation into context. Thus, this introduction begins with a historical overview highlighting the importance of the BBM equation as a mathematical model for a variety of physical phenomena. Next, to put the contribution of the aforementioned paper (ROSIER; ZHANG, 2013) into focus, we set up the problems and main results addressed in this master's thesis, whose purpose is to provide a clear and accessible exposition of the subject, aiming to facilitate the understanding of the underlying motivations and challenges faced by professional mathematicians tackling such problems. Finally, the introduction concludes with an outline of the structure of this thesis, to better guide the reader through its content.

1.1 Historical Context: Why the BBM Equation?

The BBM equation

$$u_t - u_{txx} + u_x + uu_x = 0, \tag{1.1}$$

was introduced in 1972 by T. Benjamin, J. Bona, and J. Mahony (BENJAMIN; BONA; MAHONY, 1972) as an alternative to the classical Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + u_x + uu_x = 0, \tag{1.2}$$

as a model equation governing the propagation of one-dimensional, unidirectional long waves with small amplitude in nonlinear dispersive systems. As a classical model, the BBM equation finds applications in a wide range of physical systems, including the long wavelength in liquids, hydromagnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, and acoustic waves inharmonic crystals.

In the context of shallow-water waves, in the equation (1.1), $u = u(x, t)$ represents the displacement of the water surface at location x and time t, where we shall assume $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus). Equation (1.1) is often obtained from (1.2) in the derivation of the surface equation by noticing that, in the considered regime, $u_x \sim -u_t$, so that $u_{xxx} \sim -u_{txx}$.

The authors in (BENJAMIN; BONA; MAHONY, 1972) argue that the assumptions leading to equation (1.2) equally well justify the equation (1.1) as a model for describing long-wave behavior. The equation (1.1) is also referred to as the Regularized Long Wave Equation (RLW), thanks to the strong smoothing effect of the dispersive term $-u_{txx}$, which confers (1.1) considerable technical advantages over (1.2), from the standpoint of existence, uniqueness, and stabilization theory, as proved in (BENJAMIN; BONA; MAHONY, 1972; BONA; TZVETKOV, 2009; ROUMÉGOUX, 2010) and the references therein. On the other hand, (1.1) is not integrable and it has only three invariants of motion (OLVER, 1979).

1.2 Setting Up the Problems and Main Results

We begin our study of the BBM equation (1.1) in chapter 3, directing our attention to the initial value problem (IVP) for (1.1). The existence of a solution will be established upon the assumption that the initial data $u(x,0)$ belongs to the Sobolev space $H^s(\mathbb{T})$, for any $s \geq 0$. This result turns out to be sharp in the sense that below this value of s, the initial-value problem cannot be solved by a Picard iteration. The IVP for the BBM equation was proved to be globally well-posed by Bona and Tzvetkov (BONA; TZVETKOV, 2009). On the other hand, Panthee (PANTHEE, 2010) has shown that the BBM equation is ill-posed for initial data that belong to $H^s(\mathbb{R})$ with $s < 0$. For this topic, we follow (BONA; TZVETKOV, 2009; ROUMÉGOUX, 2010; HIMONAS; PETRONILHO, 2020) in a standard procedure: first proving a local existence and uniqueness theorem (i.e., for a sufficiently small time-interval) by means of a fixed-point principle, then establishing the existence of a solution over an arbitrarily large time. Moreover, the map that associates the relevant solution to the given initial data is shown to be smooth. These well-posedness results can be summarized as stated in (ROSIER; ZHANG, 2013)

Theorem 1.2.1. *Let $s \geqslant 0, u(x,0) = u_0 \in H^s(\mathbb{T})$ and $T > 0$. Then there exists a unique solution $u \in X_T^s = C\left([-T,T]; H^s(\mathbb{T})\right)$ of the IVP associated with* (1.1). *Furthermore, for* *any $R > 0$, the map $u_0 \mapsto u$ is real analytic from $B_R(H^s(\mathbb{T}))$ into X_T^s.*

Then, some additional properties are given in the same chapter, such as the analyticity in time for the solution and the three invariants of motion.

Next, in Chapter 4, we explore the Unique Continuation Property (UCP) of the BBM equation and its applications to a control problem for (1.1). The UCP is said to hold in a given function class X if, for any nonempty open set $\omega \subset \mathbb{T}$, the only solution $u \in X$ of (1.1) satisfying

$$u(x,t) = 0 \quad \text{for } (x,t) \in \omega \times (0,T),$$

is the trivial solution $u \equiv 0$. This property is fundamental in Control Theory, as it is equivalent to the approximate controllability of linear partial differential equations (PDEs). Moreover, the UCP plays a key role in the classical uniqueness-compactness approach used in proving the stabilization of PDEs with localized damping. Despite its importance, the UCP for the BBM equation remains in its early stages of development. We begin the study of UCP by establishing the UCP for solutions of (1.1) under additional assumptions on the initial data. Specifically, the initial data must be small enough in $H^1(\mathbb{T})$ and have nonnegative mean values. The proof leverages results from Chapter 3, such as the time analyticity of solutions to the BBM equation and its invariants of motion.

Subsequently, we extend the analysis to BBM-like equations, which include the Morrison–Meiss–Carey equation and an intermediate equation between (1.1) and (1.2), called KdV-BBM equation. For the latter case, we employ the classical approach of Carleman estimates to derive the UCP.

Here we gather the main results of Chapter 4,

Theorem 1.2.2 (UCP for BBM Equation)**.** *Let $u_0 \in H^1(\mathbb{T})$ be such that*

$$\int_{\mathbb{T}} u_0(x)dx \geqslant 0,$$

and

$$\|u_0\|_{L^\infty(\mathbb{T})} < 3.$$

Assume that the solution u of the IVP associated with (1.1) *satisfies*

$$u(x,t) = 0 \quad \textit{for all } (x,t) \in \omega \times (0,T),$$

where $\omega \subset \mathbb{T}$ *is a nonempty open set and* $T > 0$. *Then* $u_0 = 0$, *and hence* $u \equiv 0$.

Theorem 1.2.3 (UCP for the KdV-BBM Equation)**.** *Let* $c \in \mathbb{R}\backslash\{0\}, T > 2\pi/|c|$, *and* $q \in L^\infty(0,T;L^\infty(\mathbb{T}))$. *Let* $\omega \subset \mathbb{T}$ *be a nonempty open set. Let* $u \in L^2(0,T;H^2(\mathbb{T})) \cup L^\infty(0,T;H^1(\mathbb{T}))$ *satisfying the KdV-BBM equation*

$$u_t - u_{txx} - cu_{xxx} + qu_x = 0, \quad x \in \mathbb{T},\ t \in (0,T),$$

where $q \in L^\infty(0,T;L^\infty(\mathbb{T}))$ *is a given potential function and* $c \neq 0$ *is a given real constant, and satisfying*

$$u(x,t) = 0 \quad \text{for a.e. } (x,t) \in \omega \times (0,T).$$

Then $u \equiv 0$ *in* $\mathbb{T} \times (0,T)$.

In the final chapter of this work, the Chapter 5, we focus on the controllability of the BBM equation, which also represents the primary goal of this endeavor. The control and stabilization of dispersive wave equations have been the subject of extensive research over the past decade. We begin by considering the linearized BBM equation with a control force

$$u_t - u_{txx} + u_x = a(x)h(x,t), \tag{1.3}$$

where $a(x)$ is supported in a subset of $\mathbb{T}$ and $h(x,t)$ represents the control input. It was shown in (MICU, 2001; ZHANG; ZUAZUA, 2003) that (1.3) is *approximately controllable* in $H^1(\mathbb{T})$. However, (1.3) is not *exactly controllable* in $H^1(\mathbb{T})$, as proved in (MICU, 2001). This stands in sharp contrast with the good control properties observed in other dispersive equations, such as the KdV equation, the nonlinear Schrödinger equation, the Benjamin–Ono equation, the Boussinesq system, and the Camassa–Holm equation.

On the other hand, the KdV-BBM equation can be derived from (1.1) by working in a moving frame $x = -ct$ with $c \in \mathbb{R}\backslash\{0\}$. Defining

$$v(x,t) = u(x - ct, t), \tag{1.4}$$

transforms (1.1) into the following KdV-BBM equation

$$v_t + (c+1)v_x - cv_{xxx} - v_{txx} + vv_x = 0. \tag{1.5}$$

The presence of the KdV term $-cv_{xxx}$ in (1.5) suggests improved control properties compared to (1.1). We establish that the KdV-BBM equation with a forcing term $a(x)k(x,t)$, supported in any given subdomain, is locally exactly controllable in $H^1(\mathbb{T})$ for any $s \geqslant 1$, provided that the control time satisfies $T > (2\pi)/|c|$. Returning to the original variables, this result implies that the equation

$$u_t + u_x - u_{txx} + uu_x = a(x+ct)h(x,t), \tag{1.6}$$

with a moving distributed control, is exactly controllable in $H^1(\mathbb{T})$ for any $s \geqslant 1$, given sufficiently large time T. The choice of T ensures that the support of the moving control, which travels at a constant velocity c, can cover the entire domain $\mathbb{T}$.

The concept of moving point control was originally introduced by J.L. Lions in (LIONS, 1992) for the wave equation. An important motivation for this approach is that exact controllability fails for the wave equation with a static pointwise control if the point corresponds to a zero of an eigenfunction of the Dirichlet Laplacian. However, exact controllability holds when the control point moves, provided it satisfies specific conditions that are easy to verify.

Similarly, for the BBM equation, applying a localized damping with a moving support leads to semiglobal exponential stabilization. This chapter demonstrates that combining local exact controllability with semi-global exponential stabilization results in the following theorem, which represents the main result of this master's thesis:

Theorem 1.2.4 (Local Exact Controllability for BBM with a Moving Control)**.** *Assume that $a \in C^\infty(\mathbb{T})$ with $a \neq 0$ is given and that $c \in \mathbb{R}\backslash\{0\}$. Let $s \geqslant 1$ and $R > 0$ be given. Then there exists a time $T = T(s,R) > 2\pi/|c|$ such that for any $u_0, u_T \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} \leqslant R, \quad \|u_T\|_{H^s} \leqslant R,$$

there exists a control $h \in L^2\left(0,T;H^{s-2}(\mathbb{T})\right)$ such that the solution $u \in C\left([0,T];H^s(\mathbb{T})\right)$ of

$$\begin{aligned} &u_t - u_{txx} + u_x + uu_x = a(x+ct)h(x,t), \quad x \in \mathbb{T}, t \in (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{T} \end{aligned}$$

satisfies

$$u(x,T) = u_T(x), \quad x \in \mathbb{T}.$$

1.3 Thesis Outline

Each chapter begins with a brief explanation of what follows therein, providing theoretical or historical information related to the topic, or outlining the chapter. In turn, the following is an outline for the entire work:

- Chapter 2, although interesting in its own right, has the primary purpose of laying the mathematical framework for the entire work. It collects useful and remarkable facts from various fields, such as Topology, Functional Analysis, Distributions, Sobolev Spaces, Semigroup Theory, and Control and Stabilization concepts;

- Chapter 3 is divided into three sections. The first section 3.1 discusses the well-posedness of the BBM equation, also providing estimates that will be useful for applying the contraction mapping principle. The following sections address the analyticity of the solution for the BBM, in Section 3.2 and the conserved quantities in Section 3.2;

- Chapter 4 first addresses the UCP property for the BBM equation in Section 4.1. The second section is split into two subsections, 4.2.1 and 4.2.2, which treat a BBM-like equation without a drift term and one with a nonlocal bilinear term, respectively. Next, the third section 4.3 of the chapter addresses the KdV-BBM equation, where the UCP is derived by means of a Carleman estimate, and due to its importance, we have dedicated the subsection 4.3.1 to it.

- Chapter 5 focuses on the controllability and stabilizability issues, each of which is addressed in a separate section. Section 5.1, which deals with controllability, is divided into subsections. The first subsection, 5.1.1, covers the exact controllability for the linearized BBM equation, while the second, 5.1.2, addresses the local aspects of controllability for the BBM equation. In turn, the section addressing stabilizability possesses two subsections. The first subsection, 5.2.1, deals with the well-posedness issue for a feedback system KdV-BBM. The last subsection, 5.2.2, addresses local and global exponential stabilization.

2 COMPENDIUM OF PRELIMINARY RESULTS

This chapter aims to lay the groundwork for the theory developed in the following chapters, as well as to establish the notation used throughout the work. The sections titled "Basic theory" and "Semigroup theory" summarize the prerequisites, while the Control and Stabilization section gives a brief summary on the specific concepts to follow the last chapter of this master's thesis.

2.1 Basic Theory

2.1.1 Elements of Topology and Functional Analysis

This subsection is inspired in (MARTEL, 2020; GRUBB, 2008; BREZIS, 2011; LIMA, 2020; KREYSZIG, 1989). Roughly speaking, a topological space is a space where one can talk about convergence and continuity, on the other hand, Functional Analysis in, say, Hilbert spaces has powerful tools to establish operators with good mapping properties and invertibility properties. A combination with Distribution Theory allows showing solvability of suitable concrete partial differential equations.

Definition 2.1.1. *We say that a sequence* $\{g_k\}_{k=1}^{\infty}$ *of a metric space* (E, d) *converges to* $g \in E$*, written* $\lim_k g_k = g$*, if* $\lim_{k\to\infty} d(g_k, g) = 0$.

Definition 2.1.2. *Let* (E, d) *be a metric space, and* A *be a subset of* E*. We say that* A *is dense in* E *if its closure* $\bar{A}$ *is* E*. Equivalently, for all* $f \in E$*, there exists a sequence* $\{a_j\}_{j=1}^{\infty}$ *of elements of* A *such that* $\lim_{j\to\infty} a_j = f$.

Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Let A be a subset of E_1 and $F : A \to E_2$ be a function. Let $g_0 \in \bar{A}$ and $h \in E_2$. We say that

$$\lim_{g\to g_0; g\in A} F(g) = h,$$

if for all neighborhood V_2 of h in E_2, there exists a neighborhood V_1 of g_0 in E_1 such that $F(V_1 \cap A) \subset V_2$. The limit, if it exists, is unique.

Definition 2.1.3. *We say that $F : A \to E_2$ is continuous at $g_0 \in A$ if*

$$\lim_{g \to g_0; g \in A} F(g) = F(g_0).$$

We say that F is continuous on A if it is continuous at any point of A. The composition of two continuous functions is continuous.

Theorem 2.1.1. *The application $F : A \to E_2$ is continuous at $g \in A$ if, and only if for any sequence $\{g_k\}_{k=1}^{\infty}$ of A converging to g, the sequence $\{F(g_k)\}_{k=1}^{\infty}$ converges to $F(g)$.*

Proof. See Proposition 9, Chapter 5 in (LIMA, 2020). ■

Definition 2.1.4 (Compactness)**.** *We say that a metric space (E, d) is compact if any sequence of E admits a subsequence that converges to an element of E. A subset A of a metric space (E, d) is compact if the metric space (A, d) is compact.*

Definition 2.1.5. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. Let $D \subset X$ and $F : D \to Y$. We say that F is uniformly continuous if the function $\omega : [0, \infty) \to [0, \infty)$ defined by*

$$\omega(\delta) = \sup_{\substack{g,h \in D \\ \|g-h\| \leq \delta}} \|F(g) - F(h)\|_Y,$$

converges to 0 as δ converges to 0.

Theorem 2.1.2 (Heine-Cantor)**.** *Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Let $F : E_1 \to E_2$ be continuous. If E_1 is compact, then F is uniformly continuous.*

Proof. See Proposition 9, Chapter 8 in (LIMA, 2020). ■

Definition 2.1.6 (Cauchy sequence)**.** *We call a Cauchy sequence in E a sequence $\{g_k\}_{k=1}^{\infty}$ such that*

$$\lim_{j,k \to \infty} d(g_j, g_k) = 0.$$

Theorem 2.1.3. *In any metric space, a converging sequence is a Cauchy sequence.*

Proof. See Theorem 1.4-5 in (KREYSZIG, 1989). ■

Definition 2.1.7 (Complete metric space)**.** *We say that a metric space (E, d) is complete if any Cauchy sequence in E is convergent.*

Definition 2.1.8 (Banach space)**.** *We say that a normed vector space* $(X, \|\cdot\|)$ *is a Banach space if any Cauchy sequence in* X *is convergent with respect to the metric* $d(x, y) = \|x-y\|$*, for all* $x, y \in X$*.*

Theorem 2.1.4. *Any finite dimensional normed vector space is a Banach space.*

Proof. See Theorem 2.4-2 in (KREYSZIG, 1989). ■

Proposition 2.1.1. *Let* A *be a subset of a metric space* (E, d)*. If* A *is complete, then* A *is closed in* E*. If* E *is complete and* A *is closed in* E*, then* A *is complete.*

Proof. See Proposition 6, Chapter 7 in (LIMA, 2020). ■

Proposition 2.1.2. *A compact metric space is complete.*

Proof. See Proposition 3 in (YOCCOZ, 1994). ■

Theorem 2.1.5 (Completion of a metric space)**.** *Let* (E, d) *be a metric space. There exists a unique (up to isometries) complete metric space* $(\tilde{E}, \tilde{d})$*, containing* E *as a dense subset and such that the restriction of* $\tilde{d}$ *to* E *is* d*. Any uniformly continuous application* $f : E \to Y$*, where* (Y, d_Y) *is a complete metric space, extends uniquely as a continuous application* $\tilde{f} : \tilde{E} \to Y$*.*

Proof. See Theorem 1.6-2 in (KREYSZIG, 1989). ■

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces on $\mathbb{R}$.

Definition 2.1.9. *A map* $A : X \to Y$ *is called a linear operator if for all* $g, h \in X, \alpha, \beta \in \mathbb{R}$*,*

$$A(\alpha g + \beta h) = \alpha A g + \beta A h.$$

The range of A *is* $R(A) = \{v \in Y : v = Ag \text{ for some } g \in X\}$*. The null space of* A *is* $N(A) = \{g \in X : Ag = 0\}$*. The graph of* A *is the set*

$$G(A) = \{(g, v) \in X \times Y : v = Ag\}.$$

Theorem 2.1.6. *Let* $A : X \to Y$ *be a linear operator. The following three properties are equivalent.*

(i) A is continuous at 0;

(ii) A is continuous on X;

(iii) there exists a constant $C \geq 0$ such that for all $g \in X$, $\|Ag\|_Y \leq C\|g\|_X$.

Proof. See Proposition 8, Chapter 2, in (LIMA, 2020) or Theorem 2.7-9 in (KREYSZIG, 1989). ■

We denote by $\mathcal{L}(X,Y)$ the vector space of linear continuous operators from X to Y equipped with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup\left\{\|Ag\|_Y ; \|g\|_X = 1\right\}.$$

Theorem 2.1.7. *Let $(X, \|\cdot\|)$ be a normed vector space, D be a dense subspace of X, and Y a Banach space. Any linear continuous linear map T from D to Y can be uniquely extended to a continuous linear map $\tilde{T}$ from X to Y, with $\|T\| = \|\tilde{T}\|$.*

Proof. See Theorem 2.7-11 in (KREYSZIG, 1989). ■

Theorem 2.1.8 (Banach-Steinhaus theorem[1])**.** *Let X be a Banach space, Y be a normed vector space, and $A_{j_{j\in\mathcal{J}}}$ be a family of linear operators from X to Y satisfying, for all $g \in X$,*

$$\sup_{j\in J} \|A_j g\|_Y < \infty.$$

Then, the bound is uniform on the unit ball of X, i.e.

$$\sup_{j\in J} \|A_j\|_{\mathcal{L}(X,Y)} < \infty.$$

Proof. See Theorem 2.2 in (BREZIS, 2011). ■

A linear operator $A : X \to Y$ is called *closed* if its graph is closed, which means that for any sequence $\{g_k\}_{k=0}^{\infty}$ of X such that $\lim_{\infty} g_k = g$ in X and $\lim_{\infty} Ag_k = v$ in Y, one has $Ag = v$.

[1]This theorem is also known as Uniform Boundedness Principle.

Theorem 2.1.9 (The Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear mapping. Then, $A \in \mathcal{L}(X, Y)$ if and only if the graph of A is a closed subspace of $X \times Y$.*

Proof. See Theorem 2.9 in (BREZIS, 2011). ■

When $Y = X$, we denote $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$ the vector space of the bounded linear operators on X. Equipped with the composition product of applications $A \circ B$, denoted simply by $AB, \mathcal{L}(X)$ is a unitary algebra, with identity element I. The norm on $\mathcal{L}(X)$, defined by

$$\|A\|_{\mathcal{L}(X)} = \sup_{\|g\| \leq 1} \|Ag\|,$$

will also be denoted by $\| \cdot \|$. It is a matter of fact that for all $A, B \in \mathcal{L}(X)$,

$$\|AB\| \leq \|A\| \|B\|.$$

An element A of $\mathcal{L}(X)$ is said to be invertible if it admits an inverse in $\mathcal{L}(X)$, i.e. if there exists $B \in \mathcal{L}(X)$ such that $AB = BA = I$.

Theorem 2.1.10 (Open Mapping theorem)**.** *Let X be a Banach space. Let $A \in \mathcal{L}(X)$ be bijective. Then the inverse of A, denoted by A^{-1}, belongs to $\mathcal{L}(X)$.*

We recall that if A and B are invertible, then AB is also invertible and it holds $(AB)^{-1} = B^{-1}A^{-1}$. We shall use the convention $A^0 = I$.

Proof. See Theorem 2.6 in (BREZIS, 2011). ■

Lemma 2.1.1 (Neumann Series Criterion)**.** *Suppose that X is a Banach space. Let $A \in \mathcal{L}(X)$ be such that $A = I - K$ with $\|K\| < 1$. Then, A is invertible and*

$$A^{-1} = \sum_{k=0}^{\infty} K^k \in \mathcal{L}(X).$$

Proof. See Example 14 in (LIMA, 2020) or see Proposition 7.1.3 in (BOTELHO; PELLEGRINO; TEIXEIRA, 2015). ■

Definition 2.1.10. *When $Y = \mathbb{K}$, $\mathcal{L}(X, Y)$ is denoted by X^* and called the dual or topological dual space of X; it is the space of continuous linear forms on X. Equipped with* *the norm*

$$\|A\|_{\mathcal{L}(X)} = \sup_{\|g\|\leq 1} |Ag|,$$

it is a Banach space.

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces and let $\mathcal{U}$ be an open set of X.

Definition 2.1.11. *For a function $f : \mathcal{U} \to \mathbb{R}$, where $0 \in \mathcal{U}$, we denote $f(h) = o(h)$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $h \in \mathcal{U}$ with $\|h\| \leq \delta$, $|f(h)| \leq \varepsilon \|h\|$.*

Definition 2.1.12. *We say that an application $F : \mathcal{U} \to Y$ is differentiable at $g \in X$ if there exists a continuous linear map, $\mathrm{d}F_g : X \to Y$ such that*

$$\|F(g+h) - F(g) - \mathrm{d}F_g(h)\|_Y = o(h).$$

If it exists, this linear map is unique and called the differential of F at g. We say that F is differentiable on $\mathcal{U}$ if it is differentiable at any point of $\mathcal{U}$. We say that F is of class C^1 on $\mathcal{U}$ if it is differentiable at any point of $\mathcal{U}$ and if the application

$$\mathrm{d}F : \mathcal{U} \to \mathcal{L}(X, Y), \quad g \mapsto \mathrm{d}F_g$$

is continuous. A linear combination of differentiable functions is differentiable, and the differential is linear. A composition of differentiable functions is differentiable.

Definition 2.1.13. *Let H be a linear vector space on $\mathbb{R}$. A (real) scalar product on H is a map $(f, g) \mapsto (f, g)$ from $H \times H$ to $\mathbb{R}$ satisfying the following properties*

(i) Bilinearity: for all $f_1, f_2, g_1, g_2 \in H, \lambda \in \mathbb{R}$,

$$(\lambda f_1 + f_2, g_1) = \lambda (f_1, g_1) + (f_2, g_1)$$
$$(f_1, \lambda g_1 + g_2) = \lambda (f_1, g_1) + (f_1, g_2),$$

(ii) Symmetry: $(f, g) = (g, f)$, for all $f, g \in H$;

(iii) Positivity: for all $f \in H, (f, f) \geq 0$ and

$$(f, f) = 0 \Longleftrightarrow f = 0.$$

Definition 2.1.14. *Let H be a linear vector space on $\mathbb{C}$. A hermitian scalar product on H is a map $(f, g) \mapsto (f, g)$ from $H \times H$ to $\mathbb{C}$ satisfying the following properties*

(i) Linearity and antilinearity: for all $f_1, f_2, g_1, g_2 \in H, \lambda \in \mathbb{C}$,

$$(\lambda f_1 + f_2, g_1) = \bar{\lambda}\,(f_1, g_1) + (f_2, g_1)$$

$$(f_1, \lambda g_1 + g_2) = \lambda\,(f_1, g_1) + (f_1, g_2)$$

(ii) Hermitian symmetry: $(f, g) = \overline{(g, f)}$, for all $f, g \in H$;

(iii) Positivity: for all $f \in H, (f, f) \geq 0$ and

$$(f, f) = 0 \Longleftrightarrow f = 0.$$

A real or complex vector space equipped with a scalar product has a natural normed space structure, by setting

$$\|f\| = (f, f)^{1/2}$$

Moreover, the Cauchy-Schwarz inequality holds

$$|(f, g)| \leq \|f\|\|g\|.$$

Definition 2.1.15. *We say that $(H, (\cdot, \cdot))$ is a Hilbert space if it is complete for the associated norm.*

Theorem 2.1.11 (Riesz Theorem)**.** *Let $(H, (\cdot, \cdot))$ be a Hilbert space. For any element $h \in H$, we associate the continuous linear form L_h on H defined by, for any $f \in H$,*

$$L_h(f) = (h, f).$$

Conversely, for any continuous linear form L on H, there exists a unique $h \in H$ such that $L = L_h$.

Proof. See Theorem 4.11 in (BREZIS, 2011). ■

Definition 2.1.16 (Riesz Basis)**.** *The set $\{f_k\} \subset \mathbb{V}$ is called a Riesz Basis if every element $s \in \mathbb{V}$ of the space can be written as*

$$s = \sum_k c_k f_k,$$

for some choice of scalars $\{c_k\}$ and if positive constants A and B exist such that

$$A\|s\|^2 \leq \sum_k |c_k|^2 \leq B\|s\|^2,$$

Riesz basis are also known as a stable basis or unconditional basis. If the Reisz basis is an orthogonal basis, then $A = B = 1$.

In infinite-dimensional Hilbert spaces, bounded sets generally do not have compact closure. Thus, it is important to weaken the notion of convergence in such spaces. Thus, we shall introduce some weak notions.

Definition 2.1.17. *Let E be a Banach space. The weak topology $\sigma(E, E')$ on E is the coarsest topology on E that makes all mappings $f \in E'$ continuous.*

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E that converges to x in E in the weak topology $\sigma(E, E')$. We shall denote this by

$$x_n \rightharpoonup x \quad \text{in } E.$$

Proposition 2.1.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E, then*

(i) $x_n \rightharpoonup x$ in E if, and only if, $\langle f, x_n\rangle \to \langle f, x\rangle, \forall f \in E'$;

(ii) if $x_n \to x$ in E, then $x_n \rightharpoonup x$ in E;

(iii) if $x_n \rightharpoonup x$ in E, then $\|x_n\|_E$ is bounded and $\|x\|_E \leqslant \lim inf \|x_n\|_E$;

(iv) if $x_n \rightharpoonup x$ in E and $f_n \to f$ in E', then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Proof. See Proposition 3.5 in (BREZIS, 2011). ■

Let E be a Banach space and let $x \in E$ be fixed. We define $J_x : E' \to \mathbb{R}$ by

$$\langle J_x, f\rangle = \langle f, x\rangle.$$

The mappings J_x are linear and continuous, therefore $J_x \in E''$, $\forall x \in E$. Now, we define $J : E \to E''$ such that $J(x) = J_x$.

Definition 2.1.18. *The weak* topology, also denoted by $\sigma(E', E)$, is the coarsest topology on E' that makes all the mappings J_x continuous.*

Proposition 2.1.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in E', then, then following holds*

(i) $f_n \overset{*}{\rightharpoonup} f$ in E' if, and only if, $\langle f_n, x\rangle \to \langle f, x\rangle$, $\forall x \in E$;

(ii) if $f_n \to f$ in E', then $f_n \rightharpoonup f$ in E';

(iii) if $f_n \rightharpoonup f$ in E', then $f_n \overset{*}{\rightharpoonup} f$ in E'.

Proof. See Proposition 3.13 in (BREZIS, 2011). ■

Proposition 2.1.5. *Let E be a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Then, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x \in E$ such that*

$$x_{n_k} \rightharpoonup x \quad \textit{weak in} \quad E.$$

Proof. See Theorem 3.18 in (BREZIS, 2011). ■

Proposition 2.1.6. *Let E be a separable Banach space and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in E', then, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and $f \in E'$ such that*

$$f_{n_k} \overset{*}{\rightharpoonup} f \quad \textit{in} \quad E'.$$

Proof. See Corollary 3.30 in (BREZIS, 2011). ■

Definition 2.1.19. *Let $\{f_j\}_{j=0}^{\infty}$ be a sequence of elements of a separable Hilbert space H and let f be an element of H. The sequence $\{f_j\}_{j=0}^{\infty}$ is said to weakly converge to f, which is denoted by $f_j \rightharpoonup f$ if*

$$\forall h \in H, \quad \lim_{j\to\infty} (h, f_j) = (h, f).$$

It is easy to see that if the weak limit exists, then it is unique.

Let $(H, (\cdot,\cdot))$ be a real Hilbert space. We denote by $\langle\cdot,\cdot\rangle$ the pairing of H with its dual space H^*.

2.1.1.1 An Overview of Differential Calculus in Banach Spaces

Here, following (ZEIDLER, 1986), our goal is to provide a generalization for Banach spaces of the Implicit Function Theorem from real variables. That is, we shall generalize the following statement: let F be a real-valued function of two real variables with $F(x_0, y_0) = 0$ for fixed (x_0, y_0). The equation

$$F(x, y) = 0$$

will have unique solution for y in a neighborhood of (x_0, y_0) if certain regularity conditions are fulfilled, and $F_y(x_0, y_0) \neq 0$.

First, some notation. For a map $r : U(0) \subseteq X \to Y$, we will write:

$$r(x) = o(\|x\|), x \to 0 \quad \text{iff } r(x)/\|x\| \to 0 \text{ as } x \to 0.$$

Definition 2.1.20. *Let $f : U(x) \subseteq X \to Y$ be a given map, with X and Y Banach spaces. Here $U(x)$ denotes a neighborhood of x.*

(i) The map f is Fréchet differentiable at x if, and only if, there exists a map $T \in \mathcal{L}(X, Y)$ such that

$$f(x + h) - f(x) = Th + o(\|h\|), \quad h \to 0, \tag{2.1}$$

for all h in some neighborhood of zero. If it exists, this T is called the Fréchet derivative of f at x. We define $f'(x) = T$. The Fréchet differential at x is defined by $df(x; h) = f'(x)h$.

(ii) If the Fréchet derivatives $f'(x)$ exist for all $x \in A$, then the mapping

$$f' : A \subseteq X \to \mathcal{L}(X, Y) \quad by \quad x \mapsto f'(x)$$

is called the Fréchet derivative of f on A.

(iii) Higher derivatives are defined successively. Thus, $f''(x)$ is the derivative of f' at x.

It is worth pointing out that we will consider derivatives at x *only* if f is defined in some neighborhood of x. Also, by (2.1) we see that are defined through linearization.

Proposition 2.1.7. *If $f'(x)$ exists as an Fréchet derivative at x, then f also is continuous* *at* x.

Proof. See Proposition 4.8 in (ZEIDLER, 1986). ■

Definition 2.1.21. *Let there be given a map* $f : D(f) \subseteq X \times Y \to Z$ *by* $(x, y) \mapsto f(x, y)$, *where* X, Y, *and* Z *are Banach spaces. Let* y *be fixed and set* $g(x) = f(x, y)$. *If* g *has an Fréchet derivative at* x, *then we define the partial Fréchet derivative of* f *at* (x, y) *with respect to the first variable* x *to be* $f_x(x, y) = g'(x)$. *The derivative* $f_y(x, y)$ *is defined similarly. Instead of* $f_x(x, y), f_y(x, y)$ *one also writes* $D_1 f(x, y), D_2 f(x, y)$, *respectively.*

We shall investigate the validity of the formula

$$f'(x, y)(h, k) = f_x(x, y)h + f_y(x, y)k \tag{2.2}$$

Proposition 2.1.8 (Partial Derivatives)**.** *We have the following properties*

(i) If f *is Fréchet differentiable at* (x, y), *then the partial Fréchet derivatives* f_x *and* f_y *exist at* (x, y) *and* (2.2) *holds for all* $h \in X$ *and* $k \in Y$.

(ii) Conversely, if f *has partial Fréchet derivatives* f_x *and* f_y *in a neighborhood of* (x, y), *and if these are continuous at* (x, y), *then* $f'(x, y)$ *exists as an Fréchet derivative and* (2.2) *holds.*

(iii) The map f *is continuously Fréchet differentiable in a neighborhood of* (x, y) *if, and only if, all partial Fréchet derivatives are continuous in a neighborhood of* (x, y).

Proof. See Proposition 4.14 in (ZEIDLER, 1986). ■

Theorem 2.1.12 (Implicit Function Theorem)**.** *Suppose that:*

(i) the mapping $F : U(x_0, y_0) \subseteq X \times Y \to Z$ *is defined on an open neighborhood* $U(x_0, y_0)$, *and* $F(x_0, y_0) = 0$, *where* X, Y, *and* Z *are Banach spaces over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$;

(ii) F_y *exists as a partial Fréchet derivative on* $U(x_0, y_0)$ *and* $F_y(x_0, y_0) : Y \to Z$ *is bijective.*

(iii) F *and* F_y *are continuous at* (x_0, y_0).

Then the following are true

(a) Existence and uniqueness. *There exist positive numbers* r_0 *and* r *such that, for every* $x \in X$ *satisfying* $\|x - x_0\| \leqslant r_0$*, there is exactly one* $y(x) \in Y$ *for which* $\|y(x) - y_0\| \leqslant r$ *and* $F(x, y(x)) = 0$.

(b) Construction of the solution. *The sequence* $(y_n(x))$ *of successive approximations, defined by* $y_0(x) \equiv y_0$*, and*

$$y_{n+1}(x) = y_n(x) - F_y(x_0, y_0)^{-1} F(x, y_n(x))$$

converges to the solution $y(x)$*, as* $n \to \infty$*, for all points* $x \in X$ *satisfying* $\|x - x_0\| \leq r_0$.

(c) Continuity. *If* F *is continuous in a neighborhood of* (x_0, y_0)*, then* $y(\cdot)$ *is continuous in a neighborhood of* x_0.

(d) Continuous differentiability. *If* F *is a* C^m*-map,* $1 \leqslant m \leqslant \infty$*, on a neighborhood of* (x_0, y_0)*, then* $y(\cdot)$ *is also a* C^m*-map on a neighborhood of* x_0.

Proof. See Theorem 4.B in (ZEIDLER, 1986). ■

Corollary 2.1.1. *If* F *is analytic at* (x_0, y_0)*, then the solution* $y(\cdot)$ *is analytic at* x_0.

Proof. See Corollary 4.23 in (ZEIDLER, 1986). ■

2.1.2 Distributions and Sobolev Spaces

Throughout this subsection we are inspired in (ADAMS, 1975; MEDEIROS; MIRANDA, 1989; BREZIS, 2011; SCHWARTZ, 1966; IORIO JÚNIOR; IORIO, 2001) and in the references therein.

We refer to a *domain*, denoted by Ω, for a nonempty open set in n-dimensional real space $\mathbb{R}^n$. We will focus on the differentiability and integrability of functions defined on the set Ω. Given $n \in \mathbb{N}$, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha^n}$, which has degree $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Moreover, if $D_j = \frac{\partial}{\partial x_j}$, then

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha^n}$$

denotes a differential operator of order $|\alpha|$. Notice that, $D^{(0,\ldots,0)}u = u$.

If α and β are two multi-indices, we say that $\beta \le \alpha$ provided $\beta_j \le \alpha_j$ for $1 \le j \le n$. Then $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. Moreover, we also denote $\alpha! = \alpha_1! \ldots \alpha_n!$ and if $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

With this, for u, v regular enough functions we state the Leibniz rule given by

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x). \tag{2.3}$$

Let $\Omega \subset \mathbb{R}^n$, we denote by $\overline{\Omega}$ the closure of Ω in $\mathbb{R}^n$. Let u a function defined on Ω, we describe the *support* of u to be the set

$$\operatorname{supp}(u) = \overline{\{x \in \Omega \colon u(x) \ne 0\}}.$$

We say that u has *compact support* in Ω if $\operatorname{supp}(u)$ is compact.

For any $m \in \mathbb{N}$, let $C^m(\Omega)$ denote the vector spaces

$$C^m(\Omega) = \{\phi \colon D^\alpha \phi, |\alpha| \le m \text{ is continuous on } \Omega\}.$$

We denote $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$. The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consists of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, that have compact support in Ω.

Definition 2.1.22. *We say that $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$, denoted by $\varphi_n \to \varphi$, if*

(i) There exists a compact K of Ω such that $\operatorname{supp}(\varphi) \subset K$ and $\operatorname{supp}(\varphi_n) \subset K$, $\forall n \in \mathbb{N}$;

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in K, for all multi-index α.

By $D(\Omega)$ we represent the space $C_0^\infty(\Omega)$, equipped with the convergence defined above and will be called *space of test functions on* Ω.

We define a *distribution over* Ω, as defined by Schwartz, to any linear form T over $\mathcal{D}(\Omega)$

that is continuous in the sense of convergence defined above, that is, for every sequence $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ that converges to $\varphi \in \mathcal{D}(\Omega)$, then $(\langle T, \varphi_n \rangle)_n \subset \mathbb{K}$ converges to $\langle T, \varphi \rangle \in \mathbb{K}$[2].

Remark 2.1.1. *The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of (Schwartz) distributions on Ω. $\mathcal{D}'(\Omega)$ is given the weak-star topology as the dual of $\mathcal{D}(\Omega)$, and is a locally convex topological vector space (TVS) with that topology.*

Let α a multi-index and $\varphi \in \mathcal{D}(\Omega)$, if $u \in C^{|\alpha|}(\Omega)$, then integrating by parts $|\alpha|$ times leads to

$$\int_\Omega (D^\alpha u(x))\, \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x)\, dx.$$

This motivates the definition of the derivative $D^\alpha T$ of a distribution $T \in \mathcal{D}'(\Omega)$

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \ \forall \varphi \in \mathcal{D}(\Omega).$$

It is notable that:

- Each distribution T over Ω has derivatives of all orders.
- $D^\alpha T$ is a distribution over Ω, where $T \in \mathcal{D}'(\Omega)$. In fact, it is easily seen that $D^\alpha T$ is linear. Now, we show that it is continuous, consider $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ converging to $\varphi \in \mathcal{D}(\Omega)$. Thus,

  $$|\langle D^\alpha T, \varphi_n \rangle - \langle D^\alpha T, \varphi \rangle| \leq |\langle T, D^\alpha \varphi_n - D^\alpha \varphi \rangle| \Rightarrow 0$$

  when $n \to \infty$.
- The map $D^\alpha : \mathcal{D}'(\Omega) \Rightarrow \mathcal{D}'(\Omega)$, such that $T \mapsto D^\alpha T$, is linear and continuous in the sense of convergence defined in $\mathcal{D}'(\Omega)$.

For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the space of (classes of) functions $u \colon \Omega \to \mathbb{R}$ measurable in Ω such that $|u|^p$ is Lebesgue integrable in Ω. This is a Banach space with the norm

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p\, dx.$$

[2]Observe that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\langle T, \varphi \rangle$ is the evaluation of T in φ, i.e. $T(\varphi)$

When $p = \infty$, $L^\infty(\Omega)$ consists of all essentially bounded functions in Ω equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)| = \inf \{C \colon |v(x)| \le C \text{ a.e. in } \Omega\}.$$

When $p = 2$ we have a Hilbert space $L^2(\Omega)$ with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx,$$

and induced norm

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2\, dx.$$

Given an integer $m > 0$, by $W^{m,p}(\Omega)$, $1 \le p \le \infty$, represents the Sobolev space of order m, over Ω of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$, for every multi-index α, with $|\alpha| \le m$. $W^{m,p}(\Omega)$ is a vector space, whatever $1 \le p < \infty$. Considering the following norm

$$\|u\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^p\, dx$$

when $1 \le p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\le m} \operatorname{supp}_{x\in\Omega} |D^\alpha u(x)|$$

when $p = \infty$, then Sobolev spaces $W^{m,p}(\Omega)$ is a Banach space.

When $p = 2$, the space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which equipped with the inner product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx$$

is a Hilbert space.

Let us denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ relative to the norm of the space $W^{m,p}(\Omega)$, i.e.

$$\overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)} = W_0^{m,p}(\Omega).$$

Whenever Ω is bounded at least in one direction x_i of $\mathbb{R}^n$, the norm of $W_0^{m,p}(\Omega)$ is given by

$$\|u\|^p_{W_0^{m,p}(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p\, dx.$$

We denote by $W^{-m,q}(\Omega)$ the topological dual of $W_0^{m,p}(\Omega)$, where $1 \leq p < \infty$ and q is the Hölder conjugated index of p[3]. We write $H^{-m}(\Omega)$ to denote the topological dual of $H_0^m(\Omega)$.

Let X and Y be two normed vector spaces such that $X \subseteq Y$. If the inclusion map $i\colon x \in X \mapsto x \in Y$ is continuous for every $x \in X$, then X is said to be *continuously embedded* in Y and will be denoted $X \hookrightarrow Y$.

Next, we aim to extend the definitions for the H^s spaces, with $s \geqslant 0$. We begin by defining the Schwartz space, which is a subspace of $L^1(\mathbb{R}^n)$ that is invariant under the Fourier transform. It consists of functions $\varphi \in C^\infty(\mathbb{R}^n)$, which, along with all their derivatives, decrease rapidly at infinity. That is, they decrease to zero at infinity faster than any power of $\|x\|^k$. More precisely:

Definition 2.1.23 (Fourier transform). *Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}$, is a function defined on $\mathbb{R}^n$ by the formula*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)dx, \quad i = \sqrt{-1},$$

where $\langle x,\xi\rangle = \sum_{i=1}^n x_i\xi_i$ is the usual inner product in $\mathbb{R}^n$.

Since $f \in L^1(\mathbb{R}^n)$, we note that $\hat{f}(\xi)$ is well defined for all $\xi \in \mathbb{R}^n$. Indeed

$$|\hat{f}(\xi)| \leq \left|\int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)dx\right| \leq \int_{\mathbb{R}^n} |f(x)|dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

Definition 2.1.24 (Schwartz space). *Schwartz space, or the space of rapidly decreasing functions, denoted by $\mathcal{S}$, is the vector subspace formed by functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\lim_{\|x\|\to\infty} \|x\|^k D^\alpha\varphi(x) = 0,$$

for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$.

We note that $C_0^\infty(\mathbb{R}^n)$ is a dense subset of $\mathcal{S}$, and for any $1 \leqslant p \leqslant \infty$, we have $\mathcal{S} \hookrightarrow L^p(\mathbb{R}^n)$.

[3] q is said to be the Hölder conjugated index of $1 \leq p \leq \infty$ if $\frac{1}{p} + \frac{1}{q} = 1$

Proposition 2.1.9 (Parseval relations)**.** *Let $f, g \in \mathcal{S}$. Then, the following holds*

$$\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\hat{g}(x)dx$$

and

$$\int_{\mathbb{R}^n} f\bar{g} = \int_{\mathbb{R}^n} \hat{f}\overline{\hat{g}}.$$

Proof. See Proposition 1.96 and Propositon 1.99 in (MEDEIROS; MIRANDA, 1989). ■

Corollary 2.1.2. *Let $f \in \mathcal{S}$. Then*

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

Proof. See Corollary 1.100 in (MEDEIROS; MIRANDA, 1989). ■

Theorem 2.1.13 (Plancherel theorem)**.** *There exists a unique isometric bijection*

$$\mathcal{P} : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right),$$

such that

$$\mathcal{P}(f) = \hat{f}, \quad \forall\, f \in \mathcal{S}.$$

Proof. See Theorem 1.101 in (MEDEIROS; MIRANDA, 1989). ■

Proposition 2.1.10. *Let $g \in \mathcal{S}$. Then*

$$g(x) = \int_{\mathbb{R}^n} e^{2\pi i\langle x,\xi\rangle}\hat{g}(\xi)d\xi.$$

Proof. See Propositon 1.97 in (MEDEIROS; MIRANDA, 1989). ■

Definition 2.1.25 (Tempered Distribution)**.** *A linear functional T defined and continuous on $\mathcal{S}$ is called a tempered distribution (or slowly increasing distribution). The set of all tempered distributions, that is, the vector space of linear and continuous functionals on $\mathcal{S}$, is denoted by $\mathcal{S}'$.*

We note that $\mathcal{S}' \subset \mathcal{D}'(\mathbb{R}^n)$.

Definition 2.1.26. *For any $s \in \mathbb{R}$, $s \geqslant 0$, we define the $H^s(\mathbb{R}^n)$ space by*

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) ; \left(1 + \|x\|^2\right)^{s/2} \hat{u} \in L^2(\mathbb{R}^n) \right\},$$

endowed with the inner product

$$(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(1 + \|x\|^2\right)^s \hat{u}(x) \overline{\hat{v}(x)} dx,$$

which turns $H^s(\mathbb{R}^n)$ into a Hilbert space.

Follows from the definition above that $H^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. Indeed, for $u \in H^s(\mathbb{R}^n)$, we have

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \|\hat{u}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{u}(x)|^2 dx \leq \int_{\mathbb{R}^n} \left(1 + \|x\|^2\right)^s |\hat{u}(x)|^2 dx = \|u\|^2_{H^s(\mathbb{R}^n)}.$$

Theorem 2.1.14 (Sobolev embeddings)**.** *We have*

(i) $H^s \hookrightarrow C^k$ *for* $s > k + \frac{n}{2}, k \in \mathbb{N}_0$

(ii) $H^s \hookrightarrow L^p$ *for* $s \geq \frac{n}{2} - \frac{n}{p}, 2 \leq p < \infty$

(iii) $L^p \hookrightarrow H^s$ *for* $s \leq \frac{n}{2} - \frac{n}{p}, 1 < p \leq 2$

(iv) $L^1 \hookrightarrow H^s$ *for* $s < -\frac{n}{2}$.

Proof. See Proposition 1.1.11 in (HERR, 2006). ■

Corollary 2.1.3. *Let $s \geq 0$ and assume that*

$$s \overset{[\leq]}{<} s_1, s_2, \quad s \overset{[<]}{\leq} s_1 + s_2 - \frac{n}{2}.$$

Then, there exists $c > 0$ such that

$$\|u_1 u_2\|_{H^s} \leq c \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}, \quad u_1 \in H^{s_1}, u_2 \in H^{s_2}.$$

In particular, H^s is a Banach algebra for $s > \frac{n}{2}$, (see Theorem 2.1.17).

Proof. See Corollary 1.1.12 in (HERR, 2006). ■

We will denote by $L^p(0,T;X),\ 1 \leq p < \infty$, the space of Banach of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X^p$ is Lebesgue integrable in $(0,T)$, with the norm

$$\|u(t)\|^p_{L^p(0,T;X)} = \int_0^T \|u(t)\|_X^p \, dt.$$

Furthermore, if $p = \infty$, $L^\infty(0,T;X)$ represents the Banach space of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X$ has supreme essential finite in $(0,T)$, with the norm

$$\|u(t)\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in(0,T)} \|u(t)\|_X.$$

Remark 2.1.2. *When $p = 2$ and X is a Hilbert space, the space $L^2(0,T;X)$ is a Hilbert space, whose inner product is given by*

$$\langle u, v\rangle_{L^2(0,T;X)} = \int_0^T \langle u(t), v(t)\rangle_X \, dt.$$

Consider the space $L^p(0,T;X),\ 1 < p < \infty$, with X being Hilbert separable space, then we can associate the topological dual space

$$[L^p(0,T;X)]' \simeq L^q(0,T;X'),$$

where p and q are Hölder conjugated index. When $p = 1$, we will associate

$$[L^1(0,T;X)]' \simeq L^\infty(0,T;X').$$

Given a Banach space X. The vector space of linear and continuous maps of $\mathcal{D}(0,T)$ on X is called the Space of Vector Distributions on $(0,T)$ with values in X and denoted by $\mathcal{D}'(0,T;X)$.

Given $S \in \mathcal{D}'(0,T;X)$, inspired on the previous derivative of distribution, we define the derivative of order m as being the vector distribution over $(0,T)$ with values in X given for

$$\left\langle \frac{\mathrm{d}^m S}{\mathrm{d}t^n}, \varphi \right\rangle = (-1)^m \left\langle S, \frac{\mathrm{d}^n \varphi}{\mathrm{d}t^n} \right\rangle, \text{ for all } \varphi \in \mathcal{D}(0,T).$$

Let us consider the Banach space

$$W^{m,p}(0,T;X) = \left\{u \in L^p(0,T;X) \colon u^{(j)} \in L^p(0,T,X), j = 1,\ldots,m\right\},$$

where $u^{(j)}$ represents the $j-$th derivative of u in the sense of distributions and the space is endowed with the norm

$$\|u\|^p_{W^{m,p}(0,T;X)} = \sum_{j=0}^{m} \|u^{(j)}\|^p_{L^p(0,T;X)}.$$

When $p = 2$ and X is a Hilbert space, the space $W^{m,2}(0,T;X)$ will be denoted by $H^m(0,T;X)$, which, equipped with the inner product

$$\langle u, v\rangle_{H^m(0,T;X)} = \sum_{j=0}^{m} \langle u^{(j)}, v^{(j)}\rangle_{L^2(0,T;X)},$$

is a Hilbert space. It is denoted by $H^m_0(0,T;X)$ the closure, in $H^m(0,T;X)$, of $\mathcal{D}(0,T;X)$ and by $H^{-m}(0,T;X)$ the topological dual of $H^m_0(0,T;X)$.

2.1.2.1 Periodic Distributions

Here, we shall introduce a class of generalized functions specially suited for the study of Fourier series and differential equations provided with periodic boundary conditions, as well as study their basic properties. We have been guided by (IORIO JÚNIOR; IORIO, 2001).

Definition 2.1.27. *A function $f : \mathbb{R} \to \mathbb{C}$ is said to be periodic with period $T \neq 0$ if*

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}.$$

Remark 2.1.3. *Note that if T is a period for f then, for any $n \in \mathbb{Z}\backslash\{0\}, nT$ is also a period for f. In particular, since $-T$ is a period, we can assume, without loss of generality, that $T > 0$. If f is constant, then f is periodic with any period. If f is continuous*[4] *and nonconstant, then there exists a smallest period $T > 0$; in this case, T is called the fundamental period of f.*

[4]This condition can be weakened but some condition is necessary. For instance, the function that is zero on the rationals and 1 elsewhere is periodic but does not have a fundamental period.

We shall denote by $\mathscr{P} = C^{\infty}_{\text{per}}$ the collection of all the functions $\varphi : \mathbb{R} \to \mathbb{C}$ which are C^{∞} and periodic with period 2π. Space $\mathscr{P}$ is a vector subspace of C^{n}_{per} for all $n \in \mathbb{N}$. Also, we shall denote by $PC^{n}_{\text{per}}\left([-\ell, \ell]\right)$ the set of all functions $f \in PC_{\text{per}}\left([-\ell, \ell]\right)$ such that there exists a partition $-\ell = x_0 < x_1 < \ldots < x_m = \ell$ of the interval $[-\ell, \ell]$ with $f \in C^{n}\left(x_j, x_{j+1}\right)$ for all $j = 0, 1, \ldots, m-1$ and $f^{(k)} \in PC_{\text{per}}\left([-\ell, \ell]\right)$ for all $k = 1, \ldots, m$. We will also use the notation $PC^{\infty}_{\text{per}}\left([-\ell, \ell]\right)$ for the set of functions that belong to $PC^{n}_{\text{per}}\left([-\ell, \ell]\right)$ for all $n \in \mathbb{Z}^{+}$.

In what follows, we introduce the Fourier transform in the context of periodic functions.

Definition 2.1.28. *Let $f \in PC_{per}$. The Fourier transform of f is the complex sequence $\mathscr{F} f = \widehat{f} = (\widehat{f}(k))_{k\in\mathbb{Z}}$ defined by*

$$\left(\mathscr{F} f\right)(k) = \widehat{f}(k) = c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$

The numbers $\widehat{f}(k) = c_k$ are the Fourier coefficients of f and the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} c_k \exp(ikx)$$

is the Fourier series generated by f.

Definition 2.1.29. *The space of rapidly decreasing sequences, denoted by $\mathscr{S}(\mathbb{Z})$, is the set of all complex-valued sequences $\alpha = \left(\alpha_k\right)_{k\in\mathbb{Z}}$ such that*

$$\sum_{k=-\infty}^{\infty} |k|^{j} \left|\alpha_k\right| < \infty \quad \forall j \in \mathbb{N}.$$

Proposition 2.1.11. *$\alpha = \left(\alpha_k\right)_{k\in\mathbb{Z}} \in \mathscr{P}(\mathbb{Z})$ if and only if*

$$\|\alpha\|_{\infty,j} = \sup_{k\in\mathbb{Z}} \left(\left|\alpha_k\right| |k|^{j}\right) < \infty \quad \forall j \in \mathbb{N}.$$

Proof. See Proposition 3.4 in (IORIO JÚNIOR; IORIO, 2001). ■

Definition 2.1.30. *Let $\alpha = \left(\alpha_k\right)_{k\in\mathbb{Z}} \in \mathscr{S}(\mathbb{Z})$. The inverse Fourier transform of α is the function*

$$\overset{\vee}{\alpha}(x) = \left(\mathscr{F}^{-1}\alpha\right)(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikx}, x \in \mathbb{R}.$$

Theorem 2.1.15. *The Fourier transform* $\wedge : \mathscr{P} \to \mathscr{S}(\mathbb{Z})$ *is an isomorphism and a homeomorphism, that is, it is linear, one to one, onto* $\mathscr{S}(\mathbb{Z})$*, and continuous with a continuous inverse.*

Proof. See Theorem 3.6 in (IORIO JÚNIOR; IORIO, 2001). ■

Proposition 2.1.12. *Let* $f, g \in C_{per}$ *. Then* $\widehat{(T_t f)}(k) = e^{-ikt}\widehat{f}(k)$ *and*

$$(T_t f) * g = T_t(f * g) = f * (T_t g)\,.$$

Proof. See Proposition 3.7 in (IORIO JÚNIOR; IORIO, 2001). ■

Proposition 2.1.13. *Let* $\varphi \in \mathscr{P}$ *and* $g \in C_{per}$ *, then*

(i) $t^{-1}\left(T_{-t}\varphi - \varphi\right) \xrightarrow{\mathscr{B}} \varphi'$ *as* $t \to 0$*,*

(ii) $\varphi * g \in \mathscr{P}$ *and* $(\varphi * g)^{(j)} = \varphi^{(j)} * g, j = 0, 1, 2\ldots$ *Moreover, if* $(\mathrm{g}_n)_{n=1}^{\infty} \subset C_{per}$ *is such that* $\|\mathrm{g}_n - \mathrm{g}\|_{\infty} \to 0$ *as* $n \to \infty$ *then* $\lim_{n\to\infty} \varphi * g_n = \varphi * g$ *in* $\mathscr{P}$*, i.e.,*

$$\left\|(\varphi * g_n)^{(j)} - (\varphi * g)^{(j)}\right\|_{\infty} \to 0 \quad \forall j \in \mathbb{N}.$$

Proof. See Proposition 3.8 in (IORIO JÚNIOR; IORIO, 2001). ■

Definition 2.1.31 (Periodic Distribution)**.** *A linear functional on* $\mathscr{P}, T : \mathscr{P} \to \mathbb{C}$*, is called a periodic distribution if there exists a sequence* $(\Psi_n)_{n\geqslant 1} \subset \mathscr{P}$ *such that*

$$T(\varphi) = \langle T, \varphi\rangle = \lim_{n\to\infty} \int_{-\pi}^{\pi} \Psi_n(x)\varphi(x)dx \forall \varphi \in \mathscr{P}.$$

The set of all periodic distributions will be denoted by $\mathscr{P}'$*, which is a complex vector space.*

Proposition 2.1.14. *Let* $f \in C_{per}$ *. Then the formula*

$$\langle T_f, \varphi\rangle = \int_{-\pi}^{\pi} f(x)\varphi(x)dx, \quad \varphi \in \mathscr{P},$$

defines a periodic distribution T_f*. The map* $f \in C_{per} \mapsto T_f \in \mathscr{P}'$ *is linear, one to one and continuous in the sense that if* $(f_n)_{n=1}^{\infty} \subset C_{per}$ *converges uniformly to* f *then* $\langle T_{f_n}, \varphi\rangle \to \langle T_f, \varphi\rangle$ *for all* $\varphi \in \mathscr{P}$*.*

Proof. See Proposition 3.17 in (IORIO JÚNIOR; IORIO, 2001). ■

Before proceeding we shall introduce a notion of convergence in $\mathscr{P}'$.

Definition 2.1.32. *We will say that a sequence $(T_n) \subset \mathscr{P}'$ converges to $T \in \mathscr{P}'$ if*

$$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \quad as \quad n \to \infty \ \ \forall \varphi \in \mathscr{P}.$$

In this case we will write $T_n \xrightarrow{\mathscr{P}'} T$.

Now we shall extend certain fundamental operations to $\mathscr{P}'$, thereby generalizing usual calculus to our context of periodic distributions.

Proposition 2.1.15. *Let $f \in PC^1_{per}$ and let $-\pi = x_0 < x_1 < \cdots < x_n = \pi$ be a partition of the interval $[-\pi, \pi]$ such that $f \in C^1(x_j, x_{j+1})$ for all $j = 0, 1, \ldots, n-1$. If we denote by $\frac{df}{dx}$ the classical derivative of f, then its distributional derivative f' is given by*

$$f' = \frac{df}{dx} + \sum_{j=1}^{n} \left[f\left(x_j^+\right) - f\left(x_j^-\right) \right] \delta_{x_j}.$$

Proof. See Proposition 3.32 in (IORIO JÚNIOR; IORIO, 2001). ■

Theorem 2.1.16. *Every periodic distribution is a continuous linear functional on $\mathscr{P}$.*

Proof. See Theorem 3.143 in (IORIO JÚNIOR; IORIO, 2001). ■

We shall further give a precise meaning to the notion that Sobolev spaces provide a classification of the elements of $\mathscr{P}'$ in terms of their smoothness.

Definition 2.1.33. *Let $s \in \mathbb{R}$. The Sobolev space $H^s_{per} = H^s_{per}([-\pi, \pi])$ is the set of all $f \in \mathscr{P}'$ such that*

$$\|f\|_s^2 = \sum_{k=-\infty}^{\infty} \left(1 + |k|^2\right)^s |\widehat{f}(k)|^2 < \infty. \tag{2.4}$$

That is, a periodic distribution f belongs to H^s_{per} if and only if

$$\left(\left(1 + |k|^2\right)^{s/2} \widehat{f}(k) \right)_{k \in \mathbb{Z}} \in \ell^2 = \ell^2(\mathbb{Z}).$$

We shall denote by $\ell_s^2 = \ell_s^2(\mathbb{Z})$ the space of all sequences $\alpha = (x_k)_{k\in\mathbb{Z}}$ with

$$\|\alpha\|_{\ell_s^2} = \left[\sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^s |\alpha_k|^2\right]^{1/2}.$$

Thus $f \in H^s_{\text{per}}$ if and only if $(\widehat{f}(k))_{k\in\mathbb{Z}} \in \ell_s^2$; in this case, $\|f\|_s = \|\hat{f}\|_{\ell_s^2}$. For all $s \in \mathbb{R}$, H^s_{per} is a Hilbert space with respect to the inner product

$$(f,g)_s = \sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^s \widehat{f}(k)\overline{\widehat{g}(k)}. \tag{2.5}$$

In the case $s = 0$, we obtain a Hilbert space that is isometrically isomorphic to $L^2([-\pi,\pi])$. In what follows, H^0_{per} will often be denoted by L^2_{per}.

Proposition 2.1.16. *Let $s, r \in \mathbb{R}, s \geqslant r$. Then $H^s_{per} \hookrightarrow H^r_{per}$, that is, H^s_{per} is continuously and densely embedded in H^r_{per} and*

$$\|f\|_r \leqslant \|f\|_s \forall f \in H^s_{per}. \tag{2.6}$$

In particular, if $s \geqslant 0, H^s_{\text{per}} \subset L^2([-\pi,\pi])$. Moreover, $\left(H^s_{\text{per}}\right)'$, the topological dual of H^s_{per}, is isometrically isomorphic to H^{-s}_{per} for all $s \in \mathbb{R}$. The duality is implemented by the pairing

$$\langle f,g\rangle_s = \sum_{k=-\infty}^{\infty} \widehat{f}(k)\widehat{g}(k), f \in H^{-s}_{\text{per}}, g \in H^s_{\text{per}}. \tag{2.7}$$

Proof. This proof follows (IORIO JÚNIOR; IORIO, 2001). We have

$$0 \leqslant \frac{\left(1+|k|^2\right)^r}{\left(1+|k|^2\right)^s} \leqslant 1,$$

whenever $s \geqslant r$. This implies (2.6). Indeed,

$$\|f\|_r^2 = \sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^r |\widehat{f}(k)|^2 = \sum_{k=-\infty}^{\infty} \frac{\left(1+|k|^2\right)^r}{\left(1+|k|^2\right)^s} \left(1+|k|^2\right)^s |\widehat{f}(k)|^2 \leqslant \|f\|_s^2.$$

It follows that H^s_{per} is continuously embedded in H^r_{per}. Next, since $\mathscr{P} \subset H^s_{\text{per}}$ for all $s \in \mathbb{R}$, to show that the embedding is dense, it is enough to show that $\mathscr{P}$ is dense in H^r_{per}. Given $g \in H^r_{\text{per}}$ let g_n be defined by $\widehat{\mathrm{g}}_n(k) = \widehat{\mathrm{g}}(k)$ if $|k| \leqslant n$, $\widehat{\mathrm{g}}_n(k) = 0$ otherwise. Then

$g_n \in \mathscr{P}$ and

$$\|g-g_n\|_r^2 = \sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^r |\widehat{g}(k)-\widehat{g}_n(k)|^2 = \sum_{|k|>n} \left(1+|k|^2\right)^r |\widehat{g}(k)|^2 \to 0,$$

as $n \to \infty$ since $g \in H^r_{\text{per}}$. If $f \in H^{-s}_{\text{per}}$, it is clear that (2.7) defines a continuous linear functional on H^s_{per} . Conversely, if $\psi : H^s_{\text{per}} \to \mathbb{C}$ is a continuous linear functional on H^s_{per} , by the Riesz lemma there exists a unique $\varphi \in H^s_{\text{per}}$ such that

$$\begin{aligned}\langle \psi, g\rangle = (g,\varphi)_s &= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \widehat{g}(k)\overline{\widehat{\varphi}(k)} \\ &= \sum_{k\in\mathbb{Z}} \widehat{g}(k)\overline{(1+k^2)^s\,\widehat{\varphi}(k)}, \quad \forall g \in H^s_{\text{per}}.\end{aligned}$$

Since $\varphi \in H^s_{\text{per}}$, $\left((1+k^2)^{s/2}\,\widehat{\varphi}(k)\right)_{k\in\mathbb{Z}} \in \ell^2$, so the periodic distribution $f \in \mathscr{P}'$ satisfying $\widehat{f}(k) = (1+k^2)^{s/2}\,\overline{\left((1+k^2)^{s/2}\,\widehat{\varphi}(k)\right)}$ belongs to H^{-s}_{per} and

$$\langle \psi, g\rangle = \sum_{k\in\mathbb{Z}} \widehat{g}(k)\widehat{f}(k) = \langle f, g\rangle_s, \quad \forall g \in H^s_{\text{per}}$$

■

Proposition 2.1.17. *Let $m \in \mathbb{N}$. Then $f \in H^m_{per}$ if and only if $\partial^j f = f^{(j)} \in L^2_{per}$, $j \in \{0,1,2,\ldots,m\}$ where the derivatives are taken in the sense of $\mathscr{P}'$. Moreover, $\|f\|_m$ and*

$$|\!|\!|f|\!|\!|_m^2 = \left[\sum_{j=0}^{m} \left\|\partial^j f_0^2\right\|\right]^{1/2}, \tag{2.8}$$

are equivalent, that is, there are positive constants C_m and C'_m such that

$$C_m\|f\|_m^2 \leqslant |\!|\!|f|\!|\!|_m^2 \leqslant C'_m\|f\|_m^2, \quad \forall f \in H^m_{\text{per}}.$$

Proof. See Proposition 3.194 in (IORIO JÚNIOR; IORIO, 2001). ■

Lemma 2.1.2 (Sobolev's lemma)**.** *If $s > \frac{1}{2}$, then $H^s_{per} \hookrightarrow C_{per}$ and*

$$\|f\|_\infty \leqslant \|\hat{f}\|_{\ell^1} \leqslant C\|f\|_s \quad \forall f \in H^s_{\text{per}}. \tag{2.9}$$

Proof. See Theorem 3.195 in (IORIO JÚNIOR; IORIO, 2001). ■

Let $f, g \in H^s_{\text{per}}$, $s > \frac{1}{2}$. Due to Sobolev's lemma 2.1.2 we may define the product $fg \in C_{\text{per}} \subset \mathscr{P}'$ by the formula

$$\langle fg, \varphi \rangle = \int_{-\pi}^{\pi} f(x)g(x)\varphi(x)dx \forall \varphi \in \mathscr{P}. \tag{2.10}$$

The important point about this product is that it turns H^s_{per} , for $s > \frac{1}{2}$, into a Banach algebra.

Definition 2.1.34 (Banach algebra). *A Banach algebra is a Banach space X together with a product $(x, y) \in X \times X \mapsto xy \in X$ such that, for all $x, y, z \in X$ and for all $s, r \in \mathbb{C}$,*

(i) $(xy)z = x(yz)$,

(ii) $r(xy) = (rx)y = x(ry)$,

(iii) $(x + y)z = xz + yz$ *and* $x(y + z) = xy + xz$,

(iv) $\|xy\| \leqslant \|x\|\|y\|$.

Lemma 2.1.3. *Let $a, b \in [0, \infty)$ and $s \geqslant 0$. Then there are positive constants m_s and M_s depending only on s such that*

$$m_s (a^s + b^s) \leqslant (a + b)^s \leqslant M_s (a^s + b^s). \tag{2.11}$$

Proof. See Lemma 3.197 in (IORIO JÚNIOR; IORIO, 2001). ■

Definition 2.1.35. *Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ and $\beta = (\beta_k)_{k\in\mathbb{Z}}$ be two sequences of complex numbers. The convolution of α and β is the sequence $\alpha * \beta$ defined by*

$$(\alpha * \beta)_k = \sum_{j=-\infty}^{\infty} \alpha_j \beta_{k-j}, \tag{2.12}$$

whenever the right hand side of this equality makes sense.

Proposition 2.1.18. *Let $\alpha \in \ell^1 = \ell^1(\mathbb{Z})$ and $\beta \in \ell^2 = \ell^2(\mathbb{Z})$. Then $\alpha * \beta \in \ell^2$ and*

$$\|\alpha * \beta\|_{\ell^2} \leqslant \|\alpha\|_{\ell^1}\|\beta\|_{\ell^2}. \tag{2.13}$$

In particular, for every fixed $\alpha \in \ell^1$, the map $\beta \longmapsto \alpha * \beta$ defines a bounded linear operator from ℓ^2 into itself.

Proof. See Proposition 3.199 in (IORIO JÚNIOR; IORIO, 2001). ■

Theorem 2.1.17 (Banach algebra)**.** *If $s > \frac{1}{2}$, H^s_{per} is a Banach algebra. In particular, there exists a constant $C_s \geqslant 0$ depending only on s such that*

$$\|fg\|_s \leqslant C_s \|f\|_s \|g\|_s, \quad \forall f, g \in H^s_{\text{per}}. \tag{2.14}$$

Proof. Following (IORIO JÚNIOR; IORIO, 2001), consider $s > \frac{1}{2}$. The Fourier series of a function in H^s_{per} converges absolutely and uniformly over $[-\pi, \pi]$. Therefore, if $f, g \in H^s_{\text{per}}$,

$$\begin{aligned} \widehat{(fg)}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x) e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijx} \right) g(x) e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} \widehat{f}(j) \int_{-\pi}^{\pi} g(x) e^{-i(k-j)x} dx \\ &= \sum_{j=-\infty}^{\infty} \widehat{f}(j) \widehat{g}(k-j). \end{aligned} \tag{2.15}$$

From Lemma 2.1.3, we have

$$\left(1 + |k|^2\right)^{s/2} \leqslant K_s \left(1 + |k|^s\right) \leqslant K_s \left(1 + |k-j|^s + |j|^s\right) \quad \forall k, j \in \mathbb{Z},$$

where K_s is a nonnegative constant. Therefore,

$$\begin{aligned} &\left(1 + |k|^2\right)^{s/2} \left| \sum_{j=-\infty}^{\infty} \widehat{f}(j) \widehat{g}(k-j) \right| \\ &\leqslant K_s \left| \sum_{j=-\infty}^{\infty} \left[\left(1 + |k-j|^s + |j|^s\right) \right] \widehat{f}(j) \widehat{g}(k-j) \right| \\ &\leqslant K_s \sum_{j=-\infty}^{\infty} \left\{ |\widehat{f}(j) \widehat{g}(k-j)| + |\widehat{f}(j)| \, |(k-j)^s \widehat{g}(k-j)| + \left| j^s \widehat{f}(j) \right| |\widehat{g}(k-j)| \right\}. \end{aligned}$$

Since $\left(m^s \widehat{f}(m)\right)_{m \in \mathbb{Z}}, (m^s \widehat{g}(m))_{m \in \mathbb{Z}} \in \ell^2$ and $(\widehat{f}(m))_{m \in \mathbb{Z}}, (\widehat{g}(m))_{m \in \mathbb{Z}} \in \ell^1 \cap \ell^2$, Proposition 2.1.18 combined with (2.15) shows that

$$\left(\left(1 + |k|^2\right)^{s/2} \left| \sum_{j=-\infty}^{\infty} \widehat{f}(j) \widehat{g}(k-j) \right| \right)_{k \in \mathbb{Z}} \in \ell^2$$

and

$$\begin{aligned}
\|fg\|_s^2 &= \sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^{s/2} |\widehat{fg}(k)|^2 \\
&= \sum_{k=-\infty}^{\infty} \left(1+|k|^2\right)^{s/2} \left| \sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{g}(k-j) \right| \\
&\leqslant K_s \left[\|\hat{f}\|_{\ell^1} \|\widehat{g}\|_{l^2} + \|(\cdot)^s \widehat{g}(\cdot)\|_{\ell^1} \|\widehat{f}\|_{l^2} + \left\| (\cdot)^s \widehat{f}(\cdot) \right\|_{l^1} \|\widehat{g}\|_{\ell^2} \right] \\
&\leqslant C_s \|f\|_s \|g\|_s,
\end{aligned}$$

which finishes the proof. ■

2.1.3 Classical Remarkable Results

Now, we shall present a series of classical results that will be used throughout this master's thesis. The proofs of these results will be omitted (see (ADAMS, 1975; BREZIS, 2011) and references therein).

Lemma 2.1.4 (Young's Inequality). *Let a and b be positive constants,* $1 \leq p, q \leq \infty$, *such that p and q are Hölder conjugated index. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, for all $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q.$$

Proof. See proof of Theorem 4.6 in (BREZIS, 2011) or see Appendix B, letters c and d in (EVANS, 2010). ■

Lemma 2.1.5 (Gronwall's Inequality - differential form). *Let* $u(t)$ *be a non-negative differentiable function on* $[0, T]$, *satisfying*

$$u'(t) \leq f(t)u(t) + g(t)$$

where $f(t)$ *and* $g(t)$ *are integrable functions over* $[0, T]$. *Then,*

$$u(t) \leq e^{\int_0^t f(\tau)d\tau} \left[u(0) + \int_0^t g(s) e^{-\int_0^s f(\tau)d\tau} ds \right], \forall t \in [0, T].$$

If $f(t)$ and $g(t)$ are non-negative functions, then the expression becomes

$$u(t) \leq e^{\int_0^t f(\tau)d\tau}\left[u(0) + \int_0^t g(s)ds\right], \forall t \in [0,T].$$

Proof. See Appendix B, letter j in (EVANS, 2010). ■

Lemma 2.1.6 (Gronwall's Inequality - integral form)**.** *Let $u(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality*

$$u(t) \leq C_1 \int_0^t u(s)ds + C_2$$

for constants $C_1, C_2 \geq 0$. Then

$$u(t) \leq C_2\left(1 + C_1 t e^{C_1 t}\right) \quad \text{for a.e. } 0 \leq t \leq T.$$

In particular, if

$$u(t) \leq C_1 \int_0^t u(s)ds$$

for a.e. $0 \leq t \leq T$, then

$$u(t) = 0 \qquad a.e.$$

Proof. See Appendix B, letter k in (EVANS, 2010). ■

Lemma 2.1.7 (Hölder's Inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, consider $1 \leq p, q \leq \infty$ such that p and q are Hölder conjugated. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} = \int_\Omega |fg| \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

Proof. See Theorem 4.6 in (BREZIS, 2011). ■

Lemma 2.1.8 (Generalized Hölder's Inequality)**.** *Let $f_j \in L^{p_j}(\Omega)$ for $0 \leq j \leq k$ such that $\frac{1}{p} = \sum_{j=1}^k \frac{1}{p_k} \leq 1$. Then $f_1 \dots f_k \in L^p(\Omega)$ and yields that*

$$\|f_1 \dots f_k\|_{L^p(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \dots \|f_k\|_{L^{p_k}(\Omega)}.$$

Proof. See Remark 2 of Theorem 4.6 in (BREZIS, 2011). ■

Lemma 2.1.9 (Poincaré-Friedrichs inequality). *Let Ω be a bounded open subset of $\mathbb{R}^n$, then for every $1 \leq p < \infty$ there exists a constant $C = C(\Omega, p) > 0$, such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Proof. See Proposition 8.13 in (BREZIS, 2011). ■

Remark 2.1.4. *Poincaré's inequality remains true if Ω has a finite measure and also if Ω has a bounded projection on some axis.*

Lemma 2.1.10 (Peetre inequality). *For any $s \in \mathbb{R}$,*

$$\langle x - y\rangle^s \leq C_s \langle x\rangle^s \langle y\rangle^{|s|} \quad \text{for } s \in \mathbb{R},$$

with a positive constant C_s, where

$$\langle x\rangle = \left(1 + |x|^2\right)^{\frac{1}{2}},$$

and its powers $\langle x\rangle^s, s \in \mathbb{R}$.

Proof. See Lemma 6.7 in (GRUBB, 2008). ■

Theorem 2.1.18 (Aubin-Lions). *Let X_0, X and X_1 be Banach spaces such that $X_0 \subset X \subset X_1$ with X_0 compactly embedded in X and $X \hookrightarrow X_1$. Suppose that $1 < p, q \leq \infty$ and*

$$W = \{u \in L^p([0,T]; X_0) \colon u_t \in L^q([0,T]; X_1)\}.$$

(i) If $p < \infty$ then W is compactly embedded into $L^p([0,T], X)$.

(ii) If $p = \infty$ and $q > 1$ then $W \hookrightarrow C([0,T]; X)$ is compact.

Proof. See Theorem II.5.16 in (BOYER; FABRIE, 2013). ■

Proposition 2.1.19. *If V is a Banach space and $v \in L^p(0, T, V)$, with $1 \leqslant p \leqslant +\infty$, then for any $h > 0$ the function given by*

$$v^{[h]}(x,t) = \frac{1}{h}\int_t^{t+h} v(x,s)ds,$$

satisfies

(i) $v^{[h]} \in W^{1,p}(0, T-h; V)$,

(ii) $\left\|v^{[h]}\right\|_{L^p(0,T-h;V)} \leqslant \|v\|_{L^p(0,T;V)}$,

(iii) $v^{[h]} \to v$ *in* $L^p\,(0, T'; V)$, *as* $h \to 0$, *for* $p < \infty$ *and* $T' < T$.

Proof. See Proposition 1.4.29 in (CAZENAVE; HARAUX, 1998). ∎

Theorem 2.1.19 (Banach's fixed point theorem.). *Let* X *be a complete metric space and* $F : X \to X$ *be a contraction. Then* F *is continuous and there exists a unique point* $x_0 \in M$ *such that* $F(x_0) = x_0$.

Proof. See Proposition 23, Chapter 7 in (LIMA, 2020) or see Theorem 24.16 in (WILLARD, 2004). ∎

Fixed-point theorems, such as the one given above, are useful in proving certain existence theorems in differential and integral equations. In our case, we shall use this theorem for well-posedness issues.

2.1.4 Semigroup Theory

The semigroup theory provides a framework for analyzing the time evolution of systems described by PDEs, conducting mainly existence and uniqueness issues through the properties of operators. Consequently, some definitions and results will be presented. The results contained here can be found in (PAZY, 1983). In the sequel, we will denote by $(X, \|\cdot\|_X)$ a Banach space.

Definition 2.1.36. *A one parameter family* $T(t)$, $0 \le t < \infty$, *of bounded linear operators from* X *into* X *is a semigroup of a bounded linear operator on* X *if*

(i) $T(0) = I$, *where* I *is the identity operator on* X*;*

(ii) $T(t+s) = T(t)T(s)$, *for all* $t, s \ge 0$;

A semigroup of a bounded linear operator $T(t)$ *is uniformly continuous if*

$$\lim_{t \to 0^+} \|(T(t) - I)x\|_X = 0, \quad \forall x \in X.$$

The linear operator A is defined by

$$D(A) = \left\{ x \in X\colon \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \left. \frac{\mathrm{d}^+ T(t)x}{\mathrm{d}t} \right|_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Corollary 2.1.4. *Let $T(t)$ be a uniformly continuous semigroup of a bounded linear operator. Then*

(i) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.

(ii) There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.

(iii) The operator A defined in item (b) is the infinitesimal generator of $T(t)$.

(iv) The application $t \mapsto T(t)$ is differentiable in norm and

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = AT(t) = T(t)A.$$

Proof. See Corollary 1.4, Chapter 1 in (PAZY, 1983). ∎

Definition 2.1.37. *A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a strongly continuous semigroup of a bounded linear operator if*

$$\lim_{t \to 0^+} T(t)x = x, \quad \forall x \in X.$$

A strongly continuous semigroup of a bounded linear operator on X will be called a semigroup of class C_0 or simply a C_0-semigroup.

Theorem 2.1.20. *Let $T(t)$ be a C_0 semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad 0 \leq t < \infty.$$

Proof. See Theorem 2.2, Chapter 1 in (PAZY, 1983). ∎

Corollary 2.1.5. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$ then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.*

Proof. See Corollary 2.5, Chapter 1 in (PAZY, 1983). ■

2.1.4.1 A Theorem That Generate Group

Definition 2.1.38. *If, in Definition 2.1.36 instead of $t \in [0, \infty)$ we consider $t \in \mathbb{R}$ and, as well, in the limits there, instead of $t \to 0^+$, we consider here $t \to 0$, $T(t)$ is called a group instead of semigroup.*

Definition 2.1.39. *We say that an operator $A \in \mathcal{L}(H)$, where H is a Hilbert space, is unitary if A is invertible and $A^* = A^{-1}$.*

Definition 2.1.40. *We say that a group T of bounded linear operators on a Hilbert space H is a unitary group if, for each $t \geq 0$, $T(t)$ is a unitary operator, that is, $T(t)^* = T(t)^{-1}$ for all $t \geqslant 0$.*

Theorem 2.1.21 (Stone's theorem)**.** *A linear operator A on a Hilbert space H is the infinitesimal generator of a unitary C_0 group if and only if $A^* = -A$.*

Proof. See Theorem 10.8, Chapter 1 in (PAZY, 1983). ■

Remark 2.1.5. *Unitary operators are isometries. Therefore, we can reformulate Stone's theorem as follow: A linear operator A on a Hilbert space H generates a group of isometries if and only if A is skew-adjoint.*

2.1.4.2 The Abstract Cauchy Problem

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. Given $x \in X$, the abstract Cauchy problem for A with initial data x consists of finding a solution $u(t)$ to the initial value problem (I.V.P.)

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t), \quad t > 0, \\ u(0) = x. \end{cases} \tag{2.16}$$

Now, let us introduce a notion of a solution to the problem (2.16).

Definition 2.1.41 (Classical solution)**.** *By a classical solution of* (2.16) *we mean a function* $u : \mathbb{R}^+ \to X$ *such that* $u(t)$ *is continuous for all* $t \geq 0$*, continuously differentiable and* $u(t) \in D(A)$ *for all* $t > 0$ *that satisfies* (2.16)*.*

Remark 2.1.6. *We want to empathize on two points about the classical solutions:*

- *Note that since* $u(t) \in D(A)$ *for* $t > 0$ *and* u *is continuous at* $t = 0$, (2.16) *cannot have solution for* $x \notin \overline{D(A)}$*.*
- *It is clear that if* A *is the infinitesimal generator of a* C_0 *semigroup* $T(t)$*, the abstract Cauchy problem for* A *has a solution, namely* $u(t) = T(t)x$*, for every* $x \in D(A)$*. Moreover, it is not difficult to show that this is the only solution of* (2.16)*.*

We turn our attention to the non-homogeneous abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \quad t > 0, \\ u(0) = x. \end{cases} \tag{2.17}$$

where $f : [0, T) \to X$. We suppose that A is the infinitesimal generator of a C_0 semigroup $T(t)$ with corresponding homogeneous equation (2.16) has a unique solution for every initial value $x \in D(A)$.

Definition 2.1.42 (Classical solution)**.** *A function* $u : [0, T) \to X$ *is a classical solution of* (2.17) *on* $[0, T)$ *if* u *is continuous on* $[0, T)$*, continuously differentiable on* $(0, T)$*,* $u(t) \in D(A)$ *for* $0 < t < T$ *and* (2.17) *is satisfied for all* $t \in [0, T)$*.*

Suppose that $u(t)$ is a classical solution of (2.17). Then $g(s) = T(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\frac{\mathrm{d}g}{\mathrm{d}s} = -AT(t-s)u(s) + T(t-s)\frac{\mathrm{d}u}{\mathrm{d}s} = T(t-s)f(s).$$

Hence, If $f \in L^1(0, T; X)$ then $S(t-s)f(s)$ is integrable on $[0, t]$ and integrating from 0 to t yields[5]

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\, ds. \tag{2.18}$$

[5]The representation of solution (2.18) is known also *Duhamel's formula*

Corollary 2.1.6. *If $f \in L^1(0,T;X)$ then for every $x \in X$ the initial value problem* (2.17) *has at most one solution. If it has a solution, this is given by* (2.18)

Proof. See Corollary 2.2, Chapter 4 in (PAZY, 1983). ■

For every $f \in L^1(0,T;X)$ the right-hand side of (2.18) is a continuous function on $[0,T)$. It is natural to consider it as a generalized solution of (2.17) even if it is not differentiable and does not strictly satisfy the equation in the classical sense. Therefore we define,

Definition 2.1.43. *Let $x \in X$ and $f \in L^1(0,T;X)$. The function $u \in C([0,T];X)$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \;\; 0 \le t \le T,$$

is the mild solution of the non-homogeneous Cauchy problem (2.17) *on* $[0,T]$.

The definition of a mild solution of the abstract Cauchy problem (2.17) coincides when $f \equiv 0$ with the definition of $T(t)x$ as the mild solution of the corresponding homogeneous equation. Moreover, not every mild solution of (2.17) is indeed a (classical) solution even in the case $f \equiv 0$.

Next, let us present another notion of solution to the abstract Cauchy problem (2.17)

Definition 2.1.44 (Strong solution)**.** *A function u which is differentiable almost everywhere on $[0,T]$ such that $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1([0,T];X)$ is called a strong solution of the abstract Cauchy problem* (2.17) *if $u(0) = x$ and*

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t),$$

almost everywhere on $[0,T]$.

Notice that if $A = 0$ and $f \in L^1([0,T];X)$, the abstract Cauchy problem (2.17) has usually no solution unless $f \in C([0,T];X)$. However, (2.17)has always a strong solution given by

$$u(t) = u(0) + \int_0^t f(s)ds.$$

Furthermore, if u is a strong solution of (2.17) and $f \in L^1([0,T];X)$, can be showed that u is a mild solution as well.

Finally, we deal with the nonlinear case. Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, u(t)), \quad t > t_0, \\ u(t_0) = u_0. \end{cases} \tag{2.19}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on a Banach space X and $f\colon [t_0, T] \times X \to X$ is a continuous in t and satisfies the Lipschitz condition[6] on u. By the aforementioned arguments can be established a solution u that satisfies the integral equation

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s, u(s))\, ds,$$

which means that is a **mild solution**. Consequently, we have the following classical result which assures the existence and uniqueness of these mild solutions

Theorem 2.1.22. *Let $f\colon [t_0, T] \times X \to X$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X. If $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X then for every $u_0 \in X$, the abstract Cauchy problem* (2.19) *has a unique mild solution $u \in C([t_0, T]; X)$. Moreover, the mapping $u_0 \mapsto u$ is Lipschitz continuous from X into $C([t_0, T]; X)$.*

Proof. See Theorem 1.2, Chapter 6 in (PAZY, 1983). ■

Additionally, can be spotlighted some points

- If $u_0, v_0 \in X$ are initial data and u, v are its respective mild solutions of (2.19), then

$$\|u(t) - v(t)\|_X \leq Me^{LMt}\|u_0 - v_0\|_X.$$

- If $u_0 \in D(A)$, then u is a strong solution of (2.19) on $[t_0, T]$, for $T > t_0$.

[6] We said that $f\colon [t_0, T] \times X \to X$ satisfies the Lipschitz condition if there exists $L > 0$ such that

$$\|f(\cdot, u) - f(\cdot, v)\|_X \leq L\|u - v\|_X, \ \forall u, v \in X.$$

2.2 Some Classical Concepts About Control and Stabilization

Here, we present some definitions, tools, as well as techniques that will be useful for the core of manuscript, the final chapter. They are inspired in (LIONS, 1988; RUSSELL, 1978; ZUAZUA, 2006; SLOTINE; LI, 1991; CORON, 2007; SLEMROD, 1974).

2.2.1 Control for Finite-Dimensional Linear Systems

Some essential concepts of control and stabilization come from finite dimensional systems (ODE) and after generalization in some sense to infinite dimensional systems (PDE). Therefore, let us consider $m, n \in \mathbb{N}^*$, $T > 0$ and the finite-dimensional system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad 0 < t < T, \\ x(0) = x^0, \end{cases} \tag{2.20}$$

where $m \leq n$, A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and $x^0 \in \mathbb{R}^n$. The function $x\colon [0,T] \to \mathbb{R}^n$ represents the *state* and $u\colon [0,T] \to \mathbb{R}^m$ are called the *control*. The most desirable goal is, of course, controlling the system using a minimum number of m of controls.

Note that, by the variations of constants formula, if $u \in L^2(0,T;\mathbb{R}^m)$, (2.20) has a unique solution $x \in H^1(0,T;\mathbb{R}^n)$ given by

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{2.21}$$

Definition 2.2.1. *We said* (2.20) *is exactly controllable in time* $T > 0$ *if given any initial and final data* $x^0, x^1 \in \mathbb{R}^n$ *there exists* $u \in L^2(0,T;\mathbb{R}^m)$ *such that the solution* (2.21) *of* (2.20) *satisfies* $x(T) = x^1$.

- The aim of the control consists in driving the solution from the initial data x^0 to the final one x^1 in time T by acting on the system through the control u.
- It is desirable to make the number of controls m to be as small as possible. However, this may affect the control properties of the system.

By making a variable change, can we consider $x^1 = 0$, this motivates the following definition

Definition 2.2.2. *We said* (2.20) *is null controllable in time* $T > 0$ *if given any initial data* $x^0 \in \mathbb{R}^n$ *there exists* $u \in L^2(0, T; \mathbb{R}^m)$ *such that the solution* (2.21) *of* (2.20) *satisfies* $x(T) = 0$.

Remark 2.2.1. *Exact and null controllability are equivalent properties in the case of finite dimensional linear systems. But this is not necessarily the case for nonlinear systems, or, for strongly time-irreversible infinite dimensional systems.*

2.2.2 Control as a Minimization Problem

Let us introduce the homogeneous *adjoint system* of (2.20)

$$\begin{cases} -\varphi' = A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{2.22}$$

where A^* denotes the adjoint matrix of A. Next, by the adjoint properties, we have a characterization for the exact controllability property,

Lemma 2.2.1. *An initial data* $x^0 \in \mathbb{R}^n$ *of* (2.20) *is driven to zero in time* T *by using a control* $u \in L^2(0, T)$ *if and only if*

$$\int_0^T \langle u, B^*\varphi \rangle \, dt + \langle x^0, \varphi(0) \rangle = 0 \tag{2.23}$$

for any $\varphi_T \in \mathbb{R}^n$, φ *being the solution of the adjoint system* (2.22)

Proof. See Lemma 2.1.1 in (ZUAZUA, 2006). ∎

Moreover, (2.23) is an optimality condition for the critical points of the functional $J \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2} \int_0^T |B^*\varphi|^2 \, dt + \langle x^0, \varphi(0) \rangle$$

with φ the solution of the adjoint system (2.22) with initial data φ_T at time $t = T$. More precisely,

Lemma 2.2.2. *Suppose that J has a minimizer $\hat{\varphi}_T \in \mathbb{R}^n$ and let $\hat{\varphi}$ be the solution of the adjoint system* (2.22) *with initial data $\hat{\varphi}_T$. Then*

$$u = B^*\hat{\varphi}$$

is a control of system (2.20) *with initial data x^0.*

Proof. See Lemma 2.1.2 in (ZUAZUA, 2006). ■

Lemma 2.2.2 gives a variational method to obtain the control[7] as a minimum of the functional J. Remark that J is continuous. Therefore, the existence of a minimum is ensured if J is coercive too, that is,

$$\lim_{|\varphi_T|\to\infty} J(\varphi_T) = \infty. \tag{2.24}$$

The coercivity of J, (2.24), follows from the next concept named as *observability*,

Definition 2.2.3. *We said that* (2.22) *is observable in time $T > 0$ if there exists $C > 0$ such that*

$$\int_0^T |B^*\varphi|^2 \, dt \geq C|\varphi(0)|^2, \quad \forall \varphi_T \in \mathbb{R}^n, \tag{2.25}$$

where φ being the solution of (2.22).

Remark 2.2.2. *The observability inequality* (2.25) *is equivalent to the following assertion: there exists $C > 0$ such that*

$$\int_0^T |B^*\varphi|^2 \, dt \geq C|\varphi_T|^2, \quad \forall \varphi_T \in \mathbb{R}^n, \tag{2.26}$$

where φ being the solution of (2.22).

Finally, the next theorem ensures that the exact controllability can be reduced to the study of observability.

Theorem 2.2.1. *The system* (2.20) *is exactly controllable in time T if and only if* (2.22) *is observable in time T.*

Proof. See Theorem 2.1.1 in (ZUAZUA, 2006). ■

[7]This is not the unique possible functional allowing to build the control.

2.2.3 A Feedback Stabilization Problem

In a practical context, the stabilization problem for a system can be defined as finding a mechanism that ensures the system's state remains close to a desired point over time. Controllability is often a prerequisite for stabilization. If a system is not controllable, it may be impossible to design a control input that drives the system to the desired equilibrium state, making stabilization unattainable. Hence, heuristically, we can see stabilization as a controllability problem when the control is exerted at any time.

Here, we suppose that A is a skew-adjoint matrix, that is $A^* = -A$. Additionally, in this case, $\langle Ax, x\rangle = 0$. Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{2.27}$$

When the control is not acting, the energy of the solutions of (2.27) is conserved, that is, is constant over the time,

$$|x(t)| = |x^0|, \quad \forall t \geq 0.$$

The *stabilization* problem can be stated in the next way. Suppose that (2.27) is controllable, then we look for a solution of the system (2.27) such that with *feedback* control

$$u(t) = Lx(t) \tag{2.28}$$

has a *exponential decay*, that is, there exists $C > 0$ and $\lambda > 0$ such that

$$|x(t)| \leq Ce^{-\lambda t}|x^0| \tag{2.29}$$

for any solution. In particular, the control u given by (2.28) acts in real-time from the state x. More precisely, we are looking for an operator L such that the solution of the system

$$x' = (A + BL)x$$

has an exponential decay rate. Observe that due to the representation of solutions, the decay can not be faster than exponential.

Theorem 2.2.2. *If A is skew-adjoint and the system* (2.27) *is controllable then $L = -B^*$ stabilizes the system, that is, the solution of*

$$\begin{cases} x' = Ax - BB^*x \\ x(0) = x^0 \end{cases} \tag{2.30}$$

has an exponential decay.

Proof. See Theorem 2.1.3 in (ZUAZUA, 2006). ■

Remark 2.2.3. *To prove the Theorem 2.2.2 a fundamental estimate is sufficient to obtain the exponential decay, that is, there exists $T > 0$ and $C > 0$ such that*

$$\int_0^T |B^*x|^2 \, dt \geq C^{-1}|x(0)|^2, \tag{2.31}$$

for any solution x of (2.30). *Note that* (2.31) *is an observability type inequality and this shows how the controllability and stabilization are related via an inequality.*

2.2.4 Control and Stabilization Extended to Infinite Dimensional Systems

All of the concepts and results mentioned above can be generalized (in some sense) to infinite dimensional systems. Let $T > 0$, H and V be real Hilbert spaces and consider the following control system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Au + Bv, \quad 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{2.32}$$

where u denotes the states and $v \in L^2(0, T; V)$ is the control. The operator $A\colon D(A) \to H$ is a linear operator and $B \in \mathcal{L}(V, D(A^*)')$[8], where $D(A^*)'$ denotes the dual space of $D(A^*)$ and A^* is the adjoint of the operator A. Additionally, A^* is associated with the homogeneous adjoint system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = -A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{2.33}$$

[8]This functional setting gives the possibility to consider boundary control operators (instead of the stronger one $B \in \mathcal{L}(V, H)$)

Now, we state the most classical notions of controllability for the abstract system (2.32),

Definition 2.2.4. *The system* (2.32) *is exactly controllable in time* $T > 0$ *if, for every initial and final data* $u_0, u_T \in H$*, there exists* $v \in L^2(0,T;V)$ *such that the solution of* (2.32) *satisfies* $u(T) = u_T$.

Definition 2.2.5. *The system* (2.32) *is null controllable in time* $T > 0$ *if, for every initial data* $u_0 \in H$*, there exists* $v \in L^2(0,T;V)$ *such that the solution of* (2.32) *satisfies* $u(T) = 0$.

Definition 2.2.6. *The system* (2.32) *is approximately controllable in time* $T > 0$ *if, for every initial and final data* $u_0, u_T \in H$*, and* $\varepsilon > 0$*, there exists* $v \in L^2(0,T;V)$ *such that the solution of* (2.32) *satisfies*

$$\|u(T) - u_T\|_H \leq \varepsilon.$$

Similar to the mentioned for finite-dimensional, a control may be obtained from the solution of the homogeneous system (2.33) with the initial data minimizing the functional $J\colon H \to \mathbb{R}$ given by

$$J(\varphi) = \frac{1}{2}\int_0^T \langle u, B^*\varphi\rangle_H \, dt + \langle u_0, \varphi(0)\rangle_H - \langle u_T, \varphi_T\rangle_H.$$

Hence, the controllability is reduced to a minimization problem. To guarantee that J has a unique minimizer, we use the next fundamental result in the calculus of variations.

Theorem 2.2.3. *Let* H *be a reflexive Banach space,* K *a closed convex subset of* H *and* $J : K \to \mathbb{R}$ *a function with the following properties:*

(i) J *is convex*

(ii) J *is lower semi-continuous*

(iii) If K *is unbounded then* J *is coercive, i.e.*

$$\lim_{\|x\|\to\infty} J(x) = \infty.$$

Then J *attains its minimum in* K*, i. e. there exists* $x_0 \in K$ *such that*

$$J(x_0) = \min_{x\in K} \varphi(x)$$

Proof. See Corollary 3.23 in (BREZIS, 2011). ■

Note that J is continuous and convex. The existence of a minimum is ensured if J is also coercive, which is obtained with the *observability inequality*

$$\int_0^T \|B^*\varphi\|_H \, dt \geq C\|\varphi(0)\|_H^2, \quad \forall \varphi(0) \in H. \tag{2.34}$$

Finally, consider the uncontrolled case, $v \equiv 0$ in (2.32). Let $\overline{u}$ be an equilibrium solution, that is, $A\overline{u} = 0$, with $\overline{u} \in D(A)$.

Definition 2.2.7. *We said that $\overline{u}$ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \overline{u}\| \leq \delta$, the unique mild solution u of* (2.32) *satisfies*

$$\|u(t) - \overline{u}\| < \varepsilon, \quad \forall t \geq 0.$$

Definition 2.2.8. *We said that $\overline{u}$ is asymptotically stable if is stable and there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \overline{u}\| \leq \delta$, the unique mild solution u of* (2.32) *satisfies*

$$\lim_{t\to\infty} \|u(t) - \overline{u}\| = 0.$$

Definition 2.2.9. *We said that $\overline{u}$ is exponentially stable if is asymptotically stable and there exists $\lambda > 0$ such that for all $u_0 \in H$ the unique mild solution u of* (2.32) *satisfies*

$$\|u(t) - \overline{u}\| < e^{-\lambda t}\|u(t) - \overline{u}\|.$$

The largest constant λ which may be utilized in the exponential stabilization is called the rate of convergence.

Definition 2.2.10. *System* (2.32) *is said to be locally uniformly exponentially stable in H if for any $r > 0$ there exist two constants $C > 0$ and $\gamma > 0$ such that for any $u_0 \in H$ with $\|u_0\|_H < r$ and for any solution u of* (2.32) *it holds that*

$$\|u(t)\|_H^2 \leqslant Ce^{-\gamma t} \|u_0\|_H^2, \quad \forall t \geqslant 0. \tag{2.35}$$

If the constant γ in (2.35) *is independent of r, the system* (2.32) *is said to be globally uniformly exponentially stable in H.*

We shall summarize, following (ROSIER, 2007), some concepts about the stabilizability of a control system

$$\Sigma_{A,B} \qquad u_t = Au + Bh, \tag{2.36}$$

where A generates a continuous semigroup of operators $\{e^{tA}\}_{t\geqslant 0} = \{W(t)\}_{t\geqslant 0} \subset H$, H is a Hilbert space, $B \in \mathcal{L}(U, H)$ and h is the control input.

Consider the following properties:

(i) for some constants $C, \gamma > 0$ and all $t \geqslant 0$, $\|W(t)\| \leqslant Ce^{-\gamma t}$;

(ii) for any $u_0 \in H, W(t)u_0 \to 0$ exponentially as $t \to 0$;

(iii) for any $u_0 \in H$, $\int_0^{+\infty} \|W(t)u_0\|_H^2 dt < +\infty$;

(iv) for any $u_0 \in H$, $W(t)u_0 \to 0$ as $t \to 0$.

Theorem 2.2.4. *We have (i) ⇔ (ii) ⇔ (iii). On the other hand, we have (i) ⇒ (iv).*

Proof. See (ROSIER, 2007). ■

Definition 2.2.11. *If* (i) (*or equivalently* (ii) *or* (iii)) *holds, then we say that the semigroup* $\{W(t)\}_{t\geqslant 0}$ *is exponentially stable. When* (iv) *holds, we say that the semigroup* $\{W(t)\}_{t\geqslant 0}$ *is strongly stable.*

For any $K \in \mathcal{L}(H, U)$, we denote by A_K the operator $A_K u = Au + BKu = (A+BK)u$, with $D(A_K) = D(A)$, and by $\{e^{tA_K}\}_{t\geqslant 0} = \{W_K(t)\}_{t\geqslant 0}$ the semigroup generated by A_K.

Definition 2.2.12. *The control system* $\Sigma_{A,B}$ *is said to be*

- *exponentially stabilizable if ∃ a feedback* $K \in L(H, U)$ *such that the operator* $A_K = A + BK$ *is exponentially stable; i.e., for some constants* $C > 0, \gamma > 0$,

$$\|W_K(t)\| \leq Ce^{-\gamma t} \quad \forall t \geqslant 0$$

- *completely stabilizable if it is exponentially stabilizable with an arbitrary exponential decay rate; i.e., for arbitrary* $\gamma \in \mathbb{R}$, *there exists a feedback* $K \in L(H, U)$ *and a constant* $C > 0$ *such that*

$$\|W_K(t)\| \leq Ce^{-\gamma t} \quad \forall t \geqslant 0$$

Theorem 2.2.5. *If the system* $(\Sigma_{A,B})$ *is null controllable, then it is exponentially stabilizable.*

Proof. See (ROSIER, 2007). ∎

Theorem 2.2.6. *Assume that* A *generates a group* $\{W(t)\}_{t\in\mathbb{R}}$ *of operators. Then the following properties are equivalent*

(i) $(\Sigma_{A,B})$ *is exactly controllable in some time* $T > 0$*;*

(ii) $(\Sigma_{A,B})$ *is null controllable in some time* $T > 0$*;*

(iii) $(\Sigma_{A,B})$ *is completely stabilizable.*

Proof. See (ROSIER, 2007). ∎

We can apply 2.2.6 to a skew-adjoint operator A, which generates a group of isometries on H. Moreover, we have the "controllability via stabilizability" principle, and explicit exponentially stabilizing feedback laws may be given.

Corollary 2.2.1 (Equivalence between controllability and stabilizability)**.** *Let* A *be skew-adjoint, i.e.,* $A^* = -A$*. Then the following propositions are equivalent*

(i) the system $(\Sigma_{A,B})$ *is exponentially stabilizable with an arbitrary prefixed exponential decay rate, that is,* $(\Sigma_{A,B})$ *is completely stabilizable;*

(ii) The system $(\Sigma_{A,B})$ *is exponentially stabilizable;*

(iii) The system $(\Sigma_{A,B})$ *is exactly controllable in some* $T > 0$*;*

(iv) The system $(\Sigma_{A,B})$ *is null controllable in some time* $T > 0$*;*

(v) For every positive definite self-adjoint operator $S \in \mathcal{L}(U)$*, the operator* $A - BSB^*$ *generates an exponentially stable semigroup on* H*.*

Proof. See (LIU, 1997; ROSIER, 2007). ∎

3 WELL-POSEDNESS FOR THE BBM EQUATION

One fundamental question when facing a Partial Differential Equation (PDE) is the solvability issue, or even more fundamental, what we mean by solving a PDE. From this subtle question arises the concept of a well-posed equation, given by Jacques Hadamard. Thus, by solving the BBM equation, we mean that, for a given initial data, the BBM equation satisfies the following conditions that characterize a well-posed problem:

(i) Existence of a solution;

(ii) Uniqueness of this solution;

(iii) Stability, meaning continuous dependence on the initial data.

For items (i) and (ii), we make use of Theorem 2.1.19 - Banach's fixed point theorem - applied in an appropriate abstract space of functions; (iii) means that the map that associates the initial data to the function on this abstract space, called the *flow map* or *solution map*, is continuous, which is particularly important for problems arising from physical applications. For the BBM, we have even more: the flow map is real analytic. All this concepts are treated in the first section of this chapter.

The remainder of the chapter, composed by two more sections, is devoted to some properties that the solution possesses. The second section addresses the time analyticity of the solution, whereas the third and final section discusses some conserved quantities.

3.1 Well-Posedness

To set forth our problem, we begin by establishing the objects and the spaces in which we will be working from now on.

For any $s \geqslant 0, H^s(\mathbb{T})$ denotes the Sobolev space

$$H^s(\mathbb{T}) = \left\{ u : \mathbb{T} \to \mathbb{R}; \|u\|_{H^s}^2 := \sum_{k \in \mathbb{Z}} \left(1 + k^2\right)^s |\widehat{u}_k|^2 < \infty \right\},$$

which is a Hilbert space with respect to the inner product

$$(u,v)_{H^s} = \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \widehat{u}_k \overline{\widehat{v}_k}.$$

Where $\hat{u}_k$ denotes the k-Fourier coefficient of u

$$\widehat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx\,,$$

where we have identified the torus $\mathbb{T}$ to $[0,2\pi)$, by using a coordinate system.

We are interested in the Cauchy problem, that is, the Initial Value Problem (IVP), associated with the BBM equation

$$\begin{cases} u_t - u_{txx} + u_x + uu_x = 0, & x\in\mathbb{T}, t\in\mathbb{R} \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}). \end{cases} \tag{3.1}$$

So, in order to use the semigroup theory (see 2.1.4.2), we put (3.1), in its integral form

$$\begin{aligned} &\partial_t u - \partial_t\partial_x^2 u + \partial_x u + u\partial_x u = 0 \\ &\partial_t\left(u - \partial_x^2 u\right) + \partial_x\left(u + \frac{u^2}{2}\right) = 0 \\ &\left(1-\partial_x^2\right)\partial_t u = -\partial_x\left(u + \frac{u^2}{2}\right) \\ &\partial_t u = -\left(1-\partial_x^2\right)^{-1}\partial_x\left(u + \frac{u^2}{2}\right). \end{aligned} \tag{3.2}$$

That is, we have the following Cauchy abstract problem (see (2.17))

$$\begin{cases} u_t = -\left(1-\partial_x^2\right)^{-1}\partial_x\left(u + \frac{u^2}{2}\right) = A(u) + A\left(\frac{u^2}{2}\right) & x\in\mathbb{T}, t\in\mathbb{R} \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}). \end{cases} \tag{3.3}$$

Claim 3.1.1. *The operator $A = -\left(1-\partial_x^2\right)^{-1}\partial_x \in \mathcal{L}\left(H^s(\mathbb{T}), H^{s+1}(\mathbb{T})\right)$ (for any $s\in\mathbb{R}$) is skew-adjoint and generates the group of isometries $\{W(t)\}_{t\in\mathbb{R}} = \{e^{tA}\}_{t\in\mathbb{R}}$.*

Proof. We want to show that $A^* = -A$, that is,

$$(Af, g)_{H^s(\mathbb{T})} = -(f, Ag)_{H^s(\mathbb{T})}, \quad \forall f, g \in H^s(\mathbb{T}).$$

Before performing the calculations for the inner product, we shall point out that

$$\widehat{(Af)}(k) = -\frac{ik}{1+k^2}\hat{f}(k).$$

Indeed, let $f \in H^s(\mathbb{T})$, and denoting $^\vee$ the Fourier inverse, we have

$$\begin{aligned}
&\widehat{(Af)}(k) =: h(k) \\
&\widehat{-(1-\partial_x^2)^{-1}\partial_x f}(k) = h(k) \\
&-\left(1-\partial_x^2\right)^{-1}\partial_x f(x) = \overset{\vee}{h}(x) \\
&\partial_x f(x) = -\left(1-\partial_x^2\right)\overset{\vee}{h}(x) \\
&ik\hat{f}(k) = -\left(1-(ik)^2\right)h(k) \\
&-\frac{ik}{1+k^2}\hat{f}(k) = h(k) = \widehat{(Af)}(k).
\end{aligned}$$

Thus, let $f, g \in H^s(\mathbb{T})$, we have

$$\begin{aligned}
(Af, g)_{H^s(\mathbb{T})} &= \left(-\left(1-\partial_x^2\right)^{-1}\partial_x f, g\right)_{H^s(\mathbb{T})} \\
&= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \widehat{(-(1-\partial_x^2)^{-1}\partial_x f)}(k)\overline{\hat{g}(k)} \\
&= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \frac{(-ik)}{1+k^2}\hat{f}(k)\overline{\hat{g}(k)} \\
&= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \hat{f}(k)\overline{\hat{g}(k)\frac{ik}{1+k^2}} \\
&= -\sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \hat{f}(k)\overline{\hat{g}(k)\frac{(-ik)}{1+k^2}} \\
&= -\sum_{k\in\mathbb{Z}} \left(1+k^2\right)^s \hat{f}(k)\overline{\widehat{(-\left(1-\partial_x^2\right)^{-1}\partial_x g)}(k)} \\
&= -\left(f, -\left(1-\partial_x^2\right)^{-1}\partial_x g\right)_{H^s(\mathbb{T})} \\
&= -(f, Ag)_{H^s(\mathbb{T})} \\
&= (f, -Ag)_{H^s(\mathbb{T})}.
\end{aligned} \tag{3.4}$$

Therefore, from Stone's Theorem 2.1.21, the operator A is the infinitesimal generator of a strongly continuous group of unitary operators, namely, $\{W(t)\}_{t\in\mathbb{R}}$ with $W(t) = e^{tA}$. ■

So, from Duhamel formula (2.18), we can put (3.1) in its integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-s)A\left(u^2/2\right)(s)ds. \tag{3.5}$$

For $s \geqslant 0$ and $T > 0$, let

$$X_T^s = C\left([-T,T]; H^s(\mathbb{T})\right).$$

As mentioned in the summary of this chapter, we shall be looking for a solution in an abstract space of functions, which turns out to be X_T^s. We see that for $u \in X_T^s$, then u solves (3.1) in $\mathcal{D}'\left(-T,T;H^{s-2}(\mathbb{T})\right)$ if, and only if, it fulfills (3.5) for all $t \in [-T,T]$. We shall apply the standard procedure of contraction map in order to prove the well-posedness of (3.1). To this end, we shall demonstrate the following two inequalities, presented in form of lemmas, which will prove to be very useful for us until the end of this work. These estimates can be found in (BONA; TZVETKOV, 2009; ROUMÉGOUX, 2010; HIMONAS; PETRONILHO, 2020).

Lemma 3.1.1. *Let $u, v \in H^s(\mathbb{T})$, with $s \geqslant 0$. Then*

$$\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(uv)\right\|_{H^s(\mathbb{T})} \leqslant C_s\|u\|_{H^s(\mathbb{T})}\|v\|_{H^s(\mathbb{T})}, \tag{3.6}$$

and

$$\left\|\left(1-\partial_x^2\right)^{-1}\partial_x u\right\|_{H^s(\mathbb{T})} \leqslant \|u\|_{H^s(\mathbb{T})}. \tag{3.7}$$

Proof. It is worth introducing some notation that is very often encountered in the literature

$$\langle k\rangle := \left(1+k^2\right)^{\frac{1}{2}}.$$

For instance, using this notation, the above definition of the H^s-norm becomes

$$\|u\|_{H^s} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|\widehat{u}_k\right|^2.$$

Estimate (3.7) follows from the definition of the $H^s(\mathbb{T})$-norm and the multiplier estimate $|k|\left(1+k^2\right)^{-1} \leq 1$. Indeed

$$\left\|\left(1-\partial_x^2\right)^{-1}\partial_x u\right\|_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}}(1+k^2)^s\frac{k^2}{(1+k^2)^2}|\hat{u}(k)|^2 \leqslant \sum_{k\in\mathbb{Z}}(1+k^2)^s|\hat{u}(k)|^2 = \|u\|_{H^s(\mathbb{T})}.$$

Now, to prove estimate (3.6), applying the definition of the $H^s(\mathbb{T})$-norm we have

$$\begin{aligned}\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(uv)\right\|^2_{H^s(\mathbb{T})} &= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\frac{k^2}{\left(1+k^2\right)^2}|\widehat{uv}(k)|^2 \\ &= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\frac{k^2}{\left(1+k^2\right)^2}\left|\sum_{\ell\in\mathbb{Z}}\widehat{u}(\ell)\widehat{v}(k-\ell)\right|^2.\end{aligned} \tag{3.8}$$

Note that for $s \geqslant 0$ we have the following inequality[1]

$$\langle k\rangle^s \leqslant 2^{s/2}\langle k-\ell\rangle^s\langle \ell\rangle^s.$$

So, from (3.8) we get

$$\begin{aligned}&\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(uv)\right\|^2_{H^s(\mathbb{T})} \\ &\leqslant 2^s\sum_{k\in\mathbb{Z}}\frac{k^2}{\left(1+k^2\right)^2}\left|\sum_{\ell\in\mathbb{Z}}\langle\ell\rangle^s\widehat{u}(\ell)\cdot\langle k-\ell\rangle^s\widehat{v}(k-\ell)\right|^2.\end{aligned} \tag{3.9}$$

Furthermore, applying Schwarz's inequality in the ℓ-sum, from the inequality above (3.9), we obtain that

$$\begin{aligned}&\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(uv)\right\|^2_{H^s(\mathbb{T})} \\ &\leqslant 2^s\sum_{k\in\mathbb{Z}}\frac{k^2}{\left(1+k^2\right)^2}\left(\sum_{\ell\in\mathbb{Z}}\langle\ell\rangle^{2s}|\widehat{u}(\ell)|^2\right)\left(\sum_{\ell\in\mathbb{Z}}\langle k-\ell\rangle^{2s}|\widehat{v}(k-\ell)|^2\right) \\ &\leqslant 2^s\|u\|^2_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}\sum_{k\in\mathbb{Z}}\frac{1}{1+k^2} \\ &\leqslant 2^s\left(1+\frac{\pi^2}{3}\right)\|u\|^2_{H^s(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})},\end{aligned} \tag{3.10}$$

which completes the proof of Lemma 3.1.1 and we got that $C_s^2 = 2^s\left(1+\frac{\pi^2}{3}\right)$. ■

[1] See Peetre inequality 2.1.10

Lemma 3.1.2. *Let $u \in H^r(\mathbb{T})$ and $v \in H^s(\mathbb{T})$ with $0 \leqslant s \leqslant r$ and $r > \frac{1}{2}$, then*

$$\| \left(1 - \partial_x^2\right)^{-1} \partial_x(uv)\|_{H^{s+1}} \leqslant C\|u\|_{H^r}\|v\|_{H^s}.$$

Proof. Since $r > \frac{1}{2}$ and $r \geq s \geq 0$, the elements of $H^r(\mathbb{T})$ are multipliers in $H^s(\mathbb{T})$ (see Theorem 2.1.17), which is to say

$$\|uv\|_{H^s} \lesssim \|u\|_{H^r}\|v\|_{H^s}.$$

Hence,

$$\begin{aligned}
\| \left(1 - \partial_x^2\right)^{-1} \partial_x(uv)\|_{H^{s+1}} &= \left\| \langle k\rangle^{s+1} \frac{k}{1+k^2}\widehat{uv}\right\|_{\ell_k^2} \\
&= \left\| \langle k\rangle^{s+1} \frac{k}{\langle k\rangle^2}\widehat{uv}\right\|_{\ell_k^2} \\
&= \left\| \langle k\rangle^{s} \frac{k}{(1+k^2)^{1/2}}\widehat{uv}\right\|_{\ell_k^2} \\
&\leqslant \|\langle k\rangle^s \widehat{uv}\|_{\ell_\xi^2} = \|uv\|_{H^s} \lesssim \|u\|_{H^r}\|v\|_{H^s}.
\end{aligned} \tag{3.11}$$

, ■

With these estimates, we are able to deal with the well-posedness of the IVP (3.1) associated with the BBM equation, this result is stated in (ROSIER; ZHANG, 2013). First, we treat the local aspects and then the global one. The proof of Theorem 3.1.1 and Theorem 3.1.2 will follow (BONA; TZVETKOV, 2009) and (ROUMÉGOUX, 2010).

Theorem 3.1.1 (Local Well-posedness)**.** *For a given initial data $u_0 \in H^s(\mathbb{T}), s \geqslant 0$ and for suitable $T > 0$ there exists a unique solution $u \in X_T^s$ of* (3.1) *(or equivalently,* (3.5)*). Furthermore, for any $R > 0$, the map $u_0 \mapsto u$ is real analytic from $B_R(X_T^s)$ into X_T^s.*

Proof. We want to show that for a given $u_0 \in H^s(\mathbb{T})$, there exists solution u of the Cauchy abstract problem (3.3), that we put here for convenience

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= Au + A\left(\frac{u^2}{2}\right), \\
u(x,0) &= u_0(x) \in H^s(\mathbb{T}),
\end{aligned}$$

in the space of functions $C\left([-T,T]; H^s(\mathbb{T})\right) = X_T^s$ for a $T > 0$ that will be determined later. For $R > 0$, let

$$B_R(X_T^s) = \left\{u \in X_T^s; \|u\|_{X_T^s} \leqslant R\right\},$$

denote the closed ball of radius R centered at the origin in X_T^s with $u_0 \in B_R(H^s(\mathbb{T}))$. Define the map Φ from $B_R(X_T^s)$ to X_T^s, by

$$\Phi(u)(t) = e^{tA}u_0 + \int_0^t e^{(t-\tau)A} A\left(\frac{u^2}{2}\right)(\tau)\, d\tau.$$

We shall prove that:

(i) there exists $R > 0$ such that $\Phi\left(B_R(X_T^s)\right) \subset B_R\left(X_T^s\right)$;

(ii) $\|\Phi(u) - \Phi(v)\|_{X_T^s} \leqslant \lambda \|u - v\|_{X_T^s}$, for some $\lambda < 1$, and for all $u, v \in B_R\left(X_T^s\right)$ (i.e., Φ is a contraction).

Thus, from (i) and (ii), the fixed-point theorem, also known as the contraction mapping principle, assures us of the existence and uniqueness of a $\tilde{u}$ such that $\Phi(\tilde{u}) = \tilde{u}$. This fixed point is our desired solution of (3.1).

Pick $u \in B_R(X_T^s)$. From Claim 3.1.1, $\{e^{tA}\}_{t\in\mathbb{R}}$ is group of isometries in H^s, that is, $\|e^{tA}u_0\|_{H^s(\mathbb{T})} = \|u_0\|_{H^s(\mathbb{T})}$ and from Lemma 3.1.1, we have, for $0 \leqslant t \leqslant T$

$$\begin{aligned}
\|\Phi(u)\|_{X_T^s} &\leqslant \left\|e^{tA}u_0\right\|_{X_T^s} + \int_0^t \left\|e^{(t-\tau)A} A\left(\frac{u^2}{2}\right)(\tau)\right\|_{X_T^s} d\tau \\
&\leqslant \|u_0\|_{X_T^s} + \int_0^t \left\|A\left(\frac{u^2}{2}\right)(\tau)\right\|_{X_T^s} d\tau \\
&\leqslant \|u_0\|_{X_T^s} + \frac{1}{2}\int_0^t \left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right)(\tau)\right\|_{X_T^s} d\tau \\
&\leqslant \|u_0\|_{X_T^s} + \frac{TC_s}{2}\|u\|_{X_T^s}^2 \\
&\leqslant \|u_0\|_{H^s(\mathbb{T})} + \frac{TC_sR^2}{2}.
\end{aligned}$$

Choose $R = 2\|u_0\|_{H^s}$. As mentioned, we must choose a convenient T so that $\|\Phi(u)\|_{X_T^s} \leqslant R$. Thus, we must have

$$\begin{aligned}
\|\Phi(u)\|_{X_T^s} \leqslant \|u_0\|_{H^s(\mathbb{T})} + \frac{TC_sR^2}{2} &\leqslant R \\
TC_sR^2 &\leqslant 2R - 2\|u_0\|_{H^s(\mathbb{T})} = 2R - R
\end{aligned}$$

$$T \leqslant \frac{1}{C_s R}.$$

Hence, setting $T = (2C_sR)^{-1}$, we have $\Phi\left(B_R(X_T^s)\right)) \subset B_R(X_T^s)$.

For the contraction property of the map Φ, pick $u, v \in B_R\left(X_T^s\right)$, then, it follows that

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{X_T^s} &= \left\| \int_0^t W(t-\tau) A\left(\frac{u^2}{2}\right)(\tau)\, d\tau - \int_0^t W(t-\tau) A\left(\frac{v^2}{2}\right)(\tau)\, d\tau \right\|_{X_T^s} \\
&= \frac{1}{2}\left\| \int_0^t W(t-\tau) A\left(u^2 - v^2\right)(\tau) d\tau \right\|_{X_T^s} \\
&= \frac{1}{2}\left\| \int_0^t W(t-\tau) A\left[(u-v)(u+v)\right](\tau) d\tau \right\|_{X_T^s} \\
&\leqslant \frac{1}{2} T C_s \|u - v\|_{X_T^s} \|u + v\|_{X_T^s} \\
&\leqslant \frac{1}{2} T C_s \|u - v\|_{X_T^s} \left(\|u\|_{X_T^s} + \|v\|_{X_T^s}\right) \\
&\leqslant \frac{1}{2} T C_s \|u - v\|_{X_T^s} 2R \\
&\leqslant T C_s R \|u - v\|_{X_T^s}.
\end{aligned}$$

Then, as we have set T by $T = (2C_sR)^{-1}$, we see that

$$\|\Phi(u) - \Phi(v)\|_{X_T^s} \leqslant \frac{1}{2}\|u - v\|_{X_T^s}.$$

Therefore, this concludes the existence and uniqueness question for Cauchy Problem (3.1), where the maximal existence time $T = T_s$ for the solution has the property that

$$T_s \geqslant \frac{1}{2C_sR} = \frac{1}{4C_s\|u_0\|_{H^s(\mathbb{T})}}.$$

Now, we turn our attention to the analyticity of the flow map Φ. This result is local in the sense that if it can be established for T sufficiently small. Let $\Lambda : H^s \times X_T^s \longrightarrow X_T^s$ be defined as

$$\Lambda\left(u_0, v(t)\right) = v(t) - W(t)u_0 - \frac{1}{2}\int_0^t W\left(t - s\right) A(v^2)\left(s\right)\, ds,$$

where the spatial variable x has been suppressed throughout. Note that, for u solution of (3.1), then $\Lambda\left(u_0, u(t)\right) = 0$. We are interested in the Fréchet derivative, see Definition 2.1.20, of Λ with respect to the second variable, that is, we want to find the linear map

$T(h)$ such that

$$\Lambda(u_0, v+h) = \Lambda(u_0, v) + T(h) + r(h), \quad \text{with } \|r(h)\| \to 0 \quad \text{as} \quad \|h\| \to 0.$$

Then, we have

$$\begin{aligned}
\Lambda(u_0, v+h) &= v(t) + h(t) - W(t)u_0 - \frac{1}{2}\int_0^t W(t-s)A\left((v+h)^2\right)(s)ds \\
&= \underbrace{v(t) - W(t)u_0 - \frac{1}{2}\int_0^t W(t-s)A(v^2)ds}_{\Lambda(u_0,v)} + \underbrace{h(t) - \int_0^t W(t-s)A(vh)ds}_{T(h)} \\
&\quad \underbrace{-\frac{1}{2}\int_0^t W(t-s)A(h^2)ds}_{r(h)} \\
&= \Lambda(u_0, v) + T(h) + r(h).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\|r(h)\|_{H^s}}{\|h\|_{X_T^s}} &= \frac{\frac{1}{2}\left\|\int_0^t W(t-s)A\left(h^2\right)ds\right\|_{H^s}}{\|h\|_{X_T^s}} \\
\frac{\|r(h)\|_{X_T^s}}{\|h\|_{X_T^s}} &\leqslant \frac{TC_s\|h\|_{X_T^s}^2}{\|h\|_{X_T^s}} \leqslant T\|h\|_{X_T^s},
\end{aligned}$$

we have that

$$\frac{\|r(h)\|_{X_T^s}}{\|h\|_{X_T^s}} \to 0 \quad \text{as} \quad \|h\|_{X_T^s} \to 0.$$

Hence, we obtain

$$\Lambda_u'\left(u_0, u(t)\right)[h] = h - \int_0^t W(t-s)A(uh)(s)ds.$$

From Lemma 3.1.1

$$\begin{aligned}
\left\|\Lambda_u'\left(u_0, u(t)\right)[h]\right\|_{X_T^s} &\leqslant \|h\|_{X_T^s} + \left\|\int_0^t W(t-s)A(uh)(s)ds\right\|_{X_T^s} \\
&\leqslant \|h\|_{X_T^s} + C_sT\|u\|_{X_T^s}\|h\|_{X_T^s}.
\end{aligned}$$

We see that Λ_u' is of the form $(I+K)$, where

$$K = -\int_0^t W(t-s)A(uh)(s)ds.$$

Then, we can take T sufficiently small so that

$$\|K\|_{\mathcal{B}\left(X_T^s, X_T^s\right)} < 1,$$

where $\mathcal{B}\left(X_T^s, X_T^s\right)$ is the Banach space of bounded linear operators on X_T^s. Since $\Lambda' = I + K$, by using the Neumann criterion, see Lemma 2.1.1, Λ' is invertible and the inverse can be expressed as a power series

$$(I + K)^{-1} = I + K + K^2 + \cdots$$

Therefore, the map Φ is real-analytic by the Implicit Function Theorem 2.1.12, which concludes the proof of Theorem 3.1.1. ■

Theorem 3.1.2 (Global Well-posedness)**.** *In Theorem 3.1.1, the solution u is global in time, that is, we can take arbitrarily large value of T.*

Proof. Fix $T > 0$. Our aim is to show that for any initial data $u_0 \in H^s$, there exists a unique solution u of (3.1) that lies in X_T^s, and that u depends continuously upon u_0. From the local well-posedness 3.1.1, we have this result for small enough data in H^s. Moreover, it is only necessary to have the existence of a solution corresponding to initial data of arbitrary size, since continuous dependence, uniqueness and the analytic dependence on the data of the flow map are all properties that are local in time.

Fix $u_0 \in H^s(\mathbb{T})$ and let $N \gg 1$ be such that

$$\sum_{|k| \geqslant N} \langle k \rangle^{2s} \left|\widehat{u_0}(k)\right|^2 \leqslant \frac{1}{T^2}.$$

Since $\langle k \rangle^s \left|\widehat{u_0}(k)\right|$ belongs to l^2, such values of N exist. Define

$$v_0(x) = \sum_{|k| \geqslant N} e^{ixk} \widehat{u_0}(k).$$

From the local well-posedness obtained in Theorem 3.1.1, there exists a unique $v \in X_T^s$

solution of the initial value problem

$$\begin{cases} u_t - u_{txx} + u_x + uu_x = 0, \quad x \in \mathbb{T} \\ u(x,0) = v_0(x) \in H^s(\mathbb{T}). \end{cases} \tag{3.12}$$

We split the initial data u_0 into two pieces, namely

$$u_0 = v_0 + w_0,$$

and we consider the following IVP (where v is now fixed)

$$\begin{cases} w_t - w_{txx} + w_x + ww_x + (vw)_x = 0, \quad x \in \mathbb{T} \\ w(x,0) = w_0(x) \in H^s(\mathbb{T}). \end{cases} \tag{3.13}$$

If there exists a solution w of (3.13) in X_T^s, then $v + w$ will be a solution of (3.1). Indeed, we would have

$$v_t + v_x + vv_x - v_{xxt} + w_t + w_x + ww_x - w_{xxt} + (vw)_x = 0$$
$$v_t + w_t + v_x + w_x + vv_x + ww_x + (vw)_x - v_{xxt} - w_{xxt} = 0,$$

that is,

$$\begin{cases} (v+w)_t + (v+w)_x + (v+w)(v+w)_x - (v+w)_{xxt} = 0 \\ (v+w)(x,0) = v_0(x) + w_0(x) = u_0(x). \end{cases}$$

Note that, as $u_0, v_0 \in H^s(\mathbb{T})$, with $s \geqslant 0$, then $(u_0 - v_0) = w_0$ is in $H^r(\mathbb{T}) \quad \forall r > 0$. In particular, $w_0 \in H^1(\mathbb{T})$. Proceeding as in (3.2), we obtain, for IVP (3.13)

$$\partial_t w = -\left(1 - \partial_x^2\right)^{-1} \partial_x \left(w + vw + \frac{w^2}{2}\right).$$

Recall that $A = -\left(1 - \partial_x^2\right)^{-1} \partial_x$ is skew-adjoint, so that, A generates a group of isometries $\{W(t)\}_{t\in\mathbb{R}} = \left\{e^{tA}\right\}_{t\in\mathbb{R}}$. So, putting (3.13) in its integral form

$$w(x,t) = e^{tA} w_0 + \int_0^t e^{(t-s)A} A \left(vw + \frac{w^2}{2}\right)(s) ds$$
$$w(x,t) = W(t) w_0 + \frac{1}{2} \int_0^t W(t-s) A \left(2vw + w^2\right)(s) ds =: \Phi(w).$$

By the same arguments used in the local well-posedness obtained in Theorem 3.1.1 and by Lemma 3.1.2 (with $r = 1, s = 0$), for any $w \in B_R(X_S^1)$

$$\begin{aligned}\|\Phi(w(x,t))\|_{H^1} &\leqslant \|w_0\|_{H^1} + \frac{1}{2}S\left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left(2vw+w^2\right)\right\|_{H^1}\\ &\leqslant \|w_0\|_{H^1} + \frac{1}{2}S\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(2vw)\right\|_{H^1} + \frac{1}{2}S\|\left(1-\partial_x^2\right)^{-1}\partial_x(w^2)\|_{H^1}\\ &\leqslant \|w_0\|_{H^1} + 2SS\left(\tilde{\tilde{C}}\|v\|_{H^0}\|w\|_{H^1}\right) + \frac{1}{2}S\tilde{C}\|w\|_{H^1}^2\\ &\leqslant \|w_0\|_{H^1} + CS\left(\|v\|_{H^0}\|w\|_{H^1} + \|w\|_{H^1}^2\right).\end{aligned}$$

Since $\|w_0\|_{H^1}, \|w\|_{H^1} \leqslant R$,

$$\|\Phi(w(x,t))\|_{H^1} \leqslant R + CS\left(\|v\|_{H^0}R + R^2\right) \leqslant CS\|v\|_{H^0}R.$$

Then,

$$\begin{aligned}\sup_{t\in[-S,S]}\|\Phi(w(x,t))\|_{H^1} &\leqslant \sup_{t\in[-S,S]} CS\|v\|_{H^0}\\ \|\Phi(w(x,t))\|_{X_S^1} &\leqslant CS\|v\|_{X_S^1}R.\end{aligned}$$

On the other hand, for $w_1, w_2 \in B_R(X_S^1)$

$$\begin{aligned}\|\Phi w_1 - \Phi w_2\|_{H^1} &\leqslant CS\left\|A\left(2vw_1 + w_1^2 - 2vw_2 - w_2^2\right)\right\|_{H^1}\\ &= CS\|A\left(2v\left(w_1-w_2\right) + \left(w_1-w_2\right)\left(w_1+w_2\right)\|_{H^1}\right)\\ &\leqslant CS\left(\|A\left(2v\left(w_1-w_2\right)\right)\|_{H^1} + \|A\left(w_1-w_2\right)\left(w_1+w_2\right)\|_{H^1}\right)\\ &\leqslant CS\left(\tilde{C}\|v\|_{H^0}\|w_1-w_2\|_{H^1} + \tilde{\tilde{C}}\|w_1-w_2\|_{H^1}\underbrace{\|w_1+w_2\|_{H^1}}_{\leqslant 2R}\right)\\ &\leqslant CS\left(\|v\|_{H^0} + 2R\right)\|w_1-w_2\|_{H^1}.\end{aligned}$$

Hence, we have,

$$\|\Phi w_1 - \Phi w_2\|_{x_S^1} \leqslant CS\left(\|v\|_{x_S^0} + 2R\right)\|w_1 - w_2\|_{x_S^1}.$$

Hence, Φ is a contraction, so Φ has a unique fixed point in X_S^1, such a point is our solution w in X_S^1 for small time S.

Multiplying (3.13) by w, we have

$$ww_t - ww_{xxt} + ww_x + w^2 w_x + w(vw)_x = 0.$$

Integrating over $\mathbb{T}$ and performing some integrations by parts, we obtain (note that w is periodic over the torus)

$$\begin{aligned}
&\int_{\mathbb{T}} ww_t dx - \int_{\mathbb{T}} ww_{xxt} dx + \int_{\mathbb{T}} ww_x dx + \int_{\mathbb{T}} w^2 w_x dx + \int_{\mathbb{T}} w\,(vw)_x\, dx = 0\\
&\frac{1}{2}\int_{\mathbb{T}} \frac{d}{dt} w^2\, dx + \int_{\mathbb{T}} w_x w_{xt} dx + \frac{1}{2}\int_{\mathbb{T}} \left(w^2\right)_x dx + \frac{1}{3}\int_{\mathbb{T}} \left(w^3\right)_x dx - \int_{\mathbb{T}} w_x v w\, dx = 0\\
&\frac{1}{2}\int_{\mathbb{T}} \frac{d}{dt} w^2\, dx + \frac{1}{2}\int_{\mathbb{T}} \frac{d}{dt} w_x^2\, dx - \int_{\mathbb{T}} w_x v w\, dx = 0\\
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} \left(w^2 + w_x^2\right)\, dx - \int_{\mathbb{T}} w_x v w\, dx = 0\\
&\frac{d}{dt}\int_{\mathbb{T}} \left(w^2 + w_x^2\right)\, dx = 2\int_{\mathbb{T}} w_x v w\, dx.
\end{aligned}$$

Then, we have

$$\frac{d}{dt}\|w(\cdot,t)\|_{H^1}^2 = 2\int_{\mathbb{T}} w_x v w\, dx.$$

By Hölder and Sobolev inequalities

$$\begin{aligned}
\left|\int_{\mathbb{T}} w_x v w\, dx\right| &\leqslant \|w_x(\cdot,t)\|_{L^2}\, \|v(\cdot,t)\|_{L^2} \|w(\cdot,t)\|_{L^\infty}\\
&\leqslant C\|v(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{H^1}^2.
\end{aligned}$$

That is,

$$\frac{d}{dt}\|w(\cdot,t)\|_{H^1}^2 \leqslant C\|v(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{H^1}^2.$$

Now, by Gronwal's inequality, for $0 \leqslant t \leqslant T$

$$\begin{aligned}
\|w(\cdot,t)\|_{H^1}^2 &\leqslant \|w(\cdot,0)\|_{H^1}^2\, e^{C\int_0^t \|v(\cdot,s)\|_{L^2} ds}\\
\|w(\cdot,t)\|_{H^1} &\leqslant \|w_0\|_{H^1}\, e^{C\int_0^t \|v(\cdot,s)\|_{L^2} ds}.
\end{aligned}$$

Therefore, we infer that w is bounded on the H^1-norm, on the interval $[-T, T]$ so, there exists solution w of (3.13) on this interval, so that, $(v + w)$ is a solution of (3.1) in X_T^s. This concludes the proof of Theorem 3.1.2. ∎

3.2 Analyticity in Time

Having established the well-posedness of the initial value problem (3.1), we now turn our attention to studying the properties of its solution. In this section, we will follow the reference (ROSIER; ZHANG, 2013) to show that the solution is analytic in time.

Proposition 3.2.1. *For $u_0 \in H^1(\mathbb{T})$, the solution $u(t)$ of the IVP (3.1) satisfies $u \in C^\omega(\mathbb{R}; H^1(\mathbb{T}))$. It means that, for each $t \in \mathbb{R}$, $u(t) \in H^1(\mathbb{T})$ such that*

$$u(t)(x) = u(x,t) = \sum_{n=0}^{\infty} t^n u_n(x),$$

where $(u_n(x))_{n\geqslant 0} \subset H^1(\mathbb{T})$.

Proof. Since the initial data $u_0 \in H^1(\mathbb{T})$, from Section 3.1, we have that $u \in C^1(\mathbb{R}; H^1(\mathbb{T}))$, so it is sufficient to check that for any $u_0 \in H^1(\mathbb{T})$ there are some numbers $b > 0, M > 0$, and some sequence $(u_n)_{n\geqslant 1}$ in $H^1(\mathbb{T})$ with

$$\|u_n\|_{H^1} \leqslant \frac{M}{b^n}, \quad n \geqslant 0, \tag{3.14}$$

such that

$$u(t) = \sum_{n\geqslant 0} t^n u_n, \quad t \in (-b,b). \tag{3.15}$$

Note that, from (3.15), the convergence ratio of the series is $(-b,b)$, then it converges uniformly in each compact subset within $(-b,b)$. That is, the series in (3.15) holds in $H^1(\mathbb{T})$ uniformly on $[-rb, rb]$ for each $r < 1$. Actually, we prove that u can be extended as an analytic function from $D_b := \{z \in \mathbb{C}; |z| < b\}$ to the space $H^1_{\mathbb{C}}(\mathbb{T}) := H^1(\mathbb{T};\mathbb{C})$, endowed with the Euclidean norm

$$\left\| \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} \right\|_{H^1} = \left(\sum_{k\in\mathbb{Z}} \left(1 + |k|^2\right) |\hat{u}_k|^2 \right)^{\frac{1}{2}}.$$

This proof is an adaptation of the classical proof of the analyticity of the flow for an ODE with an analytic vector field to our infinite dimensional framework. For $u \in H^1_{\mathbb{C}}(\mathbb{T})$, let $Au = -\left(1 - \partial_x^2\right)^{-1} \partial_x u$ and $f(u) = A\left(u + u^2\right)$. That is, our aim is to see (3.1) as the following ODE

$$\begin{cases} u_t = f(u) \\ u(0) = u_0 \in H^1(\mathbb{T}) \end{cases}$$

Since $|k| \leqslant (k^2+1)/2$ for all $k \in \mathbb{Z}$, we have $\|A\|_{\mathcal{L}\left(H^1_{\mathbb{C}}(\mathbb{T})\right)} \leqslant 1/2$. Indeed

$$\begin{aligned}
\|A\|_{\mathcal{L}\left(H^1_{\mathbb{C}}(\mathbb{T})\right)} &= \sup_{\|u\|\leqslant 1} \|Au\|_{H^1_{\mathbb{C}}(\mathbb{T})} = \sup_{\|u\|\leqslant 1} \left(\sum_{k\in\mathbb{Z}} \left(1+k^2\right) |\widehat{Au}(k)|^2 \right)^{1/2} \\
&= \sup_{\|u\|\leqslant 1} \left(\sum_{k\in\mathbb{Z}} \left(1+k^2\right) \left| \frac{(-ik)}{1+k^2} \hat{u}(k) \right|^2 \right)^{1/2} \\
&= \sup_{\|u\|\leqslant 1} \left(\sum_{k\in\mathbb{Z}} \left(1+k^2\right) \frac{k^2}{(1+k^2)^2} |\hat{u}(k)|^2 \right)^{1/2} \\
&= \sup_{\|u\|\leqslant 1} \left(\sum_{k\in\mathbb{Z}} \left(1+k^2\right) \frac{1}{4} |\hat{u}(k)|^2 \right)^{1/2} \\
&= \frac{1}{2} \sup_{\|u\|\leqslant 1} \left(\sum_{k\in\mathbb{Z}} \left(1+k^2\right) |\hat{u}(k)|^2 \right)^{1/2} \\
&= \frac{1}{2} \sup_{\|u\|\leqslant 1} \|u\|_{H^1(\mathbb{T})} \\
&\leqslant \frac{1}{2}.
\end{aligned}$$

Pick a positive constant C_1 such that

$$\left\|u^2\right\|_{H^1} \leqslant C_1 \|u\|^2_{H^1}, \qquad \text{for all } u \in H^1_{\mathbb{C}}(\mathbb{T}).$$

We define by induction on q a sequence (u^q) of analytic functions from $\mathbb{C}$ to $H^1_{\mathbb{C}}(\mathbb{T})$ which will converge uniformly on D_T, for $T > 0$ small enough, to a solution of the integral equation

$$u(z) = u_0 + \int_{[0,z]} f(u(\zeta))d\zeta = u_0 + \int_0^1 f(u(sz))zds.$$

Let

$$\begin{aligned}
&u^0(z) = u_0, \qquad \text{for } z \in \mathbb{C}, \\
&u^{q+1}(z) = u_0 + \int_{[0,z]} f\left(u^q(\zeta)\right) d\zeta, \qquad \text{for } q \geqslant 0, z \in \mathbb{C}.
\end{aligned}$$

Claim 3.2.1. *For each $q \geqslant 0$, u^q is analytic*

$$u^q(z) = \sum_{n \geqslant 0} z^n v_n^q \quad \forall z \in \mathbb{C},$$

with (v_n^q) some sequence in $H^1_{\mathbb{C}}(\mathbb{T})$ such that

$$\|v_n^q\|_{H^1} \leqslant \frac{M(q,b)}{b^n} \quad \text{for all } q, n \in \mathbb{N}, b > 0.$$

Proof. We proof Claim 3.2.1 by induction on $q \geqslant 0$. If $q = 0$ the result is clear, for

$$u^q = u_0, \quad M(0,b) = \|u_0\|_{H^1}, \quad v_0^0 = u_0 \quad v_n^0 = 0, \forall n \geqslant 1 \quad \therefore \quad u^0(z) = v_0^0 + \sum_{n \geqslant 1} z^n v_n^0$$

Assume know that Claim 3.2.1 is proved for some $q \geqslant 0$. Then, for any $r \in (0,1)$ and any $b > 0$,

$$\|z^n v_n^q\|_{H^1} \leqslant M(q,b) r^n \quad \text{for } |z| \leqslant rb.$$

So that the series $\sum_{n \geqslant 0} z^n v_n^q$ converges absolutely in $H^1_{\mathbb{C}}(\mathbb{T})$ uniformly for $z \in \overline{D_{rb}}$, since $\overline{D_{rb}}$ is compact within its convergence disk D_b. The same holds for the series $\sum_{n \geqslant 0} z^n \left(\sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q \right)$. It follows that

$$\begin{aligned} f\left(u^q(\zeta)\right) &= A\left(u^q(\zeta) + u^{2q}(\zeta)\right) \\ &= A\left(\sum_{n \geqslant 0} \zeta^n v_n^q + \sum_{n \geqslant 0} \zeta^n v_n^{2q} \right) \\ &= A\left(\sum_{n \geqslant 0} \zeta^n v_n^q + \sum_{n \geqslant 0} \zeta^n \left(\sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q \right) \right), \end{aligned}$$

converges uniformly for $\zeta \in \overline{D_{rb}}$. Thus

$$\begin{aligned} u^{q+1}(z) =& u_0 + \int_{[0,z]} f\left(u^q(\zeta)\right) d\zeta = u_0 + \int_{[0,z]} A\left(\sum_{n \geqslant 0} \zeta^n v_n^q + \sum_{n \geqslant 0} \zeta^n \left(\sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q \right) \right) d\zeta \\ &= u_0 + \int_{[0,z]} \left(\sum_{n \geqslant 0} \zeta^n A\left(v_n^q\right) + \sum_{n \geqslant 0} \zeta^n A\left(\sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q \right) \right) d\zeta \\ &= u_0 + \int_{[0,z]} \sum_{n \geqslant 0} \zeta^n \left(A\left(v_n^q\right) + A\left(\sum_{0 \leqslant l \leqslant n} v_l^q v_{n-l}^q \right) \right) d\zeta \end{aligned}$$

$$
\begin{aligned}
&= u_0 + \int_{[0,z]} \sum_{n\geqslant 0} \zeta^n A \left(v_n^q + \sum_{0\leqslant l\leqslant n} v_l^q v_{n-l}^q \right) d\zeta \\
&= u_0 + \sum_{n\geqslant 0} \frac{z^{n+1}}{n+1} A \left(v_n^q + \sum_{0\leqslant l\leqslant n} v_l^q v_{n-l}^q \right) \\
&= u_0 + \sum_{n\geqslant 1} \frac{z^n}{n} A \left(v_{n-1}^q + \sum_{0\leqslant l\leqslant n-1} v_l^q v_{n-1-l}^q \right) \\
&= \sum_{n\geqslant 0} z^n v_n^{q+1},
\end{aligned}
$$

where

$$
\begin{aligned}
v_0^{q+1} &= u_0 \\
v_n^{q+1} &= \frac{1}{n} A \left(v_{n-1}^q + \sum_{0\leqslant l\leqslant n-1} v_l^q v_{n-1-l}^q \right) \quad \text{for } n \geqslant 1.
\end{aligned}
$$

It follows that for $n \geqslant 1$

$$
\begin{aligned}
\left\| v_n^{q+1} \right\|_{H^1} &\leqslant \frac{\|A\|}{n} \left(\frac{M(q,b)}{b^{n-1}} + nC_1 \|v^q\|_{H^1_{\mathbb{C}}}^2 \right) \\
&\leqslant \frac{\|A\|}{n} \left(\frac{M(q,b)}{b^{n-1}} + nC_1 \frac{M^2(q,b)}{b^{n-1}} \right) \\
&\leqslant \frac{M(q+1,b)}{b^n},
\end{aligned}
$$

with

$$
M(q+1,b) := \sup \left\{ \|u_0\|_{H^1}, b\|A\| \left(M(q,b) + C_1 M^2(q,b) \right) \right\}.
$$

Claim 3.2.1 is proved.

■

Claim 3.2.2. *Let*

$$
T := \frac{1}{(2\|A\| \left(1 + 4C_1 \|u_0\|_{H^1}\right))}.
$$

Then

$$
\|u^q - u\|_{L^\infty\left(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T})\right)} \to 0 \textit{ as } q \to \infty \textit{ for some } u \in C\left(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T})\right).
$$

Proof. Let $Z_T = C\left(\overline{D_T}; H^1_{\mathbb{C}}(\mathbb{T})\right)$ be endowed with the norm $|||v||| = \sup_{|z|\leqslant T} \|v(z)\|_{H^1}$. Let $R > 0$, and for $v \in B_R := \{v \in Z_T; |||v||| \leqslant R\}$, let

$$
(\Gamma v)(z) = u_0 + \int_{[0,z]} f(v(\zeta)) d\zeta = u_0 + \int_{[0,z]} A\left(v(\zeta) + v^2(\zeta)\right) d\zeta.
$$

Then

$$\begin{aligned}\|\!|\Gamma v|\!\| &\leqslant \|u_0\|_{H^1} + T\|A\| \left(|\!\|v|\!\| + C_1 |\!\|v|\!\|^2\right)\\ &\leqslant \|u_0\|_{H^1} + T\|A\| \left(R + C_1 R^2\right)\end{aligned}$$

and,

$$\begin{aligned}|\!\|\Gamma v_1 - \Gamma v_2|\!\| &\leqslant T\|A\| \left(|\!\|v_1 - v_2|\!\| + \left|\!\left\|v_1^2 - v_2^2\right|\!\right\|\right)\\ &\leqslant T\|A\| \left(|\!\|v_1 - v_2|\!\| + |\!\|v_1 + v_2|\!\| \, |\!\|v_1 - v_2|\!\|\right)\\ &\leqslant T\|A\|(1 + 2RC_1) \, |\!\|v_1 - v_2|\!\| .\end{aligned}$$

Pick $R = 2\|u_0\|_{H^1}$ and $T = (2\|A\| (1 + 2C_1 R))^{-1}$. Then Γ contracts in B_R. The sequence (u^q), which is given by Picard iteration scheme, has a limit u in Z_T which fulfills

$$u(z) = u_0 + \int_{[0,z]} f(u(\zeta)) d\zeta, \quad |z| \leqslant T.$$

In particular, $u \in C^1([-T,T]; H^1(\mathbb{T}))$ (the $u^q(z)$ being real-valued for $z \in \mathbb{R}$) and it satisfies $u_t = f(u)$ on $[-T,T]$ together with $u(0) = u_0$; that is, u solves (3.1) in the class $C^1([-T,T]; H^1(\mathbb{T})) \subset X_T^1$. ■

Claim 3.2.3. *$u(z) = \sum_{n\geqslant 0} z^n v_n$ for $|z| < T$, where $v_n = \lim_{q\to\infty} v_n^q$ for each $n \geqslant 0$.*

Proof. From Claim 3.2.1, we infer that for all $n \geqslant 1$

$$v_n^q = \frac{1}{2\pi i} \int_{|z|=T} z^{-n-1} u^q(z) dz,$$

hence

$$\|v_n^p - v_n^q\|_{H^1} \leqslant T^{-n} \, |\!\|u^p - u^q|\!\| .$$

From Claim 3.2.2, we infer that (v_n^q) is a Cauchy sequence in $H^1_{\mathbb{C}}(\mathbb{T})$. Let v_n denote its limit in $H^1_{\mathbb{C}}(\mathbb{T})$. Note that

$$\|v_n - v_n^q\|_{H^1} \leqslant T^{-n} \, |\!\|u - u^q|\!\| ,$$

and hence the series $\sum_{n\geqslant 0} z^n v_n$ is convergent for $|z| < T$. Therefore, for $|z| \leqslant rT$ with $r < 1$,

$$\left\| \sum_{n\geqslant 0} z^n \left(v_n - v_n^q\right) \right\|_{H^1} \leqslant (1 - r)^{-1} \, |\!\|u - u^q|\!\| ,$$

and hence $u^q(z) = \sum_{n\geqslant 0} z^n v_n^q \to \sum_{n\geqslant 0} z^n v_n$ in Z_{rT} as $q \to \infty$. It follows that

$$u(z) = \sum_{n\geqslant 0} z^n v_n \quad \text{for } |z| < T,$$

which proves Claim 3.2.3. ■

Therefore, the proof of Proposition 3.2.1 is complete. ■

3.3 Conservation Laws and Invariants of Motion

To conclude this chapter, we present the three conservation laws and the so-called invariants of motion for the BBM equation. These laws were initially discovered in 1972 by Benjamin, Bona, and Mahony (BENJAMIN; BONA; MAHONY, 1972). However, it was not until 1979 that Peter Olvner (OLVER, 1979) proved that these three conservation laws are the only non-trivial, independent ones that the BBM equation possesses. These laws are the equivalents of the conservation of mass, momentum and energy in fluid mechanics (HAMDI et al., 2004). It is worth mentioning, given the historical linkage between the BBM and KdV equations, that in contrast to the BBM, the KdV equation possesses an infinite number of independent conservation laws (MIURA, 1976). The definitions and the theorem of this section follow (OLVER, 1979), while the Proposition 3.3.1 follows (ROSIER; ZHANG, 2013).

Definition 3.3.1 (Conservation Law)**.** *Given a general partial differential equation $F(x, t, u, u_x, u_t) = 0$ involving two independent variables x, t and one dependent variable u, a conservation law is an equation of the form*

$$T_t + X_x = 0, \tag{3.16}$$

wich is satisfied for all the solutions of the equation $F = 0$. The quantity $T = T(x, t, u, u_x, u_t)$ is called the conserved density and the $X = X(x, t, u, u_x, u_t)$ is called the conserved flux.

The conservation law (3.16) is trivilally satisfied for some G such that $T = G_x$ and $X = -G_t$. Let $T_1, \ldots, T_n$ be densities for n different conservation laws. We call these laws *dependent if there exist constants* $c_1 \dots c_n$ *such that*

$$c_1 T_1 + \cdots + c_n T_n = G_x,$$

for some G; otherwise, we call the laws independent.

Definition 3.3.2 (Invariants of Motion)**.** *For any conservation law* (3.16)*, the quantity* $\int_{-\infty}^{\infty} T dx$*, for solutions such that the integral converges, is called Invariant of motion, or constant of motion, i.e. independent of time.*

Before presenting the results of this section, we note that, if we replace u by $(-u-1)$ in the BBM equation, we have

$$\begin{aligned}
&(-u-1)_t + (-u-1)_x + (-u-1)(-u-1)_x - (-u-1)_{txx} = 0 \\
&-u_t - u_x - (u+1)(-u)_x + u_{txx} = 0 \\
&-u_t - u_x + uu_x + u_x + u_{txx} = 0 \\
&u_t - u_{txx} - uu_x = 0.
\end{aligned}$$

That is, the conservation laws of the BBM equation are in one-to-one correspondence, under the above transformation, with the following somewhat simpler equation

$$u_t - u_{txx} = uu_x. \tag{3.17}$$

Theorem 3.3.1. *The only non-trivial, independent conservation laws of* (3.17) *in which* $T(x, u, u_x, u_{xx}, \dots)$ *depends smoothly on* x, u *and the various spatial derivatives of* u *are*

$$u_t - \left(u_{xt} + \frac{1}{2}u^2\right)_x = 0, \tag{3.18}$$

$$\left(\frac{1}{2}u^2 + \frac{1}{2}u_x^2\right)_t - \left(uu_{xt} + \frac{1}{3}u^3\right)_x = 0, \tag{3.19}$$

$$\left(\frac{1}{3}u^3\right)_t + \left(u_t^2 - u_{xt}^2 - u^2 u_{xt} - \frac{1}{4}u^4\right)_x = 0. \tag{3.20}$$

Proof. The proof is based on straightforward calculations. For (3.18), assuming (3.17), we have

$$\begin{aligned} u_t - u_{txx} &= uu_x \\ u_t - (u_{txx} + uu_x) &= 0 \\ u_t - \left(u_{xt} + \frac{1}{2}u^2\right)_x &= 0. \end{aligned}$$

For (3.19), we multiply $u_t - u_{xxt} - uu_x = 0$ by u, then we have

$$\begin{aligned} &u\left(u_t - u_{xxt} - uu_x\right) = 0 \\ &uu_t - uu_{xxt} - u^2u_x = 0 \\ &uu_t + u_xu_{xt} - \left(u_xu_{xt} + uu_{xxt} + u^2u_x\right) = 0 \\ &\left(\frac{1}{2}u^2 + \frac{1}{2}u_x^2\right)_t - \left(uu_{xt} + \frac{1}{3}u^3\right)_x = 0. \end{aligned}$$

Similarly, for the third conservation law (3.20), we multiply $u_t - u_{xxt} - uu_x = 0$ by $u^2 + 2u_{xt}$, which gives

$$\begin{aligned} &\left(u_t - u_{xxt} - uu_x\right)\left(u^2 + 2u_{xt}\right) = 0 \\ &u^2\left(u_t - u_{xxt} - uu_x\right) + 2u_{xt}\left(u_t - u_{xxt} - uu_x\right) = 0 \\ &u^2u_t - u^2u_{xxt} - u^3u_x + u_{xt}\left(2u_t - 2u_{xxt} - 2uu_x\right) = 0 \\ &u^2u_t + 2u_tu_{tx} - 2u_{xt}u_{xxt} - 2uu_xu_{xt} - u^2u_{xxt} - u^3u_x = 0 \\ &u^2u_t + 2u_tu_{tx} - 2u_{xt}u_{xxt} - \left(2uu_xu_{xt} + u^2u_{xxt}\right) - u^3u_x \\ &\left(\frac{1}{3}u^3\right)_t + \left(u_t^2 - u_{xt}^2 - u^2u_{xt} - \frac{1}{4}u^4\right)_x = 0, \end{aligned}$$

and concludes the proof. ■

Proposition 3.3.1 (Invariants of Motion)**.** *For $u_0 \in H^1(\mathbb{T})$, the solution $u(t)$ of the IVP* (3.1) *is such that the three integral terms*

$$\int_{\mathbb{T}} u\,dx \quad (1) \qquad \int_{\mathbb{T}} (u^2 + u_x^2)dx \quad (2) \qquad \int_{\mathbb{T}} (u^3 + 3u^2)dx \quad (3)$$

are invariants of motion (i.e., they remain constant over time).

The invariant of motion (1) corresponds to the conservation of mass; the invariant of motion (2) represents the conservation of energy and the H^1-norm; and, hence, the invariant (3) represents the conservation of momentum.

Proof. For $u_0 \in H^1(\mathbb{T})$, from the well-posedness result, we know that there exists $u \in X_T^1, \forall\, T > 0$, hence

$$u_t = -\left(1-\partial_x^2\right)^{-1}\partial_x\left(u+\frac{u^2}{2}\right) = A\left(u+\frac{u^2}{2}\right) \in X_T^2, \quad \text{since} \quad A \in \mathcal{L}\left(H^s(\mathbb{T}), H^{s+1}(\mathbb{T})\right).$$

So, we have that all terms in the BBM equation $u_t - u_{txx} + u_x + uu_x = 0$ belongs to X_T^0. For the integral term (1), we take the integral of the BBM equation over $\mathbb{T}$ and obtain

$$\begin{aligned}
&\int_{\mathbb{T}} u_t\, dx - \int_{\mathbb{T}} u_{txx}\, dx + \int_{\mathbb{T}} u_x\, dx + \int_{\mathbb{T}} uu_x\, dx = 0\\
&\frac{d}{dt}\int_{\mathbb{T}} u\, dx - \underbrace{\int_{\mathbb{T}} (u_{tx})_x\, dx}_{(*)=0} + \underbrace{\int_{\mathbb{T}} u_x\, dx}_{(*)=0} + \frac{1}{2}\underbrace{\int_{\mathbb{T}} (u^2)_x\, dx}_{(*)=0} = 0.
\end{aligned}$$

The terms (*) are zero since the function u is periodic on the torus $\mathbb{T}$. From this we obtain $\frac{d}{dt}\int_{\mathbb{T}} u\, dx = 0$, that is, $\int_{\mathbb{T}} u(t,x)\, dx$ is cosntant in t.

For the invariant of motion (2), we multiply the BBM equation by u and proceed like the first one, integrating over $\mathbb{T}$ and noting the periodicity of u, but, for this time, we do a few manipulations, like using the chain rule and integration by parts. So we have

$$\begin{aligned}
&\int_{\mathbb{T}} uu_t\, dx - \int_{\mathbb{T}} uu_{txx}\, dx + \int_{\mathbb{T}} uu_x\, dx + \int_{\mathbb{T}} u^2u_x\, dx = 0\\
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} u^2\, dx + \int_{\mathbb{T}} u_x u_{tx}\, dx + \frac{1}{2}\underbrace{\int_{\mathbb{T}} (u^2)_x\, dx}_{(*)=0} + \frac{1}{3}\underbrace{\int_{\mathbb{T}} (u^3)_x\, dx}_{(*)=0} = 0\\
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} u^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} (u_x)^2\, dx = 0\\
&\frac{d}{dt}\int_{\mathbb{T}} \left(u^2 + u_x^2\right) dx = 0.
\end{aligned}$$

That is,

$$\int_{\mathbb{T}} u^2(x,t) + u_x^2(x,t) dx,$$

doesn't depend on t.

For the third invariant of motion, we replace u by $(-u-1)$ in the third conservation law (3.20)

$$\begin{aligned}
&\left(\frac{1}{3}(-u-1)^3\right)_t + \left((-u-1)_t^2 - (-u-1)_{xt}^2 - (-u-1)^2(-u-1)_{xt} - \frac{1}{4}(-u-1)^4\right)_x \\
&-\left(\frac{1}{3}(u+1)^3\right) + \left((-u_t)^2 - u_{xt}^2 + (u+1)^2 u_{xt} - \frac{1}{4}(u+1)^4\right)_x = 0 \\
&\left(\frac{1}{3}(u+1)^3\right)_t - \left(u_t^2 - u_{xt}^2 + (u+1)^2 u_{xt} - \frac{1}{4}(u+1)^4\right)_x = 0.
\end{aligned}$$

Then, we integrate the last expression over $\mathbb{T}$, we use the periodicity of u, which yields

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\frac{1}{3}(u+1)^3\right)_t dx - \int_{\mathbb{T}} \left(u_t^2 - u_{xt}^2 + (u+1)^2 u_{xt} - \frac{1}{4}(u+1)^4\right)_x dx = 0 \\
&\frac{d}{dt}\int_{\mathbb{T}} (u+1)^3\, dx = 0 \\
&\frac{d}{dt}\int_{\mathbb{T}} (u^3 + 3u^2 + 3u + 1)\, dx = 0.
\end{aligned}$$

From the first invariant of moviment, we know that $\int_{\mathbb{T}} u\, dx = 0$, then, we obtain that

$$\frac{d}{dt}\int_{\mathbb{T}} (u^3 + 3u^2) dx = 0.$$

Which implies the third constant integral term over time

$$\int_{\mathbb{T}} (u^3 + 3u^2) dx,$$

and concludes the proof. ∎

4 UNIQUE CONTINUATION PROPERTY

This chaper is devoted to prove the Unique Continuation Property (UCP) for the BBM equation, for some BBM-like equations and for an intermidiate equation between BBM and KdV, the KdV-BBM equation. By Unique Continuation we mean that if a solution vanishes on a subset of its domain, it is actually identically zero on the entire domain. This is an important topic in the theory of partial differential equations, with its history dating back to works of Carleman and Holmgren in the early twentieth century. Initially, most results were related to local unique continuation, however, due to its applications in control theory of PDEs, which is our main interest in this work, attention was also given to global unique continuation.

4.1 Unique Continuation Property for BBM Equation

This section derives a UCP for the solutions of the BBM equation issuing from small enough initial data in $H^1(\mathbb{T})$ with nonnegative mean values. The proof combines the analyticity in time of solutions of BBM, according to Proposition 3.2.1, the three invariants of motion presented in Proposition 3.3.1 and an appropriate Lyapunov function.

Theorem 4.1.1. *Let $u_0 \in H^1(\mathbb{T})$ be such that*

$$\int_{\mathbb{T}} u_0(x)dx \geqslant 0, \tag{4.1}$$

and

$$\|u_0\|_{L^\infty(\mathbb{T})} < 3. \tag{4.2}$$

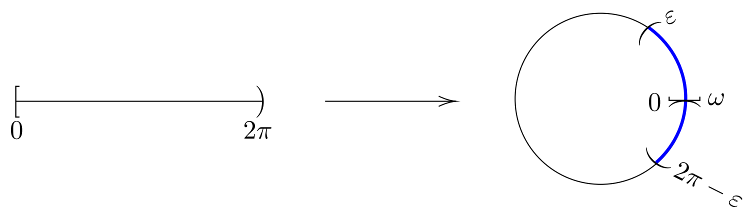
Assume that the solution u of the IVP (3.1) *satisfies*

$$u(x,t) = 0 \quad \text{for all } (x,t) \in \omega \times (0,T), \tag{4.3}$$

where $\omega \subset \mathbb{T}$ is a nonempty open set and $T > 0$. Then $u_0 = 0$, and hence $u \equiv 0$.

Proof. Using a system of coordinates in such a way that the one-dimensional torus $\mathbb{T}$ is identifying with the interval $[0, 2\pi)$ and that $\omega \supset [0, \varepsilon) \cup (2\pi - \varepsilon, 2\pi)$ for some $\varepsilon > 0$. Note that we can do this by placing ω such that it contains the origin of our coordinate system according to Figure 4.1 below,

Figure 4.1 – Coordinate system



Source: Author

We know, thanks to Proposition 3.2.1, that $u \in C^{\omega}(\mathbb{R}; H^1(\mathbb{T}))$, then we have $u(x, \cdot) \in C^{\omega}(\mathbb{R})$ for all $x \in \mathbb{T}$. (4.3) implies that[1]

$$u(x,t) = 0 \quad \text{ for } (x,t) \in \omega \times \mathbb{R}. \tag{4.4}$$

Introduce the function

$$v(x,t) = \int_0^x u(y,t)dy.$$

Since $u \in C^{\omega}(\mathbb{R}; H^1(\mathbb{T}))$, then $v \in C^{\omega}(\mathbb{R}; H^2(0, 2\pi))$ and v satisfies

$$v_t - v_{txx} + v_x + \frac{u^2}{2} = 0, \quad x \in (0, 2\pi). \tag{4.5}$$

Indeed, integrating the BBM $u_t - u_{txx} + u_x + \left(\frac{u^2}{2}\right)_x = 0$ over $(0, x)$, and noting that $v_x = u(x,t)$, and that $v_{xx} = u_x(x,t)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^x u(y,t)\,dy - \frac{d}{dt}\int_0^x u_{xx}(y,t)dy + \int_0^x u_x(y,t)dy + \int_0^x \left(\frac{u^2}{2}\right)_x dy = 0 \\
&\frac{d}{dt}v(x,t) - \frac{d}{dt}u_x(x,t) + u(x,t) + \frac{u^2}{2}(x,t) = 0 \\
&v_t - v_{txx} + v_x(x,t) + \frac{u^2}{2} = 0.
\end{aligned}$$

Denote

$$I(t) = \int_0^{2\pi} v(x,t)dx \quad \in C^{\omega}(\mathbb{R}).$$

[1]Analytic functions can be uniquely determined by their values on any open subset of its domain. So we have that if u is zero in a subregion of its domain, then u is identically zero in the whole domain. Note that it allows us to uniquely extend analytic functions. It is commonly called analytic continuation for holomorphic functions.

Integrating (4.5) over $(0, 2\pi)$, we have

$$\begin{aligned}
&\int_0^{2\pi} v_t(x,t)dx - \int_0^{2\pi} v_{txx}(x,t)dx + \int_0^{2\pi} v_x(x,t)dx + \int_0^{2\pi} \frac{u^2}{2}(x,t)dx = 0\\
&\frac{d}{dt}\int_0^{2\pi} v(x,t)dx - \frac{d}{dt}\int_0^{2\pi} v_{xx}(x,t)dx + v(2\pi,t) - v(0,t) + \frac{1}{2}\int_0^{2\pi} u^2(x,t)dx = 0\\
&I_t - \frac{d}{dt}\left[v_x(2\pi,t) - v_x(0,t)\right] + v(2\pi,t) + \frac{1}{2}\int_0^{2\pi} |u(x,t)|^2 dx = 0\\
&I_t - \frac{d}{dt}[u(2\pi,t) - u(0,t)] + \int_0^{2\pi} u(x,t)dx + \frac{1}{2}\int_0^{2\pi} |u(x,t)|^2 dx = 0\\
&I_t = -\int_0^{2\pi} u(x,t)dx - \frac{1}{2}\int_0^{2\pi} |u(x,t)|^2 dx.
\end{aligned}$$

From the invariant of motion $\int_{\mathbb{T}} u(x,t)dx$, assuming $t = 0$, the above last line can be written as

$$I_t = -\int_0^{2\pi} u_0(x)dx - \frac{1}{2}\int_0^{2\pi} |u(x,t)|^2 dx. \tag{4.6}$$

Using the assumption of non-negativity of the mean value (4.1), we obtain that the quantity in (4.6) is not greater than zero.

Now using the invariant of motion $\int_{\mathbb{T}} (u^2 + u_x^2)\, dx = \|u\|_{H^1}$ we get, by setting $t = 0$, that $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ for all $t \in \mathbb{R}$. Consequently, $v \in L^\infty(\mathbb{R}, H^2(0, 2\pi))$ and $I \in L^\infty(\mathbb{R})$. In fact, first noting that

$$\begin{aligned}
|v(x,t)| &\leqslant \int_0^x |u(y,t)|dy\\
&\leqslant \left(\int_0^x 1^2 dy\right)^{1/2} \left(\int_0^x |u(y,t)|^2 dy\right)^{1/2}\\
&\leqslant \sqrt{2\pi}\|u(\cdot,t)\|_{L^2(0,2\pi)}\\
&\leqslant \sqrt{2\pi}\|u(\cdot,t)\|_{H^1(0,2\pi)} = \sqrt{2\pi}\,\|u_0\|_{H^1(0,2\pi)}\,.
\end{aligned}$$

Then, for v, we have

$$\begin{aligned}
\|v(\cdot,t)\|_{H^2(0,2\pi)} &= \int_0^{2\pi} \left(|v(x,t)|^2 + |v_x(x,t)|^2 + |v_{xx}(x,t)|^2\right) dx\\
&= \int_0^{2\pi} |v(x,t)|^2 dx + \int_0^{2\pi} \left(|u(x,t)|^2 + |u_x(x,t)|^2\right) dx\\
&\leqslant \int_0^{2\pi} 2\pi\|u_0\|_{H^1}^2 dx + \|u(t)\|_{H^1(0,2\pi)} < \infty.
\end{aligned}$$

And, for I

$$|I(t)| \leqslant \int_0^{2\pi} |v(x,t)|dx \leqslant \int_0^{2\pi} \sqrt{2\pi}\, \|u_0\|_{t1}\, dx < \infty$$

From the boundedness and monotonicity of the function I, due to (4.6), follows that it has a finite limit as $t \to \infty$, that we denote by l. Analogously, since $\|u(t_n)\|_{H^1(\mathbb{T})} \leqslant \|u_0\|_{H^1(\mathbb{T})}$ and of the reflexivity of the Hilbert spaces, we can consider a sequence $(u(t_n))_{n\in\mathbb{N}} \subset H^1(\mathbb{T})$ with $t_n \in \mathbb{R}, \forall n \in \mathbb{N}$, such that

$$u(t_n) \rightharpoonup \tilde{u}_0 \quad \text{in } H^1(\mathbb{T}), \tag{4.7}$$

for some $\tilde{u}_0 \in H^1(\mathbb{T})$.

Letting $\tilde{u}_0$ be the initial data of the IVP for BBM and denoting $\tilde{u}$ its solution, i.e., $\tilde{u}$ solves

$$\begin{aligned} &\tilde{u}_t - \tilde{u}_{txx} + \tilde{u}_x + \tilde{u}\tilde{u}_x = 0, \quad x \in \mathbb{T}, t \in \mathbb{R}, \\ &\tilde{u}(x,0) = \tilde{u}_0(x). \end{aligned}$$

Pick any $s \in (1/2, 1)$. Since $H^1(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ is a compact embedding, we have that $u(t_n) \to \tilde{u}_0$ strongly in $H^s(\mathbb{T})$. From the well-posedness we obtain

$$u(t_n + \cdot) \to \tilde{u} \quad \text{in } C([0,1]; H^s(\mathbb{T})). \tag{4.8}$$

From (4.3) and since $\tilde{u}$ belongs to $C^\omega(\mathbb{R}, H^1(\mathbb{T}))$, follows that

$$\tilde{u}(x,t) = 0 \quad \text{for } (x,t) \in \omega \times \mathbb{R}.$$

Since $\int_0^{2\pi} u(x,t)dx$ does not depend on t, considering $t = 0$, we obtain $\int_0^{2\pi} u(x,t_n)dx = \int_0^{2\pi} u_0(x)dx, \forall t_n$. And, from the weak convergence $u(t_n) \rightharpoonup \tilde{u}_0$ in $H^1(\mathbb{T})$ we obtain, as $t_n \to \infty$,

$$\int_0^{2\pi} u(x,t_n)\, dx = \int_0^{2\pi} \tilde{u}_0(x)dx = \int_0^{2\pi} u_0(x)dx.$$

As before, we define $\tilde{v}(x,t) = \int_0^x \tilde{u}(y,t)dy$ and $\tilde{I}(t) = \int_0^{2\pi} \tilde{v}(x,t)dx$. Using a procedure analogous to the previous one, we obtain

$$\tilde{I}_t = \int_0^{2\pi} u_0(x)dx - \frac{1}{2}\int_0^{2\pi} |\tilde{u}(x,t)|^2 dx \leqslant 0. \tag{4.9}$$

However, considering $t_n \to \infty$, and using that $u(t_n + \cdot) \to \tilde{u}$ in $C([0,1]; H^s(\mathbb{T}))$, we have

$$I(t_n) \xrightarrow{n\to\infty} \tilde{I}(0).$$

Indeed, for $n \geqslant n_0$ such that, $\|u(y,t_n) - \tilde{u}_0(y)\|_{L^2(0,2\pi)} < \varepsilon/(2\pi)^{3/2}$, for a given $\varepsilon > 0$, it follows

$$\begin{aligned}
\left|I(t_n) - \tilde{I}(0)\right| &= \left|\int_0^{2\pi}\int_0^x u(y,t_n) - \tilde{u}_0(y)dydx\right| \\
&\leqslant \int_0^{2\pi}\int_0^x |u(y,t_n) - \tilde{u}_0(y)|\, dydx \\
&\leqslant \sqrt{2\pi}\int_0^{2\pi}\left(\int_0^{2\pi} |u(y,t_n) - \tilde{u}_0(y)|^2\, dy\right)^{1/2} dx < \varepsilon.
\end{aligned}$$

Analogously,

$$I(t_n + 1) \xrightarrow{n\to\infty} \tilde{I}(1).$$

On the other hand,

$$\lim_{n\to\infty} I(t_n) = \lim_{n\to\infty} I(t_n + 1) = I.$$

So, we have that $\tilde{I}(0) = \tilde{I}(1)$. But we know that $\tilde{I}_t \leqslant 0$, that is, $\tilde{I}(t)$ is a non-increasing function from 0 to 1, with $\tilde{I}(0) = \tilde{I}(1)$, what is only possible if $\tilde{I}(t) = 0$, for all $t \in [0,1]$. So, we conclude that

$$\tilde{u}(x,t) = 0 \quad (x,t) \in \mathbb{T} \times [0,1].$$

But, $\tilde{u}$ is the solution of the IVP with initial data $\tilde{u}_0$; then, $\tilde{u}_0 = 0$. But $u(t_n) \rightharpoonup \tilde{u}_0 = 0$, so, $u(t_n) \rightharpoonup 0$ what implies that

$$\int_0^{2\pi} \left(u^3(x,t_n) + 3u^2(x,t_u)\right) dx \longrightarrow 0 \text{ as } n \to \infty.$$

Knowing that $\int_0^{2\pi} (u^3 + 3u^2)\, dx$ is time independent, we consider $t = 0$ and obtain

$$\int_0^{2\pi} \left(u_0^3(x) + 3u_0^2(x)\right) dx = \int_0^{2\pi} (3 + u_0(x))\, |u_0(x)|^2\, dx = 0.$$

However, from the limitation (4.2), we know that $(3 + u_0(x)) \neq 0$, which yields $u_0 = 0$, and hence we have that $u \equiv 0$ on $\mathbb{T} \times \mathbb{R}$, which concludes the proof. ∎

4.2 Unique Continuation Property for BBM-Like Equations

This section is devoted to the UCP property for some BBM-like equations with different nonlinar terms. First, we deal with a generalized BBM equation without drift term, that is, we suppress the term u_x in the BBM equation and consider a general nonlinear term satisfying certain conditions. Next, as a corollary, we incorporate a localized damping in that generalized BBM equation. To end the section, we treat with a BBM-like equation with drift term and with a nonlocal bilinear term given by a convolution.

4.2.1 Generalized BBM Equation Without Drift Term

Consider the following generalized BBM equation

$$u_t - u_{txx} + [f(u)]_x = 0, \quad x \in \mathbb{T}, t \in \mathbb{R}, \tag{4.10}$$

$$u(x,0) = u_0(x), \tag{4.11}$$

where $f \in C^1(\mathbb{R}), f(u) \geqslant 0$ for all $u(x,t) \in \mathbb{R}$, and the only solution $u \in (-\delta, \delta)$ of $f(u) = 0$ is $u = 0$, for some number $\delta > 0$. When $f(u) = u^2/2$, we have

$$u_t - u_{txx} + uu_x = 0\,,$$

which is called the Morrison-Meiss-Carey (MMC) equation (also called width wave equation). The global well-posedness for (4.10)-(4.11) in $H^1(\mathbb{T})$ can be derived from the contraction mapping theorem as in Section 3.1 and the conservation of the H^1-norm, according to Proposition 3.3.1, invariant (2).

Theorem 4.2.1. *Let f be as above, and let ω be a nonempty open set in $\mathbb{T}$. Let $u_0 \in H^1(\mathbb{T})$ be such that the solution u of* (4.10)-(4.11) *satisfies $u(x,t) = 0$ for $(x,t) \in \omega \times (0,T)$ for some $T > 0$. Then $u_0 = 0$.*

Proof. As was done in the proof of Theorem 4.1.1, we can assume without loss of generality that $\omega = [0, \varepsilon) \cup (2\pi - \varepsilon, 2\pi)$. The prolongation of u by 0 on $(\mathbb{R}\backslash(0,2\pi)) \times (0,T)$, still denoted by u, satisfies

$$u_t - u_{txx} + [f(u)]_x = 0, \quad x \in \mathbb{R}, t \in (0,T) \tag{4.12}$$

$$u(x,t)=0, \quad x \notin (\varepsilon, 2\pi-\varepsilon), \quad t\in(0,T) \tag{4.13}$$

$$u \in C\left([0,T];H^1(\mathbb{R})\right), \quad u_t \in C\left([0,T];H^2(\mathbb{R})\right). \tag{4.14}$$

Scaling (4.12) by e^x and integrating over $\mathbb{R}$, yields for $t \in (0,T)$

$$\int_{-\infty}^{\infty} u_t e^x dx - \int_{-\infty}^{\infty} u_{txx} e^x dx + \int_{-\infty}^{\infty} [f(u)]_x e^x dx = 0$$
$$-\int_{-\infty}^{\infty} u_{tx} e^x dx + \int_{-\infty}^{\infty} u_{tx} e^x dx + f(u)e^x|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(u) e^x dx = 0.$$

Given the periodicity of u and since $u \to 0$, as $x \to \pm\infty$, we have $f(u)=0$, (which follows from the assumptions about f, that is, $f(0)=0$), then the first three terms above result in 0. Thus,

$$\int_{-\infty}^{\infty} f(u(x,t))e^x dx = 0.$$

Since $e^x > 0$ and $f \geqslant 0$ this yields

$$f(u(x,t)) = 0 \quad \text{for } (x,t) \in \mathbb{R}\times(0,T).$$

Since u is continuous and it vanishes for $x \notin (\varepsilon, 2\pi-\varepsilon)$, we again infer from the assumptions about f that $u \equiv 0$. This concludes the proof of the UCP for (4.10)-(4.11). ■

Incorporating a localized damping in (4.10), we obtain the next BBM-like equation

$$u_t - u_{txx} + [f(u)]_x + a(x)u = 0, \quad x \in \mathbb{T}, t \geqslant 0 \tag{4.15}$$

$$u(x,0) = u_0(x), \tag{4.16}$$

where $a \in C^\infty(\mathbb{T})$ is a nonnegative function with $\omega := \{x \in \mathbb{T}; a(x) > 0\}$ nonempty, and f is as above. Then we have the following weak stabilization result.

Corollary 4.2.1. *Let $u_0 \in H^1(\mathbb{T})$. Then the system* (4.15)-(4.16) *admits a unique solution $u \in C([0,T];H^1(\mathbb{T}))$ for all $T>0$. Furthermore, $u(t) \to 0$ weakly in $H^1(\mathbb{T})$, hence strongly in $H^s(\mathbb{T})$ for $s<1$, as $t \to +\infty$.*

Proof. The local well-posedness in $H^s(\mathbb{T})$ for any $s > 1/2$ is also derived from the contraction mapping theorem in a similar way as was done in Theorem 3.1.1. Our aim now is to prove the following energy identity

$$\|u(T)\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2\int_0^T \int_{\mathbb{T}} a(x)|u(x,t)|^2 dxdt = 0, \tag{4.17}$$

from which the global well-posedness in $H^1(\mathbb{T})$ is derived. Scaling each term in (4.15) by u, and integrating over $\mathbb{T}$, we obtain

$$\begin{aligned}
&uu_t - uu_{txx} + u\,[f(u)]_x + a(x)u^2 = 0\\
&\int_{\mathbb{T}} uu_t dx - \int_{\mathbb{T}} uu_{txx} dx + \int_{\mathbb{T}} u\,[f(u)]_x\, dx + \int_{\mathbb{T}} a(x)u^2 dx = 0\\
&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{T}} u^2 dx + \int_T u_x^2 dx\right) + \int_{\mathbb{T}} a(x)u^2 dx + \int_{\mathbb{T}} u\,[f(u)]_x\, dx = 0.
\end{aligned}$$

We note that the last term above is zero, indeed,

$$\int_{\mathbb{T}} u\,[f(u)]_x\, dx = -\int_{\mathbb{T}} u_x f(u) dx = F(u)|_0^{2\pi} = 0, \quad \text{since} \quad u(0) = u(2\pi),$$

where F is a primitive of f. Then, integrating the resulting expression from 0 to T

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\left(\int_{\mathbb{T}} \left(u^2 + u_x^2\right) dx\right) dt + \int_0^T \int_{\mathbb{T}} a(x)u^2 dxdt = 0\\
&\frac{1}{2}\left(\int_{\mathbb{T}} \left[u^2(x,T) + u_x^2(x,T)\right] dx - \int_{\mathbb{T}} \left[u^2(x,0) + u_x^2(x,0)\right] dx\right) + \int_0^T \int_{\mathbb{T}} a(x)u^2 dxdt = 0\\
&\frac{1}{2}\left(\|u(T)\|_{H^1(\mathbb{T})}^2 - \|u(0)\|_{H^1(\mathbb{T})}^2\right) + \int_0^T \int_{\mathbb{T}} a(x)u^2 dxdt = 0\\
&\|u(T)\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2\int_0^T \int_{\mathbb{T}} a(x)|u(x,t)|^2 dxdt = 0\,,
\end{aligned}$$

which is the desired energy identity (4.17). Therefore, we have

$$\frac{d}{dt}\|u(t)\|_{H^1}^2 + 2\int_{\mathbb{T}} a(x)|u(x,t)|^2 dx = 0\,,$$

which implies

$$\frac{d}{dt}\|u(t)\|_{H^1}^2 \leqslant 0.$$

Thus, $\|u(t)\|_{H^1}^2$ is a nonincreasing function, hence, it has a nonnegative limit l as $t \to \infty$. On the other hand, still from the application of the contraction mapping theorem, given any $s > 1/2$, any $\rho > 0$ and any $u_0, v_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s(\mathbb{T})} \leqslant \rho$, $\|v_0\|_{H^s(\mathbb{T})} \leqslant \rho$, there is some time $T = T(s, \rho) > 0$ such that the solutions u and v of (4.15)-(4.16) corresponding to the initial data u_0 and v_0, respectively, fulfill

$$\|u - v\|_{C([0,T];H^s(\mathbb{T}))} \leqslant 2\,\|u_0 - v_0\|_{H^s(\mathbb{T})}\,. \tag{4.18}$$

Pick any initial data $u_0 \in H^1(\mathbb{T})$, any $s \in (1/2, 1)$, and let $\rho = \|u_0\|_{H^1(\mathbb{T})}$ and $T = T(s, \rho)$. Let v_0 be such that, for some sequence $t_n \to \infty$ we have $u(t_n) \to v_0$ weakly in $H^1(\mathbb{T})$. Extracting a subsequence if needed, we may assume that $t_{n+1} - t_n \geqslant T$ for all n. From (4.17) we infer that

$$\lim_{n\to\infty} \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}} a(x)|u(x,t)|^2 dx dt = 0. \tag{4.19}$$

Since $u(t_n) \to v_0$ (strongly) in $H^s(\mathbb{T})$, and $\|u(t_n)\|_{H^s(\mathbb{T})} \leqslant \|u(t_n)\|_{H^1(\mathbb{T})} \leqslant \rho$, we have from (4.18) that

$$u(t_n + \cdot) \to v \quad \text{in } C([0,T]; H^s(\mathbb{T})) \quad \text{as } n \to \infty, \tag{4.20}$$

where $v = v(x,t)$ denotes the solution of

$$\begin{aligned} &v_t - v_{txx} + [f(v)]_x + a(x)v = 0, \quad x \in \mathbb{T}, t \geqslant 0, \\ &v(x,0) = v_0(x). \end{aligned}$$

Note that $v \in C([0,T]; H^1(\mathbb{T}))$ for $v_0 \in H^1(\mathbb{T})$. (4.19) combined with (4.20) yields

$$\int_0^T \int_{\mathbb{T}} a(x)|v(x,t)|^2 dx dt = 0,$$

so that $av \equiv 0$. By Theorem 4.2.1, $v_0 = 0$ and hence, as $t \to \infty$,

$$\begin{aligned} &u(t) \to 0 \quad \text{weakly in } H^1(\mathbb{T}) \\ &u(t) \to 0 \quad \text{strongly in } H^s(\mathbb{T}) \text{ for } s < 1. \end{aligned}$$

∎

4.2.2 A BBM-Like Equation With a Nonlocal Bilinear Term

Now, the third BBM-type equation we will consider here has the drift term, and a nonlocal bilinear term given by a convolution, namely

$$u_t - u_{txx} + u_x + \lambda(u * u)_x = 0, \quad x \in \mathbb{R}, \tag{4.21}$$

where $\lambda \in \mathbb{R}$ is a constant and

$$(u * v)(x) = \int_{-\infty}^{\infty} u(x-y)v(y)dy \quad \text{for } x \in \mathbb{R}.$$

A UCP can be derived without any restriction on the initial data.

Theorem 4.2.2. *Assume that $\lambda \neq 0$. Let $u \in C^1([0,T]; H^1(\mathbb{R}))$ be a solution of* (4.21) *such that*

$$u(x,t) = 0 \quad \textit{for } |x| > L, t \in (0,T). \tag{4.22}$$

Then $u \equiv 0$.

Proof. Taking the Fourier transform of each term in (4.21) yields

$$\begin{aligned} &\hat{u}_t + \xi^2\hat{u}_t + i\xi\hat{u} + \lambda i\xi\hat{u}\hat{u} = 0 \\ &\left(1+\xi^2\right)\hat{u}_t = -i\xi\hat{u} - \lambda i\xi\hat{u}^2 \\ &\left(1+\xi^2\right)\hat{u}_t = -i\xi\left(\hat{u} + \lambda\hat{u}^2\right), \quad \xi \in \mathbb{R}, \quad t \in (0,T). \end{aligned} \tag{4.23}$$

Note that, for each $t \in (0,T)$, $\hat{u}(\cdot,t)$ and $\hat{u}_t(\cdot,t)$ can be extended to $\mathbb{C}$ as entire functions of exponential type at most L. Furthermore, (4.23) is still true for $\xi \in \mathbb{C}$ and $t \in (0,T)$ by analytic continuation. To prove that $u \equiv 0$, it is sufficient to check that (see (CONWAY, 1978), 3.7 Theorem, p. 78)

$$\partial_\xi^k \hat{u}(i,t) = 0 \quad \forall k \in \mathbb{N}, \forall t \in (0,T). \tag{4.24}$$

Also, we note that

$$\partial_\xi^n \hat{u}(i,t) = \int_{-\infty}^{\infty} u(x,t)(-ix)^n e^x dx.$$

Indeed,

$$\begin{aligned}
\hat{u}(\xi,t) &= \int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}dx \\
\partial_\xi \hat{u}(\xi,t) &= \int_{-\infty}^{\infty} u(x,t)(-ix)e^{-i\xi x}dx \\
\partial_\xi^n \hat{u}(\xi,t) &= \int_{-\infty}^{\infty} u(x,t)(-ix)^n e^{-i\xi x}dx \\
\partial_\xi^n (i,t) &= \int_{-\infty}^{\infty} u(x,t)(-ix)^n e^{x}dx.
\end{aligned}$$

We will prove (4.24) by induction on k. First, from (4.23), by setting $\xi = i$

$$\begin{aligned}
\hat{u}(i,t) + \lambda \hat{u}^2(i,t) = 0 \\
\hat{u}\left(1 + \lambda \hat{u}^2\right) = 0,
\end{aligned}$$

gives that either

$$\hat{u}(i,t) = 0 \quad \forall t \in (0,T), \tag{4.25}$$

or

$$\hat{u}(i,t) = -\lambda^{-1} \quad \forall t \in (0,T). \tag{4.26}$$

Derivating with respect to ξ in (4.23) yields

$$\begin{aligned}
2\xi \hat{u}_t + \left(1+\xi^2\right)\partial_\xi \hat{u}_t &= -i\left(\hat{u} + \lambda \hat{u}^2\right) - i\xi\left(\partial_\xi \hat{u} + \lambda 2\hat{u}\partial_\xi \hat{u}\right) \\
&= -i\hat{u}(1+\lambda\hat{u}) - i\xi\partial_\xi\hat{u}(1+2\lambda\hat{u}).
\end{aligned} \tag{4.27}$$

Note that if either (4.25) or (4.26) holds, we shall have

$$\hat{u}_t(i,t) = 0.$$

So, combining with (4.27), yields

$$\begin{aligned}
2i\hat{u}_t(i,t) + (1-1)\partial_\xi \hat{u}(i,t) &= -i\hat{u}(i,t)(1+\lambda\hat{u}(i,t)) + \partial_\xi \hat{u}(i,t)\left(1+2\lambda\hat{u}\right) \\
\partial_\xi \hat{u}(i,t) &= \frac{i\hat{u}(i,t)(1+\lambda\hat{u}(i,t))}{1+2\lambda\hat{u}}.
\end{aligned}$$

Therefore, we see that

$$\begin{aligned} &\text{if} \quad \hat{u}(i,t) = 0, && \text{then} \quad \partial_\xi \hat{u}(i,t) = 0, \\ &\text{if} \quad \hat{u}(i,t) = -\lambda^{-1}, && \text{then} \quad \partial_\xi \hat{u}(i,t) = 0, \end{aligned}$$

that is,

$$\partial_\xi \hat{u}(i,t) = 0, \quad t \in (0,T).$$

Assume now that, for some $k \geqslant 2$,

$$\partial_\xi^l \hat{u}(i,t) = 0 \quad \text{ for } t \in (0,T) \text{ and any } l \in \{1, \dots, k-1\}. \tag{4.28}$$

Derivating k times with respect to ξ in (4.23) yields, from Leibniz rule (2.3),

$$\begin{gathered} \partial_\xi^k \left[\left(1+\xi^2\right) \hat{u}_t \right] = \partial_\xi^k \left[-i\xi \left(\hat{u} + \lambda \hat{u}^2 \right) \right] \\ \sum_{l=0}^{k} \binom{k}{l} \left(\partial_\xi^l \left(1+\xi^2\right) \right) \left(\partial_\xi^{(k-l)} \hat{u}_t \right) = \sum_{l=0}^{k} \binom{k}{l} \left(\partial_\xi^l (-i\xi) \right) \left(\partial_\xi^{(k-l)} \left(\hat{u} + \lambda \hat{u}^2 \right) \right). \end{gathered}$$

We note that, for $l \geqslant 3$, all the terms on the left-hand side (l.h.s) above are zero. Then, on the l.h.s., we perform the sum up to $l = 2$, which gives

$$\begin{aligned} &\left(1+\xi^2\right) \partial_\xi^k \hat{u}_t + k 2\xi \partial_\xi^{(k-1)} \hat{u}_t + \frac{k(k-1)}{2} 2 \partial_\xi^{(k-2)} \hat{u}_t \\ &= -i\xi \left(\partial_\xi^k \hat{u} + \lambda \sum_{l=0}^{k} \binom{k}{l} \partial_\xi^l \hat{u} \partial_\xi^{(k-l)} \hat{u} \right) - ik \left(\partial_\xi^{(k-1)} \hat{u} + \lambda \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_\xi^l \hat{u} \partial_\xi^{(k-1-l)} \hat{u} \right). \end{aligned}$$

From induction hypothesis and from the last line above, we obtain

$$\begin{aligned} &\partial_\xi^k \hat{u}(i,t) + 2\lambda \hat{u}(i,t) \partial_\xi^k \hat{u}(i,t) = 0 \\ &\partial_\xi^k \hat{u}(i,t) \left(1 + 2\lambda \hat{u}(i,t)\right) = 0. \end{aligned}$$

From (4.25) and (4.26), we obtain that

$$\partial_\xi^k \hat{u}(i,t) = 0.$$

Therefore,

$$\partial_\xi^k \hat{u}(i,t) = 0, \quad \forall\, k \geqslant 1. \tag{4.29}$$

If we assume (4.26) and (4.29), it would imply

$$\hat{u}(\xi, t) = -\lambda^{-1} \quad \forall \xi \in \mathbb{C},$$

which contradicts the fact that $\hat{u}(\cdot, t) \in L^2(\mathbb{R})$. Thus, (4.25) holds and combining with (4.29) implies

$$\hat{u}(\xi, t) = 0 \quad \forall \xi \in \mathbb{C}.$$

Therefore, we achieve the desired result $u \equiv 0$. ■

4.3 Unique Continuation Property for the Linearized KdV-BBM Equation

This section is concerned with the UCP for the KdV–BBM equation, which is presented as a theorem. The proof will be provided by means of a Carleman estimate, which, in turn, is presented as a proposition addressed in subsection 4.3.1. To achieve this, we first split the KdV-BBM equation into a coupled system of an elliptic equation and a transport equation. Then, we derive, for each one, a Carleman estimate, stated in the form of lemmas with the same weights for both. Afterward, we combine these lemmas to prove the proposition. Finally, we use a regularization process, as the theorem holds for a solution that is not regular enough.

In order to begin presenting the results, we shall give the KdV-BBM equation

$$u_t - u_{txx} - cu_{xxx} + qu_x = 0, \quad x \in \mathbb{T},\ t \in (0, T), \tag{4.30}$$

where $q \in L^\infty(0, T; L^\infty(\mathbb{T}))$ is a given potential function and $c \neq 0$ is a given real constant.

Theorem 4.3.1. *Let $c \in \mathbb{R}\backslash\{0\}, T > 2\pi/|c|$, and $q \in L^\infty(0, T; L^\infty(\mathbb{T}))$. Let $\omega \subset \mathbb{T}$ be a nonempty open set. Let $u \in L^2(0, T; H^2(\mathbb{T})) \cup L^\infty(0, T; H^1(\mathbb{T}))$ satisfying (4.30) and*

$$u(x, t) = 0 \quad \text{for a.e. } (x, t) \in \omega \times (0, T). \tag{4.31}$$

Then $u \equiv 0$ in $\mathbb{T} \times (0, T)$.

Proof. We first assume that u is regular enough, $u \in L^2(0, T; H^2(\mathbb{T}))$. Then, spliting the equation by setting $w = u - u_{xx} \in L^2(0, T; L^2(\mathbb{T}))$, we have

$$w_t = u_t - u_{txx} = cu_{xxx} - qu_x$$
$$cw_x = cu_x - cu_{xxx}.$$

By adding these two equations we obtain: $w_t + cw_x = (c - q)u_x$. That is, (u, w) solves the following system

$$u - u_{xx} = w \tag{4.32}$$
$$w_t + cw_x = (c - q)u_x. \tag{4.33}$$

As mentioned in the summary of this section, we shall establish some Carleman estimates for the elliptic equation (4.32) and the transport equation (4.33) with the "same weights", and combine both Carleman estimates into a single one for (4.30).

Remark 4.3.1 (The sharpness of T). *Assuming for simplicity that $q(x) = c$ for all $x \in \mathbb{T}$, where $c > 0$ is given, and that $\omega = (2\pi - \varepsilon, 2\pi)$ for a small $\epsilon > 0$, then the UCP fails in time $T \leqslant (2\pi - 2\varepsilon)/c$, which implies that there is a finite speed of propagation for KdV-BBM, since an arbitrarily large speed would produce an arbitrarily small time. Indeed, picking any nontrivial initial state $u_0 \in C_0^\infty(0, \varepsilon)$, we obtain from 4.33 that $w_t + cw_x = 0$ whose solution is $w(x,t) = w_0(x - ct)$, so, the solution (u, w) of* (4.32)-(4.33) *is $u(x,t) = u_0(x - ct), w = w_0(x - ct)$ where $w_0 = (1 - \partial_x^2)\, u_0$. Then, for $t \in \left(0, \frac{2\pi - 2\epsilon}{c}\right)$, we have that $x - ct \in \omega$, hence, the solution $u(x,t) = 0$ for $(x,t) \in \omega \times (0, (2\pi - 2\epsilon)/c)$, although $u \not\equiv 0$, since the initial data u_0 was picked nontrivial suported in $(0, \epsilon)$. Therefore, the condition $T > 2\pi/|c|$ in Theorem 4.3.1 is sharp.*

We shall introduce some notation and auxiliary functions to present the Carleman estimate in the following proposition. Once again we identify $\mathbb{T}$ with $[0, 2\pi)$ by choosing a coordinate system such that $\omega = (2\pi - \eta, 2\pi + \eta) \sim [0, \eta) \cup (2\pi - \eta, 2\pi)$ for some $\eta \in (0, \pi)$ (by choosing the origin of the coordinates inside ω). Without loss of generality, we can assume that $c > 0$ (the case $c < 0$ being similar). Assume given a time T fulfilling

$$T > \frac{2\pi}{c}. \tag{4.34}$$

So, from $cT > 2\pi$, we pick some numbers $\delta > 0$ and $\rho \in (0,1)$, such that

$$\rho c T > 2\pi + \delta, \tag{4.35}$$

and a function $\psi \in C^\infty([0,2\pi])$, such that

$$\psi(x) = |x+\delta|^2 \quad \text{for } x \in [\eta/2, 2\pi - \eta/2] \tag{4.36}$$

$$\frac{d^k\psi}{dx^k}(0) = \frac{d^k\psi}{dx^k}(2\pi) \quad \text{for } k = 1,2,3 \tag{4.37}$$

$$2\delta \leqslant \frac{d\psi}{dx}(x) \leqslant 2(2\pi+\delta) \quad \text{for } x \in [0,2\pi]. \tag{4.38}$$

Introduce the function $\varphi \in C^\infty([0,2\pi] \times \mathbb{R})$ defined by

$$\varphi(x,t) = \psi(x) - \rho c^2 t^2. \tag{4.39}$$

Now, we present the following Carleman estimate for (4.30).

4.3.1 Carleman Estimate for the KdV-BBM Equation

Proposition 4.3.1 (Carleman estimate for the KdV-BBM equation)**.** *Let ω, c and T be as above. Then there exists some positive numbers s_2 and C_2 such that for all $s \geqslant s_2$ and all $u \in L^2(0,T;H^2(\mathbb{T}))$ satisfying* (4.30)*, we have*

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}} \left[s\,|u_{xx}|^2 + s\,|u_x|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt + s \int_{\mathbb{T}} \left[|u-u_{xx}|^2\, e^{2s\varphi}\right]_{|t=0} dx \\ &\leqslant C_2 \int_0^T \int_\omega \left[s\,|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt \end{aligned} \tag{4.40}$$

Note that, from the Carleman estimate (4.40), and assuming s large enough

$$\int_{\mathbb{T}} \left[|u-u_{xx}|^2\, e^{2s\varphi}\right]_{|t=0} dx \leqslant \frac{C_2}{s} \int_0^T \int_\omega \left[s\,|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt$$

$$\inf_{x\in\mathbb{T}} \left(e^{2s\varphi(x,0)}\right) \int_{\mathbb{T}} \left[|u-u_{xx}|^2\right]_{|t=0} dx \leqslant \sup_{\substack{x\in\omega \\ t\in[0,T]}} \left(e^{2s\varphi}\right) C_2 \int_0^T \int_\omega \left(|u_{xx}|^2 + s^2|u|^2 + |u_x|^2\right) dxdt$$

$$\int_{\mathbb{T}} \left[|u-u_{xx}|^2\right]_{|t=0} dx \leqslant C \int_0^T \int_\omega \left(|u_{xx}|^2 + |u|^2 + |u_x|^2\right) dxdt.$$

Note that

$$\int_{\mathbb{T}} |u - u_{xx}|^2 dx = \int_{\mathbb{T}} \left(u^2 + u_{xx}^2\right) dx - 2\int_{\mathbb{T}} u u_{xx} dx = \int_{\mathbb{T}} \left(u^2 + u_{xx}^2\right) dx + 2\int_{\mathbb{T}} u_x u_x dx.$$

Then, we have, by adjusting the constant

$$\int_{\mathbb{T}} \left[u^2 + 2u_x^2 + u_{xx}^2\right]_{|t=0} dx \leqslant C\int_0^T \int_\omega \left(|u_{xx}|^2 + |u|^2 + |u_x|^2\right) dxdt$$
$$\int_{\mathbb{T}} \left[|u(\cdot,0)|^2 + |u_x(\cdot,0)|^2 + |u_{xx}(\cdot,0)|^2\right] dx \leqslant C\int_0^T \int_\omega \left(|u_{xx}|^2 + |u|^2 + |u_x|^2\right) dxdt.$$

Which is the observability inequality

$$\|u(\cdot,0)\|_{H^2(\mathbb{T})}^2 \leqslant C\int_0^T \|u(\cdot,t)\|_{H^2(\omega)}^2 dt.$$

For the sake of clarity, we outline more detail than in the summary as the proof of Proposition 4.3.1 is obtained. In the first step, we prove a Carleman estimate for the elliptic equation (4.32) with the weight $e^{s\psi}$. In the second step, we prove a Carleman estimate for the transport equation (4.33) with the weight $e^{5\varphi}$. Note that we are concerned here with global Carleman estimates with weights suitably chosen in the control region. Then, we combine these two Carleman estimates into a single one to obtain the desired Carleman (4.40) for the KdV-BBM equation (4.30).

4.3.1.1 Step 1: Carleman Estimate for the Elliptic Equation

Lemma 4.3.1. *There exist $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $s \geqslant s_0$ and all $u \in H^2(\mathbb{T})$, the following inequality holds*

$$\int_{\mathbb{T}} \left[s\,|u_x|^2 + s^3|u|^2\right] e^{2s\psi} dx \leqslant C_0 \left(\int_{\mathbb{T}} |u_{xx}|^2\, e^{2s\psi} dx + \int_\omega s^3|u|^2 e^{2s\psi} dx\right). \tag{4.41}$$

Proof. Let $v = e^{s\psi}u$ and $P = \partial_x^2$. Then

$$\begin{aligned} e^{s\psi}u_{xx} = e^{s\psi}Pu = e^{s\psi}P\left(e^{-s\psi}v\right) \\ &= e^{s\psi}\left[e^{-s\psi}(s\psi_x)^2 v + e^{-s\psi}(-s\psi_{xx})v - 2e^{-s\psi}s\psi_x v_x + e^{-s\psi}v_{xx}\right] \\ &= (s\psi_x)^2 v - s\psi_{xx}v - 2s\psi_x v_x + v_{xx} \\ &= P_s v + P_a v, \end{aligned}$$

where

$$P_s v = (s\psi_x)^2 v + v_{xx}, \tag{4.42}$$

$$P_a v = -2s\psi_x v_x - s\psi_{xx} v, \tag{4.43}$$

denote the (formal) self-adjoint and skew-adjoint parts of the operator $e^{s\psi}P\left(e^{-s\psi}\right)$, respectively. It follows that

$$\begin{aligned} \left\|e^{s\psi}Pu\right\|^2 &= \left(e^{s\psi}Pu, e^{s\psi}Pu\right) \\ &= (P_s v + P_a v, P_s v + P_a v) \\ &= (P_s v, P_s v) + (P_a v, P_a v) + 2(P_s v, P_a v) \\ &= \|P_s v\|^2 + \|P_a v\|^2 + 2(P_s v, P_a v). \end{aligned}$$

Where $(f, g) = \int_{\mathbb{T}} fg dx$, and $\|f\|^2 = (f, f)$. Then, for the last term above

$$\begin{aligned} (P_s v, P_a v) &= \left((s\psi_x)^2 v + v_{xx}, -2s\psi_x v_x - s\psi_{xx}v\right) \\ &= \left((s\psi_x)^2 v, -2s\psi_x v_x\right) + \left((s\psi_x)^2 v, -s\psi_{xx}v\right) + (v_{xx}, -2s\psi_x v_x) + (v_{xx}, -s\psi_{xx}v) \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For each integral, we will do some integration by parts in x, and also use (4.37)

$$\begin{aligned} I_1 = \int_{\mathbb{T}} (s\psi_x)^2 v(-2s\psi_x v_x)\, dx &= -2\int_{\mathbb{T}} s^3\psi_x^3 v v_x dx \\ &= -\int_{\mathbb{T}} s^3\psi_x^3 \left(v^2\right)_x dx = -s^3\psi_x^3 v^2\Big|_0^{2\pi} + \int_{\mathbb{T}} s^3 3\psi_x^2\psi_{xx}v^2 dx \\ &= 3\int_{\mathbb{T}} (s\psi_x)^2 s\psi_{xx}v^2 dx, \\ I_2 = \int_{\mathbb{T}} (s\psi_x)^2 v(-s\psi_{xx}v)\, dx &= -\int_{\mathbb{T}} (s\psi_x)^2 s\psi_{xx}v^2 dx, \end{aligned}$$

$$\begin{aligned}
I_3 = \int_{\mathbb{T}} v_{xx}\left(-2s\psi_x v_x\right)dx \qquad &= -\int_{\mathbb{T}} s\psi_x \left(v_x^2\right)_x dx \\
&= -s\psi_x v_x^2\big|_0^{2\pi} + \int_{\mathbb{T}} s\psi_{xx} v_x^2 dx = \int_{\mathbb{T}} s\psi_{xx} v_x^2 dx, \\
I_4 = -s\int_{\mathbb{T}} v_{xx}\left(\psi_{xx} v\right)dx \qquad &= -s v_x \psi_{xx} v\big|_0^{2\pi} + s\int_{\mathbb{T}} v_x\left(\psi_{xxx} v + \psi_{xx} v_x\right)dx \\
&= \frac{s}{2}\int_{\mathbb{T}} \psi_{xxx}\left(v^2\right)_x dx + s\int_{\mathbb{T}} \psi_{xx} v_x^2 dx \\
&= \frac{s}{2} v^2 \psi_{xxx}\Big|_0^{2\pi} - \int_{\mathbb{T}} s\psi_{xxxx}\frac{v^2}{2}dx + \int_{\mathbb{T}} s\psi_{xx} v_x^2 dx \\
&= -\int_{\mathbb{T}} s\psi_{xxxx}\frac{v^2}{2}dx + \int_{\mathbb{T}} s\psi_{xx} v_x^2 dx.
\end{aligned}$$

Gathering all together, we have

$$\begin{aligned}
2\left(I_1 + I_2 + I_3 + I_4\right) = 2\Bigg(3\int_{\mathbb{T}} \left(s\psi_x\right)^2 s\psi_{xx} v^2\, dx - \int_{\mathbb{T}} \left(s\psi_x\right)^2 s\psi_{xx} v^2\, dx + \int_{\mathbb{T}} s\psi_{xx} v_x^2\, dx \\
- \int_{\mathbb{T}} s\psi_{xxxx}\frac{v^2}{2}\, dx + \int_{\mathbb{T}} s\psi_{xx} v_x^2\, dx\Bigg) \\
= 4\int_{\mathbb{T}} \left(s\psi_x\right)^2 s\psi_{xx} v^2 dx - \int_{\mathbb{T}} s\psi_{xxxx} v^2 dx + 4\int_{\mathbb{T}} s\psi_{xxx} v_x^2 dx \\
= \int_{\mathbb{T}} \left[4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2 dx + \int_{\mathbb{T}} \left(4s\psi_{xx}\right) v_x^2 dx.
\end{aligned}$$

Therefore,

$$\left\|e^{s\psi} Pu\right\|^2 = \left\|P_s v\right\|^2 + \left\|P_a v\right\|^2 + \int_{\mathbb{T}} \left[4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2 dx + \int_{\mathbb{T}} \left(4s\psi_{xx}\right) v_x^2 dx \quad (4.44)$$

Since, for $x \in \left(\frac{\eta}{2}, 2\pi - \frac{\eta}{2}\right)$, $\psi(x) = |x+\delta|^2$ follows that

$$\begin{aligned}
&\psi_x(x) = 2|x+\delta|\frac{(x+\delta)}{|x+\delta|} = 2(x+\delta) \\
&\psi_{xx}(x) = 2 \\
&\psi_{xxxx}(x) = 0.
\end{aligned}$$

So,

$$\begin{aligned}
&4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx} = 4(s2(x+\delta))^2 s2 = 32s^3(x+\delta)^2 > 0 \\
&4s\psi_{xx} = 4s2 = 8s.
\end{aligned}$$

That is, we can infer the existence of some numbers $s_0 \geqslant 1$ and $K > 0$ such that , for all $s \geqslant s_0$

$$4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx} \geqslant Ks^3, \quad \text{for } (x,t) \in \left(\frac{\eta}{2}, 2\pi - \frac{\eta}{2}\right) \times (0,T)$$
$$4s\psi_{xx} \geqslant Ks \quad \text{for } (x,t) \in \left(\frac{\eta}{2}, 2\pi - \frac{\eta}{2}\right) \times (0,T).$$

On the other hand, setting $\omega_0 = \left[0, \frac{\eta}{2}\right) \cup \left(2\pi - \frac{\eta}{2}, 2\pi\right)$, and, since $\omega_0 \subset \omega \subset [0, 2\pi]$, with $\psi \in C^\infty([0, 2\pi])$, we can obtain a superior bound for all derivatives of ψ on ω_0, so that, we also infer that exists a number $K' > 0$ and, again for all $s \geqslant s_0 \geqslant 1$,

$$\left|4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right| \leqslant K's^3 \quad \text{for } (x,t) \in \omega_0 \times (0,T),$$
$$\left|4s\psi_{xx}\right| \leqslant K's, \quad \text{for } (x,t) \in \omega_0 \times (0,T).$$

Thus, from (4.44) we obtain

$$\|P_s v\|^2 + \int_{\mathbb{T}} \left[4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2 dx + \int_{\mathbb{T}} \left(4s\psi_{xx}\right) v_x^2 dx \leqslant \|e^{s\psi} Pu\|^2$$
$$\|P_s v\|^2 + \int_{\mathbb{T}\backslash\omega_0} \left[4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2 dx + \int_{\omega_0} \left[4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right] v^2 dx$$
$$+ \int_{\mathbb{T}\backslash\omega_0} \left(4s\psi_{xx}\right) v_x^2 dx + \int_{\omega_0} \left(4s\psi_{xx}\right) v_x^2 dx \leqslant \left\|e^{s\psi} Pu\right\|^2.$$

From the discussion about the estimates for $\left(4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right)$ and for $(4s\psi_{xx})$ in $\mathbb{T}\backslash\omega_0$ and in ω_0 we can see that, changing $\left(4\left(s\psi_x\right)^2 s\psi_{xx} - s\psi_{xxxx}\right)$ for Ks^3 in $\mathbb{T}\backslash\omega_0$, the inequality remains the same, while, in ω_0 we can change for $K's$ and add the same integral over ω_0 on the right-hand side of the inequality, preserving the inequality. The same is true for $(4s\psi_{xx})$. Thus, we conclude that, for $s \geqslant s_0$ and some constant $C > 0$

$$\|P_s v\|^2 + \int_{\mathbb{T}} \left[s\left|v_x\right|^2 + s^3|v|^2\right] dx \leqslant C\left(\left\|e^{s\psi} Pu\right\|^2 + \int_{\omega_0} \left[s\left|v_x\right|^2 + s^3|v|^2\right] dx\right). \quad (4.45)$$

Next we shall show that $\int_{\mathbb{T}} s^{-1}\left|v_{xx}\right|^2 dx$ is also less than the right hand side of (4.45), that is, we must show that

$$\int_{\mathbb{T}} s^{-1}\left|v_{xx}\right|^2\, dx < C\left(\left\|e^{s\psi} Pu\right\|^2 + \int_{\omega_0} \left[s\left|v_x\right|^2 + s^3|v|^2\right]\, dx\right).$$

From $P_s v = (s\psi_x)^2 v + v_{xx}$, we have $v_{xx} = P_s v - (s\psi_x)^2 v$. Then proceeding

$$\begin{aligned}\int_{\mathbb{T}} s^{-1} |v_{xx}|^2 dx &= \int_{\mathbb{T}} s^{-1} \left|P_s v - (s\psi_x)^2 v\right|^2 dx \\ &\leqslant 2\int_{\mathbb{T}} s^{-1} \left(|P_s v|^2 + |s\psi_x|^4 |v|^2\right) dx \\ &\leqslant 2\int_{\mathbb{T}} s^{-1} |P_s v|^2 dx + 2\int_{\mathbb{T}} s^3 |\psi_x| |v|^2 dx \\ &\leqslant 2s^{-1} \|P_s v\|^2 + 2C\int_{\mathbb{T}} s^3 |v|^2 dx \\ &\leqslant C\left(s^{-1}\|P_s v\|^2 + \int_{\mathbb{T}} s^3|v|^2 dx\right).\end{aligned}$$

Note that, as $s \geqslant s_0 \geqslant 1,\ s^{-1}\|P_s v\|^2 \leqslant \|P_s v\|^2$, thus

$$\begin{aligned}\int_{\mathbb{T}} s^{-1} |v_{xx}|^2 dx &\leqslant C\left(\|P_s v\|^2 + \int_{\mathbb{T}} \left(s^3|v|^2 + s|v_x|^2\right) dx\right) \\ &\leqslant C\left(\left\|e^{s\psi} P u\right\|^2 + \int_{\omega_0} \left[s|v_x|^2 + s^3|v|^2\right] dx\right).\end{aligned}$$

As desired. Now, we combine this with (4.45), that gives

$$\begin{aligned}&\|P_s v\|^2 + \int_{\mathbb{T}} \left\{s^{-1}|v_{xx}|^2 + s|v_x|^2 + s^3|v|^2\right\} dx \\ &\qquad \leqslant 2C\left(\left\|e^{s\psi} P u\right\|^2 + \int_{\omega_0} s^3|v|^2 dx + \int_{\omega_0} s|v_x|^2 dx\right).\end{aligned}$$

Therefore

$$\int_{\mathbb{T}} \left\{s^{-1}|v_{xx}|^2 + s|v_x|^2 + s^3|v|^2\right\} dx \leqslant C\left(\left\|e^{s\psi} P u\right\|^2 + \int_{\omega_0} s^3|v|^2 dx + \int_{\omega_0} s|v_x|^2 dx\right), \tag{4.46}$$

where C does not depend on s and v. Now we want to drop the last term in the right hand side of (4.46). Let $\xi \in C_0^\infty(\omega)$ with $0 \leqslant \xi \leqslant 1$ and $\xi(x) = 1$ for $x \in \omega_0 \subset \omega$. Then

$$\begin{aligned}\int_{\omega_0} |v_x|^2 dx &\leqslant \int_{\omega} \xi |v_x|^2 dx = \int_{\omega} (\xi v_x) v_x dx = \xi v_x v|_{\partial\omega} - \int_{\omega} (\xi_x v_x + \xi v_{xx}) v\, dx \\ &\leqslant -\frac{1}{2}\int_{\omega} \xi_x \left(v^2\right)_x dx - \int_{\omega} \xi v_{xx} v dx \\ &\leqslant \frac{1}{2}\int_{\omega} \xi_{xx} v^2 dx - \int_{\omega} \xi v_{xx} v dx.\end{aligned}$$

So that

$$\int_{\omega_0} |v_x|^2\,dx \leqslant \frac{1}{2}\int_{\omega} |\xi_{xx} v^2| dx + \int_{\omega} |v_{xx}||v|\,dx$$
$$2\int_{\omega_0} s\,|v_x|^2\,dx \leqslant \int_{\omega} |\xi_{xx}|\,|v|^2 dx + 2\int_{\omega} s\,|v_{xx}|\,|v| dx$$
$$2\int_{\omega_0} s\,|v_x|^2\,dx \leqslant \|\xi_{xx}\|_{L^\infty(\mathbb{T})}\int_{\omega} s|v|^2 dx + 2\int_{\omega}\left|\frac{v_{xx}}{s^{1/2}}\right|\left|s^{3/2} v\right|\,dx.$$

From Young inequality, $\left|\frac{v_{xx}}{s^{1/2}}\right|\left|s^{3/2}v\right| \leqslant \varepsilon\frac{1}{s}|v_{xx}|^2 + \frac{1}{4\varepsilon}s^3|v|^2$, where $\varepsilon > 0$ is a constant that can be chosen as small as desired. Thus

$$2\int_{\omega_0} s\,|v_x|^2\,dx \leqslant \|\xi_{xx}\|_{L^\infty(\mathbb{T})}\int_{\omega} s|v|^2 dx + 2\varepsilon\int_{\omega} s^{-1}|v_{xx}|^2 dx + \frac{1}{2\varepsilon}\int_{\omega} s^3|v|^2 dx.$$

Then, by setting $\varepsilon = \frac{\kappa}{2}$, we obtain

$$2\int_{\omega_0} s\,|v_x|^2\,dx \leqslant \|\xi_{xx}\|_{L^\infty(\mathbb{T})}\int_{\omega} s|v|^2 dx + \kappa\int_{\omega} s^{-1}|v_{xx}|^2 dx + \kappa^{-1}\int_{\omega} s^3|v|^2 dx. \tag{4.47}$$

For κ small enough and some constant C that does not depend on s and v, we see that

$$\int_{\omega_0} s\,|v_x|^2 \leqslant C\int_{\omega} s|v|^2 dx \leqslant C\int_{\omega} s^3|v|^2 dx.$$

Then, going back to (4.46), we obtain, with a possibly increased value of s_0

$$\int_{\mathbb{T}}\left\{s^{-1}|v_{xx}|^2 + s\,|v_x|^2 + s^3|v|^2\right\} dx \leqslant C\left(\left\|e^{s\psi}Pu\right\|^2 + \int_{\omega} s^3|v|^2 dx\right). \tag{4.48}$$

Then

$$\int_{\mathbb{T}}\left\{s\,|v_x|^2 + s^3|v|^2\right\} dx \leqslant C\left(\left\|e^{s\psi}Pu\right\|^2 + \int_{\omega} s^3|v|^2 dx\right). \tag{4.49}$$

Note that, from $v = e^{s\psi}u$, we have

$$v_x = s\psi_x e^{s\psi}u_+ e^{s\psi}u_x$$
$$v_x = s\psi_x v + e^{s\psi}u_x$$
$$v_x - s\psi_x v = e^{s\psi}u_x$$
$$e^{-s\psi}\left(v_x - s\psi_x v\right) = u_x.$$

Using (4.38), follows that

$$\begin{aligned}\int_{\mathbb{T}}\left(s\left|u_x\right|^2+s^3|u|^2\right)e^{2s\psi}dx &= \int_{\mathbb{T}}\left(s\left|e^{-s\psi}\left(v_x-s\psi_x v\right)\right|^2+s^3\left|e^{-s\psi}v\right|^2\right)e^{2s\psi}dx\\ &= \int_{\mathbb{T}}\left(s\left|v_x-s\psi_x v\right|^2+s^3|v|^2\right)dx\\ &\leqslant C_1\int_{\mathbb{T}}\left[s\left(|v_x|^2+s^2\psi_x^2|v|^2\right)+s^3|v|^2\right]dx\\ &\leqslant C_2\int_{\mathbb{T}}\left[s|v_x|^2+s^3|v|^2\right]dx\end{aligned} \tag{4.50}$$

To conclude the proof, we come back to the variable u in (4.49) and we use (4.50), which gives

$$\begin{aligned}\int_{\mathbb{T}}\left(s\left|u_x\right|^2+s^3|u|^2\right)e^{2s\psi}dx &\leqslant C_2\int_{\mathbb{T}}\left[s|v_x|^2+s^3|v|^2\right]dx\\ &\leqslant C\left(\left\|e^{s\psi}Pu\right\|^2+\int_{\omega}s^3|v|^2dx\right)\\ &\leqslant C\left(\int_{\mathbb{T}}\left|u_{xx}\right|^2e^{2s\psi}dx+\int_{\omega}s^3\left|u\right|^2e^{2s\psi}dx\right),\end{aligned}$$

which is the desired Carleman estimate (4.41). ■

4.3.1.2 Step 2: Carleman Estimate for the Transport Equation

Lemma 4.3.2. *There exist $s_1 \geqslant s_0$ and $C_1 > 0$ such that for all $s \geqslant s_1$ and all $w \in L^2(\mathbb{T}\times(0,T))$ with $w_t+ cw_x \in L^2(\mathbb{T}\times(0,T))$, the following holds*

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}s|w|^2e^{2s\varphi}dxdt+\int_{\mathbb{T}}s\left[|w|^2e^{2s\varphi}\right]_{|t=0}dx+\int_{\mathbb{T}}s\left[|w|^2e^{2s\varphi}\right]_{|t=T}dx\\ &\qquad\leqslant C_1\left(\int_0^T\int_{\mathbb{T}}\left|w_t+cw_x\right|^2e^{2s\varphi}dxdt+\int_0^T\int_{\omega}s|w|^2e^{2s\varphi}dxdt\right).\end{aligned} \tag{4.51}$$

Proof. We first assume that w is regular enough, that is $w \in H^1(\mathbb{T}\times(0,T))$. For this case, the proof will follow the same outline as the proof of Lemma 4.3.1. Thus, let $v = e^{s\varphi}w$ and $P = \partial_t + c\partial_x$. Then,

$$\begin{aligned}e^{s\varphi}(w_t+cw_x) = e^{s\varphi}Pw &= e^{s\varphi}P\left(e^{-s\varphi}v\right)\\ &= e^{s\varphi}\left(-s\varphi_t e^{-s\varphi}v+e^{-s\varphi}v_t+c\left(-s\varphi_x e^{-s\varphi}v+e^{-s\varphi}v_x\right)\right)\\ &= \left(-s\varphi_t v+v_t-cs\varphi_x v+cv_x\right)\\ &= \left(-s\varphi_t v-\mathrm{cs}\,\varphi_x v\right)+\left(v_t+cv_x\right) =: P_s v+P_a v.\end{aligned}$$

It follows that

$$\begin{aligned}\|e^{s\varphi}Pw\|^2_{L^2(\mathbb{T}\times(0,T))} &= (e^{s\varphi}Pw, e^{s\varphi}Pw)_{L^2(\mathbb{T}\times(0,T))}\\ &= (P_sv + P_av, P_sv + P_av)_{L^2(\mathbb{T}\times(0,T))}\\ &= \|P_sv\|^2_{L^2(\mathbb{T}\times(0,T))} + \|P_av\|^2_{L^2(\mathbb{T}\times(0,T))} + 2\,(P_sv, P_av)_{L^2(\mathbb{T}\times(0,T))}\,.\end{aligned} \tag{4.52}$$

For the last term above

$$\begin{aligned}(P_sv, P_av)_{L^2(\mathbb{T}\times(0,T))} &= (-s\varphi_t v - cs\varphi_x v, v_t + cv_x)_{L^2(\mathbb{T}\times(0,T))}\\ &= (-s\varphi_t v, v_t)_{L^2(\mathbb{T}\times(0,T))} + (-s\varphi_t v, cv_x)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + (-cs\varphi_x v, v_t)_{L^2(\mathbb{T}\times(0,T))} + (-cs\varphi_x v, cv_x)_{L^2(\mathbb{T}\times(0,T))}\\ &= \int_0^T\int_{\mathbb{T}} -s\varphi_t v v_t\,dxdt + \int_0^T\int_{\mathbb{T}} -s\varphi_t v c v_x\,dxdt\\ &\quad + \int_0^T\int_{\mathbb{T}} -cs\varphi_x v v_t\,dxdt + \int_0^T\int_{\mathbb{T}} -c^2 s\varphi_x v v_x\,dxdt\\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We will treat each integral separately

$$I_1 = \frac{1}{2}\int_0^T\int_{\mathbb{T}} -s\varphi_t\left(v^2\right)_t\,dxdt = \frac{1}{2}\int_0^T\int_{\mathbb{T}} s\varphi_{tt}v^2dxdt - \frac{1}{2}\int_{\mathbb{T}} s\varphi_t v\Big|_0^T\,dx,$$

$$I_2 = \frac{1}{2}\int_0^T\int_{\mathbb{T}} -sc\varphi_t\left(v^2\right)_x\,dxdt = \frac{1}{2}\int_0^T\int_{\mathbb{T}} sc\varphi_{tx}v^2dxdt - \frac{1}{2}\int_0^T sc\varphi_t v^2\Big|_0^{2\pi},$$

$$I_3 = \frac{1}{2}\int_0^T\int_{\mathbb{T}} -cs\varphi_x\left(v^2\right)_t\,dxdt = \frac{1}{2}\int_0^T\int_{\mathbb{T}} cs\varphi_{xt}v^2dxdt - \frac{1}{2}\int_{\mathbb{T}} cs\varphi_x v^2\Big|_0^T\,dx,$$

$$I_4 = \frac{1}{2}\int_0^T\int_{\mathbb{T}} -c^2s\varphi_x\left(v^2\right)_x\,dxdt = \frac{1}{2}\int_0^T\int_{\mathbb{T}} c^2s\varphi_{xx}v^2dxdt - \frac{1}{2}\int_0^T c^2s\varphi_x v^2\Big|_0^{2\pi}\,dt.$$

Then, gathering all together, we have

$$\begin{aligned}2\,(P_sv, P_av)_{L^2(\mathbb{T}\times(0,T))} &= \int_0^T\int_{\mathbb{T}}\left(s\varphi_{tt} + 2sc\varphi_{tx} + c^2s\varphi_{xx}\right)v^2dxdt\\ &\quad - \int_{\mathbb{T}}\left(s\varphi_t + cs\varphi_x\right)v^2\Big|_0^T\,dx - \int_0^T\left(sc\varphi_t + c^2s\varphi_x\right)v^2\Big|_0^{2\pi}\,dt.\end{aligned} \tag{4.53}$$

Now, since $\varphi(x,t) = \psi(x) - \rho c^2t^2$, then $\varphi_x = \psi_x$ and $\varphi_t = -2\rho c^2 t$ and, according to (4.37), we have that $\varphi_x(0,t) = \varphi_x(2\pi,t), \varphi_t(0,t) = \varphi_t(2\pi,t)$, as well as $v(0,t) = v(2\pi,t)$, since $v(x,t) = e^{s\varphi}w$, with w periodic in x. Thus, the last term in (4.53) is null. From, (4.52) and (4.53), we have

$$\int_0^T\int_{\mathbb{T}}\left|e^{s\varphi}Pw\right|^2dxdt = \int_0^T\int_{\mathbb{T}} s^2\left|\varphi_t + c\varphi_x\right|^2|v|^2dxdt + \int_0^T\int_{\mathbb{T}}\left|v_t + cv_x\right|^2dxdt$$
$$+\int_0^T\int_{\mathbb{T}} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)v^2dxdt - \int_{\mathbb{T}} s\left(\varphi_t + c\varphi_x\right)v^2\Big|_{t=T} + \int_T s\left(\varphi_t + c\varphi_x\right)v^2\Big|_{t=0}.$$

Now, we shall make estimates for $(\varphi_{tt} + 2c\varphi_{xt} + c^2\varphi_{xx})$ and for $(\varphi_t + c\varphi_x)$. First we note that $(\varphi_t + c\varphi_x) = -2\rho c^2 t + c\psi_x$, then, for $x \in (0, 2\pi), t = T$ and from (4.35)-(4.38), we have that

$$-(\varphi_t + c\varphi_x) \geqslant 2c(\rho cT - 2\pi - \delta) > 0.$$

Analogously, for $x \in (0, 2\pi), t = 0$, we obtain

$$\varphi_t + c\varphi_x \geqslant 2c\delta > 0.$$

So, we have

$$\int_0^T\int_{\mathbb{T}} s^2\left|\varphi_t + c\varphi_x\right|^2\left|v^2\right|dxdt + \int_0^T\int_{\mathbb{T}}\left|v_t + cv_x\right|^2dxdt$$
$$+\int_0^T\int_{\mathbb{T}} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)v^2dxdt + \int_{\mathbb{T}} s(2c(\rho cT - 2\pi - \delta))|v|^2_{|t=T}$$
$$+\int_{\mathbb{T}} s(2c\delta)|v|^2_{|t=0} \leqslant \int_0^T\int_{\mathbb{T}} |e^{s\varphi Pw}|^2dxdt.$$

Then

$$\int_0^T\int_{\mathbb{T}} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)|v|^2dxdt + \int_{\mathbb{T}} s\left(|v|^2_{t=0} + |v|^2_{t=T}\right)dx$$
$$\leqslant C\int_0^T\int_{\mathbb{T}}\left|e^{s\varphi}Pw\right|^2dxdt.$$

But

$$\int_0^T\int_{\mathbb{T}} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)|v|^2dxdt = \int_0^T\int_{\mathbb{T}\setminus\omega_0} s(\overbrace{\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}}^{2(1-\rho)c^2})|v|^2dxdt$$
$$+\int_0^T\int_{\omega_0} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)|v|^2dxdt.$$

For $x \in \mathbb{T}\backslash\omega_0 = \left(\frac{\eta}{2}, 2\pi - \frac{\eta}{2}\right), t \in (0,T)$

$$\begin{aligned}
&\varphi(x,t) = |x+s|^2 - \rho c^2 t^2 \\
&\varphi_t = -2\rho c^2 t \\
&\varphi_{tt} = -2\rho c^2 \\
&\varphi_{xt} = 0 \\
&\varphi_{xx} = \psi_{xs} = 2, \quad \varphi_x = 2(x+\delta).
\end{aligned}$$

Then, $\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx} = 2(1-\rho)c^2 > 0$, for $\rho \in (0,1)$ and, as $\omega_0 \subset \omega$

$$\int_0^T \int_{\omega_0} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)|v|^2 dxdt \leqslant \int_0^T \int_{\omega} s\left(\varphi_{tt} + 2c\varphi_{tx} + c^2\varphi_{xx}\right)|v|^2 dxdt.$$

Therefore

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}} s|v|^2 dxdt + \int_{\mathbb{T}} s\left(|v|^2_{t=0} + |v|^2_{t=T}\right) dx \\
&\leqslant C\left(\int_0^T \int_{\mathbb{T}} \left|e^{s\varphi} Pw\right|^2 dxdt + \int_0^T \int_{\omega} s|v|^2 dxdt\right).
\end{aligned}$$

To finish the proof of the lemma 4.3.2 for $w \in H^1(\mathbb{T} \times (0,T))$, we replace v by $e^{s\varphi}w$, which produces (4.51).

We now claim that Lemma 4.3.2 is still true when w and $f := w_t + cw_x$ are in $L^2(0,T;L^2(\mathbb{T}))$. Indeed, in that case, from Aubin-Lions Theorem 2.1.18, we have $w \in C([0,T];L^2(\mathbb{T}))$, and, from density we can consider two sequences (w_0^n) and (f^n) in $H^1(\mathbb{T})$ and $L^2(0,T;H^1(\mathbb{T}))$, respectively, such that

$$\begin{aligned}
&w_0^n \to w(0) \quad \text{in } L^2(\mathbb{T}) \\
&f^n \to f \text{ in } L^2\left(0,T;L^2(\mathbb{T})\right)
\end{aligned}$$

then the solution $w^n \in C([0,T];H^1(\mathbb{T}))$ of

$$\begin{aligned}
&w_t^n + cw_x^n = f^n, \\
&w^n(0) = w_0^n
\end{aligned}$$

satisfies $w^n \in H^1(\mathbb{T} \times (0,T))$ and $w^n \to w$ in $C([0,T];L^2(\mathbb{T}))$, so that we can apply (4.51) to w^n and next pass to the limit $n \to \infty$ in (4.51). The proof of Lemma 4.3.2 is complete. ■

Proof. Finally, we are able to proof the Proposition 4.3.1, that is, the Carleman estimate for the KdV-BBM equation. Let $u \in L^2(0,T;H^2(\mathbb{T}))$ satisfying $u_t - u_{txx} - cu_{xxx} + qu_x = 0$, and let $w = u - u_{xx} \in L^2(0,T,L^2(\mathbb{T}))$. Then $w_t + cw_x = (c-q)u_x \in L^2(0,T;L^2(\mathbb{T}))$. Our task is to combine the following results proved so far

$$u - u_{xx} = w, \qquad (4.32)$$

$$w_t + cw_x = (c-q)u_x, \qquad (4.33)$$

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}} s|w|^2e^{2s\varphi}dxdt + \int_{\mathbb{T}} s\left[|w|^2e^{2s\varphi}\right]_{|t=0}dx + \int_{\mathbb{T}} s\left[|w|^2e^{2s\varphi}\right]_{|t=T}dx\\ &\leqslant C_1\left(\int_0^T\int_{\mathbb{T}}|w_t+cw_x|^2e^{2s\varphi}dxdt + \int_0^T\int_\omega s|w|^2e^{2s\varphi}dxdt\right), \qquad (4.51)\end{aligned}$$

and (4.41) multiplied by $e^{-2s\rho c^2t^2}$ and integrated over $(0,T)$

$$\int_0^T\int_{\mathbb{T}}\left[s|u_x|^2 + s^3|u|^2\right]e^{2s\varphi}dxdt \leqslant C_0\left(\int_0^T\int_{\mathbb{T}}|u_{xx}|^2e^{2s\varphi}dxdt + \int_0^T\int_\omega s^3|u|^2e^{2s\varphi}dxdt\right). \qquad (4.54)$$

Replacing $w_t + cw_x$ by $(c-q)u_x$ in (4.51) and adding to (4.54), we obtain, for $s \geqslant s_1$

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\left[s|u_x|^2 + s^3|u|^2 + s|u-u_{xx}|^2\right]e^{2s\varphi}dxdt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0}dx\\ &\leqslant C\left(\int_0^T\int_{\mathbb{T}}\left[|u_{xx}|^2 + |(c-q)u_x|^2\right]e^{2s\varphi}dxdt + \int_0^T\int_\omega\left[s|u-u_{xx}|^2 + s^3|u|^2\right]e^{2s\varphi}dxdt\right). \qquad (4.55)\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\left(s|u_{xx}|^2 + s|u_x|^2 + s^3|u|^2\right)e^{2s\varphi}dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0}dx\\ &= \int_0^T\int_{\mathbb{T}}\left(s|(u-u_{xx}) - u|^2 + s|u_x|^2 + s^3|u|^2\right)e^{2s\varphi}dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0}dx.\end{aligned}$$

Then, from (4.55), knowing that $(a-b)^2 \leqslant 2(a^2+b^2)$ and noting that $s \leqslant s^3$, since $s \geqslant 1$, we have that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \left(s|u_{xx}|^2 + s|u_x|^2 + s^3|u|^2\right) e^{2s\varphi} dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0} dx \\
&= \int_0^T \int_{\mathbb{T}} \left(s|(u-u_{xx})-u|^2 + s|u_x|^2 + s^3|u|^2\right) e^{2s\varphi} dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0} dx \\
&\leqslant C \int_0^T \int_{\mathbb{T}} \left[s\left|u-u_{xx}\right|^2 + s\left|u_x\right|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt + s\int_{\mathbb{T}} \left[\left|u-u_{xx}\right|^2 e^{2s\varphi}\right]_{|t=0} dx \\
&\leqslant C_1 \left(\int_0^T \int_{\mathbb{T}} \left[|u_{xx}|^2 + |(c-q)u_x|^2\right] e^{2s\varphi} dxdt + \int_0^T \int_{\omega} \left[s|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt\right) \\
&\leqslant \frac{\tilde{C}_1}{s} \left(\int_0^T \int_{\mathbb{T}} \left[s|u_{xx}|^2 + s|u_x|^2\right] e^{2s\varphi} dxdt\right) + C_1 \int_0^T \int_{\omega} \left[s|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt.
\end{aligned}
$$

Increasing the values of s if necessary, so that, $1 - \frac{\tilde{C}_1}{s} > 0$ we have

$$
\begin{aligned}
&\left(1 - \frac{\tilde{C}_1}{s}\right) \int_0^T \int_{\mathbb{T}} \left(s|u_{xx}|^2 + s|u_x|^2 + s^3|u|^2\right) e^{2s\varphi} dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0} dx \\
&\qquad C_1 \int_0^T \int_{\omega} \left[s|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \left(s|u_{xx}|^2 + s|u_x|^2 + s^3|u|^2\right) e^{2s\varphi} dt + \int_{\mathbb{T}} s\left[|u-u_{xx}|^2\right]_{|t=0} dx \\
&\qquad \leqslant C_1 \left(\frac{s}{s-\tilde{C}_1}\right) \int_0^T \int_{\omega} \left[s|u_{xx}|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt.
\end{aligned}
$$

For a sufficiently large s and $s_2 \geqslant s_1$ and $C_2 > \tilde{C}_1$ large enough we obtain the desired Carleman estimate

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \left[s\left|u_{xx}\right|^2 + s\left|u_x\right|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt + s\int_{\mathbb{T}} \left[\left|u-u_{xx}\right|^2 e^{2s\varphi}\right]_{|t=0} dx \\
&\leqslant C_2 \int_0^T \int_{\omega} \left[s\left|u_{xx}\right|^2 + s^3|u|^2\right] e^{2s\varphi} dxdt.
\end{aligned}
$$

■

We are now in a position to prove the UCP for the KdV-BBM equation stated in Theorem 4.3.1. We recall that u was set to belong to $L^2(0,T;H^2(\mathbb{T})) \cup L^\infty(0,T;H^1(\mathbb{T}))$. For $u \in L^2(0,T;H^2(\mathbb{T}))$ fulfilling (4.30) and (4.31), we obtain the UCP at once from the Carleman estimate (4.40).

On the other hand, for $u \in L^{\infty}(0,T;H^1(\mathbb{T}))$ we have that u and $w := u - u_{xx}$ are not regular enough to apply the Carleman estimates proved so far. So, we need to smooth them by using some convolution in time. For any function $v = v(x,t)$ and any number $h > 0$, we set

$$v^{[h]}(x,t) = \frac{1}{h}\int_t^{t+h} v(x,s)ds.$$

We recall that, from Proposition 2.1.19, if $v \in L^p(0,T;V)$, where $1 \leqslant p \leqslant +\infty$ and V denotes any Banach space, then $v^{[h]} \in W^{1,p}(0,T-h;V)$, $\left\|v^{[h]}\right\|_{L^p(0,T-h;V)} \leqslant \|v\|_{L^p(0,T;V)}$, and for $p < \infty$ and $T' < T$

$$v^{[h]} \to v \quad \text{in } L^p\left(0,T';V\right) \quad \text{as } h \to 0.$$

Then, in our context, $u^{[h]} \in W^{1,\infty}\left(0,T';H^1(\mathbb{T})\right)$, for any positive number $h < h_0 := T - T'$, with $T' \in \left(\frac{2\pi}{c}, T\right)$. In the sequel, $u_t^{[h]}$ denotes $\left(u^{[h]}\right)_t$, $u_x^{[h]}$ denotes $\left(u^{[h]}\right)_x$, etc. We assume again $c > 0$, the pair (ρ,δ) satisfying (4.35) with T replaced by T' and we define the functions ψ and φ as before. Thus, in that conditions, since

$$u_t - u_{txx} - cu_{xxx} + (qu_x) = 0,$$

we have that $u^{[h]}$ solves

$$u_t^{[h]} - u_{txx}^{[h]} - cu_{xxx}^{[h]} + (qu_x)^{[h]} = 0 \quad \text{in } \mathcal{D}'\left(0,T';H^{-2}(\mathbb{T})\right), \tag{4.56}$$

$$u^{[h]}(x,t) = 0, \quad (x,t) \in \omega \times (0,T'). \tag{4.57}$$

Since $u^{[h]} \in W^{1,\infty}\left(0,T';H^1(\mathbb{T})\right)$, from (4.56), we infer that

$$u_{xxx}^{[h]} = c^{-1}\left(u_t^{[h]} - u_{txx}^{[h]} + (qu_x)^{[h]}\right) \in L^{\infty}\left(0,T';H^{-1}(\mathbb{T})\right),$$

hence

$$u^{[h]} \in L^{\infty}\left(0,T';H^2(\mathbb{T})\right) \tag{4.58}$$

This yields, with (4.32) and (4.33)

$$w^{[h]} = u^{[h]} - u_{xx}^{[h]} \in L^{\infty}\left(0,T';L^2(\mathbb{T})\right) \tag{4.59}$$

$$w_t^{[h]} + cw_x^{[h]} = ((c-q)u_x)^{[h]} \in W^{1,\infty}\left(0, T; L^2(\mathbb{T})\right) \tag{4.60}$$

Thus, from lemmas 4.3.1 and 4.3.2 and from (4.57) and (4.59), we infer that there exists some constants $s_1 > 0$ and $C_1 > 0$ such that for all $s \geqslant s_1$ and all $h \in (0, h_0)$, we have

$$\begin{aligned}
&\int_0^{T'} \int_{\mathbb{T}} \left[s\left|u_x^{[h]}\right|^2 + s^3 |u^{[h]}|^2 + s\left|w^{[h]}\right|^2 \right] e^{2s\varphi} dxdt \\
&\qquad \leqslant C_0 \left(\int_0^{T'} \int_{\mathbb{T}} \left|u_{xx}^{[h]}\right|^2 + \left|((c-q)u_x)^{[h]}\right|^2 e^{2s\varphi} dxdt \right) \\
&\qquad \leqslant C_0 \left(\int_0^{T'} \int_{\mathbb{T}} \left|u^{[h]} - w^{[h]}\right|^2 + \left|((c-q)u_x)^{[h]}\right|^2 e^{2s\varphi} dxdt \right).
\end{aligned}$$

But, from (4.58)-(4.60), we infer that

$$\begin{aligned}
&\int_0^{T'} \int_{\mathbb{T}} \left(s\left|u_x^{[h]}\right|^2 + s^3 \left|u^{[h]}\right|^2 + s\left|w^{[h]}\right|^2 \right) e^{2s\varphi} dxdt \\
&\qquad \leqslant C_1 \int_0^{T'} \int_{\mathbb{T}} \left(\left|u^{[h]}\right|^2 + \left|w^{[h]}\right|^2 + \left|((c-q)u_x)^{[h]}\right|^2 \right) e^{2s\varphi} dxdt \\
&\qquad \leqslant C_1 \int_0^{T'} \int_{\mathbb{T}} \left(\left|u^{[h]}\right|^2 + \left|w^{[h]}\right|^2 + 2\left|(c-q)u_x^{[h]}\right|^2 \right. \\
&\qquad\quad \left. + 2\left|((c-q)u_x)^{[h]} - (c-q)u_x^{[h]}\right|^2 \right) e^{2s\varphi} dxdt.
\end{aligned} \tag{4.61}$$

Comparing the powers of s in (4.61), we deduce that, by increasing the the constants s_1 and C_1 in a convenient way, for $s \geqslant s_3 > s_1, h \in (0, h_0)$ and some constant $C_3 > C_1$ (that does not depend on s, h), we have that

$$\begin{aligned}
&\int_0^{T'} \int_{\mathbb{T}} \left(s\left|u_x^{[h]}\right|^2 + s^3 \left|u^{[h]}\right|^2 + s\left|w^{[h]}\right|^2 \right) e^{2s\varphi} dxdt \\
&\qquad \leqslant C_3 \int_0^{T'} \int_{\mathbb{T}} \left|((c-q)u_x)^{[h]} - (c-q)u_x^{[h]}\right|^2 e^{2s\varphi} dxdt.
\end{aligned}$$

Fix s to the value s_3, and let $h \to 0$. We claim that

$$\int_0^{T'} \int_{\mathbb{T}} \left|((c-q)u_x)^{[h]} - (c-q)u_x^{[h]}\right|^2 e^{2s_3\varphi} dxdt \to 0 \quad \text{as } h \to 0.$$

Indeed, as $h \to 0$,

$$((c-q)u_x)^{[h]} \to (c-q)u_x \quad \text{in } L^2\left(0,T';L^2(\mathbb{T})\right),$$
$$(c-q)u_x^{[h]} \to (c-q)u_x \quad \text{in } L^2\left(0,T';L^2(\mathbb{T})\right),$$

while $e^{2s_3\varphi} \in L^\infty(\mathbb{T}\times(0,T'))$. Therefore,

$$\int_0^{T'}\int_{\mathbb{T}}\left(s_3\left|u_x^{[h]}\right|^2 + s_3^3\left|u^{[h]}\right|^2 + s_3\left|w^{[h]}\right|^2\right)e^{2s_3\varphi}dxdt \to 0 \text{ as } h \to 0.$$

In particular,

$$\int_0^{T'}\int_{\mathbb{T}}\left|u^{[h]}\right|^2 e^{2s_3\varphi}dxdt \to 0 \text{ as } h \to 0.$$

On the other hand, $u^{[h]} \to u$ in $L^2(0,T';L^2(\mathbb{T}))$, hence

$$\int_0^{T'}\int_{\mathbb{T}}\left|u^{[h]}\right|^2 e^{2s_3\varphi}dxdt \to \int_0^{T'}\int_{\mathbb{T}}|u|^2 e^{2s_3\varphi}dxdt \text{ as } h \to 0.$$

From uniqueness of limit, we conclude that

$$\int_0^{T'}\int_{\mathbb{T}}|u|^2 e^{2s_3\varphi}dxdt = 0.$$

Therefore, $u \equiv 0$ in $\mathbb{T}\times(0,T')$. As T' may be taken arbitrarily close to T, we infer that $u \equiv 0$ in $\mathbb{T}\times(0,T)$, as desired. The proof of Theorem 4.3.1 is complete. ■

5 CONTROL AND STABILIZATION OF THE KDV-BBM EQUATION

This chapter, devoted to the controllability of the BBM equation, is the core of this work. When dealing with PDE we have at our disposal three concepts of controllability; namely the *exact controllability* (any pair of state vectors may be connected by a trajectory), *the null controllability* (any state vector may be steered to 0) and the *approximate controllability* (any state vector may be steered arbitrarily close to another state vector). One of the main concerns of control theory is the relationship between controllability and stabilizability. We begin the chapter by treating the controllability concept, while the latter is addressed in the second section. We refer the reader to section 2.2, for more details.

We consider the following system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x + uu_x = a(x)h, \quad x \in \mathbb{T}, t \geqslant 0 \tag{5.1}$$

$$u(x,0) = u_0(x), \tag{5.2}$$

where $c \in \mathbb{R}\backslash\{0\}$ and $a \in C^\infty(\mathbb{T})$ is a given nonzero function. Let

$$\omega = \{x \in \mathbb{T}; a(x) \neq 0\} \neq \emptyset. \tag{5.3}$$

5.1 Exact controllability

We begin with a local controllability result, in sufficiently large time, for the system (5.1)-(5.2). However, to prove this, we first treat the linear case.

Theorem 5.1.1. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0, s \geqslant 0$ and $T > 2\pi/|c|$. Then there exists a $\delta > 0$ such that for any $u_0, u_T \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} + \|u_T\|_{H^s} < \delta,$$

one can find a control input $h \in L^2\left(0,T;H^{s-2}(\mathbb{T})\right)$ such that the system (5.1)-(5.2) admits a unique solution $u \in C\left([0,T],H^s(\mathbb{T})\right)$ satisfying $u(\cdot,T) = u_T$.

Proof. The result is first proved for the linearized equation, and next extended to the nonlinear one by a fixed-point argument.

5.1.1 Exact Controllability of the Linearized System

We first consider the exact controllability of the linearized system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x = a(x)h, \tag{5.4}$$

$$u(x,0) = u_0(x), \tag{5.5}$$

in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. Note that (5.4) can be written as

$$\begin{aligned} \partial_t u &= \left(1-\partial_x^2\right)^{-1}\left(c\partial_x^3 - (c+1)\partial_x\right)u + \left(1-\partial_x^2\right)^{-1}[a(x)h] \\ \partial_t u &= Au + \left(1-\partial_x^2\right)^{-1}[a(x)h], \end{aligned}$$

in which $A = (1-\partial_x^2)^{-1}(c\partial_x^3 - (c+1)\partial_x)$ is the operator with domain $D(A) = H^{s+1}(\mathbb{T}) \subset H^s(\mathbb{T})$. We claim that A is skew-adjoint, that is, $A^* = -A$. Indeed, let $f, g \in H^{s+1}(\mathbb{T})$, then

$$\begin{aligned} (Af,g)_{H^{s+1}(\mathbb{T})} &= \left(\left(1-\partial_x^2\right)^{-1}\left(c\partial_x^3 - (c+1)\partial_x\right)f, g\right)_{H^{s+1}(\mathbb{T})} \\ &= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{s+1} \widehat{(1-\partial_x^2)^{-1}(c\partial_x^3-(c+1)\partial_x)f}(k)\overline{\hat{g}(k)} \\ &= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{s+1} \frac{(-i)(ck^3+(c+1)k)}{1+k^2}\hat{f}(k)\overline{\hat{g}(k)} \\ &= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{s+1} \hat{f}(k)\overline{\hat{g}(k)\frac{i(ck^3+(c+1)k)}{1+k^2}} \\ &= -\sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{s+1} \hat{f}(k)\overline{\hat{g}(k)\frac{(-i)(ck^3+(c+1)k)}{1+k^2}} \\ &= -\sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{s+1} \hat{f}(k)\overline{\widehat{(1-\partial_x^2)^{-1}(c\partial_x^3-(c+1)\partial_x)g}(k)} \\ &= -\left(f, \left(c\partial_x^3-(c+1)\partial_x\right)g\right)_{H^{s+1}(\mathbb{T})} \\ &= -(f,Ag)_{H^{s+1}(\mathbb{T})} \\ &= (f,-Ag)_{H^{s+1}(\mathbb{T})} \end{aligned}$$

Therefore, from Stone's Theorem, the operator A generates a group of isometries

$\{W(t)\}_{t\in\mathbb{R}} = \{e^{tA}\}_{t\in\mathbb{R}}$ in $H^s(\mathbb{T})$. Note that

$$\mathcal{F}(Av)(k) = \widehat{Av}(k) = \frac{(-i)(ck^3+(c+1)k)}{1+k^2}\hat{f}(k),$$

so that, we have the following representation in Fourier series for any $v \in H^s(\mathbb{T})$

$$W(t)v = e^{tA}v = \sum_{k=-\infty}^{\infty} e^{-it\frac{ck^3+(c+1)k}{k^2+1}}\hat{v}_k e^{ikx}, \tag{5.6}$$

$$v = \sum_{k=-\infty}^{\infty} \hat{v}_k e^{ikx} \in H^s(\mathbb{T}).$$

The system (5.4)-(5.5) may be cast into the following integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}[a(x)h(\tau)]d\tau.$$

Take $h(x,t)$ in (5.4) to have the following form

$$h(x,t) = a(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}, \tag{5.7}$$

where f_j and $q_j(t)$ are to be determined later. Then the solution u of equation (5.4) can be written as

$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t)e^{ikx},$$

that we substitute it into (5.4) to obtain

$$\left(\partial_t - \partial_t\partial_x^2 - c\partial_x^3 + (c+1)\partial_x\right)\left(\sum_{k=-\infty}^{\infty}\hat{u}_k(t)e^{ikx}\right) = a(x)a(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}$$

For each k, we have

$$\left(1+k^2\right)\frac{d}{dt}\hat{u}_k(t)e^{ikx} + (cik^3+(c+1)ik)\hat{u}_k(t)e^{ikx} = a^2(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}$$

$$\frac{d}{dt}\hat{u}_k(t)e^{ikx} + \frac{ik(ck^2+c+1)}{1+k^2}\hat{u}_k(t)e^{ikx} = \frac{1}{1+k^2}a^2(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}$$

$$\frac{d}{dt}\hat{u}_k(t) + \frac{ik(ck^2+c+1)}{1+k^2}\hat{u}_k(t) = \frac{1}{1+k^2}a^2(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}e^{-ikx}$$

$$\frac{d}{dt}\hat{u}_k(t) + ik\sigma(k)\hat{u}_k(t) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j q_j(t) a^2(x) e^{i(j-k)x}.$$

Then $\hat{u}_k(t)$ solves the following ODE

$$\frac{d}{dt}\hat{u}_k(t) + ik\sigma(k)\hat{u}_k(t) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j q_j(t) m_{j,k}, \tag{5.8}$$

with $\sigma(k) = \dfrac{ck^2+c+1}{1+k^2}$, and

$$m_{j,k} = \frac{1}{2\pi}\int_{\mathbb{T}} a^2(x) e^{i(j-k)x} dx.$$

Thus, from the theory of ODE, its solution at $t = T$ is

$$\hat{u}_k(T) = e^{-ik\sigma(k)T}\hat{u}_k(0) + \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{-ik\sigma(k)(T-\tau)} q_j(\tau) d\tau$$

$$\hat{u}_k(T) - e^{-ik\sigma(k)T}\hat{u}_k(0) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{-ik\sigma(k)(T-\tau)} q_j(\tau) d\tau.$$

Or, multiplying by $e^{ik\sigma(k)T}$

$$\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{ik\sigma(k)\tau} q_j(\tau) d\tau.$$

It may occur that the eigenvalues

$$\lambda_k = ik\sigma(k), \quad k \in \mathbb{Z},$$

are not all different. If we count only the distinct values, we obtain the sequence $(\lambda_k)_{k\in\mathbb{I}}$, where $\mathbb{I} \subset \mathbb{Z}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1, k_2 \in \mathbb{I}$ with $k_1 \neq k_2$. For each $k_1 \in \mathbb{Z}$ set

$$I(k_1) = \{k \in \mathbb{Z}; k\sigma(k) = k_1\sigma(k_1)\},$$

and $m(k_1) = |I(k_1)|$ (the number of elements in $I(k_1)$). We note that there exists some integer k^* such that $k \in \mathbb{I}$ if $|k| > k^*$. Thus there are only finite many integers in $\mathbb{I}$, say

$k_j, j = 1, \ldots, n$, such that one can find another integer $k \neq k_j$ with $\lambda_k = \lambda_{k_j}$. Let

$$\mathbb{I}_j = \left\{k \in \mathbb{Z}; k \neq k_j, \lambda_k = \lambda_{k_j}\right\}, \quad j = 1, 2, \ldots, n.$$

Then

$$\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \cdots \cup \mathbb{I}_n.$$

Note that $\mathbb{I}_j$ contains at most two integers, for $m(k_j) \leqslant 3$. This is a consequence of the fact that $m(k_j)$ is less than the number of entire roots of the equation

$$\begin{aligned} x\sigma(x) &= \alpha \\ x\sigma(x) &= x\frac{cx^2 + c + 1}{1 + x^2} = \alpha, \end{aligned}$$

where α is an arbitrary real number. The roots of this equation are also roots of a polynomial of degree less or equal to 3. Then, as $\mathbb{I}_j = I(k_j) \backslash \{k_j\}$, we have that $|\mathbb{I}_j|$, the number of elements of $\mathbb{I}_j$, is $m(k_j) - 1$, that is, $|\mathbb{I}_j| \leqslant 2$. We write

$$\mathbb{I}_j = \left\{k_{j,1}, k_{j,m(k_j)-1}\right\}, \quad j = 1, 2, \ldots, n$$

and rewrite k_j as $k_{j,0}$. That is, the n elements of $\mathbb{I}$ will be denote by $k_{j,0}$ for $j = 1, \ldots, n$. Let

$$p_k(t) := e^{-ik\sigma(k)t} = e^{-\lambda_k t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Then the set

$$\mathcal{P} := \{p_k(t); k \in \mathbb{I}\}$$

forms a Riesz basis (see definition 2.1.16) for its closed span, $\mathcal{P}_T$, in $L^2(0, T)$ if

$$T > \frac{2\pi}{|c|}.$$

Let $\mathcal{L} := \{q_j(t); j \in \mathbb{I}\}$ be the unique dual Riesz basis for $\mathcal{P}$ in $\mathcal{P}_T$; that is, the functions in $\mathcal{L}$ are the unique elements of $\mathcal{P}_T$ such that

$$(q_j(t), p_k(t))_{L^2(0,T)} = \int_0^T q_j(t)\overline{p_k(t)}dt = \delta_{kj} = \begin{cases} 1, \text{ if } j = k \\ 0, \text{ if } j \neq k \end{cases}, \quad j, k \in \mathbb{I}.$$

In addition, we choose

$$q_k = q_{k_j} \quad \text{if } k \in \mathbb{I}_j.$$

From the solution $\hat{u}_k(T)$ of the ODE (5.8)

$$\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{ik\sigma(k)\tau}q_j(\tau)d\tau$$
$$\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T \overline{p_k(\tau)}q_j(\tau)d\tau.$$

Then, for such choice of $q_j(t)$, we have, for any $k \in \mathbb{Z}$

$$\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0) = \frac{1}{1+k^2}f_k m_{k,k}, \quad \text{if } k \in \mathbb{I}\backslash\{k_1,\ldots,k_n\}; \tag{5.9}$$

$$\hat{u}_{k_{j,q}}(T)e^{ik_j\sigma(k_j)T} - \hat{u}_{k_{j,q}}(0) = \frac{1}{1+k_{j,q}^2}\sum_{l=0}^{m(k_j)-1} f_{k_{j,l}}m_{k_{j,l},k_{j,q}}, \quad \text{if } k = k_{j,q}, j = 1,\ldots,n, q = 0,\ldots,m(k_j)-1. \tag{5.10}$$

It is known that for any finite set $\mathcal{J} \subset \mathbb{Z}$, the Gram matrix $A_{\mathcal{J}} = (m_{p,q})_{p,q\in\mathcal{J}}$ is definite positive, hence invertible. It follows that the system (5.9)-(5.10) admits a unique solution $\vec{f}(\ldots,f_{-2},f_{-1},\ f_0,f_1,f_2,\ldots)$. Since

$$m_{k,k} = \frac{1}{2\pi}\int_{\mathbb{T}} a^2(x)e^{i(k-k)x}dx = \frac{1}{2\pi}\int_{\mathbb{T}} a^2(x)dx =: \mu \neq 0,$$

we have, from (5.9),

$$\left(1+k^2\right)\left(\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0)\right) = f_k\mu$$
$$f_k = \frac{1+k^2}{\mu}\left(\hat{u}_k(T)e^{ik\sigma(k)T} - \hat{u}_k(0)\right) \quad \text{for } |k| > k^*.$$

Note that

$$\|h\|^2_{L^2(0,T;H^{s-2}(\mathbb{T}))} = \int_0^T \|h(\cdot,t)\|^2_{H^{s-2}}\,dt$$
$$= \int_0^T \left\|a(x)\sum_{j=-\infty}^{\infty} f_j q_j(t)e^{ijx}\right\|^2_{H^{s-2}} dt$$

$$\begin{aligned}
&\leq C\int_0^T \left\| \sum_{j=-\infty}^{\infty} f_j q_j(t) e^{ijx} \right\|_{H^{s-2}}^2 dt \\
&\leqslant C\int_0^T \sum_{k=-\infty}^{\infty} \left(1+k^2\right)^{s-2} \left| \widehat{\sum_{j=-\infty}^{\infty} f_j q_j(t) e^{ijx}}(k) \right|^2 dt \\
&\leqslant C\int_0^T \sum_{k=-\infty}^{\infty} \left(1+k^2\right)^{s-2} \left| \sum_{j=-\infty}^{\infty} f_j q_j(t) \widehat{e^{ijx}}(k) \right|^2 dt \\
&\leqslant C\int_0^T \sum_{k=-\infty}^{\infty} \left(1+k^2\right)^{s-2} \left| \sum_{j=-\infty}^{\infty} f_j q_j(t) \delta_{jk} \right|^2 dt \\
&\leqslant C\int_0^T \sum_{j=-\infty}^{\infty} \left(1+j^2\right)^{s-2} \left| f_j q_j(t) \right|^2 dt \\
&\leqslant C \sum_{j=-\infty}^{\infty} \left(1+j^2\right)^{s-2} \left| f_j \right|^2 dt \\
&\leqslant C \left(\sum_{j=-\infty}^{\infty} \left(1+j^2\right)^{s} \left| \hat{u}_j(0) \right|^2 + \sum_{j=-\infty}^{\infty} \left(1+j^2\right)^{s} \left| \hat{u}_j(T) \right|^2 \right) \\
&\leqslant C \left(\|u(0)\|_{H^s}^2 + \|u(T)\|_{H^s}^2 \right).
\end{aligned}$$

This analysis leads us to the following controllability result for the linear system (5.4)-(5.5).

Proposition 5.1.1. *Let $s \in \mathbb{R}$ and $T > \frac{2\pi}{|c|}$ be given. For any $u_0, u_T \in H^s(\mathbb{T})$, there exists a control $h \in L^2\left(0,T; H^{s-2}(\mathbb{T})\right)$ such that the system (5.4)-(5.5) admits a unique solution $u \in C\left([0,T]; H^s(\mathbb{T})\right)$ satisfying*

$$u(x,T) = u_T(x).$$

Moreover, there exists a constant $C > 0$ depending only on s and T such that

$$\|h\|_{L^2(0,T;H^{s-2}(\mathbb{T}))} \leqslant C\left(\|u_0\|_{H^s} + \|u_T\|_{H^s}\right).$$

Introduce the (bounded) operator $\Phi : H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to L^2\left(0,T; H^{s-2}(\mathbb{T})\right)$ defined by

$$\Phi\left(u_0, u_T\right)(t) = h(t)$$

where h is given by (5.7) and $\vec{f}$ is the solution of (5.9)-(5.10) with $\left(\hat{u}_0\right)_k$ and $\left(\hat{u}_T\right)_k$ substituted to $\hat{u}_k(0)$ and $\hat{u}_k(T)$, respectively. Then $h = \Phi\left(u_0, u_T\right)$ is a control driving the solution u of (5.4)-(5.5) from u_0 at $t=0$ to u_T at $t=T$.

5.1.2 Local Exact Controllability of the BBM Equation

Pick any time $T>2\pi/|c|$, and any $u_0, u_T \in H^s(\mathbb{T})(s \geqslant 0)$ satisfying

$$\|u_0\|_{H^s} \leqslant \delta, \quad \|u_T\|_{H^s} \leqslant \delta,$$

with δ to be determined. For any $u \in C\left([0,T]; H^s(\mathbb{T})\right)$, we set

$$\omega(u) = -\int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\left(uu_x\right)(\tau)d\tau.$$

Then, for any $u, v \in C\left([0,T]; H^s(\mathbb{T})\right)$, it follows that

$$\begin{aligned}
\omega(u)-\omega(v) &= \int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\left(vv_x - uu_x\right)(\tau)d\tau \\
&= \frac{1}{2}\int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(v^2-u^2\right)d\tau \\
&= \frac{1}{2}\int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left[(v-u)(v+u)\right]d\tau.
\end{aligned}$$

That is,

$$\begin{aligned}
\|\omega(u)-\omega(v)\|_{H^s} &= \frac{1}{2}\left\|\int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left[(v-u)(v+u)\right]d\tau\right\|_{H^s} \\
&\leqslant CT\|u+v\|_{L^\infty(0,T;H^s(\mathbb{T}))}\|u-v\|_{L^\infty(0,T;H^s(\mathbb{T}))}
\end{aligned}$$

where we have aplied Lemma 3.1.1.

Furthermore,

$$\begin{aligned}
&W(t)u_0 + \int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T-\omega(u)\right)-uu_x\right](\tau)d\tau \\
&= \begin{cases} u_0 & \text{if } t=0, \\ \omega(u)+\left(u_T-\omega(u)\right)=u_T & \text{if } t=T. \end{cases}
\end{aligned}$$

Indeed, for the case $t = T$ we have

$$\begin{aligned}
&W(T)u_0 + \int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right) - uu_x\right](\tau)d\tau \\
&= W(T)u_0 + \int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right)\right](\tau)d\tau \\
&\qquad - \int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}(uu_x)(\tau)d\tau \\
&= W(T)u_0 + \int_0^T W(T-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right)\right](\tau)d\tau + \omega(u) \\
&= (u_T - \omega(u)) + \omega(u) \\
&= u_T.
\end{aligned}$$

We are led to consider the nonlinear map

$$\Gamma(u) = W(t)u_0 + \int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right) - uu_x\right](\tau)d\tau.$$

The proof of Theorem 5.1.1 will be complete if we can show that the map Γ has a fixed point in some closed ball of the space $C\left([0,T]; H^s(\mathbb{T})\right)$. For any $R > 0$, let

$$B_R = \left\{u \in C\left([0,T]; H^s(\mathbb{T})\right); \|u\|_{C([0,T];H^s(\mathbb{T}))} \leqslant R\right\}.$$

So, for $u \in B_R$ and for $t \in [0,T]$ we have

$$\begin{aligned}
\|\Gamma(u)(t)\|_{H^s} &\leqslant \|u_0\|_{H^s} + \left\|\int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right) - uu_x\right](\tau)d\tau\right\|_{H^s} \\
&\leqslant \|u_0\|_{H^s} + \int_0^t \left\|\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right) - uu_x\right](\tau)\right\|_{H^s} d\tau \\
&\leqslant \|u_0\|_s + T\left\|\left(1-\partial_x^2\right)^{-1}\left[a(x)\Phi\left(u_0, u_T - \omega(u)\right)\right]\right\|_s + T\left\|\left(1-\partial_x^2\right)^{-1}(uu_x)\right\|_s \\
&\leqslant \|u_0\|_{H^s} + C\left\|a(x)\Phi\left(u_0, u_T - \omega(u)\right)\right\|_{H^{s-2}} + \frac{T}{2}\left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right)\right\|_{H^s} \\
&\leqslant \|u_0\|_{H^s} + C\left(\|u_0\|_{H^s} + \|u_T\|_{H^s}\right) + C_2\|u\|_{H^s}^2 \\
&\leqslant C_1\left(\|u_0\|_{H^s} + \|u_T\|_{H^s}\right) + C_2R^2.
\end{aligned}$$

Therefore,

$$\sup_{t\in[0,T]} \|\Gamma(u)(t)\|_{H^s} \leqslant C_1\left(\|u_0\|_{H^s} + \|u_T\|_{H^s}\right) + C_2R^2,$$

that is

$$\|\Gamma(u)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant C_1 \left(\|u_0\|_{H^s} + \|u_T\|_{H^s}\right) + C_2 R^2,$$

with C_1, C_2 depending on s and T, but not on R, $\|u_0\|_{H^s}$ or $\|u_T\|_{H^s}$.

Thus, picking $R = (2C_2)^{-1}$ and $\delta = (8C_1C_2)^{-1}$, we obtain for u_0, u_T satisfying

$$\|u_0\|_{H^s} \leqslant \delta, \quad \|u_T\|_{H^s} \leqslant \delta,$$

and $u \in B_R$ that

$$\|\Gamma(u)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant R.$$

That is, $\Gamma(B_R) \subset B_T$.

To conclude, we shall show that the map Γ is a contraction map, that is, that

$$\|\Gamma(u) - \Gamma(v)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant \lambda \|u - v\|_{C([0,T];H^s(\mathbb{T}))},$$

for $u, v \in B_R$, for some $\lambda \in (0,1)$. So, we have that

$$\begin{aligned}\Gamma(u) - \Gamma(v) &= \int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}(vv_x - uu_x)(\tau)d\tau \\ &= \frac{1}{2}\int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left[(v-u)(v+u)\right](\tau)d\tau.\end{aligned}$$

From an analogous calculation we get that

$$\begin{aligned}\|\Gamma(u) - \Gamma(v)\|_{H^s} &\leqslant \frac{T}{2}\left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left[(v-u)(v+u)\right]\right\|_{H^s} \\ &\leqslant C\left\|u-v\right\|_{H^s}\left\|u+v\right\|_{H^s} \\ &\leqslant C_2 R\left\|u-v\right\|_{H^s}.\end{aligned}$$

Taking the supremum and picking $R = (2C_2)^{-1}$ we obtain

$$\|\Gamma(u) - \Gamma(v)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant \frac{1}{2}\|u - v\|_{C([0,T];H^s(\mathbb{T}))}.$$

To sum it up, we have seen that, for all $u, v \in B_R$, then

$$\|\Gamma(u)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant R, \tag{5.11}$$

and,

$$\|\Gamma(u) - \Gamma(v)\|_{C([0,T];H^s(\mathbb{T}))} \leqslant \frac{1}{2}\|u - v\|_{C([0,T];H^s(\mathbb{T}))}. \tag{5.12}$$

From the contraction mapping theorem, it follows that Γ has a unique fixed point u in B_R. Then u satisfies (5.1)-(5.2) with $h = \Phi(u_0, u_T - \omega(u))$ and $u(T) = u_T$, as desired. The proof of Theorem 5.1.1 is complete. ■

5.2 Exponential Stabilizability

We are now concerned with the stabilization of (5.1)-(5.2) with a feedback law $h = h(u)$. To guess the expression of h, we first write the linearized system (5.4)-(5.5) in a convenient way

$$\begin{aligned}
&u_t - u_{txx} - cu_{xxx} + (c+1)u_x = a(x)h \\
&\left(1 - \partial_x^2\right)\partial_t u = \left(c\partial_x^3 - (c+1)\partial_x\right)u + a(x)h \\
&\partial_t u = \left(1 - \partial_x^2\right)^{-1}\left(c\partial_x^3 - (c+1)\partial_x\right)u + \left(1 - \partial_x^2\right)^{-1} ah \\
&\partial_t u = \underbrace{\left(1 - \partial_x^2\right)^{-1}\left(c\partial_x^3 - (c+1)\partial_x\right)}_{A} u + \underbrace{\left(1 - \partial_x^2\right)^{-1} a\left(1 - \partial_x^2\right)}_{B}\underbrace{\left(1 - \partial_x^2\right)^{-1} h}_{k}.
\end{aligned}$$

Then, we have

$$u_t = Au + Bk, \tag{5.13}$$

$$u(0) = u_0, \tag{5.14}$$

where $A = (1 - \partial_x^2)^{-1}(c\partial_x^3 - (c+1)\partial_x) \in \mathcal{L}(H^{s+1}; H^s)$ as before in section (5.1), $k(t) = (1 - \partial_x^2)^{-1} h(t) \in L^2(0, T; H^s(\mathbb{T}))$ is the new control input, and

$$B = \left(1 - \partial_x^2\right)^{-1} a\left(1 - \partial_x^2\right) \in \mathcal{L}\left(H^s(\mathbb{T})\right). \tag{5.15}$$

We already noticed that A is skew-adjoint in $H^s(\mathbb{T})$, and that (5.13)-(5.14) is exactly controllable in $H^s(\mathbb{T})$, with some control functions $k \in L^2(0, T; H^s(\mathbb{T}))$, for any $s \geqslant 0$. If we choose the simple feedback law

$$k = -B^{*,s}u, \tag{5.16}$$

the resulting closed-loop system

$$u_t = Au - BB^{*,s}u = \left(A - BB^{*,s}\right)u, \tag{5.17}$$

$$u(0) = u_0, \tag{5.18}$$

is exponentially stable in $H^s(\mathbb{T})$ (see Corollary 2.2.1). In (5.16), $B^{*,s}$ denotes the adjoint of B in $\mathcal{L}\left(H^s(\mathbb{T})\right)$, that is, $B^{*,s}$ is the operator in $\mathcal{L}\left(H^s(\mathbb{T})\right)$ such that for all $u, v \in H^s(\mathbb{T})$, we have $(Bu, v)_{H^s(\mathbb{T})} = (u, B^{*,s}v)_{H^s(\mathbb{T})}$.

Then, by computing $B^{*,s}$ we obtain that

$$\begin{aligned}
(Bu, v)_{H^s(\mathbb{T})} &= \int_{\mathbb{T}} \left(1+x^2\right)^s \mathcal{F}\left(\left(1-\partial_x^2\right)^{-1} a(x)\left(1-\partial_x^2\right)u(x)\right)\overline{\mathcal{F}(v(x))}dx \\
&= \int_{\mathbb{T}} \left(1+x^2\right)^s \frac{1}{1+x^2}\mathcal{F}\left(a(x)\left(1-\partial_x^2\right)u(x)\right)\overline{\mathcal{F}(v(x))}dx \\
&= \int_{\mathbb{T}} \left(1+x^2\right)^{s-1} \mathcal{F}\left(a(x)\left(1-\partial_x^2\right)u(x)\right)\overline{\mathcal{F}(v(x))}dx \\
&= \int_{\mathbb{T}} \mathcal{F}\left(a(x)\left(1-\partial_x^2\right)u(x)\right)\overline{\mathcal{F}\left(1-\partial_x^2\right)^{s-1}v(x)}dx \\
&= \left(a(x)\left(1-\partial_x^2\right)u(x), \left(1-\partial_x^2\right)^{s-1}v(x)\right)_{L^2(\mathbb{T})} \\
&= \left(\left(1-\partial_x^2\right)u(x), a(x)\left(1-\partial_x^2\right)^{s-1}v(x)\right)_{L^2(\mathbb{T})} \\
&= \int_{\mathbb{T}} \mathcal{F}\left(1-\partial_x^2\right)u(x)\overline{\mathcal{F}\left(a(x)\left(1-\partial_x^2\right)^{s-1}v(x)\right)}dx \\
&= \int_{\mathbb{T}} \left(1+x^2\right)^s\left(1+x^2\right)^{1-s}\mathcal{F}u(x)\overline{\mathcal{F}\left(a(x)\left(1-\partial_x^2\right)^{s-1}v(x)\right)}dx \\
&= \int_{\mathbb{T}} \left(1+x^2\right)^s\mathcal{F}(u(x))\overline{\mathcal{F}\left(\left(1-\partial_x^2\right)^{1-s}a(x)\left(1-\partial_x^2\right)^{s-1}v(x)\right)}dx \\
&= \left(u, B^{*,s}v\right)_{H^s(\mathbb{T})}.
\end{aligned}$$

That is

$$B^{*,s}u = \left(1-\partial_x^2\right)^{1-s} a\left(1-\partial_x^2\right)^{s-1}u. \tag{5.19}$$

In particular

$$B^{*,1}u = au.$$

Let $\tilde{A} = A - BB^{*,1}$, where $\left(BB^{*,1}\right)u = \left(1-\partial_x^2\right)^{-1}\left[a\left(1-\partial_x^2\right)(au)\right]$. Since $BB^{*,1} \in \mathcal{L}\left(H^s(\mathbb{T})\right)$ and A is skew-adjoint in $H^s(\mathbb{T}), \tilde{A}$ is the infinitesimal generator of a group $\{W_a(t)\}_{t\in\mathbb{R}}$ on $H^s(\mathbb{T})$ (see e.g. (PAZY, 1983), Theorem 1.1, p. 76). Our first aim is to show that the closed-loop system (5.17)-(5.18) is exponentially stable in $H^s(\mathbb{T})$ for all $s \geqslant 1$.

Lemma 5.2.1. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$. Then there exists a constant $\gamma > 0$ such that for any $s \geqslant 1$, one can find a constant $C_s > 0$ for which the following holds for all $u_0 \in H^s(\mathbb{T})$*

$$\|W_a(t)u_0\|_{H^s} \leqslant C_s e^{-\gamma t} \|u_0\|_{H^s} \qquad \textit{for all } t \geqslant 0. \tag{5.20}$$

Proof. (5.20) is well known for $s = 1$ (see e.g. (LIU, 1997)). Assume that it is true for some $s \in \mathbb{N}^*$, and pick any $u_0 \in H^{s+1}(\mathbb{T})$. Let $v_0 = \tilde{A}u_0 \in H^s(\mathbb{T})$. Then

$$\|W_a(t)v_0\|_{H^s} \leqslant C_s e^{-\gamma t} \|v_0\|_{H^s}.$$

We have

$$W_a(t)v_0 = W_a(t)\tilde{A}u_0 = \tilde{A}W_a(t)u_0 = AW_a(t)u_0 - BB^{*,1}W_a(t)u_0,$$

hence

$$\|AW_a(t)u_0\|_{H^s} \leqslant \|W_a(t)v_0\|_{H^s} + \left\|BB^{*,1}\right\|_{\mathcal{L}(H^s)} \|W_a(t)u_0\|_{H^s} \leqslant Ce^{-\gamma t} \|u_0\|_{H^{s+1}}.$$

Therefore

$$\|W_a(t)u_0\|_{H^{s+1}} \leqslant C_{s+1} e^{-\gamma t} \|u_0\|_{H^{s+1}},$$

as desired. The estimate (5.20) is thus proved for any $s \in \mathbb{N}^*$. It may be extended to any $s \in [1, +\infty\ )$ by interpolation. ■

Plugging the feedback law $k = -B^{*,1}u = -au$ in the nonlinear equation gives the following closed-loop system

$$u_t - u_{txx} - cu_{xxx} + (c+1)u_x + uu_x = -a\left(1 - \partial_x^2\right)[au], \tag{5.21}$$

$$u(x,0) = u_0(x) \tag{5.22}$$

The rest of the section is described as follow: in subsection 5.2.1 we prove the global well-posedness for the system (5.21)-(5.22) in $H^s(\mathbb{T})$ for any $s \geqslant 0$ and, for the subsection 5.2.2 we turn to the stabilization issue, first showing that (5.21)-(5.22) is locally exponentially stable in $H^s(\mathbb{T})$, for $s \geqslant 1$, and, next, the global exponential stabilization is treated for $s = 1$, and, to end, for $s \geqslant 1$.

5.2.1 Well-Posedness of the Feedback-Controlled KdV-BBM System

Theorem 5.2.1. *Let $s \geqslant 0$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the system* (5.21)-(5.22) *admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$.*

Before presenting the proof of Theorem 5.2.1, we recall, for the sake of completeness and convenience, the following bilinear estimate from (ROUMÉGOUX, 2010), which we already encountered in Chapter 3 (lemma 3.1.2), which, once again, will prove to be very helpful.

Lemma 5.2.2. *Let $w \in H^r(\mathbb{T})$ and $v \in H^{r'}(\mathbb{T})$ with $0 \leqslant r \leqslant s, 0 \leqslant r' \leqslant s$ and $0 \leqslant 2s - r - r' < \frac{1}{4}$. Then*

$$\left\| \left(1 - \partial_x^2\right)^{-1} \partial_x(wv) \right\|_{H^s} \leqslant c_{r,r',s} \|w\|_{H^r} \|v\|_{H^{r'}}.$$

In particular, if $w \in H^r(\mathbb{T})$ and $v \in H^s(\mathbb{T})$ with $0 \leqslant r \leqslant s < r + \frac{1}{4}$, then

$$\left\| \left(1 - \partial_x^2\right)^{-1} \partial_x(wv) \right\|_{H^s} \leqslant c_{r,s} \|w\|_{H^r} \|v\|_{H^s}$$

Proof of Theorem 5.2.1. This proof is divided into three steps:

- Step 1 is the local well-posedness for the system (5.21)-(5.22) in $H^s(\mathbb{T})$ for $s \geqslant 0$, where we will make use of the fixed-point theorem. To this end, we will prove that a convenient map is a contraction map from a closed ball to itself;
- Step 2 is the global well-posedness for the system (5.21)-(5.22) in $H^s(\mathbb{T})$ for $s \geqslant 1$, where we use a global *a priori* estimate, which is proved using iteratively, the Lemma 5.2.2 and Gronwall's lemma;
- Step 3 is the Global Well-Posedness for the system (5.21)-(5.22), in $H^s(\mathbb{T})$, but, this time, for $0 \leqslant s < 1$.

Step 1. The system is locally well-posed in the space $H^s(\mathbb{T})$.

Let $s \geqslant 0$ and $R > 0$ be given. There exists a T^* depending only on s and R such that for any $u_0 \in H^s(\mathbb{T})$ with

$$\|u_0\|_{H^s} \leqslant R,$$

the system (5.21)-(5.22) admits a unique solution $u \in C\left([0,T^*];H^s(\mathbb{T})\right)$. Moreover, $T^* \to \infty$ as $R \to 0$.

In order to rewrite (5.21)-(5.22) in its integral form, we proceed as follows

$$\begin{aligned}
&\left(1-\partial_x^2\right)u_t = \left(c\partial_x^3-(c+1)\partial_x\right)u - uu_x + a(x)h \\
&u_t = \left(1-\partial_x^2\right)^{-1}\left(c\partial_x^3-(c+1)\partial_x\right)u - \left(1-\partial_x^2\right)^{-1}uu_x + \left(1-\partial_x^2\right)^{-1}ah \\
&u_t = Au - \left(1-\partial_x^2\right)^{-1}uu_x + \underbrace{\left(1-\partial_x^2\right)^{-1}a\left(1-\partial_x^2\right)}_{B}\underbrace{\left(1-\partial_x^2\right)^{-1}h}_{k} \\
&u_t = Au - \left(1-\partial_x^2\right)^{-1}uu_x + Bk \\
&u_t = Au - BB^{*,1}u - \left(1-\partial_x^2\right)^{-1}uu_x \\
&u_t = \left(A - BB^{*,1}\right)u - \left(1-\partial_x^2\right)^{-1}uu_x \\
&u_t = \tilde{A}u - \left(1-\partial_x^2\right)^{-1}uu_x.
\end{aligned}$$

Thus, from Duhamel formula, its integral form, or its mild solution, becomes

$$u(t) = W_a(t)u_0 - \int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}(uu_x)(\tau)d\tau. \tag{5.23}$$

For given $\theta > 0$, define a map Γ on $C\left([0,\theta];H^s(\mathbb{T})\right)$ by

$$\Gamma(v) = W_a(t)u_0 - \int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}(vv_x)(\tau)d\tau,$$

for any $v \in C\left([0,\theta];H^s(\mathbb{T})\right)$. Since $\{W_a(t)\}_{t\in\mathbb{R}}$ is a group of isometries, we have

$$\|W_a(t)u_0\|_{H^s(\mathbb{T})} = \|u_0\|_{H^s(\mathbb{T})},$$

then,

$$\sup_{t\in[0,\theta]}\|W_a(t)u_0\|_{H^s(\mathbb{T})} \leqslant \|u_0\|_{H^s(\mathbb{T})},$$

that is,

$$\|W_a(t)u_0\|_{C([0,\theta];H^s(\mathbb{T}))} \leqslant \|u_0\|_{H^s(\mathbb{T})}.$$

On the other hand, from Lemma 5.2.2

$$\int_0^t \left\| W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}(vv_x)(\tau)\right\|_{H^s(\mathbb{T})} d\tau = \frac{1}{2}\int_0^t \left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left(v^2\right)(\tau)\right\|_{H^s} d\tau$$
$$\leqslant \frac{c_{s,s}}{2}\int_0^t \|v(\tau)\|^2_{H^s(\mathbb{T})}d\tau.$$

Thus,

$$\left\|\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}(vv_x)(\tau)d\tau\right\|_{C([0,\theta];H^s(\mathbb{T}))} \leqslant \frac{c_{s,s}}{2}\theta\|v\|^2_{C([0,\theta];H^s(\mathbb{T}))}.$$

Then, we obtain

$$\|\Gamma(v)\|_{C([0,\theta],H^s(\mathbb{T}))} \leqslant \|u_0\|_{H^s(\mathbb{T})} + \frac{c_{s,s}}{2}\theta\|v\|^2_{C([0,\theta],H^s(\mathbb{T}))}.$$

Now, for $v, w \in C\left([0,\theta], H^s(\mathbb{T})\right)$

$$\Gamma(v)-\Gamma(w) = \int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}(ww_x - vv_x)(\tau)d\tau,$$

hence,

$$\begin{aligned}
\|\Gamma(v)-\Gamma(w)\|_{C([0,\theta],H^s(\mathbb{T}))} &\leqslant \frac{\theta}{2}\sup_{0\leqslant t\leqslant\theta}\left\|\left(1-\partial_x^2\right)^{-1}\partial_x\left(w^2-v^2\right)(t)\right\|_{H^s}\\
&\leqslant \frac{\theta}{2}\sup_{0\leqslant t\leqslant\theta}\left\|\left(1-\partial_x^2\right)^{-1}\partial_x(w-v)(w+v)(t)\right\|_{H^s(\mathbb{T})}\\
&\leqslant \frac{\theta c_{s,s}}{2}\sup_{0\leqslant t\leqslant\theta}\left(\|(w-v)(t)\|_{H^s}\|(w+v)(t)\|_{H^s}\right)\\
&\leqslant \frac{\theta c_{s,s}}{2}\|w-v\|_{C([0,\theta],H^s(\mathbb{T}))}\|w+v\|_{C([0,\theta],H^s(\mathbb{T}))}.
\end{aligned}$$

Therefore, for given $R>0$ and $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R$, one can choose $T^* = (4c_{s,s}R)^{-1}$ such that Γ is a contraction mapping in the ball

$$B := \left\{v \in C\left([0,T^*];H^s(\mathbb{T})\right); \|v\|_{C([0,T^*];H^s(\mathbb{T}))} \leqslant 2R\right\},$$

whose fixed point u is the desired solution.

Step 2. The system is globally well-posed in the space $H^s(\mathbb{T})$ for any $s \geqslant 1$.

To this end, it suffices to establish the following global a priori estimate for smooth solutions of the system (5.21)-(5.22).

Let $s \geqslant 1$ and $T > 0$ be given. There exists a continuous nondecreasing function

$$\alpha_{s,T} : \mathbb{R}^+ \to \mathbb{R}^+,$$

such that any smooth solution u of the system (5.21)-(5.22) satisfies

$$\|u\|_{C([0,T];H^s)} = \sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{H^s} \leqslant \alpha_{s,T}\left(\|u_0\|_{H^s}\right). \tag{5.24}$$

Estimate (5.24) holds immediately when $s = 1$ because of the energy identity

$$\|u(t)\|_{H^1}^2 - \|u_0\|_{H^1}^2 = -2\int_0^t \|au(\tau)\|_{H^1}^2 d\tau \quad \forall t \geqslant 0$$

Now we begin the iterative process mentioned earlier in the proof outline. Consider $1 < s \leqslant s_1 := 1 + \frac{1}{8}$, applying Lemma 5.2.1 and Lemma 5.2.2 to the equation (5.23) yields that for any $0 < t \leqslant T$,

$$\begin{aligned}\|u(\cdot,t)\|_{H^s} &\leqslant C_s \|u_0\|_{H^s} + \frac{C_s c_{1,s}}{2}\int_0^t \|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{H^s} d\tau \\ &\leqslant C\|u_0\|_{H^s} + C\alpha_{1,T}\left(\|u_0\|_{H^1}\right)\int_0^t \|u(\cdot,\tau)\|_{H^s} d\tau.\end{aligned}$$

By using Gronwall's lemma 2.1.6

$$\begin{aligned}&\|u(\cdot,t)\|_{H^s} \leqslant C\|u_0\|_{H^s}\left(1 + C\alpha_{1,T}(\|u_0\|_{H^1})te^{C\alpha_{1,T}(\|u_0\|_{H^1})t}\right) \quad \text{for } 0 \leqslant t \leqslant T \\ &\sup_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_{H^s} \leqslant C\|u_0\|_{H^s}\left(1 + C\alpha_{1,T}(\|u_0\|_{H^1})Te^{C\alpha_{1,T}(\|u_0\|_{H^1})T}\right) \\ &\leqslant \alpha_{s,T}\left(\|u_0\|_{H^s}\right),\end{aligned}$$

which is estimate (5.24) for $1 < s \leqslant s_1$. Similarly, for $s_1 < s \leqslant s_2 := 1 + \frac{2}{8}$,

$$\begin{aligned}\|u(\cdot,t)\|_{H^s} &\leqslant C_s \|u_0\|_{H^s} + \frac{C_s c_{s_1,s}}{2}\int_0^t \|u(\cdot,\tau)\|_{H^{s_1}}\|u(\cdot,\tau)\|_{H^s} d\tau \\ &\leqslant C\|u_0\|_{H^s} + C\alpha_{s_1,T}\left(\|u_0\|_{H^{s_1}}\right)\int_0^t \|u(\cdot,\tau)\|_{H^s} d\tau.\end{aligned}$$

Then,

$$\sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{H^s} \leqslant \alpha_{s,T}\left(\|u_0\|_{H^s}\right) \qquad \text{for } s_1 \leqslant s \leqslant s_2.$$

Therefore, estimate (5.24) holds for $1 < s \leqslant s_2$. Continuing this argument, we can show that the estimate (5.24) holds for $1 < s \leqslant s_k := 1 + \frac{k}{8}$ for any $k \geqslant 1$. Indeed, suppose $k \geqslant 3$. We shall prove that estimate (5.24) holds for $s_{k-1} \leqslant s \leqslant s_k := 1 + \frac{k}{8}$. It suffices to show that $s < s_{k-1} + \frac{1}{4}$ so that we can use Lemma 5.2.2. Hence, we must have

$$s < s_{k-1} + \frac{1}{4} = 1 + \frac{k-1}{8} + \frac{1}{4} = 1 + \frac{k}{8} + \frac{1}{8} = s_k + \frac{1}{8}.$$

As $s \leqslant s_k$, we have $s < s_k + \frac{1}{8}$. This concludes Step 2.

Step 3. The system (5.21)-(5.22) is globally well-posed in the space $H^s(\mathbb{T})$ for any $0 \leqslant s < 1$.

As in (ROUMÉGOUX, 2010) and in the proof of Theorem 3.1.2 we decompose any $u_0 \in H^s(\mathbb{T})$ as

$$u_0 = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} = \sum_{|k|\leqslant k_0} + \sum_{|k|>k_0} =: w_0 + v_0,$$

with $v_0 \in H^s(\mathbb{T})$ small enough

$$\|v_0\|_{H^s} \leqslant \delta,$$

for some small $\delta > 0$ to be chosen, and $w_0 \in H^1(\mathbb{T})$. We consider the following two initial value problems

$$\begin{cases} v_t - v_{txx} - cv_{xxx} + (c+1)v_x + vv_x = -a\left(1-\partial_x^2\right)[av], \\ v(x,0) = v_0(x), \end{cases} \tag{5.25}$$

and

$$\begin{cases} w_t - w_{txx} - cw_{xxx} + (c+1)w_x + ww_x + (vw)_x = -a\left(1-\partial_x^2\right)[aw] \\ w(x,0) = w_0(x). \end{cases} \tag{5.26}$$

By the local well-posedness established in Step 1, for given $T > 0$, if δ is small enough, then (5.25) admits a unique solution $v \in C\left([0,T]; H^s(\mathbb{T})\right)$. For (5.26), with

$v \in C([0,T]; H^s(\mathbb{T}))$, by using Lemma 5.2.1, the estimate

$$\left\| \left(1-\partial_x^2\right)^{-1} \partial_x(wv) \right\|_{H^1} \leqslant C\|wv\|_{L^2} \leqslant C\|w\|_{H^1}\|v\|_{H^s},$$

and the contraction mapping principle, one can show first that it is locally well-posed in the space $H^1(\mathbb{T})$. Then, for any smooth solution w of (5.26) it holds that

$$\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|_{H^1}^2 - \int_{\mathbb{T}} v(x,t)w(x,t)w_x(x,t)dx = -\|a(\cdot)w(\cdot,t)\|_{H^1}^2,$$

which implies that

$$\|w(\cdot,t)\|_{H^1}^2 \leqslant \|w_0\|_{H^1}^2 \exp\left(C\int_0^t \|v(\cdot,\tau)\|_{L^2} d\tau\right),$$

for any $t \geqslant 0$. The above estimate can be extended to any $w_0 \in H^1(\mathbb{T})$ by a density argument. Consequently, for $w_0 \in H^1(\mathbb{T})$ and $v \in C([0,T]; H^s(\mathbb{T}))$, (5.26) admits a unique solution $w \in C([0,T]; H^1(\mathbb{T}))$. Thus, $u = w + v \in C([0,T]; H^s(\mathbb{T}))$ is the desired solution of system (5.21)-(5.22). The proof of Theorem 5.2.1 is complete. ■

5.2.2 Local and Global Exponential Stabilization

The next proposition shows that the system (5.21)-(5.22) is locally exponentially stable in $H^s(\mathbb{T})$ for any $s \geqslant 1$. Whereas the global exponential stabilization results are addressed in the theorems, first for $s = 1$, and then for $s \geqslant 1$ (see Definition 2.2.10). In addition, an observability inequality will be derived in order to prove the global result for $s = 1$.

Proposition 5.2.1. *Let $s \geqslant 1$ be given and $\gamma > 0$ be as given in Lemma 5.2.1. Then there exist two numbers $\delta > 0$ and C_s' depending only on s such that for any $u_0 \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s} \leqslant \delta,$$

the corresponding solution u of the system (5.21)-(5.22) *satisfies*

$$\|u(\cdot,t)\|_{H^s} \leqslant C_s' e^{-\gamma t}\|u_0\|_{H^s} \quad \forall t \geqslant 0.$$

Proof. As in the proof of Theorem 5.2.1, rewrite the system (5.21)-(5.22) in its integral form

$$u(t) = W_a(t)u_0 - \frac{1}{2}\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right)(\tau)d\tau,$$

and consider the map

$$\Gamma(v) := W_a(t)u_0 - \frac{1}{2}\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(v^2\right)(\tau)d\tau.$$

For given $s \geqslant 1$, by Lemma 5.2.1 and Lemma 5.2.2, there exists a constant $C_s > 0$ such that

$$\begin{aligned}
\|\Gamma(v)(\cdot,t)\|_{H^s} &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2}\int_0^t e^{-\gamma(t-\tau)}\|v(\cdot,\tau)\|_{H^s}^2 d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2}\int_0^t e^{2\gamma\tau}\|v(\cdot,\tau)\|_{H^s}^2 e^{-\gamma(t+\tau)} d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2}\sup_{0\leqslant\tau\leqslant t}\|e^{\gamma\tau}v(\cdot,\tau)\|_{H^s}^2\int_0^t e^{-\gamma(t+\tau)} d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma}e^{-\gamma t}\left(1-e^{-\gamma t}\right)\sup_{0\leqslant\tau\leqslant t}\|e^{\gamma\tau}v(\cdot,\tau)\|_{H^s}^2,
\end{aligned}$$

for any $t \geqslant 0$. Let us introduce the Banach space

$$Y_s := \left\{v \in C\left([0,\infty); H^s(\mathbb{T})\right) : \|v\|_{Y_s} := \sup_{0\leqslant t<\infty}\left\|e^{\gamma t}v(\cdot,t)\right\|_{H^s} < \infty\right\}.$$

For any $v \in Y_s$, and for any $t \geqslant 0$

$$\begin{aligned}
e^{\gamma t}\|\Gamma(v)(\cdot,t)\|_{H^s} &\leqslant C_s e^{\gamma t}e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma}e^{\gamma t}e^{-\gamma t}\left(1-e^{-\gamma t}\right)\sup_{0\leqslant t<\infty}\left\|e^{\gamma t}v(\cdot,t)\right\|_{H^s}^2 \\
&\leqslant C_s\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma}\left(1-e^{-\gamma t}\right)\|v\|_{Y_s}^2.
\end{aligned}$$

Then,

$$\|\Gamma(v)\|_{Y_s} \leqslant C_s\|u_0\|_{H^s} + \frac{C_s c_{s,s}}{2\gamma}\|v\|_{Y_s}^2.$$

Choose

$$\delta = \frac{\gamma}{4C_s^2 c_{s,s}}, \quad R = 2C_s\delta.$$

Then, if $\|u_0\| \leqslant \delta$, for any $v \in Y_s$ with $\|v\|_{Y_s} \leqslant R$,

$$\|\Gamma(v)\|_{Y_s} \leqslant C_s\delta + \frac{C_s c_{s,s}}{2\gamma}\left(2C_s\delta\right)R \leqslant R.$$

Moreover, for any $v_1, v_2 \in Y_s$ with $\|v_1\|_{Y_s} \leqslant R$ and $\|v_2\|_{Y_s} \leqslant R$,

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_s} \leqslant \frac{1}{2} \|v_1 - v_2\|_{Y_s}.$$

The map Γ is a contraction whose fixed point $u \in Y_s$ is the desired solution satisfying

$$\|u(\cdot, t)\|_{H^s} \leqslant 2C_s e^{-\gamma t} \|u_0\|_{H^s},$$

for any $t \geqslant 0$. ■

Now we turn to the issue of the global stabilization of the system (5.21)-(5.22). First we show that the system (5.21)-(5.22) is globally exponentially stable in the space $H^1(\mathbb{T})$.

Theorem 5.2.2. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$, and let $\gamma > 0$ be as in Lemma 5.2.1. Then for any $R_0 > 0$, there exists a constant $C^* > 0$ such that for any $u_0 \in H^1(\mathbb{T})$ with $\|u_0\|_{H^1} \leqslant R_0$, the corresponding solution u of* (5.21)-(5.22) *satisfies*

$$\|u(\cdot, t)\|_{H^1} \leqslant C^* e^{-\gamma t} \|u_0\|_{H^1} \qquad \text{for all } t \geqslant 0. \tag{5.27}$$

Theorem 5.2.2 is a direct consequence of the following observability inequality.

Proposition 5.2.2. *Let $R_0 > 0$ be given. Then there exist two positive numbers T and β such that for any $u_0 \in H^1(\mathbb{T})$ satisfying*

$$\|u_0\|_{H^1} \leqslant R_0, \tag{5.28}$$

the corresponding solution u of (5.21)-(5.22) *satisfies*

$$\|u_0\|_{H^1}^2 \leqslant \beta \int_0^T \|au(t)\|_{H^1}^2 dt. \tag{5.29}$$

First, we use Proposition 5.2.2 to prove Theorem 5.2.2, and then, we provide a proof of the proposition.

Proof of the Theorem 5.2.2. If (5.29) holds, then it follows from the energy identity

$$\|u(t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 - 2\int_0^t \|au(\tau)\|_{H^1}^2 d\tau \quad \forall t \geqslant 0, \tag{5.30}$$

that

$$\begin{aligned}\|u(T)\|_{H^1}^2 &\leqslant \|u_0\|_{H^1}^2 - 2\beta^{-1}\|u_0\|_{H^1}^2 \\ &\leqslant \left(1-2\beta^{-1}\right)\|u_0\|_{H^1}^2 .\end{aligned}$$

Applying the same argument on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots,$ we have

$$\|u(mT)\|_{H^1}^2 \leqslant \left(1-2\beta^{-1}\right)\|u((m-1)T)\|_{H^1}^2 \leqslant \ldots \leqslant \left(1-2\beta^{-1}\right)^m \|u_0\|_{H^1}^2 ,$$

which gives, for $t > 0$, such that $(m-1)T < t < mT$

$$\begin{aligned}\|u(t)\|_{H^1}^2 &\leqslant \left(1-2\beta^{-1}\right)^{m-1}\|u_0\|_{H^1}^2 \\ &\leqslant e^{(m-1)\log\left(1-2\beta^{-1}\right)}\|u_0\|_{H^1}^2 \\ &\leqslant e^{(1-m)\log\left(1-2\beta^{-1}\right)^{-1}}\|u_0\|_{H^1}^2 \\ &\leqslant e e^{\frac{-mt}{t}\log\left(1-2\beta^{-1}\right)^{-1}}\|u_0\|_{H^1}^2 \\ &\leqslant e e^{\frac{-t}{T}\log\left(1-2\beta^{-1}\right)^{-1}}\|u_0\|_{H^1}^2 ,\end{aligned}$$

that is,

$$\|u(t)\|_{H^1} \leqslant Ce^{-\kappa t}\|u_0\|_{H^1} \qquad \text{for all } t \geqslant 0, \tag{5.31}$$

for some positive constants $C = C(R_0), \kappa = \kappa(R_0)$. Finally, we can replace κ by the γ given in Lemma 5.2.1. Indeed, let $t' = \kappa^{-1}\log\left[1 + CR_0\delta^{-1}\right]$, where δ is as given in Proposition 5.2.1. Then for $\|u_0\|_{H^1} \leqslant R_0, \|u(t')\|_{H^1} < \delta$, hence for all $t \geqslant t'$

$$\|u(t)\|_{H^1} \leqslant C_1'\|u(t')\|_{H^1} e^{-\gamma(t-t')} \leqslant (C_1'\delta/R_0)\|u_0\|_{H^1} e^{-\gamma(t-t')} \leqslant C^* e^{-\gamma t}\|u_0\|_{H^1} ,$$

where $C^* = (C_1'\delta/R_0)\, e^{\gamma t'}$. ■

Now we return to the proof of Proposition 5.2.2.

Proof of Proposition 5.2.2. Pick for the moment any $T > 2\pi/|c|$ (its value will be specified later on). We prove the estimate (5.29) by contradiction. If (5.29) is not true, then for any $n \geqslant 1$ (5.21)-(5.22) admits a solution $u_n \in C([0,T]; H^1(\mathbb{T}))$ satisfying

$$\|u_n(0)\|_{H^1} \leqslant R_0, \tag{5.32}$$

and

$$\int_0^T \|au_n(t)\|_{H^1}^2 \, dt < \frac{1}{n} \|u_{0,n}\|_{H^1}^2 , \tag{5.33}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n := \|u_{0,n}\|_{H^1} \leqslant R_0$, we can choose a subsequence of (α_n), which we still denote by (α_n), such that $\lim_{n\to\infty} \alpha_n = \alpha$. We note that $\alpha_n > 0$ for all n, by (5.33). So we can set $v_n = u_n/\alpha_n$ for all $n \geqslant 1$. Then, multiplying (5.21) by $1/\alpha_n$ we get

$$v_{n,t} - v_{n,txx} - cv_{n,xxx} + (c+1)v_{n,x} + \alpha_n v_n v_{n,x} = -a\left(1 - \partial_x^2\right)[av_n] , \tag{5.34}$$

and

$$\int_0^T \|av_n\|_{H^1}^2 \, dt < \frac{1}{n}\|v_{0,n}\|_{H^1}^2 = \frac{1}{n}\frac{\|u_{0,n}\|_{H^1}^2}{\alpha_n^2} = \frac{1}{n}. \tag{5.35}$$

Because of

$$\|v_n(0)\|_{H^1} = 1, \tag{5.36}$$

we have that the sequence (v_n) is bounded in $L^\infty(0,T;H^1(\mathbb{T})\ )$, while $(v_{n,t})$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}))$. Since, for $s<1$, $H^1 \hookrightarrow H^s$ and $H^1 \subset H^s \subset L^2$ we can use Aubin-Lions' theorem 2.1.18 and a diagonal process, to infer that we can extract a subsequence of (v_n), still denoted (v_n), such that

$$v_n \to v \text{ in } C([0,T];H^s(\mathbb{T})) \qquad \forall s < 1, \tag{5.37}$$

$$v_n \to v \text{ in } L^\infty\left(0,T;H^1(\mathbb{T})\right) \qquad \text{weak }^*, \tag{5.38}$$

for some $v \in L^\infty(0,T;H^1(\mathbb{T})) \cap C([0,T];H^s(\mathbb{T}))$ for all $s<1$. Note that, by (5.37)-(5.38), we have that

$$\alpha_n v_n v_{n,x} \to \alpha v v_x \quad \text{ in } L^\infty\left(0,T;L^2(\mathbb{T})\right) \quad \text{ weak*}. \tag{5.39}$$

Furthermore, by (5.35),

$$\int_0^T \|av\|_{H^1}^2 dt \leqslant \liminf_{n\to\infty} \int_0^T \|av_n\|_{H^1}^2 \, dt = 0. \tag{5.40}$$

Thus, v solves

$$v_t - v_{txx} - cv_{xxx} + (c+1)v_x + \alpha v v_x = 0 \quad \text{ on } \mathbb{T}\times(0,T), \tag{5.41}$$

$$v = 0 \quad \text{ on } \omega\times(0,T), \tag{5.42}$$

where ω is given in (5.3). According to Theorem 4.3.1, $v \equiv 0$ on $\mathbb{T} \times (0, T)$. We claim that (v_n) is linearizable in the following sense: if (w_n) denotes the sequence of solutions to the linear KdV-BBM equation with the same initial data

$$w_{n,t} - w_{n,txx} - cw_{n,xxx} + (c+1)w_{n,x} = -a\left(1 - \partial_x^2\right)[aw_n] \tag{5.43}$$

$$w_n(x,0) = v_n(x,0), \tag{5.44}$$

then

$$\sup_{0 \leqslant t \leqslant T} \|v_n(t) - w_n(t)\|_{H^1} \to 0 \quad \text{as } n \to \infty. \tag{5.45}$$

Indeed, by (5.34) and (5.43), if $d_n = v_n - w_n$, then d_n solves

$$d_{n,t} - d_{n,txx} - cd_{n,xxx} + (c+1)d_{n,x} = -a\left(1 - \partial_x^2\right)[ad_n] - \alpha_n v_n v_{n,x},$$

$$d_n(0) = 0.$$

Since $\|W_a(t)\|_{\mathcal{L}(H^1(\mathbb{T}))} \leqslant 1$ and $d_{0,n} = 0$, we have, from Duhamel formula, that for $t \in [0, T]$

$$\|d_n(t)\|_{H^1} \leqslant \int_0^T \left\|\left(1 - \partial_x^2\right)^{-1}\left(\alpha_n v_n v_{n,x}\right)(\tau)\right\|_{H^1} d\tau.$$

Combined with (5.37) and the fact that $v \equiv 0$, this gives (5.45). Then, returning to the linearized equation (5.43), by Lemma 5.2.1, we have that

$$\|w_n(t)\|_{H^1} \leqslant C_1 e^{-\gamma t}\|w_n(0)\|_{H^1} \quad \text{for all } t \geqslant 0. \tag{5.46}$$

From (5.46) and the energy identity for (5.43)-(5.44), namely

$$\|w_n(t)\|_{H^1}^2 - \|w_n(0)\|_{H^1}^2 = -2\int_0^t \|aw_n(\tau)\|_{H^1}^2 d\tau, \tag{5.47}$$

we have for $Ce^{-\lambda T} < 1$

$$\begin{aligned} \|w_n(0)\|_{H^1}^2 &= \|w_n(t)\|_{H^1}^2 + 2\int_0^t \|aw_n(\tau)\|_{H^1}^2 d\tau \\ &\leqslant C_1^2 e^{-2\gamma T}\|w_n(0)\|_{H^1}^2 + 2\int_0^T \|aw_n(\tau)\|_{H^1}^2 d\tau \\ &\leqslant 2\left(1 - C_1^2 e^{-2\gamma T}\right)^{-1}\int_0^T \|aw_n(\tau)\|_{H^1}^2 d\tau, \end{aligned} \tag{5.48}$$

where, $Ce^{-\lambda T} < 1$ ensures us that $\left(1 - C_1^2 e^{-2\gamma T}\right) > 0$. Thus, combining (5.48) with (5.35) and (5.45), this yields $\|v_n(0)\|_{H^1} = \|w_n(0)\|_{H^1} \to 0$, which contradicts (5.36). This completes the proof of Proposition 5.2.2 and Theorem 5.2.2. ■

Next we show that the system (5.21)-(5.22) is exponentially stable in the space $H^s(\mathbb{T})$ for any $s \geqslant 1$.

Theorem 5.2.3. *Let $a \in C^\infty(\mathbb{T})$ with $a \neq 0$ and $\gamma > 0$ be as given in Lemma 5.2.1. For any given $s \geqslant 1$ and $R_0 > 0$, there exists a constant $C > 0$ depending only on s and R_0 such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R_0$, the corresponding solution u of* (5.21)-(5.22) *satisfies*

$$\|u(\cdot,t)\|_{H^s} \leqslant Ce^{-\gamma t} \|u_0\|_{H^s} \qquad \text{for all } t \geqslant 0. \tag{5.49}$$

Proof. As before, rewrite the system in its integral form

$$u(t) = W_a(t)u_0 - \frac{1}{2}\int_0^t W_a(t-\tau)\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right)(\tau)d\tau.$$

For $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leqslant R_0$, applying Lemma 5.2.1, Lemma 5.2.2 and Theorem 5.2.2 yields that, for any $1 \leqslant s \leqslant 1 + \frac{1}{10}$,

$$\begin{aligned}
\|u(\cdot,t)\|_{H^s} &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{1,1,s}}{2}\int_0^t e^{-\gamma(t-\tau)}\|u(\cdot,\tau)\|_{H^1}^2 d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{1,1,s}\left(C^*\right)^2}{2}\int_0^t e^{-\gamma(t-\tau)}e^{-2\gamma\tau}\|u_0\|_{H^1}^2 d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{1,1,s}\left(C^*\right)^2}{2}e^{-\gamma t}\|u_0\|_{H^1}^2\int_0^t e^{-\gamma\tau}d\tau \\
&\leqslant \left(C_s + \frac{C_s c_{1,1,s}\left(C^*\right)^2}{2\gamma}\|u_0\|_{H^1}\right)e^{-\gamma t}\|u_0\|_{H^s},
\end{aligned}$$

for any $t \geqslant 0$. Thus, the estimate (5.49) holds for $1 \leqslant s \leqslant m_1 := 1 + \frac{1}{10}$. Similarly, for $m_1 \leqslant s \leqslant m_2 := 1 + \frac{2}{10}$, we have for $\|u_0\|_{H^s} \leqslant R_0$

$$\begin{aligned}
\|u(\cdot,t)\|_{H^s} &\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + \frac{C_s c_{m_1,m_1,s}}{2}\int_0^t e^{-\gamma(t-\tau)}\|u(\cdot,\tau)\|_{H^{m_1}}^2 d\tau \\
&\leqslant C_s e^{-\gamma t}\|u_0\|_{H^s} + C\left(s,m_1,R_0\right)\int_0^t e^{-\gamma(t-\tau)}e^{-2\gamma\tau}\|u_0\|_{H^{m_1}}^2 d\tau \\
&\leqslant \left(C_s + C\left(s,m_1,R_0\right)\|u_0\|_{H^{m_1}}\gamma^{-1}\right)e^{-\gamma t}\|u_0\|_{H^s}.
\end{aligned}$$

Thus the estimate (5.49) holds for $1 \leqslant s \leqslant m_2 := 1 + \frac{2}{10}$. Repeating this argument yields that the estimate (5.49) holds for $1 \leqslant s \leqslant m_k := 1 + \frac{k}{10}$ for $k = 1, 2, \ldots$, which concludes the proof. ■

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