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Título: Hilbert functions and the Lefschetz properties for Artinian Gorenstein algebras

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I dedicate this thesis to my entire family, especially to my mother Mércia Nunes.

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ABSTRACT

This work is divided into two parts. In the first one, we studied minimal Hilbert functions for Artinian Gorenstein algebras, we conjecture that for certain algebras the Hilbert vector is always minimal, and prove this conjecture for a particular case. In the second part, we studied the Lefschetz locus for Artinian Gorenstein algebras with Hilbert vector $(1, N+1, N+1, 1)$.

Keywords: Hilbert functions; Lefschetz properties; Artinian Gorenstein algebra

CONTENTS

1 INTRODUCTION

The study of standard graded Artinian Gorenstein algebras has attracted the attention of many mathematicians, and many papers have been written on the subject. These algebras play a crucial role in commutative algebra, algebraic geometry, combinatorics, and algebraic topology since they represent a class of algebras that possess both finiteness and symmetry properties. In this thesis, we are interested in the study of the Hilbert function and the Lefschetz properties for some of these algebras.

In the study of Hilbert functions of standard graded algebras, Macaulay's and Green's Theorem 2.1.4 stand out as being of fundamental importance. Macaulay's Theorem regulates the growth of the Hilbert function from one degree to the next, and Green's Theorem regulates the Hilbert functions of the restriction modulo a general linear form. One area where both theorems have been applied is in the problem to classifying Hilbert functions of Artinian Gorenstein algebras. More precisely we have the following question: *Given a symmetric sequence $H = (1, h_1, \dots, h_d = 1)$, under what conditions is this sequence the Hilbert function of an Artinian Gorenstein algebra?*

Stanley (1975) conjectured that, if H is an SI-sequence 2.1.9, then H is the Hilbert vector of an Artinian Gorenstein algebra. Later, he provides a counter-example for this conjecture in Stanley (1978). He showed that the vector $(1, 13, 12, 13, 1)$ is the Hilbert vector of an Artinian Gorenstein algebra, and Migliore and Zanello (2017) showed that this vector is optimal, in other words, they showed that the vectors $(1, 13, 11, 13, 1)$ and $(1, 12, 11, 12, 1)$ do not come from Artinian Gorenstein algebras. Therefore, to answer the previous question, many mathematicians started to study symmetric vectors, non-unimodal such that this vector is the Hilbert function of an Artinian Gorenstein algebra. Recall that a sequence of integers is *unimodal* if it does not increase after decreasing. Much work has been done over the years trying to determine conditions under which a Gorenstein Hilbert vector can be non-unimodal, see for example Ahn, Migliore and Shin (2018), Cerminara et al. (2020), Migliore, Nagel and Zanello (2008), Zanello (2009). In particular, we know that non-unimodal Hilbert vectors for Artinian Gorenstein algebras exist in any codimension $r \geq 5$ (i.e., 5 or more variables). The existence of non-unimodal Hilbert vectors in codimension 4 remains an open question.

Thus we have the central question to the first part of this thesis *given fixed codimension r and socle degree d, what is the smallest possible non-unimodal Hilbert vector for an Artinian*

Gorenstein algebra with codimension r and socle degree d? This question is connected to the first one since all compressed Hilbert functions were understood in Iarrobino (1984), they have maximal Hilbert vector.

In our study, the full Perazzo algebras play a crucial role. These algebras are Artinian Gorenstein algebras whose Macaulay generator is a full Perazzo polynomial, see 3.1.3. In the first part, in collaboration with Rodrigo Gondim, Giovanna Ilardi, and Giuseppe Zappalà, we study the minimal Hilbert vectors of Artinian Gorenstein algebras with small codimension and socle degree, and we conjectured that the Hilbert vector of a full Perazzo algebra is always minimal. This paper was published, see Bezerra et al. (2024).

The study of Lefschetz properties is of great importance in algebra, combinatorics, geometry, and topology; it has been crucial in the investigation of graded Artinian algebras. The study of Lefschetz properties started with the hard Lefschetz Theorem in Lefschetz (1950) on the cohomology of smooth projective complex varieties. A standard graded Artinian algebra A is said to satisfy the Weak Lefschets property (WLP) if the multiplication map in each degree by a generic linear form L has maximal rank. A is said to satisfy the Strong Lefschets property (SLP) if the same holds for powers of L.

The hard Lefschtz Theorem implies that the Hilbert function of the cohomology rings of compact Kahler manifolds are unimodal and symmetric. The cohomology rings of compact Kahler manifolds are Poincaré duality algebras. It is well known (see for example Maeno and Watanabe (2009)) that commutative Poincaré duality algebras are exactly Artinian Gorenstein algebras. So it is natural to investigate their Hilbert functions and Lefschetz properties. It is known that if an Artinian Gorenstein algebra satisfies WLP, then its Hilbert function is unimodal, that is, the Lefschetz properties affects the behavior of the Hilbert function of Artinian Gorenstein algebra.

It is important study all Artinian Gorenstein algebras with a fixed Hilbert function H, we denote by $\mathrm{Gor}(H)$ this space. In Iarrobino and Kanev (1999), the authors studied this space deeply. In their work, they proved, for example, that there is a close relation between graded Artinian Gorenstein algebras of codimension three and finite length Cohen-Macaulay subschemes of $\mathbb{P}^2$. They show that whenever the Hilbert function H is equal to s for at least three degrees, there is a fibration $\mathrm{Gor}(H) \to \mathfrak{H}(H) \subset \mathrm{Hilb}^s(\mathbb{P}^2)$ which takes the form f to the initial part of the ideal $\mathrm{Ann}_Q(f)$. On the other hand, Boij (1999) gives geometric constructions of families of graded Artinian Gorenstein algebras, some of which span a component of the space $\mathrm{Gor}(H)$ parametrizing Artinian Gorenstein algebras with a given Hilbert function H. This gives a lot of examples where $\mathrm{Gor}(H)$ is reducible. The author also shows that the Hilbert function of an Artinian Gorenstein algebra with codimension four can have an arbitrary long constant part without having the weak Lefschetz property. Fassarella, Ferrer and Gondim (2021) provided a classification of developable cubic hypersurfaces in $\mathbb{P}^4$, and using the Macaulay-Matlis duality they describe the Locus in $\mathrm{Gor}(1,5,5,1)$ corresponding to those algebras which satisfy the Strong Lefschetz property.

The second part of the thesis, in collaboration with Rodrigo Gondim and Viviana Ferrer, was inspired by Fassarella, Ferrer and Gondim (2021) and Gondim and Russo (2015). We use the Macaulay-Matlis duality to describe the Lefschetz locus in $\mathrm{Gor}(1,n,n,1)$. We parametrize the space of cubics, not cones with vanishing hessian, and calculate its dimension and degree using techniques of intersection theory. As an application, we parametrize the cubics, not cones with vanishing hessian in $\mathbb{P}^5$ and $\mathbb{P}^6$, and we calculate its dimension and degree.

2 FUNDAMENTAL RESULTS

In this chapter, we summarise some classical definitions and results that will be useful throughout this thesis.

In all work, we consider $\mathbb{K}$ a field of characteristic zero and denote by $Q = \mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring. We consider Q as a standard graded ring with $\deg(X_i) = 1$. We denote by A the standard graded $\mathbb{K}$-algebra given by the quotient $A = Q/I$, where I is a homogeneous ideal of Q. Each graded part of A is $A_i = Q_i/I_i$.

2.1 HILBERT FUNCTIONS

The first classical definition we will see is the definition of Hilbert Functions. This topic concerns calculating dimensions of graded modules that are finitely generated. The definition is the following.

Definition 2.1.1. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded Q-module. The *Hilbert function* of M is the function $h_M : \mathbb{Z} \to \mathbb{Z}$ given by $h_M(i) = \dim_{\mathbb{K}} M_i$. We write the Hilbert function of M like $\mathrm{Hilb}(M) = (h_0, h_1, \ldots, h_i, \ldots)$.

We are interested in the case where $M = A$ and A is Artinian. In this case, $A = \bigoplus_{i=0}^{d} A_i$ for some integer $d \geq 0$ and we can write $\mathrm{Hilb}(A)$ as a vector

$$\mathrm{Hilb}(A) = (h_0, h_1, \ldots, h_d)$$

and we refer to the Hilbert function of an Artinian algebra as a *Hilbert vector* of A, or simply *h-vector* of A.

Example 2.1.2. *Let $Q = \mathbb{K}[X, Y, Z]$ be the polynomial ring and consider the ideal $I = (Y^2Z - XZ^2, XY^2 - X^2Z, X^3, Y^4, Z^3)$. We have that $A = Q/I$ is an Artinian $\mathbb{K}$-algebra with h-vector $(1, 3, 6, 6, 3, 1)$.*

Some classical results given bounds for the growth of the Hilbert function of Artinian $\mathbb{K}$-algebras are due to Macaulay, Gotzmann, and Green. Before starting them, we need to recall the following definition:

Definition 2.1.3. Let k and i be positive integers. The *i-binomial expansion of k*, denoted by $k_{(i)}$, is

$$k = k_{(i)} = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \cdots + \binom{k_j}{j} \tag{2.1}$$

where $k_i > k_i - 1 > \cdots > k_j \geq j \geq 1$.

An expansion 2.1 always exists and is unique (see, e.g., Bruns and Herzog (1998)). Following the notation in Bruns and Herzog (1998), we define for any integers a and b,

$$(k_{(i)})_a^b = \binom{k_i + b}{i + a} + \binom{k_{i-1} + b}{i - 1 + a} + \cdots + \binom{k_j + b}{j + a}$$

where we set $\binom{s}{c} = 0$ whenever $s < c$ or $c < 0$.

Theorem 2.1.4. *Let $A = Q/I$ be a standard graded $\mathbb{K}$-algebra and $L \in A$ a general linear form (according to the Zariski topology). Denote by h_c the degree c entry of the Hilbert function of A and by h'_c the degree c entry of the Hilbert function of $A/(L)$. Then:*

- *(Macaulay) $h_{c+1} \leq ((h_c)_{(c)})_{+1}^{+1}$;*
- *(Gotzmann) If $h_{c+1} = ((h_c)_{(c)})_{+1}^{+1}$ and I is generated in degrees $\leq c$, then*

  $$h_{c+s} \leq ((h_c)_{(c)})_s^s$$

  for all $s \geq 1$;
- *(Green) $h'_c \leq ((h_c)_{(c)})_0^{-1}$.*

Proof. For Macaulay and Gotzmann, see Bruns and Herzog (1998), Theorems 4.2.10 and 4.3.3, respectively. For Green, see Green (2006), Theorem 1. □

Definition 2.1.5. We say that a sequence of non-negative integers $(h_0, h_1, \ldots, h_i, \ldots)$ is an *$\mathcal{O}$-sequence* if it satisfies Macaulay's Theorem.

Remark 2.1.6. *An interesting fact is that if we have an $\mathcal{O}$-sequence, then there is a graded algebra such that its Hilbert function is the given sequence. For more details see Migliore (2007).*

In the next definitions, we fix A an Artinian $\mathbb{K}$-algebra and $H = \mathrm{Hilb}(A) = (1, h_1, \ldots, h_d)$ its Hilbert vector.

Definition 2.1.7. We say that $\mathrm{Hilb}(A)$ is *unimodal* if there is a t such that

$$h_0 \leq h_1 \leq \ldots \leq h_t \geq h_{t+1} \geq \ldots \geq h_d$$

.

Definition 2.1.8. We define the *difference* of $\mathrm{Hilb}(A)$, denoted by $\triangle(H)$, as the sequence $\triangle H = (1, h_1 - 1, \ldots, h_t - h_{t-1})$, where $t = \min\{i \mid h_i \geq h_{i+1}\}$.

Definition 2.1.9. A sequence of non-negative integers satisfying the definitions 2.1.5, 2.1.7 and 2.1.8 is called *SI-sequence*.

2.2 ARTINIAN GORENSTEIN ALGEBRAS

Definition 2.2.1. Let $A = \bigoplus_{i=0}^{d} A_i$ be an Artinian $\mathbb{K}$-algebra. We say that A is a *Poincaré duality algebra* if $\dim_{\mathbb{K}} A_d = 1$ and the pairing $A_i \times A_{d-i} \to A_d$ given by multiplication is a perfect pairing for every $i = 0, \ldots, d$.

An important point is that Artinian Gorenstein algebras are characterized to be Poincaré duality algebras.

Proposition 2.2.2. *A graded Artinian $\mathbb{K}$-algebra A is Gorenstein if and only if it is a Poincaré duality algebra.*

Proof. Maeno and Watanabe (2009), Proposition 2.1. □

Notice that, for an Artinian Gorenstein algebra, by Poincare duality we have the isomorphisms

$$A_{d-i} \simeq \mathrm{Hom}_{\mathbb{K}}(A_i, A_d)$$

therefore, its Hilbert vector is symmetric, that is, $h_i = h_{d-i}$, for every $i = 0, \ldots, [\frac{d}{2}]$.

When A is Artinian, without loss of generality, we can suppose that $I_1 = 0$. Since $A = Q/I$ is Artinian, its Krull dimension is zero, therefore, the codimension of I is n. By abuse of notation, we give for A a property of the ideal I, we say that A has *codimension* $n = h_1$.

Definition 2.2.3. For a graded algebra $A = Q/I$, the *socle* of A is $\mathrm{Soc}(A) = 0 : \overline{\mathfrak{m}}$, where $\overline{\mathfrak{m}} = (\overline{X_1}, \ldots, \overline{X_n})$ is the homogeneous maximal ideal of A. If $\mathrm{Hilb}(A) = (1, h_1, \ldots, h_d)$, the integer d is called *socle degree* of A.

In particular, when A is Gorenstein the socle of A has dimension 1.

2.3 MACAULAY-MATLIS DUALITY

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a new polynomial ring. We make R into a Q-module, where Q acts in R by differentiation $X_i \circ x_j = \delta_{ij}$. This action is sometimes called *apolarity*. In other words, the polynomials of Q act as derivatives upon the polynomials of R.

Example 2.3.1. *Let $f = x^3 + y^2 z \in R = \mathbb{K}[x, y, z]$ and $G = X^2 + Z^2 \in Q = \mathbb{K}[X, Y, Z]$. Then $G \circ f = 6x$.*

For a homogeneous ideal $I \subset Q$, the *inverse system* of I, denoted I^{-1} is the Q-submodule of R consisting of all elements of R annihilated by I, that is

$$I^{-1} = \{f \in R;\ f \circ g = 0,\ \text{for all } g \in I\}.$$

Remark 2.3.2. *The following information is well-known for the inverse system:*

1. *In general, I^{-1} is not an ideal of R;*
2. $\dim_{\mathbb{K}}(I^{-1})_i = \dim_{\mathbb{K}}(R_i/I_i)$*;*
3. *I^{-1} is a finitely generated Q-module if and only if I is an Artinian ideal;*
4. *If I is a monomial ideal, $(I^{-1})_i$ is generated by monomials in R_i corresponding to the monomials in Q_i but not in I_i.*

Definition 2.3.3. For an ideal $I \subset R$, we define the *annihilator* of I in Q as the ideal

$$\text{Ann}_Q(I) = \{\alpha(X_1, \ldots, X_n) \in Q; \alpha(X_1, \ldots, X_n)I = 0\}.$$

In terms of generators, if $I = (f_1, \ldots, f_r)$, then $\alpha(X_1, \ldots, X_n) \in \text{Ann}_Q(I)$ if and only if $\alpha(X_1, \ldots, X_n) \circ f_i = 0$ for every $i = 0, \ldots, r$. If I is generated by a single element $f \in R$, we may write $\text{Ann}_Q(f)$ for $\text{Ann}_Q(I)$.

The Macaulay-Matlis duality gives us the bijection:

{Homogeneous ideals of Q}	$\leftrightarrow$	{Graded Q submodules of R}
$\text{Ann}_Q(M)$	$\leftarrow$	M
I	$\rightarrow$	I^{-1}

For Artinian Gorenstein algebras, the duality gives us the following Theorem, which is a machinery to construct Artinian Gorenstein algebras.

Theorem 2.3.4. *(Maeno and Watanabe (2009)) Let $A = Q/I$ be a graded Artinian algebra. Then A is Gorenstein if and only if there exists a homogeneous polynomial $f \in R_d$ such that $I = \mathrm{Ann}_Q(f)$.*

Example 2.3.5. *The Artinian algebra A given in example 2.1.2 is Gorenstein. In that case, we have $I = \mathrm{Ann}_Q(f)$, with $f = xy^3z + x^2yz^2 \in R = \mathbb{K}[x, y, z]_5$.*

For simplicity, sometimes we write A_f to denote the Artinian Gorenstein algebra associated with the homogeneous polynomial $f \in R_d$, i.e.,

$$A_f = \frac{Q}{\mathrm{Ann}_Q(f)}.$$

2.4 LEFSCHETZ PROPERTIES

Definition 2.4.1. Let $A = \bigoplus_{i=0}^{d} A_i$ be an Artinian algebra. We say that A satisfies the *weak Lefschetz property*, briefly WLP, if there exists $L \in A_1$ such that the multiplication map

$$\times L : A_i \to A_{i+1}$$

has maximal rank for every i. In this case, L is said to be a *weak Lefschetz element* of A.

We say that A satisfies the *strong Lefschetz property*, briefly SLP, if there exists $L \in A_1$ such that the multiplication map

$$\times L^k : A_i \to A_{i+k}$$

has maximal rank for all i and k. In this case, L is said to be a *strong Lefschetz element* of A.

Lefschetz properties are an open condition in the sense that, Lefschetz elements of Artinian algebras A form a Zariski open subsets. Therefore, A has WLP (SLP) if and only if a generic linear form L is a weak Lefschetz element (strong Lefschetz element).

It is known that all Artinian Gorenstein algebras with codimension 2 have SLP, see Proposition 3.15 in Harima et al. (2013). All complete intersection algebras in codimension 3 have WLP (Theorem 3.48, in Harima et al. (2013)), but still open when Artinian Gorenstein algebras with codimension 3 have SLP in general.

Another interesting problem is to study when the Hilbert vector of an Artinian Gorenstein algebra forces WLP or SLP. Many works are doing this study, for example, Abdallah et al. (2023), Boij et al. (2014), Boij et al. (2024), among others.

When $A = \bigoplus_{i=0}^{d}$ is an Artinian Gorenstein algebra, to verify WLP it is sufficient to verify the injectivity of the multiplication maps $\times L : A_k \to A_{k+1}$, if $d = 2k+1$, and $\times L : A_{k-1} \to A_k$ if $d = 2k$, see Migliore, Miró-Roig and Nagel (2011). For SLP it is sufficient to verify that all multiplication maps $\times L^{d-2i} : A_i \to A_{d-i}$ are isomorphisms for $i \in \{1, \dots, [\frac{d}{2}]\}$. In this way, it is natural to ask about the matrix of these linear maps. Ahead of this, we may define Higher Hessian and Mixed Hessian.

Let $f \in R_d$ be a homogeneous polynomial, let $A = Q/\operatorname{Ann}_Q(f) = \bigoplus_{i=0}^{d} A_i$ its associated Artinian Gorenstein algebra.

Definition 2.4.2. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_k\} \subset A_k$ be an ordered $\mathbb{K}$-basis. The *kth Hessian matrix* of f with respect to $\mathcal{B}$ is

$$\operatorname{Hess}_f^k = [\alpha_i \alpha_j(f)]_{i,j}.$$

The kth *Hessian* of f with respect to $\mathcal{B}$ is

$$\operatorname{hess}_f^k = \det(\operatorname{Hess}_f^k).$$

Remark 2.4.3.

- *The Hessian of order $k = 0$ it is just $\operatorname{hess}_f^0 = f$. For $k = 1$, hess_f^1 is the classical Hessian.*
- *The Hessian matrix depends on the choice of the $\mathbb{K}$-basis of A_k, although its vanishing does not depend on this choice.*

Following the notation in Gondim and Zappala (2019), let $t \leq l$ be two integers, take $L \in A_1$ and let us consider the $\mathbb{K}$-vector space map

$$\mu_L : A_t \to A_l, \ \mu_L(\alpha) = L^{l-t}\alpha.$$

Let $\mathcal{B}_t = (\alpha_1, \dots, \alpha_r)$ be a vector basis as above and $\mathcal{B}_l = (\beta_1, \dots, \beta_s)$ be a $\mathbb{K}$-linear basis of A_l.

Definition 2.4.4. We call the matrix

$$\operatorname{Hess}_f^{(t,l)} = [\alpha_i \beta_j(f)]$$

the *mixed Hessian matrix* of f of mixed order (t, l) with respect to the bases $\mathcal{B}_t$ and $\mathcal{B}_l$. Moreover, we have $\operatorname{Hess}_f^k = \operatorname{Hess}_f^{(k,k)}$.

With the two definitions above, we can state the Hessian criterion for an Artinian Gorenstein algebra to satisfy the weak and strong Lefschetz properties.

Theorem 2.4.5. *Let $A = Q/\operatorname{Ann}_Q(f)$ be a standard graded Artinian Gorenstein algebra of codimension n and socle degree d and let $L = a_1X_1 + \cdots + a_nX_n \in A_1$, such that $f(a_1, \dots, a_n) \neq 0$. The map $\mu_L : A_t \to A_l$, for $t < l \leq \frac{d}{2}$, has maximal rank if and only if the (mixed) Hessian matrix $\operatorname{Hess}_f^{(t,d-l)}(a_1, \dots, a_n)$ has maximal rank. In particular, we have the following:*

1. *(Strong Lefschetz Hessian criterion, Maeno and Watanabe (2009)) L is a stong Lefschetz element of A if and only if $\operatorname{hess}_f^k(a_1, \dots, a_n) \neq 0$ for all $t = 1, \dots, [\frac{d}{2}]$.*

2. *(Weak Lefschetz Hessian criterion, Gondim and Zappala (2019)) $L \in A_1$ is a weak Lefschetz element of A if and only if either $d = 2q+1$ is odd and $\operatorname{hess}_f^q(a_1, \dots, a_n) \neq 0$ or $d = 2q$ is even and $\operatorname{Hess}_f^{(q-1,q)}(a_1, \dots, a_n)$ has maximal rank.*

Example 2.4.6. *The Artinian algebra in the example 2.1.2 has SLP. We saw in example 2.3.5 that $A = Q/\operatorname{Ann}_Q(f)$, where $f = xy^3z + x^2yz^2$, and that, A is an Artinian Gorenstein algebra. Let us use the Hessian criterion to verify SLP.*

We have that A has socle degree 5, so we need to check hess_f and hess_f^2. As before, a linear $\mathbb{K}$-basis for A_1 and A_2 are, respectively (X, Y, Z) and $(X^2, XY, XZ, Y^2, YZ, Z^2)$. Then the respective Hessian matrices are

$$\operatorname{Hess}_f = \begin{pmatrix} 2yz^2 & 3y^2z + 2xz^2 & y^3 + 4xyz \\ 3y^2z + 2xz^2 & 6xyz & 3xy^2 + 2x^2z \\ y^3 + 4xyz & 3xy^2 + 2x^2z & 2x^2y \end{pmatrix},$$

$$\operatorname{Hess}_f^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 4z & 4y \\ 0 & 0 & 4z & 6z & 6y & 4x \\ 0 & 4z & 4y & 6y & 4x & 0 \\ 0 & 6z & 6y & 0 & 6x & 0 \\ 4z & 6y & 4x & 6x & 0 & 0 \\ 4y & 4x & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since hess_f and hess_f^2 are both non null, then A has the strong Lefschetz property.

3 ON MINIMAL GORENSTEIN HILBERT FUNCTIONS

We deal with standard graded Artinian Gorenstein $\mathbb{K}$-algebras over a field of characteristic zero. A natural and classical problem consists in understanding their possible Hilbert function, sometimes also called Hilbert vector. When the codimension of the algebra is less than or equal to 3, all possible Hilbert vectors were characterized in Stanley (1978); in particular, they are unimodal, i.e. they never strictly increase after a strict decrease. While it is known that non unimodal Gorenstein h-vectors exist in every codimension greater than or equal to 5 (see Bernstein and Iarrobino (1992), Boij (1995), Boij and Laksov (1994)), it is open whether non unimodal Gorenstein h-vectors of codimension 4 exist. For algebras with codimension 4 having a small initial degree the Hilbert vector is unimodal (see Seo and Srinivasan (2012), Migliore, Nagel and Zanello (2007)).

Consider the family $\mathcal{AG}_{\mathbb{K}}(r,d)$ of standard graded Artinian Gorenstein $\mathbb{K}$-algebras of socle degree d and codimension r. By Poncaré duality, the Hilbert function of $A \in \mathcal{AG}_{\mathbb{K}}(r,d)$ is a symmetric vector $\mathrm{Hilb}(A) = (1, r, h_2, \ldots, h_{d-2}, r, 1)$, that is $h_k = h_{d-k}$. There is a natural partial order in this family given by:

$$(1, r, h_2, \ldots, h_{d-2}, r, 1) \preceq (1, r, \tilde{h}_2, \ldots, \tilde{h}_{d-2}, r, 1),$$

if $h_i \leq \tilde{h}_i$, for all $i \in \{2, \ldots, d-2\}$. The maximal Hilbert functions are associated with compressed algebras and completely described in Iarrobino and Kanev (1999). In fact, the Hilbert vector of a compressed Gorenstein algebra is a maximum in $\mathcal{AG}_{\mathbb{K}}(r,d)$. On the other hand, classifying minimal Hilbert functions is a hard problem. We do not know in general if there is a minimum. Moreover, given two comparable Gorenstein Hilbert functions, it is not true that any symmetric vector between them is Gorenstein. Some partial results in this direction were obtained in Zanello (2009) and called the interval conjecture.

The first example of a non-unimodal Gorenstein h-vector was given by Stanley (see Stanley (1978, Example 4.3)). He showed that the h-vector $(1, 13, 12, 13, 1)$ is indeed a Gorenstein h-vector. In Migliore and Zanello (2017) the authors showed that Stanley's example is optimal, i.e. if we consider the $h-$vector $(1, 12, 11, 12, 1)$, it is not Gorenstein. We say that a vector is totally non unimodal if

$$h_1 > h_2 > \ldots > h_k \text{ for } k = \lfloor d/2 \rfloor.$$

A totally non unimodal Gorenstein Hilbert vector exists for every socle degree $d \geq 4$ when the codimension r is large enough. It is related to a conjecture posed by Stanley and proved in

Migliore, Nagel and Zanello (2008), Migliore, Nagel and Zanello (2009) and also a consequence of our Proposition 3.1.4, see Corollary 3.1.5.

From Macaulay-Matlis duality, every standard graded Artinian Gorenstein $\mathbb{K}$-algebra can be presented by a quotient of a ring of differential operators by a homogeneous ideal that is the annihilator of a single form in the dual ring of polynomials. Full Perazzo algebras are associated with full Perazzo polynomials, they are the family that we will study in detail. Perazzo polynomials are related to the Gordan-Noether theory of forms with vanishing Hessian (see Russo (2016, Chapter 7) and Gondim (2017)). In Gondim (2017) the author introduced the terminology Perazzo algebras to denote the Artinian Gorenstein algebra associated with a Perazzo polynomial. In Fiorindo, Mezzetti and Miró-Roig (2023) and Abdallah et al. (2022) the authors study the Hilbert vector and the Lefschetz properties for Perazzo algebras in codimension 5. In Cerminara et al. (2020) the authors study full Perazzo algebras focusing on socle degree 4, showing that they have minimal Hilbert vector in some cases. In this paper, we deal with codimension greater than 13 and we are more interested in full Perazzo algebras. In the case of socle degree 4, we recall the known results.

We now describe the contents of the thesis in more detail. In the first section, we recall the definition of full Perazzo algebras and we pose the full Perazzo Conjecture (see Conjecture 3.1.7). A full Perazzo polynomial of type m and degree d is a bigraded polynomial of bidegree $(1, d-1)$ given by $f = \sum x_j M_j$ where $\{M_j | j = 1, \ldots, \binom{m+d-2}{d-1}\}$ is a basis for $\mathbb{K}[u_1, \ldots, u_m]_{(d-1)}$. The associated Artinian Gorenstein algebra is called full Perazzo algebra.

Conjecture. *Let H be the Hilbert vector of a full Perazzo algebra of type $m \geq 3$ and socle degree $d \geq 4$ and let $r = r(m,d)$ its codimension. Then H is minimal in the family of Hilbert vectors of Artinian Gorenstein algebras of codimension r and socle degree d, that is, if $\hat{H}$ is a comparable Artinian Gorenstein Hilbert vector such that $\hat{H} \preceq H$, then $\hat{H} = H$.*

In the second section, we prove special cases of the Conjecture in socle degree 4 and we try to fill the gaps in order to classify all possible Hilbert functions up to codimension 25 (see Theorem 3.2.6, Corollary 3.2.7 and Proposition 3.2.8). In socle degree 5 we prove the Conjecture for $m \in \{3, 4, 5, 6, 7, 8, 9, 10\}$ (see Theorem 3.2.15) and a stronger version of the conjecture for $m = 3$ (see Corollary 3.2.16).

In the third section, we prove our main result that the full Perazzo Conjecture is true for arbitrary socle degree $d \geq 4$ and type $m = 3$.

Theorem. *Every full Perazzo algebra with socle degree $d \geq 4$ of type $m = 3$ has minimal Hilbert function.*

In the last section, we give a new proof of part of a result originally proved in Migliore, Nagel and Zanello (2009), concerning the asymptotic behavior of the minimum entry of a Gorenstein Hilbert function (see Theorem 3.4.2).

3.1 MINIMAL GORENSTEIN HILBERT FUNCTIONS

We will deal with standard bigraded Artinian Gorenstein algebras $A = \bigoplus_{i=0}^{d} A_i,\ A_d \neq 0$, with $A_k = \bigoplus_{i=0}^{k} A_{(i,k-i)}$, $A_{(d_1,d_2)} \neq 0$ for some d_1, d_2 such that $d_1 + d_2 = d$, we call (d_1, d_2) the socle bidegree of A. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A^*_{(i,j)} \simeq A_{(d_1-i,d_2-j)}$.
Let $R = \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring viewed as a standard bigraded ring in the sets of variables $\{x_1, \ldots, x_n\}$ and $\{u_1, \ldots, u_m\}$ and let $Q = \mathbb{K}[X_1, \ldots, X_n, U_1, \ldots, U_m]$ be the associated ring of differential operators.

We want to stress that the bijection given by Macaulay-Matlis duality preserves bigrading, that is, there is a bijection:

$$\begin{array}{ccc} \{\text{Bihomogeneous ideals of } Q\} & \leftrightarrow & \{\text{Bigraded } Q \text{ submodules of } R\} \\ \operatorname{Ann}_Q(M) & \leftarrow & M \\ I & \rightarrow & I^{-1} \end{array}$$

If $f \in R_{(d_1,d_2)}$ is a bihomogeneous polynomial of total degree $d = d_1 + d_2$, then $I = \operatorname{Ann}_Q(f) \subset Q$ is a bihomogeneous ideal and $A = Q/I$ is a standard bigraded Artinian Gorenstein algebra of socle bidegree (d_1, d_2) and codimension $r = m + n$ if we assume, without loss of generality, that $I_1 = 0$.

Notice that being A the associated algebra of a bihomogeneous polynomial $f \in R_{(d_1,d_2)}$, for all $\alpha \in Q_{(i,j)}$ with $i > d_1$ or $j > d_2$ we get $\alpha(f) = 0$, therefore, in these conditions $I_{(i,j)} = Q_{(i,j)}$. As a consequence, we have the following decomposition for all A_k:

$$A_k = \bigoplus_{i+j=k, i \leq d_1, j \leq d_2} A_{(i,j)}.$$

Furthermore, for $i < d_1$ and $j < d_1$, the evaluation map $Q_{i,j} \to A_{(d_1-i,d_2-j)}$ given by $\alpha \mapsto \alpha(f)$

provides the following short exact sequence:

$$0 \to I_{(i,j)} \to Q_{(i,j)} \to A_{(d_1-i,d_2-j)} \to 0.$$

Remark 3.1.1. *With the previous notation, all bihomogeneous polynomials of bidegree* $(1, d-1)$ *can be written in the form*

$$f = x_1 g_1 + \cdots + x_n g_n,$$

where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$. *The associated algebra,* $A = Q/\operatorname{Ann}_Q(f)$, *is bigraded, has socle bidegree* $(1, d-1)$ *and we assume that* $I_1 = 0$, *so* $\operatorname{codim} A = m + n$.

We recall the construction of full Perazzo algebras, introduced in Cerminara et al. (2020).

Definition 3.1.2. Let $\mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring in the n variables $x_1, \ldots, x_n$ and in the m variables $u_1, \ldots, u_m$. A *Perazzo polynomial* is a reduced bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$, of degree d, of the form

$$f = \sum_{i=1}^{n} x_i g_i \tag{3.1}$$

with $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$, for $i = 1, \ldots, n$, linearly independent and algebraically dependent polynomials in the variables $u_1, \ldots, u_m$. The associated algebra is called a *Perazzo algebra*, it has codimension $m + n$ and socle degree d.

Now we fix $m \geq 2$ and we consider the m variables $u_1, \ldots, u_m$. For a multi-index $\alpha = (e_1, \ldots, e_m)$ with $e_1 + \cdots + e_m = d - 1$, let

$$M_\alpha = u_1^{e_1} \cdots u_m^{e_m} \in Q_{d-1}$$

the be a $\mathbb{K}$-linear basis for Q_{d-1} and denote $\tau_m = \dim Q_{d-1} = \binom{m+d-2}{d-1}$.

Definition 3.1.3. Let $f \in \mathbb{K}[x_1, \ldots, x_{\tau_m}, u_1, \ldots, u_m]_{(1,d-1)}$ be a Perazzo polynomial of degree d of form:

$$f = \sum_{j=1}^{\tau_m} x_j M_j. \tag{3.2}$$

In this case, f is called *full Perazzo polynomial* of type m and degree d. The associated algebra is a *full Perazzo algebra* of socle degree d and codimension $m + \tau_m$.

Proposition 3.1.4. *Let* A *be a full Perazzo algebra of type* $m \geq 2$ *and socle degree* d. *Then for* $k = 0, \ldots, \lfloor \frac{d}{2} \rfloor$

$$h_k = \dim A_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}.$$

In particular, its Hilbert function is totally non-unimodal for $r >> 0$.

Proof. Using the bigrading of A and considering that the polynomial f has degree 1 in the variables $x_1, \dots, x_{\tau_m}$, fixed $k = 0, \dots, \lfloor \frac{d}{2} \rfloor$, we have the following decomposition:

$$A_k = A_{(0,k)} \oplus A_{(1,k-1)}.$$

(i) It is clear that $A_{(0,k)} = Q_{(0,k)}$, hence $\dim A_{(0,k)} = \dim Q_{(0,k)} = \binom{m+k-1}{k}$.

(ii) We have $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$ and $A_{(0,d-k)} = Q_{(0,d-k)}$, hence $\dim A_{(1,k-1)} = \dim Q_{(0,d-k)} = \binom{m+d-k-1}{d-k}$.

To verify that the Hilbert vector is asymptotically totally non unimodal it is enough to see that as a function of m, $h_k(m) \simeq \frac{1}{(d-k)!} m^{d-k}$ for $k \leq d/2$. □

Corollary 3.1.5. *For every $d \geq 4$ there is a positive integer r_0 such that for all $r \geq r_0$ there is an Artinian Gorenstein algebra with socle degree d and codimension r having a totally non unimodal Hilbert vector.*

Proof. Let m be large enough in order to guarantee that the Hilbert vector of the full Perazzo algebra $A = Q/\operatorname{Ann}(f)$, of type m and socle degree d has a totally non unimodal Hilbert vector. For every $r > m + \binom{m+d-2}{d-1}$, let $s = r - [m + \binom{m+d-2}{d-1}]$ and consider the algebra $A' = Q'/\operatorname{Ann}(g)$ where $Q' = Q[Y_1, \dots, Y_s]$ and $g = f + \sum_{i=1}^{s} Y_i^d$. It is easy to see that the Hilbert vector of A' is given by $h'_k = h_k + s$ for $k \neq 0, d$, therefore, it is totally non-unimodal and the result follows. □

Let $d \geq 4$, $r \geq 3$. Consider the family $\mathcal{AG}(r, d)$ of standard graded Artinian Gorenstein $\mathbb{K}$-algebras of socle degree d and codimension r. As always, we consider $\mathbb{K}$, a fixed field of characteristic 0. We know that the Hilbert function of $A \in \mathcal{AG}(r, d)$ is a symmetric vector $\operatorname{Hilb}(A) = (1, r, h_2, \dots, h_{d-2}, r, 1)$, with $h_i = h_{d-i}$ by Poincaré duality.

Consider the family of length d symmetric vectors of type $(1, r, h_2, \dots, h_{d-2}, r, 1)$, where $h_i = h_{d-i}$. There is a natural partial order in this family

$$(1, r, h_2, \dots, h_{d-2}, r, 1) \preceq (1, r, \tilde{h}_2, \dots, \tilde{h}_{d-2}, r, 1).$$

If $h_i \leq \tilde{h}_i$, for all $i \in \{2, \dots, d-2\}$. This order can be restricted to $\mathcal{AG}(r, d)$ which becomes a *poset*.

Definition 3.1.6. Let r, d be fixed positive integers and let H be a length $d+1$ symmetric vector $(1, r, h_2, \ldots, h_{d-2}, r, 1)$. We say that H is a *minimal Artinian Gorenstein Hilbert function* of socle degree d and codimension r if there is an Artinian Gorenstein algebra such that $\mathrm{Hilb}(A) = H$ and H is minimal in $\mathcal{AG}(r, d)$ with respect to $\preceq$. To be precise, if $\hat{H}$ is a comparable Artinian Gorenstein Hilbert vector such that $\hat{H} \preceq H$, then $\hat{H} = H$.

We now present the full Perazzo Conjecture.

Conjecture 3.1.7. *Let H be the Hilbert vector of a full Perazzo algebra of type m and socle degree d. Then H is minimal in $\mathcal{AG}(r, d)$.*

3.2 MINIMAL GORENSTEIN HILBERT FUNCTIONS IN LOW SOCLE DEGREE

In this section we study Gorenstein Hilbert functions of algebras with socle degree 4 and 5. Part of the results in socle degree 4 can be found in Cerminara et al. (2020).

3.2.1 Minimal Gorenstein Hilbert functions in socle degree 4

In socle degree 4, a Gorenstein sequence is of the form

$$(1, r, h, r, 1).$$

Let $\mu(r)$ be the integer such that $(1, r, \mu(r), r, 1)$ is a Gorenstein sequence, but $(1, r, \mu(r) - 1, r, 1)$ is not a Gorenstein sequence. Then $\mu(r) \leq h \leq \binom{r+1}{2}$.

It is well known that $(1, r, h, r, 1)$ is a Gorenstein sequence if and only if $\mu(r) \leq h \leq \binom{r+1}{2}$ (see Zanello (2009)). We set $\delta(r) = r - \mu(r)$. This function was introduced in Migliore, Nagel and Zanello (2008) and also studied in Cerminara et al. (2020). The function $\delta(r)$ is not decreasing, so $\delta(r) \leq \delta(r+1)$, for every r (see Migliore, Nagel and Zanello (2008, Proposition 8))

By Remark 5.4 in Cerminara et al. (2020), if $\delta(r-1) < \delta(r)$ then $\delta(r) = \delta(r-1) + 1$.

Definition 3.2.1. We say that the Gorenstein sequence $(1, r, \mu(r), r, 1)$ is *minimal*. Moreover we say that the Gorenstein sequence $(1, r, \mu(r), r, 1)$ is *strongly minimal* if $\delta(r-1) < \delta(r)$.

By Remark 5.4 in Cerminara et al. (2020), if $(1, r, \mu(r), r, 1)$ is strongly minimal, then $\delta(r) = \delta(r-1) + 1$.

The minimal r such that $(1, r, \mu(r), r, 1)$ is not unimodal is $r = 13$ (Migliore and Zanello (2017)). So $\delta(r) = 0$ for $r \leq 12$.

Proposition 3.2.2. $\delta(r) = 1$ *iff* $13 \leq r \leq 19$.

Consequently the sequence $(1, 13, 12, 13, 1)$ *is strongly minimal.*

Proof. The sequence $(1, 13, 12, 13, 1)$ is a Gorenstein sequence. This was originally proved by Stanley in Stanley (1978). This sequence is also the Hilbert Function of the full Perazzo algebra with $m = 3$. In Migliore and Zanello (2017, Proposition 3.1), it was proved that $(1, 12, 11, 12, 1)$ is not a Gorenstein sequence. Consequently $\delta(12) = 0$, therefore $\delta(r) = 0$ for every $r \leq 12$. In Ahn, Migliore and Shin (2018), Theorem 4.1, was shown that $(1, 19, 17, 19, 1)$ is not a Gorenstein sequence, so $\delta(r) = 1$ for $13 \leq r \leq 19$. In Migliore and Zanello (2017), Remark 3.5, was observed that $(1, 20, 18, 20, 1)$ is a Gorenstein sequence, so for $r \geq 20$ we have that $\delta(r) \geq 2$. □

Corollary 3.2.3. $\delta(20) = 2$. *Consequently the sequence* $(1, 20, 18, 20, 1)$ *is strongly minimal.*

Proof. In Migliore and Zanello (2017), Remark 3.5, it was observed that $(1, 20, 18, 20, 1)$ is a Gorenstein sequence. So, by Remark 5.4 in Cerminara et al. (2020), $\delta(20) = 2$. □

Proposition 3.2.4. *Let* $m \geq 3$. *We have that*

$$\delta\left(m + \binom{m+2}{3}\right) \geq \binom{m}{3}.$$

Proof. For $r = m + \binom{m+2}{3}$ there exists the full Perazzo Algebra. It realizes the Gorenstein sequence

$$(1, m + \binom{m+2}{3}, m(m+1), m + \binom{m+2}{3}, 1).$$

So $\delta\left(m + \binom{m+2}{3}\right) \geq \binom{m+2}{3} + m - m(m+1) = \binom{m+2}{3} - m^2 = \binom{m}{3}$. □

Lemma 3.2.5. *Let* $(1, r, h, r, 1)$ *be a Gorenstein sequence. Let* $u = r - h$, *with* $u \geq 0$. *Then*

$$\left(\left((r_{(3)})_0^{-1} - u\right)_{(2)}\right)_1^1 \geq (r_{(3)})_0^{-1}.$$

Proof. Let A be a Gorenstein algebra with Hilbert function $(1, r, h, r, 1)$ and let L be a general linear form. Using the same argument as in Proposition 3.1 in Migliore and Zanello (2017), we get that the Hilbert function of $A/(L)$ is of the type

$$(1, r - 1, s - u, s).$$

By the Theorems of Green and of Macaulay we have $s \leq (r_{(3)})_0^{-1}$ and

$$\left((s-u)_{(2)}\right)_1^1 \geq s.$$

Consequently

$$\left((s+t-u)_{(2)}\right)_1^1 \geq s+t, \text{ for every } t \geq 0.$$

In particular, for $t = (r_{(3)})_0^{-1} - s$ we are done. □

Theorem 3.2.6. $\delta(24) = 4$ *and* $\delta(40) = 10$.

Proof. By Proposition 3.2.4, $\delta(24) \geq \binom{4}{3} = 4$. We have to prove that $(1, 24, 19, 24, 1)$ is not a Gorenstein sequence. Indeed $24_{(3)} = \binom{6}{3} + \binom{3}{2} + \binom{1}{1}$, so $(24_{(3)})_0^{-1} = 11$. Since $u = 5$ we have that

$$\left((11-5)_{(2)}\right)_1^1 = 10 < 11.$$

By Lemma 3.2.5, $(1, 24, 19, 24, 1)$ is not a Gorenstein sequence.

By Proposition 3.2.4, $\delta(40) \geq \binom{5}{3} = 10$. We have to prove that $(1, 40, 29, 40, 1)$ is not a Gorenstein sequence. Indeed $40_{(3)} = \binom{7}{3} + \binom{3}{2} + \binom{2}{1}$, so $(40_{(3)})_0^{-1} = 22$. Since $u = 11$ we have that

$$\left((22-11)_{(2)}\right)_1^1 = 21 < 22.$$

By Lemma 3.2.5, $(1, 40, 29, 40, 1)$ is not a Gorenstein sequence. □

Corollary 3.2.7. $\delta(25) = 4$.

Proof. By Theorem 3.2.6 and by Theorem 2.5 in Migliore and Zanello (2017), $(1, 25, 21, 25, 1)$ is a Gorenstein sequence, so $\delta(25) \geq 4$. We have to prove that $(1, 25, 20, 25, 1)$ is not a Gorenstein sequence. Indeed $25_{(3)} = \binom{6}{3} + \binom{3}{2} + \binom{2}{1}$, so $(25_{(3)})_0^{-1} = 12$. Since $u = 5$ we have that

$$\left((12-5)_{(2)}\right)_1^1 = 11 < 12.$$

By Lemma 3.2.5, $(1, 25, 20, 25, 1)$ is not a Gorenstein sequence. □

Proposition 3.2.8. $2 \leq \delta(21) \leq \delta(22) \leq \delta(23) \leq 4$.

Proof. This follows trivially by the fact that $\delta(20) = 2$ and $\delta(24) = 4$. □

Proposition 3.2.9. $20 \leq \delta(62) \leq 21$.

Proof. By Proposition 3.2.4, for $m = 6$, we get that $(1, 62, 42, 62, 1)$ is a Gorenstein sequence.

On the other hand, $(1, 62, 40, 62, 1)$ is not a Gorenstein sequence by Lemma 3.2.5. Indeed

$$(62_{(3)})_0^{-1} = \binom{7}{3} + \binom{3}{2} = 38$$

and

$$\left((38 - 22)_{(2)}\right)_1^1 = 36 < 38.$$

□

Proposition 3.2.10. $\delta(26) = 4 = \delta(27)$.

Proof. For $r = 26$, we have to prove that $(1, 26, 21, 26, 1)$ is not a Gorenstein sequence. Indeed, let $A = R/I$ be a Gorenstein algebra with Hilbert function $(1, 26, 21, 26, 1)$, L be a general linear form. We set $J = (I_{\leq 3})$, $\bar{J} = (J, L)/(L)$ and $S = R/(L)$. By the Theorems of Green and of Macaulay (2.1.4) and repeating the above method, $R/\bar{J}$ has Hilbert function $(1, 25, 8, 13)$. As $R/\bar{J}$ has maximal growth from degree 2 to degree 3 and $\bar{J}$ has no new generators in degree 4, by Gotzmann's Theorem we get $h_{R/\bar{J}}(t) = \binom{t+2}{2} + t$. Therefore, $\bar{J}$ is the saturated ideal, in all degrees ≥ 2 of the union of a plane and a line in $\mathbb{P}^{24}$. It follows that, up to saturation, J is the ideal of a scheme T given by the union in $\mathbb{P}^{25}$ of a $3-$dimensional linear variety, a plane and m points (possibly embedded). Hence, $50 \leq h_{R/\bar{J}}(4) \leq 45$, which is absurd.

Now, for $r = 27$, following the same argument as above, we prove that the sequence $(1, 27, 22, 27, 1)$ is not Gorenstein. In this case we conclude $50 \leq h_{R/\bar{J}}(4) \leq 46$. □

3.2.2 Minimal Gorenstein Hilbert functions in socle degree 5

In socle degree 5, a Gorenstein sequence is of the form

$$(1, r, h, h, r, 1).$$

Let $\mu(r)$ be the integer such that $(1, r, \mu(r), \mu(r), r, 1)$ is a Gorenstein sequence, but $(1, r, \mu(r) - 1, \mu(r) - 1, r, 1)$ is not a Gorenstein sequence. Then $\mu(r) \leq h \leq \binom{r+1}{2}$.

It is well known that $(1, r, h, h, r, 1)$ is a Gorenstein sequence if and only if $\mu(r) \leq h \leq \binom{r+1}{2}$ (see Zanello (2009)). We set $\delta(r) = r - \mu(r)$.

Definition 3.2.11. We say that the Gorenstein sequence $(1, r, \mu(r), \mu(r), r, 1)$ is *minimal*. Moreover we say that the Gorenstein sequence $(1, r, \mu(r), \mu(r), r, 1)$ is *strongly minimal* if $\delta(r - 1) < \delta(r)$.

Proposition 3.2.12. *In socle degree* 5 *we have*

$\delta(r) = 0$ *if and only if* $r \leq 16$.

$\delta(r) = 1$ *if and only if* $r = 17$.

For $18 \leq r \leq 25$, $\delta(r) = 2$

Proof. For $18 \leq r \leq 25$, see Theorem 3.4 and Remark 3.5 in Migliore and Zanello (2017). □

Lemma 3.2.13. *Let* $(1, r, h, h, r, 1)$ *be a Gorenstein sequence. Let* $u = r - h$. *Then*

$$\left(\left((r_{(4)})_0^{-1} - u\right)_{(2)}\right)_2^2 \geq (r_{(4)})_0^{-1}.$$

Proof. Analogous to Lemma 3.2.5. □

Proposition 3.2.14. *Let* $m \geq 3$. *We have that*

$$\delta\left(m + \binom{m+3}{4}\right) \geq \frac{m+5}{4}\binom{m}{3}.$$

Proof. For $r = m + \binom{m+3}{4}$ there exists the full Perazzo Algebra. It realizes the Gorenstein sequence

$$\left(1, m + \binom{m+3}{4}, \binom{m+1}{2} + \binom{m+2}{3}, \binom{m+1}{2} + \binom{m+2}{3}, m + \binom{m+3}{4}, 1\right).$$

So

$$\delta\left(m + \binom{m+3}{4}\right) \geq \binom{m+3}{4} + m - \binom{m+1}{2} - \binom{m+2}{3} = \frac{m+5}{4}\binom{m}{3}.$$

□

Theorem 3.2.15. *For* $m \in \{3, 4, 5, 6, 7, 8, 9, 10\}$, *we have* $\delta\left(m + \binom{m+3}{4}\right) = \frac{m+5}{4}\binom{m}{3}$. *That is, the full Perazzo conjecture is true in these cases.*

Proof. By Proposition 3.2.14, we have to prove that for $m \in \{3, 4, 5, 6, 7, 8, 9, 10\}$, the sequence

$$\left(1, m + \binom{m+3}{4}, \binom{m+1}{2} + \binom{m+2}{3} - 1, \binom{m+1}{2} + \binom{m+2}{3} - 1, m + \binom{m+3}{4}, 1\right)$$

is not a Gorenstein sequence.

The case $m = 3$ will be dealt with in general in the next section. We can assume $m \geq 4$.

For $m = 4$ we have to prove that the sequence $(1, 39, 29, 29, 39, 1)$ is not a Gorenstein sequence. Indeed, using Lemma 3.2.13, we have:

$$(39_{(4)})_0^{-1} = 16; \; u = 10; \; 16 - 10 = 6; \; (6_{(2)})_2^2 = 15 < 16.$$

For $m = 5$ we have to prove that the sequence $(1, 75, 49, 49, 75, 1)$ is not a Gorenstein sequence. Indeed, by Lemma 3.2.13, we have:

$$(75_{(4)})_0^{-1} = 36;\ u = 26;\ 36 - 26 = 10;\ (10_{(2)})_2^2 = 35 < 36.$$

For $m = 6$ we have to prove that the sequence $(1, 132, 76, 76, 132, 1)$ is not a Gorenstein sequence. Indeed, using Lemma 3.2.13, we have:

$$(132_{(4)})_0^{-1} = 71;\ u = 56;\ 71 - 56 = 15;\ (15_{(2)})_2^2 = 70 < 71.$$

For $m = 7$ we have to prove that the sequence $(1, 217, 111, 111, 217, 1)$ is not a Gorenstein sequence. Indeed, by Lemma 3.2.13, we have:

$$(217_{(4)})_0^{-1} = 128;\ u = 106;\ 128 - 106 = 22;\ (22_{(2)})_2^2 = 127 < 128.$$

For $m = 8$ we have to prove that the sequence $(1, 338, 155, 155, 338, 1)$ is not a Gorenstein sequence. Indeed, using Lemma 3.2.13, we have:

$$(338_{(4)})_0^{-1} = 212;\ u = 183;\ 212 - 183 = 29;\ (29_{(2)})_2^2 = 211 < 212.$$

For $m = 9$ we have to prove that the sequence $(1, 504, 209, 209, 504, 1)$ is not a Gorenstein sequence. Indeed, using symmetry, Green's Theorem, and Macaulay's Theorem, the following diagram represents the Hilbert functions of R/I, $R/(I : L)$ and $R/(I, L)$

1	504	209	209	504	1
	1	171	54	171	1
1	503	38	155	333	

By Lemma 3.2.5 the middle line is not a Gorenstein sequence.

For $m = 10$ we have to prove that the sequence $(1, 725, 274, 274, 725, 1)$ is not a Gorenstein sequence. Indeed, using symmetry, Green's Theorem, and Macaulay's Theorem, the following diagram represents the Hilbert functions of R/I, $R/(I : L)$ and $R/(I, L)$

1	725	274	274	725	1
	1	226	65	226	1
1	724	48	209	499	

By Lemma 3.2.5 the middle line is not a Gorenstein sequence. $\square$

Corollary 3.2.16. *The Gorenstein vector*

$$(1, 18, 16, 16, 18, 1)$$

is strongly minimal.

Proof. By the previous Theorem, we know that it is a minimal Gorenstein Hilbert vector. We have to prove that $(1, 17, 15, 15, 17, 1)$ is not a Gorenstein sequence. Indeed, by Proposition 3.2.12, $\delta(17) = 1$. □

3.3 A FAMILY OF MINIMAL GORENSTEIN HILBERT FUNCTIONS

Consider the family of full Perazzo algebras of type $m = 3$ and socle degree $d \geq 4$. Its Hilbert function is given by $h_k = \binom{k+2}{k} + \binom{2+2q-k}{2q-k}$, for $k \leq \lfloor d/2 \rfloor$ and by symmetry we get $h_{d-k} = h_k$.

Lemma 3.3.1. *Let $k \leq \lfloor d/2 \rfloor$. Then we have:*

$$\left(\left(\binom{k+1}{2}\right)_{(d-k)}\right)_0^{-1} \leq k - 2.$$

Proof. First of all, consider $d > 2k - 1$. In this case, $d - k + 1 > k$, i.e. $(d - k + 1) + (d - k) + \ldots + (d - 2k + 3) > k + (k - 1) + \ldots + 2 + 1 = k(k + 1)/2 = \binom{k+1}{2}$.

We have: $(\binom{k+1}{2})_{(d-k)} < \binom{d-k+1}{d-k} + \binom{d-k}{d-k-1} + \ldots \binom{d-2k+3}{d-2k+2}$. Then

$$\left(\left(\binom{k+1}{2}\right)_{(d-k)}\right)_0^{-1} \leq k - 2.$$

Now there are only two other cases to consider: 1) $d = 2k - 1$, 2) $d = 2k$.

They are similar, we will do the details for $d = 2k$. In this case, we have:

$$\left(\left(\binom{k+1}{2}\right)_{(k)}\right)_0^{-1} \leq k - 2.$$

Indeed we know that the k-binomial expansion of $\binom{k+1}{2}$ has two blocks

$(\binom{k+1}{2})_{(k)} = [\binom{k+1}{k} + \binom{k}{k-1} + \ldots + \binom{j+1}{j}] + [\binom{j-1}{j-1} + \ldots + \binom{i}{i}]$. The first block consists of binomials of type $\binom{s+1}{s}$ and the second one of type $\binom{s}{s}$.

Therefore:

$$\left(\left(\binom{k+1}{2}\right)_{(k)}\right)_0^{-1} = k-j+1.$$

If $k-j+1 > k-2$, then $j \le 2$, but the cases $j=1$ and $j=2$ are not possible. In fact, suppose, $j=2$, since:

$$\binom{k+1}{k} + \binom{k}{k-1} + \ldots + \binom{3}{2} = (k+4)(k-1)/2 \ge \binom{k+1}{2}.$$

It is absurd for $k > 2$. The case $j=1$ is analogous, the result follows. □

Theorem 3.3.2. *Every full Perazzo algebra with socle degree $d \ge 4$ of type $m=3$ has minimal Hilbert function.*

Proof. We want to show that the Hilbert vector of the full Perazzo algebra

$$H = (1, h_1, h_2, h_3, \ldots, h_{d-1}, h_d = 1)$$

with $h_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}$ is a minimal Gorenstein Hilbert vector. Let

$$\hat{H} = (1, \hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots, \hat{h}_{d-1}, 1)$$

be a comparable Artinian Gorenstein Hilbert vector $\hat{H} \preceq H$ of length $d+1$ and $\hat{h}_1 = h_1$. We will proceed in steps to show that $\hat{H} = H$. Consider, on the contrary, one of the following situations:

1. For some $k \in \{2, \ldots, \lfloor d/2 \rfloor - 1\}$, $\hat{h}_k < h_k$;
2. For $d = 2q$, suppose that $\hat{h}_t = h_t$ for all $t < q$ and $\hat{h}_q < h_q$;
3. For $d = 2q+1$, suppose that $\hat{h}_t = h_t$ for all $t < q$ and $\hat{h}_q < h_q$.

We will show that all of these situations give rise to a contradiction.

(1). Let $A = Q/I$ with $I = \operatorname{Ann}(f)$ be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra such that $H_A = \hat{H}$ with $\hat{h}_k = \dim A_k < h_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}$ for some $k \in \{2, \ldots, \lfloor d/2 \rfloor - 1\}$. Suppose that k is minimal satisfying this property, that is, for $t < k$ we get $\hat{h}_t = h_t$, by the comparability hypothesis. Let $L \in A_1$ be a generic linear form and let $S = Q/(L)$. We get the following exact sequence:

$$0 \longrightarrow Q/(I:L)(-1) \longrightarrow Q/I \longrightarrow S/\overline{I} \longrightarrow 0$$

with $\overline{I} = \dfrac{(I,L)}{L}$ and $(I : L) = Ann(f)$ and $f' = L(f)$ denoting the derivative of f with respect to $L \in Q$. Therefore $Q/(I : L)$ is also Gorenstein. We get the following diagram:

$$\begin{array}{ccccccccc} 1 & \hat{h}_1 & \dots & \hat{h}_k & \dots & \hat{h}_{d-k} & \hat{h}_{d-k+1} & \dots & 1 \\ & 1 & \dots & a_{k-1} & \dots & & a_{k-1} & \dots & 1 \\ \hline 1 & \hat{h}_1 - 1 & \dots & h'_k & \dots & & h'_{d-k+1} & & \end{array}$$

We have $\hat{h}_{d-k+1} = \hat{h}_{k-1} = h_{k-1} = h_{d-k+1} = \binom{k+1}{k-1} + \binom{d-k+3}{d-k+1}$. The $(d-k+1)$-binomial decomposition of h_{d-k+1} is $(h_{d-k+1})_{(d-k+1)} = \binom{d-k+3}{d-k+1} + (\binom{k+1}{k-1})_{(d-k)}$. By Green's Theorem we have

$$h'_{d-k+1} \leq ((\hat{h}_{d-k+1})_{(d-k+1)})_0^{-1} = ((h_{d-k+1})_{(d-k+1)})_0^{-1} = \binom{d-k+2}{d-k+1} + \left(\left(\binom{k+1}{k-1}\right)_{(d-k)}\right)_0^{-1}.$$

By Lemma 3.3.1, we have

$$h'_{d-k+1} \leq d - k + 2 + k - 2 = d.$$

We consider only the case $h'_{d-k+1} = d$, the other cases are similar.

We have $a_{k-1} = \hat{h}_{d-k+1} - d$, $h'_k = \hat{h}_k - (\hat{h}_{d-k+1} - d)$. Since $\hat{h}_k \leq h_k - 1$ we have $h'_k \leq h_k - h_{d-k+1} + d - 1$.

We recall that

$$h_k = \binom{k+2}{2} + \binom{d-k+2}{2}, \; h_{d-k+1} = \binom{d-k+3}{2} + \binom{k+1}{2}.$$

Therefore

$$\begin{aligned} h_k - h_{d-k+1} &= \left[\binom{k+2}{2} - \binom{k+1}{2}\right] - \left[\binom{d-k+3}{2} - \binom{d-k+2}{2}\right] \\ &= (k+1) - (d-k+2) \\ &= 2k - d - 1 \end{aligned}$$

We obtain $h'_k \leq 2k - 2$. Thence $h'_k \leq 2k - 2 = k + 1 + k - 3$ which implies that $(h'_k)_k \leq \binom{k+1}{k} + \binom{k-1}{k-1} + \dots + \binom{3}{3}$.

By Macaulay's Theorem applied $d - 2k + 1$ times we have

$$h'_{d-k+1} \leq ((h'_k)_k)^{d-2k+1}_{d-2k+1} \leq \binom{k+1+d-2k+1}{k+d-2k+1} + k - 3 = k + 1 + d - 2k + 1 + k - 3$$

therefore $d \leq d - 1$,a contradiction.

(2). Case $d = 2q$ is even. Suppose that $\hat{h}_t = h_t$ for all $t < q$ and $\hat{h}_q < h_q$. Let $L \in Q$ be a generic linear form and $S = Q/(L)$. We have the following exact sequence:

$$0 \longrightarrow Q/(I:L)(-1) \longrightarrow Q/I \longrightarrow S/\overline{I} \longrightarrow 0$$

where $\overline{I} = \dfrac{(I,L)}{L}$ and $(I:L) = Ann(f)$, $f' = L(f)$, that is, $Q/(I:L)$ is also Gorenstein. In the middle we get the following diagram:

$$\begin{array}{cccccccc} 1 & h_1 & \dots & h_{q-1} & \hat{h}_q & h_{q+1} & \dots & 1 \\ & & & & & & & \\ & 1 & \dots & a_{q-2} & a_{q-1} & a_{q-1} & \dots & 1 \\ \hline & & & & & & & \\ 1 & h_1 - 1 & \dots & & h'_q & h'_{q+1} & & \end{array}$$

Since $(h_{q+1})_{(q+1)} = \binom{q+3}{q+1} + (\binom{q+1}{2})_q$, from Green's Theorem

$$h'_{q+1} \leq ((h_{q+1})_{q+1})_0^{-1} = \binom{q+2}{q+1} + ((\binom{q+1}{2})_q)_0^{-1}.$$

By Lemma 3.3.1 we have

$$((h_{q+1})_{q+1})_0^{-1} = \binom{q+2}{q+1} + ((\binom{q+1}{2})_q)_0^{-1} \leq q + 2 + q - 2 = 2q.$$

We study the case $h'_{q+1} = 2q$, the other cases are similar.

We have $a'_{q-1} = h_{q+1} - 2q$, $h'_q = \hat{h}_q - a_{q-1} \leq h_q - h_{q+1} + 2q - 1$, then $h'_q \leq 2q - 2 = (q+1) + (q-3)$.

Therefore

$$(h'_q)_q \leq \binom{q+1}{q} + \binom{q-1}{q-1} + \binom{q-2}{q-2} + \dots + \binom{3}{3},$$

with $\binom{q-1}{q-1} + \binom{q-2}{q-2} + \dots + \binom{3}{3}$ being counted $q - 3$ times.

From Macaulay's Theorem we have $h'_{q+1} \leq ((h'_q)_{(q)})_{+1}^{+1}$, hence

$$\begin{aligned} 2q &\leq \binom{q+2}{q+1} + \binom{q}{q} + \binom{q-1}{q-1} + \dots + \binom{4}{4} \\ &\leq q + 2 + q - 3 = 2q - 1. \end{aligned}$$

It is a contradiction.

(3). If $d = 2q + 1$ is odd. Suppose that $\hat{h}_t = h_t$ for all $t < q$ and $\hat{h}_q < h_q$. By the same argument:

$$\begin{array}{ccccccccc} 1 & h_1 & \dots & \hat{h}_q & \hat{h}_{q+1} & h_{q+2} & \dots & 1 \\ & 1 & \dots & a_{q-1} & a_q & a_{q-1} & \dots & 1 \\ \hline 1 & h_1-1 & \dots & h'_q & h'_{q+1} & h'_{q+2} & & \end{array}$$

Since $h_{q+2} = \binom{q+4}{q+2} + \binom{q+1}{2}$ and $(h_{q+2})_{q+2} = \binom{q+4}{q+2} + (\binom{q+1}{2})_{q+1}$, by Green's Theorem

$$h'_{q+2} \leq ((h_{q+2})_{(q+2)})_0^{-1} = \binom{q+3}{q+2} + (\binom{q+1}{2})_{(q+1)})_0^{-1} \leq q+3+q-2 = 2q+1.$$

We consider only the case $h'_{q+2} = 2q+1$. We have $a_{q-1} = h_{q+2} - (2q+1)$, $h'_q = \hat{h}_q - a_{q-1}$. Then $h'_q \leq h_q - 1 - a_{q-1}$, $h'_q \leq h_q - 1 - (h_{q+2} - (2q+1))$, thence $h'_q \leq 2q-2$. We have

$$h_q - h_{q+2} = \binom{q+2}{2} - \binom{q+1}{2} + \binom{q+3}{2} - \binom{q+4}{2} = -2.$$

Therefore

$$\begin{aligned} h'_q &\leq (q+1)+(q-3) \\ &\leq \binom{q+1}{q} + \binom{q-1}{q-1} + \cdots + \binom{3}{3} \end{aligned}$$

where the terms $\binom{q-1}{q-1} + \cdots + \binom{3}{3}$ are $q-3$.

By Macaulay's Theorem, we have

$$h'_{q+1} \leq ((h'_q)_q)_{+1}^{+1} = \binom{q+2}{q+1} + \binom{q}{q} + \cdots + \binom{4}{4},$$

the last terms are $q-3$.

By Macaulay's Theorem, we have

$$h'_{q+2} \leq \binom{q+3}{q+2} + \binom{q+1}{q+1} + \cdots + \binom{5}{5} = q+3+q-3 = 2q,$$

then $2q+1 \leq 2q$.

It is a contradiction. The result follows. □

3.4 ASYMPTOTIC BEHAVIOR OF THE MINIMUM

In this section, we give a new proof of part of Theorem 3.6 in Migliore, Nagel and Zanello (2009).

Let $P_m = m + \binom{m+d-2}{d-1}$ be the codimension of a full Perazzo algebra of type m. Denote by $\mu_{d,k}(r)$ the minimal entry in degree k of a Gorenstein h-vector with codimension r and socle degree d.

Lemma 3.4.1. $\mu_{d,k}(P_m) \geq \binom{m+d-k-1}{d-k}$.

Proof. We proceed by induction on k.

For $k = 1$, we have $\mu_{d,1}(P_m) = P_m = m + \binom{m+d-2}{d-1} > \binom{m+d-2}{d-1}$.

Now, suppose that

$$\mu_{d,k-1}(P_m) > \binom{m+d-k}{d-k+1}.$$

From Theorem 2.4 in Migliore, Nagel and Zanello (2009), we get

$$\mu_{d,k}(P_m) \geq \left((\mu_{d,k-1}(P_m))_{(d-k+1)}\right)^{-1}_{-1} + \left((\mu_{d,k-1}(P_m))_{(d-k+1)}\right)^{-d+2k}_{-d+2k+1}.$$

By inductive hypothesis, and by basic properties of binomial expansions, we have:

$$\left((\mu_{d,k-1}(P_m))_{(d-k+1)}\right)^{-1}_{-1} > \binom{m+d-k-1}{d-k} \text{ and } \left((\mu_{d,k-1}(P_m))_{(d-k+1)}\right)^{-d+2k}_{-d+2k+1} > \binom{m+k}{k+2}.$$

So,

$$\mu_{d,k}(P_m) \geq \binom{m+d-k-1}{d-k} + \binom{m+k}{k+2} > \binom{m+d-k-1}{d-k}.$$

as we wanted.

□

Theorem 3.4.2 (Migliore, Nagel and Zanello (2009))**.** *Let A be a Gorenstein algebra of codimension r and socle degree d. Then, for all $k < \lfloor d/2 \rfloor$*

$$\lim_{r\to\infty} \frac{\mu_{d,k}(r)}{r^{\frac{d-k}{d-1}}} = \frac{((d-1)!)^{\frac{d-k}{d-1}}}{(d-k)!}.$$

Proof. For any integer $r >> 0$ there is a unique integer $P_m = m + \binom{m+d-2}{d-1}$ such that

$$P_m \leq r \leq P_{m+1}.$$

Applying the function $\mu_{d,k}$ we have

$$\mu_{d,k}(P_m) \leq \mu_{d,k}(r) \leq \mu_{d,k}(P_{m+1}).$$

By Lemma 3.4.1

$$\binom{m+d-k-1}{d-k} \leq \mu_{d,k}(r) \leq \binom{m+d-k}{d-k} + \binom{m+k}{k}.$$

Therefore

$$\frac{m^{d-k}}{(d-k)!} + o(m^{d-k-1}) \leq \mu_{d,k}(r) \leq \frac{m^{d-k}}{(d-k)!} + o(m^{d-k-1})$$

where $o(m^s)$ denote all terms of degree less than s.

On other hand, since $P_m \leq r \leq P_{m+1}$, then

$$\begin{aligned}
\frac{m^{d-1}}{(d-1)!} + o(m^{d-2}) \leq &\quad r &\leq \frac{m^{d-1}}{(d-1)!} + o(m^{d-2}) \\
\frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1}) \leq &\quad r^{\frac{d-k}{d-1}} &\leq \frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1}) \\
\frac{1}{\frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1})} \leq &\quad \frac{1}{r^{\frac{d-k}{d-1}}} &\leq \frac{1}{\frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1})}
\end{aligned}$$

Multiplying, we get

$$\frac{\frac{m^{d-k}}{(d-k)!} + o(m^{d-k-1})}{\frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1})} \leq \frac{\mu_{d,k}(r)}{r^{\frac{d-k}{d-1}}} \leq \frac{\frac{m^{d-k}}{(d-k)!} + o(m^{d-k-1})}{\frac{m^{d-k}}{((d-1)!)^{\frac{d-k}{d-1}}} + o(m^{d-k-1})}.$$

Since on both sides the limit exists and are the same, the result follows. $\square$

4 ON LEFSCHETZ LOCUS IN GOR$(1, N+1, N+1, 1)$

Given a standard graded Artinian Gorenstein $\mathbb{K}$-algebra $A = \oplus_{i=0}^{d} A_i$, its Hilbert vector is $H = \mathrm{Hilb}(A) = (1, h_1, \ldots, h_d)$, where $h_i = \dim_{\mathbb{K}} A_i$. We denote by $\mathrm{Gor}(H)$ the space which parametrizes the Artinian Gorenstein algebras with Hilbert vector H. Iarrobino and Kanev (1999) studied deeply this space. Later, Boij (1999) studied this space giving examples where this space is reducible.

Fassarella, Ferrer and Gondim (2021), studied the Lefschetz locus in $\mathrm{Gor}(1, 5, 5, 1)$. Here, we are interested in the study of the locus of standard graded Artinian Gorenstein algebras with codimension $N+1$ and socle degree 3 in $\mathrm{Gor}(1, N+1, N+1, 1)$, which satisfy SLP. By the Macaulay-Matlis duality, these algebras are associated with a cubic homogeneous polynomial f, such that $A = Q/\operatorname{Ann}_Q(f)$, and by 2.4.5, these algebras has SLP if and only if $\mathrm{hess}_f \neq 0$. Perazzo (1900) and Gondim and Russo (2015) studied deeply these cubics with vanishing hessian.

Inspired in Gondim and Russo (2015), we parametrize the space of cubics, not cones with vanishing hessian, and we calculate its dimension and degree using techniques of intersection theory. We analyze all possible Jordan types for these cubics as an algebraic description. To finish, we calculate the dimension and degree of the cubics with vanishing hessian in $\mathrm{Gor}(1, 6, 6, 1)$ and $\mathrm{Gor}(1, 7, 7, 1)$.

4.1 CUBICS WITH VANISHING HESSIAN

We will work over an algebraically closed field of characteristic zero.

Definition 4.1.1. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. The *vertex* of X is the closed subset

$$\mathrm{Vert}(X) = \{p \in X | \langle p, q \rangle \subset X, \ \forall q \in X\}$$

where if $p, q \in X$, $\langle p, q \rangle$ denotes the line join p, q. A projective variety $X \subset \mathbb{P}^N$ is a *cone* if $\mathrm{Vert}(X) \neq \emptyset$. In this case, $\mathrm{Vert}(X)$ is a linear subspace of $\mathbb{P}^N$.

If X is a hypersurface, we have the following well-known equivalence.

Proposition 4.1.2. *Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface of degree d. Then the following conditions are equivalent:*

(i) X is a cone;

(ii) The partial derivatives $f_0, f_1, ..., f_N$ of f are linearly dependent;

(iii) Up to a projective transformation f depends on at most N variables;

Cones have vanishing Hessian, but the converse is not true.

Definition 4.1.3. The *polar map* of a hypersurface $X = V(f) \subset \mathbb{P}^N$ is the rational map given by the derivatives of f.

$$\Phi_f : \mathbb{P}^N \dashrightarrow (\mathbb{P}^N)^*$$

$$\Phi_f(p) = (\tfrac{\partial f}{\partial x_0}(p), \tfrac{\partial f}{\partial x_1}(p), ..., \tfrac{\partial f}{\partial x_N}(p)).$$

The *polar image* of X is $Z = \overline{\Phi_f(\mathbb{P}^N)}$.

The next proposition makes clear the difference between being a cone and the vanishing of its Hessian.

Proposition 4.1.4. *Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface, Φ_f the associated Polar map and $Z = \overline{\Phi_X(\mathbb{P}^N)}$ the Polar image.*

1. *X has vanishing Hessian $\Leftrightarrow$ $Z \subsetneq \mathbb{P}^{N*}$ $\Leftrightarrow$ The partial derivatives of f are algebraically dependent.*

2. *X is a cone $\Leftrightarrow$ $Z \subset H = \mathbb{P}^{N-1*} \subset \mathbb{P}^N$ (is degenerated) $\Leftrightarrow$ The partial derivatives of f are Linearly dependent.*

Proof. See Ciliberto, Russo and Simis (2008). □

The following example appears in the work of Gordan and Noether (1876) and Perazzo Perazzo (1900), where it is called *un essempio semplicissimo*.

Example 4.1.5. *Let $X = V(f) \subset \mathbb{P}^4$ be the irreducible hypersurface given by*

$$f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^4$$

An easy calculation shows that X is not a cone. On the other hand, $f_0f_2 = f_1^2$ is a trivial algebraic relation among the derivatives of f, so $\text{hess}_f = 0$, by Proposition 4.1.4.

The next result will be useful. Its proof can be found in the original work of Perazzo, see Perazzo (1900) in the cubic case and for a general degree in Zak (2004, pg.21).

Proposition 4.1.6. *Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface with vanishing hessian, $Y = \mathrm{Sing}(X)_{red}$ the singular locus and Z^* the dual of the Polar image of X. Then*

$$Z^* \subset Y.$$

4.1.1 Two families of cubics with vanishing Hessian

The notion of the Perazzo map was implicitly introduced in Perazzo (1900), see also Gondim and Russo (2015).

Definition 4.1.7. Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface with vanishing hessian, let $\Phi_X : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be its polar map and let $Z = \Phi_X(\mathbb{P}^N) \subsetneq (\mathbb{P}^N)^*$ be its polar image. *The Perazzo map of X* is the rational map:

$$\begin{array}{rccc} \mathcal{P}_X : & \mathbb{P}^N & \dashrightarrow & \mathbb{G}(\mathrm{codim}(Z)-1, N) \\ & p & \mapsto & (T_{\Phi_f(p)}Z)^* \end{array}$$

defined in the open set $\mathcal{U} = \Phi_f^{-1}(Z_{\mathrm{reg}})$, where Z_{reg} is the locus of smooth points of Z.

The image of the Perazzo map will be denoted by $W_X = \overline{\mathcal{P}_X(\mathbb{P}^N)} \subset \mathbb{G}(\mathrm{codim}(Z)-1, N)$, or simply by W, and its dimension $\mu = \dim W$ is called the *Perazzo rank of X*.

We are particularly interested in the $\mathrm{codim}(Z) = 1$ case. In this case:

$$\begin{array}{ccccc} \mathbb{P}^N & \dashrightarrow & (\mathbb{P}^N)^* & \dashrightarrow & \mathbb{P}^N \\ & & Z & \dashrightarrow & W = Z^* \end{array}$$

The general fiber of the Perazzo map is linear, see Gondim and Russo (2015, Theorem 2.5).

Definition 4.1.8. An irreducible cubic hypersurface $X \subset \mathbb{P}^N$ with vanishing hessian, not a cone, will be called a *Special Perazzo Cubic Hypersurface* if the general fibers of its Perazzo map determine a congruence of linear spaces passing through a fixed $\mathbb{P}^{N-\mu-1}$.

From Gondim and Russo (2015, Theorem 3.3), if $X = V(f) \subset \mathbb{P}^N$ is a special Perazzo form with $\mathrm{codim}(Z) = 1$ and $\mu = \dim Z^*$ then $Z^* \subset \langle Z^* \rangle = \mathbb{P}^k \subset \mathrm{Sing}(X)$ is a hypersurface, that is $k = \mu + 1$.

Conversely, for all cubic hypersurface $X = V(f) \subset \mathbb{P}^N$, if $\mathbb{P}^k \subset \mathrm{Sing}(X)$, then from Gondim and Russo (2015, Proposition 4.1), we have:

$$f = \sum_{i=0}^{k} x_i g_i + h.$$

Here $g_i \in \mathbb{K}[x_{N-m+1}, \ldots, x_N]_2$ and $h \in \mathbb{K}[x_{k+1}, \ldots, x_N]_3$ and $k + m \leq N$. Notice that $2 \leq m \leq k$ implies $\mathrm{hess}_f = 0$ since in this case the partial derivatives of f are algebraically dependent. We say that the form f is *minimal* if $m = k = 2$ and that f is *maximal* if $N = 2k = 2m$. From Gondim and Russo (2015, Lemma 2.9, Proposition 2.12) these two families correspond to special Perazzo cubic forms.

4.1.1.1 ***The minimal family***

The minimal family of special Perazzo cubics consists of $f \in \mathbb{K}[x_0, \ldots, x_N]$ with $\dim Z^* = 1$. In this case:

$$f = x_0 g_0 + x_1 g_1 + x_2 g_2 + h \tag{4.1}$$

Here $g_i \in \mathbb{K}[x_{N-1}, x_N]$ and $h \in \mathbb{K}[x_3, \ldots, x_N]$. Since X is a special Perazzo hypersurface, $Z^* \subset \langle Z^* \rangle = \mathbb{P}^2 \subset \mathrm{Sing}(X)$.

4.1.1.2 ***The maximal family***

The maximal family of special Perazzo cubics consists of $X = V(f) \subset \mathbb{P}^{2k}$ with $\dim Z^* = k - 1$. Therefore, $Z^* \subset \langle Z^* \rangle = \mathbb{P}^k \subset \mathrm{Sing}(X)$. Putting $N = 2k$, we have, $f \in \mathbb{K}[x_0, \ldots, x_N]$

$$f = x_0 g_0 + x_1 g_1 + \ldots + x_k g_k + h. \tag{4.2}$$

Here $g_i, h \in \mathbb{K}[x_{k+1}, \ldots, x_N]$.

Remark 4.1.9. *Notice that there are cubics whose canonic form is of the maximal family but belongs to the minimal family. Consider the cubic*

$$f = x_0 x_4^2 + x_1 x_4 x_5 + x_2 x_5^2 + x_3 x_6^2 \ \in \mathbb{K}[x_0, \ldots, x_6].$$

We have that $f \in (x_4, x_5, x_6)^2$, *then we have* $\mathbb{P}^3 = V(x_4, x_5, x_6) \subset \mathrm{Sing}(X)$, *therefore* $\mathrm{hess}_f = 0$. *On the other hand,* $\dim Z^* = 1$ *and* $Z^* \subset \mathbb{P}^2 = V(x_3, x_4, x_5, x_6) \subset \mathbb{P}^3$.

4.2 JORDAN TYPES

In this section, we give an algebraic description of the minimal and maximal families by calculating the Jordan types of the Artinian Gorenstein algebra associated with the cubic polynomial of each family. Before that, let us recall some definitions and results about the Jordan type of Artinian algebra.

Let $A = \bigoplus_{i=0}^{d} A_i$ be a standard graded Artinian $\mathbb{K}$-algebra. We get A as a module over itself. Given $L \in A_1$ consider the map $\times L : A \to A$ given by $\times L(u) = Lu$. Since A is Artinian, the map $\times L$ is nilpotent and its eigenvalues are only 0. The Jordan decomposition of such a map is given by Jordan blocks with 0 in the diagonal; therefore it induces a partition of $\dim_{\mathbb{K}}(A)$ that we denote $\mathcal{J}_{A,L}$ and we call the *Jordan type* of A with respect to L. Without loss of generality, we consider the partition in a non-increasing order.

If $L = a_0X_0 + \cdots + a_NX_N \in A_1$ is a generic linear form, then it is known that the Jordan type of any standard graded Artinian Gorenstein $\mathbb{K}$-algebra $A = Q/\operatorname{Ann}_Q(f)$ depends only on the rank of the mixed Hessians of f, see Costa and Gondim (2019). If L is not generic, to compute the Jordan type we consider the rank of the mixed Hessian $\operatorname{Hess}_f^{(i,j)}(L^{\perp})$, where $L^{\perp} = (a_0 : \ldots : a_N) \in \mathbb{P}^N$.

We are interested in the Jordan type of a standard graded Artinian Gorenstein $\mathbb{K}$-algebra with socle degree 3 and $\operatorname{char}(\mathbb{K}) = 0$. In Costa and Gondim (2019), the authors proved the following result.

Proposition 4.2.1. *Let $f \in S_3$ be a cubic form, A_f its associated Artinian Gorenstein algebra, and consider $L = a_0X_0 + \cdots + a_NX_N \in A_1$ a linear form. Consider $rk(\operatorname{Hess}_f(L^{\perp})) = r \leq N+1$. The Jordan type of A_f with Hilbert vector $(1, N+1, N+1, 1)$ with respect to L is*

$$\mathcal{J}_{A_f,L} = 4^1 \oplus 2^{r-1} \oplus 1^{2(N+1-r)}.$$

4.2.1 Jordan types for the minimal family

Let $X = V(f) \subset \mathbb{P}^N$ be a cubic hypersurface having vanishing hessian belonging to the minimal family. Then, by 4.1 we have

$$f = x_0g_0 + x_1g_1 + x_2g_2 + h$$

where $g_i \in \mathbb{K}[x_{N-1}, x_N]_2$ and $h \in \mathbb{K}[x_3, \ldots, x_N]_3$. In this case, we have $\operatorname{codim}(Z) = 1$, $\dim(Z^*) = 1$ and being h general, we have $\dim(X^*) = N - 2$.

Since X has vanishing hessian, the Artinian Gorenstein algebra associated with the polynomial f fails SLP. Let $L = a_0X_0 + \cdots a_NX_N \in A_1$ be a linear form and consider, $L^\perp = (a_0 : \ldots : a_N) \in \mathbb{P}^N$. We will analyze all possible Jordan types of A_f with respect to L. By Proposition 4.2.1, we need to study the rank of $\mathrm{Hess}_f(L^\perp)$.

The Hessian matrix Hess_f is

$$\left(\begin{array}{ccc|c} 0 & \ldots & 0 & \\ \vdots & \ddots & \vdots & P \\ 0 & \ldots & 0 & \\ \hline & P^T & & H \end{array}\right)$$

where P is the $3 \times (N-2)$ matrix given by

$$\left(\begin{array}{ccc|cc} 0 & \ldots & 0 & 2x_{N-1} & 0 \\ 0 & \ddots & 0 & x_N & x_{N-1} \\ 0 & \ldots & 0 & 0 & 2x_N \end{array}\right)$$

and H is the $(N-2) \times (N-2)$ matrix

$$\left(\begin{array}{c|cc} h_{ij} & h_j & \\ \hline h_j^T & 2x_0 + h_{N-1,N-1} & x_1 + h_{N-1,N} \\ & x_1 + h_{N,N-1} & 2x_2 + h_{N,N} \end{array}\right)$$

with $i, j \in \{3, \ldots, N-2\}$.

Let us denote by r the rank of the Hessian matrix. We will analyze some cases.

- If $L^\perp \in \mathbb{P}^N$ is generic, then $r = N$;
- If $L^\perp \in X = V(f)$, by Lemma 7.2.8 in Russo (2016), we have $\dim X^* \leq r-2 \leq N-2$. Since h is general, $\dim X^* = N-2$. Therefore, $r = N$;

- If $L^\perp \in \langle Z^* \rangle = V(x_3, \dots, x_N) = \mathbb{P}^2 \subset \operatorname{Sing} X$, then the rank of Hess_f is the rank of the matrix
$$\begin{bmatrix} 2x_0 & x_1 \\ x_1 & 2x_2 \end{bmatrix}$$
Therefore, if $L^\perp \in \langle Z^* \rangle$ is generic, then $r = 2$, and if $L^\perp \in Z^*$, $r = 1$.

Therefore
$$r = \begin{cases} N, & \text{if } L^\perp \in \mathbb{P}^N \setminus X \\ N, & \text{if } L^\perp \in X_{\text{reg}} \\ 2, & \text{if } L^\perp \in \mathcal{U} \subset \langle Z^* \rangle \\ 1, & \text{if } L^\perp \in Z^*. \end{cases}$$

So, by Proposition 4.2.1, we have the following possible Jordan types to the minimal family: $\mathcal{J}_{A_f,L} = 4^1 \oplus 2^{N-1} \oplus 1^2$ if $L^\perp \in \mathbb{P}^N \setminus X$ (respec. in X); $\mathcal{J}_{A_f,L} = 4^1 \oplus 2^1 \oplus 1^{2(N-1)}$ if $L^\perp \in \mathcal{U} \subset \langle Z^* \rangle$; and $\mathcal{J}_{A_f,L} = 4^1 \oplus 1^{2N}$ if $L^\perp \in Z^*$.

4.2.2 Jordan types for the maximal family

Let $X = V(f) \subset \mathbb{P}^N$, $N = 2k$, be a cubic hypersurface having vanishing hessian, not a cone with codim$(Z) = 1$, dim$Z^* = k-1$ and such that $M = \langle Z^* \rangle = \mathbb{P}^k = V(x_{k+1}, \dots, x_{2k}) \subset Y = (\operatorname{Sing} X)_{red}$. Then, by (4.2), we have
$$f = \sum_{i=0}^{k} x_i g^i + h$$
where $h, g^i \in \mathbb{K}[x_{k+1}, \dots, x_{2k}]$, $\deg(h) = 3$ and $\deg(g^i) = 2$.

We know that its associated Artinian Gorenstein algebra A_f does not have strong Lefschetz property. Let $L = a_0 X_0 + \cdots a_N X_N \in A_1$ be a linear form and consider, $L^\perp = (a_0 : \dots : a_N) \in \mathbb{P}^N$. We will analyze all possible Jordan types of A_f with respect to L. By Proposition 4.2.1, we need to study the rank of $\operatorname{Hess}_f(L^\perp)$.

The Hessian matrix Hess$_f$ is given by

$$\left(\begin{array}{ccc|c} 0 & \dots & 0 & \\ \vdots & \ddots & \vdots & g_j^i \\ 0 & \dots & 0 & \\ \hline & & & \\ & g_t^i & & l_{tj}(x_i) + h_{tj}(x_j) \\ & & & \end{array}\right)$$

where $i \in \{0, \dots, k\}, j, t \in \{k+1, \dots, 2k\}$ and $l_{tj}(x_i)$, $h_{tj}(x_j)$ are linear forms.

Let's denote by $r = \mathrm{rk}(\mathrm{Hess}_f)$. Since $\mathrm{hess}_f = 0$, for any point $L^\perp \in \mathbb{P}^N \setminus X$, $r = N$. If $L^\perp \in X$ is a general point, by Lemma 7.2.8 in Russo (2016), we have $r = N$. Now, if $L^\perp \in \mathcal{U} \subset M$, with $\mathcal{U}$ is a open subset, we have $r = k$. Moreover, $r \leq k-1$ if $L^\perp \in \triangle \subset M$, where $\triangle$ denotes the zero locus of a divisor of det($[l_{tj}(x_i) + h_{tj}(x_j)]_{k \times k}$). Therefore

$$r = \begin{cases} N, & \text{if } L^\perp \in \mathbb{P}^{2n} \setminus X \\ N, & \text{if } L^\perp \in X_{\mathrm{reg}} \\ k, & \text{if } L^\perp \in \mathcal{U} \subset M \\ \leq k-1, & \text{if } L^\perp \in \triangle \subset M. \end{cases}$$

So, by Proposition 4.2.1, we have the following possible Jordan types to the maximal family: $\mathcal{J}_{A_f,L} = 4^1 \oplus 2^{N-1} \oplus 1^2$ if $L^\perp \in \mathbb{P}^N \setminus X$ (respec. in X); $\mathcal{J}_{A_f,L} = 4^1 \oplus 2^{k-1} \oplus 1^{2(N+1-k)}$ if $L^\perp \in \mathcal{U} \subset M$; and $\mathcal{J}_{A_f,L} = 4^1 \oplus 2^{\leq k-2} \oplus 1^{\leq 2(N-k+2)}$ if $L^\perp \in \triangle \subset M$.

4.3 PARAMETER SPACES

Theorem 2.4.5 states that for $f \in S_3$, A_f fails to have SLP if and only if hess$_f = 0$. Let denote by $\mathcal{H}$ the locus of cubics with vanishing hessian, and by $\mathcal{C}$ the locus of cubics cones. In the previous sections, we have shown two families of cubics in $\overline{\mathcal{H} \setminus \mathcal{C}}$ that in low dimension ($n \leq 7$) exhaust the algebras failing SLP.

In this section, we construct parameter spaces for these two families. Let denote by $\mathbb{X}$ each family, the parameter space that we will construct describes $\mathbb{X}$ as the birational image of the projectivization of a vector bundle. With this description, we can compute the dimension and degree of $\mathbb{X}$ using techniques of Intersection Theory.

To compute the degree of $\mathbb{X}$ our principal tools are the Segre and Chern classes of a vector bundle. We refer the reader to Fulton (2013, Chapter 3) or Eisenbud and Harris (2016, §10.1 and Ch 5) for a systematic treatment of Segre and Chern classes.

In the following proposition, we give an enumerative interpretation for the degree of a variety $\mathbb{X}$ of $\mathrm{Gor}(1, N+1, N+1, 1)$.

Proposition 4.3.1. *Let* $\mathbb{X} \subset \mathrm{Gor}(1, N+1, N+1, 1)$ *be a subvariety of dimension* m *and degree* d*. Given* $L_1, \dots, L_m$ *generic linear forms in* Q_1*, the degree* d *is the number of Algebras in* $\mathbb{X}$ *that have* $L_1, \dots, L_m$ *as nilpotents of index* 3*.*

Proof. As $\dim \mathbb{X} = m$, the degree of $\mathbb{X}$ is the number of points in the intersection of $\mathbb{X}$ with a generic codimension m linear space of $\mathrm{Gor}(1, N+1, N+1, 1)$, and such a linear subspace is the intersection of m generic hyperplanes.

Using the correspondence of $\mathrm{Gor}(1, N+1, N+1, 1)$ with $\mathbb{P}(S_3)$ of Theorem 2.3.4, we can describe the hyperplanes in $\mathrm{Gor}(1, N+1, N+1, 1)$. Recall that an hyperplane on $\mathbb{P}(S_3)$ corresponds to a point $P = [a_0 : \cdots : a_N] \in \mathbb{P}^N$ as follows

$$H_P = \{f \in \mathbb{P}(S_3) \mid f(P) = 0\}.$$

On the other hand, the generalized Euler formula:

$$\text{if } L = a_0 X_0 + \cdots + a_N X_N \in Q_1 \text{ then } L^3(f) = 3! f(P),$$

implies that $f(P) = 0$ is equivalent to $L^3 \in Ann(f)$. We conclude that an hyperplane in $\mathrm{Gor}(1, N+1, N+1, 1)$ is of the form

$$H_L = \{A_f \mid \bar{L} \in A_f \text{ is nilpotent of index } 3\}$$

So the degree of $\mathbb{X} \subset \mathrm{Gor}(1, N+1, N+1, 1)$ has the following interpretation:

Given $L_1, \dots, L_m$ generic linear forms in Q_1, there exists $\deg(\mathbb{X})$ Algebras in $\mathbb{X}$ that have $L_1, \dots, L_m$ has nilpotents of index 3.

□

Recall that S_d denote $\mathrm{Sym}_d(S_1)$.

4.3.1 The parameter space for the Minimal family

In this section, we denote the minimal family by $\mathbb{X}$. Recall that according with 4.1, a generic element of $\mathbb{X}$ is, up to change of coordinates, equals to

$$f = x_0 g_0 + x_1 g_1 + x_2 g_2 + h, \tag{4.3}$$

with $g_i \in \mathbb{K}[x_{N-1}, x_N]$ and $h \in \mathbb{K}[x_3, \ldots, x_N]$. So, to parametrize these cubics, we must choose an ideal J such that $V(J)$ has dimension 2. Afterward, we have to construct three quadrics in the variables in J (i.e. three elements of $\mathrm{Sym}_2(J)$) and a cubic in the variables of I (i.e. an element of $\mathrm{Sym}_3(I)$).

Observe that $f \in \langle x_3, \ldots, x_N \rangle^2$, so $V(f)$ contains a plane in its singular locus.

We describe the parameter space in the following Theorem.

Theorem 4.3.2. *The minimal family in $\mathbb{P}^N$ is a rational subvariety of $\mathbb{P}(S_3)$ of dimension $5(N-2) + \binom{N}{3} + 4$. The degree of this family is given by the top Segre class $s_m(\mathcal{E})$ of a vector bundle over the flag variety $\mathbb{F} = \mathbb{F}(2, N-2, N+1)$ where, $m = dim\mathbb{F} = 5(N-2) - 4$, and can be computed using the Script in A.1.1.*

Proof. Start with the Grassmannian of 2-planes in $\mathbb{P}^N$: $\mathbb{G}(N-2, N+1)$, name $\mathcal{T}_1$ the tautological vector bundle of rank $N-2$, that is, for this Grassmannian, we consider the tautological sequence

$$0 \to \mathcal{T}_1 \to \mathcal{O}_{\mathbb{G}(N-2,N+1)} \otimes S_1 \to \mathcal{Q}_1 \to 0 \tag{4.4}$$

where $\mathcal{T}_1$ is a vector bundle of rank $N-2$, whose fiber over the plane $V(x_3, \ldots, x_N)$ is the subspace $I = [x_3, \ldots, x_N]_{\mathbb{K}}$.

Now consider $\mathbb{G}(2, \mathcal{T}_1)$, the Grassmannian of rank 2 subundles of $\mathcal{T}_1$ with structure map $\rho : \mathbb{G}(2, \mathcal{T}_1) \to \mathbb{G}(N-2, N+1)$. For this variety, we have the following tautological sequence

$$0 \to \mathcal{T}_2 \to \rho^*\mathcal{T}_1 \to \mathcal{Q}_2 \to 0$$

where $\mathcal{T}_2$ is a vector bundle of rank 2, whose fiber over $(I, J) \in \mathbb{G}(2, \mathcal{T}_1)$ is J.

Observe that $\mathbb{G}(2, \mathcal{T}_1)$ is in fact the flag variety $\mathbb{F} := \mathbb{F}(2, N-2, N+1)$. It has dimension $5(N-2) - 4$.

Consider the multiplication map:

$$\varphi : \mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1 \to S_3$$

given by $\varphi\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i b_i$. It defines a map of vector bundles over $\mathbb{F}$.

Let V_1 be the image of φ, we have an exact sequence

$$0 \to ker(\varphi) \to \mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1 \overset{\varphi}{\to} V_1 \subset S_3 \to 0 \tag{4.5}$$

where $ker(\varphi) = \wedge^2 \mathcal{T}_2 \otimes \mathcal{T}_2$ (c.f. Fassarella, Ferrer and Gondim (2021)). So we get an isomorphism

$$\frac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1}{ker(\varphi)} \overset{\overline{\varphi}}{\simeq} V_1.$$

Next, we consider the following map of vector bundles:

$$T : V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to S_3$$

defined by $T\left(\overline{\sum_i a_i \otimes b_i}, h\right) = \overline{\varphi}\left(\overline{\sum_i a_i \otimes b_i}\right) + h = \sum_i a_i b_i + h$. It is not difficult to see that $ker(T) = \overline{\varphi}\left(\dfrac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1}{ker(\varphi)}\right)$.

Defining $\mathcal{E} = im(T)$, we obtain that $\mathcal{E}$ parametrizes the cubics of the normal form (4.3), and we have an exact sequence

$$0 \to \frac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1}{ker(\varphi)} \to V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to \mathcal{E} \to 0. \tag{4.6}$$

In this way, we obtain a fiber bundle $\mathcal{E}$ over $\mathbb{F}$ of rank $9 + \binom{N}{3}$.

By considering the projectivization $\mathbb{P}(\mathcal{E})$ of the vector bundle $\mathcal{E}$, we conclude that $\mathbb{X}$ is the image by the second projection p_2:

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{E}) & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ \mathbb{F} := \mathbb{G}(2, \mathcal{T}_1) & & \mathbb{X} \subset \mathbb{P}(S_3) \end{array}$$

From this, we conclude that $\mathbb{X}$ is irreducible. We claim that p_2 is generically injective. Indeed, for a generic $f \in \mathbb{X}$, the singular set of f contains a unique plane, from which we recover I. Consider now the differential of f, $df \in H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(3)) \subset S_2 \otimes S_1$. Projecting df from $S_2 \otimes I$ we get $f_{x_0} dx_0 + f_{x_1} dx_1 + f_{x_2} dx_2$, and by construction, $f_{x_0}, f_{x_1}, f_{x_2} \in Sym_2([u, v])$ for some $u, v \in I$, then we recover J.

To compute the degree of $\mathbb{X} \subset \mathbb{P}(S_3)$ first we prove that, in the present setting, $\deg \mathbb{X} = \int s_m(\mathcal{E}) \cap [\mathbb{F}]$, the m-Segre class of $\mathcal{E}$, with $m = \dim\mathbb{F}$. A similar equality will be used in the following sections, so we prove it in more generality. Indeed, by definition of push forward of cycles, we have $p_{2*}[\mathbb{P}(\mathcal{E})] = \deg(p_2)[\mathbb{X}]$. As in our case $\deg(p_2) = 1$, putting $\nu = \dim \mathbb{P}(\mathcal{E}) = \dim \mathbb{X}$ and $H = c_1(\mathcal{O}_{\mathbb{P}(S_3)}(1))$, we obtain

$$\deg \mathbb{X} = \int H^\nu \cap [\mathbb{X}] = \int H^\nu \cap p_{2*}[\mathbb{P}(\mathcal{E})] = \int p_2^* H^\nu \cap [\mathbb{P}(\mathcal{E})]$$

where the last equality was obtained from the projection formula. Now,

$$\int p_2^* H^\nu \cap [\mathbb{P}(\mathcal{E})] = \int \widetilde{H}^\nu \cap [\mathbb{P}(\mathcal{E})]$$

where $\widetilde{H} = c_1(\mathcal{O}_\mathcal{E}(1))$.

Set $e = \mathrm{rk}(\mathcal{E})$. Thus $\dim \mathbb{P}(\mathcal{E}) = e - 1 + m$. Hence projection onto the basis $\mathbb{F}$ gives

$$\int \widetilde{H}^\nu \cap [\mathbb{P}(\mathcal{E})] = \int p_{1*}(\widetilde{H}^\nu \cap p_1^*[\mathbb{F}]) = \int s_m(\mathcal{E}) \cap [\mathbb{F}].$$

Observe that for the minimal family we have $\dim \mathbb{X} = \dim \mathbb{P}(\mathcal{E}) = e - 1 + m = 4 + \binom{N}{3} + 5(N-2)$.

To compute $s_m(\mathcal{E})$, using sequence (4.6) and Whitney formula we have:

$$s(\mathcal{E}) = c\left(\frac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1}{ker(\varphi)}\right) s(V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1)) \tag{4.7}$$

Other applications of the Whitney formula give us:

$$c\left(\frac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1}{ker(\varphi)}\right) = c(\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1) s(ker(\varphi))$$

and

$$s(V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1)) = s(V_1) s(\mathrm{Sym}_3(\mathcal{T}_1))$$

On the other hand, by sequence (4.5) we have

$$s(V_1) = c(ker(\varphi)) s(\mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1)$$

Substituting the above equalities in (4.7) and using the fact that Segre and Chern classes are inverses to each other we obtain:

$$s(\mathcal{E}) = c(\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1) s(\mathrm{Sym}_3(\mathcal{T}_1)) s(\mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1) \tag{4.8}$$

To simplify (4.8) we twist equation (4.4) by $\mathrm{Sym}_2(\mathcal{T}_2)$ and use Whitney formula to obtain $s(\mathrm{Sym}_2(\mathcal{T}_2) \otimes S_1) c(\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1) = s(\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{Q}_1)$. Finally, we obtain

$$s(\mathcal{E}) = s(\mathrm{Sym}_3(\mathcal{T}_1)) s(\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{Q}_1) = s(\mathrm{Sym}_3(\mathcal{T}_1) \oplus \mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{Q}_1).$$

□

Next, we parametrize and compute the dimension and degree of the intersection of the minimal family with cones: $\mathbb{X} \cap \mathcal{C}$.

There are three types of cones in $\mathbb{X}$, tha will be denoting by $\mathcal{C}_i;\ i = 1, 2, 3$. Denoting the vertex by $p = V(I_0)$ we have that $N - 3 \leq \dim(I \cap I_0) \leq N - 2$ and $1 \leq \dim(J \cap I_0) \leq 2$.

1. If $\dim(I \cap I_0) = N - 3$ and $\dim(J \cap I_0) = 1$, we can suppose that $I_0 = \langle x_3, \ldots, x_{N-1} \rangle$. In this case $f \in \mathbb{X} \cap \mathcal{C}_1$ is up to change of coordinates, $x_0 x_{N-1}^2 + h(x_3, \ldots, x_{N-1})$.

2. If $\dim(I \cap I_0) = N - 3$ and $J \subset I_0$, we can suppose that $I_0 = \langle x_4, \ldots, x_{N-1}, x_N \rangle$, and $f \in \mathbb{X} \cap \mathcal{C}_2$ is up to change of coordinates, $x_0 x_{N-1}^2 + x_1 x_{N-1} x_N + x_2 x_N^2 + h(x_4, \ldots, x_{N-1}, x_N)$. It is easy to see that $\mathbb{X} \cap \mathcal{C}_1 \subset \mathbb{X} \cap \mathcal{C}_2$.

3. If $J \subset I \subset I_0$, then the vertex $p = V(I_0) \subset V(I) = \mathbb{P}^2$. In this case we can suppose that $I_0 = \langle x_1, x_2, \ldots, x_{N-1}, x_N \rangle$. Thus $f \in \mathbb{X} \cap \mathcal{C}_3$ is up to change of coordinates, to $x_1 x_{N-1} x_N + x_2 x_N^2 + h(x_3, \ldots, x_N)$.

   We have $\mathbb{X} \cap \mathcal{C}_1 \subset \mathbb{X} \cap \mathcal{C}_2 \subset \mathbb{X} \cap \mathcal{C}_3$.

To parametrize $\mathbb{X} \cap \mathcal{C}_3$ consider the grassmannian $\mathbb{G}(N, N+1)$ with taulotogical bundle $\mathcal{T}_0$, then consider the grassmannian $\mathbb{G}(N-2, \mathcal{T}_0)$ with taulotogical bundle $\mathcal{T}_1$ and the grassmannian $\mathbb{G}(2, \mathcal{T}_1)$ with taulotogical bundle $\mathcal{T}_2$. Construct the tower of fibrations

$$\mathbb{G}(2, \mathcal{T}_1) \to \mathbb{G}(N-2, \mathcal{T}_0) \to \mathbb{G}(N, N+1).$$

We have that $\mathbb{G}(2, \mathcal{T}_1)$ is the flag variey $\mathbb{F} := \mathbb{F}(2, N-2, N, N+1)$ and $\dim \mathbb{F} = N + 2(N-2) + 2(N-4) = 5(N-2) - 2$.

The construction is completely analogous to what we did above. Consider

$$\wedge^2 \mathcal{T}_2 \otimes \mathcal{T}_2 \to \mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_0 \to V_1$$

and

$$\frac{\mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{T}_1}{\wedge^2 \mathcal{T}_2 \otimes \mathcal{T}_2} \to V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to \mathcal{F}$$

We obtain a fiber bundle $\mathcal{F}$ such that $\mathrm{rk}(\mathcal{F}) = 3N - 2 + \binom{N}{3} - (3(N-2) - 2) = \binom{N}{3} + 6$, and $\mathbb{X} \cap \mathcal{C}_3$ is the projection on the second factor of $\mathbb{P}(\mathcal{F})$. This projection is generically injective, therefore

$$\dim(\mathbb{X} \cap \mathcal{C}_3) = 5(N-2) - 2 + \binom{N}{3} + 5 = 5(N-2) + \binom{N}{3} + 3.$$

Observe that $\mathbb{X} \cap \mathcal{C}_3$ is a divisor in $\mathbb{X}$. From the construction of $\mathcal{F}$ we can obtain the degree of this divisor. We get the following result.

Proposition 4.3.3. *The variety $\mathbb{X} \cap \mathcal{C}_3$ is a divisor in $\mathbb{X}$ of degree given by the top Segre class $s_m(Sym_3(\mathcal{T}_1) \oplus Sym_2(\mathcal{T}_2) \otimes \mathcal{Q}_1)$, where $m = 5(N-2) - 2$, and can be computed using the Script in A.1.2.*

4.3.2 The parameter space of the Maximal family

Next we consider $N = 2k$ and denote by $\mathbb{X}$ the family such that a general element is given by

$$f := \sum_{i=0}^{k} x_i g_i(x_{k+1}, \dots, x_N) + h(x_{k+1}, \dots, x_N) \tag{4.9}$$

where $g_0, g_1, \dots g_k$ are quadratic forms and h is a cubic form in $x_{k+1}, \dots, x_N$.

Observe that the derivatives $[f_{x_0}, \dots, f_{x_k}] = [g_0, \dots, g_k]$ defines a map $\mathbb{P}^{k-1} \dashrightarrow \mathbb{P}^k$, so they are algebrically dependent and f has vanishing hessian.

Theorem 4.3.4. *For even $N = 2k$, the maximal family of cubics in $\mathbb{P}^N$ is a rational projective irreducible variety of dimension*

$$\dim \mathbb{X} = (k+1)(\binom{k+1}{2} + k) + \binom{k+2}{3} - 1$$

and the degree is given by $s_m(\mathcal{E}_k)$, where $\mathcal{E}_k$ is a vector bundle over a variety $\mathbb{G}_k$ of dimension $m = (k+1)(\binom{k+1}{2} - 1)$. The degree of this family can be computed using the Script in A.1.3.

Proof. We begin by consider $\mathbb{G}(k, S_1)$, the Grassmannian of k-planes in $\mathbb{P}^N$ with tautological sequence

$$0 \to \mathcal{T}_1 \to \mathbb{G}(k, S_1) \times S_1 \to \mathcal{Q}_1 \to 0.$$

For the choice of $k+1$ forms $g_0, \dots, g_k$ of degree two in $x_{k+1}, \dots, x_N$, consider the grassmannian $\mathbb{G}(k+1, \mathrm{Sym}_2(\mathcal{T}_1))$ with structure map

$$\rho : \mathbb{G}_k := \mathbb{G}(k+1, \mathrm{Sym}_2(\mathcal{T}_1)) \to \mathbb{G}(k, S_1).$$

From this it is easy to see that $\dim \mathbb{G}_k = (k+1)(\binom{k+1}{2} - 1)$

Over $\mathbb{G}_k$ we have the tautological sequence

$$0 \to \mathcal{T}_2 \to \rho^*(\mathrm{Sym}_2(\mathcal{T}_1)) \to \mathcal{Q}_2 \to 0$$

Next, we construct a map T_k whose image parametrizes these cubics with normal form (4.9).

Using the natural injective maps

$$\mathcal{T}_2 \to \rho^*(\mathrm{Sym}_2(\mathcal{T}_1)) \to \mathbb{G}_k \times S_2,$$

we construct the following exact sequence of vector bundles over $\mathbb{G}_k$:

$$0 \to ker(\varphi) \to \mathcal{T}_2 \otimes S_1 \xrightarrow{\varphi} V_1 \subset S_3 \to 0 \tag{4.10}$$

where $\varphi\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i b_i$. Therefore we have $\dfrac{\mathcal{T}_2 \otimes S_1}{ker(\varphi)} \overset{\overline{\varphi}}{\simeq} V_1 \subset S_3$.

Now define

$$T_k : V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to S_3$$

by $T_k\left(\overline{\sum_i a_i \otimes b_i}, h\right) = \overline{\varphi}\left(\overline{\sum_i a_i \otimes b_i}\right) + h = \sum_i a_i b_i + h$.

It is easy to see that $ker(\varphi) \subset \mathcal{T}_2 \otimes \mathcal{T}_1$, so we obtain that $\overline{\varphi}\left(\dfrac{\mathcal{T}_2 \otimes \mathcal{T}_1}{ker(\varphi)}\right)$ is a subvector bundle of both V_1 and $\mathrm{Sym}_3(\mathcal{T}_1)$. This vector bundle coincides with $ker(T_k)$.

In this way, we obtain the following exact sequence

$$0 \to \overline{\varphi}\left(\frac{\mathcal{T}_2 \otimes \mathcal{T}_1}{ker(\varphi)}\right) \to V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to \mathcal{E}_k \to 0 \tag{4.11}$$

where $\mathcal{E}_k = \mathrm{Im}(T_k)$ is the required vector bundle.

Following the above construction is not difficult to see that $\mathrm{rk}\,\mathcal{E}_k = \binom{k+2}{3} + (k+1)^2$.

We have the following projections

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{E}_k) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbb{G} & & \mathbb{X} \subset \mathbb{P}(S_3) \end{array}$$

We claim that p_2 is generically injective. So $\mathbb{X} = p_2(\mathbb{P}(\mathcal{E}_k))$, has dimension $\binom{k+2}{3} + (k+1)[k + \binom{k+1}{2}] - 1$.

The proof of the claim follows the same arguments as the proof of the Theorem 4.3.2: a generic cubic $f \in \mathbb{X}$ has a unique 3-plane on its singular set, so we recover I. Hence, projecting $df \in S_2 \otimes S_1$ from $S_2 \otimes I$ we recover $[g_0, g_1, g_2, g_3]_{\mathbb{K}}$.

To compute the degree we proceed exactly as in the proof of Theorem 4.3.2. By sequence (4.11) we have

$$s(\mathcal{E}_k) = s(V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1))c(\frac{\mathcal{T}_2 \otimes \mathcal{T}_1}{ker(\varphi)}) =$$

$$s(V_1)s(\mathrm{Sym}_3(\mathcal{T}_1))c(\mathcal{T}_2\otimes\mathcal{T}_1)s(ker(\varphi))$$

On the other hand, by sequence (4.10) we have

$$s(V_1)=c(ker(\varphi))s(\mathcal{T}_2\otimes S_1).$$

Therefore

$$s(\mathcal{E}_k)=s(\mathcal{T}_2\otimes S_1)s(\mathrm{Sym}_3(\mathcal{T}_1))c(\mathcal{T}_2\otimes\mathcal{T}_1)=s(\mathcal{T}_2\otimes\mathcal{Q}_1)s(\mathrm{Sym}_3(\mathcal{T}_1)).$$

□

In what follows, we construct a parameter space for the intersection of $\mathbb{X}$ with cubic cones in $\mathbb{P}^N$.

Let denote the vertex of the cone by $p=V(I_0)$, then $\dim I\cap I_0\geq k-1$. There are two types of cones:

1. The case where $\dim I\cap I_0=k-1$, i.e. $p\notin V(x_{k+1},\dots,x_N)=\mathbb{P}^k$. We can suppose that $I_0=\langle x_0,\dots,x_{N-1}\rangle$, thus f is up to change of coordinates, $\sum_{i=0}^k x_ig_i(x_{k+1},\dots,x_{N-1})+h(x_{k+1},\dots,x_{N-1})$ where g_i,h do not depend on x_N. We write $\mathbb{X}\cap\mathcal{C}_1$ the intersection with these cones.

2. The case where $\dim I\cap I_0=k$, i.e $I\subset I_0$ and $p=V(I_0)\in\mathbb{P}^k$. In this case we can assume that $I_0=\langle x_1,\dots,x_N\rangle$ and f is up to change of coordinates, $\sum_{i=1}^k x_ig_i(x_{k+1},\dots,x_N)+h(x_{k+1},\dots,x_N)$. We write $\mathbb{X}\cap\mathcal{C}_2$ the intersection with these cones. Observe that $\mathbb{X}\cap\mathcal{C}_1\subset\mathbb{X}\cap\mathcal{C}_2$.

Next, we parametrize $\mathbb{X}\cap\mathcal{C}_2$. Consider the grassmannian $\mathbb{G}(N,N+1)$ with taulotogical bundle $\mathcal{T}_0$, then consider the grassmannian $\mathbb{G}(k,\mathcal{T}_0)$ with tautological bundle $\mathcal{T}_1$ and the grassmannian $\mathbb{G}(k,\mathrm{Sym}_2(\mathcal{T}_1))$ with tautological bundle $\mathcal{T}_2$. We obtain the following tower of fibrations

$$\mathbb{G}(k,\mathrm{Sym}_2(\mathcal{T}_1))\to\mathbb{G}(k,\mathcal{T}_0)\to\mathbb{G}(N,N+1).$$

We have that $\mathbb{F}:=\mathbb{G}(k,\mathrm{Sym}_2(\mathcal{T}_1))$ has dimension $m:=\dim\mathbb{F}=N+k(N-k)+k(\binom{k+1}{2}-k)=k(2+\binom{k+1}{2})$.

With basis $\mathbb{F}$, consider the following exact sequence, where the third map is the multiplication map:

$$0\to ker(\varphi)\to\mathcal{T}_2\otimes\mathcal{T}_0\to V_1\to 0$$

and

$$0 \to \frac{\mathcal{T}_2 \otimes \mathcal{T}_1}{ker(\varphi)} \to V_1 \oplus \mathrm{Sym}_3(\mathcal{T}_1) \to \mathcal{F} \to 0$$

From the above sequences we have $\mathrm{rk}(\mathcal{F}) = kN + \binom{k+2}{3} - k^2 = k^2 + \binom{k+2}{3}$.

We obtain that $\mathbb{X} \cap \mathcal{C}_2$ is the projection on the second factor of $\mathbb{P}(\mathcal{F})$, and the projection is generically injective. Therefore

$$\dim(\mathbb{X} \cap \mathcal{C}_2) = k(2 + \binom{k+1}{2}) + k^2 + \binom{k+2}{3} - 1.$$

From the construction of $\mathcal{F}$ we obtain that the degree of $\mathbb{X} \cap \mathcal{C}_2$ is the Segre class $s_m(\mathrm{Sym}_3(\mathcal{T}_1) \oplus \mathcal{T}_2 \otimes \mathcal{Q}_1)$.

Therefore, we have the proposition.

Proposition 4.3.5. *The variety $\mathbb{X} \cap \mathcal{C}_2 \subset \mathbb{X}$ has codimension $\binom{k}{2}$ and its degree is given by the top Segre class $s_m(Sym_3(\mathcal{T}_1) \oplus \mathcal{T}_2 \otimes \mathcal{Q}_1)$, where $m = k(2 + \binom{k+1}{2})$. The degree can be computed using the Script in A.1.4.*

4.4 THE LEFSCHETZ LOCUS IN $\mathrm{Gor}(1, N+1, N+1, 1)$ FOR $N \leq 6$

Applying the previous section, we compute the dimension and degree to the minimal and maximal family in $\mathbb{P}^5$ and $\mathbb{P}^6$. Gondim and Russo (2015) showed that in $\mathbb{P}^5$ there exists only the minimal family while in $\mathbb{P}^6$ there are the minimal and maximal family. Furthermore, we discuss the strong Lefschetz property for Artinian Gorenstein algebras of socle degree 3 and codimension 6 and 7.

4.4.1 The Lefschetz locus in $\mathrm{Gor}(1, 6, 6, 1)$

Given an Artinian Gorenstein algebra $A \in \mathrm{Gor}(1, 6, 6, 1)$, by Macaulay-Matlis duality, there exists a homogeneous polynomial $f \in S_3$ such that $A = A_f$, and $\mathrm{Hilb}(A) = (1, 6, 6, 1)$. By Theorem 2.4.5, we know that A_f fails to have SLP if and only if $\mathrm{hess}_f = 0$. Let us denote by $\mathcal{H}$ the locus of cubics with vanishing hessian, and by $\mathcal{C}$ the locus of cubics cones.

As the authors show in Gondim and Russo (2015), a cubic $f \in \mathcal{H} \setminus \mathcal{C}$ is protectively equivalent to

$$f = x_0 x_4^2 + x_1 x_4 x_5 + x_2 x_5^2 + h(x_3, x_4, x_5) \tag{4.12}$$

where $h(x_3, x_4, x_5)$ is a cubic. So, by Macaulay-Matlis duality, $A_f \in \mathrm{Gor}(1,6,6,1)$ fails to have SLP if and only if f can be written as in (4.12).

Proposition 4.4.1. *The locus in* $\mathrm{Gor}(1,6,6,1)$ *of algebras satisfying SLP is* $\mathrm{Gor}(1,6,6,1) \setminus (\mathcal{H} \setminus \mathcal{C})$ *where* $\mathcal{H} \setminus \mathcal{C}$ *is the locus of cubics, not a cone, with vanishing hessian in* $\mathbb{P}^5$.

Using the results in Theorem 4.3.2 we obtain

Theorem 4.4.2. *The locus* $\mathbb{X} := \overline{\mathcal{H} \setminus \mathcal{C}} \subset \mathbb{P}^{55}$ *is a rational irreducible projective variety of dimension* 29 *and degree* 51847992.

Proof. Applying the construction done in Theorem 4.3.2 we obtain a fiber bundle $\mathcal{E}$ of rank $19 = \binom{5}{3} + 9$ over the flag variety $\mathbb{F} = \mathbb{F}(2,3,6)$. From this data we obtain $\dim(\mathbb{P}(\mathcal{E})) = 29$. The degree is given by $s_{11}(\mathcal{E})$, and we compute it using the Scripts in §A.1 for $N = 5$: $s_{11}(\mathcal{E}) = 51847992$. ☐

Proposition 4.4.3. *In* $\mathbb{P}^5$ *the intersection of the minimal family with cones* $\mathbb{X} \cap \mathcal{C}_3$ *is a divisor in* $\mathbb{X}$ *of degree* 98048160.

4.4.2 The Lefschetz locus in $\mathrm{Gor}(1,7,7,1)$

Gondim and Russo (2015) showed that there exist exactly two families of cubics hypersurfaces in $\mathbb{P}^6$ not cones with vanishing hessian: we name these families *the minimal family* and *the maximal family*, according to the dimension of the linear space that a generic member of each family has in its singular set.

The cubics in the minimal family are given by $X = V(f) \subset \mathbb{P}^6$, where f is up to change of coordinates,

$$f_{min} := x_0 x_5^2 + x_1 x_5 x_6 + x_2 x_6^2 + h(x_3, x_4, x_5, x_6) \tag{4.13}$$

with $h(x_3, x_4, x_5, x_6)$ a cubic form in the variables x_3, x_4, x_5, x_6.

On the other hand, if $X = V(f) \subset \mathbb{P}^6$ is a cubic hypersurface in the maximal family, f is up to change of coordinates,

$$f_{max} := \sum_{i=0}^{3} x_i g_i(x_4, x_5, x_6) + h(x_4, x_5, x_6) \tag{4.14}$$

where g_0, g_1, g_2, g_3 are quadratic forms and h is a cubic form in x_4, x_5, x_6.

In our context, the results in Gondim and Russo (2015) can be stated as follows:

Proposition 4.4.4. *The Lefschetz locus in* $\mathrm{Gor}(1,7,7,1)$ *is* $\mathrm{Gor}(1,7,7,1)\backslash(\mathcal{H}\backslash\mathcal{C})$ *where* $\mathcal{H}\backslash\mathcal{C}$ *is the locus of cubics not cone with vanishing hessian in* $\mathbb{P}^6$*. Furthermore* $\overline{\mathcal{H}\setminus\mathcal{C}} = \mathbb{X}_{min}\cup\mathbb{X}_{max}$*. Where a generic cubic in* $\mathbb{X}_{min}$ *(respectively in* $\mathbb{X}_{max}$*) has normal form as in (4.13) (respectively as in (4.14)).*

Next, we describe parameter spaces for each family of cubics hypersurfaces in $\mathbb{P}^6$.

4.4.2.1 Minimal family in $\mathbb{P}^6$

Using the results in 4.3.2 we obtain

Theorem 4.4.5. *The locus* $\mathbb{X}_{min}$ *is a rational projective irreducible variety of* $\mathbb{P}^{83}$ *of dimension* 44 *and degree* 229416381544*.*

Proof. From Theorem 4.3.2, we obtain the dimension of the family. The degree is computed by the Segree class $s_{16}(\mathcal{E}) = s_{16}(\mathrm{Sym}_3(\mathcal{T}_1) \oplus \mathrm{Sym}_2(\mathcal{T}_2) \otimes \mathcal{Q}_1)$. We compute these Segre classes using the Script in A.1.1) with $N = 6$. ☐

Proposition 4.4.6. *In* $\mathbb{P}^6$*, the intersection of the minimal family with cones* $\mathbb{X}_{min} \cap \mathcal{C}_3$ *is a divisor in* $\mathbb{X}_{min}$ *of degree* 378294450492*.*

4.4.2.2 Maximal family in $\mathbb{P}^6$

Theorem 4.4.7. *The locus* $\mathbb{X}_{max} \subset \mathbb{P}^{83}$ *is a rational projective irreducible variety of dimension* 45 *and degree* 5792937080*.*

Proof. By Theorem 4.3.4 the degree is computed using the Script in A.1.3) with $N = 6$. ☐

From Proposition 4.3.5 we obtain:

Proposition 4.4.8. *In* $\mathbb{P}^6$*, the intersection of the maximal family with cones* $\mathbb{X}_{max} \cap \mathcal{C}_2$ *has codimension* 3 *in* $\mathbb{X}_{max}$ *and degree* 51258091892*.*

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ANNEX A – SCRIPTS

A.1 SCRIPTS.

A.1.1 In $\mathbb{P}^N$, minimal family

```
loadPackage "Schubert2"
--choose your N
N= ;
-- Grassmannian of planes in N-space.
G1=flagBundle ({3,N-2});
-- names the sub and quotient bundles on G1
(Q1,Tau1)=G1.Bundles;

-- define F=grass(2,Tau1), the quotient has rank 2
F=flagBundle ({N-4,2},Tau1);
-- names the sub and quotient bundles on F
(Q2,Tau2) = F.Bundles ;

--Define E1 and E2, such that E=E1+E2
E1=(symmetricPower(2,Tau2))*Q1;
E2=symmetricPower(3,Tau1);

--compute the dimF-Segre class of the proof of the Theorem.
integral (segre (5*(N-2)-4,E1+E2))
```

A.1.2 In $\mathbb{P}^N$, minimal family intersection with cones

```
loadPackage "Schubert2"
```

```
--give a value for N
N=;

--define grass(N,N+1),
--the quociente Tau0=Q0 has rk N.
G0=flagBundle ({1,N});
(S0,Q0) = G0.Bundles;

-- define  grass(N-2,Tau0),
--the  quocient Tau1=Q1 has rk  N-2.
F1=flagBundle ({2,N-2},Q0);
(S1,Q1) = F1.Bundles;

-- define grass(2,Tau1),
--the quotient  Tau2=Q2 has rk  2.
F2=flagBundle ({N-4,2},Q1);
(S2,Q2) = F2.Bundles;



E1=(symmetricPower(2,Q2))*S1;
E2=symmetricPower(3,Q1);
rank (E1+E2);
integral(segre (dim F2,E1+E2))
```

A.1.3 In $\mathbb{P}^N$, $N = 2k$, maximal family

```
loadPackage "Schubert2"
--choose your k=N/2
k=;
```

```
-- Grassmannian of k planes in N-space.
G1=flagBundle ({k+1,k})

-- names the sub and quotient bundles on G1
(Q1,tau1) = G1.Bundles



k1=substitute((k+1)*(k-2)/2,ZZ);
k2=substitute((k+1),ZZ);
-- define F=grass(k+1,s_2(Tau1))
F=flagBundle ({k1,k2},symmetricPower(2,tau1))

-- names the sub and quotient bundles on F
(Q2,tau2) = F.Bundles

E1=tau2*Q1
E2=symmetricPower(3,tau1)

--compute the dimF-Segre class of the proof of the
integral(segre (dim(F),E1+E2))
```

A.1.4 In $\mathbb{P}^N$, maximal family intersection with cones

```
--intersection of maximal family with cones in PN, N=2k

--choose your k=N/2
k=;
N=substitute(2*k,ZZ);
-- Grassmannian of N planes in N+1-space.
G0=flagBundle ({1,N}); --define  grass(N,N+1) with Q0=Tau0
(Q0,tau0) = G0.Bundles;
```

```
-- Grassmannian of k planes in Q0-space.
G1=flagBundle ({k,k},tau0); --define  grass(k,Q0) with Q1=Tau1
(Q1,tau1) = G1.Bundles;

k1=substitute(k*(k-1)/2,ZZ);
F=flagBundle ({k1,k},symmetricPower(2,tau1));
--define  grass of quotient of rk k of S2(Tau1),Q2=Tau2
(Q2,tau2) = F.Bundles;

E1=tau2*Q1;
E2=symmetricPower(3,tau1);

integral(segre (dim(F),E1+E2))
```