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HAGEDORN INFLATION IN STRING GAS COSMOLOGY

por

Tiago José Nunes da Silva

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Banca Examinadora:
Prof. Bruno Geraldo Carneiro da Cunha (Orientador-UFPE)
Prof. Flávio Menezes de Aguiar (DF-UFPE)
Prof. Claudio Benedito Silva Furtado (DF-UFPB)

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Universidade Federal de Pernambuco
Departamento de Física – CCEN
Programa de Pós-Graduação em Física
Cidade Universitária - 50670-901 Recife PE Brasil
Fone (++ 55 81) 2126-8449/2126-8450 - Fax (++ 55 81) 3271-0359
http://www.df.ufpe.br/pg e-mail: posgrad@df.ufpe.br

Parecer da Banca Examinadora de Defesa de Dissertação de Mestrado

Tiago José Nunes da Silva

INFLAÇÃO DE HAGEDORN EM COSMOLOGIA DE GÁS DE CORDAS

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To all of those who will risk going too far just to find out how far one can go.

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There is a theory which states that if ever anybody discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

—DOUGLAS ADAMS

Resumo

Durante o século XX, avanços teóricos e experimentais jogaram uma nova luz sobre o estudo da história e evolução do universo, a Cosmologia. A partir dos trabalhos de Edwin Hubble, a cosmologia moderna pôde ser vista como ciência. Mas foi nas últimas décadas, sobretudo, com o desenvolvimento da cosmologia de precisão e devido a uma melhor compreensão da natureza em seu nível mais fundamental, que a Cosmologia despertou maior interesse científico. Uma das fronteiras da área diz respeito ao universo muito jovem: modelos cosmológicos são baseados em uma teoria de gravitação; no entanto, o paradigma atual de teoria de gravitação, a Relatividade Geral de Einstein, quebra para regimes de energia acima da escala de Planck. Assim, para descrever o universo primitivo, quando a densidade de energia era muito alta, precisamos de uma nova teoria de gravitação. Uma teoria de gravitação capaz de lidar com os efeitos quânticos.

Hoje nós temos uma candidata à tal teoria quântica de gravitação: Teoria de Cordas. Nesta dissertação, analisaremos um cenário cosmológico construído sobre Teoria de Cordas, o cenário cosmológico do gás de cordas, proposto originalmente por Robert Brandenberger e Cumrum Vafa. O cenário faz uso de simetrias e dualidades próprias de Teoria de Cordas e do fato de que um gás de cordas possui uma temperatura limitante para descrever o universo primordial e propôr respostas à questões abertas de cosmologia, como a formação da estrutura causal e a dimensionalidade do espaço-tempo. Uma das questões em aberto no cenário é a ocorrência ou não de inflação, uma era de crescimento exponencial do universo, que produz a estrutura causal observada experimentalmente e dilui relíquias produzidas no universo primordial para os níveis observados. Propondo uma interação entre as cordas do gás proporcional a seu acoplamento, estudamos a evolução resultante do universo e sob quais condições podemos ter um período inflacionário.

Palavras-chave: Cosmologia, Teoria de Cordas, Cosmologia de Cordas, Gás de Cordas, Inflação.

Abstract

During the 20th century both theoretical and experimental advances shone a light over the study of the history and evolution of the universe, the so-called Cosmology. Since the works of Edwin Hubble, modern Cosmology gained the status of science. Especially in the last decades, marked by the birth of precision cosmology and by the development of tools that led to better comprehension of nature in its most fundamental level, have seen a growing interest in Cosmology. One of the research frontiers in this area is the study of the very early universe: cosmological models are constructed over a theory of gravitation; however, our current paradigm of such theory, Einstein's theory of General Relativity, breaks down at energy scales beyond Planck Scale. Because of that, a proper description of the early universe, which had an extremely high energy density, requires a new theory of gravitation, one that is able to account for the quantum effects. Our best candidate for such quantum theory of gravitation is String Theory. In this work we are going to study a cosmological model built over String Theory, the string gas cosmological model that was originally purposed by Robert Brandenberger and Cumrum Vafa. The scenario makes use of stringy symmetries and dualities and from the fact that a string gas has a maximum limiting temperature, the Hagedorn temperature, to describe the very early universe and purpose an answer to some open questions in Cosmology, like the formation of the universe causal structure, and the dimensionality of space-time. One of the open questions in this scenario is related to existence or not of inflation, an era when the universe scales grew exponentially. This era would generate the causal sptructre we observe and dilute the density of primordial relics to the measured levels. We purpose the interaction between strings in the gas to be proportional to their coupling and study the resulting evolution of the universe, focusing on the wheter inflation takes place or not.

Keywords: Cosmology, String Theory, String Cosmology, String Gas, Inflation.

Contents

1 Overview **1**
1.1 A Brief History of All Things 2
1.1.1 Early Days: The birth of Modern Cosmology 2
1.1.2 The Cosmic Microwave Background and the Inflationary Paradigm 3
1.1.3 Late Time Evolution and the Dark Sector 4
1.2 String Theory 9
1.2.1 Why String Theory 9
1.3 A New Paradigm: String Cosmology 11
1.3.1 Cosmology with string gases 11

2 Standard Big Bang Cosmology and the Inflationary Scenario **14**
2.1 FRW Cosmology 14
2.1.1 Einstein Equations 14
2.1.2 The FRW Metric 15
2.1.3 FRW Dynamics 17
2.2 Phenomenological Aspects of Standard Big Bang Cosmology 20
2.2.1 Hubble Law 20
2.2.2 The Cosmic Microwave Background 22
2.3 Inflation 24
2.3.1 The Flatness Problem 25
2.3.2 The Horizon Problem 26
2.3.3 The Mechanism of Inflation 26
2.3.4 Fluctuations and Inflation 30

3 String Cosmology **32**
3.1 Strings in time-dependent backgrounds 32
3.2 Thermodynamics of a String Gas 34
3.2.1 The Random Walk Picture 35
3.2.2 Highly Excited Strings: Explicit Calculation 38
3.3 Toroidal compactification and T-duality 40
3.3.1 Kaluza-Klein Reduction and Winding Modes 40
3.3.2 The Compactified Spectrum and T-Duality 42
3.4 The String Gas Scenario 44
3.4.1 Winding Modes and the Dimensionality of Spacetime 46
3.4.2 Causal Structure and Inflation 48

4 Hagedorn Inflation of String Gases **51**
4.1 The Relic Problem 51
4.2 String Gas Thermodynamics Revisited 52
4.3 A New Mechanism for Inflation 54
4.3.1 The Setup 54
4.3.2 The Equations of Motion 55
4.4 The Dynamics 56
4.4.1 A possible realization: Decay of winding modes 57
4.5 Parameter Tuning and Initial Conditions 60
4.5.1 Initial Conditions for the Modes 60

5 Conclusion and Perspectives **63**

A String Quantization **65**
65
A.1 The Relativistic Point Particle 65
A.1.1 Lagrangian and Symmetries 65
A.1.2 Ein Einbein 67
A.2 Action Principles 67
A.2.1 The Nambu-Goto Action 67
A.2.2 The Polyakov Action 69
A.2.3 Mode Expansion 71
A.3 String Quantization 73
A.3.1 The Lightcone Gauge 73
A.3.2 Quantization 76
A.3.3 Analysis of Spectrum 77

List of Figures

1.1 APM Survey picture of a large part of the sky, about 30 degrees across, showing almost a million galaxies out to a distance of about 2 billion light years. Credits: Steve Maddox, Will Sutherland, George Efstathiou and Jon Loveday. 3
1.2 Luminosity versus redshift for supernovae Credits: Supernova Cosmology Project 5
1.3 Dark Matter ring observed by Hubble Space Telescope Credits: NASA, ESA, M.J. Jee and H. Ford (Johns Hopkins University). 6
1.4 Evolution of the Λ-CDM Cosmological Model. Credits: NASA/WMAP Science Team 9

2.1 Possibilities for the geometry of FRW universes depending on the value of K 17
2.2 Possibilities for the evolution of FRW universes depending on the value of K. Credits: European Space Agency 20
2.3 Experimental data supporting Hubble Law. The first one contains the original data set obtained by Edwin Hubble and his linear fit. The second one contains recent data with higher precision. Credits for the second picture: Wendy Freedman, Observatories of the Carnegie Institution of Washington and NASA. 21
2.4 Full sky map of CMB from 7 year WMAP data. Credits: NASA/WMAP Science Team 23
2.5 Power spectrum of CMB anisotropies. Credits: NASA/WMAP Science Team 24
2.6 Cosmology of an inflationary universe. Inflation goes from t_i to t_R 28
2.7 Inflationary Potential Phase Transition 30

3.1 Possibilities for winding states. Figure Credits: Barton Zwiebach. *A first course in string theory*; 2nd ed. Cambridge Univ. Press, Cambridge, 2009. 41
3.2 Closed oriented strings of winding numbers w=-1,0,-1 and the transition from a w=0 string to w=-1 and w=+1 strings. In this case, we have one compact dimension X and one non-compact dimension Y. Figure Credits: Joseph Polchinski. *String Theory: An Introduction to the Bosonic String (Cambridge Monographs on Mathematical Physics)*, volume 1. 1998. 48
3.3 Causal structure of an inflationary universe and of String Gas Cosmology 49

4.1 Plots from the numerical solution for the case $D = 2$. We have set as initial conditions $a(t=0) = 10$ and ρ_0=1. We have also tuned $E_{1_0} = 1000$ and $k = 0.001$. 59
4.2 Plots of a(t) and $\ddot{a}$(t) for the limiting case $D = 2.99$ 60
4.3 Plots of a(t) and ξ(t) for the case $D = 4$ and for the same parameters of Figure 4.1 60

A.1 Worldline and worldsheet generated by a non-dimensional and by a one-dimensional object that propagate on space-time. Figure Credits: Joseph Polchinski. *String Theory: An Introduction to the Bosonic String (Cambridge Monographs on Mathematical Physics)*, volume 1. 1998. 68

CHAPTER 1

Overview

A poet once said, "The whole universe is in a glass of wine." We will probably never know in what sense he meant that, for poets do not write to be understood. But it is true that if we look at a glass of wine closely enough we see the entire universe.

—RICHARD FEYNMAN

The comprehension of the universe is a dream whose age matches the age of mankind itself. From this point of view, Cosmology may be seen as one of the oldest subject of study of men. From deities to Einstein field equations, a long way has been pursued in this quest. The last century, notably, has seen the emergence of accurate observational data and finally has Cosmology gained the status of science.

Side by side with the experimental developments, the last century has also seen a tremendous development of theoretical models. A better comprehension of the evolution of the universe strongly depends on a fundamental comprehension of the laws of nature. The last century saw the formulation of the Einstein's theory of General Relativity and of the Standard Model of Particle Physics. Our current paradigm of modern cosmology, the Λ-CDM model was based over these two pillars of Physics, each one on its regime of validity. The challenge that presents itself to us right now is to extend the regime of validity of cosmological models by conciliating these two pillars in a quantum theory of gravity. String Theory has proven so far to be our best candidate for such theory. In this state of affairs, string cosmology holds importance for both Cosmology and String Theory: on one hand, stringy cosmological models may help solving some open questions of modern cosmology, as the dimensionality of space-time and the generation of the causal structure we observe today; on the other hand, cosmological experiments may be the perfect site for probing String Theory itself. It is with this motivation that we shall proceed on this work.

The outline will be as given: in this chapter, a brief history of modern cosmology will be described, as an introduction of its successes and open questions. In the second chapter, a more detailed view of modern cosmology will be given, focusing on the inflationary scenario. The third chapter will be devoted to the presentation of string cosmology, focusing on the string gas scenario. The fourth chapter will contain our study of inflation in the string gas scenario. There is also an appendix on String Theory. We strongly encourage those not familiar with this subject to devote some time on the concepts developed on this appendix, since some key points of our work depend such concepts.

1.1 A Brief History of All Things

1.1.1 Early Days: The birth of Modern Cosmology

We can trace the beginning of the so called *modern cosmology* back to the first years of the 20th century. The first notable fact was the formulation of the theory of General Relativity by Albert Einstein [1]. The theory describes the universe as a four-dimensional manifold on which is defined a metric of Lorentzian signature. Einstein's equation

$$G_{ab} = 8\pi T_{ab} \tag{1.1}$$

relates the geometry of space-time (through the Einstein tensor G_{ab}) to the matter distribution in space-time (related to the stress-energy tensor T_{ab}). In the words of Wheeler, *"Matter tells space how to curve and space tells matter how to move."*.

In this scenario, we can restate the fundamental question of cosmology, *how does the universe evolves?* as *which solution of Einstein's equations describes our universe, at least as an idealized model?*

To answer this question, one needs other inputs about the universe. The Standard Big Bang Scenario (SBB), which led ultimately to our current paradigm of cosmological model the Λ-CDM model, was originally constructed over three pillars:

- General Relativity is the correct theory of gravitation;
- our universe is homogeneous and isotropic on large distance scales;
- the matter content of the universe may be described as a perfect fluid.

As we will see in more detail in the next chapter, these assumptions led to the construction of the Friedmann-Robertson-Walker model (or FRW Cosmology). The statement that our universe is homogeneous and isotropic on large distance scales is called *cosmological principle*. Since there are no preferred directions in space, it is natural to assume isotropy.

The assumption of homogeneity corroborates the idea that we occupy no special area of the universe and is supported by experimental data, as shown in Figure 1.1.

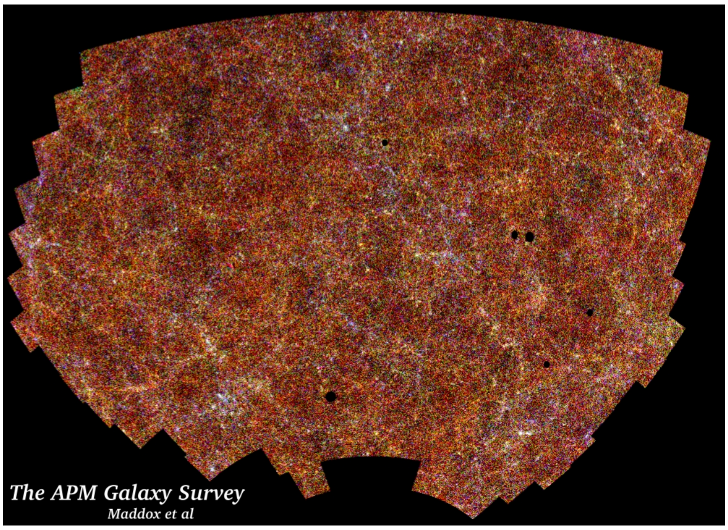


Figure 1.1 APM Survey picture of a large part of the sky, about 30 degrees across, showing almost a million galaxies out to a distance of about 2 billion light years. Credits: Steve Maddox, Will Sutherland, George Efstathiou and Jon Loveday.

The picture shows the galaxy distribution as a density map in equal area projection on the sky. Note how uniform is the distribution of measured galaxies.

The FRW scenario suggested the picture of a universe that was extremely small and hot in the past, and that expanded and cooled down during its evolution. This is the basis of the Standard Big Bang (SBB) scenario. The idea of a expanding universe was somewhat controversial at that time. It took the work of the american astronomer Edwin Hubble to experimentally support this idea.

Measuring the redshift of galaxies outside the Milky Way, Hubble managed to show that they were moving away from us with a speed that was proportional to their distance from us [2]: the farther the galaxy, the faster it is receding from us. This work is considered to mark the beginning of modern cosmology.

1.1.2 The Cosmic Microwave Background and the Inflationary Paradigm

As we shall see in more detail in the next chapter, another important experimental support for SBB is the Cosmic Microwave Background (CMB), measured for the first time in 1965 by Arno Penzias and Robert Wilson at the Bell Labs [3]. Is consists of a radiation that fills the entire universe and resembles the universe as it was approximately 300,000 years after the big bang. It tells us a lot about the universe on very large scales, since the radiation we see today has traveled over such a large distance.

Even though the CMB is in accordance with an isotropic universe, it presented cosmologists with an intriguing question: points that are too far apart to be causally connected by the time of CMB emission are in thermal equilibrium with each other. If the initial conditions are to be believed to do not be special, some mechanism must be responsible for justifying such thermal equilibrium. This is known as the horizon problem.

Possible answers for this question were purposed by Starobinsky [4] and Alan Guth [5] under the name of *inflation*. The idea is that an era of exponential expansion of the universe might justify the causal structure we observe nowadays. It was purposed as a solution to another problem tough: an extremely low density of magnetic monopoles is measured. However, it is believed that the primordial universe had a huge number of monopoles. Inflation also manages to dilute the density of monopoles and other cosmological relics.

As a third success, inflationary mechanisms predict an almost invariant spectrum of fluctuations. Just as it is measured by the CMB. These three phenomenological facts give Inflation a strong support. There is, however, a drawback: even though there are several inflationary mechanisms, there is no theory of inflation. Many inflationary models, for example, are driven by scalar fields (the inflaton). However it is not clear what exactly is this scalar field.

In spite of this drawback, by the end of the last century a global view of the cosmological "standard model" was somewhat clear: the universe started extremely small and hot, passed through an inflationary phase and then kept expanding in a non-inflationary regime that would resemble FRW cosmology. The matter content of the universe on late times would be dominated by radiation. It was not clear what the inflaton was and singularities (specially the big bang singularity for cosmologists) required more attention, but is was common sense that the answer for these questions would only be given by a more fundamental theory of gravitation.

1.1.3 Late Time Evolution and the Dark Sector

This picture dramatically changed in 1999, when a group lead by Saul Perlmutter, from the Supernova Cosmology Project, conducted measurements for luminosity distance as a function of the redshift for 42 Type Ia supernovae [6]. The most recent results of the project are shown in Figure 1.2.

In the description of the energy content of the universe as a perfect fluid, it is usual to di-

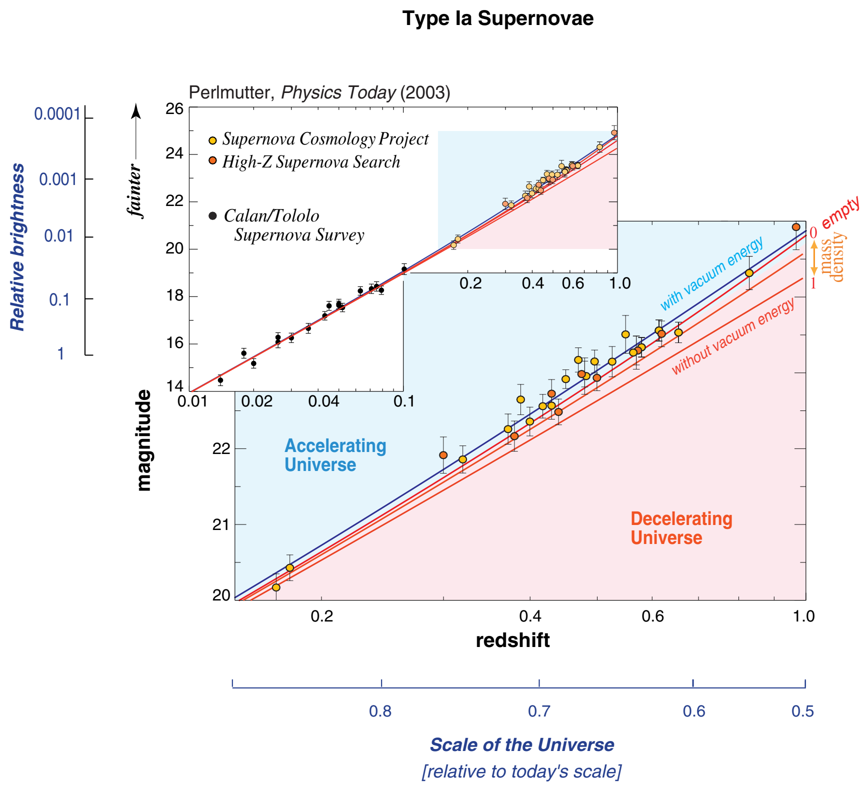


Figure 1.2 Luminosity versus redshift for supernovae Credits: Supernova Cosmology Project

vide this content in three species: ordinary matter, radiation (composed mostly by photons and neutrinos) and vacuum energy. Since they are described as perfect fluids, one is able to write down their equations of state. In fact, matter is pressureless ($P_M = 0$), radiation has an equation of state given by $p_R = (1/3)\rho_r$ and vacuum energy is associated with a negative pressure $(p_\Lambda) = -\rho_\Lambda$. According to FRW-dynamics, this implies that radiation density goes as a^{-4} while matter density goes as $^{-3}$, that is, the energy in the universe should have been dominated in early times by radiation, but since it decays faster than matter, we expect it to be pretty small after sufficient amount of time. The late behavior should, at first glance, be dominated by matter.

Surprisingly, the data obtained by Perlmutter et al. indicates with 99% of confidence that in current days the energy density of the universe is dominated by vacuum energy! According to the data, approximately 30% of the energy density is under the form of mass and around 70% is under the form of vacuum energy. What is even most surprisingly it that according to the dynamics of FRW cosmology, if ρ_Λ is sufficiently large compared to ρ_M, the universe is not only expanding, but doing it with a positive acceleration [7]!

As a last ingredient to the cosmological mess, there is the fact that some part of the matter in the universe (actually most part) seems to be missing. By ordinary matter in the paragraph above we mean anything made from atoms and their constituents (stars, planets, gas, dust). The technical name for this is *baryonic matter*, since the majority of this mass comes from baryons.

It turns out that baryonic matter is not even nearly enough to account for the observed total matter energy density in the universe. Estimates from direct counting of baryons and consistency with the CMB spectrum indicate that baryonic matter only accounts for around 4% of the total energy density of the universe. The rest of the matter energy comes from some kind of unknown non-baryonic matter. Even though we are still not able to relate this matter with the ordinary matter we know from our experience on Earth, its gravitational effects may be verified. One example of this is the dark matter ring observed by Hubble telescope.

Figure 1.3 Dark Matter ring observed by Hubble Space Telescope Credits: NASA, ESA, M.J. Jee and H. Ford (Johns Hopkins University).

Such ring is produced by an effect called gravitational lensing: gravity bends the light of distant background galaxies, generating this ring-like structure.

These unknown energy and matter in the universe are usually called *dark energy* and *dark matter*. It is important to remark that even tough they both have *dark* in their terminology, this is only to reflect the fact that we know very little about them. They are not related. Actually, it should be noted that the effect of dark energy on the evolution of the universe is opposite to that of dark matter: while the first one accelerates the expansion, the last one tends to slow it down.

Some possibilities have been purposed to explain both the existence of dark matter and of dark energy. However, we will not focus on them in this work. The reader is referred to the references for more details on this topic. Let us just mention that the most common way to describe dark energy is to treat vacuum energy as a cosmological constant in Einstein's equation. Three are the main questions regarding dark energy: the first one is that it is possible to roughly estimate the value of vacuum energy from the sum of zero-point fluctuations (the energies of quantum fields in their vacuum state). This estimative results in an energy density of order

$$\rho_\Lambda \approx 10^{112} \mathrm{erg/cm}^3.$$

Cosmological observations, however, indicate a much smaller value:

$$\rho_\Lambda^{\mathrm{obs}} \approx 10^{-8} \mathrm{erg/cm}^3.$$

This is a difference of 120 orders of magnitude between theoretical and experimental values. Why is the cosmological constant so much smaller than we expected? The second question is even deeper: what is the origin of all this energy? Finally, there is the coincidence problem: the ratio between vacuum energy and matter evolution in the universe is given by

$$\frac{\Omega_\Lambda}{\Omega_M} \propto a^3. \tag{1.2}$$

However, our observations show that nowadays they have the same order of magnitude. This means in the past vacuum energy was really small and that in the future matter density will become negligible. A possible explanation is that our observations have not detected a cosmological constant, but some kind of dynamical component that mimics the properties of vacuum energy. It is in this sense that the term *dark energy* is more used than *cosmological constant*.

Some have argued that a solution for this problem would require the anthropic principle [8]: existence of intelligent life is most likely to occur where the absolute magnitude of vacuum energy is not too large, since large positive Λ would tear particles apart even before galaxies could exist and large negative Λ would drive a fast collapse of the universe before life could take place. This argument, however, seems to be of little use towards understanding the mechanism.

As for dark matter, the important question that poses itself is: what is the origin of this non-baryonic dark matter [9]? What we already know is that essencialy every particle in the Standard Model of Particle Physics has been ruled out as a candidate for dark matter. This means we must seek it outside the Standard Model.

One possible way to extend the Standard Model is through Supersymmetry. In supersymmetric theories, every particle has a supersymmetric partner: a corresponding particle that has the same quantum numbers of the original particle, except for the spin, that is changed by a factor of 1/2. One of such particles is the neutralino, a mix of states of the Zino, the photino and the Higgsino (which are superpartners of the Z boson, the photon and the Higgs). Even tough the neutralino is the lightest of the predicted supersymmetric partner, it has enough mass ($m_{\text{neutralino}} > 100GeV$) and stability to account for the observed dark matter abundance [10].

What is certain is that dark matter must be cold: it must have been non-relativistic for a long time, for if it was hot it would have free-streamed out of overdense regions, suppressing the formation of galaxies. Because of that, is common to call it Cold Dark Matter (CDM). It must be very weakly interaction with ordinary matter, since is has not been detected yet. Because of that, candidates for dark matter as the neutralinos are also called WIMPS: weakly interacting massive particles.

With all that in mind, we may define what cosmologists understand as the Cosmological Standard Model nowadays: the universe started out extremely dense and hot, as in Standard Big Bang (SBB) cosmology. It has passed through an inflationary phase while it was still young, generating the causal structure we observe nowadays and diluting the density of transplanckian relics. Inflation should have taken place after $t = 10^{-42}s$ and have lasted at least 60 e-folds to generate the correct spectrum of fluctuations (one e-fold is the time necessary for the universe to grow by a factor of e). Far in the future from the inflationary era, the universe obeys the dynamics of FRW cosmology with a cosmological constant Λ and the presence of cold dark matter as sources of dark energy and dark matter. This is the so called $\Lambda - CDM$ model. The evolution of this scenario is depicted in Figure 1.4. Standard references for modern cosmology include Weinberg [11], Dodelson [12] and Liddle [13].

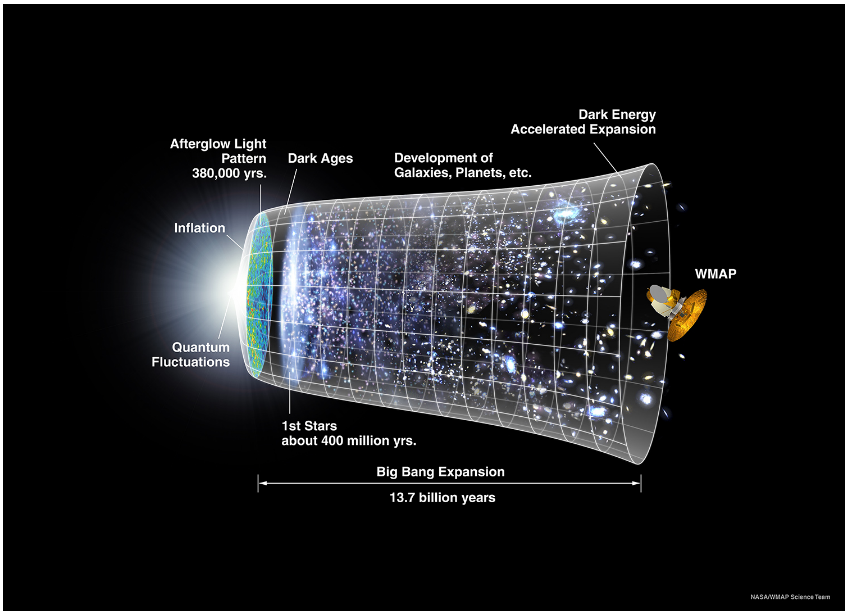


Figure 1.4 Evolution of the Λ-CDM Cosmological Model. Credits: NASA/WMAP Science Team

1.2 String Theory

Another serious issue in Modern Cosmology is the fact that General Relativity, as any physical theory, has a regime of validity. Thus, a description of the universe based in General Relativity will be limited to this regime. In special, General Relativity breaks down in regimes of extreme energy density (in a technical fashion, it is said that GR is not an UV complete theory). In such regimes, we need to replace it by a quantum theory of gravity. Let us see why.

1.2.1 Why String Theory

General Relativity predicts the existence of singularities. These can be thought as a region of infinite curvature on the space-time. Remember, for example, FRW flat space-time:

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{1.3}$$

where the scale factor $a(t)$ goes to zero at some finite time $t = 0$ where space-time is infinitely curved. General Relativity is not able to explain what happens when $t = 0$. The same happens inside black holes: even tough General Relativity predicts their existence, it breaks down at their singularities. This means we need a more fundamental theory of gravity if we want to describe what happens when a particle reaches a space-time singularity. This should be a quantum theory of gravity, that manages to account for the quantum effects that become relevant when curvature reaches planckian values. It is known that quantizing gravity is a difficult problem, since some of its divergences are not renormalizable.

Consider, for example, Quantum Electrodynamics (QED). One is able to make some parameters of the theory, such as the mass and charge of the electron, absorb divergences in QED loop diagrams. Since these parameters are measurable, the predictions of the theory for all other physical observables are finite (in this sense, we measure the already renormalized mass and charge of the electron).

The above procedure can not be made in General Relativity (GR): renormalizing all the divergences would require an infinite number of such parameters. Therefore, we can only trust the theory in the regime where all these terms become negligible, namely the low energy regime. In this context, General Relativity can be thought as a low-energy effective theory of a more fundamental theory: String Theory, which contains additional heavy degrees of freedom. These heavy degrees of freedom may be integrated out to yield the effective theory:

$$e^{iS_{\rm eff}(l)} = \int Dh\, e^{iS(l,h)} \tag{1.4}$$

where l denotes the light (super)gravity modes and h the heavy additional modes. The ratio E^2/M^2 between (low) energy scale of interest and the mass of the heavy modes serves as an expansion parameter for the effective action, through which quantum effects may be computed. However, this expansion is only valid when the parameter is small. It may happen that the mass of a heavy mode depends on the light mode and may be small in certain regimes. In extreme cases, when the mass of a heavy mode goes to zero, the effective theory can contain divergent couplings and then become singular. This can be solved by taking this heavy mode that becomes light and including it in the low energy description instead of integrating it out with the heavy modes. The additional stringy degrees of freedom resolve the singularity of the original gravity theory.

The above discussion illustrates the breakdown of GR at high energy densities. A more accurate cosmological model, thus, should be constructed over the more fundamental underlying theory. We have a candidate for such a theory: String Theory.

1.3 A New Paradigm: String Cosmology

In spite of the recent fame String Theory has acquired even in the non-scientific world for being a "Great Unification Theory" (GUT), this is not the main reason why physicists are so interested in strings. The main reasons are: i) that String Theory provides a quantum theory of gravity that has General Relativity as a low-energy effective theory; ii) the Gauge/Gravity duality.

Once we are equipped with a quantum theory of gravity, we can use it to construct cosmological models not limited by the regime of validity of General Relativity and try to solve some questions that could not be answered before: one of them is the fact that the initial singularity of space is an issue that may not be totally believable from a physical point of view. As is the infinity of the temperature of the primordial universe. And there is also the issue of the dimensionality of spacetime: General Relativity sets it to be four by hand.

Some attempts have been made using string-inspired ideas, namely the Pre-Big Bang Scenario [14], Ekyprotic Scenario [15],[16] the Randall-Sundrum Scenario [17], Cyclic Cosmologies [18], AdS Cosmologies [19] and last but not least, the String Gas Scenario [20],[21] in which we shall focus on this work.

1.3.1 Cosmology with string gases

The String Gas Scenario was originally proposed by Robert Brandenberger and Cumrum Vafa. The idea is to study how highly excited strings behave when they are put in small spaces (compactified dimensions), focusing on some features that are stringy in nature, namely the Hagedorn temperature and T-Duality.

The Hagedorn temperature is a limiting temperature for a string gas that emerges from the fact that the asymptotic density of energy levels of a string grows as

$$\rho(m) \approx \exp(\pi\sqrt{\alpha'}m). \tag{1.5}$$

As a consequence, the energy of a string diverges as

$$E \sim \frac{1}{\beta - \beta_H} \tag{1.6}$$

where $\beta_H = 1/4\pi\alpha'$ is the Hagedorn temperature. This stringy feature eliminates the divergence on the universe temperature as it approaches $t \to 0$ in SBB.

The other feature comes from considerations about the dimensionality of spacetime.

(Super)String Theory predicts the existence of (10) 26 spacetime dimensions. These are higher than the usual 4 dimensions we know from everyday life. A natural hypothesis is that the remaining dimensions are compactified in scales much smaller than the 4 large dimensions we observe.

T-Duality is the fact that the energy spectrum of strings on compactified dimensions

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N_L + N_R - 2) \tag{1.7}$$

is invariant under $R \leftrightarrow R' = (\alpha'/R)$, $n \leftrightarrow w$. On the right-hand side of the equation above, the first term is contribution from momentum modes of the string to the spectrum, the second term is the contribution from winding modes and the last term is the contribution from oscillators modes. The reader not familiar with the equation above is referred to the appendix and to Chapter 3, where it will be treated in detail.

This fact implies that strings are not sensible to distances smaller than the string scale: if we start downsizing the distance scale, at some point strings will cross a self-dual radius from which point they will start to behave as if the distance scale was growing again. Since the spectrum will be the same if we have a too large radius or a too small dual radius, we can not distinguish too large from too small!

Winding modes will also play an important role in determining the dimensionality of spacetime. In the Brandenberger-Vafa scenario the universe starts close to the Hagedorn temperature, with all spatial dimensions compactified on a string-scale torus. Momentum and winding modes of the string spectrum have opposite pressure contributions to the gas: winding modes prevent expansion and momentum modes prevent shrinking.

Fluctuations might lead to an annihilation of winding modes, making it possible for compactified dimensions to expand. In the picture, we should not ask ourselves why are there compact dimensions. Instead we must reverse the question and ask why some of them have expanded. The number of large spatial dimensions would not be input by hand, but a possible outcome of string interactions. Some works have been made to determine whether decompactification on 3 spatial dimensions is an statistically favored outcome.

In this work, we will focus on studying the dynamics of the String Gas scenario close to the Hagedorn temperature. One important question in this scenario is about the existence of an inflationary era in the evolution of a universe filled with a string gas. As we shall see, the fact that the universe starts in a thermal state (a Hagedorn phase) creates a causal structure in this scenario without the need of an inflationary phase. The String Gas scenario has no horizon problem. However, the fact that and inflationary era is not required to solve these problems does not mean it has not existed. More than that, some other problems that arise in Standard Big Bang cosmology (SBB), as the requirement of an explanation for the extremely low density of monopoles still arise in this scenario. One can either search for an inflationary phase that solves these other problems or find a new mechanism to explain them.

As a last remark, it is important to note that the appendix was thought as an important part of this work, specially for those not familiar with String Theory. Major concepts that will be used through this work are developed on the appendix, so we really suggest that all of those who are not familiar with String Theory devote some time reading it. Previous knowledge on field quantization is recommended.

CHAPTER 2

Standard Big Bang Cosmology and the Inflationary Scenario

Parallel lines, move so fast toward the same point, infinity is as near as it is far.

—KINGS OF CONVENIENCE

2.1 FRW Cosmology

One notable fact about the evolution of the universe is that it appears to be the same in all directions and in all positions around us, at least when the relevant distances are bigger than 300 million light years. The assumption that the universe is homogeneous and isotropic, the so called *cosmological principle*, is one of the ingredients of the FRW cosmological model, the base model of most research done on modern cosmology, at least as a first approximation.

The FRW metric was first obtained by Friedmann, as a solution to Einstein field equations [22]. After that, Robertson [23] and Walker [24] obtained it again in the context of homogeneity and isotropy.

The FRW model is derived from three pillars. The first one the already mentioned cosmological principle. The second one is the assumption that the evolution of a cosmological system is dominated by gravity, and that the theory of gravitation is Einstein General Relativity. At last, the universe matter content is described as a perfect fluid.

Now, let's use these ingredients and construct the model.

2.1.1 Einstein Equations

The evolution of the universe occurs in large distance scales (cosmological scales). On such scales, the evolution of the system will be determined by gravitational interaction. The first assumption of FRW cosmology is that these interactions are described by Einstein's theory of General Relativity.

In particular, the action that describes these interactions reads

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_\Sigma + \int d^4x\sqrt{-g}\mathscr{L}_m. \tag{2.1}$$

The first term is just the Einstein-Hilbert term, comprising the determinant of the metric g and the Ricci scalar R, related to the geometry of space-time. The last term is the action due to matter fields, acting as gravitational sources. The second term is the Gibbons-Hawking boundary term [25] and is required in order to reproduce the standard Einstein equations (originally Einstein did not deduced his equations from an action principle; this term makes the equations of motion of this action reproduce the ones deduced by Einstein). Varying this actions with respect to the metric gives Einstein's field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{2.2}$$

The tensor $G_{\mu\nu}$ is the so-called Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter sources defined by

$$\delta_g(\sqrt{-g}\mathscr{L}_m) = \frac{1}{2}\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}. \tag{2.3}$$

Now we must give some observational input and look for solutions of equation (2.2). This input will be the cosmological principle of homogeneity and isotropy.

2.1.2 The FRW Metric

The objective here is to obtain metrics that are homogeneous and isotropic solutions of Einstein field equations. Firstly, one can think of the obvious flat space metric:

$$ds^2 = dx^2$$

Note that three dimensional translations and rotations leave this invariant.

One can also think of a three-dimensional spherical surface in a four-dimensional Euclidean space with some radius a, in which case the line element would be:

$$ds^2 = dx^2 + dz^2, \qquad z^2 + x^2 = a^2$$

Now, the transformations that leave this invariant are four-dimensional rotations.

There is only one other possibility [26] of such line element. It is a hyperspherical surface in a four-dimensional pseudo-Euclidean space, with line element

$$ds^2 = dx^2 - dz^2, \qquad z^2 - x^2 = a^2$$

where a^2 is an arbitrary positive constant. Now, four-dimensional pseudo-rotations, like Lorentz transformations with z instead of time, leave this invariant.

We can rescale coordinates

$$x' \equiv ax, \qquad z' \equiv az,$$

so that the spherical and hyperspherical cases can be written as:

$$ds^2 = a^2[dx^2 \pm dz^2], \qquad z^2 \pm x^2 = 1$$

Differentiating $z^2 \pm x^2 = 1$ and inserting the result, we obtain:

$$ds^2 = a^2 \left[dx^2 \pm \frac{(x \cdot dx)^2}{1 \mp x^2} \right].$$

This can be extended to the Euclidean case by rewriting it as:

$$ds^2 = a^2 \left[dx^2 + K \frac{(x \cdot dx)^2}{1 - Kx^2} \right]$$

where we have three possible cases for the value of K:

$$K = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ \ \ 0 & \text{Euclidean} \end{cases}$$

Now, if we promote a to an arbitrary function of time (the FRW scale factor), we can insert a term in the spacetime element, so that we may finally arrive at the FRW spacetime metric:

$$d\tau^2 \equiv -g_{\mu\nu}(x)dx^\mu dx^\nu = dt^2 - a^2(t) \left[dx^2 + K \frac{(x \cdot dx)^2}{1 - Kx^2} \right]. \tag{2.4}$$

It is possible to rewrite this metric in spherical polar coordinates:

$$dx^2 = dr^2 + r^2 d\Omega, \qquad d\Omega \equiv d\theta^2 + \sin^2\theta \, d\phi^2$$

so that:

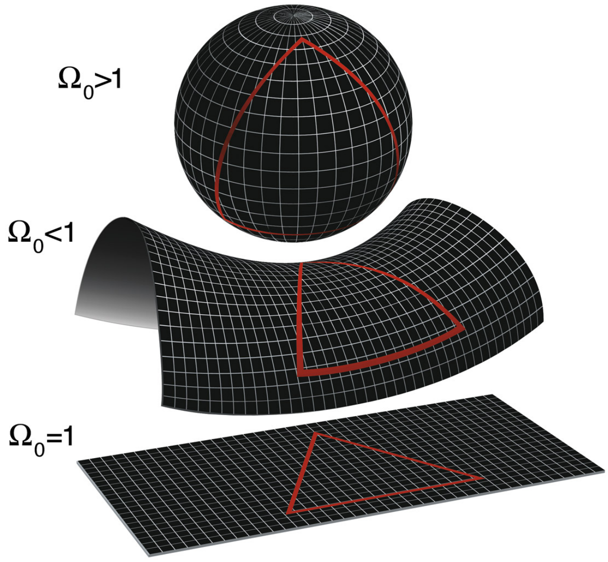


Figure 2.1 Possibilities for the geometry of FRW universes depending on the value of K

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right]. \tag{2.5}$$

Note that in deriving this metric, we have used two of the three basic assumptions of the FRW modes, namely the conditions of isotropy and homogeneity, and the usage of Einstein General Relativity as a theory of gravitation. Now, we may go further: describing the universe matter content as a perfect fluid we are going to obtain the dynamics of the model.

2.1.3 FRW Dynamics

As was previously stated, in the FRW model the matter content is described as a perfect fluid. In a perfect fluid, the components of the energy momentum tensor must take the form

$$T^{ij} = p\delta_{ij}, \quad T^{i0} = T_{i0} = 0, \quad T^{00} = \rho,$$

where ρ e p are, respectively, the energy density and pressure. Note that if the statement above was not true, there would exist preferential directions in space.

Now, suppose a locally inertial Cartesian frame with an arbitrary velocity. Then, if p and ρ are the same as in the comoving inertial frame, the energy momentum tensor has the form

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)u^\alpha u^\beta$$

Here, u^α is defined to have the components $u^0 = 1$, $u^i = 0$ in the locally comoving Cartesian inertial frame and to transform as a four-vector under Lorentz transformations. It is known as the velocity vector and it is normalized so that, in any inertial frame, $\eta_{\alpha\beta}u^\alpha u^\beta = -1$. In general, therefore, the energy-momentum tensor of a perfect fluid in a gravitational field is given by:

$$T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)u^\mu u^\nu, \quad g_{\mu\nu}u^\mu u^\nu = -1$$

Note that this formula is generally covariant and is true in locally inertial Cartesian coordinate systems. The assumption of homogeneity and isotropy, ergo, implies that the components of the energy momentum tensor take, everywhere, the form

$$T^{00} = \rho(t), \quad T^{0i}, \quad T^{ij} = \tilde{g}^{ij}(x)a^{-2}(t)p(t). \tag{2.6}$$

Note that the momentum conservation $\nabla_\mu T^{i\mu} = 0$ is satisfied for the FRW metric and the energy momentum tensor above. On the other hand, from the energy conservation law we get:

$$\nabla_\mu T^{0\mu} = \partial_\mu T^{0\mu} + \Gamma^0_{\mu\nu}T^{\nu\mu} + \Gamma^\mu_{\mu\nu}T^{0\nu} =$$
$$= \frac{\partial T^{00}}{\partial t} + \Gamma^0_{ij}T^{ij} + \Gamma^\mu_{i0}T^{00} = \frac{d\rho}{dt} + \frac{3\dot{a}}{a}(p+\rho) = 0$$

So that

$$\frac{d\rho}{dt} + \frac{3\dot{a}}{a}(p+\rho) = 0. \tag{2.7}$$

A perfect fluid has a equation of state of the form

$$p = w\rho.$$

Inserting this equation of state in the energy conservation equation, we get:

$$\rho \propto a^{-3-3w} \tag{2.8}$$

One can also arrive at the above conclusion by noting that, for the FRW metric, the Ricci tensor is given by

$$R_{ij} = -[2K + 2\dot{a}^2 + a\ddot{a}]\tilde{g}_{ij}$$

which, in conjunction with 2.6, reduces Einstein equations to:

$$
\begin{aligned}
-\frac{2K}{a^2} - \frac{2\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} &= -4\pi G(\rho - p), \\
\frac{3\ddot{a}}{a} &= -4\pi G(3p + \rho).
\end{aligned} \tag{2.9}
$$

These are known as the FRW cosmological equations, and determine the dynamics of the FRW cosmological evolution. One can combine them in order to obtain:

$$
\dot{a}^2 + K = \frac{8\pi G\rho a^2}{3} \tag{2.10}
$$

This was the equation originally obtained by Friedmann. Its derivative combined with the second FRW equation yields:

$$
\frac{d\rho}{dt} = -\frac{3\dot{a}}{a}(p + \rho)
$$

which is exactly the conservation of energy equation we have obtained before.

Now we may return to the analysis of equation (2.8). It has, in principle, two particular cases:

- Radiation: $p = \rho/3 \Rightarrow \rho \propto a^{-4}$
- Dust (cold matter): $p = 0 \Rightarrow \rho \propto a^{-3}$

With this information, we are ready to advance in the details of dynamics of the universe evolution. The first thing to be noted is that the value of K, the constant that determines the geometry of the FRW universe, also determines qualitative features of the future evolution of the universe. We can eliminate the $\ddot{a}$ factor in FRW equations 2.9 to obtain:

$$
\frac{3\dot{a}^2}{a^2} = 8\pi G\rho - \frac{3K}{a^2} \tag{2.11}
$$

From this, it becomes clear that if $K = -1$ or $K = 0$, then $\dot{a}$ can never become zero. Thus, if $\dot{a} > 0$ in present time, then $\dot{a}$ will be non-negative forever and the universe can never stop expanding.

On the other hand, if $K = +1$, one can note that the first term on the right-hand side decreases more rapidly with a than the second (since $\rho \propto a^{-4}$ or $\rho \propto a^{-3}$). Since the left-hand side of the equation must be positive, then a must have some upper bound a_c. Also, the fact that a is bounded, in conjunction with the second FRW equation, imposes a lower bound on $\ddot{a}$. The result, is that the universe can not asymptote a_c when time goes to infinity,

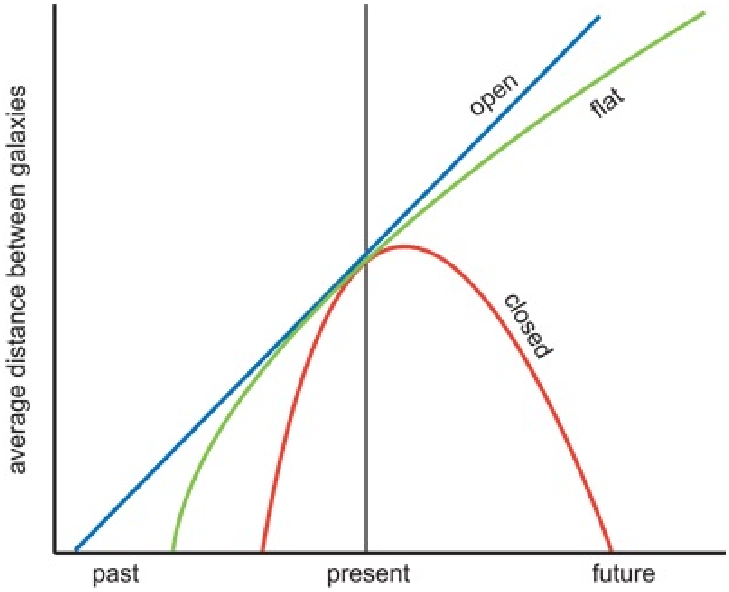


Figure 2.2 Possibilities for the evolution of FRW universes depending on the value of K. Credits: European Space Agency

but rather will achieve a maximum value a_c and then begin to contract, coming to a "big crunch" sometime in the future.

2.2 Phenomenological Aspects of Standard Big Bang Cosmology

The description of the universe through standard Big Bang cosmology, as we have seen so far, is based on a classical treatment of both space-time (General Relativity) and matter (perfect fluids). We are going to present experimental evidence that give phenomenological support for SBB, namely the Hubble Law and the isotropy and black body nature of CMB.

2.2.1 Hubble Law

The FRW model was formulated in the first half of the 1920's decade. It was not, however, the only purposed cosmological model. It was not clear, at that time, that the expansion of the universe predicted by the model was true. Other scientists, including Albert Einstein himself, believed that the universe was a steady system.

It was only by 1929 that an experiment was able to shine a light on this doubt. The american astronomer Edwin Hubble measured the spectra of many galaxies. He was able, from this data, using the Doppler shifts of spectral lines, to obtain the relative velocities of these galaxies. Eventually, he was able to correlate the velocities of the galaxies with their distances to the Earth through a "roughly linear" relation [2]. The result is what is now known as the Hubble Law:

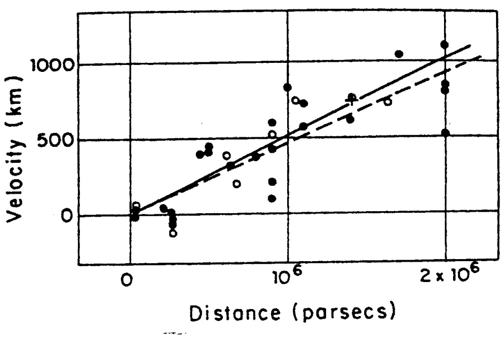
$$v_r = H_0 d \tag{2.12}$$

The slope of the curve is known as Hubble's constant, and characterizes the rate of expansion of the universe. It has the form

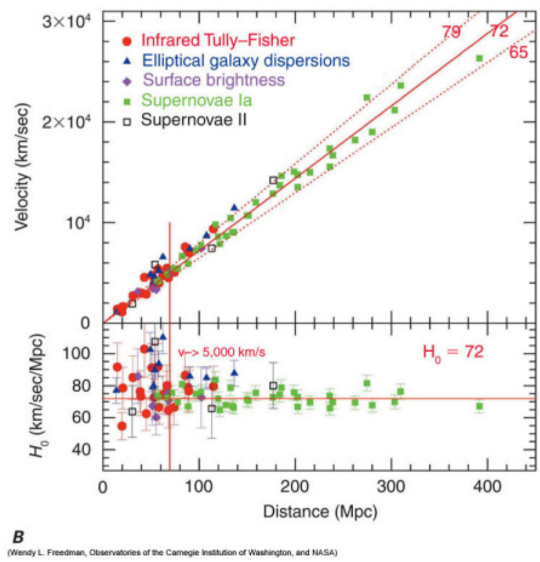
$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \tag{2.13}$$

where the 0 subscript denotes that the values are measured in the present epoch. Even though measurement of the Doppler shift in galaxies spectra have been obtained since the 1910's, only the relation obtained from the work of Hubble was able to show that the universe is expanding.

Figure 2.3 exhibits the original set of data obtained by Edwin Hubble together with data from more recent experiments.



(a) Hubble's original data



(b) Recent data

Figure 2.3 Experimental data supporting Hubble Law. The first one contains the original data set obtained by Edwin Hubble and his linear fit. The second one contains recent data with higher precision. Credits for the second picture: Wendy Freedman, Observatories of the Carnegie Institution of Washington and NASA.

The work of Edwin Hubble usually is cited as the beginning of Modern Cosmology.

2.2.2 The Cosmic Microwave Background

Diffuse photon backgrounds, coming in all wavelengths, are observed in the universe. Most of these photons are found to be in a nearly isotropic background with a thermal spectrum at a temperature of $2,73K$. This spectrum is the so called Cosmic Microwave Background (CMB) and was first detected, by accident, by Arno Penzias and Robert Wilson, from the Bell Labs, in 1965 [3].

The original objective of Penzias and Wilson was to detect radio waves bounced off echo balloons satellites. After removing the effects of radar and radio broadcasting and suppressing interference from the heat in the receiver itself, they found a low and steady noise that persisted in the receiver. This was the first experimental measurement of the CMB.

We know that, as the universe evolves and expands, it energy density increases, and so its temperature. At sufficiently high temperatures, the photons start to interact. At temperatures above approximately 13 eV hydrogen was ionized, and the photons were coupled to charged particles. The rapid collisions of photons with free electrons established a thermal equilibrium between radiation and hot dense matter. The number density of photons in equilibrium with matter at temperature T at photon frequency between ν and $\nu + d\nu$ is given by the black body spectrum:

$$n_T(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_B T) - 1} \tag{2.14}$$

where h is Planck's constant and k_B is Boltzmann's constant. We use natural units, where $c = 1$.

When the temperature became low enough for hydrogen to be stable, the universe became transparent. This moment, when the decoupling occurred, is called the "last scattering surface". The CMB is to a good approximation a "photography" of the this surface.

Since the time of decoupling the matter became cooler and less dense and the radiation began a free expansion. Due to redshift effects, a photon that has frequency ν at some time t, had a frequency $\nu a(t)/a(t_L)$ at recombination time t_L. So, if we assume that the decoupling occurred in a definite instant, the number density of photons at time t is given by:

$$n(\nu,t)d\nu = \Big(a(t_L)/a(t)\Big)^3 n_{T_{(t_L)}}\Big(\nu a(t)/a(t_L)\Big) d\Big(\nu a(t)/a(t_L)\Big) \tag{2.15}$$

where the factor $(a(t_L)/a(t))^3$ arises from the dilution of photons due to the cosmic expansion. Using 2.14 on this last equation, we get the number density at time t:

$$n(\nu,t)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_B T(t)) - 1} = n_{T(t)}(\nu)d\nu \tag{2.16}$$

where

$$T(t) = T(t_L)\frac{a(t_L)}{a(t)}. \tag{2.17}$$

This shows that the spectrum keeps the same form after the universe expansion. The photon density is given by the black-body form even after the photons went out of equilibrium with matter, but with a redshifted temperature. This conclusion holds if instead of a definite instant the decoupling occurred in a finite time interval, since the interactions between photons and matter are dominated by elastic scattering, which is true for the temperature at which the decoupling takes place ($T \approx 3000K$).

One key success of standard big bang cosmology is that this radiation is characterized by a surprising isotropy. Anisotropies arise only at a fractional level of a bit less than 10^{-4}. This is illustrated by Figure 2.4.

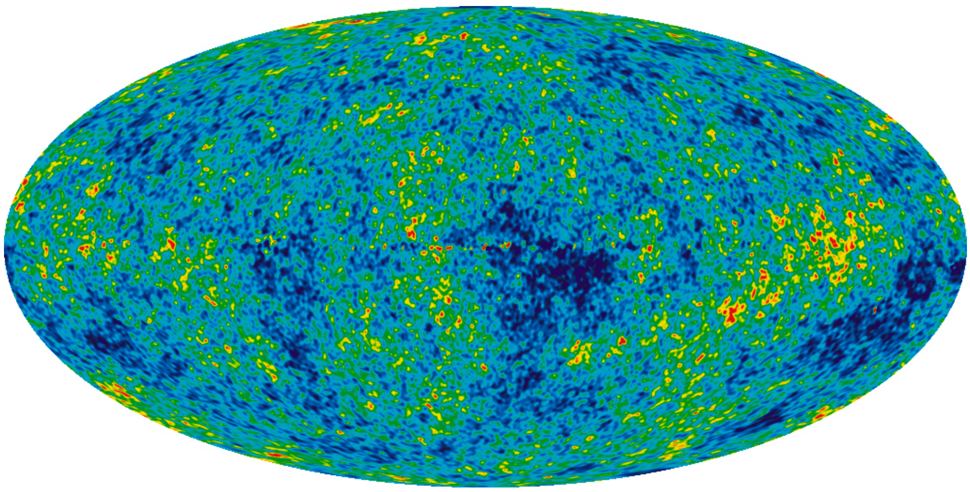


Figure 2.4 Full sky map of CMB from 7 year WMAP data. Credits: NASA/WMAP Science Team

We may study the CMB anisotropies in terms of their angular power spectrum. The typical way is to decompose the temperature fluctuations in spherical harmonics:

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi) \tag{2.18}$$

so that the amount of anisotropy at multipole moment l is expressed via the power spectrum

$$C_l = <|a_{lm}|^2> . \tag{2.19}$$

Note that $\theta = 180^o/l$, so that higher multipoles correspond to smaller angular separations on the sky. The coefficients C_l define the angular power spectrum of CMB anisotropies. Figure 2.5 shows the resulting angular power spectrum from the 5 year data of the WMAP experiment.

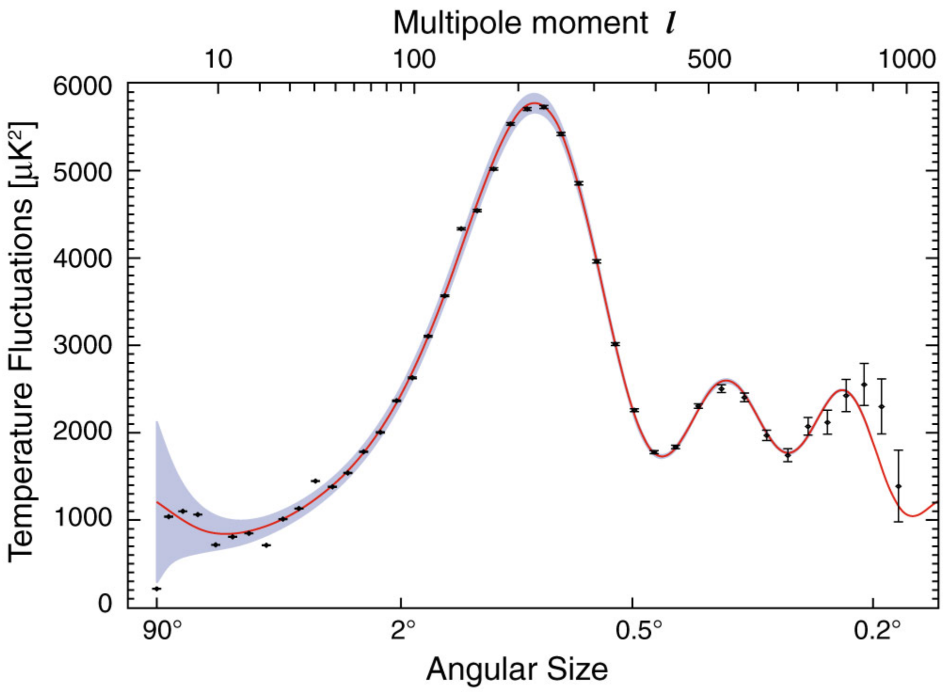


Figure 2.5 Power spectrum of CMB anisotropies. Credits: NASA/WMAP Science Team

The spectrum of CMB anisotropies is relevant in several areas of cosmology. We will get back to it later, when we introduce the mechanism of inflation.

2.3 Inflation

Standard Big Bang (SBB) cosmology has had success in explaining several aspects of the universe evolution. Notably the two phenomenological aspects reviewed so far: Hubble Law and the existence and black-body nature of CMB radiation. However, SBB is not able to account for some conceptual puzzles: the flatness of the universe and the large scale homogeneity of CMB. These questions impel us to introduce a mechanism called *inflation* through which these (and other) issues in SBB may be solved.

We are going to review flatness and homogeneity issues and then introduce the mechanism of *inflation*.

2.3.1 The Flatness Problem

The first issue we are going to address is the so called flatness problem. For that, let us first introduce some useful quantities.

We begin by defining the density parameter in a species of matter i:

$$\Omega_i = \frac{8\pi G}{3^2}\rho_i = \frac{\rho_i}{\rho_{\text{crit}}} \tag{2.20}$$

where

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G} \tag{2.21}$$

is the critical density corresponding to the energy density of a flat universe. In terms of the total density parameter

$$\Omega = \sum_i \Omega_i$$

we can rewrite 2.11 as

$$\Omega - 1 = \frac{K}{H^2 a^2} \tag{2.22}$$

Now he have a clearer view of the dependence of a FRW universe geometry with its energy density:

- $\rho < \rho_{\text{crit}} \leftrightarrow \Omega < 1 \leftrightarrow K = -1 \leftrightarrow$ open universe
- $\rho = \rho_{\text{crit}} \leftrightarrow \Omega = 1 \leftrightarrow K = 0 \leftrightarrow$ flat universe
- $\rho > \rho_{\text{crit}} \leftrightarrow \Omega > 1 \leftrightarrow K = +1 \leftrightarrow$ closed universe

Note from equation (2.22) that as the universe expands $|\Omega - 1|$ increases. Thus, in an expanding universe $\Omega = 1$ is a repulsive fixed point and any deviation from this value will grow with time.

It happens that measurements indicate $\Omega \approx 1$. In order to explain this present small value, the initial energy density had to be extremely close to critical density. For example, at $T = 10^{15}$ GeV, we have:

$$\frac{\rho - \rho_{\text{crit}}}{\rho_{\text{crit}}} \approx 10^{-50} \tag{2.23}$$

To understand the origin of the fine tuning of these initial conditions it the so called *flatness problem* of Standard Big Bang Cosmology.

2.3.2 The Horizon Problem

In FRW cosmology, there is a finite time interval since the Big Bang singularity. Because of that, photons can only have traveled a finite distance since the beginning of the universe. This generates a particle horizon in FRW cosmologies.

A radial null path in a flat spacetime obeys:

$$0 = ds^2 = -dt^2 + a^2 dr^2 \tag{2.24}$$

The comoving distance traveled by a photon that follows this trajectory between times t_1 and t_2 is:

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)} \tag{2.25}$$

The horizon problem is the fact that, even though widely separated points on the last scattering surface were completely outside each other's horizons, the CMB is isotropic to a high degree of precision (temperature differences between the coolest and hottest points in the CMB thermal spectrum are of the order of $10^{-4}K$).

Regions of the universe that were outside causal contact distance are observed to be in the same temperature. So, it is necessary to modify the causal structure of SBB in order to explain how did these regions came to thermal equilibrium. Two possible solutions arise: i) some mechanism occurring during the evolution of the universe connected causally these regions or ii) the universe has began in some special (already in thermal equilibrium) state.

The most accepted solution is *Inflation*, a mechanism through which the universe grows exponentially during a finite time interval in its early evolution. As we are going to see, this may solve both the flatness and horizon problems.

2.3.3 The Mechanism of Inflation

Since Standard Big Bang Cosmology presents the problems listed above, we must somehow modify it in order to provide the universe an accurate causal structure and a solution for the flatness problem. The main idea in inflation is to consider that early universe was dominated neither by matter nor by radiation, but by vacuum energy.

As we have seen in Section 1.2, the scale factor falls as a^{-4} for radiation and as a^{-3} for matter. Vacuum energy has a equation of state given by:

$$p = -\rho_\Lambda$$

So that, in virtue of equation (2.8), during vacuum energy dominated era, we have $\rho_\Lambda/3M_P^2 \propto a^0$. We may rewrite equation (2.8) for this era:

$$H^2 = \frac{\rho_\Lambda}{3M_P^3} - \frac{K}{a^2} \tag{2.26}$$

The first term on the right-hand side grows rapidly with respect to the second one, that is, the density parameter is driven to unity, so the universe becomes flatter with time. If the vacuum energy dominated era lasts long enough, the universe will enter radiation dominated era already with the density parameter very close to unity, what would explain the platitude issue.

A period of exponential growth of the universe, driven by such vacuum energy dominance, on the other hand, would make the physical horizon much larger than the Hubble radius H_0^{-1}. This would provide a causal structure for the isotropy of CMB.

The *inflationary era* is this era of exponential growth of the universe:

$$a(t) \propto e^{Ht} \tag{2.27}$$

where the Hubble parameter H is supposed to remain constant through this era. The figure below shows represents the evolution of an universe with an inflationary era that lasts between times t_i and t_R. Phenomenology restrains inflation to last for 60 or more *e-folds* so that it generates the correct CMB spectrum. One e-fold is the time necessary for the universe to increase by a factor of e its scale factor.

The universe exits inflation, at time t_R, very hot and dense. From this point, it follows SBB evolution. Inflation exponentially dilutes the number of particles initially in thermal equilibrium, lowering matter temperature also exponentially. The energy responsible for inflation is released as thermal energy at time t_r, in a non-adiabatic process that increases the entropy of the universe called *reheating*. Time t_R is called *reheating time*.

The vacuum energy necessary for an early universe inflationary phase may be provided by the potential of a scalar field (which we call *inflaton*). The action of such scalar field that

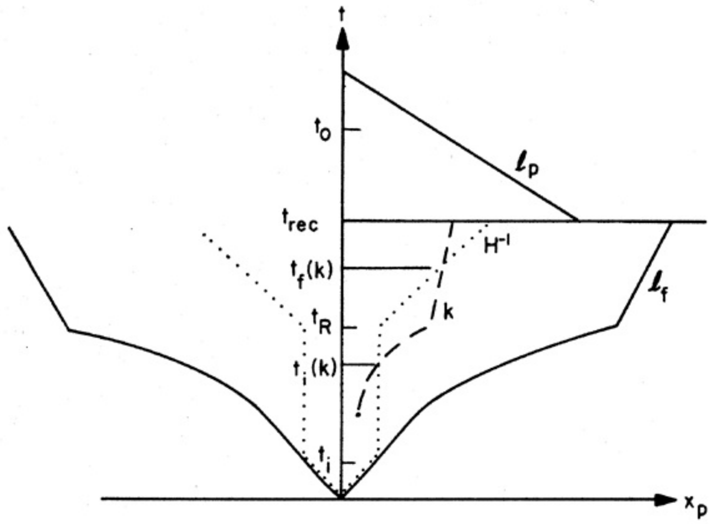


Figure 2.6 Cosmology of an inflationary universe. Inflation goes from t_i to t_R

dominates the universe in a curved spacetime is given by:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{2.28}$$

If we assume the field to be homogeneous and the metric of the space to be that of FRW, the equation of motion of the field is given by:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.29}$$

where a dot indicates time derivatives and a prime indicates derivatives with respect to the field. Ignoring the curvature term on 2.26 since inflation will flatten the universe, it becomes:

$$H^2 = \frac{1}{3M_P^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \tag{2.30}$$

Required conditions for inflation are a dominance of the field potential energy over its kinetic term and a slow time evolution of the scalar field, which are called the slow-roll conditions:

$$\begin{gathered} \dot{\phi}^2 << V(\phi) \\ |\ddot{\phi}| << |3H\dot{\phi}|, |V'| \end{gathered} \tag{2.31}$$

Another way to say that is to require that the *slow-roll parameters*

$$\epsilon = \frac{1}{2} M_P^2 \left(\frac{V'}{V} \right)^2$$
$$\eta = M_P^2 \left(\frac{V''}{V} \right) \tag{2.32}$$

to be small. Note, however, that the existence of an inflationary phase still depends on the initial conditions of the scalar field, like $|\dot{\phi}|$. What can be assured is that if these parameters are small, most initial conditions will be attracted to an inflationary phase.

A simple example of potential that satisfies these conditions is the toy model $V(\phi) = 1/2m^2\phi^2$. Starting with initial conditions such that $\dot{\phi} = 0$ (and neglecting the $\ddot{\phi}$ term), our set of equations reduce to:

$$3H\dot{\phi} = -V'(\phi)$$
$$H^2 = \frac{8\pi}{3} G V(\phi) \tag{2.33}$$

which can be solved to give:

$$\dot{\phi} = -\frac{1}{\sqrt{12\pi}} m M_P \tag{2.34}$$

The slow-roll conditions 2.31 break down when

$$\phi = \frac{1}{\sqrt{12\pi}} M_P \tag{2.35}$$

thus yielding a condition for the end of inflation and the onset of reheating.

In our toy model, reheating will be driven by the the $3H\dot{\phi}$ damping term on the equation of motion (Hubble friction). More generally, reheating has been described in terms of energy transfer from coherent oscillations of ϕ into other particles via parametric resonance [27] [28]. Since many relics may be produced during reheating, a proper understanding of it is necessary. In the context of particle Physics, for example, it is necessary that inflation lasts long enough to dilute the density of magnetic monopoles generated by GUT breaking to less than observational limits. Once inflation has ended, it is important that reheating does not produce too many monopoles. Also, it is important that reheating does not become so hot that it reproduces baryogenesis.

The standard inflationary scenario was proposed by Albrecht and Steinhardt [29] and Linde [30] independently and is made of a scalar field theory in which a double well potential undergoes a second order phase transition.

The critical temperature is characterized by a vanishing second derivative of the poten-

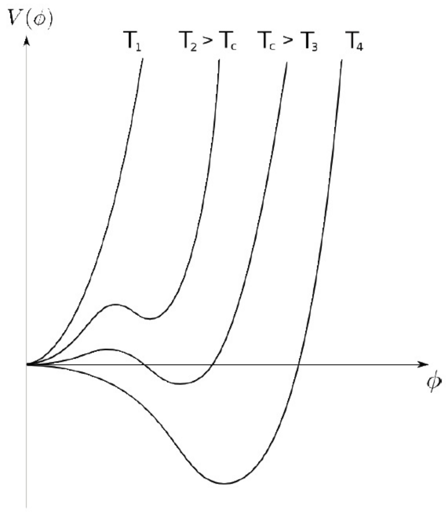


Figure 2.7 Inflationary Potential Phase Transition

tial at $\phi = 0$. For fairly general initial conditions, the field is trapped in a metastable state at $\phi = 0$ by finite temperature effects. As the universe expands, the temperature cools down. When the temperature is lower than the critical, $\phi = 0$ is a local maximum of the potential. Its instability is triggered by thermal fluctuations that drive $\phi(x)$ towards one of the global minima at $\phi = \pm a$. Inflation occurs while the false vacuums decay.

Many realizations of potential-driven inflation have been developed as Hybrid Inflation [31], Chaotic Inflation [32] and Eternal Inflation [33]. The reader is referred for the references for more details.

2.3.4 Fluctuations and Inflation

The major success of the inflationary paradigm is that it is able to account for the observed power spectrum of fluctuations in the CMB.

If the inhomogeneities are small, they can be described by a linear theory, so that the Fourier modes k evolve independently. Any perturbation will have its wavelength inflated by the exponential growth of the universe, while the Hubble radius remains constant. For those perturbations generated early in the inflationary phase, the wavelength will become larger than (exit) the Hubble radius. After the end of inflation, the Hubble radius will increase as t, while the wavelengths of fluctuations will grow as $a(t)$. After some time, the modes will become, once again, smaller than (enter) the Hubble radius. In this way, inflation is able to generate cosmological fluctuations with a causal structure.

When the fluctuations generated by physical process cross the Hubble radius, they satisfy

$$\frac{\delta M}{M}(k, t_i(k)) = \text{constant}, \tag{2.36}$$

where δM is the mass fluctuation on a length scale k^{-1} at time t. The general assumption that super-Hubble scale fluctuations are no affected by causal Physics implies that the magnitude of the fluctuation can only change by a factor independent of k, so that a time t_f we have:

$$\frac{\delta M}{M}(k, t_f(k)) = \text{constant}. \tag{2.37}$$

This is precisely the definition of a scale-invariant spectrum. Both experiments led with galaxy redshift surveys and the CMB spectrum give a power spectrum of density fluctuations that is consistent with a scale-invariant primordial spectrum as illustrated by Figure 2.5. This prediction is the most important experimental support for Inflation.

Until the end of the last century the standard picture the universe was that of universe with an early inflationary phase, that lasts long enough to generate the correct causal structure and spectrum of density perturbations, followed by a radiation dominated era. The late time evolution of the system would be described by a matter dominated FRW universe.

CHAPTER 3

String Cosmology

Sweet are the uses of adversity

—SHAKESPEARE

A proper investigation of the cosmological properties of String Theory requires string models in models that are compatible with our understanding of the early universe, namely time-dependent backgrounds at nonzero temperature. The approach we are going to follow is the so called String Gas Cosmology (SGC) a scenario has been originally proposed by Brandenberger and Vafa [20] in 1989. It has the beautiful properties of generating a causal structure for the observed universe (thus, the horizon problem does not exist in this scenario) and an explanation for the dimensionality of space-time.

The model makes use of symmetries and dualities that are inherent of String Theory, namely the T-Duality and the Hagedorn Temperature. The structure of the remainder of this chapter is as follows: first we will discuss the evolution of strings in curved space-time. This will be our theory of gravity, instead of General Relativity. Then we will discuss String Thermodynamics and the emergence of the Hagedorn phase. Finally, we will discuss toroidal compactification and T-duality. With all these ingredients, we will be able to, at the end of this chapter, give a general view of the model, questions that it resolves and open problems.

3.1 Strings in time-dependent backgrounds

A closed string in a background generated by its bosonic, massless modes is described by a nonlinear sigma model:

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,[\sqrt{-\gamma}\gamma^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu] \tag{3.1}$$

where γ^{ab} is the world-sheet metric, $(2\pi\alpha')$ is the inverse string tension, $G_{\mu\nu}$ is the background space-time metric and $B_{\mu\nu}$ is the background antisymmetric tensor. The X^μ with $\mu = 0,...,D-1$ are the full D-dimensional space-time coordinates. The σ^a with $\sigma^0 \equiv \tau$ and

$\sigma^1 \equiv \sigma$ are the world-sheet coordinates.

To the above action, one may add a topological term

$$S_\phi = -\frac{1}{4\pi}\int d^2\sigma\,\sqrt{\gamma}\phi(X)R^{(2)} \tag{3.2}$$

where ϕ is the background dilaton and $R^{(2)}$ is the world-sheet Ricci scalar. With this setup, the string coupling is given in terms of the vacuum expectation value of the dilaton $g_s = e^{\phi_0}$. The background fields $G_{\mu\nu}, B_{\mu\nu}$ and ϕ are realized as couplings of the nonlinear sigma model.

Keeping terms at the tree level in α', the equations of motion of the above metric can be derived from those of the low-energy effective action of supergravity D space-time dimensions:

$$S_0 = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-G}e^{-2\phi}\left(R + c + 4(\nabla\phi)^2\right) \tag{3.3}$$

where the constant c is defined as:

$$\frac{2}{3}(d_c - N - N_c). \tag{3.4}$$

In this definition, d_c is the critical spatial dimension and N is the number of spatial dimensions in which strings propagate. In case N is not the critical spatial dimension of strings (that is, if N<25 for strings or N<9 for superstrings), either we have N_c compact internal dimensions or we are dealing with non-critical strings. Thus, c reflects the criticality of the theory.

For the sake of simplicity, we assume spatial dimensions to be toroidal with lengths $a_i = \exp\lambda_i$ (this condition can be relaxed, as shown by Easther et al. [34]. We also assume the background fields to be evolving slowly. This guarantees that we can ignore higher derivatives (α') corrections.

With all this in mind, let us consider the metric and the dilaton field to have to form:

$$\begin{gathered} ds^2 = -dt^2 + \sum_{i=1}^{N} a_i^2(t)dx_i{}^2 \\ a_i = e^{\lambda_i(t)}, \phi = \phi(t). \end{gathered} \tag{3.5}$$

It is also useful to introduce the shifted and rescaled dilaton φ that absorbs the space volume factor:

$$\varphi \equiv 2\phi - \sum_{i=1}^{N} \lambda_i, \sqrt{-G}e^{-2\phi} = e^{-\varphi} \tag{3.6}$$

With these definitions, the action takes the form

$$S_0 = -\int dt\, e^{-\varphi}\sqrt{-G_{00}}[c - G^{00}\sum_{i=1}^{N}\dot{\lambda}^2 + G^{00}\dot{\varphi}^2] \tag{3.7}$$

The term G_{00} was kept as we will need to vary the action with respect to it in order to obtain the full set of equations of motion (in these equations, G_{00} will be set to -1). Note the invariance of the action under the duality transformation

$$\lambda_i \to -\lambda_i, \phi \to \phi - \lambda_i, \varphi \to \varphi. \tag{3.8}$$

At this point a pause is required to make the important remark that we are working in the adiabatic approximation, that is, the background fields are evolving slowly enough that corrections from higher derivatives (α') terms can be neglected. In this regime, the global aspects of the dynamics of the theory may be described by focusing on massless modes, which justifies the truncation of the action on tree level terms.

So far, we have worked in the absence of stringy matter. In the same way that we replace GR by String Theory in our attempt to construct a quantum cosmological model, we must also replace our classical description of the matter content of the universe as a perfect fluid by some kind of stringy content. In our case, this content will be a gas of almost free strings modes in thermal equilibrium at temperature β^{-1}.

Before proceeding to coupling the gravitational model we have constructed this far with a gas of strings, let us study such gas in detail.

3.2 Thermodynamics of a String Gas

An important aspect of String Theory is that it contains an upper bound on the allowed value of the temperature of a string gas, the Hagedorn Temperature [35], above which the canonical ensemble approach to thermodynamics is invalid as it leads to a divergent partition function. Such a limiting temperature would play a very important role in cosmology, since it prohibits, at first glance, an "‘infinitely hot"’ big bang. Other consequences are not so obvious, so this subject deserves a detailed study.

Historically, this limiting temperature was first observed in the dual theory of hadrons.

Quantum Chromodynamics (QCD) offered a physical interpretation of this phenomena: instead of being an actual limiting temperature, its presence suggests a change in the relevant degrees of freedom in terms of which the system is relatively simple: is was related to a "‘deconfinement"’ transition in which the hadrons liberate their quark-gluon constituents [36], [37]. Once we stop describing the system in terms of the hadrons and start describing it in terms of the quarks and gluons there is no bound on the temperature.

Indeed, in the case of ten dimensional compactified strings, the divergent partition function does not necessarily mean that the Hagedorn temperature is physically unsurpassable: physical quantities such as energy density and specific heat are finite at Hagedorn temperature. However, for strings with some uncompactified space-time dimensions, the Hagedorn temperature seems to be in fact the maximally attainable temperature, as in such cases the energy diverges as we approach the Hagedorn temperature. This happens because by "‘putting strings in a box"’, no matter how large the box is, excites the winding modes of the string, which are not present in the infinite volume limit. From a cosmological point of view, it also makes more sense to work with very large but finite boxes than with infinite space. As an examples, note that to put an infinite box in a thermal bath with an arbitrarily small temperature one would need infinite total energy.

In this case, what happens is that close to the Hagedorn temperature, all the energy added to the system is utilized to create the large number of new particles becoming available as the energy increases instead of increasing the energy of the particles already present al lower energies, keeping the temperature constant.

Let us move to a more detailed view of these stringy phenomena.

3.2.1 The Random Walk Picture

Let us begin with an intuitive geometrical approach for the density of states of a highly excited string from a random walk, following the approach by Abel et al. [38].

Think of a highly excited closed string as a random walk in target space, so that the energy ϵ of the string is proportional to the length of the random walk. The number of random walks that start in the same fixed point grows as $\exp(\beta_s \epsilon)$. This term is responsible for the bulk of energy of highly excited strings. Since we are working with closed strings, the random walk must close on itself. This overcounts by a factor of roughly the volume of the walk, which we will denote by $V_{walk} = W$. Since the random walk may be translated in the volume V of space, we must add a V term. Finally, we must insert a factor of $1/\epsilon$ since any point in the string can be the starting point of the random walk. This gives us:

$$\omega_{cl} \approx V \cdot \frac{1}{\epsilon} \cdot \frac{e^{\beta_s \epsilon}}{W(\epsilon)} \tag{3.9}$$

This expression has two characteristic limiting cases: the volume of the random walk is of order $\epsilon^{d/2}$ when it is well-contained in d spatial dimensions (that is, $L >> \sqrt{\epsilon}$), whereas it saturates at order V when it is space-filling (that is, $L << \sqrt{\epsilon}$). The first case gives us, for d non-compact dimensions:

$$\omega_{cl}(\epsilon)/V \approx \frac{e^{\beta_s \epsilon}}{\epsilon^{1+d/2}} \tag{3.10}$$

whereas the second case gives us, in a compact space

$$\omega_{cl}(\epsilon) = \frac{e^{\beta_s \epsilon}}{\epsilon}. \tag{3.11}$$

equation (3.11) gives the exact leading term of the density of states for highly excited strings. With these results, we may now proceed to the transition from a single long string to a gas of long strings.

Consider the formal partition function of the gas:

$$Z(\beta) = \int_0^\infty dE\, \Omega(E) e^{-\beta E} \tag{3.12}$$

A consistent canonical ensemble may only be defined for temperatures that lead to $\beta > \beta_S$, since for temperatures above the Hagedorn temperature β_S the partition functions diverges.

Defining $Z(\beta)$ by analytic continuation in the complex β plane, the density of states may be written as the inverse Laplace transform

$$\Omega(E) = \int_{C_\beta} \frac{d\beta}{2\pi i}\, e^{\beta E} Z(\beta). \tag{3.13}$$

The contour C_β is taken parallel to the imaginary axis and to the right of all singularities of $Z(\beta)$. One possible route, from this point is to explicitly evaluate the integral equation (3.13) in the one loop approximation [39]. However, near the Hagedorn phase the dominating long strings are macroscopic and behave quasi-classically. With this in mind, we may assume Maxwell-Boltzmann statistics and then write $Z(\beta) = \exp z(\beta)$, where $z(\beta)$ is the single-string partition function, which can be calculated from the single string density of states:

$$z(\beta) = \int_0^\infty d\epsilon\, \omega(\epsilon) e^{-\beta\epsilon} \tag{3.14}$$

Direct calculation gives the behavior of $z(\beta)$ near the Hagedorn singularity $\beta = \beta_S$:

$$z(\beta) \approx f(\beta - \beta_S)^{\gamma} [\log(\beta - \beta_S)]^{\delta} \tag{3.15}$$

where $\delta = 1$ if γ is a non-negative integer and $\delta = 0$ otherwise. This is precisely a critical behavior of the Hagedorn density as a function of the formal canonical temperature $1/\beta$ with a critical exponent given by γ.

The integral equation (3.13) may be evaluated in different approximations, depending on the different regimes one is interested. Whenever a saddle-point approximation is applicable, an equivalence between the canonical and microcanonical ensembles is find, with positive and large specific heat. A necessary condition for this is that $\gamma \leq 1$, ensuring that the canonical internal energy $E(\beta) \approx \partial_{\beta} z(\beta)$ diverges at the Hagedorn singularity.

When such approximation is not available, one can either proceed to a direct evaluation of the integral in special marginal cases or find a complementary approximation.

In particular, systems with close-packing of random walks (high energy in a fixed volume) have $\gamma = -1$ for open strings and $\gamma = 0$ for closed strings. In the first case, the saddle point approximation applies and the gas of open strings has a canonical behavior, positive specific heat and

$$\Omega(E)_{\text{open}} \approx \exp(\beta_S E + C\sqrt{E}) \tag{3.16}$$

where C is a constant. For closed strings, the leading singularity at very high energy and finite volume $V = L^d$ is always a simple pole of the partition function at the Hagedorn singularity

$$Z(\beta) = (\beta - \beta_S)^{-1} \cdot Z(\beta)_{\text{regular}} \tag{3.17}$$

This pole alone generates a multi-string density

$$\Omega(E)_{\text{closed}} \approx \exp(\beta_S E + \rho_S V) \tag{3.18}$$

where $\rho_S = O(1)$ in string units. Thus, for closed strings the specific heat is still infinite in this approximation. The contribution of the sub-leading singularities turns the thermodynamics into a weakly limiting behavior with positive specific heat and exponentially suppressed corrections to the linear entropy law.

A more technical remark that must be made is about the principle of asymptotic darkness [40]. This principle states that black holes dominate the extreme high-energy regime of theories that incorporate gravity. String Theory has its own correspondence principle, the Horowitz-Polchinski Correspondence Principle [41] [42]. In particular, the entropy of Schwarzschild black holes in d dimensions scales as

$$S \approx E(g_s^2 E)^{\frac{1}{d-2}}$$

eventually dominating over the Hagedorn degeneracy for $g_s^2 E > 1$. At this point the Hagedorn plateau must end and drop to lower temperatures $T(E) \approx E^{1/(2-d)}$ which is a phase of negative specific heat for $d < 2$. However, the black hole will eventually acquire the size of the box, so this phase can not continue to arbitrarily high energies. The threshold coincides with the Jeans length entering inside the box, coinciding to an energy

$$E_L \approx \frac{L^{d-2}}{g_s^2}.$$

The bound $E < E_L$ implies that no thermodynamical limit (large volume with constant energy density) is these systems, since $E_l/V \to 0$. An exit out of this is provided by an appropriate infrared regularization or by cosmological particularities, such as T-duality.

As a particular and important case of what has been discussed so far through this session, we are going to compute the density of levels for strings.

3.2.2 Highly Excited Strings: Explicit Calculation

Let us take, initially, the total number of open string states with $\alpha' M^2 = n-1$, which we shall denote d_n. For more details on the String Theory spectrum, the reader is referred for the appendix. If we define our number operator N as

$$N = \sum_{n=1}^{\infty} n\alpha_{-n} \cdot \alpha_n$$

so d_n is the coefficient of w^n on $tr\, w^N$. Actually, there are $\alpha_n^i, i = 1,2,...,24$, corresponding to the 24 transverse oscillators related to physical states. Instead of calculating each d_n individually, our strategy to obtain the density of levels will be to calculate the generating function

$$G(w) = \sum_{n=0}^{\infty} d_n w^n = tr\, w^N. \tag{3.19}$$

Note that:

$$tr\, w^N = \prod_{n=1}^{\infty} tr\, w^{\alpha_{-n}\cdot\alpha_n} = \prod_{n=1}^{\infty} (1-w^n)^{-24} = [f(w)]^{-24} \tag{3.20}$$

with

$$f(w) = \prod_{n=1}^{\infty} (1-w^n). \tag{3.21}$$

The exchange $w \to e^{2\pi i\tau}$, relates the function $f(w)$ with Dedekind's Eta function

$$\eta(\tau) = e^{(i\pi\tau/12)} \prod_{n=1}^{\infty} (1-e^{2\pi i n\tau}). \tag{3.22}$$

Dedekind's Eta function is known to obey the modular transformation:

$$\eta(-1/\tau) = (-i\tau)^{(1/2)}\eta(\tau). \tag{3.23}$$

By applying this transformation to $f(w)$, we are able to obtain the so called Hardy-Ramanujan formula:

$$\begin{gathered} f(w) = \left(\frac{-2\pi}{\log w}\right)^{1/2} w^{-1/24} q^{1/12} f(q^2) \\ q = \exp\left(\frac{2\pi^2}{\log w}\right) \end{gathered} \tag{3.24}$$

Now we may find an asymptotic expression for $f(w)$ when $w \to 1 (q \to 0)$:

$$f(w) \sim A(1-w)^{-1/2} \exp\left(-\frac{\pi^2}{6(1-w)}\right) \tag{3.25}$$

Once we have found the asymptotic expression for $f(w)$, we may return to our initial problem and find out the behavior of d_n for large values of n. From the generating function $G(w) = \sum d_n w^n$, we obtain d_n through a contour integral over a small circle centered at the origin

$$d_n = \frac{1}{2\pi i}\oint \frac{G(w)}{w^{n+1}} dw. \tag{3.26}$$

Since $G(w)$ goes to zero very rapidly for $w \to 1$ and w^{n+1} is very small for large values of n and $w < 1$, there is a well defined saddle-point for w close to 1. A saddle point evaluation of this integral gives for the limit $n \to \infty$

$$d_n = (\text{const})\, n^{-27/4} \exp(4\pi\sqrt{n}) \tag{3.27}$$

Using $n \sim \alpha' m^2$, we arrive at the asymptotic density of levels:

$$\rho(m) \sim m^{-25/2} \exp\left(\frac{4\pi}{\sqrt{\alpha'}} m\right) = m^{-25/2} \exp\left(\beta_H m\right). \tag{3.28}$$

This result will be used in the next chapter for an explicit calculation of the partition function of a gas of strings. Now we shall take a break on strings thermodynamics and investigate another aspect of String Theory that is important for the String Gas scenario: T-Duality.

3.3 Toroidal compactification and T-duality

String Theory predicts the existence of more than the 4 observed space-time dimensions. A possible explanation for why we don't observe the extra dimensions is that they may be compactified. The simplest compactification of string theory is the so called "‘toroidal compactification"’, in which one or more dimensions are periodically identified.

3.3.1 Kaluza-Klein Reduction and Winding Modes

For the sake of simplicity, we will begin with the compactification of a single space dimension, the so called *Kaluza-Klein Reduction*. The space dimensions will take the form

$$\mathbf{R}^{1,24} \times \mathbf{S}^1 \tag{3.29}$$

with the coordinate from the compactified dimension having the form

$$X^{25} \equiv X^{25} \tag{3.30}$$

The goal is to understand the Physics viewed by on observer who lives in the non-compact $\mathbf{R}^{24}$ space, so we will be initially interested in the Physics at length scales $>> R$.

From the point of view of the string world-sheet, the compactification will affect the dynamics in two ways.

First, the requirement that the string wave function is single valued will result in a quantization of the spatial momentum of the string. Since the string wavefunction includes factors of $e^{ip\cdot X}$, the momentum can no longer take any value, but has to obey

$$p^{25} = \frac{n}{R} \qquad n \in \mathbf{Z} \tag{3.31}$$

Second, we will be able to allow more general boundary conditions for the mode expansion of X than the ones imposed without compactified spatial dimensions (refer for the appendix for string quantization in no compactfied space-times). In particular, we may relax the boundary condition $X(\sigma+2\pi) = X(\sigma)$ to

$$X^{25}(\sigma+2\pi) = X^{25}(\sigma) + 2\pi mR \qquad m \in \mathbf{Z} \tag{3.32}$$

This integer m tells us how many times the string winds around $\mathbf{S}^1$ and is called winding number. This winding phenomenon reflects the fact that when there are compactified dimensions not all closed strings can be reduced continuously to zero size. Imagine, for example, a world with only two spatial dimensions, one of which is compactified, just like the surface of an infinitely long cylinder.

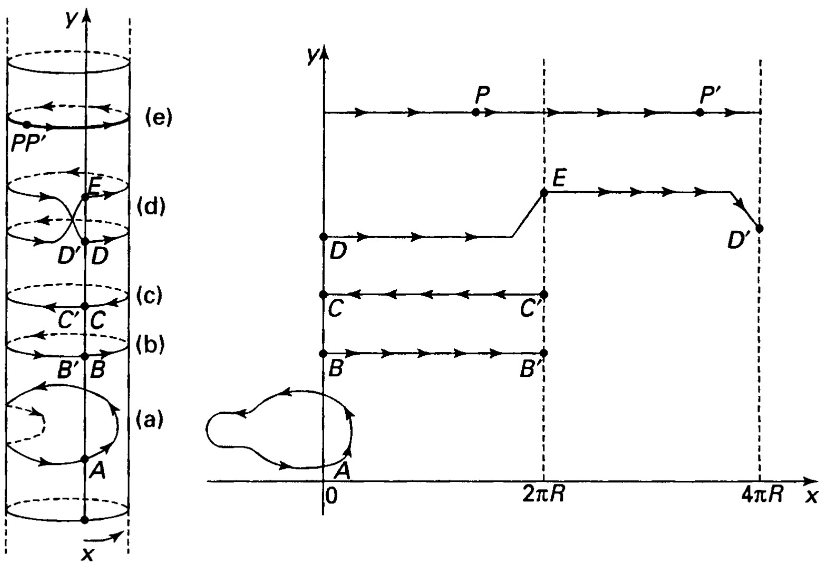


Figure 3.1 Possibilities for winding states. Figure Credits: Barton Zwiebach. *A first course in string theory*; 2nd ed. Cambridge Univ. Press, Cambridge, 2009.

In this imaginary world, we call x the coordinate on the compactified dimension and y the remaining coordinate. The right-hand side of Figure 3.1 shows the (x, y) planification of the cylinder, known as the *covering space* of the cylinder.

Consider, for example, the string labeled as A. It is clear from the figure that it may be continuously contracted down to a point, since it does not wind around the compact dimension. So, we say that is has winding number zero.

Now, consider the strings labeled as B and C. They can not be contracted to a point without a cut, since they do wind the the compact dimension once. Because of that, they have

±1 as winding numbers, depending on the orientation of the winding: b winds the circle in the direction of positive x so it has winding number $+1$, while c winds the circle in the opposite direction so it has winding number -1. Note that strings that wind around the cylinder appear in the covering space as open strings. However, not every open string in the covering space represents a wound closed string, some represent open strings on the cylinder.

Strings d and e both wind the cylinder twice, with the difference that the y coordinate of d is not constant. However, they both have winding number $+2$. It is important to note that even though the points P and P' appear to be the same on the cylinder, they are the same point on the string. This becomes clear on the covering space.

In general, we say that a string has winding number m, with m integer, if it wraps m times around the compactified dimension in the positive direction of the dimension coordinate.

3.3.2 The Compactified Spectrum and T-Duality

In a similar fashion to what is made through string quantization (again the reader is referred for the appendix for more details), we make the mode expansion of the periodic field X^{25}:

$$X^{25}(\sigma,\tau) = x^{25} + \frac{\alpha' n}{R}\tau + mR\sigma + \text{oscillator modes.} \tag{3.33}$$

Note that this expansion incorporates the possibility of winding modes together with the oscillator modes and the now quantized momentum. The next step is to split $X^{25}(\sigma,\tau)$ into right-moving and left-moving parts. But before that, let us introduce the quantities

$$\begin{aligned} p_L &= \frac{n}{R} + \frac{mR}{\alpha'} \\ p_R &= \frac{n}{R} - mR\alpha'. \end{aligned} \tag{3.34}$$

With these definitions, we may write

$$X^{25}(\sigma,\tau) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-) \tag{3.35}$$

where

$$\begin{aligned} X_L^{25}(\sigma^+) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_L\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\tilde{\alpha}_n^{25}e^{-in\sigma^+} \\ X_R^{25}(\sigma^-) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_R\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\alpha_n^{25}e^{-in\sigma^-} \end{aligned} \tag{3.36}$$

The mode expansion is not changed for the other X^i over $\mathbf{R}^{1,24}$.

What will the spectrum of this theory look like to an observer living in $d = 25$ non-compact directions? Each particle state will be described by a momentum $p^\mu, \mu = 0, ..., 24$ and will have mass given by

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu \tag{3.37}$$

Again, the mass of the particles will be fixed in terms of the oscillator modes of the string by the L_0 and $\tilde{L}_0$ equations. These now give

$$M^2 = p_L^2 = \frac{4}{\alpha'}(N-1) = p_R^2 + \frac{4}{\alpha'}(\tilde{N}-1) \tag{3.38}$$

where N and $\tilde{N}$ are the levels defined in light-cone quantization and the -1 factors are the necessary normal ordering coefficients. Note the presence of the momentum and winding terms around $\mathbf{S}^1$ on the right-hand side of these equations. In this case, level matching no longer tells us that $N = \tilde{N}$, but instead

$$N - \tilde{N} = nm \tag{3.39}$$

A full expansion of the mass formula, thus, yields

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.40}$$

The extra terms in this equation tell us that winding and momentum modes will contribute to the energy spectrum of the theory.

A remarkable property of this spectrum it that it is invariant under the exchange

$$R \leftrightarrow \frac{\alpha'}{R} \qquad m \leftrightarrow n \tag{3.41}$$

As a result, strings with radius R will have the same spectrum of strings with radius α'/R! In the limit of $R \to \infty$, the winding modes become highly massive and thus irrelevant for the low-energy dynamics. The momentum modes, on the other hand, become very light, forming a continuum in the strict limit. This continuum of energy states is what is meant by the existence of a non-compact direction in space.

In the opposite limit, $R \to 0$, the momentum modes become heavy and thus can be ignored (it takes too much energy to get anything moving on $\mathbf{S}^1$. The winding modes, however, become light and start to form a continuum.

This equivalence of the string spectrum on circles of radii R and α'/R extends to the full conformal field theory and hence to string interactions. Strings can not tell the difference between very large and very small circles! Instead, it exhibits a minimum length scale: as you try to probe distance scales smaller and smaller, at $R = \sqrt{\alpha'}$ the theory starts to act as if the circle is growing again, with winding modes playing the role of momentum modes. This is the so called *T-duality of String Theory*.

It is important to note that this symmetry is not just a symmetry of the spectrum, but is a symmetry of the whole String Theory: any physical process computed for the strings in a box of size R is identical to a dual physical process computed for a box of dual radius. Not only the spectrum is invariant, but also all the scattering amplitudes for dual processes.

Winding modes and T-Duality will be important in the description of the string gas scenario.

3.4 The String Gas Scenario

No we can go ahead with the construction of the cosmological scenario and couple a gas of strings (our matter content) to the gravitational action. The contribution of this gas for the action is given by

$$S_m = \int dt \sqrt{-G_{00}} F(\lambda_i, \beta\sqrt{-G_{00}}) \tag{3.42}$$

where F is the one loop free energy of the gas. With this, we finally arrive at the full action of the model:

$$S = S_0 + S_m = -\int dt e^{-\varphi} \sqrt{-G_{00}} [c - G^{00} \sum_{i=1}^{N} \dot{\lambda}^2 + G^{00} \dot{\varphi}^2 + \sqrt{-G_{00}} F(\lambda_i, \beta\sqrt{-G_{00}})] \tag{3.43}$$

The variation of the above action with respect to λ_i, ϕ and G_{00} yields the following equations of motion:

$$c - \sum_{i=1}^{N} \dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi} E \tag{3.44}$$

$$\ddot{\lambda}_i - \dot{\varphi}\dot{\lambda}_i = \frac{1}{2} e^{\varphi} P_i \tag{3.45}$$

$$\ddot{\varphi} - \sum_{i=1}^{N} \dot{\lambda_i}^2 = \frac{1}{2} e^{\varphi} E \tag{3.46}$$

where

$$E = -2\frac{\delta S_m}{\delta G_{00}} = F + \beta \frac{\partial F}{\partial \beta} \tag{3.47}$$

$$P_i = -\frac{\delta S_m}{\delta \lambda_i} = -\frac{\partial F}{\partial \lambda_i} \tag{3.48}$$

are the total energy of the matter and the pressure in the ith direction, respectively. Note the modified conservation law for the energy that arises from these equations:

$$\dot{E} + \sum_{i=1}^{N} \dot{\lambda}_i P_i = 0. \tag{3.49}$$

Since the free energy F is a function of λ and $\beta(t)$, this is equivalent to the conservation of the entropy $S = \beta^2 \partial F / \partial \beta$, so that the temperature will "‘adjust"’ itself to the radius $\lambda_i(t)$ in order to maintain the entropy constant. This means it is possible to express β in terms of λ_i if we solve the adiabaticity condition, what makes it possible to express the Energy as a function of λ alone

$$E(\lambda) = E(\lambda, \beta(\lambda)).$$

Now we may turn back to our original dilaton field ϕ and rewrite the equations of motion:

$$c - \sum_{i=1}^{N} \dot{\lambda_i}^2 + (2\dot{\phi} - \sum_{i=1}^{N} \dot{\lambda}_i)^2 = e^{2\phi} \rho \tag{3.50}$$

$$\ddot{\lambda}_j - (2\dot{\phi} - \sum_{i=1}^{N} \dot{\lambda}_i)\dot{\lambda}_j = \frac{1}{2} e^{2\phi} p_j \tag{3.51}$$

$$2\ddot{\phi} - \sum_{i=1}^{N} \ddot{\lambda}_i - \sum_{i=1}^{N} \dot{\lambda_i}^2 = \frac{1}{2} e^{2\phi} \rho \tag{3.52}$$

with the definitions of the pressure density $p_i = P_i / V$, the energy density $\rho = E/V$ and the space volume $V = \exp \sum \lambda_i$.

We may now try to compare these with the standard FRW scenario. For that, we begin by setting the dilaton ϕ to be a constant. Then, equation (3.50) for flat space becomes

$$\left(\frac{\dot{a}}{a}\right) = \dot{\lambda}^2 = \frac{-c}{N^2 - N} + G\rho \tag{3.53}$$

where G is a constant and the first term plays the role of a cosmological constant! For $c = 0$, the compatibility of solutions of equation (3.53) with equation (3.51) and equation (3.52) require

$$\sum_{i=1}^{N} p_i = \rho. \tag{3.54}$$

That is: matter with vanishing energy momentum trace, like a gas of massless particles in thermodynamical equilibrium, which is just like FRW radiation. At high temperatures this conditions is not satisfied and the stringy model departures from General Relativity.

More generally, we can develop a further study of the critical case by setting $c = 0$ directly in the equations (3.44)-(3.46) above. If we assume isotropy (that is, all λ_i are equal to each other and denoted by λ) so that $a = \exp\lambda$ is the cosmological scale factor, we get

$$-N\dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi}E, \tag{3.55}$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = \frac{1}{2}e^{\varphi}P, \tag{3.56}$$

$$\ddot{\varphi} - N\dot{\lambda}^2 = \frac{1}{2}e^{\varphi}E, \tag{3.57}$$

where $P = -N^{-1}\partial E/\partial\lambda$. Therefore, if we want to solve this system of equations, we need to specify the function $E(\lambda)$ as well as provide initial conditions for λ, φ and $\dot{\varphi}$. In the next chapter we are going to study a similar system motivated by this set up. Before that, let us focus on some features of the SGC scenario.

3.4.1 Winding Modes and the Dimensionality of Spacetime

As we have seen, in string theory models there is possibility that some dimensions are compactified. Compactified dimensions make it possible to justify the difference between the spacetime dimensionality predicted by the theory and the dimensionality with we observe in common life. However, even tough one could construct a model where some of the dimensions are large and some are compact, there was a lack of explanation for this separation into large and compactified dimensions.

For a long time, the community has tried to answer why are there compact dimensions. The string gas scenario subverts this question by purposing that all dimensions started small (compactified) and through annihilation of winding modes they became large. The original argument is as follows.

First, note that an increase on the volume of the system (increase of R) will increase the energy of the winding modes. That means a smaller fraction of the phase space will be available to them. This means winding modes will prevent expansion (more energy will be necessary for expansion when winding modes are present). These modes will contribute with negative pressure for the gas.

If we assume that the gas is in thermal equilibrium, an expansion of the radius will cause the winding modes in the gas to decay. Expansion will continue as long as there is thermal equilibrium to favor less and less of winding states with larger and larger radii. If thermal equilibrium is not maintained, winding modes will not decay and will still be present, what would slow down the expansion (eventually stop it).

Now let us think of the universe as a box. Actually, since superstring theory predicts the existence of 9 spatial dimensions, let us thing of the universe as a nine-dimensional box with equal size in each direction. Consider that the box has already expanded a little but we still have a large number of winding modes. As we increase the size of the box, the winding modes will start to behave like classical strings with a width of the order of one in Planck units, which go from one side of the box to the other (they will have a width at the same order of the size of the box).

Thermal equilibrium will be maintained through processes like

$$w + \overline{w} \leftrightarrow \text{unwound states}$$

where $\overline{w}$ denotes an state with the opposite winding number of state w. Such processes occur if both the winding states have the same Planck length of one another.

As the winding modes propagate, they will span a two-dimensional worldsheet (which will have an effective thickness of the order of one in Planck units). A required condition for two winding modes to interact is that there is an intersection between their worldsheets. In this scenario, a large number of dimensions will make it harder for winding modes to interact. If the universe has, as in our example, 9 spatial dimensions, we are left with $2+2 < 9+1$ and the worldsheets will generically not intersect. The strings will just miss one another. If winding modes are not able to find one another, they can not annihilate each other and maintain the thermal equilibrium. Since equilibrium is not maintained, winding modes will stop the expansion.

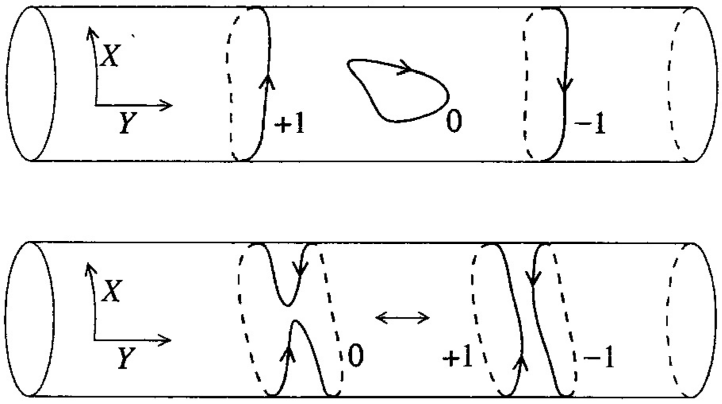


Figure 3.2 Closed oriented strings of winding numbers w=-1,0,-1 and the transition from a w=0 string to w=-1 and w=+1 strings. In this case, we have one compact dimension X and one non-compact dimension Y. Figure Credits: Joseph Polchinski. *String Theory: An Introduction to the Bosonic String (Cambridge Monographs on Mathematical Physics)*, volume 1. 1998.

In this mechanism, the largest extended spacetime dimensionality consistent with maintaining thermal equilibrium is 4, for which 2+2=3+1 and therefore strings will generically find each other and equilibrate. In this case, winding modes will annihilate and 3 spatial dimensions may expand, generating a box in which six dimensions are of the order of the Planck length and three of the dimensions decompactify, giving rise to the universe as observed today.

This was the original argument by Brandenberger and Vafa in the paper that gave birth to this scenario [20]. As was noted by Easther et al., this scenario has a problem: there is a singularity in past finite time and the dilaton rolls monotonically towards weak coupling, making it unlikely for strings to annihilate if one waits too much, even if they do happen to intersect. Because of that, there is only a small window of time for the necessary fluctuations to take place and as a result three large dimensions are no statistically favored [43]. In [44] Greene et al. study the dynamics of a string gas coupled to a modified gravity action set up to avoid singularities, what results in bouncing and cyclic cosmologies. However, even in these scenarios the BV decompactification scenario is not operative. It still an open challenge to construct a stringy model that does preferentially decompactify in three dimensions.

3.4.2 Causal Structure and Inflation

One notable aspect of String Gas Cosmology is that it is able to account for the causal structure we observe in the universe. As we have seen, points that seem to be causally disconnected nowadays happen to appear in the CMG spectrum in thermal equilibrium. One of the successes of the mechanism of inflation was its ability to explain that. String Gas Cosmology provides a different but also as effective way to explain such thermal equilibrium.

First, let us recall how the causal structure is generated in inflationary cosmology. During inflation, the Hubble radius $H^{-1}(t)$ is constant. The Hubble radius is an important divisory in the theory of cosmological perturbations [45], [46]: is scales are smaller than the Hubble radius, they will oscillate; if they are larger than the Hubble radius they will be frozen and will not be affected by microphysics. Because of that causal microphysical processes can generate fluctuations only on sub-Hubble scales.

In the inflationary scenario, fixed comoving scales that are probed in nowadays experiments have wave-lengths which start out smaller than the Hubble radius at the beginning of inflation and expand exponentially. They exit the Hubble radius at some time $t_i(k)$ and propagate with a wavelength larger than the Hubble radius until the reenter $H^{-1}(t)$ at $t_f(k)$ and become in causal contact again.

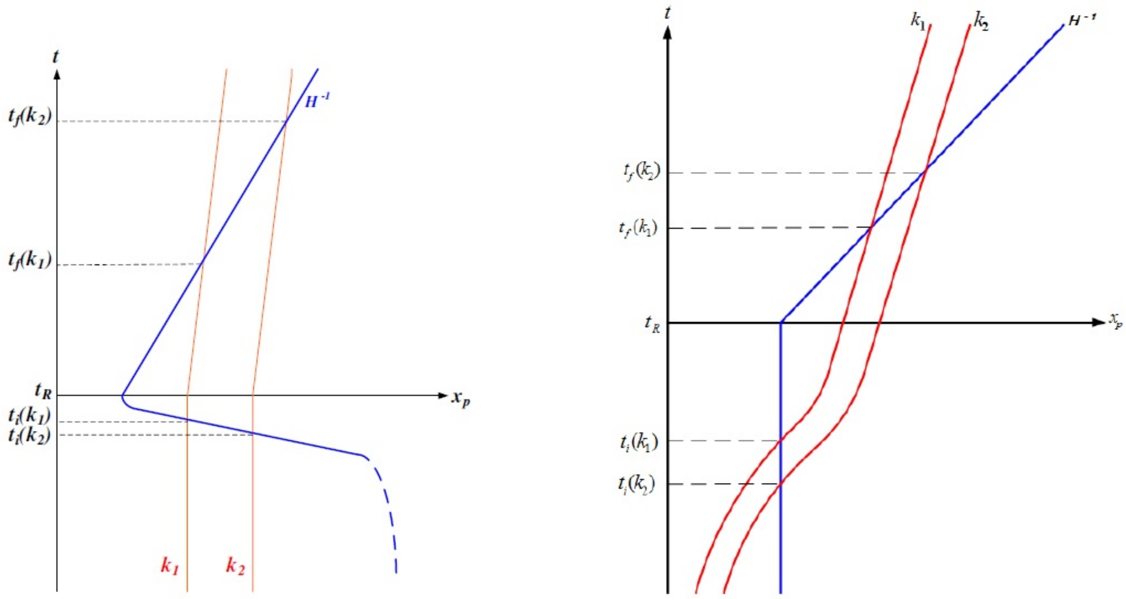


Figure 3.3 Causal structure of an inflationary universe and of String Gas Cosmology

On the other hand, in the String Gas scenario (suppose for now no inflationary phase in SGC), the Hubble radius is infinite deep in the Hagedorn phase. During the expansion of the universe, the Hubble radius decreases to microscopic value and then turns around and starts growing linearly. Physical wavelengths corresponding to fixed comoving scales are constant during the Hagedorn phase. Because of that, all scales on which fluctuations are measured nowadays were inside Hubble radius on the Hagedorn phase. When in radiation era, physical wavelengths will grow as $\sqrt{t}$, entering Hubble radius at a late time $t_f(k)$.

Note that in inflationary cosmology any thermal fluctuations that existed before inflation will suffer red-shift, leaving a quantum vacuum state, while in the Hagedorn phase of SGC matter is in a thermal state. Thus, the SGC and the inflationary scenario have different generation mechanisms for fluctuations: while in inflationary cosmology fluctuations originate as quantum vacuum perturbations [47], [48], in SGC we find thermal fluctuations of the string gas as sources of inhomogeneities.

It has already been shown [49], [50], [51] that the spectrum of scalar and tensor modes predicted by string thermodynamics in the Hagedorn phase of SGC is almost scale-invariant, as measured by CMB and also predicted by the inflationary scenario.

CHAPTER 4

Hagedorn Inflation of String Gases

4.1 The Relic Problem

We have seen in the last chapter that the string gas scenario is capable of generating a causal structure for the universe that is inflation free. However, the inflationary mechanism is also capable of explaining other issues in Standard Big Bang Cosmology, like why is the current universe so close to be flat and why do we measure such a low density of cosmological relics. At high temperatures, such as the Hagedorn temperature ($T_H \simeq 10^{16}$ GeV, which is close to GUT scale), an overproduction of unwanted relics take place.

Magnetic monopoles are an example of such relics. As shown in [52], [53], any Grand Unification Theory (GUT) that includes electromagnetism will produce super-heavy magnetic monopoles. In fact, had the universe simply cooled down from a temperature of $T \simeq 10^{15}$ GeV, an abundance of order $\Omega_M \simeq 10^{13}$ would be observed for magnetic monopoles [52].

Another example is the gravitino. The gravitino is a spin 3/2 fermion, that is the supersymmetric partner of the graviton, the particle that mediates supergravity interactions [54]. It is also a good candidate for dark matter, due to its mass (that ranges from TeV to GeV) and long lifetime. The gravitino has the interesting fact that it is may not be produced in the early universe, but it may also be produced during radiation-dominated era, as a thermal relic. The presence of thermally produced gravitinos impose bounds on the allowed maximal temperature in the radiation dominated era. It occurs that a simple cooldown from the Hagedorn phase in SGC to the radiation dominated era would violate theses bounds.

In such scenario, one is left with two alternatives: either find some other mechanism responsible for solving these issues or try to incorporate inflation in the SGC scenario. A mechanism for inflation in the Hagedorn phase was purposed by Abel et al. [55]. In this work, we propose a new mechanism, based on the decay of excited strings.

4.2 String Gas Thermodynamics Revisited

We want to evaluate the partition function for a single string placed in a box of volume V. We know that string states are obtained by the action of the light-cone creation operators on the momentum eigenstates (see Appendix A).

We are going to label string states by their occupation numbers $\lambda_{n,i}$ and their momenta p, so that a set of basis states can be written as

$$|\lambda, p\rangle = \prod_{n=1}^{\infty} \prod_{i=2}^{25} (a_n^i \dagger)^{\lambda_{n,i}} |p\rangle, \tag{4.1}$$

and the energy of such states is given by

$$E(\{\lambda_{n,i}\}, \vec{p}) = \sqrt{M^2(\{\lambda_{n,i}\}) + \vec{p}^2}. \tag{4.2}$$

The string partition function is obtained by summing over all states $|\lambda, p\rangle$, that is, over all spatial momenta $\vec{p}$ and all occupation numbers $\lambda_{n,i}$:

$$Z_{str} = \sum_{\alpha} \exp(-\beta E_\alpha) = \sum_{\lambda_{n,i}} \sum_{\vec{p}} \exp\left[-\beta \sqrt{M^2(\{\lambda_{n,i}\}) + \vec{p}^2}\right]. \tag{4.3}$$

The sum over the momentum results in the partition function Z for a relativistic particle of mass-squared $M^2(\{\lambda_{n,i}\})$, leaving us with

$$Z_{str} = \sum_{\lambda_{n,i}} Z(M^2(\{\lambda_{n,i}\})). \tag{4.4}$$

We can exchange the sum in the above expression for a integral over all levels dN if we know the correct level density. Since we know the energy density equation (3.28), we can use the relation that comes from string quantization $\alpha' M^2 \approx N$, so that $dN = 2\alpha' M dM$ to integrate over the mass M.

The partition function of a relativistic particle is know [56] to be given by

$$Z(M^2) \approx V e^{-\beta M} \left(\frac{M}{2\pi\beta}\right)^{\frac{d}{2}}, \tag{4.5}$$

that in our system of interest becomes

$$Z(M^2) \approx 2^{25/2} V (kTkT_H)^{25/2} (\sqrt{\alpha'} M)^{25/2} \exp\left(-4\pi\sqrt{\alpha'} M \frac{T_H}{T}\right). \tag{4.6}$$

We are then left with

$$Z_{\text{str}} \approx Z_0 + \int_{M_0}^{\infty} (\sqrt{\alpha'}M)^{-25/2} \exp(\beta_H M) Z(M^2)\, d(\sqrt{\alpha'}M). \tag{4.7}$$

Note that we have separated this integral by assuming that there is a mass M_0 for which equation (3.28) is valid (remember this result is valid for highly excited strings). We are interested in a regime where Z_0 is negligible compared to the integral term.

The integral equation (4.7) converges only for $T < T_H$, where:

$$Z_{\text{str}} \approx Z_0 + \frac{2^{11}}{\pi} V (kTkT_H)^{25/2} \left(\frac{T}{T_H - T}\right) \exp\left(-4\pi\sqrt{N_0}\left[\frac{T_H}{T} - 1\right]\right). \tag{4.8}$$

For T sufficiently close to T_H, thereof, the partition function of a single string will be approximated by

$$Z_{str} \approx \frac{2^{11}}{\pi} V (kT_H)^{25} \left(\frac{T_H}{T_H - T}\right). \tag{4.9}$$

From this partition function, we may compute some thermodynamical quantities. For example, the entropy will be given by

$$S \equiv -\left(\frac{\partial A}{\partial T}\right)_V \tag{4.10}$$

where

$$A = \left[\log V + \log\left(\frac{2^{11}}{\pi} V k^{25} T_H^{26}\right) - \log(T_H - T)\right](-kT) \tag{4.11}$$

so that

$$S = k\left[\log V + \log \nu V - \log(T_H - T) + \frac{T}{T_H - T}\right] \tag{4.12}$$

where $\nu = 2^{11} k^{25} T_H^{26}/\pi$. Note the divergence when $T \to T_H$.

The next natural step is to evolve from a single string to a gas of N strings. Since we are working near the Hagedorn temperature, where we can use Maxwell-Boltzmann, we can assume that our strings have very small coupling, so that we may use

$$Z_{gas} = \frac{(Z_{str})^N}{N!}$$

which yields

$$Z_{gas} = \left(\frac{2^{11}}{\pi}\right)^N \frac{V^N}{N!} \left(\frac{T_H}{T_H - T}\right)^N = \frac{C^N}{N!} \left(\frac{T_H}{T_H - T}\right)^N \tag{4.13}$$

With the gas partition function in hands, we are again able to calculate thermodynamical quantities. Our main interest right now will be in the gas energy.

From equation (4.13) we obtain

$$\begin{aligned} \log Z &= N\log C - \log N! + N\log\left(\frac{T_H}{T_H - T}\right) \\ &= N\log C - \log N! + N\log\left(\frac{\beta}{\beta-\beta_H}\right) \\ &= N\log C - \log N! + N[\log\beta - \log(\beta-\beta_H)] \\ &. \end{aligned} \tag{4.14}$$

So that

$$E \equiv -\frac{\partial}{\partial\beta}\log[Z(\beta)] = N\left[\frac{1}{\beta-\beta_H} - \frac{1}{\beta}\right] = N\left[\frac{\beta_H}{\beta(\beta-\beta_H)}\right] \tag{4.15}$$

Since we are working with β very close to β_H, the behavior of the energy will be that of

$$E \propto \frac{1}{\beta-\beta_H} \tag{4.16}$$

4.3 A New Mechanism for Inflation

4.3.1 The Setup

We are going to consider a universe with a matter content described by a gas of strings. The initial state of this gas is a thermal state in equilibrium at a temperature very close to the Hagedorn temperature. Strings in the gas will populate different levels of the string spectrum. The gas of strings, then, will be a mixture of different species.

The fundamental level of the string spectrum is massless. Because of that, a gas of non-excited strings can be thought as a gas of radiation. Excited levels are a mixture of modes that will satisfy specific state functions.

Note that the probability of a certain level being occupied at a temperature β^* is given by:

$$P(E_i) = \frac{e^{-\beta^* E_i}}{Z(\beta^*)}, \tag{4.17}$$

that is, is decreases exponentially as we consider highly excited levels. For the sake of simplicity we will only consider the fundamental level E_0 and the first excited level E_1.

Two effects will drive the evolution of this system: the first one is the decay of excited strings to the fundamental level; the second one is the cosmological redshift of the modes. By considering these effects together with FRW equation, we will be able to study the evolution of such universe.

4.3.2 The Equations of Motion

Let us begin by considering the cosmological redshift effect. First consider the mixture of two matter species. One is the radiation-like species corresponding to strings in their fundamental level. These will satisfy the state function $p_0 = w_{\mathrm{rad}}\rho_0 = (1/D)\rho_0$. The second one corresponds to string in the first excited level and will have a state function of the form $p_1 = w\rho_1$.

Taking the continuity equation for these species in D spatial dimensions yields:

$$\begin{aligned} \nabla_{\mu\nu} T^{\mu\nu} &= 0 \\ \Rightarrow \frac{d\rho_0}{dt} + D\frac{\dot{a}}{a}(P_0+\rho_0) + \frac{d\rho_1}{dt} + D\frac{\dot{a}}{a}(P_1+\rho_1) &= 0 \\ \Rightarrow \frac{d\rho_0}{dt} + (D+1)\frac{\dot{a}}{a}\rho_0 + \frac{d\rho_1}{dt} + D(1+w)\frac{\dot{a}}{a}\rho_1 &= 0 \end{aligned} \tag{4.18}$$

From the last equation above, we can extract the dynamical equation for $\dot{\rho}_0$:

$$\dot{\rho}_0 = -(D+1)H\rho_0 - \dot{\rho}_1 - D(1+w)\rho_1 H. \tag{4.19}$$

The next effect we must consider is the decay of excited strings to the fundamental level. In such process, energy associated with the matter species corresponding to excited levels will flow to radiation-like matter.

$$\frac{dE_1}{dt} = -KE_1. \tag{4.20}$$

This can be rewritten as

$$\Rightarrow \frac{d}{dt}(\rho_1 a^D) = -K\rho_1 a^d. \tag{4.21}$$

We finish the description of the system with FRW equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi}{3}(\rho_0+\rho_1). \tag{4.22}$$

We can take the time derivative of this last equation, to obtain:

$$2H\dot{H} = \frac{8\pi}{3}(\dot{\rho_0} + \dot{\rho_1}). \tag{4.23}$$

4.4 The Dynamics

Let us begin our study of the system with equation (4.21). One can solve it to obtain the typical decay behavior.

$$\rho_1(t)a^D(t) = E_1(t=0)e^{-Kt} = \rho_1(t=0)a(t=0)^d e^{-Kt}. \tag{4.24}$$

However, for now let us just use it to extract an equation for $\dot{\rho_1}(t)$. Expanding the left-hand side of this equation, we obtain:

$$\frac{d}{dt}(\rho_1 a^d) = \dot{\rho_1}a^d + Da^{(D-1)}\dot{a}\rho_1 \tag{4.25}$$

Plugging this back into equation (4.21), we obtain:

$$\dot{\rho_1} = -K\rho_1 - D\frac{\dot{a}}{a}\rho_1 = -K\rho_1 - DH\rho_1. \tag{4.26}$$

We can now plug equation (4.19) and equation (4.26) into equation (4.23). This yields

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi}{3}(-(D+1)H\rho_0 - D(w+1)H\rho_1) \\ \Rightarrow \dot{H} &= -\frac{4\pi}{3}((D+1)\rho_0 + D(1+w)\rho_1) \end{aligned} \tag{4.27}$$

We are interested in finding out if it is possible that inflation occurs in such setup, that is, if it is possible to obtain a growth of the scale factor as $\ddot{a}$(t)> 0. This condition can be rewritten in terms of a slow-roll parameter ξ as

$$\xi = \frac{\dot{H}}{H^2} > -1. \tag{4.28}$$

We can find out how to obtain this condition by dividing equation (4.27) by FRW equation equation (4.22):

$$\frac{\dot{H}}{H^2} = -\frac{(D+1)\rho_0 + D(1+w)\rho_1}{2(\rho_0+\rho_1)} \tag{4.29}$$

so that the condition equation (4.28) becomes

$$\frac{(D+1)\rho_0 + D(1+w)\rho_1}{2(\rho_0+\rho_1)} < 1. \tag{4.30}$$

Rearranging the terms, we find out that for inflation to occur the following relation between ρ_1 and ρ_0 must be respected:

$$\rho_0 < \frac{2 - D(1+w)}{D-1}\rho_1 \tag{4.31}$$

It is left to us, then to find out in which regime of the cosmological evolution is this condition satisfied.

4.4.1 A possible realization: Decay of winding modes

Recall from the string spectrum equation (3.40) that the contribution of the winding modes to the energy is given by

$$E_w^2 = \frac{1}{\alpha'}\sum_i m_i^2 R_i^2 \tag{4.32}$$

where m_i is the excitation level of the winding mode and R_i is radius of the compact dimension, both for the ith dimension. Considering an isotropic initial situation in which the gas of winding modes is uniformly distributed in all spatial dimensions so that $R_i = a \forall i$ and $V \sim a^D$. We can obtain the equation of state of the gas by assuming an adiabatic evolution

$$T\,dS = d(\rho_w) + p\,dV = 0 \tag{4.33}$$

which gives

$$p = -\left(\frac{\partial \rho V}{\partial V}\right)_{S=\text{const}}. \tag{4.34}$$

For the winding modes, we can take the square root of the spectrum to find $E_w \sim a$. But since we also have $E_w \sim \rho_w a^d$, we must have $\rho_w \sim a^{1-d}$ and

$$p_w = -\frac{\partial E_w}{\partial a}\frac{da}{dV} = -\frac{E_w}{DV} = -\frac{1}{D}\rho_w. \tag{4.35}$$

That is, for winding modes we have $w = -1/D$. Inserting this equation of state into equation (4.31), we obtain

$$\rho_0 < \frac{2 - D(1 - \frac{1}{D})}{D - 1}\rho_1$$
$$\Rightarrow \rho_0 < \frac{3 - D}{D - 1}\rho_1. \tag{4.36}$$

We are then left with three possible situations:

- D<3: In this case, the inflation condition will be satisfied provided that there is enough energy on the excited winding modes. Since our system begins at a very high temperature (close to β_H), a high number of excited strings is expected. Inflation will take place during the decays of such modes. Once most of the energy has decayed onto radiation-like matter, the condition is no longer satisfied and inflation ends, leaving us with the cosmological evolution of a standard FRW universe dominated by radiation.

- D=3: This is a marginal case. We would need $\rho_0 \approx 0$, that is, a huge amount of energy would be required to lie in the winding modes for inflation to take place.

- D>3: In this case, inflation is not possible, since a negative energy density would be required for the radiation-like modes. In this sense, we have obtained $D = 3$ as a critical dimensionality for decay-driven inflation to take place on the scenario.

In what follows we will obtain a numerical solution of such realization as an example. First, consider equation equation (4.24). Plugging this result into equation (4.26), we obtain

$$\dot{\rho} = -(D+1)H\rho + K\frac{E_{1_0}e^{-Kt}}{a^D} - DwH\frac{E_{1_0}e^{-Kt}}{a^D}. \tag{4.37}$$

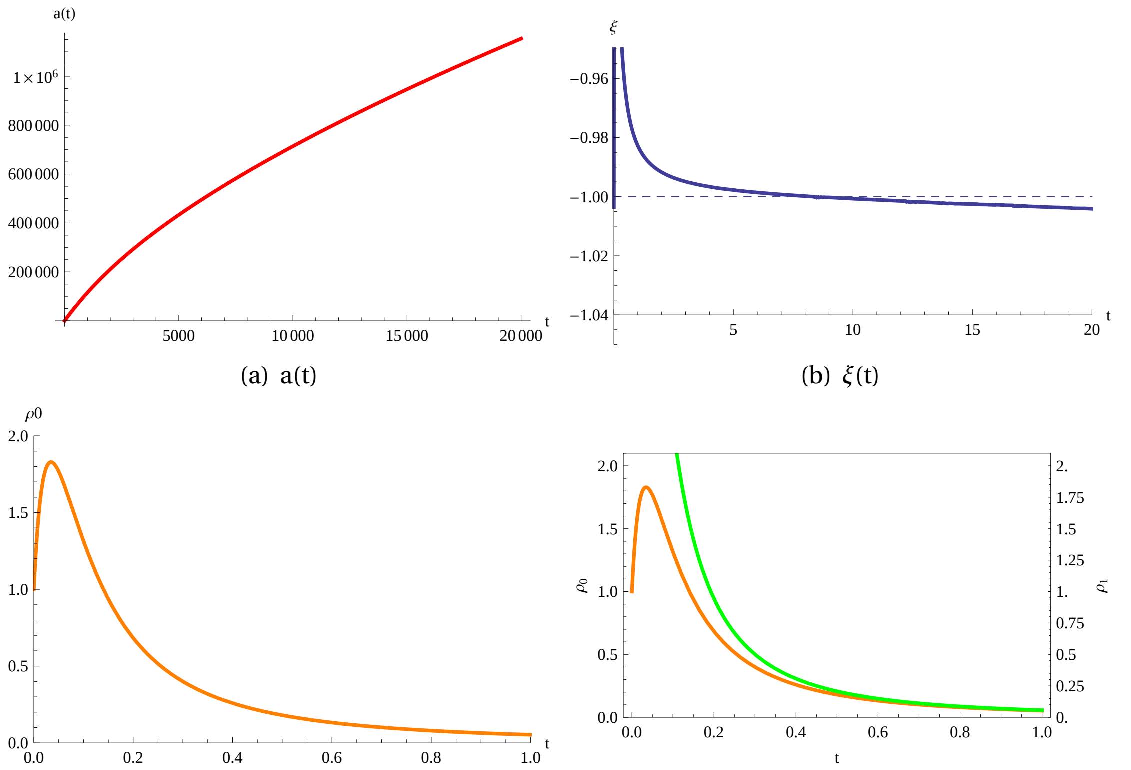
where we have used the notation $E_{1_0} = E_1(t = 0)$.

We can also plug equation (4.24) into equation (4.22) to obtain:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\left(\rho_0 + \frac{E_{1_0}e^{-Kt}}{a^d}\right). \tag{4.38}$$

We can insert back the conditions for inflation we have obtained and numerically solve the system comprised of equations equation (4.37) and equation (4.38). In this example, let us take the case $D = 2$, $w = -(1/2)$. Plots of a numerical solution presented in Figure 4.1.

Note from Figure 4.1 the behavior of the scale factor a(t): in the beginning there will be a region where the condition $\xi < -1$ is satisfied. In this region there is inflation. At a posterior time ξ decreases and becomes smaller than -1.



(a) a(t) (b) ξ(t)

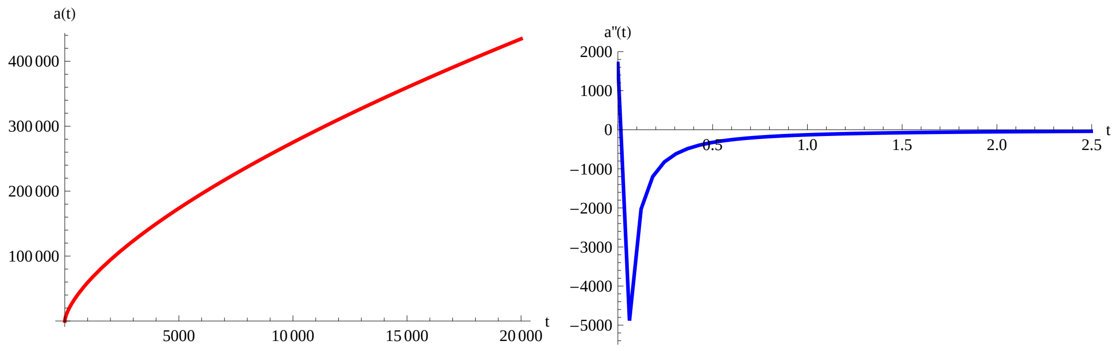
(c) ρ_0(t) (d) ρ_0(t) and ρ_1(t)

Figure 4.1 Plots from the numerical solution for the case $D = 2$. We have set as initial conditions $a(t = 0) = 10$ and ρ_0=1. We have also tuned $E_{1_0} = 1000$ and $k = 0.001$.

Note also the behavior of ρ_0: first it will increase as a result of the decay from excited levels. Then, after it reaches a peak, it will start to decrease, as a result of the cosmological redshift. If we plug $D = 2$ into equation (4.31) we obtain $\rho_0 < \rho_1$ as the inflation condition for $D = 2$. Since ρ_1(t) falls off more rapidly than ρ_0(t), at some time radiation will dominate. At this point ξ becomes smaller than -1, the concavity of the scale factor a(t) becomes negative and inflation ceases. The late time evolution is that of a radiation dominated universe.

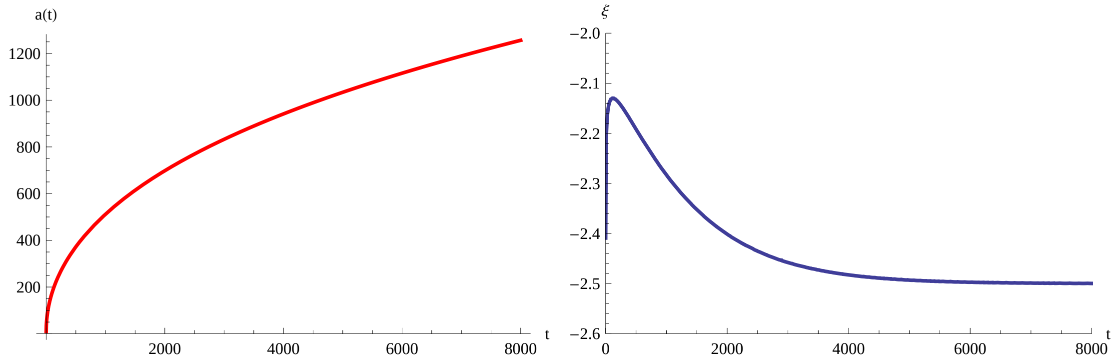
A similar behavior is obtained for the limiting case $D = 2.99$ as shown in Figure 4.2.

On the other side, let us now obtain the evolution of the a(t) and ξ(t) for the case $D = 4$. These are presented in Figure 4.3. Note that this time the parameter ξ is always smaller than -1 and there is no inflationary epoch.



(a) a(t) (b) $\ddot{a}$(t)

Figure 4.2 Plots of a(t) and $\ddot{a}$(t) for the limiting case $D = 2.99$



(a) a(t) (b) ξ(t)

Figure 4.3 Plots of a(t) and ξ(t) for the case $D = 4$ and for the same parameters of Figure 4.1

4.5 Parameter Tuning and Initial Conditions

To establish contact between our model and phenomenology, we must tune two parameters of the system. The first one is the Hagedorn Temperature $\beta_H = 1/4\pi\alpha'$, that controls the initial conditions for both $\rho_0(t)$ and $rho_1(t)$. The other parameter is the decay rate K. It will depend on the value of the dilaton field zeroth mode ϕ_0 and the string coupling constant g_c.

Tuning the value of α', ϕ_0 and g_c, we tune β_H and K and hence adjust the initial conditions for ρ_0 and ρ_1 and the decay rate of the excited modes.

In what follows we shall obtain the initial conditions for ρ_0 and ρ_1.

4.5.1 Initial Conditions for the Modes

As we have already seen, string theory has a natural limiting temperature, so the SBB picture of a infinitely hot early universe must be replaced by a finite temperature early universe.

Since at this limiting temperature string energy diverges, some kind of cutoff for the temperature is required. A natural cutoff when speaking of strings is to require that the average energy per string is in the Planck scale. With this ansatz, one is able to calculate the probabilities $P_1(t \to 0)$ and $P_i(t \to 0)$.

For a single string, we have

$$Z = C\left(\frac{\beta}{\beta - \beta_H}\right) \Rightarrow \log Z = \log C + \log \beta - \log(\beta - \beta_H)$$

so that

$$\overline{E} = -\frac{\partial \log Z}{\partial \beta} = \frac{\beta_H}{\beta(\beta - \beta_H)} \tag{4.39}$$

which for $\beta \approx \beta_H$ acquires the same form of the full gas

$$\overline{E}_{str} \sim \frac{1}{\beta - \beta_H}. \tag{4.40}$$

Our cutoff condition is that $\overline{E} \sim E_{pl}$. This yields a cutoff temperature β_c

$$E_{pl} = \frac{1}{\beta_c - \beta_H} \Rightarrow \beta_c = \beta_H + \frac{1}{E_{pl}}. \tag{4.41}$$

The next step is to calculate the string partition function at this cutoff temperature. It is given by

$$\begin{aligned} Z_{str}(\beta = \beta_c) &= C\left(\frac{\beta_c}{\beta_c - \beta_H}\right) = C\left[\frac{\beta_H + 1/E_{pl}}{1/E_{pl}}\right] \\ \Rightarrow Z_{str} &= C[1 + \beta_H E_{pl}] \end{aligned} \tag{4.42}$$

.

At a fixed temperature (or equivalently in this scenario at fixed time), the probability of a state E_n being occupied is given by

$$P(E_n) = \frac{e^{-\beta E_n}}{Z}. \tag{4.43}$$

Thus at cutoff temperature we will find

$$\begin{aligned} P(E_0)_{\beta_c} &= \frac{e^{-\beta_c E_1}}{C[1 + \beta_H E_{pl}]} \\ P(E_1)_{\beta_c} &= \frac{e^{-\beta_c E_i}}{C[1 + \beta_H E_{pl}]} \end{aligned} \tag{4.44}$$

The energy amount in each level i at time $t = 0$ will be given by

$$E_i(t=0) = P_i(t=0)E = P_i(t=0)\overline{E}N. \tag{4.45}$$

Before proceeding, let us make the definition

$$\gamma = \frac{1}{C[1+\beta_H E_{pl}]}.$$

With this definition, the two components of the content matter at time $t = 0$ will have the energies given by

$$\begin{aligned} E_0(t=0) = P_0(t=0)\overline{E}N = \left[e^{-\beta_c E_0}\gamma_0\right]\left[\frac{1}{\beta_c - \beta_H}\right]N \\ E_1(t=0) = P_1(t=0)\overline{E}N = \left[e^{-\beta_c E_1}\gamma_0\right]\left[\frac{1}{\beta_c - \beta_H}\right]N \end{aligned} \tag{4.46}$$

where the notation γ_0 was used to represent γ taken at $t = 0$ [1]. These are the initial conditions for the system.

[1] Note that γ depends on the volume V through the term C

CHAPTER 5

Conclusion and Perspectives

The last century has witnessed the birth and development of Cosmology as a science. Since the pioneer works by Einstein, Hubble and Friedmann, Robertson and Walker in the beginning of the century to the astonishing discovery that we live in an universe with an accelerated expansion, huge advances have been made in an attempt to understand the most fundamental nature of the cosmos. The past decades have seen the emergence of the precision cosmology era. Together with theoretical developments in fundamental Physics, new satellites as the WMAP and the Planck open a window to even larger developments in Cosmology in the next decades.

This work was devoted to the study of the very young universe. In such regime, the energy density was so high that our current paradigm of theory of gravity breaks down. Since the large scale structure of the universe is driven by gravity, a cosmological model that is to be trusted on very early times must be constructed over a theory that remains valid on such times. Right now, we have a candidate for such theory: String Theory. Stringy-based cosmological models may shine a new light on cosmological questions that have been open for years, like singularities, the dimensionality of spacetime and a theory of inflation (or an alternative for it). Also, precision experiments in Cosmology may prove to be the perfect site for probing String Theory. Motivated by this, we used a stringy description of matter and String Theory as a theory of gravity to construct a model for the very early universe.

After a brief overview of Modern Cosmology and the Λ-CDM model on the first chapter, the second chapter was devoted to a proper study of Standard Big Bang Cosmology and the Inflationary Scenario. Under the assumptions of homogeneity and isotropy the FRW model was constructed and shown to agree with the experimental results obtained by Edwin Hubble, what marked the birth of Modern Cosmology. The problems of flatness, abundance of relics and horizon were presented and the inflationary scenario was presented as a possible solution for these issues.

The inflationary scenario, however, has its own issues, namely the fact that even though we have a handful of inflationary mechanisms, we haven't found so far a theory of inflation. Moreover, inflationary scenarios based on General Relativity have the same limitations of the theory itself. This makes a more fundamental theory needed. The third chapter of this work was devoted to introduce string cosmological models. After a motivation for choosing String Theory as a quantum theory of gravity, the evolution of strings in curved space-times was presented. Important attention was payed to the thermodynamics of strings at high temperatures, since strings have a limiting maximal temperature: the Hagedorn Temperature, which is relevant for cosmology. Other cosmologically relevant feature of String Theory, T-Duality, was presented. Then, the String Gas Scenario, that was the basis of our work, was introduced.

The fourth chapter was devoted to the study of our setup: the matter content of the universe is modeled as a gas of strings that begins very close to the Hagedorn temperature. This gas splits into two matter species: a gas of radiation, made of strings at the fundamental level and a gas of the excited string modes. The dynamics of the system is driven by the decay of the excited modes and by the cosmological redshift. The setup shows that the decay of winding modes may drive inflation in the scenario provided that the universe inflates in less than the critical spatial dimensionality $D = 3$. This may indicate a new clue to justify decompactification in 3 dimensions and explain the dimensionality of space-time.

As perspectives of further developments, one may try to conciliate this inflationary mechanism with the Brandenberger-Vafa decompactification scenario in the context of String Gas Cosmology and phenomenologically fine-tune the above results so that they yield the correct observed spectrum of particles. Then one should be able to construct an effective model that comprises both a possible explanation for the dimensionality problem of spacetime and an inflationary mechanism.

APPENDIX A

String Quantization

O binômio de Newton é tão belo quanto a Vênus de Milo. O que há é pouca gente para dar por isso. (Newton's binomial is as beautiful as the Venus de Milo. The truth is few people notice it.)

—FERNANDO PESSOA

In this appendix we shall discuss the quantization of String Theory. This is intended as a fundamental part of this work, specially aimed at those who are not familiar with String Theory, to whom this will work as an introduction to the subject. For a comprehensive review, the reader is referred to the canonical references, the books by Polchinski [57], Green, Scharwz and Witten[58] and Becker, Becker and Schwarz [59].

A.1 The Relativistic Point Particle

Let us begin studying the dynamics of a relativistic point particle. It is a pedagogical start point, since it will introduce some of the features that appear on String Theory.

A.1.1 Lagrangian and Symmetries

Consider a point particle of mass m moving over a D-dimensional Minkowski space $\mathbf{R}^{1,D-1}$ with signature

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, ..., +1)$$

By fixing an a frame with coordinates $X^\mu = (t, \vec{x})$ the action of the particle may be written as

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}} \tag{A.1}$$

One can check that this Lagrangian yields the conjugate momentum and energy known for a relativistic point particle

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}}, \qquad E = \sqrt{m^2 + \vec{p}^2} \tag{A.2}$$

However, although this Lagrangian is correct, is does not put time and space on equal foot: while the position is a dynamical degree of freedom, the time is merely a parameter that labels the position. Since Lorentz transformations mix up space and time, it would be good to find a new Lagrangian where such symmetries become evident.

The way to find such action is to use gauge symmetry to promote time to a degree of freedom of our action without it really being a true dynamical degree of freedom.

Take the action

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \tag{A.3}$$

where $\mu = 0, ..., D-1$ and $\dot{X}^\mu = dX^\mu / d\tau$. The new parameter τ we have introduced labels the position of the particle along the worldline described by its motion. Note that this action is simply the proper time $\int ds$ along the worldline.

It is important to remark that not all degrees of freedom in this action are physical. Note that this action is invariant under reparameterization: one can pick a parameter $\tilde{\tau} = \tilde{\tau}(t)$ on the worldline, so that $d\tau = d\tilde{\tau}|d\tau)d\tilde{\tau}$, $dX^\mu / d\tau = (dX^\mu / d\tilde{\tau})(d\tilde{\tau}/d\tau)$ and the action may be rewritten in the $\tilde{\tau}$ reparameterization

$$S = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}} \frac{dX^\nu}{d\tilde{\tau}} \eta_{\mu\nu}} \tag{A.4}$$

This reparameterization invariance is a gauge symmetry of the system and so it is really a redundancy in our description: although we seem to be dealing with D degrees of freedom X^μ, one of them is fake.

Also, the Poincaré symmetry of the particle becomes evident on this action, as a global symmetry on the worldline

$$X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu \tag{A.5}$$

where Λ is a Lorentz transformation that satisfies $\lambda^\mu_\nu \eta^{\nu\rho} \Lambda^\sigma_\rho = \eta^{\mu\sigma}$ and c^μ is a constant translation. Introducing a gauge symmetry into the system was the price to be paied to make all the symmetries of the action evident.

A.1.2 Ein Einbein

It is possible to describe the free relativistic particle using another action

$$S' = \frac{1}{2}\int d\tau(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2) \tag{A.6}$$

by introducing an additional field on the worldline, an independent worldline metric $\gamma_{\tau\tau}(\tau)$ and inserting in the action the einbein $\eta(\tau) = (-\gamma_{\tau\tau}(\tau))^{1/2}$ which is defined to be positive.

Note that this system is also invariant under Poincaré transformations and reparameterization, thus having the same symmetries as the action S.

In fact, it is possible to return to the action S by varying the action with respect to the einbein to obtain the equation of motion

$$\eta^2 = -\dot{X}^{\mu}\dot{X}_{\mu}/m^2 \tag{A.7}$$

and using this to eliminate $\eta(\tau)$ in S'.

What's the difference between S and S' since they are classically equivalent and exhibit the same symmetries? The fact is that S has a complicated form with derivatives inside a square root, what makes it hard to quantize. It is difficult to make sense of an action of this form on a path integral. On the other hand, S' is quadratic in the derivatives and its path integral is not as near as complicated as that of S, even though their quantum theories will be equivalent. This is a good reason to define the quantum theory from S', and not from S. A similar procedure will have to be carried with strings.

A.2 Action Principles

A.2.1 The Nambu-Goto Action

We have seen that a non-dimensional object (a point particle) will span a worldline while moving on space-time. In a similar fashion, a one-dimensional object will span two-dimension worldsheet. We will describe this surface in terms of two parameters, $X^{\mu}(\tau,\sigma)$ as in Figure A.1, where τ is a timelike coordinate and σ is a spacelike coordinate.

In this language, we usually refer to space-time as the *target space* to distinguish it from

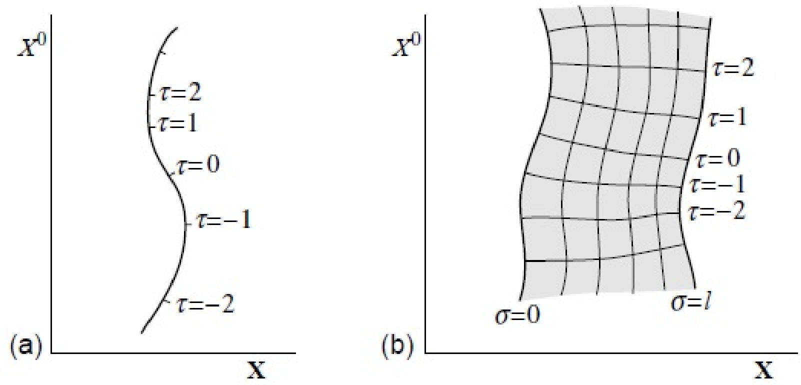


Figure A.1 Worldline and worldsheet generated by a non-dimensional and by a one-dimensional object that propagate on space-time. Figure Credits: Joseph Polchinski. *String Theory: An Introduction to the Bosonic String (Cambridge Monographs on Mathematical Physics)*, volume 1. 1998.

the worldsheet.

If we keep we mind what we have learned from the point particle case, it is natural to think that the simplest action that preserves the symmetries of equation (A.6) must be proportional to the area of the worldsheet. We define the induced metric h_{ab}

$$h_{ab} = \partial_a X^\mu \partial_B X_\mu \tag{A.8}$$

and write the Nambu-Goto action as

$$\begin{aligned} S_{NG} &= \int_M d\tau\, d\sigma \mathscr{L}_{NG} \\ \mathscr{L}_{NG} &= -\frac{1}{2\pi\alpha'}\sqrt{-\det h_{ab}} \end{aligned} \tag{A.9}$$

where M denotes the world-sheet and the term $1/2\pi\alpha'$ is the tension T of the string in terms of the Regge slope α' that has units of spacetime-length-squared.

Note that this action has two symmetries. The first one is the D-dimensional Poincaré group (the isometry group of flat spacetime)

$$X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu \tag{A.10}$$

where Λ^μ_ν is a Lorentz transformation and a^μ is a translation. The second one is diffeomorphism invariance: if we take new coordinates $(\tau'(\tau,\sigma), \sigma'(\tau,\sigma))$ the transformation is given by

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma). \tag{A.11}$$

A.2.2 The Polyakov Action

Just like the point particle action, the Nambu-Goto action has derivatives in the square root. To simplify it, we may introduce a world-sheet metric γ_{ab} of Lorentzian signature $(-,+)$, in order to obtain the metric

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{1/2}\gamma^{ab}\partial_a X^{\mu}\partial_b X_{\mu}. \tag{A.12}$$

Here, $\gamma = \det\gamma_{ab}$. This is the Polyakov action. By varying this action to the metric, and requiring $\delta_{\gamma}S_P[X,\gamma] = 0$ once can show that it is equivalent to the Nambu-Goto action. The Polyakov action preserves the symmetries of the Nambu-Goto action: the D-dimensional Poincaré invariance

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= \Lambda^{\mu}_{\nu}X^{\nu}(\tau,\sigma) + a^{\mu} \\ \gamma'_{ab}(\tau,\sigma) &= \gamma_{ab}(\tau,\sigma); \end{aligned} \tag{A.13}$$

and diffeomorphism invariance:

$$\begin{aligned} X'^{\mu}(\tau',\sigma') &= X^{\mu}(\tau,\sigma) \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^c}\gamma'_{cd}(\tau',\sigma') &= \gamma_{ab}(\tau,\sigma). \end{aligned} \tag{A.14}$$

In addition, the Polyakov action also has two-dimensional Weyl invariance, a local rescaling of the worldsheet metric:

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= X^{\mu}(\tau,\sigma) \\ \gamma'_{ab}(\tau,\sigma) &= \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma) \end{aligned} \tag{A.15}$$

where $\omega(\tau,\sigma)$ is arbitrary.

We turn our interest now to the equations of motion of this action. To simplify our calculations we may take advantage of the redundancies in gauge symmetries to choose proper coordinates. First, we use reparameterization invariance to fix a gauge for the Polyakov action. Since the worldsheet metric has three independent components, we can choose a valor to any two of them. Our choice will be to make the metric locally conformally flat:

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \tag{A.16}$$

where $\phi(\tau,\sigma)$ is some function on the worldsheet. This is known as the conformal gauge. We are left with Weyl invariance, so we can use this to remove the last independent component of the metric. We choose $g_{\alpha\beta}$, to obtain a flat metric on the worldsheet in Minkowski coordinates:

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \tag{A.17}$$

Note the importance of this result: one may use Weyl invariance to make any two-dimensional metric flat! With this choice, the Polyakov action will simplify to a theory of D free scalar fields:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X. \tag{A.18}$$

The equations of motion for X^μ reduce to the free wave equation:

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{A.19}$$

Since we have made a choice for the metric $g_{\alpha\beta}$, we must make sure that the equation of motion for $g_{\alpha\beta}$ is satisfied. In particular, the variation of the action with respect to the metric gives rise to its stress-energy tensor $T_{\alpha\beta}$:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}. \tag{A.20}$$

Setting $g_{\alpha\beta} = \eta_{\alpha\beta}$ we get

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X \tag{A.21}$$

and the associated equation of motion is simply $T_{\alpha\beta}$ or, more explicitly:

$$\begin{aligned} T_{01} &= \dot{X} \cdot X' = 0 \\ T_{00} = T_{11} &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \end{aligned} \tag{A.22}$$

These work as constraints on the equations of motion, that is, the equations of motion of the string are the free wave equation (A.19) subject to the constraints equation (A.22).

A.2.3 Mode Expansion

Now we turn to solve the equations of motion. We begin by introducing lightcone coordinates on the worldsheet

$$\sigma^{\pm} = \tau \pm \sigma \tag{A.23}$$

so that the equations of motion will read

$$\partial_+ \partial_- X^{\mu} = 0. \tag{A.24}$$

This equation has a general solution given by

$$X^{\mu}(\sigma, \tau) = X_L^{\mu}(\sigma^+) + X_R^{\mu}(\sigma^-) \tag{A.25}$$

where X_L^{μ} and X_R^{μ} are arbitrary functions that describe left-moving and right-moving waves respectively. Beyond the constraints (A.22) these must obey the periodicity condition

$$X^{\mu}(\sigma, \tau) = X^{\mu}(\sigma + 2\pi, \tau). \tag{A.26}$$

The natural next step, therefore, is to expand the most general periodic solution in Fourier modes:

$$\begin{aligned} X_L^{\mu}(\sigma^+) &= \frac{1}{2} x^{\mu} + \frac{1}{2} \alpha' p^{\mu} \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^{\mu} e^{-in\sigma^+}, \\ X_R^{\mu}(\sigma^-) &= \frac{1}{2} x^{\mu} + \frac{1}{2} \alpha' p^{\mu} \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-in\sigma^-} \end{aligned} \tag{A.27}$$

where α' and $1/n$ are normalization factors chosen for later convenience and the variables x^{μ} and p^{μ} are the center of mass position and momentum of the string. Note that due to the reality of X^{μ} the coefficients of the Fourier modes α_n^{μ} and $\tilde{\alpha}_n^{\mu}$ must obey

$$\alpha_n^{\mu} = (\alpha_{-n}^{\mu})^*; \qquad \tilde{\alpha}_n^{\mu} = (\tilde{\alpha}_{-n}^{\mu})^*. \tag{A.28}$$

We are still required to satisfy the constraints (A.22). Again making use of lightcone coordinates on the worldsheet,

$$\sigma^{\pm} \tau \pm \sigma$$

they become

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0. \tag{A.29}$$

An explicit calculation gives

$$\partial_- = \partial_- X_R^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-}. \tag{A.30}$$

If we take the sum above to be over all $n \in \mathbb{Z}$ and define $\alpha_0^\mu \equiv \sqrt{\alpha'/2} p^\mu$, this may be rewritten as

$$\partial_- X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^-} \tag{A.31}$$

so that

$$\begin{aligned}(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} \\ \equiv \alpha' \sum_n L_n e^{-in\sigma^-} &= 0\end{aligned} \tag{A.32}$$

where the sum of the oscillator modes is defined as

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m. \tag{A.33}$$

Analogous definitions can be made for the left moving modes:

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m, \qquad \tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{A.34}$$

Thus, we are left with an infinite number of constraints

$$L_n = \tilde{L}_n = 0 \qquad n \in \mathbb{Z}. \tag{A.35}$$

Note that L_0 and $\tilde{L}_0$ both include the square of the spacetime momentum p^μ. Since in Minkowski space the square of the spacetime momentum is the square of the rest mass of a particle

$$p_\mu p^\mu = -M^2 \tag{A.36}$$

the L_0 and $\tilde{L}_0$ constraints tell us the effective mass of a string in terms of the excited oscillator modes:

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}. \tag{A.37}$$

Since $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}p^\mu$, we have two expressions for the invariant mass (one in terms of the right-moving oscillators α_n^μ and one in terms of the left-moving oscillators $\tilde{\alpha}_n^\mu$) that must be equal to each other. This is known as *level matching* and will be important in the string quantization.

A.3 String Quantization

A.3.1 The Lightcone Gauge

In the following, we will develop the quantization of String Theory. Since String Theory is a gauge theory, we may do this in some different ways. The approach we are going to follow is the so called lightcone gauge quantization. The idea here is to first solve all the constraints of the system to determine the space of physically distinct classical solutions and then quantize these solutions.

The first point to be noted is that even after we fix the gauge to the worldsheet to $g_{\alpha\beta} = \eta_{\alpha\beta}$ we still have some gauge freedom: any coordinate transformation $\sigma \to \tilde{\sigma}(\sigma)$ that changes the metric by $\eta_{\alpha\beta} \to \Omega^2(\sigma)\eta_{\alpha\beta}$ can be undone by a Weyl transformation. Consider, for example, the lightcone coordinates on the worldsheet

$$\sigma^\pm = \tau \pm \sigma.$$

The flat metric on the worldsheet will take the form

$$ds^2 = -d\sigma^+ d\sigma^-. \tag{A.38}$$

The result of transformations of the form $\sigma^+ \to \tilde{\sigma}^+(\sigma^+)$ or $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$ will be an additional overall multiplicative factor on the flat metric, that may be undone by a proper Weyl transformation.

Our objective is to fix this reparameterization invariance. We begin by introducing spacetime lightcone coordinates:

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \tag{A.39}$$

so that the spacetime Minkowski metric reads

$$ds^2 = -2dX^+ dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \tag{A.40}$$

With this notation, the solution to the equation of motion for X^+ is given by

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-). \tag{A.41}$$

We obtain the *lightcone gauge* by choosing coordinates such that

$$\begin{aligned} X_L^+ &= \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+, \\ X_R^+ &= \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-, \end{aligned} \tag{A.42}$$

so that

$$X^+ = x^+ + \alpha' p^+ \tau. \tag{A.43}$$

More than just fix the reparameterization invariance, this choice of gauge will also simplify the constraint equations. The equation of motion of X^+, for instance, will read

$$\partial_+ \partial_- X^- = 0, \tag{A.44}$$

and can be solved through the usage of our usual ansatz

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-). \tag{A.45}$$

A notable fact about the lightcone gauge is that X^- will be completely determined by the constraints. Constraints (A.29), for example, which read

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{A.46}$$

in the lightcone gauge (A.43) become

$$\begin{aligned} \partial_+ X_L^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \\ \partial_- X_R^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i. \end{aligned} \tag{A.47}$$

The function $X^-(\sigma^+, \sigma^-)$ will be completely determined in terms of the other fields, except for an integration constant. To see this more clearly, let us write the mode expansion for X_L^- and X_R^-:

$$
\begin{aligned}
X_L^-(\sigma^+) &= \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+}, \\
X_R^-(\sigma^-) &= \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^- e^{-in\sigma^-}
\end{aligned}
\tag{A.48}
$$

Note that p^-, α_n^- and $\tilde{\alpha}_n^-$ are all fixed by the constraints (A.47) and that x^- is the undetermined integration constant. Let us use this to obtain the level matching condition.

We can read off p^- from α_0^- by two forms. In the first one, we use the fact that

$$
\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i \tag{A.49}
$$

which applied to our definition $\alpha_0^- = \sqrt{\alpha'/2p^-}$ yields

$$
\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i \right). \tag{A.50}
$$

The second way is to consider the α_0^- equation arising from the first of equation (A.47), which reads

$$
\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \right). \tag{A.51}
$$

These two equations yield the level matching conditions

$$
M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i. \tag{A.52}
$$

Note that this time the sum is over oscillators α^i and $\tilde{\alpha}^i$ only for $i = 1, ..., D-2$. This means that if the α^i modes, all the other oscillator modes are determined. In this sense, these are the physical excitations of the string. They are called the *transverse oscillators*. So the most general solution is described in terms of $2(D-2)$ transverse oscillator modes together with a number of zero modes describing the center of mass and momentum of the string. Note, however, that p^- is not a dynamical variable, but it is fixed by the constraints.

Once these degrees of freedom have been identified, we may now proceed to quantization.

A.3.2 Quantization

String quantization is made by imposing commutation relations:

$$\left[x^i, p^j\right] = i\delta^{ij}, \quad [x^-, p^+] = -i, \quad [x^+, p^-] = -i$$
$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0} \tag{A.53}$$

To construct the Hilbert space of states we define a vacuum state $|0; p>$ such that

$$\hat{p}^\mu|0; p >= p^\mu|0; p^\mu >, \quad \alpha_n^i|0; p >= \tilde{\alpha}_n^i|0; p >= 0 \text{ for } n > 0 \tag{A.54}$$

and build a Fock space by acting with the creation operators α_{-n}^i and $\tilde{\alpha}_{-n}^i$ with $n > 0$. These operators will be only the transverse operators defined above.

Remember, however, that p^- is not an independent variable as stated above. Because of that, constraints equation (A.50) and equation (A.51) must be imposed by hand as operator equations that define the physical states.

In the classical theory, these constraints are equivalent to the mass-shell condition equation (A.52). However, in the quantum theory there's an ordering ambiguity in the sum over oscillator modes. The way out of this is to choose all operators to be normal ordered. Then, this ambiguity in the sum will appear as an overall constant a that we will determine. With this considerations, the spectrum of states in the lightcone gauge is given by

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i - a\right). \tag{A.55}$$

We define the operators

$$N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i, \quad \tilde{N} = \sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i \tag{A.56}$$

which are related to the number operators of the harmonic oscillator, so that

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a). \tag{A.57}$$

The value and N and $\tilde{N}$ is usually called the level (and now the expression *level matching* makes sense). Now we are left to determine the value of a.

If we forget about normal ordering for a second, we may rewrite

$$\frac{1}{2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i = \frac{1}{2}\sum_{n<0}\alpha_{-n}^i\alpha_n^i + \frac{1}{2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i \tag{A.58}$$

where the summation over $i = 1, ..., D-2$ is implicit. To put this in normal ordered form, we need to correct the second term in the right-hand side of the equation above. Normal ordering gives

$$\begin{aligned}\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n &= \frac{1}{2}\sum_{n<0}[\alpha^i_n\alpha^i_{-n} - n(D-2)] + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \\ &= \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \frac{D-2}{2}\sum_{n>0} n.\end{aligned} \tag{A.59}$$

The last term diverges and need to be regularized. This may be done through a method known as *zeta function regularization*: consider the more general sum

$$\sum_{n=1}^{\infty} n^{-s}. \tag{A.60}$$

For Re(s) > 1, this sum converges to a function known as the Riemann Zeta function $\zeta(s)$. This function has a unique analytic continuation to the point $s = -1$, where we get $\zeta(-1) = -1/12$.

Inserting

$$\sum_{n=1}^{\infty} n = -1/12$$

in equation (A.59), we we obtain the ordering constant a to be

$$a = \frac{D-2}{24}.$$

With this result, we are finally able to write the string spectrum:

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right). \tag{A.61}$$

A.3.3 Analysis of Spectrum

Now we shall proceed to the analysis of the string spectrum. We begin by the ground state $|0;p\rangle$. No oscillators are excited, and then

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6}. \tag{A.62}$$

This is a negative mass-squared! These are called tachyons.

The first excited states by acting on the ground state with the creation operators, obeying the level matching condition. This gives us $(D-2)^2$ particle states

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0;p\rangle. \tag{A.63}$$

For each of these states the mass is given by

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) \tag{A.64}$$

The requirement of Lorentz invariance will determine the number of dimensions D: for a massive particle, one can always go to the rest frame $p^\mu = (m, 0, ..., 0)$ and the internal states will form a representation of the spatial rotation group SO(D-1). For a massless particle, however, there is no rest frame. We can, however, choose a spacetime momentum for the particle of the form $p^\mu = (p, 0, ..., 0, p)$. In this case, the particles fill out a representation of the little group SO(D-2).

Thus, in D dimensions a massive vector particle has $D-1$ spin states while a massless vector needs only $D-2$. At first level, we have only the $D-2$ states $\alpha^i_{-1}|0; k>$, so these must be massless, what implies

$$D = 26.$$

The states (A.63) will then transform in the $24 \otimes 24$ representation of $SO(24)$. These can be decomposed into three irreducible representations:

traceless symmetric $\oplus$ anti-symmetric $\oplus$ singlet (=trace).

Each on these modes can be associated to a massless field in spacetime such that the string oscillation may be identified with a quantum of these fields. Namely, the anti-symmetric tensor field is usually called anti-symmetric field, 2-form field or *Kalb-Ramond* field $B_{\mu\nu}(X)$. The scalar field is the dilaton $\phi(X)$. The particle in the symmetric traceless representation of $SO(24)$ is a massless spin 2 particle. There are arguments by Feynman and Weinberg [60] that any interacting massless spin two particles must be equivalent to General Relativity. Because of that, this field is identified with the $G_{\mu\nu}$ metric field of spacetime.

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